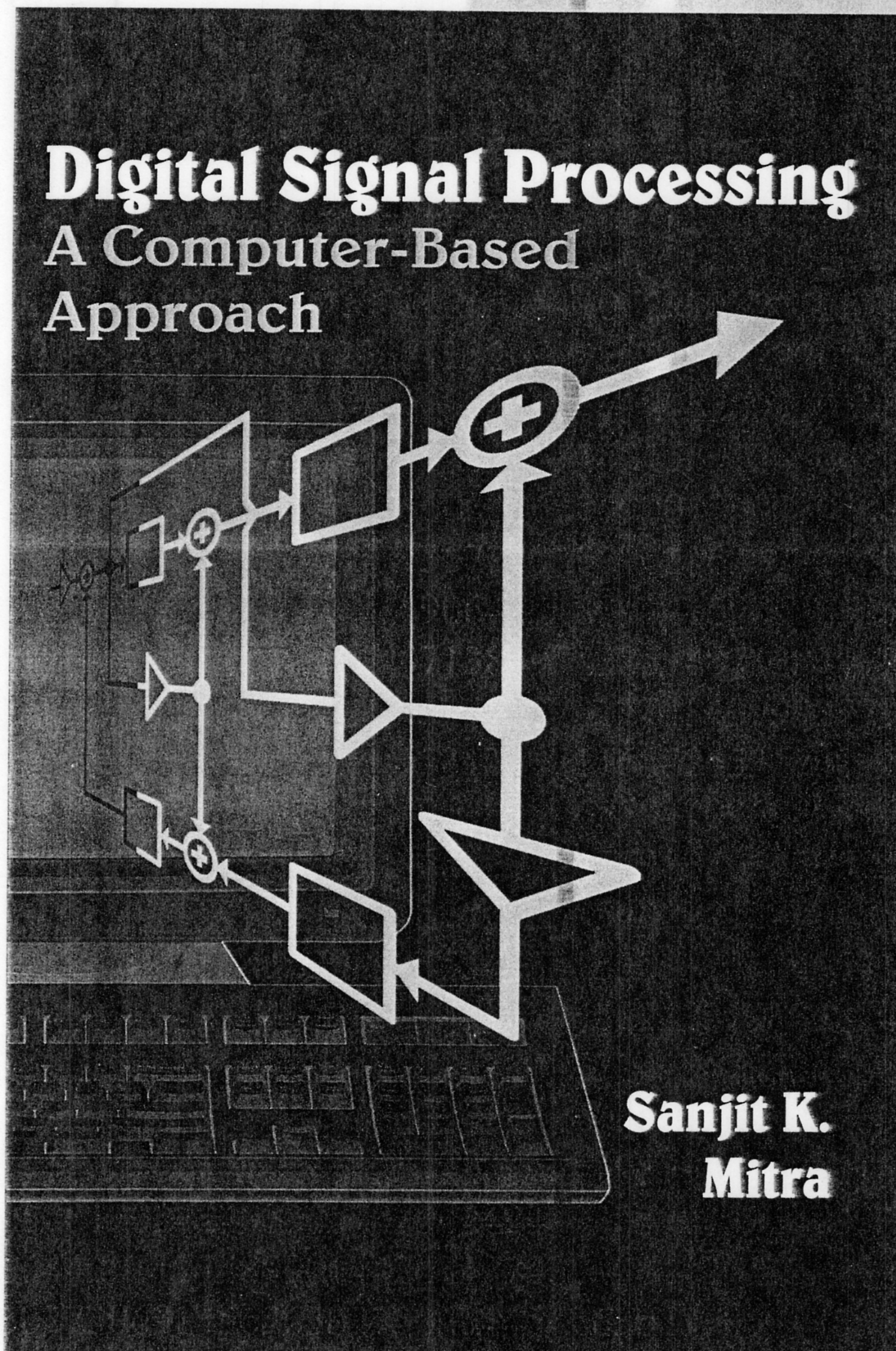
Solutions Manual

to accompany



Prepared by
Rajeev Gandhi

DIGITAL SIGNAL PROCESSING
A Computer-Based Approach

Sanjit K. Mitra

Text Errata List

1. Page xii, Line 17: Replace "Short-term" with "Short-time".
2. Page xiv, Line 4: Replace "**ece.ucsb.edu**" with "**iplserv.ece.ucsb.edu**".
3. Page xvi, Line 6: Replace "5.5" with "8.5".
4. Page 13, Line 2: Replace "$A\sin(\Omega_{0t})$" with "$A\sin(\Omega_0 t)$".
5. Page 25, Interchange Figure 1.22 (a) and Figure 1.22 (b).
6. Page 125, Eq. (3.10): Replace "$\sum_{n = mult\ of\ L}^{\infty}$" with "$\sum_{n = mult\ of\ L}$".
7. Page 127, Table 3.2, First line: Replace " **Length-N sequence** " with " **Sequence** " and replace " **N-point DFT** " with " **DTFT** ".
8. Page 127, Table 3.2, Line 5 from top: Replace "$e^{-j\omega n_0}G(j\omega)$" with "$e^{-j\omega n_0}G\left(e^{j\omega}\right)$".
9. Page 131, Eq. (3.13) Replace "$e^{-2\pi kn/N}$" with "$e^{-j2\pi kn/N}$".
10. Page 148, Eq. (3.75) and Eq. (3.76): Replace "L" with "$L-1$".
11. Page 155, Eq. (3.88) Replace "$(N-M)$" with $(N-M+1)$".
12. Page 175, Line 13 from bottom: Replace "$\frac{-1}{3}/\left(1+\frac{1}{3}z^{-1}\right)$" with "$\frac{-1}{3}z^{-1}/\left(1+\frac{1}{3}z^{-1}\right)$".
13. Page 175, Line 12 from bottom: Replace "$-3n\left(\frac{-1}{3}\right)^n\mu[n]$" with "$-3(n-1)\left(\frac{-1}{3}\right)^{(n-1)}\mu[n-1]$".
14. Page 175, Line 10 from bottom: Replace the equation with "$g[n] = \left[0.24\left(\frac{-1}{3}\right)^n + 0.36\left(\frac{1}{2}\right)^n\right]\mu[n] + 0.36(n-1)\left(\frac{-1}{3}\right)^n\mu[n-1]$".
15. Page 183, Eq. (3.149): Replace "$\tilde{h}[k]$" with "$\tilde{h}[n]$".
16. Page 184, Replace "Figure P3.2 shows four" with "Figure P3.2 shows two".
17. Page 184, Line 3 from bottom: Replace "length-sequence" with "length-N sequence".
18. Page 197, Problem M3.3: In the numerator of expression in (a), replace "$0.1915e^{j\omega}$" with "$0.1915e^{-j\omega}$", and "$0.1915e^{-j\omega}$" with "$0.1915e^{-3j\omega}$".
19. Page 198, Problem M3.17: Part (iii) - Replace "$|z| > 0.4$" with "$|z| > 0.9486833$".
20. Page 198, Problem M3.17: Part (iv) - Replace "$|z| > 0.4$" with "$|z| > 0.5$".
21. Page 211, Line below Eq. (4.48) Replace "$e^{j2\pi kM}$" with "$e^{j2\pi k/M}$".
22. Page 211, Line 2 from bottom: Replace "3.20" with "3.21".
23. Page 219, Line 8 from bottom: Replace "$H_{LP}[n]$" with "$h_{LP}[n]$".
24. Page 220, Line 8 from bottom: Replace "$\log_{10}|H_1(e^{j0})|$" with "$\log_{10}|H_0(e^{j0})|$".
25. Page 222, Eq. (4.68): Insert a "j" in front of "$e^{-j\omega/2}\sin\left(\frac{\omega}{2}\right)$".

26. Page 230, Figure 4.19: Replace "$v[\pm n]$" with "$v[-n]$" and replace "$w[\pm n]$" with "$w[-n]$".

27. Page 247, Eq. (4.139b): Replace "$A_0(z) + A_1(z)$" with "$A_0(z) - A_1(z)$".

28. Page 252, Line 5 from the begining of section 4.8: Replace "zero locations" with "pole locations".

29. Page 254 After Line 4: Replace "$\prod_{i=1}^{M} \lambda_i$" with "$(-1)^M \prod_{i=1}^{M} \lambda_i$".

30. Page 271 Problem 4.61 (b) line 3: Replace "$(4\pi + \omega_s)/M$" with "$(2\pi + \omega_s)/M$".

31. Page 274 Problem 4.67 (c) Replace "$\widehat{G}(\omega)$" with "$\widehat{H}(\omega)$".

32. Page 275 Problem 4.76 (a), (b) and (c): Replace "H_{BS}" with "$H_{BS}(z)$".

33. Page 276 Problem 4.82 (d) Replace "H_3" with "$H_3(z)$".

34. Page 277, Problem 4.88: Replace "H_a" with "$H_a(z)$", and "H_b" with "$H_b(z)$".

35. Page 277, Problem 4.88 (b): Replace "$6\left(1 + z^{-1}\right)^3$" in the numerator with "$3\left(1.5 + 6.5z^{-1} + 6.5z^{-2} + 1.5z^{-3}\right.$".

36. Page 280, Line 5 from bottom: Replace "4.84" with "4.83".

37. Page 280, Line 2 from bottom: Replace "4.97" with "4.96".

38. Page 290, Last Line of Figure caption: Replace "$\cos(2\pi t)$" with "$\cos(26\pi t)$".

39. Page 340, Eq. (6.3b): Replace "d" with "D".

40. Page 314, Line 12 from bottom: Replace "9.4" with "6.4".

41. Page 327, Eq. (5.65): Replace "$2R_L + R$" in the denominator by "$2(R_L + R)$".

42. Page 374, Line 3 from bottom: Replace "$A_m(z)$" with "$A_M(z)$".

43. Page 393, Line 1: Replace "0.39" by "0.3π".

44. Page 415, Problem M 6.2 (b) and Problem M 6.3 (b) : Replace "four" by "two".

45. Page 415, Problem M 6.8 (c) Replace numerator of G(z) with "$0.04934436\left(1 + 1.3z^{-1} + 2.21z^{-2} + 2.21z^{-3} + 1.3z^{-4} + z^{-5}\right)$".

46. Page 547, Example 8.20: Replace "$A = -0.325_{10}$" with "$A = -0.375_{10}$".

47. Page 557, Line 5 from bottom: Replace "Eq. (8.84)" with "Eq.(8.88)".

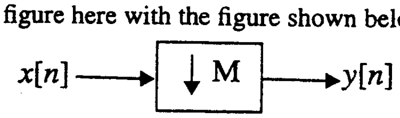
48. Page 568, Problem 8.44: Replace "$\sum_{i=1}^{b} a_{-1}2^{-i}$" with "$\sum_{i=1}^{b} a_{-i}2^{-i}$".

49. Page 570, Figure P8.5(a): Replace "$\frac{1+z^{-1}}{2}$" with "$\frac{1+z^{-2}}{2}$".

50. Page 577, Line 2 from top: Replace $\sum_{i=1}^{\beta} a_{-i}2^{-1}$ with $\sum_{i=1}^{\beta} a_{-i}2^{-i}$.

51. Page 579, Last Line: Replace "[Mit74a]" with "[Mit74c]".

52. Page 659, Figure 10.4: Replace the figure here with the figure shown below:



53. Page 669, Figure 10.15(b): Replace "L" with "M".

54. Page 678, Example 10.7 line 5: Replace "$f_1 = 0.31$" with "$f_2 = 0.31$".

55. Page 695, Lines 1-3 below Eq. (10.61): Replace this sentence with "As a result, $y[Ln+k] = \alpha x[n]$, i.e., the input samples appear at the output without any distortion at all values of n, whereas the in-between $(L-1)$ samples are determined by interpolation."

56. Page 707, Equation 10.109 and the equation two lines below: Replace "$\left(1-\left|H_0\left(e^{j\omega}\right)\right|^2-\left|H_1\left(e^{j\omega}\right)\right|^2\right)$" with "$\left(1-\left|H_0\left(e^{j\omega}\right)\right|^2-\left|H_1\left(e^{j\omega}\right)\right|^2\right)^2$".

57. Page 731, Problem 10.14 (b), (c): In the denominator, replace "$0.8z^{-1}$" with "$0.8z^{-2}$"

58. Page 732, Problem 10.19: Replace "$\sum_{k=0}^{M-1}\left|H_k\left(e^{j\omega}W^k\right)\right|^2$" with "$\sum_{k=0}^{M-1}\left|H_k\left(e^{j\omega}\right)\right|^2$".

59. Page 736, Problem 10.40(b): Replace "$\frac{1}{2}\{H(z)-H(-z)\}$" with "$\frac{1}{2}\{H(z)-H(-z)\}z$".

60. Page 755, Section 11.5 Title: Replace "Short-Term" with "Short-Time".

61. Page 755, Line 10 from bottom: Replace "$\omega_0 n$" with "$2\omega_0 n$".

62. Page 798, Lines 11, 13, and 15: Replace "z^{-3}" with "z^{-1}", and "[n-3]" with "[n-1]".

63. Page 829, Lines 4 and 5: Replace "f_1 =0.68" with "f_1 =0.18" and "f_2 =0.8" with "f_2 =0.3".

64. Page 829, Lines 7 and 8: Replace "f_1 =0.68" with "f_1 =0.18" and "f_2 =0.8,0.77,0.74 and 0.71" with "f_2 =0.3,0.27,0.24 and 0.21".

65. Page 829, Line 11: Replace "f_1 =0.68" with "f_1 =0.18" and "f_2 =0.71" with "f_2 =0.21".

66. Page 829, Problem M11.11, line 2: Replace "0.46" with "0.36".

67. Page 830, Problem M11.13, line 2: Replace "Program 11_4" with "Program 11_5".

68. Page 830, Problem M11.14, line 2: Replace "Program 11_5" with "Program 11_6".

69. Page 830, Problem M11.15, line 2: Replace "Program 11_6" with "Program 11_7".

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Sanjit K. Mitra
Department of Electrical and Computer Engineering
University of California, Santa Barbara

Prepared by
Rajeev Gandhi

WCB McGraw-Hill

Boston Burr Ridge, IL Dubuque, IA Madison, WI New York San Francisco St. Louis
Bangkok Bogotá Caracas Lisbon London Madrid
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Chapter 2

2.1 **(a)** $x[n] = \left\{3 \quad -4 \quad \underset{\uparrow}{2} \quad 0 \quad 6 \quad 3 \quad 9 \quad 5\right\}$. Thus $x[-n] = \left\{5 \quad 9 \quad 3 \quad 6 \quad 0 \quad \underset{\uparrow}{2} \quad -4 \quad 3\right\}$

Hence, $x_{ev}[n] = \dfrac{x[n]+x[-n]}{2} = \left\{\frac{5}{2} \quad \frac{9}{2} \quad \frac{3}{2} \quad \frac{9}{2} \quad -2 \quad \underset{\uparrow}{2} \quad -2 \quad \frac{9}{2} \quad \frac{3}{2} \quad \frac{9}{2} \quad \frac{5}{2}\right\}$,

and $x_{od}[n] = \dfrac{x[n]-x[-n]}{2} = \left\{-\frac{5}{2} \quad -\frac{9}{2} \quad -\frac{3}{2} \quad -\frac{3}{2} \quad -2 \quad \underset{\uparrow}{0} \quad 2 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{9}{2} \quad \frac{5}{2}\right\}$

(b) $u[n] = \left\{-2 \quad 1 \quad 2 \quad 5 \quad 0 \quad \underset{\uparrow}{0.1} \quad 6 \quad 3\right\}$

Thus, $u[-n] = \left\{3 \quad 6 \quad \underset{\uparrow}{0.1} \quad 0 \quad 5 \quad 2 \quad 1 \quad -2\right\}$

Hence $u_{ev}[n] = \left\{-1 \quad \frac{1}{2} \quad 1 \quad 4 \quad 3 \quad \underset{\uparrow}{0.1} \quad 3 \quad 4 \quad 1 \quad \frac{1}{2} \quad -1\right\}$

and $u_{od}[n] = \left\{-1 \quad \frac{1}{2} \quad 1 \quad 1 \quad -3 \quad \underset{\uparrow}{0} \quad 3 \quad -1 \quad -1 \quad -\frac{1}{2} \quad 1\right\}$

(c) $v[n] = A\cos(\omega_0 n) + B\sin(\omega_0 n)$. Thus, $v[-n] = A\cos(\omega_0 n) - B\sin(\omega_0 n)$.

Hence $v_{ev}[n] = A\cos(\omega_0 n)$ and $v_{od}[n] = B\sin(\omega_0 n)$.

(d) $g[n] = n^2$. Thus, $g[-n] = n^2$.

Hence $g_{ev}[n] = n^2$ and $g_{od}[n] = 0$.

(e) $h[n] = n^3$. Thus, $h[-n] = -n^3$.

Hence $h_{ev}[n] = 0$ and $h_{od}[n] = n^3$.

2.2 **(a)** $\{x[n]\} = \left\{A\alpha^n\right\}$, $0 \le n \le N-1$, where A and α are complex numbers. Hence,

$$x_{pcs}[n] = \frac{1}{2}\left\{x[n] + x^*[\langle -n \rangle_N]\right\} = \frac{1}{2}\left\{A\alpha^n + A^*(\alpha^*)^{N-n}\right\}$$

where A^* and α^*, represent the complex conjugates of A and α, respectively.

$$x_{pca}[n] = \frac{1}{2}\left\{x[n] - x[\langle -n \rangle_N]\right\} = \frac{1}{2}\left\{A\alpha^n - A^*(\alpha^*)^{N-n}\right\}$$

(b) $\{h[n]\} = \left\{3+j2 \quad -4-j5 \quad \underset{\uparrow}{2+j3} \quad 5+j \quad -6+j4\right\}$.

Hence, $\{h^*[-n]\} = \left\{-6-j4 \quad 5-j \quad \underset{\uparrow}{2-j3} \quad -4+j5 \quad 3-j2\right\}$.

Thus, $h_{pcs}[n] = \left\{-\frac{3}{2}-j \quad \frac{1}{2}-j3 \quad \underset{\uparrow}{2} \quad \frac{1}{2}+j3 \quad -\frac{3}{2}+j\right\}$,

and $h_{pca}[n] = \left\{\frac{9}{2}+j3 \quad -\frac{9}{2}-j2 \quad \underset{\uparrow}{j3} \quad \frac{9}{2}-j2 \quad -\frac{9}{2}+j3\right\}$.

2.3 **(a)** $\{x[n]\} = \left\{A\alpha^n\right\}$ where A and α are complex numbers, with $|\alpha| < 1$.

Since for $n < 0$, $|\alpha|^n$ can become arbitrarily large hence $\{x[n]\}$ is not a bounded sequence.

(b) $\{y[n]\} = A\alpha^n\mu[n]$ where A and α are complex numbers, with $|\alpha| < 1$.
In this case $|y[n]| \le |A|$ $\forall n$ hence $\{y[n]\}$ is a bounded sequence.

(c) $\{h[n]\} = C\beta^n\mu[n]$ where C and β are complex numbers, with $|\beta| > 1$.
Since $|\beta|^n$ becomes arbitrarily large as n increases hence $\{h[n]\}$ is not a bounded sequence.

(d) $\{g[n]\} = 4\sin^2(\omega_a n)$. Since $|g[n]| \le 4$ $\forall n$ hence $\{g[n]\}$ is a bounded sequence.

(e) $\{v[n]\} = 3\cos(\omega_b n^2)$. Since $|v[n]| \le 3$ $\forall n$ hence $\{v[n]\}$ is a bounded sequence.

2.4 Recall, $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$.

Since x[n] is a causal sequence, thus $x[-n] = 0$ $\forall n > 0$. Hence,
$x[n] = x_{ev}[n] + x_{ev}[-n] = 2x_{ev}[n]$, $\forall n > 0$. For n = 0, $x[0] = x_{ev}[0]$.

Thus x[n] can be completely recovered from its even part.

Likewise, $x_{od}[n] = \frac{1}{2}(x[n] - x[-n]) = \begin{cases} \frac{1}{2}x[n], & n > 0, \\ 0, & n = 0. \end{cases}$

Thus x[n] can be recovered from its odd part $\forall n$ except n = 0.

2.5 $2y_{ca}[n] = y[n] - y^*[-n]$. Since y[n] is a causal sequence $y[n] = 2y_{ca}[n]$ $\forall n > 0$.
For n = 0, $\mathrm{Im}\{y[0]\} = y_{ca}[0]$. Hence real part of y[0] cannot be fully recovered from $y_{ca}[n]$.
Therefore y[n] cannot be fully recovered from $y_{ca}[n]$.
$2y_{cs}[n] = y[n] + y^*[-n]$. Hence, $y[n] = 2y_{cs}[n]$ $\forall n > 0$.
For n = 0, $\mathrm{Re}\{y[0]\} = y_{cs}[0]$. Hence imaginary part of y[0] cannot be recovered from $y_{cs}[n]$.
Therefore y[n] cannot be fully recovered from $y_{cs}[n]$.

2.6 $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$. This implies, $x_{ev}[-n] = \frac{1}{2}(x[-n] + x[n]) = x_{ev}[n]$.

Hence even part of a real sequence is even.
$x_{od}[n] = \frac{1}{2}(x[n] - x[-n])$. This implies, $x_{od}[-n] = \frac{1}{2}(x[-n] - x[n]) = -x_{od}[n]$.

Hence the odd part of a real sequence is odd.

2.7 RHS of Eq. (2.170a) is $x_{cs}[n] + x_{cs}[n-N] = \frac{1}{2}(x[n] + x^*[-n]) + \frac{1}{2}(x[n-N] + x^*[N-n])$.

Since $x[n] = 0$ $\forall n < 0$, Hence
$x_{cs}[n] + x_{cs}[n-N] = \frac{1}{2}(x[n] + x^*[N-n]) = x_{pcs}[n]$, $0 \le n \le N-1$.

RHS of Eq. (2.170b) is

$$x_{ca}[n]+x_{ca}[n-N]=\frac{1}{2}(x[n]-x^*[-n])+\frac{1}{2}(x[n-N]-x^*[n-N])$$
$$=\frac{1}{2}(x[n]-x^*[N-n])=x_{pca}[n], \qquad 0\le n\le N-1.$$

2.8 $x_{pcs}[n]=\frac{1}{2}\left(x[n]+x^*[<-n>_N]\right)$ for $0\le n\le N-1$, Since, $x[<-n>_N]=x[N-n]$, it follows that $x_{pcs}[n]=\frac{1}{2}(x[n]+x^*[N-n]),\quad 1\le n\le N-1.$

For n = 0, $x_{pcs}[0]=\frac{1}{2}(x[0]+x^*[0])=\mathrm{Re}\{x[0]\}$.

Similarly $x_{pca}[n]=\frac{1}{2}\left(x[n]-x^*[<-n>_N]\right)=\frac{1}{2}(x[n]-x^*[N-n]),\quad 1\le n\le N-1$. Hence,

for n = 0, $x_{pca}[0]=\frac{1}{2}(x[0]-x^*[0])=\mathrm{Im}\{x[0]\}$.

2.9 $x[n]=\cos(2\pi kn/N), \qquad 0\le n\le N-1$. Hence,

$$E_x=\sum_{n=0}^{N-1}\cos^2(2\pi kn/N)=\frac{1}{2}\sum_{n=0}^{N-1}(1+\cos(4\pi kn/N))=\frac{N}{2}+\frac{1}{2}\sum_{n=0}^{N-1}\cos(4\pi kn/N).$$

Let $C=\sum_{n=0}^{N-1}\cos(4\pi kn/N)$, and $S=\sum_{n=0}^{N-1}\sin(4\pi kn/N)$.

Therefore $C+jS=\sum_{n=0}^{N-1}e^{j4\pi kn/N}=\dfrac{e^{j4\pi k}-1}{e^{j4\pi k/N}-1}=0$. Thus, $C=\mathrm{Re}\{C+jS\}=0$.

As $C=\mathrm{Re}\{C+jS\}=0$, it follows that $E_x=\frac{N}{2}$.

2.10 **(a)** Average power of sequence $x[n]=\mu[n]$ is

$$P_{av}=\lim_{M\to\infty}\frac{1}{M+1}\sum_{\ell=0}^{\ell=M}\mu^2[\ell]=\lim_{M\to\infty}\frac{1}{M+1}(M+1)=1.$$

(b) The sequence $x[n]=A\cos(\frac{2\pi n}{M}+\phi)$ is periodic with a period M. Hence

$$P_{av}=\frac{1}{M}\sum_{n=0}^{M-1}A^2\cos^2(\frac{2\pi n}{M}+\phi)=\frac{A^2}{2M}\sum_{n=0}^{M-1}\left(1+\cos\left(\frac{4\pi n}{M}+\phi\right)\right)=\frac{A^2}{2}.$$

2.11 **(a)** Consider a sequence defined by $x[n]=\sum_{k=-\infty}^{n}\delta[k]$.

If n < 0 then k = 0 is not included in the sum and hence x[n] = 0, whereas for $n\ge 0$, k = 0 is included in the sum hence $x[n]=1\ \forall\, n\ge 0$. Thus $x[n]=\sum_{k=-\infty}^{n}\delta[k]=\begin{cases}1, & n\ge 0,\\ 0, & n<0,\end{cases}=\mu[n].$

(b) Since $\mu[n]=\begin{cases}1, & n\ge 0,\\ 0, & n<0,\end{cases}$ it follows that $\mu[n-1]=\begin{cases}1, & n\ge 1,\\ 0, & n\le 0.\end{cases}$

Hence, $\mu[n]-\mu[n-1]=\begin{cases}1, & n=0,\\ 0, & n\ne 0,\end{cases}=\delta[n].$

2.12 Now $x[n] = A\sin(\omega_0 n + \phi)$.

(a) Given $x[n] = \{0 \quad 1.5 \quad 0 \quad -1.5\}$, it follows that the fundamental period of x[n] is N = 4. Hence $\omega_0 = 2\pi/4 = \pi/2$. Solving $x[0] = A\sin(\phi) = 0$ we get $\phi = 0$, and solving $x[1] = A\sin(\pi/2) = 1.5$, we get A = 1.5.

(b) Given $x[n] = \{\sqrt{2} \quad \sqrt{2} \quad -\sqrt{2} \quad -\sqrt{2}\}$. Again the fundamental period is N = 4, hence $\omega_0 = 2\pi/4 = \pi/2$. Next from $x[0] = A\sin(\phi) = \sqrt{2}$ and $x[1] = A\sin(\pi/2 + \phi) = A\cos(\phi) = \sqrt{2}$ it can be seen that A = 2 and $\phi = \pi/4$ is one solution satisfying the above equations.

(c) $x[n] = \{3 \quad -3\}$. Here the fundamental period is N = 2, hence $\omega_0 = \pi$. Next from x[0] = $A\sin(\phi) = 3$ and $x[1] = A\sin(\phi + \pi) = -A\sin(\phi) = -3$ observe that $A = 3$ and $\phi = \pi/2$ that A = 3 and $\phi = \pi/2$ is one solution satisfying these two equations.

(d) $x[n] = \{2 \quad \sqrt{2} \quad 0 \quad -\sqrt{2} \quad -2 \quad -\sqrt{2} \quad 0 \quad 2\}$. Thus the fundamental period is N = 8, hence $\omega_0 = \pi/4$. From $x[0] = A\sin(\phi) = 2$ and $x[1] = A\sin(\phi + \pi/4) = \sqrt{2}$ we note that A = 2 and $\phi = \pi/2$ is one solution satisfying these two equations.

2.13 (a) As $\tilde{x}_1[n] = e^{j0.25\pi n}$, $\tilde{x}_1[n+8] = e^{j(0.25\pi n + 2\pi)} = \tilde{x}_1[n]$. Hence N = 8 is the fundamental period of the sequence $\tilde{x}_1[n]$.

(b) $\tilde{x}_2[n] = \cos(0.2\pi n)$. Let N be the fundamental period of the sequence. This implies that $\tilde{x}_2[n+N] = \cos(0.2\pi n + 0.2\pi N)$. For $\tilde{x}_2[n]$ to be periodic , we must have $0.2\pi N = 2\pi$. Hence N = 10 is the fundamental period of the sequence $\tilde{x}_2[n]$.

(c) $\tilde{x}_3[n] = 2\cos(0.1\pi n) + 2\sin(0.2\pi n)$. Let N be the fundamental period of the sequence. This implies that $\tilde{x}_3[n+N] = 2\cos(0.1\pi n + 0.1\pi N) + 2\sin(0.2\pi n + 0.2\pi N) = \tilde{x}_3[n]$. For to be periodic in N we must satisfy $0.1\pi N = 2\pi r$ and $0.2\pi N = 2\pi k$. Therefore N = 20r and N = 10k. Hence N = 20 is the fundamental period of the sequence $\tilde{x}_3[n]$.

(d) $\tilde{x}_4[n] = 3\sin(0.8\pi n) - 4\cos(0.1\pi n)$. Let N be the period of the sequence. Then $\tilde{x}_4[n+N] = 3\sin(0.8\pi n + 0.8\pi N) - 4\cos(0.1\pi n + 0.1\pi N) = \tilde{x}_4[n]$. Hence N = 20 is the fundamental period of the sequence $\tilde{x}_4[n]$.

(e) $\tilde{x}_5[n] = 5\sin(0.1\pi n) + 4\cos(0.9\pi n) - \cos(0.8\pi n)$. Let N be the period of the sequence. Then $\tilde{x}_5[n+N] = 5\sin(0.1\pi n + 0.1\pi N) + 4\cos(0.9\pi n + 0.9\pi N) - \cos(0.8\pi n + 0.8\pi N) = \tilde{x}_5[n]$. Hence, N = 20 is the fundamental period of the sequence $\tilde{x}_5[n]$.

(f) $\tilde{x}_6[n]$ = n modulo 5. Since $\tilde{x}_6[n+5]$ = (n+5) modulo 5 = n modulo 5 = $\tilde{x}_6[n]$. Hence N = 5 is the fundamental period of the sequence $\tilde{x}_6[n]$.

2.14 $x[n] = A\cos(\omega_0 n)$

(a) $\omega_o = 0.15\pi$ and $N = 2\pi r/\omega_o$. Substituting the value of ω_o we get $N = 40r/3$. N takes an integer value when r = 3. Hence N = 40 is the fundamental period of this sequence.

(b) $\omega_o = 0.18\pi$ and $N = 2\pi r/\omega_o$. It can be shown here N = 100 is the fundamental period of the sequence.

(c) $\omega_o = 0.225\pi$ and $N = 2\pi r/\omega_o$. It can be shown here N = 80 is the fundamental period of the sequence.

(d) $\omega_o = 0.3\pi$ and $N = 2\pi r/\omega_o$. It can be shown here N = 20 is the fundamental period of the sequence.

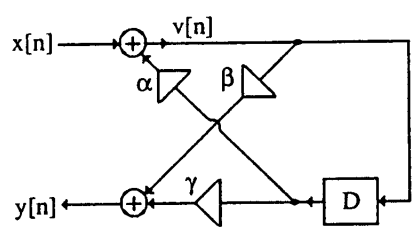
(e) $\omega_o = 0.45\pi$ and $N = 2\pi r/\omega_o$. It can be shown here N = 40 is the fundamental period of the sequence.

2.15 **(a)** $x[n] = g[n]g[n]$. Hence $x[-n] = g[-n]g[-n]$. Since g[n] is even, hence $g[-n] = g[n]$. Therefore $x[-n] = g[-n]g[-n] = g[n]g[n] = x[n]$. Hence x[n] is even.

(b) $u[n] = g[n]h[n]$. Hence, $u[-n] = g[-n]h[-n] = g[n](-h[n]) = -g[n]h[n] = -u[n]$. Hence u[n] is odd.

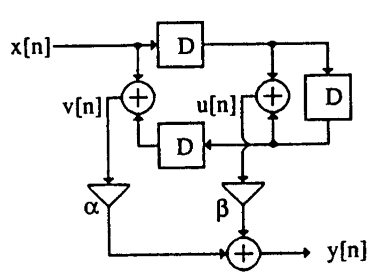
(c) $v[n] = h[n]h[n]$. Hence, $v[-n] = h[n]h[n] = (-h[n])(-h[n]) = h[n]h[n] = v[n]$. Hence v[n] is even.

2.16 (a)



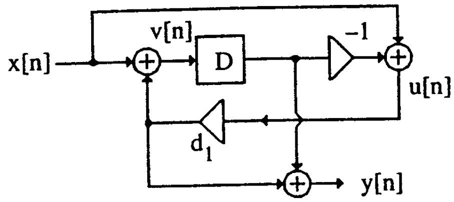
From figure, $v[n] = \alpha\, v[n-1] + x[n]$ and $y[n] = \beta\, v[n] + \gamma\, v[n-1]$. Hence,
$\alpha\, y[n-1] = \alpha\beta\, v[n-1] + \alpha\gamma\, v[n-2]$, and
$y[n] - \alpha\, y[n-1] = \beta(v[n] - \alpha\, v[n-1]) + \gamma(v[n-1] - \alpha\, v[n-2])$. Thus,
$y[n] = \alpha y[n-1] + \beta x[n] + \gamma x[n-1]$

(b)



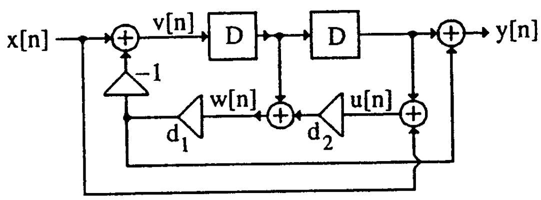
It follows from the figure, v[n] = x[n] + x[n–3], u[n] = x[n–1] + x[n–2], and y[n] = $\alpha v[n]+\beta u[n]$. Hence, $y[n] = \alpha(x[n]+x[n-3])+\beta(x[n-1]+x[n-2])$.

(c)



Here $v[n] = x[n] + d_1u[n]$ and $u[n] = x[n] - v[n-1] = x[n] - x[n-1] - d_1u[n-1]$. Thus,
$y[n] = v[n-1]+d_1u[n] = x[n]-u[n]+d_1u[n]$. As a result,
$d_1y[n-1] = d_1x[n-1]-d_1u[n-1]+d_1^2u[n-1]$
and $y[n]+d_1y[n-1] = d_1x[n-1]+x[n]+(d_1-1)(x[n]-x[n-1])$. Hence,
$y[n] = d_1x[n]+x[n-1]-d_1y[n-1]$.

(d)



It follows that $v[n] = x[n] - d_1w[n]$, $w[n] = v[n-1] + d_2u[n]$, and $u[n] = v[n-2] + x[n]$. Thus,
$w[n] = x[n-1]-d_1w[n-1]+d_2x[n]+d_2x[n-2]-d_1d_2w[n-2]$, and
$w[n]+d_1w[n-1]+d_1d_2w[n-2] = x[n-1]+d_2x[n]+d_2x[n-2]$. However,
$y[n] = v[n-2]+d_1w[n]$,
Hence, $y[n] = x[n-2]-d_1w[n-2]+d_1w[n]$ from which we get
$d_1y[n-1] = d_1x[n-3]-d_1^2w[n-3]+d_1^2w[n-1]$ and
$d_1d_2y[n-2] = d_1d_2x[n-4]-d_1^2d_2w[n-4]+d_1^2d_2w[n-2]$. Adding the last three equations we arrive at

$$y[n]+d_1y[n-1]+d_1d_2y[n-2] = x[n-2]+d_1x[n-3]+d_1d_2x[n-4]$$
$$-d_1(w[n-2]+d_1w[n-3]+d_1d_2w[n-4]) + d_1(w[n]+d_1w[n-1]+d_1d_2w[n-2])$$
$$= d_1d_2x[n]+d_1x[n-1]+x[n-2].$$

2.17 $x[n] = 0.5\delta[n+2]+1.5\delta[n-1]-\delta[n-2]+\delta[n-4]+0.75\delta[n-6]$.
Since $\delta[n] = \mu[n]-\mu[n-1]$, Hence

$$x[n] = 0.5\mu[n+2]-0.5\mu[n+1]+1.5\mu[n-1]-2.5\mu[n-2]+\mu[n-3]+\mu[n-4]$$
$$-\mu[n-5]+0.75\mu[n-6]-0.75\mu[n-7].$$

2.18 **(a)** $w[n] = -\delta[n+3]+\delta[n]+4\delta[n-1]-2\delta[n-3]+3\delta[n-6]$

(b) Using the fact that $\delta[n] = \mu[n]-\mu[n-1]$, we get

$$w[n] = -\mu[n+3] + \mu[n+2] + \mu[n] + 3\mu[n-1] - 4\mu[n-2] - 2\mu[n-3]$$
$$+2\mu[n-4] + 3\mu[n-6] - 3\mu[n-7].$$

2.19 Let $y_1[n]$ and $y_2[n]$ be the outputs of the system corresponding to the inputs $x_1[n]$ and $x_2[n]$, respectively. Let $y_3[n]$ be the output corresponding to the input $x_3[n] = ax_1[n] + bx_2[n]$. For the discrete-time system to be linear, we need to show that $y_3[n] = ay_1[n] + by_2[n]$.

(a) $y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$. Hence,

$$y_3[n] = a\left(\alpha_1 x_1[n] + \alpha_2 x_1[n-1] + \alpha_3 x_1[n-2] + \alpha_4 x_1[n-3]\right)$$
$$+ b\left(\alpha_1 x_2[n] + \alpha_2 x_2[n-1] + \alpha_3 x_2[n-2] + \alpha_4 x_2[n-3]\right)$$
$= ay_1[n] + by_2[n]$. Hence the system is linear.

(b) $y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$. Hence,

$$y_3[n] = b_0\left(ax_1[n] + bx_2[n]\right) + b_1\left(ax_1[n-1] + bx_2[n-1]\right)$$
$$+ b_2\left(ax_1[n-2] + bx_2[n-2]\right) + a_1 y_3[n-1] + a_2 y_3[n-2]$$
$$= a\left(b_0 x_1[n] + b_1 x_1[n-1] + b_2 x_1[n-2]\right)$$
$$+ b\left(b_0 x_2[n] + b_1 x_2[n-1] + b_2 x_2[n-2]\right) + a_1 y_3[n-1] + a_2 y_3[n-2]$$
$$y_3[n] = a\left(y_1[n] - a_1 y_1[n-1] - a_2 y_1[n-2]\right)$$
$$+ b\left(y_2[n] - a_1 y_2[n-1] - a_2 y_2[n-2]\right) + a_1 y_3[n-1] + a_2 y_3[n-2]$$

$y_3[n] = ay_1[n] + by_2[n]$, i.e. the system is linear for a causal input only if $y_1[-1] = y_1[-2] = y_2[-1] = y_2[-2] = 0$.

(c) $y[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots, \\ 0, & \text{otherwise.} \end{cases}$

$$y_3[n] = \begin{cases} ax_1[n/L] + bx_2[n/L], & n = 0, \pm L, \pm 2L, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$
$= ay_1[n] + by_2[n]$. Hence the system is linear.

(d) $y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$. Hence, $y_3[n] = \frac{1}{M}\sum_{k=0}^{M-1} x_3[n-k] = \frac{a}{M}\sum_{k=0}^{M-1} x_1[n-k] + \frac{b}{M}\sum_{k=0}^{M-1} x_2[n-k]$

$= ay_1[n] + by_2[n]$. Hence the system is linear.

(e) $y[n] = x_u[n] + \frac{1}{2}\left(x_u[n-1] + x_u[n+1]\right)$.

$$y_3[n] = x_{3u}[n] + \frac{1}{2}\left(x_{3u}[n-1] + x_{3u}[n+1]\right) = a x_{1u}[n] + b x_{2u}[n]$$
$$+ \frac{1}{2}\left(a x_{1u}[n-1] + b x_{2u}[n-1] + a x_{1u}[n+1] + b x_{2u}[n+1]\right) = a y_1[n] + b y_2[n].$$

Hence the system is linear.

(f) $y[n] = x_u[n] + \frac{1}{3}\left(x_u[n-1] + x_u[n+2]\right) + \frac{2}{3}\left(x_u[n-2] + x_u[n+1]\right)$.

$$y_3[n] = x_{3u}[n] + \frac{1}{3}\left(x_{3u}[n-1] + x_{3u}[n+2]\right) + \frac{2}{3}\left(x_{3u}[n-2] + x_{3u}[n+1]\right)$$

$$= ax_{2u}[n] + bx_{2u}[n] \; + \frac{1}{3}\left(ax_{1u}[n-1] + bx_{2u}[n-1] + ax_{1u}[n+2] + bx_{2u}[n+2]\right)$$

$$+ \frac{2}{3}\left(ax_{1u}[n-2] + bx_{2u}[n-2] + ax_{1u}[n+1] + bx_{2u}[n+1]\right) \; = ay_1[n] + by_2[n].$$

Hence the system is linear.

2.20 **(a)** $y[n] = nx[n]$.

For an input $x_1[n]$ the output is $y_1[n] = n\, x_1[n]$, and for an input $x_1[n]$ the output is $y_2[n] = n\, x_2[n]$. Thus, for an input $x_3[n] = \alpha\, x_1[n] + \beta\, x_2[n]$, the output $y_3[n]$ is given by

$$y_3[n] = n\left(\alpha x_1[n] + \beta x_2[n]\right) = \alpha y_1[n] + \beta y_2[n].$$

Hence the system is linear.

Since there is no output before the input hence the system is causal. However, $|y[n]|$ being proportional to n, it is possible that a bounded input can result in an unbounded output. Let $x[n] = 1 \;\forall\, n$, then $y[n] = n$. Hence $y[n] \to \infty$ as $n \to \infty$, hence not BIBO stable.

Let $y[n]$ be the output for an input $x[n]$, and let $y_1[n]$ be the output for an input $x_1[n]$. If $x_1[n] = x[n-n_0]$ then $y_1[n] = nx_1[n] = nx[n-n_0]$. However, $y[n-n_0] = (n-n_0)x[n-n_0]$. Since $y_1[n] \neq y[n-n_0]$, the system is not time-invariant.

(b) $y[n] = x^3[n]$.

For an input $x_1[n]$ the output is $y_1[n] = x_1^3[n]$, and for an input $x_1[n]$ the output is $y_2[n] = x_1^3[n]$. Thus, for an input $x_3[n] = \alpha\, x_1[n] + \beta\, x_2[n]$, the output $y_3[n]$ is given by

$$y_3[n] = (\alpha x_1[n] + \beta x_2[n])^3 \neq \alpha^3 x_1^3[n] + \beta^3 x_2^3[n]$$

Hence the system is not linear.

Since there is no output before the input hence the system is causal.

Here, a bounded input produces bounded output hence the system is BIBO stable too.

Also following an analysis similar to that in part (a) it is easy to show that the system is time-invariant.

(c) $y[n] = \beta + \sum_{\ell=0}^{5} x[n-\ell]$

Since $\beta \neq 0$ hence the system is not linear. However, here a bounded input produces bounded output. Thus, the system is BIBO stable. Also it is easy to check that the system is causal and time-invariant.

(d) $y[n]=\beta+\sum_{\ell=-5}^{5} x[n-\ell]$

The system is non-linear, BIBO stable, non-causal and time-invariant.

(e) $y[n]=\alpha x[-n]$

The system is linear, stable, non causal. Let y[n] be the output for an input x[n] and $y_1[n]$ be the output for an input $x_1[n]$. Then $y[n]=\alpha x[-n]$ and $y_1[n]=\alpha x_1[-n]$.

Let $x_1[n]=x[n-n_0]$, then $y_1[n]=\alpha x_1[-n]=\alpha x[-n-n_0]$, whereas $y[n-n_0]=\alpha x[n_0-n]$. Hence the system is time-varying.

(f) y[n] = x[n+6]

The given system is linear, non-causal, stable and time-invariant.

2.21 y[n] = x[n + 1] − 2x[n] + x[n − 1].

Let $y_1[n]$ be the output for an input $x_1[n]$ and $y_2[n]$ be the output for an input $x_2[n]$. Then for an input $x_3[n]=\alpha x_1[n]+\beta x_2[n]$ the output $y_3[n]$ is given by

$$\begin{aligned} y_3[n] &= x_3[n+1]-2x_3[n]+x_3[n-1] \\ &= \alpha x_1[n+1]-2\alpha x_1[n]+\alpha x_1[n-1]+\beta x_2[n+1]-2\beta x_2[n]+\beta x_2[n-1] \\ &= \alpha y_1[n]+\beta y_2[n]. \end{aligned}$$

Hence the system is linear. If $x_1[n]=x[n-n_0]$ then $y_1[n]=y[n-n_0]$. Hence the system is time-inavariant. Also the system's impulse response is given by

$$h[n]=\begin{cases} -2, & n=0, \\ 1, & n=1,-1, \\ 0, & \text{elsewhere.} \end{cases}$$

Since $h[n] \neq 0 \ \forall\, n<0$ the system is non-causal.

2.22 Median filtering is a nonlinear operation. Consider the following sequences as the input to a median filter: $x_1[n]=\{3,\ 4,\ 5\}$ and $x_2[n]=\{2,\ -2,\ -2\}$. The corresponding outputs of the median filter are $y_1[n]=4$ and $y_2[n]=-2$. Now consider another input sequence $x_3[n] = x_1[n] + x_2[n]$. Then the corresponding output is $y_3[n] = 3$, On the other hand, $y_1[n]+y_2[n]=2$. Hence median filtering is not a linear operation. To test the time-invariance property, let x[n] and $x_1[n]$ be the two inputs to the system with correponding outputs y[n] and $y_1[n]$. If $x_1[n]=x[n-n_0]$ then $y_1[n]=\text{median}\{x_1[n-k],\dots\dots,x_1[n],\dots\dots x_1[n+k]\}$

$$= \text{median}\{x[n-k-n_0],\dots\dots,x[n-n_0],\dots\dots x[n+k-n_0]\}=y[n-n_0].$$

Hence the system is time invariant.

2.23 $y[n]=\frac{1}{2}\left(y[n-1]+\frac{x[n]}{y[n-1]}\right)$

Now for an input $x[n] = \alpha\,\mu[n]$, the ouput $y[n]$ converges to some constant K as $n \to \infty$. The above difference equation as $n \to \infty$ reduces to $K = \frac{1}{2}\left(K + \frac{\alpha}{K}\right)$ which is equivalent to $K^2 = \alpha$ or in other words, $K = \sqrt{\alpha}$.

It is easy to show that the system is non-linear. Now assume $y_1[n]$ be the output for an input $x_1[n]$. Then $y_1[n] = \frac{1}{2}\left(y_1[n-1] + \frac{x_1[n]}{y_1[n-1]}\right)$

If $x_1[n] = x[n-n_0]$. Then, $y_1[n] = \frac{1}{2}\left(y_1[n-1] + \frac{x[n-n_0]}{y_1[n-1]}\right)$.

Thus $y_1[n] = y[n-n_0]$. Hence the above system is time invariant.

2.24 As $\delta[n] = \mu[n] - \mu[n-1]$, $T\{\delta[n]\} = T\{\mu[n]\} - T\{\mu[n-1]\} \Rightarrow h[n] = s[n] - s[n-1]$

For a discrete LTI system

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty}(s[k]-s[k-1])x[n-k] = \sum_{k=-\infty}^{\infty} s[k]x[n-k] - \sum_{k=-\infty}^{\infty} s[k-1]x[n-k]$$

2.25 Now $y[n] = x[n] \circledast h[n] = = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$

Substituting $h[n] = 0.5\delta[n-2] + \delta[n-2] + 0.5\delta[n-1]$, and $x[n] = \delta[n-3] + \delta[n-2] + \delta[n-1] + \delta[n]$, we obtain $y[n] =$

$$= \sum_{k=-\infty}^{\infty}(0.5\delta[k-2] + \delta[k-1] + 0.5\delta[k])(\delta[n-3-k] + \delta[n-2-k] + \delta[n-1-k] + \delta[n-k]).$$ Hence

$$y[n] = \{\underset{\uparrow}{0.5} \quad 1.5 \quad 2 \quad 2 \quad 1.5 \quad 0.5\}.$$

2.26 $x_1[n] = \{\underset{\uparrow}{0} \quad 1 \quad 0 \quad 2\}$, $h_1[n] = \{\underset{\uparrow}{2} \quad 0 \quad 1\}$,

$x_2[n] = \{\underset{\uparrow}{2} \quad -1 \quad 0 \quad 3\}$, $h_2[n] = \{\underset{\uparrow}{-1} \quad 2 \quad 1\}$

(a) $y_1[n] = x_1[n] \circledast h_1[n] = \{\underset{\uparrow}{0} \quad 2 \quad 0 \quad 5 \quad 0 \quad 2\}$

(b) $y_2[n] = x_2[n] \circledast h_2[n] = \{\underset{\uparrow}{-2} \quad 5 \quad 0 \quad -4 \quad 6 \quad 3\}$

(c) $y_3[n] = x_1[n] \circledast h_2[n] = \{\underset{\uparrow}{0} \quad -1 \quad 2 \quad -1 \quad 4 \quad 2\}$

(d) $y_4[n] = x_2[n] \circledast h_1[n] = \{\underset{\uparrow}{4} \quad -2 \quad 2 \quad 5 \quad 0 \quad 3\}$

2.27 $y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$

Substituting k by n-m in the above expression, we get $y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = h[n] \circledast x[n]$.

Hence convolution operation is commutative.

Let $y[n] = x[n] \circledast \left(h_1[n] + h_2[n]\right) = = \sum_{k=-\infty}^{\infty} x[n-k]\left(h_1[k] + h_2[k]\right)$

$= \sum_{k=-\infty}^{\infty} x[n-k]h_1[k] + \sum_{k=-\infty}^{k=\infty} x[n-k]h_2[k] = x[n] \circledast h_1[n] + x[n] \circledast h_2[n]$.

Hence convolution is distributive.

2.28 $x_3[n] \circledast x_2[n] \circledast x_1[n] = x_3[n] \circledast (x_2[n] \circledast x_1[n])$

As $x_2[n] \circledast x_1[n]$ is an unbounded sequence hence the result of this convolution cannot be determined. But $x_2[n] \circledast x_3[n] \circledast x_1[n] = x_2[n] \circledast (x_3[n] \circledast x_1[n])$. Now $x_3[n] \circledast x_1[n] = 0$ for all values of n hence the overall result is zero. Hence for the given sequences $x_3[n] \circledast x_2[n] \circledast x_1[n] \neq x_2[n] \circledast x_3[n] \circledast x_1[n]$.

2.29 $w[n] = x[n] \circledast h[n] \circledast g[n]$. Define $y[n] = x[n] \circledast h[n] = \sum_k x[k]h[n-k]$ and $f[n] =$

$h[n] \circledast g[n] = \sum_k g[k]h[n-k]$.

Consider $w_1[n] = (x[n] \circledast h[n]) \circledast g[n] = y[n] \circledast g[n] = \sum_m g[m] \sum_k x[k]h[n-m-k]$.

Next consider $w_2[n] = x[n] \circledast (h[n] \circledast g[n]) = x[n] \circledast f[n] = \sum_k x[k] \sum_m g[m]h[n-k-m]$.

Difference between the expressions for $w_1[n]$ and $w_2[n]$ is that the order of the summations is changed.

A) Assumptions: h[n] and g[n] are causal filters, and $x[n] = 0$ for $n < 0$. This implies

$$y[m] = \begin{cases} 0, & \text{for } m < 0, \\ \sum_{k=0}^{m} x[k]h[m-k], & \text{for } m \geq 0. \end{cases}$$

Thus, $w[n] = \sum_{m=0}^{n} g[m]y[n-m] = \sum_{m=0}^{n} g[m] \sum_{k=0}^{n-m} x[k]h[n-m-k]$.

All sums have only a finite number of terms. Hence, interchange of the order of summations is justified and will give correct results.

B) Assumptions: h[n] and g[n] are stable filters, and x[n] is a bounded sequence with $|x[n]| \leq X$.

Here, $y[m] = \sum_{k=-\infty}^{\infty} h[k]x[m-k] = \sum_{k=k_1}^{k_2} h[k]x[m-k] + \varepsilon_{k_1,k_2}[m]$ with $\left|\varepsilon_{k_1,k_2}[m]\right| \leq \varepsilon_n X$.

In this case all sums have effectively only a finite number of terms and the error can be reduced by choosing k_1 and k_2 sufficiently large. Hence in this case the problem is again effectively reduced to that of the one-sided sequences. Here, again an interchange of summations is justified and will give correct results.

Hence for the convolution to be associative, it is sufficient that the sequences be stable and single-sided.

2.30 $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$. Since h[k] is of length M and defined for $0 \le k \le M-1$, the convolution sum reduces to $y[n] = \sum_{k=0}^{M-1} x[n-k]h[k]$.

y[n] will be non-zero for all those values of n and k for which n – k satisfies $0 \le n-k \le N-1$. Minimum value of n – k = 0 and occurs for lowest n at n = 0 and k = 0. Maximum value of n – k = N–1 and occurs for maximum value of k at M – 1. Thus n – k = M – 1 $\Rightarrow n = N + M - 2$. Hence the total number of non-zero samples = N + M – 1.

2.31 $y[n] = x[n] \circledast x[n] = \sum_{k=-\infty}^{\infty} x[n-k]x[k]$.

Since x[n – k] = 0 if n – k < 0 and x[k] = 0 if k < 0 hence the above summation reduces to

$$y[n] = \sum_{k=n}^{N-1} x[n-k]x[k] = \begin{cases} n+1, & 0 \le n \le N-1, \\ 2N-n, & N \le n \le 2N-1. \end{cases}$$

Hence the output is a triangular sequence with a maximum value of N. Locations of the output samples with values $\frac{N}{4}$ are n = $\frac{N}{4}$ – 1 and $\frac{7N}{4}$ – 1. Locations of the output samples with values $\frac{N}{2}$ are n = $\frac{N}{2}$ – 1 and $\frac{3N}{2}$ – 1. Note: It is tacitly assumed that N is divisible by 4 otherwise $\frac{N}{4}$ is not an integer.

2.32 Maximum value will occur when all positive samples are superimposed over themselves. This occurs at n = $\frac{N}{2}$ – 1 and at n = $\frac{3N}{2}$ – 1 and the corresponding maximum value is $\frac{N}{2}$. Minimum value will occur if all the positive values are superimposed over the negative values and vice versa. This happens at n = N – 1 and the corresponding minimum value is –N.

2.33 (a) $y[n] = g_{ev}[n] \circledast h_{ev}[n] = \sum_{k=-\infty}^{\infty} h_{ev}[n-k]g_{ev}[k]$. Hence, $y[-n] = \sum_{k=-\infty}^{\infty} h_{ev}[-n-k]g_{ev}[k]$.

Replace k by – k. Then above summation becomes

$$y[-n]=\sum_{k=-\infty}^{\infty} h_{ev}[-n+k]g_{ev}[-k] = \sum_{k=-\infty}^{\infty} h_{ev}[-(n-k)]g_{ev}[-k] = \sum_{k=-\infty}^{\infty} h_{ev}[(n-k)]g_{ev}[k]$$

$= y[n]$.

Hence $g_{ev}[n] \circledast h_{ev}[n]$ is even.

(b) $y[n] = g_{ev}[n] \circledast h_{od}[n] = \sum_{k=-\infty}^{\infty} h_{od}[(n-k)]g_{ev}[k]$. As a result,

$$y[-n] = \sum_{k=-\infty}^{\infty} h_{od}[(-n-k)]g_{ev}[k] = \sum_{k=-\infty}^{\infty} h_{od}[-(n-k)]g_{ev}[-k] = -\sum_{k=-\infty}^{\infty} h_{od}[(n-k)]g_{ev}[k].$$

Hence $g_{ev}[n] \circledast h_{od}[n]$ is odd.

(c) $y[n] = g_{od}[n] \circledast h_{od}[n] = \sum_{k=-\infty}^{\infty} h_{od}[n-k]g_{od}[k]$. As a result,

$$y[-n] = \sum_{k=-\infty}^{\infty} h_{od}[-n-k]g_{od}[k] = \sum_{k=-\infty}^{\infty} h_{od}[-(n-k)]g_{od}[-k] = \sum_{k=-\infty}^{\infty} h_{od}[(n-k)]g_{od}[k].$$

Hence $g_{od}[n] \circledast h_{od}[n]$ is even.

2.34 $y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k]=\sum_{k=0}^{M-1} x[n-k]h[k]$. Now, $y[0] = x[0]h[0] \quad \Rightarrow x[0] = y[0]/h[0]$

In general, $x[n]=\frac{1}{h[0]}\left\{y[n]-\sum_{k=1}^{M-1} h[k]x[n-k]\right\}$,

where $x[n-k]$, $k = 1, 2, \ldots, n$ have been computed before.

2.35 **(a)** Given $\{y[n]\}=\{2 \quad 8 \quad 20 \quad 40 \quad 60 \quad 68 \quad 62 \quad 40\}$ and $\{h[n]\}=\{2 \quad 4 \quad 6 \quad 8\}$.

Therefore length of $x[n] = 8 - 4 + 1 = 5$

Using $x[n]=\frac{1}{h[0]}\left\{y[n]-\sum_{k=1}^{8} h[k]x[n-k]\right\}$, values of $x[n]$ for $n = 0, 1, \ldots, 4$ can be computed

and is given by $\{x[n]\}=\{1 \quad 2 \quad 3 \quad 4 \quad 5\}$.

(b) Here $x[0] = y[0]/h[0] = 3$, $x[1] = \frac{y[1]-x[0]h[1]}{h[0]} = \frac{8-6}{-1} = -2$. Continuing further we obtain $\{x[n]\}=\{3 \quad -2 \quad 4 \quad -6 \quad 7\}$.

(c) $x[0] = \frac{5j}{2+j} = 1 + j2$. Following the procedure outlined above, we arrive at

$\{x[n]\}=\{1+j2 \quad 2-j3 \quad 3+j\}$.

2.36 $y[n] = ay[n-1] + bx[n]$. Hence, $y[0] = ay[-1] + bx[0]$. Next,

$y[1] = ay[0] + bx[1] = a^2y[-1]+ab\,x[0]+b\,x[1]$. Continuing further in similar way we obtain

$$y[n] = a^{n+1}y[-1] + \sum_{k=0}^{n} a^{n-k} b\, x[k]..$$

(a) Let $y_1[n]$ be the output due to an input $x_1[n]$. Then $y_1[n] = a^{n+1}y[-1] + \sum_{k=0}^{n} a^{n-k} b\, x_1[k].$

If $x_1[n] = x[n - n_0]$ then

$$y_1[n] = a^{n+1}y[-1] + \sum_{k=n_0}^{n} a^{n-k} b\, x[k - n_0] = a^{n+1}y[-1] + \sum_{r=0}^{n-n_0} a^{n-n_0-r} b\, x[r].$$

However, $y[n - n_0] = a^{n-n_0+1}y[-1] + \sum_{r=0}^{n-n_0} a^{n-n_0-r} b\, x[r].$

Hence $y_1[n] \neq y[n - n_0]$ unless $y[-1] = 0$. For example, if $y[-1] = 1$ then the system is time variant. However if $y[-1] = 0$ then the system is time -invariant.

(b) Let $y_1[n]$ and $y_2[n]$ be the outputs due to the inputs $x_1[n]$ and $x_2[n]$. Let $y[n]$ be the output for an input $\alpha x_1[n] + \beta x_2[n]$. However,

$$\alpha y_1[n] + \beta y_2[n] = \alpha a^{n+1}y[-1] + \beta a^{n+1}y[-1] + \alpha \sum_{k=0}^{n} a^{n-k} b\, x_1[k] + \beta \sum_{k=0}^{n} a^{n-k} b\, x_2[k]$$

whereas

$$y[n] = a^{n+1}y[-1] + \alpha \sum_{k=0}^{n} a^{n-k} b\, x_1[k] + \beta \sum_{k=0}^{n} a^{n-k} b\, x_2[k].$$

Hence the system is not linear if $y[-1] = 1$ but is linear if $y[-1] = 0$.

(c) Generalising the above result it can be shown that for an N-th order causal discrete time system to be linear and time invariant we require $y[-N] = y[-N+1] = \cdots = y[-1] = 0$.

2.37 The difference equation representation is
$y[n] = y[n-1] - x_1[n] + x_2[n]$ with $x_1[n] \le y[n]$ and initial deposit $= y[-1]$.

2.38 (a) $f[n] = f[n-1] + f[n-2]$. Let $f[n] = \alpha r^n$, then the difference equation reduces to $\alpha r^n - \alpha r^{n-1} - \alpha r^{n-2} = 0$ which reduces to $r^2 - r - 1 = 0$ resulting in $r = \frac{1 \pm \sqrt{5}}{2}$.

Thus, $f[n] = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$.

Since $f[0] = 0$ hence $\alpha_1 + \alpha_2 = 0$. Also $f[1] = 1$ hence $\frac{\alpha_1 + \alpha_2}{2} + \sqrt{5}\,\frac{\alpha_1 - \alpha_2}{2} = 1$.

Solving for α_1 and α_2, we get $\alpha_1 = -\alpha_2 = \frac{1}{\sqrt{5}}$. Hence, $f[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

(b) $y[n] = y[n-1] + y[n-2] + x[n-1]$. As system is LTI, the initial conditions are equal to zero. Let $x[n] = \delta[n]$. Then, $y[n] = y[n-1] + y[n-2] + \delta[n-1]$. Hence,

$y[0] = y[-1] + y[-2] = 0$ and $y[1] = 1$. For $n > 1$ the corresponding difference equation is $y[n] = y[n-1] + y[n-2]$ with initial conditions $y[0] = 0$ and $y[1] = 1$, which are the same as those for the solution of Fibonacci's sequence. Hence the solution for $n > 1$ is given by

$$y[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Thus f[n] denotes the impulse response of a causal LTI system described by the difference equation $y[n] = y[n-1] + y[n-2] + x[n-1]$.

2.39 $y[n] = y[n-1] - x[n] + R$ is the difference equation representing the system.

$y[0] = D$ and $x[n] = P$. Hence $y[n] = y[n-1] - (P - R)$ or $y[n] - y[n-1] + P - R = 0$.

Thus, $y[1] = y[0] - (P - R)$, and $y[2] = y[1] - (P - R) = y[0] - 2(P - R)$. Continuing the process, we get $y[n] = y[0] - n(P - R)$.

Let N be the number of years needed to repay the loan fully then $y[N] = 0$.

Then $N = y[0]/ (P - R) = D/(P - R)$.

2.40 $y[n] = \alpha y[n-1]+x[n]$. Denoting, $y[n] = y_{re}[n] + j\, y_{im}[n]$, and $\alpha = a + jb$, we get,

$$y_{re}[n] + jy_{im}[n] = (a+jb)(y_{re}[n-1] + jy_{im}[n-1]) + x[n].$$

Equating the real and the imaginary parts , and noting that x[n] is real, we get

$$y_{re}[n] = a\,y_{re}[n-1] - b\,y_{im}[n-1] + x[n], \qquad (1)$$

$$y_{im}[n] = b\,y_{re}[n-1] + a\,y_{im}[n-1]$$

Therefore

$$y_{im}[n-1] = \frac{1}{a}y_{im}[n] - \frac{b}{a}y_{re}[n-1]$$

Hence a single input, two output difference equation is

$$y_{re}[n] = ay_{re}[n-1] - \frac{b}{a}y_{im}[n] + \frac{b^2}{a}y_{re}[n-1] + x[n]$$

thus $b\,y_{im}[n-1] = -a\,y_{re}[n-1] + (a^2+b^2)y_{re}[n-2] + a\,x[n-1]$.

Substituting the above in Eq. (1) we get

$$y_{re}[n] = 2a\,y_{re}[n-1] - (a^2+b^2)y_{re}[n-2] + x[n] - a\,x[n-1]$$

which is a second-order difference equation representing $y_{re}[n]$ in terms of x[n].

2.41 The impulse response of the cascade is given by $h[n] = h_1[n] \circledast h_2[n]$ where

$h_1[n] = \alpha^n \mu[n]$ and $h_2[n] = \beta^n \mu[n]$. Hence, $h[n] = \left(\sum_{k=0}^{n} \alpha^k \beta^{n-k}\right)\mu[n]$.

2.42 $g[n] = h[n] \circledast h[n]$ where $h[n] = \begin{cases} n+1, & 0 \le n \le 3, \\ 0, & \text{otherwise.} \end{cases}$

Hence $g[n]=\{\underset{\uparrow}{1} \quad 4 \quad 10 \quad 20 \quad 25 \quad 24 \quad 16\}$,

and $\hat{g}[n]=\{\underset{\uparrow}{1/25} \quad 4/25 \quad 2/5 \quad 4/5 \quad 1 \quad 24/25 \quad 16/25\}$.

2.43 For a filter with complex impulse response, the first part of the proof is same as that for a filter with real impulse response. Since, $y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$,

$$|y[n]|=\left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|.$$

Since the input is bounded hence $0 \le |x[n]| \le B_x$. Therefore, $|y[n]| \le B_x \sum_{k=-\infty}^{\infty} |h[k]|$.

So if $\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$ then $|y[n]| \le B_x S$ indicating that y[n] is also bounded.

To proove the converse we need to show that if a bounded output is produced by a bounded input then $S < \infty$. Consider the following bounded input defined by $x[n]=\frac{h^*[-n]}{|h[-n]|}$.

Then $y[0]=\sum_{k=-\infty}^{\infty} \frac{h^*[k]h[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| = S$. Now since the output is bounded thus $S < \infty$.

Thus for a filter with complex response too is BIBO stable if and only if $\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$.

2.44 $g[n] = h_1[n] \circledast h_2[n]$ or equivalently, $g[k]=\sum_{r=-\infty}^{\infty} h_1[k-r]h_2[r]$. Thus,

$$\sum_{k=-\infty}^{\infty} |g[k]| = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} |h_1[k-r]||h_2[r]| \le \left(\sum_{k=-\infty}^{\infty} |h_1[k]|\right)\left(\sum_{r=-\infty}^{\infty} |h_2[r]|\right).$$

Since $h_1[n]$ and $h_2[n]$ are stable, $\sum_k |h_1[k]| < \infty$ and $\sum_k |h_2[k]| < \infty$. Hence, $\sum_k |g[k]| < \infty$.

Hence the cascade of two stable LTI systems is also stable.

2.45 Consider a cascade of two passive systems. Let $y_1[n]$ be the output of the first system which is the input to the second system in the cascade. Let y[n] be the overall output of the cascade.

The first system being passive we have $\sum_{n=-\infty}^{\infty} |y_1[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2$.

Likewise the second system being also passive we have $\sum_{n=-\infty}^{\infty} |y[n]|^2 \le \sum_{n=-\infty}^{\infty} |y_1[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2$, indicating that cascade of two passive systems is also a passive system. Similarly one can prove that cascade of two lossless systems is also a lossless system.

2.46 Now, $h[n] = \alpha^n \mu[n]$. Therefore $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} \alpha^k x[n-k]$

$$= x[n] + \sum_{k=1}^{\infty} \alpha^k x[n-k] = x[n] + \alpha \sum_{k=0}^{\infty} \alpha^k x[n-1-k] = x[n] + \alpha y[n-1].$$

Hence, $x[n] = y[n] - \alpha y[n-1]$. Thus the inverse system is given by $y[n] = x[n] - \alpha x[n-1]$.

The impulse response of the inverse system is given by $g[n] = \{\underset{\uparrow}{1} \quad -\alpha\}$.

2.47 $y[n] = y[n-1] + y[n-2] + x[n-1]$. Hence, $x[n-1] = y[n] - y[n-1] - y[n-2]$, i.e. $x[n] = y[n+1] - y[n] - y[n-1]$. Hence the inverse system is characterised by $y[n] = x[n+1] - x[n] - x[n-1]$ with an impulse response given by $g[n] = \{1 \quad \underset{\uparrow}{-1} \quad -1\}$.

2.48 $y[n] = p_0 x[n] + p_1 x[n-1] - d_1 y[n-1]$ which leads to $x[n] = \frac{1}{p_0} y[n] + \frac{d_1}{p_0} y[n-1] - \frac{p_1}{p_0} x[n-1]$

Hence the inverse system is characterised by the difference equation

$$y_1[n] = \frac{1}{p_0} x_1[n] + \frac{d_1}{p_0} x_1[n-1] - \frac{p_1}{p_0} y_1[n-1].$$

2.49 Let the first order causal system be $y[n] = a\,y[n-1] + b\,x[n] + c\,x[n-1]$.

Let the input to the system be $x[n] = \delta[n]$. Then its impulse response samples are given by $h[0] = b$, $h[1] = a\,b + c$, $h[2] = ah[1]$. Solving these three equations we get

$$a = \frac{h[2]}{h[1]}, \quad b = h[0], \quad c = h[1] - \frac{h[2]h[0]}{h[1]}.$$

2.50 $\sum_{k=0}^{M} p_k x[n-k] = \sum_{k=0}^{N} d_k y[n-k]$.

Let the input to the system be $x[n] = \delta[n]$. Then, $\sum_{k=0}^{M} p_k \delta[n-k] = \sum_{k=0}^{N} d_k h[n-k]$. Thus,

$p_r = \sum_{k=0}^{N} d_k h[r-k]$. Since the system is assumed to be causal, $h[r-k] = 0 \;\; \forall k > r$.

$$p_r = \sum_{k=0}^{r} d_k h[r-k] = \sum_{k=0}^{r} h[k] d_{r-k}.$$

2.51 $v[n] = \sum_{k=0}^{M} p_k x[n-k]$. Define another causal FIR system with input and output related by

$y[n] = \sum_{k=0}^{N} d_k v[n-k]$. The inverse system of the new system is defined by

$v[n] = \sum_{k=0}^{N} d_k y[n-k]$. This implies $\sum_{k=0}^{M} p_k x[n-k] = \sum_{k=0}^{N} d_k y[n-k]$.

Hence a causal IIR digital filter can be realised as a cascade of an FIR filter and inverse of a causal FIR fliter.

2.52 Let $\sum_{k=0}^{M} p_k x[n-k] = y[n] + \sum_{k=1}^{N} d_k y[n-k]$ be the difference equation representing the causal IIR digital filter. For an input $x[n] = \delta[n]$, the corresponding output is then y[n] = h[n], the impulse response of the filter. As there are M+1 $\{p_k\}$ coefficients, and N $\{d_k\}$ coefficients, there are a total of N+M+1 unknowns. To determine these coefficients from the impulse response samples, we compute only the first N+M+1 samples. To illstrate the method, without any loss of generality, we assume N = M = 3. Then , from the difference equation reprsentation we arrive at the following N+M+1 = 7 equations:

$$h[0] = p_0,$$
$$h[1] + h[0]d_1 = p_1,$$
$$h[2] + h[1]d_1 + h[0]d_2 = p_2,$$
$$h[3] + h[2]d_1 + h[1]d_2 + h[0]d_3 = p_3,$$
$$h[4] + h[3]d_1 + h[2]d_2 + h[1]d_3 = 0,$$
$$h[5] + h[4]d_1 + h[3]d_2 + h[2]d_3 = 0,$$
$$h[6] + h[5]d_1 + h[4]d_2 + h[3]d_3 = 0.$$

Writing the last three equations in matrix form we arrive at

$$\begin{bmatrix} h[4] \\ h[5] \\ h[6] \end{bmatrix} + \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and hence, $$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = -\begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix}^{-1} \begin{bmatrix} h[4] \\ h[5] \\ h[6] \end{bmatrix}.$$

Substituting these values of $\{d_i\}$ in the first four equations written in matrix form we get

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} 1 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

2.53 **(a)** $v[n] = (h_1[n] + h_3[n] \circledast h_5[n]) \circledast x[n]$ and $y[n] = h_2[n] \circledast v[n] + h_3[n] \circledast h_4[n] \circledast x[n]$.

Thus, $y[n] = (h_2[n] \circledast h_1[n] + h_2[n] \circledast h_3[n] \circledast h_5[n] + h_3[n] \circledast h_4[n]) \circledast x[n]$.

Hence the impulse response is given by

$h[n] = h_2[n] \circledast h_1[n] + h_2[n] \circledast h_3[n] \circledast h_5[n] + h_3[n] \circledast h_4[n]$

(b) $v[n] = h_4[n] \circledast x[n] + h_1[n] \circledast h_2[n] \circledast x[n]$.

Thus, $y[n] = h_3[n] \circledast v[n] + h_1[n] \circledast h_5[n] \circledast x[n]$

$$= h_3[n] \circledast h_4[n] \circledast x[n] + h_3[n] \circledast h_1[n] \circledast h_2[n] \circledast x[n] + h_1[n] \circledast h_5[n] \circledast x[n]$$

Hence the impulse response is given by

$$h[n] = h_3[n] \circledast h_4[n] + h_3[n] \circledast h_1[n] \circledast h_2[n] + h_1[n] \circledast h_5[n]$$

2.54 $u[n] = h_1[n] \circledast h_2[n] + h_3[n]$.

But, $h_1[n] \circledast h_2[n] = 6\,\delta[n]+2\,\delta[n-1]+3\,\delta[n-2]+\delta[n-3]$. Hence,

$$h[n]=7\,\delta[n]-\delta[n-1]+3\,\delta[n-2]+\delta[n-3]+7\,\delta[n-4]+6\,\delta[n-6].$$

2.55 The state equations are given by

$$s_2[n] = s_1[n-1],$$
$$s_3[n] = s_2[n-1],$$
$$s_1[n+1] = x[n] - 0.8 s_1[n] - 0.6 s_2[n] - 0.4 s_3[n]$$
$$y[n] = 0.4\big(x[n] - 0.8 s_1[n] - 0.6 s_2[n] - 0.4 s_3[n]\big) + 0.6 s_1[n] + s_2[n] + 0.8 s_3[n]$$

i.e., $$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \\ s_3[n+1] \end{bmatrix} = \begin{bmatrix} -0.8 & -0.6 & -0.4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x[n]$$

$$y[n] = \begin{bmatrix} 0.28 & 0.76 & 0.64 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} + 0.4 x[n]$$

2.56 For a second-order system, the difference equation is of order 2, and is of the form

$$p_0 x[n] + p_1 x[n-1] + p_2 x[n-2] = y[n] + d_1 y[n-1] + d_2 y[n-2].$$

We thus need to determine 5 coefficients, p_0, p_1, p_2, d_1 and d_2 from the state-space description. To this end, we first determine the first 5 samples of the impulse response samples from the state-space description of the causal LTI discrete-time system:

$$h[n] = \begin{cases} D, & \text{for } n = 0, \\ \mathbf{C}\mathbf{A}^{n-1}\mathbf{B}, & \text{for } n > 0. \end{cases}$$

Here, $\mathbf{A} = \begin{bmatrix} 0.5 & 0.4 \\ -0.2 & 0.3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 3 \end{bmatrix}$, and $D = 0.4$.

Thus, $h[0] = 0.4$, $h[1] = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 11$, $h[2] = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0.5 & 0.4 \\ -0.2 & 0.3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 5.5$,

$h[3] = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0.5 & 0.4 \\ -0.2 & 0.3 \end{bmatrix}^2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1.87$, and $h[4] = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0.5 & 0.4 \\ -0.2 & 0.3 \end{bmatrix}^3\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0.231$.

Next, following the results of Problem 2.52, we arrive at

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -\begin{bmatrix} h[2] & h[1] \\ h[3] & h[2] \end{bmatrix}^{-1}\begin{bmatrix} h[3] \\ h[4] \end{bmatrix} = -\begin{bmatrix} 5.5 & 11 \\ 1.87 & 5.5 \end{bmatrix}^{-1}\begin{bmatrix} 1.87 \\ 0.231 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 0.23 \end{bmatrix},$$

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} 1 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0 & 0 \\ 11 & 0.4 & 0 \\ 5.5 & 11 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ 0.23 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 10.68 \\ -3.208 \end{bmatrix}.$$

Hence, the difference equation is given by

$$0.4\,x[n] + 10.68\,x[n-1] - 3.208\,x[n-2] = y[n] - 0.8\,y[n-1] + 0.23\,y[n-2].$$

2.57 $\sigma_x^2 = E\left\{(x - m_x)^2\right\} = E\left\{x^2 + m_x^2 - 2xm_x\right\} = E\{x^2\} + E\{m_x^2\} - 2E\{xm_x\}.$

Since m_x is a constant, hence $E\{m_x^2\} = m_x^2$ and $E\{xm_x\} = m_x E\{x\} = m_x^2$. Thus,

$\sigma_x^2 = E\{x^2\} + m_x^2 - 2m_x^2 = E\{x^2\} - m_x^2.$

2.58 $E\{x+y\} = \iint (x+y) f_{xy}(x,y)\,dx\,dy = \iint x f_{xy}(x,y)\,dx\,dy + \iint y f_{xy}(x,y)\,dx\,dy$

$= \int x\left(\int f_{xy}(x,y)\,dy\right)dx + \int y\left(\int f_{xy}(x,y)\,dx\right)dy = \int x f_x(x)\,dx + \int y f_y(y)\,dy$

$= E\{x\} + E\{y\}.$

2.59 mean $= m_x = E\{x\} = \int_{-\infty}^{\infty} x p_X(x)\,dx$

variance $= \sigma_x^2 = E\{(x - m_x)^2\} = \int_{-\infty}^{\infty} (x - m_x)^2 p_X(x)\,dx$

(a) $p_X(x) = \dfrac{\alpha}{\pi}\left(\dfrac{1}{x^2 + \alpha^2}\right)$. Now, $m_x = \dfrac{\alpha}{\pi}\int_{-\infty}^{\infty} \dfrac{x\,dx}{x^2 + \alpha^2}$. Since $\dfrac{x}{x^2 + \alpha^2}$ is odd hence the integral is zero. Thus $m_x = 0$.

$\sigma_x^2 = \dfrac{\alpha}{\pi}\int_{-\infty}^{\infty} \dfrac{x^2\,dx}{x^2 + \alpha^2}$. Since this integral does not converge hence variance is not defined for this density function.

(b) $p_X(x) = \dfrac{\alpha}{2} e^{-\alpha|x|}$. Now, $m_x = \dfrac{\alpha}{2}\int_{-\infty}^{\infty} x\, e^{-\alpha|x|}\,dx = 0$. Next,

$$\sigma_x^2 = \frac{\alpha}{2}\int_{-\infty}^{\infty} x^2 e^{-\alpha|x|}\,dx = \alpha\int_0^{\infty} x^2 e^{-\alpha x}\,dx = \alpha\left\{\left.\frac{x^2 e^{-\alpha x}}{-\alpha}\right|_0^{\infty} + \int_0^{\infty} \frac{2x}{\alpha} e^{-\alpha x}\,dx\right\}$$

$$= \alpha\left\{0 + \left.\frac{2x}{\alpha}\frac{e^{-\alpha x}}{-\alpha}\right|_0^{\infty} + \int_0^{\infty} \frac{2}{\alpha^2} e^{-\alpha x}\,dx\right\} = \frac{2}{\alpha^2}.$$

(c) $p_X(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} p^{\ell}(1-p)^{n-\ell}\delta(x-\ell)$. Now,

$$m_x = \int_{-\infty}^{\infty} x \sum_{\ell=0}^{n} \binom{n}{\ell} p^{\ell}(1-p)^{n-\ell}\delta(x-\ell)\,dx = \sum_{\ell=0}^{n} \binom{n}{\ell} p^{\ell}(1-p)^{n-\ell} = np$$

$$\sigma_x^2 = E\{x^2\} - m_x^2 = \int_{-\infty}^{\infty} x^2 \sum_{\ell=0}^{n} \binom{n}{\ell} p^\ell (1-p)^{n-\ell} \delta(x-\ell)dx - (np)^2$$

$$= \sum_{\ell=0}^{n} \ell^2 \binom{n}{\ell} p^\ell (1-p)^{n-\ell} - n^2p^2 = n\,p(1-p).$$

(d) $p_x(x) = \sum_{\ell=0}^{\infty} \frac{e^{-\alpha}\alpha^\ell}{\ell!}\delta(x-\ell)$. Now,

$$m_x = \int_{-\infty}^{\infty} x \sum_{\ell=0}^{\infty} \frac{e^{-\alpha}\alpha^\ell}{\ell!}\delta(x-\ell)dx = \sum_{\ell=0}^{\infty} \ell \frac{e^{-\alpha}\alpha^\ell}{\ell!} = \alpha.$$

$$\sigma_x^2 = E\{x^2\} - m_x^2 = \frac{\alpha}{2}\int_{-\infty}^{\infty} x^2 \sum_{\ell=0}^{\infty} \frac{e^{-\alpha}\alpha^\ell}{\ell!}\delta(x-\ell)\,dx - \alpha^2 = \sum_{\ell=0}^{\infty} \ell^2 \frac{e^{-\alpha}\alpha^\ell}{\ell!} - \alpha^2 = \alpha.$$

(e) $p_x(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2}\mu(x)$. Now,

$$m_x = \frac{1}{\alpha^2}\int_{-\infty}^{\infty} x^2 e^{-x^2/2\alpha^2}\mu(x)dx = \frac{1}{\alpha^2}\int_0^{\infty} x^2 e^{-x^2/2\alpha^2}dx = \alpha\sqrt{\pi/2}.$$

$$\sigma_x^2 = E\{x^2\} - m_x^2 = \frac{1}{\alpha^2}\int_{-\infty}^{\infty} x^3 e^{-x^2/2\alpha^2}\mu(x)dx - \frac{\alpha^2\pi}{2} = \left(2 - \frac{\pi}{2}\right)\alpha^2.$$

2.60 Recall that random variables x and y are linearly independent if and only if

$E\left\{|a_1x + a_2y|^2\right\} > 0 \quad \forall\, a_1, a_2$ except when $a_1 = a_2 = 0$, Now,

$$E\left\{|a_1|^2|x|^2\right\} + E\left\{|a_2|^2|y|^2\right\} + E\{(a_1)^* a_2 x y^*\} + E\{a_1(a_2)^* x^* y\} = |a_1|^2 E\left\{|x|^2\right\} + |a_2|^2 E\left\{|y|^2\right\} > 0$$

$\forall\, a_1$ and a_2 except when $a_1 = a_2 = 0$.

Hence if x,y are statistically independent they are also linearly independent.

2.61 $\phi_{xy}[\ell] = E\{x[n+\ell]y^*[n]\}$, $\quad \phi_{xy}[-\ell] = E\{x[n-\ell]y^*[n]\}$, $\quad \phi_{yx}[\ell] = E\{y[n+\ell]x^*[n]\}$.

Therefore, $\phi_{yx}^*[\ell] = E\{y^*[n+\ell]x[n]\} = E\{x[n-\ell]y^*[n]\} = \phi_{xy}[-\ell]$.

Hence $\phi_{xy}[-\ell] = \phi_{yx}^*[\ell l]$.

Since $\gamma_{xy}[\ell] = \phi_{xy}[\ell] - m_x(m_y)^*$. Thus, $\gamma_{xy}[\ell] = \phi_{xy}[\ell] - m_x(m_y)^*$. Hence, $\gamma_{xy}[\ell] = \phi_{xy}[\ell] - m_x(m_y)^*$. As a result, $\gamma_{xy}^*[\ell] = \phi_{xy}^*[\ell] - (m_x)^* m_y$.

Hence, $\gamma_{xy}[-\ell] = \gamma_{yx}^*[\ell]$.

The remaining two properties can be proved in a similar way by letting x = y.

2.62 $E\left\{|x[n] - x[n-\ell]|^2\right\} \ge 0.$

$E\left\{|x[n]|^2\right\} + E\left\{|x[n-\ell]|^2\right\} - E\{x[n]x^*[n-\ell]\} - E\{x^*[n]x[n-\ell]\} \ge 0$

$2\phi_{xx}[0] - 2\phi_{xx}[\ell] \ge 0$

$\phi_{xx}[0] \geq \phi_{xx}[\ell]$

Using Cauchy's inequality $E\{|x|^2\}E\{|y|^2\} \leq E^2\{xy\}$. Hence, $\phi_{xx}[0]\phi_{yy}[0] \leq |\phi_{xy}[\ell]|^2$.

One can show similarly $\gamma_{xx}[0]\,\gamma_{yy}[0] \leq |\gamma_{xy}[\ell]|^2$.

2.63 Since there are no periodicities in $\{x[n]\}$ hence $x[n]$, $x[n+\ell]$ become uncorrelated as $\ell \to \infty$.. Thus $\lim_{\ell\to\infty} \gamma_{xx}[\ell] = \lim_{\ell\to\infty} \phi_{xx}[\ell] - |m_x|^2 \to 0$. Hence $\lim_{\ell\to\infty} \phi_{xx}[\ell] = |m_x|^2$.

2.64 $C = E\{(X-\kappa)^2\}$. To find the value of κ that minimize the mean square error C, we differentiate C with respect to κ and set it to zero.

Thus $\frac{dC}{d\kappa} = E\{2(X-\kappa)\} = 0$ which results in $\kappa = E\{X\}$, and the minimum value of C is σ_x^2.

2.65 $V = aX + bY$. Therefore, $E\{V\} = a\,E\{X\} + b\,E\{Y\} = a\,m_x + b\,m_y$. and

$\sigma_v^2 = E\{(V-m_v)^2\} = E\{(a(X-m_x) + b(Y-m_y))^2\}$.

Since X,Y are statistically independent hence $\sigma_v^2 = a^2\sigma_x^2 + b^2\sigma_y^2$.

2.66 **(a)** $r[n] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{\ell=-M}^{M} x[\ell]x[\ell+n]$. Since x[n] is periodic with period N, hence x[n+N] = x[n]. Now, $r[n+N] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{\ell=-M}^{M} x[\ell]x[\ell+n+N] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{\ell=-M}^{M} x[\ell]x[\ell+n] = r[n]$.

Hence r[n] is also periodic with a period N.

(b) $r[-n] = \sum_{\ell=-\infty}^{\infty} x[\ell]x[\ell-n] = \sum_{\ell=-\infty}^{\infty} x[\ell+n]x[\ell] = r[n]$. Hence r[n] is an even function of n.

(c) Since $\sum_{\ell=-\infty}^{\infty} (x[\ell] - x[\ell+n])^2 \geq 0$, it follows that $2r[0] - 2r[n] \geq 0$. Hence $r[0] \geq r[n]$ for all values of n.

(d) For an aperiodic sequences r[0] = $\sum_{\ell=-\infty}^{\infty} x^2[\ell]$ which is equal to the energy of the sequence.

For an periodic sequence, the average power is defined by $P_{av} = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{\ell=-M}^{M} x^2[\ell]$.

Substituting n = 0 in the autocorrelation sequence of a periodic random sequence we get r[0] is equal to its average power.

(e) Since the autocorrelation sequence of a periodic sequence is also periodic with the same period, hence an way to determine the period of the actual sequence is by measuring the distance between two maxima's say r[0] and r[N] in the autocorrelation sequence. Note that due to part (c) r[0] will always have the maximum value.

2.67 **(a)** $x_1[n] = \sin\left(\frac{2\pi n}{M}\right)$. Since this sequence is periodic with a period M it follows that

$$r[n] = \frac{1}{M}\sum_{\ell=0}^{M-1} x_1[\ell]x_1[n+\ell] = \frac{1}{M}\sum_{\ell=0}^{M-1}\sin\left(\frac{2\pi\ell}{M}\right)\sin\left(\frac{2\pi(\ell+n)}{M}\right)$$

$$= \frac{1}{M}\sum_{\ell=0}^{M-1}\left\{\sin\left(\frac{2\pi\ell}{M}\right)\sin\left(\frac{2\pi\ell}{M}\right)\cos\left(\frac{2\pi n}{M}\right)+\cos\left(\frac{2\pi\ell}{M}\right)\sin\left(\frac{2\pi\ell}{M}\right)\sin\left(\frac{2\pi n}{M}\right)\right\}$$

$$= \frac{1}{2}\cos\left(\frac{2\pi n}{M}\right).$$

Thus, r[n] is periodic. The period of the sequences r[n] and x[n] is M.

(b) $x_2[n] = n \text{ mod ulo} 7$. Since the above sequence is periodic with period 7, it follows that

$$r[n] = \frac{1}{7}\sum_{\ell=0}^{6} x_2[\ell]x_2[n+\ell] = \{\underset{\uparrow}{13} \quad 10 \quad 8 \quad 7 \quad 7 \quad 8 \quad 10\},$$ r[n] is periodic with a period 7.

(c) The sequence $x_3[n] = (-1)^n$ is periodic with a period 2. Hence

$$r[n] = \frac{1}{2}\sum_{\ell=0}^{1} x_3[\ell]x_3[n+\ell] = \{\underset{\uparrow}{1} \quad -1\}.$$ Thus, r[n] is periodic with a period 2.

M2.1
```
L = input('Desired length = '); n = 1:L;
FT = input('Sampling frequency = ');T = 1/FT;
imp = [1 zeros(1,L-1)];step = ones(1,L);
ramp = (n-1).*step;
subplot(3,1,1);
stem(n-1,imp);
xlabel(['Time in ',num2str(T), ' sec']);ylabel('Amplitude');
subplot(3,1,2);
stem(n-1,step);
xlabel(['Time in ',num2str(T), ' sec']);ylabel('Amplitude');
subplot(3,1,3);
stem(n-1,ramp);
xlabel(['Time in ',num2str(T), ' sec']);ylabel('Amplitude');
```

M2.2
```
A=input('The peak value =');
L=input('Length of sequence =');
N=input('The period of sequence =');
FT=input('The desired sampling frequency =');
T=1/FT;
t=0:L-1;
x=A*sawtooth(2*pi*t/N);
y=A*square(2*pi*(t/N),40);
subplot(211)
```

```
stem(t,x);
ylabel('Amplitude');
xlabel(['Time in ',num2str(T),'sec']);
subplot(212)
stem(t,y);
ylabel('Amplitude');
xlabel(['Time in ',num2str(T),'sec']);
```

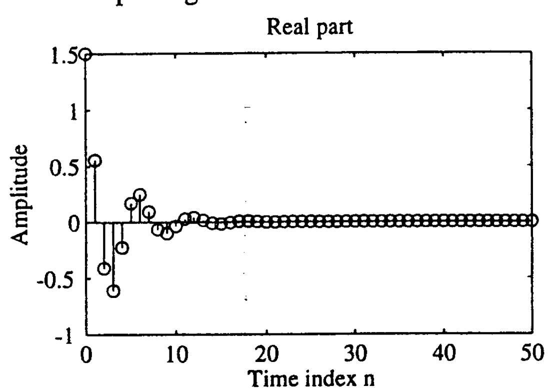
M2.3 (a) The input data entered during the execution of Program 2_1 are

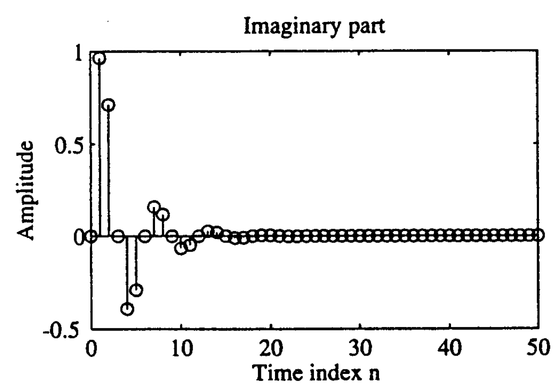
```
Type in real exponent = -1/12
Type in imaginary exponent = pi/6
Type in the gain constant = 1
Type in length of sequence = 41
```

(b) The input data entered during the execution of Program 2_1 are

```
Type in real exponent = -0.3
Type in imaginary exponent = pi/3
Type in the gain constant = 1.5
Type in length of sequence = 51
```

The plots generated are shown below:

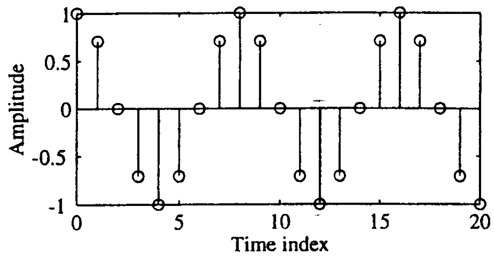




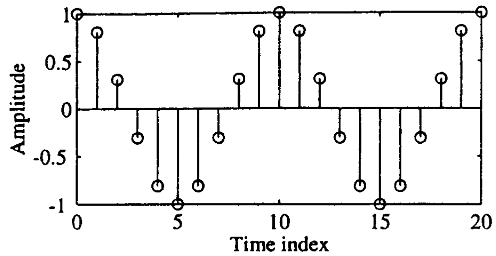
M2.4 (a)

```
L = input('Desired length = ');
A = input('Amplitude = ');
omega = input('Angular frequency = ');
phi = input('Phase = ');
n = 0:L-1;
x = A*cos(omega*n + phi);
stem(n,x);
xlabel('Time index');ylabel('Amplitude');
```

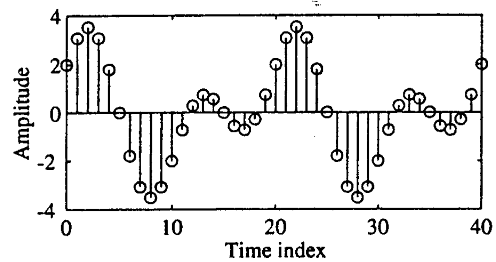
(b)



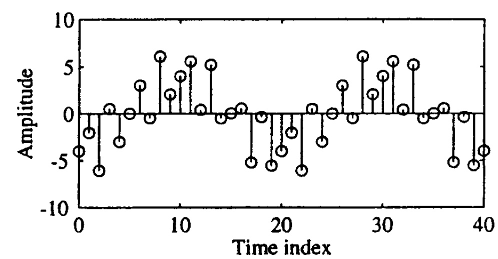
(i) $\cos(0.25\pi n)$



(ii) $\cos(0.2\pi n)$



(iii) $2\cos(0.1\pi n) + 2\sin(0.2\pi n)$



(iv) $3\sin(0.8\pi n) - 4\cos(0.1\pi n)$

M2.5
```
t = 0:0.001:1;
fo = input('Frequency of sinusoid in Hz = ');
FT = input('Samplig frequency in Hz = ');
g1 = cos(2*pi*fo*t);
plot(t,g1,'-')
xlabel('time');ylabel('Amplitude')
hold
n = 0:1:FT;
gs = cos(2*pi*fo*n/FT);
plot(n/FT,gs,'o')
hold off
```

M2.6
```
t = 0:0.001:0.85;
g1 = cos(6*pi*t);g2 = cos(14*pi*t);g3 = cos(26*pi*t);
plot(t/0.85,g1,'-',t/0.85,g2,'--',t/0.85,g3,':')
xlabel('time');ylabel('Amplitude')
hold
n = 0:1:8;
gs = cos(0.6*pi*n);
plot(n/8.5,gs,'o')
```

M2.7 As the length of the moving average filter is increased, the output of the filter gets more smoother. However, the delay between the input and the output sequences also increases (This can be seen from the plots generated by Program 2_4 for various values of the filter length.)

M2.8
```
alpha = input('Alpha = ');
yo = 1;y1 = 0.5*(yo + (alpha/yo));
while abs(y1 - yo) > 0.00001
y2 = 0.5*(y1 + (alpha/y1));
yo = y1; y1 = y2;
end
disp('Square root of alpha is'); disp(y1)
```

M2.9

```
alpha = input('Alpha = ');
yo = 1;y1 = 0.5*(yo + (alpha/yo));
while abs(y1 - yo) > 0.00001
y2 = 0.5*(y1 + (alpha/y1));
yo = y1; y1 = y2;
end
disp('Square root of alpha is'); disp(y1)
```

M2.10

```
y = input('First sequence = ');
h = input('Second sequence = ');
[x,r] = deconv(y,h);
disp(x)
```

(a) `1 2 3 4 5`, **(b)** `3 -2 4 -6 7`,

(c) `1 + i2 2 - i3 3 + i1`

Chapter 3

3.1 $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. If $\{x[n]\}$ is absolutely summable then $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$. Thus,

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]|\left|e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Hence, if $\{x[n]\}$ is absolutely summable then $X(e^{j\omega})$ converges.

3.2 **(a)** $x[n] = \delta[n]$. Then, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = e^{-j\omega 0} = 1.$

(b) $y[n] = \mu[n] = y_{ev}[n] + y_{od}[n]$, where $y_{ev}[n] = \frac{1}{2}(y[n] + y[-n]) = \frac{1}{2}(\mu[n] + \mu[-n]) = \frac{1}{2} + \frac{1}{2}\delta[n]$,

and $y_{od}[n] = \frac{1}{2}(y[n] - y[-n]) = \frac{1}{2}(\mu[n] - \mu[-n]) = \mu[n] - \frac{1}{2} - \frac{1}{2}\delta[n].$

Now, $Y_{ev}(e^{j\omega}) = \frac{1}{2}\left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)\right] + \frac{1}{2} = \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k) + \frac{1}{2}.$

Since $y_{od}[n] = \mu[n] - \frac{1}{2} + \frac{1}{2}\delta[n]$, $y_{od}[n] = \mu[n-1] - \frac{1}{2} + \frac{1}{2}\delta[n-1]$. As a result,

$y_{od}[n] - y_{od}[n-1] = \mu[n] - \mu[n-1] + \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$. Taking the DTFT of both sides we then get $Y_{od}(e^{j\omega}) - e^{-j\omega}\, Y_{od}(e^{j\omega}) = \frac{1}{2}\left(1 + e^{-j\omega}\right)$. or

$Y_{od}(e^{j\omega}) = \frac{1}{2}\frac{1 + e^{-j\omega}}{1 - e^{-j\omega}} = \frac{1}{1 - e^{-j\omega}} - \frac{1}{2}$. Hence,

$$Y(e^{j\omega}) = Y_{ev}(e^{j\omega}) + Y_{od}(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k).$$

(c) Let x[n] be the sequence with the DTFT $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$. Its inverse DTFT is then given by $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_o)e^{j\omega n}d\omega = e^{j\omega_o n}$.

(d) $w[n] = \alpha^n \mu[n]$. Then, $W(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-\alpha e^{-j\omega}}$.

3.3 **(a)** Let $f[n] = \alpha g[n] + \beta h[n]$, Then $F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\omega n}$.

$$= \sum_{n=-\infty}^{\infty} (\alpha g[n] + \beta h[n])e^{-j\omega n} = \alpha G(e^{j\omega}) + \beta H(e^{j\omega}).$$

(b) Let $y[n] = g[n - n_o]$, Then $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} g[n - n_o]e^{-j\omega n}$

$$= e^{-j\omega n_o} \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} = e^{-j\omega n_o} G(e^{j\omega}).$$

(c) Let $h[n] = e^{j\omega_o n} g[n]$, then $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j\omega_o n} g[n]e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} g[n]e^{-j(\omega-\omega_o)n} = G(e^{j(\omega-\omega_o)}).$$

(d) $G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$. Hence $\frac{d(G(e^{j\omega}))}{d\omega} = \sum_{n=-\infty}^{\infty} -jng[n]e^{-j\omega n}$.

Therefore, $j\frac{d(G(e^{j\omega}))}{d\omega} = \sum_{n=-\infty}^{\infty} ng[n]e^{-j\omega n}$. Thus the DTFT of ng[n] is $j\frac{d(G(e^{j\omega}))}{d\omega}$.

(e) $y[n] = g[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$. Hence $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g[k]h[n-k]e^{-j\omega n}$

$$= \sum_{k=-\infty}^{\infty} g[k]H(e^{j\omega})e^{-j\omega k} = H(e^{j\omega}) \sum_{k=-\infty}^{\infty} g[k]e^{-j\omega k} = H(e^{j\omega})G(e^{j\omega}).$$

(f) $y[n] = g[n]h[n]$. Hence $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]h[n]e^{-j\omega n}$

Since $g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})e^{j\theta n}d\theta$ we can rewrite the above DTFT as

$$Y(e^{j\omega}) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} h[n]e^{-j\omega n}G(e^{j\theta})e^{j\theta n}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})\sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega-\theta)n}d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta.$$

(g) $y[n] = \sum_{n=-\infty}^{\infty} g[n]h^*[n] = \sum_{n=-\infty}^{\infty} g[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})e^{-j\omega n}d\omega\right)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})\left(\sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}\right)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})G(e^{j\omega})d\omega .$$

3.4 $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

(a) $X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} = \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}$. Thus $x[-n] \Leftrightarrow X(e^{-j\omega})$.

(b) $X^*(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x^*[n]e^{j\omega n} = \sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n}$. Thus $x^*[-n] \Leftrightarrow X^*(e^{j\omega})$.

(c) $\text{Re}\{x[n]\} = \frac{1}{2}\{x[n] + x^*[n]\}$. Now the DTFT of $x^*[n]$ is $X^*(e^{-j\omega})$. Hence, using the linearity property of the DTFT we obtain $\text{Re}\{x[n]\} \Leftrightarrow \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$.

(d) $\text{Im}\{x[n]\} = \frac{1}{2j}\{x[n] - x^*[n]\}$. Hence, using the linearity property of the DTFT we obtain $j\,\text{Im}\{x[n]\} \Leftrightarrow \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$.

(e) $x_{cs}[n] = \frac{1}{2}(x[n] + x^*[-n])$. Using the linearity property and the results of part (b) we get $x_{cs}[n] \Leftrightarrow \frac{1}{2}\left\{X(e^{j\omega}) + X^*(e^{j\omega})\right\} = \text{Re}\{X(e^{j\omega})\}$.

(f) $x_{ca}[n] = \frac{1}{2}(x[n] - x^*[-n])$. Using the linearity property and the results of part (b) we get $x_{ca}[n] \Leftrightarrow \frac{1}{2}\left(X(e^{j\omega}) - X^*(e^{j\omega})\right) = j\,\text{Im}\{X(e^{j\omega})\}$.

3.5 (a) $x_{ev}[n] = \frac{1}{2}\{x[n] + x[-n]\}$. Since $x[n]$ is real, $x[-n] = x^*[-n]$. Thus $x_{ev}[n] = \frac{1}{2}\{x[n] + x^*[-n]\}$. Now using the linearity property and the results of Problem 3.4 part (b) we get $x_{ev}[n] \Leftrightarrow \frac{1}{2}\left\{X(e^{j\omega}) + X^*(e^{j\omega})\right\} = \text{Re}\{X(e^{j\omega})\}$

(b) $x_{od}[n] = \frac{1}{2}\{x[n] - x[-n]\} = \frac{1}{2}\{x[n] - x^*[-n]\}$. Thus $x_{od}[n] \Leftrightarrow \frac{1}{2}\left(X(e^{j\omega}) - X^*(e^{j\omega})\right) = j\,\text{Im}\{X(e^{j\omega})\}$

(c) Since $x[n] = x^*[n]$. This implies $X(e^{j\omega}) = X^*(e^{-j\omega})$. Thus $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ and $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$. Also $\left|X(e^{j\omega})\right| = \sqrt{X_{re}^2(e^{j\omega}) + X_{im}^2(e^{j\omega})} = \sqrt{X_{re}^2(e^{-j\omega}) + X_{im}^2(e^{-j\omega})}$ $= \left|X(e^{-j\omega})\right|$. Likewise, $\arg X(e^{j\omega}) = \tan^{-1}\frac{X_{im}(e^{j\omega})}{X_{re}(e^{j\omega})} = -\tan^{-1}\frac{X_{im}(e^{-j\omega})}{X_{re}(e^{-j\omega})} = -\arg X(e^{j\omega})$.

3.6 $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$. Hence, $x^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})e^{-j\omega n}d\omega$.

(a) Since x[n] is real and even, hence $X(e^{j\omega}) = X^*(e^{j\omega})$. Thus,

$$x[-n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{-j\omega n}d\omega,$$

Therefore, $x[n] = \frac{1}{2}(x[n] + x[-n]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cos(\omega n)d\omega.$

Now x[n] being even, $X(e^{j\omega}) = X(e^{-j\omega})$. As a result, the term inside the above integral is even,

and hence $x[n] = \frac{1}{\pi}\int_{0}^{\pi} X(e^{j\omega})\cos(\omega n)d\omega$

(b) Since x[n] is odd hence x[n] = – x[– n].

Thus $x[n] = \frac{1}{2}(x[n] - x[-n]) = \frac{j}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\sin(\omega n)d\omega$. Again, since x[n] = – x[– n],

$X(e^{j\omega}) = -X(e^{-j\omega})$. The term inside the integral is even, hence $x[n] = \frac{j}{\pi}\int_{0}^{\pi} X(e^{j\omega})\sin(\omega n)d\omega$

3.7 $x[n] = \alpha^n \cos(\omega_o n + \phi)\mu[n] = A\alpha^n\left(\frac{e^{j\omega_o n}e^{j\phi} + e^{-j\omega_o n}e^{-j\phi}}{2}\right)\mu[n]$

$$= \frac{A}{2}e^{j\phi}\left(\alpha e^{j\omega_o}\right)^n\mu[n] + \frac{A}{2}e^{-j\phi}\left(\alpha e^{-j\omega_o}\right)\mu[n].$$ Therefore,

$$X(e^{j\omega}) = \frac{A}{2}e^{j\phi}\frac{1}{1-\alpha e^{-j\omega}e^{j\omega_o}} + \frac{A}{2}e^{-j\phi}\frac{1}{1-\alpha e^{-j\omega}e^{-j\omega_o}}.$$

3.8 **(a)** $x_1[n] = \alpha^n\mu[n]$, $|\alpha| < 1$. Thus, $X_1(e^{j\omega}) = \sum_{n=0}^{\infty}(\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$

(b) $x_2[n] = \alpha^n\mu[-n]$, $|\alpha| > 1$. Thus, $X_2(e^{j\omega}) = \sum_{n=-\infty}^{0}\alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty}\alpha^{-n}e^{j\omega n} = \frac{1}{1-\alpha^{-1}e^{j\omega}}$.

(c) $x_3[n] = \begin{cases}\alpha^{|n|}, & |n| \le M,\\ 0, & \text{otherwise.}\end{cases}$ Then, $X_3(e^{j\omega}) = \sum_{n=0}^{M}\alpha^n e^{-j\omega n} + \sum_{n=-M}^{-1}\alpha^{-n}e^{-j\omega n}$

$$= \frac{1-\alpha^{M+1}e^{-j\omega(M+1)}}{1-\alpha e^{-j\omega}} + \alpha^M e^{j\omega M}\frac{1-\alpha^{-M}e^{-j\omega M}}{1-\alpha^{-1}e^{-j\omega}}.$$

(d) $x_4[n] = \alpha^n\mu[n+3]$, $|\alpha| < 1$. Note $x_4[n] = \alpha^{-3}x_1[n+3]$. Hence,

$$X_4(e^{j\omega}) = \alpha^{-3}e^{j3\omega}X_1(e^{j\omega}) = \frac{\alpha^{-3}e^{j3\omega}}{1-\alpha e^{-j\omega}}.$$

(e) $x_5[n] = n\alpha^n\mu[n]$. Note that $x_5[n] = nx_1[n]$. Hence,

$$X_5(e^{j\omega}) = j\frac{dX_1(e^{j\omega})}{d\omega} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}.$$

3.9 (a) $y_1[n]=\begin{cases}1, & -N\le n\le N,\\ 0, & \text{otherwise.}\end{cases}$ Then $Y_1(e^{j\omega})=\sum_{n=-N}^{N}e^{-j\omega n}=e^{j\omega N}\dfrac{(1-e^{-j\omega(2N+1)})}{(1-e^{-j\omega})}=\dfrac{\sin(\omega[N+\frac{1}{2}])}{\sin(\omega/2)}$

(b) $y_2[n]=\begin{cases}1-\dfrac{|n|}{N}, & -N\le n\le N,\\ 0, & \text{otherwise.}\end{cases}$ Now $y_2[n]=y_0[n]\circledast y_0[n]$ where

$y_0[n]=\begin{cases}1, & -N/2\le n\le N/2,\\ 0, & \text{otherwise.}\end{cases}$ Thus $Y_2(e^{j\omega})=Y_0^2(e^{j\omega})=\dfrac{\sin^2\left(\omega\left[\frac{N+1}{2}\right]\right)}{\sin^2(\omega/2)}$.

(c) $y_3[n]=\begin{cases}\cos(\pi n/2N), & -N\le n\le N,\\ 0, & \text{otherwise.}\end{cases}$ Then,

$$Y_3(e^{j\omega})=\frac{1}{2}\sum_{n=-N}^{N}e^{-j(\pi n/2N)}e^{-j\omega n}+\frac{1}{2}\sum_{n=-N}^{N}e^{j(\pi n/2N)}e^{-j\omega n}$$
$$=\frac{1}{2}\sum_{n=-N}^{N}e^{-j\left(\omega-\frac{\pi}{2N}\right)n}+\frac{1}{2}\sum_{n=-N}^{N}e^{-j\left(\omega+\frac{\pi}{2N}\right)n}$$
$$=\frac{1}{2}\frac{\sin\left((\omega-\frac{\pi}{2N})(N+\frac{1}{2})\right)}{\sin\left((\omega-\frac{\pi}{2N})/2\right)}+\frac{1}{2}\frac{\sin\left((\omega+\frac{\pi}{2N})(N+\frac{1}{2})\right)}{\sin\left((\omega+\frac{\pi}{2N})/2\right)}.$$

3.10 (a) $X_a(e^{j\omega})=\sum_{k=-\infty}^{\infty}\delta(\omega+2\pi k)$. Hence, $x[n]=\dfrac{1}{2\pi}\int_{-\pi}^{\pi}\delta(\omega)\,e^{j\omega n}d\omega=1.$

(b) $X_b(e^{j\omega})=\dfrac{1-e^{j\omega(N+1)}}{1-e^{-j\omega}}=\sum_{n=0}^{N}e^{-j\omega n}$. Hence, $x[n]=\begin{cases}1, & 0\le n\le N,\\ 0, & \text{otherwise.}\end{cases}$

(c) $X_c(e^{j\omega})=1+2\sum_{\ell=0}^{N}\cos(\omega\ell)=\sum_{\ell=-N}^{N}e^{-j\omega\ell}$. Hence $x[n]=\begin{cases}1, & -N\le n\le N,\\ 0, & \text{otherwise.}\end{cases}$

(d) $X_d(e^{j\omega})=\dfrac{-j\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$, $|\alpha|<1$. Now we can rewrite $X_d(e^{j\omega})$ as

$$X_d(e^{j\omega})=\frac{d}{d\omega}\frac{1}{(1-\alpha e^{-j\omega})}=\frac{d}{d\omega}\left(X_o(e^{j\omega})\right)\text{ where }X_o(e^{j\omega})=\frac{1}{1-\alpha e^{-j\omega}}.$$

Now $x_o[n]=\alpha^n\mu[n]$. Hence, from Table 3.2, $x_d[n]=-jn\alpha^n\mu[n]$.

3.11 (a) $X(e^{j0})=\sum_{n=-2}^{6}x[n]=8.$ (b) $X(e^{j\pi})=\sum_{n=-2}^{6}e^{-j\pi n}x[n]=8.$

(c) $x[n]=\dfrac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{-j\omega n}d\omega$. Hence $\int_{-\pi}^{\pi}X(e^{j\omega})d\omega=2\pi x[0]=-2\pi.$

(d) By Parseval's thorem $\dfrac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2d\omega=\sum_{n=-\infty}^{\infty}|x[n]|^2.$

Hence $\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 88\pi.$

(e) The inverse DTFT of $j\frac{dX(e^{j\omega})}{d\omega}$ is $n\,x[n] = \{-4 \quad -1 \quad \underset{\uparrow}{0} \quad 0 \quad 6 \quad 6 \quad 0 \quad -15 \quad 24\}.$

Hence $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega = 2\pi\{16+1+36+36+225+576\} = 1780\pi.$

3.12 $\{x[n]\} = \{1 \quad 5 \quad -2 \quad 1 \quad \underset{\uparrow}{3} \quad 4 \quad 2 \quad 0 \quad 5\}.$

(a) $X(e^{j0}) = \sum_{n=-4}^{4} x[n] = 19.$ **(b)** $X(e^{j\pi}) = \sum_{n=-4}^{4} e^{-j\pi n} x[n] = -1.$

(c) $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi x[0] = 6\pi.$

(d) $\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 170\pi.$

(e) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} n^2 |x[n]|^2 = 1380\pi.$

3.13 **(a)** The sequence in Figure P3.1(b) can be expressed as $g_2[n] = g_1[n] + g_1[n-4].$
Hence $G_2(e^{j\omega}) = (1 + e^{-j4\omega}) G_1(e^{j\omega}).$

(b) The sequence in Figure P3.1(c) can be expressed as $g_3[n] = g_1[-n+3] + g_1[n-4]$ $= g_1[-(n-3)] + g_1[n-4].$ Hence $G_3(e^{j\omega}) = e^{j3\omega} G_1(e^{-j\omega}) + e^{-j4\omega} G_1(e^{j\omega}).$

(c) The sequence in Figure P3.1(d) can be expressed as $g_4[n] = g_1[n] + g_1[-n+7].$
Hence $G_4(e^{j\omega}) = G_1(e^{j\omega}) + e^{j7\omega} G_1(e^{-j\omega}).$

3.14 $y[n] = g[n] \circledast h[n] = \sum_{k=-\infty}^{k=\infty} g[k] h[n-k]$

(a) $\sum_{n=-\infty}^{\infty} y[n] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g[k] h[n-k] = \sum_{k=-\infty}^{\infty} g[k] \sum_{n=-\infty}^{\infty} h[n-k].$ Substituting $n-k$ by k in the second term, we get $\sum_{k=-\infty}^{\infty} y[k] = \left(\sum_{k=-\infty}^{\infty} g[k]\right)\left(\sum_{k=-\infty}^{\infty} h[k]\right).$

(b) $(-1)^n y[n] = \sum_{k=-\infty}^{\infty} g[k] (-1)^n h[n-k].$ Therefore,

$$\sum_{n=-\infty}^{\infty} (-1)^n y[n] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g[k] (-1)^{n-k} (-1)^k h[n-k] = \left(\sum_{n=-\infty}^{\infty} (-1)^n g[n]\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n h[n]\right).$$

3.15 **(a)** $x[n] = x_{ev}[n] + x_{od}[n]$. Now, for a causal x[n], from the results of Problem 2.4, we observe

$$x[n] = 2x_{ev}[n]\mu[n] - x[0]\delta[n] = h[n] - x[0]\delta[n], \quad (2)$$

$$x[n] = 2x_{od}[n]\mu[n] + x[0]\delta[n]. \quad (3)$$

Taking the DTFT of both sides of Eq. (2) we get

$$X(e^{j\omega}) = H(e^{j\omega}) - x[0], \quad (4)$$

where $H(e^{j\omega}) = \text{DTFT}\{2x_{ev}[n]\mu[n]\} = \frac{1}{\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\,\mu(e^{j(\omega-\theta)})\,d\theta,$ (5)

Note: The DTFT of $x_{ev}[n]$ is $X_{re}(e^{j\omega})$, and the DTFT of $\mu[n]$ is $\mu(e^{j\omega})$.

Now, from Table 3.1, $\mu(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega+2\pi k) = \frac{1}{2} - \frac{j}{2}\cot\left(\frac{\omega}{2}\right) + \pi\sum_{k=-\infty}^{\infty}\delta(\omega+2\pi k).$

Substituting the above in Eq. (5) we get

$$H(e^{j\omega}) = \frac{1}{\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\left\{\frac{1}{2} - \frac{j}{2}\cot\left(\frac{\theta}{2}\right) + \pi\sum_{k=-\infty}^{\infty}\delta(\theta+2\pi k)\right\}d\theta$$

$$= X_{re}(e^{j\omega}) + \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\,d\theta - \frac{j}{2\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta.$$

Substituting the above in Eq. (4) we get

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega}) = H(e^{j\omega}) - x[0]$$

$$= X_{re}(e^{j\omega}) + \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\,d\theta - \frac{j}{2\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta - x[0]$$

$$= X_{re}(e^{j\omega}) - \frac{j}{2\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta, \quad (6)$$

since $\frac{1}{2\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\,d\theta = x[0]$, as x[n] is real. Comparing the imaginary part of both sides of Eq. (6) we therefore get $X_{im}(e^{j\omega}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} X_{re}(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta.$

(b) Taking the DTFT of both sides of Eq. (3) we get

$$X(e^{j\omega}) = G(e^{j\omega}) + x[0], \quad (7)$$

where, $G(e^{j\omega}) = \text{DTFT}\{2x_{od}[n]\mu[n]\} = \frac{j}{\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})\,\mu(e^{j(\omega-\theta)})\,d\theta,$ (8)

as $jX_{im}(e^{j\omega})$ is the DTFT of $x_{od}[n]$. Substituting the expression for $\mu(e^{j\omega})$ given above in Eq. (8) we get $G(e^{j\omega}) = \frac{j}{\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})\left\{\frac{1}{2} - \frac{j}{2}\cot\left(\frac{\theta}{2}\right) + \pi\sum_{k=-\infty}^{\infty}\delta(\theta+2\pi k)\right\}d\theta$

$$= jX_{im}(e^{j\omega}) + \frac{j}{2\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta$$

Substituting the above in Eq. (7) we get

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega}) = G(e^{j\omega}) + x[0]$$
$$= jX_{im}(e^{j\omega}) + \frac{j}{2\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta + x[0]$$
$$= jX_{im}(e^{j\omega}) + \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta + x[0] \qquad (9)$$

as $\frac{1}{2\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})d\theta = 0$ since $X_{im}(e^{j\omega})$ is an odd function of ω. Comparing the real parts of both sides of Eq. (9) we finally arrive at $X_{re}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{im}(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta + x[0]$.

3.16 $S = \sum_{n=0}^{N-1} W_N^{-(k-l)n} = \sum_{n=0}^{N-1} e^{j2\pi n(k-l)/N}$

If $k - l \neq rN$ then $S = \frac{1-e^{j2\pi n(k-l)}}{1-e^{j2\pi n(k-l)/N}} = \frac{1-1}{1-e^{j2\pi n(k-l)/N}} = 0$.

If $k - l = rN$ then $S = \sum_{n=0}^{N-1} W_N^{-rnN} = \sum_{n=0}^{N-1} e^{-j2\pi nr} = \sum_{n=0}^{N-1} 1 = N$.

Hence, $\sum_{n=0}^{N-1} W_N^{-(k-l)n} = \begin{cases} N, & \text{for } k-l = rN, \text{ r an integer,} \\ 0, & \text{otherwise.} \end{cases}$

3.17 $\tilde{y}[n] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n-r]$. Then $\tilde{y}[n+kN] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n+kN-r]$. Since $\tilde{h}[n]$ is periodic in n with a period N, $\tilde{h}[n+kN-r] = \tilde{h}[n-r]$. Therefore $\tilde{y}[n+kN] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n-r] = \tilde{y}[n]$, hence $\tilde{y}[n]$ is also periodic in n with a period N.

3.18 $\tilde{x}[n] = \{0 \ \ 1 \ \ 0 \ \ 2\}$ and $\tilde{h}[n] = \{2 \ \ 0 \ \ 1 \ \ 0\}$

Now $\tilde{y}[0] = \sum_{r=0}^{3} \tilde{x}[r]\tilde{h}[0-r] = \tilde{x}[0]\tilde{h}[0] + \tilde{x}[1]\tilde{h}[3] + \tilde{x}[2]\tilde{h}[2] + \tilde{x}[3]\tilde{h}[1] = 0$.

Similarly $\tilde{y}[1] = \sum_{r=0}^{3} \tilde{x}[r]\tilde{h}[1-r] = \tilde{x}[0]\tilde{h}[1] + \tilde{x}[1]\tilde{h}[0] + \tilde{x}[2]\tilde{h}[3] + \tilde{x}[3]\tilde{h}[2] = 4$.

Continuing the process we can show that $\tilde{y}[2] = 0$ and $\tilde{y}[3] = 5$.

3.19 $\tilde{x}[n] = \{2 \ \ -1 \ \ 0 \ \ 3\}$ and $\tilde{h}[n] = \{-1 \ \ 2 \ \ 1 \ \ 0\}$

Following a procedure similar to the above problem, we can show that $\tilde{y}[n] = \{4 \ \ 8 \ \ 0 \ \ -4\}$.

3.20 Since $\tilde{\psi}_k[n+rN] = \tilde{\psi}_k[n]$, hence all the terms which are not in the range 0,1,....N-1 can be accumulated to $\tilde{\psi}_k[n]$, where $0 \le k \le N-1$. Hence in this case the Fourier series representation involves only N complex exponential sequences. Let

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j2\pi kn/N} \text{ then}$$

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi rn/N}=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j2\pi(k-r)n/N}=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\sum_{n=0}^{N-1}e^{j2\pi(k-r)n/N}.$$

Now from the results of Problem 3.16, the inner summation is equal to N if k = r, otherwise it is equal to 0. Thus $\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi rn/N}=\tilde{X}[r]$.

3.21 **(a)** $\tilde{x}_1[n]=\sin(\frac{\pi n}{4})=\frac{1}{2j}\{e^{j\pi n/4}-e^{-j\pi n/4}\}$. In this case N = 8. Thus,

$$\tilde{X}_1[k]=\sum_{n=0}^{7}\frac{1}{2j}\{e^{j\pi n/4}-e^{-j\pi n/4}\}e^{-j2\pi kn/8}=\frac{1}{2j}\left\{\sum_{n=0}^{7}e^{-j\pi n(k-1)/4}-\sum_{n=0}^{7}e^{-j\pi n(k+1)/4}\right\}$$

From the result of Problem 3.16 we note that the first sum is non-zero (and equal to 8) only for k = 1 while the second sum is nonzero only for k = 7.

$$\text{Thus, } \tilde{X}_1[k]=\begin{cases}-4j, & k=1,\\ 4j, & k=7,\\ 0, & \text{elsewhere.}\end{cases}.$$

(b) $\tilde{x}_2[n]=\frac{2}{2j}\left(e^{j\pi n/4}-e^{-j\pi n/4}\right)+\frac{e^{j\pi n/3}+e^{-j\pi n/3}}{2}$. In this case N = 24.

$$\tilde{X}_2[k]=\sum_{n=0}^{23}\left\{-je^{-2\pi n(k-3)/24}+je^{-j2\pi n(k+3)/24}+0.5e^{-j2\pi n(k-4)/24}+0.5e^{-j2\pi n(k+4)/24}\right\}$$

$$\text{Hence, } \tilde{X}_2[k]=\begin{cases}-24j, & k=3,\\ 24j, & k=21,\\ 12, & k=4,\\ 12, & k=20,\\ 0, & \text{elsewhere.}\end{cases}$$

3.22 Since $\tilde{p}[n]$ is periodic with period N, then from Problem 3.20, $\tilde{p}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{P}[k]e^{-j2\pi kn/N}$

where $\tilde{P}[k]=\sum_{n=0}^{N-1}\tilde{p}[n]e^{-j2\pi kn/N}=1$. Hence $\tilde{p}[n]=\frac{1}{N}\sum_{k=0}^{N-1}e^{-j2\pi kn/N}$.

3.23 $x[n]=\{\underset{\uparrow}{1}\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\}$. Then, $X(z)=\sum_{n=0}^{5}x[n]z^{-n}$. Thus, $\tilde{X}[k]=\sum_{n=0}^{5}x[n]e^{-j\pi kn/2}$

Hence, $\tilde{x}[n]=\frac{1}{4}\sum_{k=0}^{3}\tilde{X}[k]e^{j2\pi kn/4}=\frac{1}{4}\sum_{k=0}^{3}\sum_{r=0}^{5}x[r]e^{j2\pi kn/4}e^{-j\pi rk/2}=\frac{1}{4}\sum_{r=0}^{5}x[r]\sum_{k=0}^{3}e^{j2\pi k(n-r)/4}$.

Therefore, $\tilde{x}[0]=x[0]+x[4]=6$, $\tilde{x}[1]=x[1]+x[5]=8$, $\tilde{x}[2]=x[2]=3$, $\tilde{x}[3]=x[3]=4$.

3.24 **(a)** $\tilde{G}[k]=\sum_{n=0}^{N-1}\tilde{g}[n]e^{-j2\pi kn/N}=\sum_{n=0}^{N-1}\tilde{x}[n]\tilde{y}[n]e^{-j2\pi kn/N}$. Now, $\tilde{x}[n]=\frac{1}{N}\sum_{r=0}^{N-1}\tilde{X}[r]e^{j2\pi rn/N}$

Therefore, $\tilde{G}[k]=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{r=0}^{N-1}\tilde{X}[r]\tilde{y}[n]e^{-j2\pi(k-r)n/N}=\frac{1}{N}\sum_{r=0}^{N-1}\tilde{X}[r]\sum_{n=0}^{N-1}\tilde{y}[n]e^{-j2\pi(k-r)n/N}$

$$=\frac{1}{N}\sum_{r=0}^{N-1}\tilde{X}[r]\tilde{Y}[k-r].$$

(b) $\tilde{h}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\tilde{Y}[k]e^{j2\pi kn/N}=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{r=0}^{N-1}\tilde{x}[r]\tilde{Y}[k]e^{j2\pi k(n-r)/N}$

$$=\sum_{r=0}^{N-1}\tilde{x}[r]\left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{Y}[k]e^{j2\pi k(n-r)/N}\right)=\sum_{r=0}^{N-1}\tilde{x}[r]\tilde{y}[n-r].$$

3.25 (a) $y[n]=\alpha g[n]+\beta h[n]$. Therefore

$$Y[k]=\sum_{n=0}^{N-1}y[n]W_N^{nk}=\alpha\sum_{n=0}^{N-1}g[n]W_N^{nk}+\beta\sum_{n=0}^{N-1}h[n]W_N^{nk}=\alpha G[k]+\beta H[k]$$

(b) $x[n]=g[<n-n_0>_N]$. Therefore $X[k]=\sum_{n=0}^{N-1}g[<n-n_0>_N]W_N^{nk}$

$$=\sum_{n=0}^{n_0-1}g[N+n-n_0]W_N^{nk}+\sum_{n=n_0}^{N-1}g[n-n_0]W_N^{nk}$$

$$=\sum_{n=N-n_0}^{N-1}g[n]W_N^{(n+n_0-N)k}+\sum_{n=0}^{N-n_0-1}g[n]W_N^{(n+n_0)k}=W_N^{n_0k}\sum_{n=0}^{N-1}g[n]W_N^{nk}=W_N^{n_0k}G[k].$$

(c) $u[n]=W_N^{-k_0n}g[n]$. Hence $U[k]=\sum_{n=0}^{N-1}u[n]W_N^{nk}=\sum_{n=0}^{N-1}g[n]W_N^{(k-k_0)n}$

$$=\begin{cases}\sum_{n=0}^{N-1}W_N^{(k-k_0)n}g[n], & \text{if } k\ge k_0,\\ \sum_{n=0}^{N-1}W_N^{(N+k-k_0)n}g[n], & \text{if } k<k_0.\end{cases}$$

Thus, $U[k]=\begin{cases}G[k-k_0], & \text{if } k\ge k_0,\\ G[N+k-k_0], & \text{if } k<k_0,\end{cases}=G[<k-k_0>_N].$

(d) $h[n]=G[n]$. Therefore, $H[k]=\sum_{n=0}^{N-1}h[n]W_N^{nk}=\sum_{n=0}^{N-1}G[n]W_N^{nk}=\sum_{n=0}^{N-1}\sum_{r=0}^{N-1}g[r]W^{nr}W^{kr}$

$$=\sum_{r=0}^{N-1}g[r]\sum_{n=0}^{N-1}W^{(k+r)n}.$$

The second sum is non-zero only if $k=r=0$ or else if $r=N-k$ and $k\neq 0$. Hence,

$$H[k]=\begin{cases}Ng[0], & \text{if } k=0,\\ Ng[N-k], & \text{if } k>0,\end{cases}=Ng[<-k>_N].$$

(e) $u[n]=\sum_{m=0}^{N-1}g[m]h[<n-m>_N]$. Therefore, $U[k]=\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}g[m]h[<n-m>_N]W_N^{nk}$

$$= \sum_{m=0}^{N-1} g[m] \sum_{n=0}^{N-1} h[<n-m>_N] W_N^{nk} = \sum_{m=0}^{N-1} g[m] H[k] W_N^{mk} = H[k]G[k].$$

(f) $v[n] = g[n]h[n]$. Therefore, $V[k] = \sum_{n=0}^{N-1} g[n]h[n]W_N^{nk} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{r=0}^{N-1} h[n]G[r]W^{nk}W^{-nr} =$

$$\frac{1}{N}\sum_{r=0}^{N-1} G[r] \sum_{n=0}^{N-1} h[n]W^{(k-r)n} = \frac{1}{N}\sum_{r=0}^{N-1} G[r]H[<k-r>_N].$$

3.26 $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W^{-nk}$. Thus $x^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]W^{nk}$. Therefore,

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} \frac{1}{N^2}\left(\sum_{r=0}^{N-1} X[r]W^{-nr}\right)\left(\sum_{\ell=0}^{N-1} X^*[\ell]W^{n\ell}\right) = \frac{1}{N^2}\sum_{r=0}^{N-1}\sum_{\ell=0}^{N-1} X[r]X^*[\ell]\sum_{n=0}^{N-1} W^{n(\ell-r)}.$$

Since the inner sum is non-zero only if $\ell = r$, we get $\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$.

3.27 $X[k] = \sum_{n=0}^{N-1} x[n]W^{nk}$.

(a) $X^*[k] = \sum_{n=0}^{N-1} x^*[n]W^{-nk}$. Replacing k by N – k on both sides we obtain

$X^*[N-k] = \sum_{n=0}^{N-1} x^*[n]W^{-n(N-k)} = \sum_{n=0}^{N-1} x^*[n]W^{nk}$. Thus $x^*[n] \Leftrightarrow X^*[N-k] = X^*[<-k>_N]$.

(b) $X^*[k] = \sum_{n=0}^{N-1} x^*[n]W^{-nk}$. Replacing n by N – n in the summation we get

$$X^*[k] = \sum_{n=0}^{N-1} x^*[N-n]W^{-(N-n)k} = \sum_{n=0}^{N-1} x^*[N-n]W^{nk}.$$

Thus $x^*[N-n] = x^*[<-n>_N] \Leftrightarrow X^*[k]$.

(c) $Re\{x[n]\} = \frac{1}{2}\{x[n] + x^*[n]\}$. Now taking DFT of both sides and using results of part (:)

we get $Re\{x[n]\} \Leftrightarrow \frac{1}{2}\{X[k] + X^*[<-k>_N]\}$.

(d) $j\,Im\{x[n]\} = \frac{1}{2}\{x[n] - x^*[n]\}$ this imples $j\,Im\{x[n]\} \Leftrightarrow \frac{1}{2}\{X[k] - X^*[<-k>_N]\}$.

(e) $x_{pcs}[n] = \frac{1}{2}\{x[n] + x^*[<-n>_N]\}$ Using linearity and results of part (b) we get

$x_{pcs}[n] \Leftrightarrow \frac{1}{2}\{X[k] + X^*[k]\} = Re\{X[k]\}$.

(f) $x_{pca}[n] = \frac{1}{2}\{x[n] - x^*[<-n>_N]\}$. Again using results of part (b) and linearity we get

$x_{pca}[n] \Leftrightarrow \frac{1}{2}\{X[k] - X^*[k]\} = j\,Im\,\{X[k]\}$.

3.28 $X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$.

(a) $x_{pe}[n] = \frac{1}{2}\{x[n] + x[\langle -n\rangle_N]\}$. From Table 3.6, $x^*[\langle -n\rangle_N] \overset{DFT}{\Leftrightarrow} X^*[k]$. Since x[n] is real, $x[\langle -n\rangle_N] = x^*[\langle -n\rangle_N] \overset{DFT}{\Leftrightarrow} X^*[k]$. Thus, $X_{pe}[k] = \frac{1}{2}\{X[k] + X^*[k]\} = \text{Re}\{X[k]\}$.

(b) $x_{po}[n] = \frac{1}{2}\{x[n] - x[\langle -n\rangle_N]\}$. As a result, $X_{po}[k] = \frac{1}{2}\{X[k] - X^*[k]\} = j\,\text{Im}\{X[k]\}$.

(c) Since for a real sequence, x[n] = x*[n], taking DFT of both sides we get $X[k] = X^*[\langle -k\rangle_N]$. This implies, $\text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\} = \text{Re}\{X[\langle -k\rangle_N]\} - j\,\text{Im}\{X[\langle -k\rangle_N]\}$.

Comparing real and imaginary parts we get

$$\text{Re}\{X[k]\} = \text{Re}\{X[\langle -k\rangle_N]\} \text{ and } \text{Im}\{X[k]\} = -\text{Im}\{X[\langle -k\rangle_N]\}.$$

Also $|X[k]| = \sqrt{(\text{Re}\{X[k]\})^2 + (\text{Im}\{X[k]\})^2}$

$$= \sqrt{\left(\text{Re}\{X[\langle -k\rangle_N]\}\right)^2 + \left(-\text{Im}\{X[\langle -k\rangle_N]\}\right)^2} = |X[\langle -k\rangle_N]|$$

and $\arg\{X[k]\} = \tan^{-1}\left(\frac{\text{Im}\{X[k]\}}{\text{Re}\{X[k]\}}\right) = \tan^{-1}\left(\frac{-\text{Im}\{X[\langle -k\rangle_N]\}}{\text{Re}\{X[\langle -k\rangle_N]\}}\right) = -\arg\{X[\langle -k\rangle_N]\}$

3.29 (a) For a sequence to have a real DFT, the sequence must be periodic conjugate symmetric. As $x_4[n]$ is the only periodic conjugate symmetric sequence, it will have real-valued DFT's.

(b) For a sequence to have a imaginary valued DFT, it must be periodic conjugate anti-symmetric. Neither sequence is periodic conjugate anti-symmetric.

3.30 (a) Now, $X[N/2] = \sum_{n=0}^{N-1} x[n]W_N^{nN/2} = \sum_{n=0}^{N-1}(-1)^n x[n]$. Hence if x[n] = x[N − 1 − n] and N is even, then $\sum_{n=0}^{N-1}(-1)^n x[n] = 0$ or X[N/2] = 0.

(b) $X[0] = \sum_{n=0}^{N-1} x[n]$ so if x[n] = − x[N − 1 − n], then X[0] = 0.

(c) $X[2\ell] = \sum_{n=0}^{N-1} x[n]W^{2n\ell} = \sum_{n=0}^{\frac{N}{2}-1} x[n]W^{2n\ell} + \sum_{n=N/2}^{N-1} x[n]W^{2n\ell}$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n]W^{2n\ell} + \sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]W^{2n\ell} = \sum_{n=0}^{\frac{N}{2}-1}(x[n] + x[n+\tfrac{N}{2}])W^{2n\ell}.$$

Hence if $x[n] = -x[n+\frac{N}{2}] = -x[n+M]$, then $X[2\ell] = 0$, for $\ell = 0, 1, \ldots, M-1$.

3.31 Let x[n] = a (where a is some real constant), for $0 \le n \le N-1$.

Now, $X[k] = \sum_{n=0}^{N-1} aW_N^{nk} = a\sum_{n=0}^{N-1} e^{-j2\pi nk/N}$. Since $\sum_{n=0}^{N-1} e^{-j2\pi nk/N} = \begin{cases} N, & \text{if } k=0, \\ 0, & \text{elsewhere,} \end{cases}$

then $X[k] = \begin{cases} Na, & \text{if } k=0, \\ 0, & \text{elsewhere.} \end{cases}$ Thus the N-point DFT X[k] is real valued.

Next, $X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = a\sum_{n=0}^{N-1} e^{-j\omega n}$. If N is even then

$$X(e^{j\omega}) = ae^{-j\omega(N-1)/2}\left(\sum_{n=0}^{\frac{N}{2}-1}\cos\left(\frac{\omega(N-1-2n)}{2}\right)\right).$$ Hence in this case $X(e^{j\omega})$ has linear phase.

If N is odd then $X(e^{j\omega}) = a\,e^{-j\omega(N-1)/2}\left(1+\sum_{n=0}^{\frac{N-1}{2}}\cos\left(\frac{\omega(N-1-2n)}{2}\right)\right)$. In this case also $X(e^{j\omega}$ has linear phase.

3.32 **(a)** $\{G[k]\} = \{1+j \quad -2.1+j3.2 \quad -1.2-j2.4 \quad 0 \quad 0.9+j3.1 \quad -0.3+j1.1\}$.

$h[n] = g[\langle n-4\rangle_6]$, and hence $H[k] = W_6^{4k}G[k]$, k = 0, 1, . . ., 5. Therefore ,

$\{H[k]\} = \{W_6^0G[0] \quad W_6^4G[1] \quad W_6^8G[2] \quad W_6^{12}G[3] \quad W_6^{16}G[4] \quad W_6^{20}G[5]\}$

$= \{1+j, \quad -1.7213-j3.4187, \quad -1.4785+j2.2392, \quad 0, \quad -3.1347-j0.7706, \quad 1.1026-j0.2902\}$

(b) In this case $h[n] = W_6^{-3n}g[n] = (-1)^n g[n]$. Therefore,

$\{h[n]\} = \{W_6^{-0}g[0] \quad W_6^{-3}g[1] \quad W_6^{-6}g[2] \quad W_6^{-9}g[3] \quad W_6^{-12}g[4] \quad W_6^{-15}g[6]\}$

$= \{4.1 \quad -3.5 \quad 1.2 \quad -5 \quad 2 \quad -3.3\}$.

3.33 $Y[k] = \sum_{n=0}^{MN-1} y[n]W_{MN}^{nk} = \sum_{n=0}^{N-1} x[n]W_{MN}^{nk}$. Thus, $Y[kM] = \sum_{n=0}^{N-1} x[n]W_N^{nk} = X[k]$.
Hence, X[k] = Y[kM].

3.34 **(a)** $X[0] = \sum_{n=0}^{9} x[n] = 22$.

(b) $X[5] = \sum_{n=0}^{9} (-1)^n x[n] = -2$.

(c) $\sum_{k=0}^{9} X[k] = 10\,x[0] = 20$.

(d) $S = \sum_{k=0}^{9} e^{-j(4\pi k/5)}X[k]$. The IDTFT of $e^{-j(4\pi k/5)}X[k]$ is $x[\langle n-4\rangle_{10}]$. Thus
$S = 10\,x[\langle 0-4\rangle_{10}] = 10\,x[6] = 0$.

(e) From Parseval's theorem $\sum_{k=0}^{9} |X[k]|^2 = 10\sum_{k=0}^{9} |x[k]|^2$. Thus $\sum_{k=0}^{9} |X[k]|^2 = 800$.

3.35 Since x[n] is real, $X[k] = X^*[\langle -k \rangle_N]$. Thus

$X[11] = X^*[1] = -5 + j4$, $X[10] = X^*[2] = 3 + j2$, $X[9] = X^*[3] = 1 - j3$,

$X[8] = X^*[4] = 2 - j5$, $X[7] = X^*[5] = 6 + 2j$.

(a) $x[0] = \frac{1}{12}\sum_{k=0}^{11} X[k] = \frac{36}{12} = 3$

(b) $x[6] = \frac{1}{12}\sum_{k=0}^{11} (-1)^k X[k] = \frac{28}{12}$.

(c) $\sum_{n=0}^{11} x[n] = X[0] = 10.$

(d) $S = \sum_{n=0}^{11} e^{j2\pi n/3} x[n]$. The DFT of $e^{j2\pi n/3}x[n]$ is $X[\langle k-4 \rangle_{12}]$.

Thus $S = X[\langle 0-4 \rangle_{12}] = X[8] = 2 - j5$.

(e) $\sum_{n=0}^{11} |x[n]|^2 = \frac{1}{12}\sum_{k=0}^{11} |X[k]|^2 = \frac{510}{12}$.

3.36 Now, $y_c[n] = \sum_{k=0}^{3} g[k]h[\langle n-k \rangle_4]$. Hence,

$$y_c[0] = g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1],$$
$$y_c[1] = g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2],$$
$$y_c[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3], \text{ and}$$
$$y_c[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0].$$

Likewise, $y_L[n] = \sum_{k=0}^{3} g[k]h[n-k]$. Hence,

$$y_L[0] = g[0]h[0],$$
$$y_L[1] = g[0]h[1] + g[1]h[0],$$
$$y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0],$$
$$y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0],$$
$$y_L[4] = g[1]h[3] + g[2]h[2] + g[3]h[1],$$
$$y_L[5] = g[2]h[3] + g[3]h[2], \text{ and}$$
$$y_L[6] = g[3]h[3].$$

Comparing $y_c[n]$ with $y_L[n]$ we observe that

$$y_c[0] = y_L[0] + y_L[4],$$
$$y_c[1] = y_L[1] + y_L[5],$$
$$y_c[2] = y_L[2] + y_L[6], \text{ and}$$
$$y_c[3] = y_L[3] + y_L[7].$$

3.37 (a) $y_L[n] = \sum_{k=0}^{3} g[k]h[n-k]$. Thus, $y_L[0] = g[0]h[0] = 5\times(-3) = -15$,

$$y_L[1] = g[0]h[1] + g[1]h[0] = 5\times4 + 2\times(-3) = 14,$$
$$y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] = 5\times0 + 2\times4 + 4(-3) = -4,$$

$y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0] = 5\times2 + 2\times0 + 4\times4 + (-1)\times(-3) = 29,$
$y_L[4] = g[0]h[4] + g[1]h[3] + g[2]h[2] + g[3]h[1] = 5\times(-1) + 2\times2 + 4\times0 + (-1)\times4 = -5,$
$y_L[5] = g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] = 5\times2 + 2\times(-1) + 4\times2 + (-1)\times0 = 16,$
$y_L[6] = g[1]h[5] + g[2]h[4] + g[3]h[3] = 2\times2 + 4\times(-1) + (-1)\times2 = -2,$
$y_L[7] = g[2]h[5] + g[3]h[4] = 4\times2 + (-1)\times(-1) = 9,$
$y_L[8] = g[3]h[5] = (-1)\times2 = -2.$

Hence, $\{y_L[n]\} = \{-15 \quad 14 \quad -4 \quad 29 \quad -5 \quad 16 \quad -2 \quad 9 \quad -2\}$

(b) $y_c[n] = \sum_{k=0}^{5} g_e[k]h[\langle n-k \rangle_6]$, where $\{g_e[n]\} = \{\underset{\uparrow}{5} \quad 2 \quad 4 \quad -1 \quad 0 \quad 0\}$. Thus,

$y_C[0] = g_e[0]h[0] + g_e[1]h[5] + g_e[2]h[4] + g_e[3]h[3] + g_e[4]h[2] + g_e[5]h[1]$
$= g[0]h[0] + g[1]h[5] + g[2]h[4] + g[3]h[3] = 5\times(-3) + 2\times2 + 4\times(-1) + (-1)\times2 = -17,$
$y_C[1] = g_e[0]h[1] + g_e[1]h[0] + g_e[2]h[5] + g_e[3]h[4] + g_e[4]h[3] + g_e[5]h[2]$
$= g[0]h[1] + g[1]h[0] + g[2]h[5] + g[3]h[4] = 5\times4 + 2\times(-3) + 4\times2 + (-1)\times(-1) = 23,$
$y_C[2] = g_e[0]h[2] + g_e[1]h[1] + g_e[2]h[0] + g_e[3]h[5] + g_e[4]h[4] + g_e[5]h[3]$
$= g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[5] = 5\times0 + 2\times4 + 4\times(-3) + (-1)\times2 = -6,$
$y_C[3] = g_e[0]h[3] + g_e[1]h[2] + g_e[2]h[1] + g_e[3]h[0] + g_e[4]h[5] + g_e[5]h[4]$
$= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0] = 5\times2 + 2\times0 + 4\times4 + (-1)\times(-3) = 29,$
$y_C[4] = g_e[0]h[4] + g_e[1]h[3] + g_e[2]h[2] + g_e[3]h[1] + g_e[4]h[0] + g_e[5]h[5]$
$= g[0]h[4] + g[1]h[3] + g[2]h[2] + g[3]h[1] = 5\times(-1) + 2\times2 + 4\times0 + (-1)\times4 = -5,$
$y_C[5] = g_e[0]h[5] + g_e[1]h[4] + g_e[2]h[3] + g_e[3]h[2] + g_e[4]h[1] + g_e[5]h[0]$
$= g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] = 5\times2 + 2\times(-1) + 4\times2 + (-1)\times0 = 16.$

Hence, $\{y_C[n]\} = \{-17 \quad 23 \quad -6 \quad 29 \quad -5 \quad 16\}$

(c) Using MATLAB command `fft` we determine the two 6-point DFTs,
$G_e[k] = \{10 \quad 5-j5.1962 \quad 1+j1.7322 \quad 8 \quad 1-j1.7322 \quad 5.5+j5.1962\}$ and
$H[k] = \{4 \quad -1.5-j2.5981 \quad -3.5-j0.866 \quad -12 \quad -3.5+j0.866 \quad -1.5+j2.5981\}$
Thus,
$G_e[k]H[k] = \{40 \quad -21-j5.1962 \quad -2-j6.9282 \quad -96 \quad -2+j6.9282 \quad -21+j5.1962\}$
Next, using the MATLAB command `ifft` we compute the IDFT of $G_e[k]H[k]$ resulting in
$\{y_C[n]\} = \{-17 \quad 23 \quad -6 \quad 29 \quad -5 \quad 16\}.$

3.38 Let $u[n] = g_e[n] + j\,h[n]$, where $g_e[n]$ is given in Part (b) of the above problem with $U[k]$ denoting its 6-point DFT. Now, $u[n] = \{5-j3, \ 2+j4, \ 4, \ -1+j2, \ -j \ \ j2\}$. Using the MATLAB command `fft` we compute its 6-point DFT which is given by

$U[k]$
$= \{10+j4, \ 7.5981-j6.6962, \ 1.866-j1.7679, \ 8-j12, \ 0.134-j5.232, \ 2.4019+j3.6962\}$

Thus, $U[\langle -k \rangle_N] = U[N-k]$
$= \{10+j4, \ 2.4019+j3.6962, \ 0.134-j5.232, \ 8-j12, \ 1.866-j1.7679, \ 7.5981-j6.6962\}$

Therefore, $G_e[k] = \frac{1}{2}\left(U[k] + U^*[\langle -k \rangle_N]\right),$
$= \{10 \quad 5-j5.1962 \quad 1+j1.7322 \quad 8 \quad 1-j1.7322 \quad 5+j5.1962\}$, and

$H[k] = \frac{1}{2}\left(U[k] - U^*[\langle -k \rangle_N]\right),$

$= \{4 \quad -1.5-j2.5981 \quad -3.5-j0.866 \quad -12 \quad -3.5+j0.866 \quad -1.5+j2.5981\}$

3.39 We need to show $g[n] \circledN h[n] = h[n] \circledN g[n]$.

Let $x[n] = g[n] \circledN h[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N]$

and $y[n] = h[n] \circledN g[n] = \sum_{m=0}^{N-1} h[m]g[\langle n-m \rangle_N]$

$$= \sum_{m=0}^{n} h[m]g[n-m] + \sum_{m=n+1}^{N-1} h[m]g[N+n-m]$$

$$= \sum_{m=0}^{n} h[n-m]g[m] + \sum_{m=n+1}^{N-1} h[N+n-m]g[m] = \sum_{m=0}^{N-1} h[\langle n-m \rangle_N]g[m] = x[n].$$

Hence circular convolution is commutative.

3.40 **(a)** $y[n] = g[n] \circledN h[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N]$. Thus,

$$\sum_{n=0}^{N-1} y[n] = \sum_{m=0}^{N-1} g[m] \sum_{n=0}^{N-1} h[\langle n-m \rangle_N] = \left(\sum_{n=0}^{N-1} h[n]\right)\left(\sum_{m=0}^{N-1} g[m]\right).$$

(b) $y[n] = g[n] \circledN h[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N]$. Thus,

$$\sum_{n=0}^{N-1} (-1)^n y[n] = \sum_{m=0}^{N-1} g[m] \sum_{n=0}^{N-1} h[\langle n-m \rangle_N](-1)^n$$

$$= \sum_{m=0}^{N-1} g[m]\left(\sum_{n=0}^{m-1} h[N+n-m](-1)^n + \sum_{n=m}^{N-1} (-1)^n h[n-m]\right)$$

Replacing n by N+n−m in the first sum and by n−m in the second we obtain

$$\sum_{n=0}^{N-1} (-1)^n y[n] = \sum_{m=0}^{N-1} g[m]\left(\sum_{n=N-m}^{N-1} h[n](-1)^{n-N+m} + \sum_{n=0}^{N-m-1} (-1)^{n+m} h[n]\right)$$

$$= \left(\sum_{n=0}^{N-1} (-1)^n h[n]\right)\left(\sum_{n=0}^{N-1} (-1)^n g[n]\right).$$

3.41 $y[n] = \cos\left(\frac{2\pi \ell n}{N}\right)x[n] = \frac{x[n]}{2}\left(e^{-j2\pi \ell n/N} + e^{j2\pi \ell n/N}\right) = \frac{1}{2}x[n]W_N^{n\ell} + \frac{1}{2}x[n]W_N^{-n\ell}$.

Hence $Y[k] = \frac{1}{2}X[\langle k+\ell \rangle_N] + \frac{1}{2}X[\langle k-\ell \rangle_N]$.

3.42 Since x[n] is real hence $X[k] = X^*[\langle -k \rangle_N]$. Thus,

$X[1] = X^*[\langle -1 \rangle_9] = X^*[8] = 5.5 + j8.0,$

$X[3] = X^*[\langle -3 \rangle_9] = X^*[6] = 9.3 - j6.3,$

$X[5] = X^*[<-5>_9] = X^*[4] = -1.7 - j5.2$, and
$X[7] = X^*[<-7>_9] = X^*[2] = 2.5 - j4.6$.

3.43 Since x[n] is real hence $X[k] = X^*[<-k>_N]$. Thus,
$X[2] = X^*[<-2>_9] = X^*[7] = -4.1527 - j\,0.2645$,
$X[3] = X^*[<-3>_9] = X^*[6] = 6.5 - j\,2.5981$,
$X[5] = X^*[<-5>_9] = X^*[4] = -6.3794 - j\,4.1212$ and
$X[8] = X^*[<-8>_9] = X^*[1] = 2.2426 + j$.

3.44 Since X[k]'s are purely real, $x[n] = x^*[<-n>_N]$. Thus
$x[5] = x^*[<-5>_8] = x^*[3] = -0.1098 - j1.6705$,
$x[6] = x^*[<-6>_8] = x^*[2] = 0.25 - j0.125$, and
$x[7] = x^*[<-7>_8] = x^*[1] = -0.642 + j0.0795$.

3.45 Since $Y[k] = W_7^{4k}X[k]$ hence $y[n] = x[<n-4>_7]$. Therefore,

$$y[n] = \left\{\underset{\uparrow}{x[4]} \quad x[5] \quad x[6] \quad x[0] \quad x[1] \quad x[2] \quad x[3]\right\} = \left\{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 5 \quad 3 \quad -2 \quad -4\right\}.$$

3.46 $x[0] = 1,\ x[1] = -1,\ x[2] = 2,\ x[3] = 3,\ x[4] = 0,\ x[5] = 0$.
Since $G[k] = W_6^{3k}X[k]$ hence $g[n] = x[<n-3>_6]$. Thus,

$$g[n] = \left\{\underset{\uparrow}{x[3]} \quad x[4] \quad x[5] \quad x[0] \quad x[1] \quad x[2]\right\} = \{3 \quad 0 \quad 0 \quad 1 \quad -1 \quad 2\}.$$

3.47 $y[n] = x[5n], \quad 0 \le n \le \frac{N}{5}-1$. Therefore, $Y[k] = \sum_{n=0}^{\frac{N}{5}-1} y[n]W_{N/5}^{nk} = \sum_{n=0}^{\frac{N}{5}-1} x[5n]W_{N/5}^{nk}$.

Now, $x[5n] = \frac{1}{N}\sum_{m=0}^{N-1} X[m]W_N^{-5mn} = \frac{1}{N}\sum_{m=0}^{N-1} X[m]W_{N/5}^{-mn}$. Hence,

$$Y[k] = \frac{1}{N}\sum_{n=0}^{N/5-1}\sum_{m=0}^{N-1} X[m]W_{N/5}^{-mn}\,W_{N/5}^{nk} = \frac{1}{N}\sum_{m=0}^{N-1} X[m]\sum_{n=0}^{N/5-1} W_{N/5}^{(k-m)n}$$

Since $\sum_{n=0}^{N/5-1} W_{N/5}^{(k-m)n} = \begin{cases} \frac{N}{5}, & m = k, k+\frac{N}{5}, k+\frac{2N}{5}, k+\frac{3N}{5}, k+\frac{4N}{5}; \\ 0, & \text{elsewhere,} \end{cases}$

Thus $Y[k] = \frac{1}{5}\left(X[k] + X[k+\frac{N}{5}] + X[k+\frac{2N}{5}] + X[k+\frac{3N}{5}] + X[k+\frac{4N}{5}]\right)$.

3. 48 $v[n] = x[n] + j\,y[n]$. Now the DFT of the real part of the sequence v[n] is
$X[k] = \frac{1}{2}\{V[k] + V^*[<-k>_8]\}$ and the DFT of the imaginary part of the sequence is
$Y[k] = \frac{1}{2j}\{V[k] - V^*[<-k>_8]\}$. Therefore,

$$X[k] = \left\{1,\ -1-j,\ \frac{7}{2}+j\frac{15}{2},\ -\frac{1}{2}-j2,\ 2,\ -\frac{1}{2}+j2,\ \frac{7}{2}-j\frac{15}{2},\ -1+j\right\}, \text{ and}$$

$$Y[k] = \left\{-3,\ 5+j,\ -\frac{1}{2}+j\frac{1}{2},\ -3+j\frac{7}{2},\ 5,\ -3-j\frac{7}{2},\ -\frac{1}{2}-j\frac{1}{2},\ 5-j\right\}$$

3.49 $v[n] = g[n] + jh[n] = \{3+j2 \quad 2+j \quad 1+j \quad 4+j3\}$.

Now, $V[k] = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}\begin{bmatrix}3+j2\\ 2+j\\ 1+j\\ 4+j3\end{bmatrix} = \begin{bmatrix}10+j7\\ j3\\ -2-j\\ 4-j\end{bmatrix}$. Therefore,

$V^*[k] = [10-j7,\ \ -j3,\ \ -2+j,\ \ 4+j]$ and $V^*[\langle 4-k\rangle_4] = [10-j7,\ \ 4+j,\ \ -2+j,\ \ -j3]$.

Hence, $G[k] = \frac{1}{2}\{V[k] + V^*[\langle 4-k\rangle_4]\} = [10,\ \ 2+j2,\ \ -2,\ \ 2-j2]$, and

$$H[k] = \frac{1}{2j}\{V[k] - V^*[\langle 4-k\rangle_4]\} = [7,\ \ 1+j2,\ \ -1,\ \ 1-j2].$$

3.50 (a) Let p[n] = IDFT of P[k]. Hence

$$\begin{bmatrix}p[0]\\ p[1]\\ p[2]\\ p[3]\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}\begin{bmatrix}9\\ -2-j3\\ 3\\ -2+j3\end{bmatrix} = \begin{bmatrix}2\\ 3\\ 4\\ 0\end{bmatrix},$$

Similarly d[n] = IDFT of D[k]. Hence,

$$\begin{bmatrix}d[0]\\ d[1]\\ d[2]\\ d[3]\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}\begin{bmatrix}0.7\\ 0.4+j0.9\\ 2.5\\ 0.4-j0.9\end{bmatrix} = \begin{bmatrix}1\\ -0.9\\ 0.6\\ 0\end{bmatrix}.$$

Now, $P(e^{j\omega}) = p[0] + p[1]e^{-j\omega} + p[2]e^{-2j\omega} + p[3]e^{-3j\omega}$ and
$D(e^{j\omega}) = d[0] + d[1]e^{-j\omega} + d[2]e^{-2j\omega} + d[3]e^{-3j\omega}$.

Therefore $\quad X(e^{j\omega}) = \dfrac{P(e^{j\omega})}{D(e^{j\omega})} = \dfrac{2+3e^{-j\omega}+4e^{-2j\omega}}{1-0.9e^{-j\omega}+0.6e^{-2j\omega}}$.

(b) Here, p[n] = IDFT of P[k] = {0.75 2.25 2.75 3.25} and
d[n] = IDFT of D[k] = {1 −0.6 0.5 −0.4}.

Therefore $X(e^{j\omega}) = \dfrac{P(e^{j\omega})}{D(e^{j\omega})} = \dfrac{0.75+2.25e^{-j\omega}+2.75e^{-2j\omega}+3.25e^{-3j\omega}}{1-0.6e^{-j\omega}+0.5e^{-2j\omega}-0.4e^{-3j\omega}}$.

3.51 $X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$ and $\hat{X}[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}$.

Now $\hat{x}[n] = \frac{1}{M}\sum_{k=0}^{M-1}\hat{X}[k]W_M^{-nk} = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{m=0}^{N-1} x[m]e^{-j2\pi km/M}W_M^{-nk}$

$= \frac{1}{M}\sum_{m=0}^{N-1} x[m]\sum_{k=0}^{M-1} e^{-j2\pi k(m-n)/M} = \sum_{r=-\infty}^{\infty} x[n+rM]..$

Thus $\hat{x}[n]$ is obtained by shifting x[n] by multiples of M and adding the shifted copies. Since the new sequence is obtained by shifting in multiples of M, hence to recover the original sequence take any M consecutive samples. This would be true only if the shifted copies of x[n] did not overlap with each other, that is, if only if $M \geq N$.

3.52 Since $\mathcal{F}^{-1} = \frac{1}{N}\mathcal{F}$ thus $\mathcal{F} = N\mathcal{F}^{-1}$. Thus

$y[n] = \mathcal{F}\{\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{x[n]\}\}\}\} = N\mathcal{F}^{-1}\{\mathcal{F}\{N\mathcal{F}^{-1}\{\mathcal{F}\{x[n]\}\}\}\} = N^2 x[n]$.

3.53 $y[n] = x[n] \circledast h[n] = \sum_{k=0}^{29} x[k]h[n-k] = \sum_{k=0}^{29} h[k]x[n-k] = \sum_{k=10}^{19} h[k]x[n-k].$

$u[n] = x[n] \, \textcircled{N} \, h[n] = \sum_{k=0}^{29} h[k]x[\langle n-k \rangle_{30}] = \sum_{k=10}^{19} h[k]x[\langle n-k \rangle_{30}].$

Now for $n \geq 19$, $x[\langle n-k \rangle_{30}] = x[n-k]$. Thus $u[n] = y[n]$ for $19 \leq n \leq 29$.

3.54 Overlap and add method: Since the impulse response is of length 60 and the DFT size to be used is 128, hence the number of data samples required for each convolution will be 128 – 59 = 69. Thus the total number of DFT's required for data samples is $\left\lceil \frac{1200}{69} \right\rceil = 18.$
Also the DFT of the impulse response needs to be computed once. Hence, the total number of DFT's used are = 18 + 1 = 19. The total number of IDFT's used are = 18.

(b) Overlap and save method: In this since the first 60 – 1 = 59 points are lost, we need to pad the data sequence with 59 zeros. Again each convolution will result in 128 - 59 = 69 correct values. Thus the total number of DFT's required for the data are $\left\lceil \frac{1259}{69} \right\rceil = 19.$

Again 1 DFT is required for the impulse response. Thus
The total number of DFT's used are = 19 + 1 = 20.
The total number of IDFT's used are = 19.

3.55 $X(z) = \sum_{n=0}^{9} x[n]z^{-n}$, hence $X_0[k] = \sum_{n=0}^{9} x[n]e^{-j2\pi kn/7}$. Thus,

$$x_0[n] = \frac{1}{7}\sum_{k=0}^{6} X_0[k]e^{j2\pi kn/7} = \frac{1}{7}\sum_{m=0}^{9} x[m]\sum_{k=0}^{6} e^{j2\pi k(n-m)/7} = \sum_{r=-\infty}^{\infty} x[n+7r].$$

$x_0[n] = \{-1 \quad 5 \quad -7 \quad 0 \quad 3 \quad 2 \quad 5\}.$

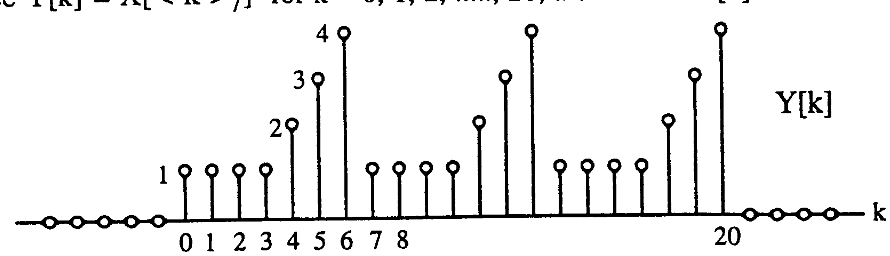
3.56 (a) $y[n] = \begin{cases} x[n/L], & n = 0, L, 2L, \ldots\ldots, (N-1)L, \\ 0, & \text{elsewhere.} \end{cases}$

$$Y[k] = \sum_{n=0}^{NL-1} y[n]W_{NL}^{nk} = \sum_{n=0}^{N-1} x[n]W_{NL}^{nLk} = \sum_{n=0}^{N-1} x[n]W_{N}^{nk}.$$

For $k \geq N$, let $k = k_0 + rN$ where $k_0 = \langle k \rangle_N$. Then,

$$Y[k] = Y[k_0 + rN] = \sum_{n=0}^{N-1} x[n]W_N^{n(k_0+rN)} = \sum_{n=0}^{N-1} x[n]W_N^{nk_0} = X[k_0] = X[\langle k \rangle_N].$$

(b) Since $Y[k] = X[\langle k \rangle_7]$ for k = 0, 1, 2, ..., 20, a sketch of Y[k] is thus as shown below.



3.57 $x_0[n] = x[2n+1] + x[2n]$, $x_1[n] = x[2n+1] - x[2n]$, $y_1[n] = y[2n+1] + y[2n]$, and $y_0[n] = y[2n+1] - y[2n]$, $0 \leq n \leq \frac{N}{2} - 1$. Since x[n] and y[n] are real, symmetric sequences, it

follows that $x_0[n]$ and $y_0[n]$ are real, symmetric sequences, and $x_1[n]$ and $y_1[n]$ are real, antisymmetric sequences. Now consider, the (N/2)-length sequence
$u[n] = x_0[n] + y_1[n] + j\left(x_1[n] + y_0[n]\right)$. Its conjugate sequence is given by
$u^*[n] = x_0[n] + y_1[n] - j\left(x_1[n] + y_0[n]\right)$. Next we observe that

$$u[\langle -n\rangle_{N/2}] = x_0[\langle -n\rangle_{N/2}] + y_1[\langle -n\rangle_{N/2}] + j\left(x_1[\langle -n\rangle_{N/2}] + y_0[\langle -n\rangle_{N/2}]\right)$$
$$= x_0[n] - y_1[n] + j\left(-x_1[n] + y_0[n]\right).$$ Its conjugate sequence is given by
$$u^*[\langle -n\rangle_{N/2}] = x_0[n] - y_1[n] - j\left(-x_1[n] + y_0[n]\right).$$

By adding the last 4 sequences we get
$4x_0[n] = u[n] + u^*[n] + u[\langle -n\rangle_{N/2}] + u^*[\langle -n\rangle_{N/2}]$.
From Table 3.6, if U[k] = DFT{u[n]}, then $U^*[\langle -k\rangle_{N/2}] = \text{DFT}\{u^*[n]\}$,
$U^*[k] = \text{DFT}\{u^*[\langle -n\rangle_{N/2}]\}$, and $U[\langle -k\rangle_{N/2}] = \text{DFT}\{u[\langle -n\rangle_{N/2}]\}$. Thus,
$X_0[k] = \text{DFT}\{x_0[n]\} = \frac{1}{4}\left(U[k] + U^*[\langle -k\rangle_{N/2}] + U[\langle -k\rangle_{N/2}] + U^*[k]\right)$. Similarly,
$j4x_1[n] = u[n] - u^*[n] - u[\langle -n\rangle_{N/2}] + u^*[\langle -n\rangle_{N/2}]$. Hence,
$X_1[k] = \text{DFT}\{x_1[n]\} = \frac{1}{4j}\left(U[k] - U^*[\langle -k\rangle_{N/2}] - U[\langle -k\rangle_{N/2}] + U^*[k]\right)$. Likewise,
$4y_1[n] = u[n] - u[\langle -n\rangle_{N/2}] + u^*[n] - u^*[\langle -n\rangle_{N/2}]$. Thus,
$Y_1[k] = \text{DFT}\{y_1[n]\} = \frac{1}{4}\left(U[k] - U[\langle -k\rangle_{N/2}] + U^*[\langle -k\rangle_{N/2}] - U^*[k]\right)$. Finally,
$j4y_0[n] = u[n] + u[\langle -n\rangle_{N/2}] - u^*[n] - u^*[\langle -n\rangle_{N/2}]$. Hence,
$Y_0[k] = \text{DFT}\{y_0[n]\} = \frac{1}{4j}\left(U[k] + U[\langle -k\rangle_{N/2}] - U^*[\langle -k\rangle_{N/2}] - U^*[k]\right)$.

3.58 $g[n] = \frac{1}{2}(x[2n] + x[2n+1])$, $h[n] = \frac{1}{2}(x[2n] - x[2n+1])$, $0 \le n \le \frac{N}{2} - 1$. Solving for x[2n] and x[2n+1], we get x[2n] = g[n] + h[n] and x[2n + 1] = g[n] − h[n], $0 \le n \le \frac{N}{2} - 1$. Therefore,

$$X(z) = \sum_{n=0}^{N-1} x[n]z^{-n} = \sum_{n=0}^{\frac{N}{2}-1} x[2n]z^{-n} + z^{-1}\sum_{n=0}^{\frac{N}{2}-1} x[2n+1]z^{-n}$$
$$= \sum_{n=0}^{\frac{N}{2}-1} (g[n]+h[n])z^{-n} + z^{-1}\sum_{n=0}^{\frac{N}{2}-1} (g[n]+h[n])z^{-n} = (1+z^{-1})\sum_{n=0}^{\frac{N}{2}-1} g[n]z^{-n} + (1-z^{-1})\sum_{n=0}^{\frac{N}{2}-1} h[n]z^{-n}.$$

Hence, $X[k] = X(z)\big|_{z=W_N^k} = (1+W_N^{-k})G[\langle k\rangle_{N/2}] + (1-W_N^{-k})H[\langle k\rangle_{N/2}]$, $0 \le k \le N-1$.

3.59 $g[n] = a_1x[2n] + a_2x[2n+1]$ and $h[n] = a_3x[2n] + a_4x[2n+1]$, with $a_1a_4 \ne a_2a_3$. Solving for x[2n] and x[2n+1], we get $x[2n] = \frac{a_4g[n] - a_2h[n]}{a_1a_4 - a_2a_3}$, and $x[2n+1] = \frac{-a_3g[n] + a_1h[n]}{a_1a_4 - a_2a_3}$. Therefore

$$X(z) = \sum_{n=0}^{N-1} x[n]z^{-n} = \sum_{n=0}^{\frac{N}{2}-1} x[2n]z^{-n} + z^{-1}\sum_{n=0}^{\frac{N}{2}-1} x[2n+1]z^{-n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left(\frac{a_4 g[n]-a_2 h[n]}{a_1 a_4 - a_2 a3}\right) z^{-n} + z^{-1}\sum_{n=0}^{\frac{N}{2}-1}\left(\frac{-a_3 g[n]+a_1 h[n]}{a_1 a_4 - a_2 a3}\right) z^{-n}$$

$$= \frac{1}{a_1 a_4 - a_2 a_3}(a_4 - a_3 z^{-1})\sum_{n=0}^{\frac{N}{2}-1} g[n]z^{-n} + (-a_2 + a_1 z^{-1})\sum_{n=0}^{\frac{N}{2}-1} h[n]z^{-n}.$$ Hence,

$$X[k] = \frac{1}{a_1 a_4 - a_2 a_3}(a_4 - a_3 W_N^{-nk})G[<k>_{N/2}] + (-a_2 + a_1 W_N^{-nk})G[<k>_{N/2}], \quad 0 \le k \le N-1.$$

3.60 $G(z)$ has poles at $z = -1 \pm j\,2$, and $2 \pm j\,3$. Hence there are three possible ROCs.

(i) $\Re_1$: $|z| \le \sqrt{5}$. The inverse z-transform $g[n]$ in this case is a left-sided sequence.

(ii) $\Re_2$: $\sqrt{5} \le |z| \le \sqrt{13}$. The inverse z-transform $g[n]$ in this case is a two-sided sequence.

(iii) $\Re_3$: $|z| \ge \sqrt{13}$. The inverse z-transform $g[n]$ in this case is a right-sided sequence.

3.61 **(a)** (i) $x_1[n] = (0.3)^n \mu[n]$, Thus, $X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=0}^{\infty}(0.3)^n z^{-n} = \frac{1}{1-0.3z^{-1}}$, $\quad |z| > 0.3$

The ROC of $X_1(z)$ is given by $\Re_1$: $|z| > 0.3$.

(ii) $x_2[n] = (-0.5)^n \mu[n]$, Thus, $X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} = \sum_{n=0}^{\infty}(-0.5)^n z^{-n} = \frac{1}{1+0.5z^{-1}}$, $\quad |z| > 0.5$

The ROC of $X_2(z)$ is given by $\Re_2$: $|z| > 0.5$.

(iii) $x_3[n] = (0.2)^n \mu[n-5]$. Thus, $X_3(z) = \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} = \sum_{n=5}^{\infty}(0.2)^n z^{-n}$

$= \frac{(0.2)^5 z^{-5}}{1-0.2z^{-1}}$, $\quad |z| > 0.2$. The ROC of $X_3(z)$ is given by $\Re_3$: $|z| > 0.2$.

(iv) $x_4[n] = (-0.2)^n \mu[-n-1]$. Thus,

$$X_4(z) = \sum_{n=-\infty}^{\infty} x_4[n]z^{-n} = \sum_{n=-\infty}^{-1}(-0.2)^n z^{-n} = \sum_{m=1}^{\infty}(-0.2)^{-m} z^m = \frac{-1}{1+0.2z^{-1}}, \quad |z| < 0.2.$$

The ROC of $X_4(z)$ is given by $\Re_4$: $|z| < 0.2$

(b) (i) Now, the ROC of $X_1(z)$ is given by $\Re_1$: $|z| > 0.3$ and the ROC of $X_2(z)$ is given by $\Re_2$: $|z| > 0.5$. Hence, the ROC of $Y_1(z)$ is given by $\Re_1 \cap \Re_2 = \Re_2$: $|z| > 0.5$

(ii) The ROC of $Y_2(z)$ is given by $\Re_1 \cap \Re_3 = \Re_1$: $|z| > 0.3$.

(iii) The ROC of $Y_3(z)$ is given by $\Re_1 \cap \Re_4 = \varnothing$. Hence, the z-transform of the sequence $y_3[n]$ does not converge anywhere in the z-plane.

(iv) The ROC of $Y_4(z)$ is given by $\Re_2 \cap \Re_3 = \Re_2$: $|z| > 0.5$.

(v) The ROC of $Y_5(z)$ is given by $\Re_2 \cap \Re_4 = \varnothing$. Hence, the z-transform of the sequence $y_5[n]$ does not converge anywhere in the z-plane.

3.62 (i) $\mathcal{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \delta[0] = 1,$ which converges everywhere in the z-plane.

(ii) See Example 3.15.

(iii) $\mathcal{Z}\{\alpha^n \mu[n]\} = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}}, \quad \forall\ |z| > |\alpha|$

(iv) See Example 3.29.

(v) $x[n] = r^n \sin(\omega_0 n)\mu[n] = \frac{r^n}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)\mu[n].$ Using the results of (iii) and the linearity property of the z-transform we obtain

$$\mathcal{Z}\{ r^n \sin(\omega_0 n)\mu[n]\} = \frac{1}{2j}\left(\frac{1}{1-re^{j\omega_0}z^{-1}}\right) - \frac{1}{2j}\left(\frac{1}{1-re^{-j\omega_0}z^{-1}}\right)$$

$$= \frac{\frac{r}{2j}\left(e^{j\omega_0} - e^{-j\omega_0}\right)z^{-1}}{1-rz^{-1}\left(e^{j\omega_0}+e^{-j\omega_0}\right)+r^2z^{-2}} = \frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}, \qquad \forall\ |z| > |r|$$

3.63 **(a)** $x_1[n] = \alpha^n \mu[n], \quad |\alpha| < 1.$ Thus, $X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n}$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|.$$

The ROC of $X_1(z)$ is $|z| > |\alpha|$ which includes the unit circle. On the unit circle $z = e^{j\omega}$. Thus $X_1(e^{j\omega}) = X_1(z)\big|_{z=e^{j\omega}} = \frac{1}{1-\alpha e^{-j\omega}}$ which is the same as the DTFT of $x_1[n]$.

(b) $x_2[n] = \alpha^n \mu[-n], \quad |\alpha| > 1.$ Thus, $X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} = \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n] z^{-n}$

$$= \sum_{n=-\infty}^{0} \alpha^n z^{-n} = \frac{1}{1-\alpha^{-1}z}, \quad |z| < |\alpha|.$$

The ROC of $X_2(z)$ is $|z| < |\alpha|$. now since $|\alpha| > 1$, the ROC includes the unit circle. On the unit circle $z = e^{j\omega}$. Thus, $X_2(e^{j\omega}) = X_2(z)\big|_{z=e^{j\omega}} = \frac{1}{1-\alpha^{-1}e^{j\omega}}$

(c) $x_3[n] = \alpha^{|n|}\mu[n] = \alpha^n \mu[n]$ which is the same sequence as that in part (a).

Hence $X_3(z) = X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \frac{1}{1-\alpha z^{-1}}, \ |z| > |\alpha|.$

(d) $x_4[n] = \begin{cases} \alpha^{|n|}, & |n| \le M, \\ 0, & \text{elsewhere.} \end{cases}$ Therefore, $X_4(z) = \sum_{n=-\infty}^{\infty} x_4[n] z^{-n}$

$$= \sum_{n=-M}^{M} \alpha^{|n|} z^{-n} = \sum_{n=-M}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{M} \alpha^n z^{-n} = \frac{\alpha^M z^M (1-\alpha^{-M} z^{-M})}{(1-\alpha^{-1}z^{-1})} + \frac{1-\alpha^{M+1}z^{-(M+1)}}{1-\alpha z^{-1}}.$$

ROC of $X_4(z)$ is the entire z-plane. Thus, the DTFT of $x_4[n]$ is given by

$$X_4(e^{j\omega}) = X_4(z)\big|_{z=e^{j\omega}} = \frac{\alpha^M e^{j\omega M} - 1}{(1-\alpha^{-1}e^{-j\omega})} + \frac{1-\alpha^{M+1}e^{-j\omega(M+1)}}{1-\alpha e^{-j\omega}}.$$

(e) $x_5[n] = \alpha^n \mu[n+3]$, $|\alpha| < 1$. Thus,

$$X_5(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n+3] z^{-n} = \sum_{n=-3}^{\infty} \alpha^n z^{-n} = \frac{\alpha^{-3}z^3}{1-\alpha z^{-1}} = \frac{\alpha^{-3}z^4}{z-\alpha}, \quad |z| > |\alpha|.$$

Since the ROC of $X_5(z)$ includes the unit circle thus $X_5(e^{j\omega}) = X_5(z)\big|_{z=e^{j\omega}} = \dfrac{\alpha^{-3}e^{j4\omega}}{e^{j\omega}-\alpha}$.

(f) $x_6[n] = n\alpha^n \mu[n]$. Now, $X_6(z) = \sum_{n=-\infty}^{\infty} n\alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} n\alpha^n z^{-n}$.

Now differentiating $X_1(z)$ (from part (a)) we get $\dfrac{dX_1(z)}{dz} = \sum_{n=0}^{\infty} (-n)\alpha^n z^{-n-1}$. Thus,

$$z\frac{dX_1(z)}{dz} = -\sum_{n=0}^{\infty} n\,\alpha^n z^{-n} = -X_6(z).$$

Hence $X_6(z) = -z\dfrac{dX_1(z)}{dz} = -z\dfrac{d}{dz}\left(\dfrac{1}{1-\alpha z^{-1}}\right) = \dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $|z| > |\alpha|$..

The ROC of $X_6(z)$ includes the unit circle. Hence, $X_6(e^{j\omega}) = X_6(z)\big|_{z=e^{j\omega}} = \dfrac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$.

3.64 (a) $y_1[n] = \begin{cases} 1, & -N \le n \le N, \\ ,0 & \text{elsewhere.} \end{cases}$

$$Y_1(z) = \sum_{n=-\infty}^{\infty} y_1[n] z^{-n} = \sum_{n=-N}^{N} z^{-n} = \frac{z^N(1-z^{-(2N+1)})}{1-z^{-1}} = \frac{z^{N+1}-z^{-N}}{z-1}.$$

ROC of $Y_1(z)$ is the entire z-plane except $z = \infty$. Since the ROC includes the unit circle, we get

$$Y_1(e^{j\omega}) = Y_1(z)\big|_{z=e^{j\omega}} = \frac{e^{j\omega N} - e^{-j\omega(N+1)}}{1-e^{-j\omega}} = \frac{\sin\left(\omega(N+\frac{1}{2})\right)}{\sin(\omega/2)}.$$

(b) $y_2[n] = \begin{cases} 1 - \dfrac{|n|}{N}, & -N \le n \le N, \\ 0, & \text{elsewhere.} \end{cases}$

As in Problem 3.9(b), $y_2[n] = y_0[n] \circledast y_0[n]$ where $y_0[n] = \begin{cases} 1, & -N/2 \le n \le N/2, \\ 0, & \text{elsewhere.} \end{cases}$

Now, $Y_0(z) = \sum_{n=-N/2}^{N/2} z^{-n} = z^{N/2}\dfrac{(1-z^{-(N+1)})}{1-z^{-1}}$. Thus, $Y_2(z) = Y_0^2(z) = z^N\left(\dfrac{(1-z^{-(N+1)})}{1-z^{-1}}\right)^2$.

The ROC of $Y_2(z)$ is the entire z-plane except $z = \infty$. The DTFT of $y_2[n]$ is therefore given by $Y_2(e^{j\omega}) = Y_2(z)\big|_{z=e^{j\omega}} = e^{j\omega N}\left(\dfrac{(1-e^{-j\omega(N+1)})}{1-e^{-j\omega}}\right)^2 = \dfrac{\sin^2(\omega(N+1)/2)}{\sin^2(\omega/2)}$.

(c) $y_3[n] = \begin{cases} \cos(\pi n/2N), & -N \le n \le N, \\ 0, & \text{elsewhere.} \end{cases}$

Thus, $Y_3(z) = \sum_{n=-\infty}^{\infty} y_3[n]z^{-n} = \sum_{n=-N}^{n=N} \frac{1}{2}\left(e^{j\pi n/2N} + e^{-j\pi n/2N}\right)z^{-n}$

$$= \frac{1}{2}\frac{e^{-j\pi/2}z^{N}\left(1 - e^{j\pi(2N+1)/2N}z^{-(2N+1)}\right)}{1 - e^{j\pi/2N}z^{-1}} + \frac{1}{2}\frac{e^{j\pi/2}z^{N}\left(1 - e^{-j\pi(2N+1)/2N}z^{-(2N+1)}\right)}{1 - e^{-j\pi/2N}z^{-1}}$$

$$= \frac{1}{2}\frac{\left(e^{-j\pi(N+1/2)/2N}z^{(N+1/2)} - e^{j\pi(N+1/2)/2N}z^{-(N+1/2)}\right)}{(z^{1/2}e^{-j\pi/4N} - e^{j\pi/4N}z^{-1/2})}$$

$$+ \frac{1}{2}\frac{\left(e^{j\pi(N+1/2)/2N}z^{(N+1/2)} - e^{-j\pi(N+1/2)/2N}z^{-(N+1/2)}\right)}{(z^{1/2}e^{j\pi/4N} - e^{-j\pi/4N}z^{-1/2})}.$$

The ROC of $Y_3(z)$ is the entire z-plane except $z = \infty$. The DTFT of $y_3[n]$ is therefore given by $Y_3(e^{j\omega}) = Y_3(z)\big|_{z=e^{j\omega}} = \frac{1}{2}\frac{\left(e^{-j\pi(N+1/2)/2N}e^{j\omega(N+1/2)} - e^{j\pi(N+1/2)/2N}e^{-j\omega(N+1/2)}\right)}{(e^{j\omega/2}e^{-j\pi/4N} - e^{j\pi/4N}e^{-j\omega/2})}$

$$+ \frac{1}{2}\frac{\left(e^{j\pi(N+1/2)/2N}e^{j\omega(N+1/2)} - e^{-j\pi(N+1/2)/2N}e^{-j\omega(N+1/2)}\right)}{(e^{j\omega/2}e^{j\pi/4N} - e^{-j\pi/4N}e^{-j\omega/2})}$$

$$= \frac{1}{2}\frac{\sin\left((\omega - \frac{\pi}{2N})(N + \frac{1}{2})\right)}{\sin\left((\omega - \frac{\pi}{2N})/2\right)} + \frac{1}{2}\frac{\sin\left((\omega + \frac{\pi}{2N})(N + \frac{1}{2})\right)}{\sin\left((\omega + \frac{\pi}{2N})/2\right)}.$$

3.65 **(a)** $Y_1(z) = \frac{1}{1-4z}$, $|z| < \frac{1}{4}$. Consider $X_1(z) = \frac{1}{1-4z^{-1}}$, $|z| > 4$. From Table 3.8, the inverse z-transform $x_1[n]$ of $X_1(z)$ is given by $x_1[n] = 4^n\mu[n]$. From the time-reversal property of the z-transform given in Table 3.9, the inverse z-transform $y_1[n]$ of $Y_1(z)$ is thus given by $y_1[n] = x_1[-n] = 4^{-n}\mu[-n] = \left(\frac{1}{4}\right)^n \mu[-n]$.

(b) $Y_2(z) = \frac{1}{1 - z^{-1} + 0.5z^{-2}} = \frac{1}{2}\frac{1-j}{1-\left(\frac{1+j}{2}\right)z^{-1}} + \frac{1}{2}\frac{1+j}{1-\left(\frac{1-j}{2}\right)z^{-1}}$, Thus, from Table 3.8 and the linearity property of the z-transform given in Table 3.9, we obtain

$$y_2[n] = \frac{1}{2}(1-j)\left(\frac{1+j}{2}\right)^n \mu[n] + \frac{1}{2}(1+j)\left(\frac{1-j}{2}\right)^n \mu[n].$$

Let $r\cos(\theta) = \frac{1}{2}$ and $r\sin(\theta) = \frac{1}{2}$. Then, $r = \frac{1}{\sqrt{2}}$ and $\tan(\theta) = 1$, or, $\theta = \frac{\pi}{4}$. Substituting these in $y_2[n]$ we get

$$y_2[n] = (r\cos(\theta) - jr\sin(\theta))(r\cos(\theta) + jr\sin(\theta))^n \mu[n]$$
$$+ (r\cos(\theta) + jr\sin(\theta))(r\cos(\theta) - jr\sin(\theta))^n \mu[n]$$
$$= \left(r^{n+1}e^{-j\theta}e^{jn\theta} + r^{n+1}e^{j\theta}e^{-jn\theta}\right)\mu[n] = 2\left(2^{-(n+1)/2}\right)\cos\left(\frac{(n-1)\pi}{4}\right)\mu[n].$$

(c) $Y_3(z) = \frac{12 + 8z^{-1} - 3z^{-2}}{12 - 7z^{-1} + z^{-2}} = \frac{3}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 - 0.25z^{-1}} - 3$, $|z| > \frac{1}{3}$.

Since the ROC is $|z|>\frac{1}{3}$, hence $y_3[n]=3\left(\frac{1}{3}\right)^n\mu[n]+\left(\frac{1}{4}\right)^n\mu[n]-3\delta[n]$.

(d) $Y_4(z)=\dfrac{1-z^{-1}+\frac{1}{6}z^{-2}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}=1-\dfrac{1}{1-\frac{1}{2}z^{-1}}+\dfrac{1}{1-\frac{1}{3}z^{-1}},\qquad \frac{1}{3}<|z|<\frac{1}{2}.$

Since ROC is given by $\frac{1}{3}<|z|<\frac{1}{2}$, the ROC of the middle term in the partial-fraction expansion above has a ROC $|z|<\frac{1}{2}$, and the last term has a ROC given by $|z|>\frac{1}{3}$. Therefore,

$$y_4[n]=\left(\frac{1}{2}\right)^n\mu[-n-1]+\left(\frac{1}{3}\right)^n\mu[n]+\delta[n].$$

3.66 (a) We first determine the inverse z-transform $x_a[n]$ by a partial-fraction expansion.

$$X_a(z)=z^2+3z+5+\frac{2}{z^2+5z+4}=z^2+3z+5+\frac{\frac{1}{6}}{1+4z^{-1}}-\frac{\frac{2}{3}}{1+z^{-1}}+\frac{1}{2},\qquad |z|>4.$$

Rewriting we get $X_a(z)=z^2+3z+\frac{11}{2}+\dfrac{\frac{1}{6}}{1+4z^{-1}}-\dfrac{\frac{2}{3}}{1+z^{-1}},\qquad |z|>4$. Thus

$$x_a[n]=\delta[n+2]+3\delta[n+1]+\frac{11}{2}\delta[n]+\frac{1}{6}(-4)^n\mu[n]-\frac{2}{3}(-1)^n\mu[n]..$$

We next determine the inverse z-transform $x_a[n]$ by the long division approach. To this end we rewrite $X_a(z)$ as $X_a(z)=z^2+3z+5+\dfrac{2}{z^2+5z+4}=z^2+3z+5+\dfrac{2z^{-2}}{1+5z^{-1}+4z^{-2}}$.

Dividing $2z^{-2}$by $1+5z^{-1}+4z^{-2}$ we get $\dfrac{2z^{-2}}{1+5z^{-1}+4z^{-2}}=2z^{-2}-10z^{-3}+42z^{-4}-170z^{-5}+\cdots$

Therefore, $X_a(z)=z^2+3z+5+2z^{-2}-10z^{-3}+42z^{-4}-170z^{-5}+\cdots$, and hence,

$$\{x_a[n]\}=\{1\quad 3\quad \underset{\uparrow}{5}\quad 0\quad 2\quad -10\quad 42\quad -170\quad \cdots\}$$

(b) We first determine the inverse z-transform $x_b[n]$ by a partial-fraction expansion

$X_b(z)=\dfrac{1+0.5z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}=\dfrac{4}{1-0.5z^{-1}}-\dfrac{3}{1-0.25z^{-1}},\qquad |z|>0.5$. Therefore,

$x_b[n]=4(0.5)^n\mu[n]-3(0.25)^n\mu[n]$.

We next determine the inverse z-transform $x_b[n]$ by the long division approach:

$X_b(z)=\dfrac{1+0.5z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}=1+1.25z^{-1}+0.8125\,z^{-2}+0.4531z^{-3}+0.2383z^{-4}+\cdots$

Therefore, $\{x_b[n]\}=\{1\quad 1.25\quad 0.8125\quad 0.4531\quad 0.2388\quad \cdots\}$.

(c) We first determine the inverse z-transform $x_c[n]$ by a partial-fraction expansion

$$X_c(z)=\frac{2}{(z+0.5)(z+1)}=4+\frac{4}{1+z^{-1}}-\frac{8}{1+0.5z^{-1}},\qquad 0.5<|z|<1.$$

Since the ROC is an annular strip, $x_c[n]$ is a two-sided sequence. The second term in the partial fraction expansion has an ROC given by $|z|<1$, and has an inverse z-transform which is a left-

sided sequence $-4(-1)^n\mu[-n-1]$, whereas, the last term has an ROC given by $|z|>0.5$, and has an inverse z-transform $-8(-0.5)^n\mu[n]$ which is a right-sided sequence.
Therefore, $x_c[n] = 4\delta[n] - 4(-1)^n\mu[-n-1] - 8(-0.5)^n\mu[n]$.

We next determine the inverse z-transform $x_c[n]$ by the long division approach. To this end we need to develop a power series expansion of $4/(1+z^{-1})$ in a positive power of z, and, a power series expansion of $8/(1+0.5z^{-1})$ in a positive power of z^{-1}. Now,

$$\frac{4}{1+z^{-1}} = \frac{4z}{1+z} = 4z(1-z+z^2-z^3+z^4-\cdots) = 4z-4z^2+4z^3-4z^4+4z^5-\cdots,\text{ whereas,}$$

$$\frac{8}{1+0.5z^{-1}} = 8-4z^{-1}+2z^{-2}-z^{-3}+0.5z^{-4}-0.25z^{-5}+\cdots.\ \text{Therefore,}$$

$$\{x_c[n]\} = \{\cdots\ 4\ \ -4\ \ 4\ \ -4\ \ 4\ \ \underset{\uparrow}{-4}\ \ 4\ \ -2\ \ 1\ \ -0.5\ \ 0.25\ \ \cdots\}.$$

(d) $$X_d(z) = \frac{z^4}{(z-0.5)^2(z-0.2)(z+0.6)} = \frac{1}{(1-0.5z^{-1})^2(1-0.2z^{-1})(1+0.6z^{-1})},\quad |z|>0.6.$$

We first determine the inverse z-transform $x_d[n]$ by a partial-fraction expansion

$$X_d(z) = \frac{0.2231}{1+0.6z^{-1}} - \frac{0.0918}{1-0.5z^{-1}} + \frac{0.7576}{(1-0.5z^{-1})^2} + \frac{0.1111}{1-0.2z^{-1}},\qquad |z|>0.6.\ \text{Therefore,}$$

$$x_d[n] = 0.2231(-0.6)^n\mu[n] + 0.1111(0.2)^n\mu[n] - 0.0918(0.5)^n\mu[n] + 0.7576(n+1)(0.5)^n\mu[n+1]$$

We next determine the inverse z-transform $x_d[n]$ by the long division approach.

$$X_d(z) = \frac{z^4}{(z-0.5)^2(z-0.2)(z+0.6)} = \frac{1}{1-0.6z^{-1}-0.27z^{-2}+0.22z^{-3}-0.3z^{-4}}$$
$$= 1+0.6z^{-1}+0.63z^{-2}+0.32z^{-3}+0.2601z^{-4}+0.1219z^{-5}+\cdots$$

Therefore, $\{x_d[n]\} = \{\underset{\uparrow}{1}\ \ 0.6\ \ 0.63\ \ 0.32\ \ 0.2601\ \ 0.1219\ \ \cdots\}$

3.67 $H(z) = \dfrac{1}{1-2r\cos(\theta)z^{-1}+r^2z^{-2}}$, $\quad |z|>r>0$. By using partial-fraction expansion we write

$$H(z) = \frac{1}{\left(e^{j\theta}-e^{-j\theta}\right)}\left\{\frac{e^{j\theta}}{1-re^{j\theta}z^{-1}} - \frac{e^{-j\theta}}{1-re^{-j\theta}z^{-1}}\right\} = \frac{1}{2j\sin(\theta)}\left\{\frac{e^{j\theta}}{1-re^{j\theta}z^{-1}} - \frac{e^{-j\theta}}{1-re^{-j\theta}z^{-1}}\right\}.\ \text{Thus,}$$

$$h[n] = \frac{1}{2j\sin(\theta)}\left\{e^{j\theta}r^ne^{jn\theta}\mu[n] - r^ne^{-j\theta}e^{-jn\theta}\mu[n]\right\} = \frac{r^n}{\sin(\theta)}\left\{\frac{e^{j\theta(n+1)}-e^{-j\theta(n+1)}}{2j}\right\}\mu[n]$$
$$= \frac{r^n\sin((n+1)\theta)}{\sin(\theta)}\mu[n].$$

3.68 $G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$ with a ROC given by $\mathcal{R}_g$.

(a) Therefore $G^*(z) = \sum_{n=-\infty}^{\infty} g^*[n](z^*)^{-n}$ and $G^*(z^*) = \sum_{n=-\infty}^{\infty} g^*[n]z^{-n}$.

Thus the z-transform of $g^*[n]$ is $G^*(z^*)$.

(b) Replace n by −m in the summation. This leads to $G(z)=\sum_{m=-\infty}^{\infty} g[-m]z^{m}$. Therefore

$G(1/z)=\sum_{m=-\infty}^{\infty} g[-m]z^{-m}$. Thus the z-transform of g[−n] is G(1/z). Note that since z has been replaced by 1/z, the ROC of G(1/z) will be $1/\mathcal{R}_g$.

(c) Let $y[n]=\alpha g[n]+\beta h[n]$. Then,

$$Y(z)=\sum_{n=-\infty}^{\infty}(\alpha g[n]+\beta h[n])z^{-n}=\alpha\sum_{n=-\infty}^{\infty} g[n]z^{-n}+\beta\sum_{n=-\infty}^{\infty} h[n]z^{-n}=\alpha G(z)+\beta H(z)$$

In this case Y(z) will converge wherever both G(z) and H(z) converge. Thus the ROC of Y(z) is $\mathcal{R}_g \cap \mathcal{R}_h$, where is $\mathcal{R}_g$ the ROC of G(z) and $\mathcal{R}_h$ is the ROC of H(z).

(d) $y[n]=g[n-n_0]$. Hence $Y(z)=\sum_{n=-\infty}^{\infty} y[n]z^{-n}=\sum_{n=-\infty}^{\infty} g[n-n_0]z^{-n}=\sum_{m=-\infty}^{\infty} g[m]z^{-(m+n_0)}$

$$=z^{-n_0}\sum_{m=-\infty}^{\infty} g[m]z^{-m}=z^{-n_0}G(z).$$

In this case the ROC of Y(z) is the same as that of G(z) except for the possible addition or elimination of the point z = 0 or z = ∞ (due to the factor z^{-n_0}).

(e) $y[n]=\alpha^n g[n]$. Hence, $Y(z)=\sum_{n=-\infty}^{\infty} y[n]z^{-n}=\sum_{n=-\infty}^{\infty} g[n](z\alpha^{-1})^{-n}=G(z/\alpha)$.

The ROC of Y(z) is $|\alpha|\mathcal{R}_g$.

(f) y[n] = ng[n]. Hence $Y(z)=\sum_{n=-\infty}^{\infty} ng[n]z^{-n}$.

Now $G(z)=\sum_{n=-\infty}^{\infty} g[n]z^{-n}$. Thus, $\frac{dG(z)}{dz}=-\sum_{n=-\infty}^{\infty} ng[n]z^{-n-1} \Rightarrow z\frac{dG(z)}{dz}=-\sum_{n=-\infty}^{\infty} ng[n]z^{-n}$.

Thus $Y(z)=-z\frac{dG(z)}{dz}$.

(g) $y[n]=g[n]\circledast h[n]=\sum_{k=-\infty}^{\infty} g[k]h[n-k]$. Hence,

$$Y(z)=\sum_{n=-\infty}^{\infty} y[n]z^{-n}=\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} g[k]h[n-k]\right)z^{-n}=\sum_{k=-\infty}^{\infty} g[k]\sum_{n=-\infty}^{\infty} h[n-k]z^{-n}$$

$$=\sum_{k=-\infty}^{\infty} g[k]H(z)z^{-k}=H(z)G(z).$$

In this case also Y(z) will converge wherever both H(z) and G(z) converge. Thus ROC of Y(z) is $\mathcal{R}_g \cap \mathcal{R}_h$.

(h) y[n] = g[n]h[n]. Hence, $Y(z)=\sum_{n=-\infty}^{\infty} g[n]h[n]z^{-n}$. From Eq. (3.107),

$$g[n]=\frac{1}{2\pi j}\oint_C G(v)v^{n-1}dv.\ \text{Thus,}\ Y(z)=\sum_{n=-\infty}^{\infty}h[n]\left(\frac{1}{2\pi j}\oint_C G(v)v^{n-1}dv\right)z^{-n}$$

$$=\frac{1}{2\pi j}\oint_C G(v)\left(\sum_{n=-\infty}^{\infty}h[n]z^{-n}v^{n-1}\right)dv=\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}dv.$$

(i) $$\sum_{n=-\infty}^{\infty}g[n]h^*[n]=\frac{1}{2\pi j}\oint_C G(v)\sum_{n=-\infty}^{\infty}h^*[n]v^n\ v^{-1}dv=\frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv.$$

3.69 (a) Expnading in a power series we get $X_1(z)=\dfrac{1}{1+z^{-3}}=\sum_{n=0}^{\infty}(-1)^n z^{-3n}$, $\quad |z|>1.$

Thus, $x_1[n]=\begin{cases}(-1)^k, & \text{if } n=3k \text{ and } n\ge 0,\\ 0, & \text{elsewhere.}\end{cases}$

Using partial fraction, we get

$$X_1(z)=\frac{1}{1+z^{-3}}=\frac{\frac{1}{3}}{1+z^{-1}}+\frac{\frac{1}{3}}{1-(\frac{1}{2}+j\frac{\sqrt{3}}{2})z^{-1}}+\frac{\frac{1}{3}}{1-(\frac{1}{2}-j\frac{\sqrt{3}}{2})z^{-1}}.\ \text{Therefore,}$$

$$x_1[n]=\frac{1}{3}(-1)^n\mu[n]+\frac{1}{3}\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^n\mu[n]+\frac{1}{3}\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)^n\mu[n]$$

$$=\frac{1}{3}(-1)^n\mu[n]+\frac{1}{3}e^{jn\pi/3}\mu[n]+\frac{1}{3}e^{-jn\pi/3}\mu[n]=\frac{1}{3}(-1)^n\mu[n]+\frac{2}{3}\cos(n\pi/3)\mu[n].$$

Thus $x_1[n]=\begin{cases}(-1)^k, & \text{if } n=3k \text{ and } n\ge 0,\\ 0, & \text{elsewhere.}\end{cases}$

(b) Expnading in a power series we get $X_2(z)=\dfrac{1}{1-z^{-2}}=\sum_{n=0}^{\infty}z^{-2n}$, $\quad |z|>1.$

Thus , $x_2[n]=\begin{cases}1, & \text{if } n=2k \text{ and } n\ge 0,\\ 0, & \text{elsewhere.}\end{cases}$

Using partial fraction, we get $X_2(z)=\dfrac{1}{1-z^{-2}}=\dfrac{\frac{1}{2}}{1+z^{-1}}+\dfrac{\frac{1}{2}}{1-z^{-1}}$. Therefore,

$$x_2[n]=\frac{1}{2}\mu[n]+\frac{1}{2}(-1)^n\mu[n]$$

Thus, $x_2[n]=\begin{cases}1, & \text{if } n=2k \text{ and } n\ge 0,\\ 0, & \text{elsewhere.}\end{cases}$

3.70 (a) $X_1(z)=\log\left(1-\alpha z^{-1}\right)$, $\quad |z|>|\alpha|$. Expanding $\log\left(1-\alpha z^{-1}\right)$ in a power series we get

$$X_1(z)=-\alpha z^{-1}-\frac{\alpha^2z^{-2}}{2}-\frac{\alpha^3z^{-3}}{3}+.....=-\sum_{n=1}^{\infty}\frac{\alpha^n}{n}z^{-n}.$$

Thus, $x_1[n]=-\dfrac{\alpha^n}{n}\mu[n-1].$

(b) $X_2(z)=\log(1-\beta z)$, $\quad |z|<1/|\beta|$. Expanding $\log(1-\beta z)$ in a power series we get

$$X_2(z)=-\beta z-\frac{\beta^2z^2}{2}-\frac{\beta^3z^3}{3}-......=-\sum_{n=1}^{\infty}\frac{\beta^n}{n}z^n.$$

Thus $x_2[n] = \frac{\beta^{-n}}{n}\mu[-n-1]$.

3.71 $H(z) = \frac{(z+1)}{(z-0.5)(z+0.25)}$. Since the inverse z-transform h[n] of H(z) is a right-sided sequence, the ROC of H(z) is given by $|z| > 0.5$. By partial fraction expansion we get $H(z) = -8 + \frac{4}{1-0.5z^{-1}} + \frac{4}{1+0.25z^{-1}}$. Thus, $h[n] = -8\delta[n] + 4(0.5)^n\mu[n] + 4(-0.25)^n\mu[n]$.

3.72 From Eq. (3.155), for N = 3, we get

$\mathbf{D}_3 = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 1 & z_1^{-1} & z_1^{-2} \\ 1 & z_2^{-1} & z_2^{-2} \end{bmatrix}$. The determinant of $\mathbf{D}_3$ is given by

$$\det(\mathbf{D}_3) = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 1 & z_1^{-1} & z_1^{-2} \\ 1 & z_2^{-1} & z_2^{-2} \end{vmatrix} = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 0 & z_1^{-1}-z_0^{-1} & z_1^{-2}-z_0^{-2} \\ 0 & z_2^{-1}-z_0^{-1} & z_2^{-2}-z_0^{-2} \end{vmatrix} = \begin{vmatrix} z_1^{-1}-z_0^{-1} & z_1^{-2}-z_0^{-2} \\ z_2^{-1}-z_0^{-1} & z_2^{-2}-z_0^{-2} \end{vmatrix}$$

$$= \left(z_1^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_0^{-1}\right)\begin{vmatrix} 1 & z_1^{-1}+z_0^{-1} \\ 1 & z_2^{-1}+z_0^{-1} \end{vmatrix} = \left(z_1^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_1^{-1}\right) = \prod_{2\ge k>\ell\ge 0}\left(z_k^{-1}-z_\ell^{-1}\right).$$

From Eq. (3.155), for N = 4, we get

$\mathbf{D}_4 = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 1 & z_1^{-1} & z_1^{-2} & z_1^{-3} \\ 1 & z_2^{-1} & z_2^{-2} & z_2^{-3} \\ 1 & z_3^{-1} & z_3^{-2} & z_3^{-3} \end{bmatrix}$. The determinant of $\mathbf{D}_4$ is given by

$$\det(\mathbf{D}_4) = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 1 & z_1^{-1} & z_1^{-2} & z_1^{-3} \\ 1 & z_2^{-1} & z_2^{-2} & z_2^{-3} \\ 1 & z_3^{-1} & z_3^{-2} & z_3^{-3} \end{vmatrix} = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 0 & z_1^{-1}-z_0^{-1} & z_1^{-2}-z_0^{-2} & z_1^{-3}-z_0^{-3} \\ 0 & z_2^{-1}-z_0^{-1} & z_2^{-2}-z_0^{-2} & z_2^{-3}-z_0^{-3} \\ 0 & z_3^{-1}-z_0^{-1} & z_3^{-2}-z_0^{-2} & z_3^{-3}-z_0^{-3} \end{vmatrix}$$

$$= \begin{vmatrix} z_1^{-1}-z_0^{-1} & z_1^{-2}-z_0^{-2} & z_1^{-3}-z_0^{-3} \\ z_2^{-1}-z_0^{-1} & z_1^{-2}-z_0^{-2} & z_2^{-3}-z_0^{-3} \\ z_3^{-1}-z_0^{-1} & z_1^{-2}-z_0^{-2} & z_3^{-3}-z_0^{-3} \end{vmatrix} = \left(z_1^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_0^{-1}\right)\left(z_3^{-1}-z_0^{-1}\right)\begin{vmatrix} 1 & z_1^{-1}+z_0^{-1} & z_1^{-2}+z_1^{-1}z_0^{-1}+z_0^{-2} \\ 1 & z_2^{-1}+z_0^{-1} & z_2^{-2}+z_2^{-1}z_0^{-1}+z_0^{-2} \\ 1 & z_3^{-1}+z_0^{-1} & z_3^{-2}+z_3^{-1}z_0^{-1}+z_0^{-2} \end{vmatrix}$$

$$= \left(z_1^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_0^{-1}\right)\left(z_3^{-1}-z_0^{-1}\right)\begin{vmatrix} 1 & z_1^{-1}+z_0^{-1} & z_1^{-2}+z_1^{-1}z_0^{-1}+z_0^{-2} \\ 0 & z_2^{-1}-z_1^{-1} & (z_2^{-1}-z_1^{-1})(z_2^{-1}+z_1^{-2}+z_0^{-1}) \\ 0 & z_3^{-1}-z_1^{-1} & (z_3^{-1}-z_1^{-1})(z_3^{-1}+z_1^{-2}+z_0^{-1}) \end{vmatrix}$$

$$= \left(z_1^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_0^{-1}\right)\left(z_3^{-1}-z_0^{-1}\right)\begin{vmatrix} z_2^{-1}-z_1^{-1} & (z_2^{-1}-z_1^{-1})(z_2^{-1}+z_1^{-1}+z_0^{-1}) \\ z_3^{-1}-z_1^{-1} & (z_3^{-1}-z_1^{-1})(z_3^{-1}+z_1^{-1}+z_0^{-1}) \end{vmatrix}$$

$$= \left(z_1^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_0^{-1}\right)\left(z_3^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_1^{-1}\right)\left(z_3^{-1}-z_1^{-1}\right)\begin{vmatrix} 1 & z_2^{-1}+z_1^{-1}+z_0^{-1} \\ 1 & z_3^{-1}+z_1^{-1}+z_0^{-1} \end{vmatrix}$$

$$= \left(z_1^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_0^{-1}\right)\left(z_3^{-1}-z_0^{-1}\right)\left(z_2^{-1}-z_1^{-1}\right)\left(z_3^{-1}-z_1^{-1}\right)\left(z_3^{-1}-z_2^{-1}\right) = \prod_{3\ge k>\ell\ge 0}\left(z_k^{-1}-z_\ell^{-1}\right).$$

Hence, in the general case, $\det(\mathbf{D}_N) = \prod_{N-1\geq k>\ell\geq 0} \left(z_k^{-1} - z_\ell^{-1}\right)$. It follows from this expression that the determinant is non-zero, i.e. $\mathbf{D}_N$ is non-singular, if the sampling points z_k are distinct.

3.73 $X_{NDFT}[0] = 2 - 4 + 4 - 8 = -6$, $X_{NDFT}[1] = 2 + 2 + 1 + 1 = 6$,
$X_{NDFT}[2] = 2 + 4 + 4 + 8 = 18$, $X_{NDFT}[3] = 2 + 6 + 9 + 27 = 44$.
$I_0(z) = (1 - z^{-1})(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})$. Thus, $I_0\left(-\frac{1}{2}\right) = 10$,
$I_1(z) = (1 + \frac{1}{2}z^{-1})(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})$. Thus, $I_1(1) = 0.5$,
$I_2(z) = (1 + \frac{1}{2}z^{-1})(1 - z^{-1})(1 - \frac{1}{3}z^{-1})$. Thus, $I_2\left(\frac{1}{2}\right) = -0.6662$ and
$I_3(z) = (1 + \frac{1}{2}z^{-1})(1 - z^{-1})(1 - \frac{1}{2}z^{-1})$. Thus, $I_3\left(\frac{1}{3}\right) = 2.5$. Therefore,

$$X(z) = -\frac{6}{10}I_0(z) + \frac{6}{0.5}I_1(z) - \frac{18}{0.6662}I_2(z) + \frac{44}{2.5}I_3(z) = 2 + 2z^{-1} + z^{-2} + z^{-3}.$$

3.74 $c_0 = X_{NDFT}[0] = -6$, $c_1 = \dfrac{X_{NDFT}[1] - X_{NDFT}[0]}{1 - (-\frac{1}{2})(1)} = 8$, $c_2 = \dfrac{18 + 6 - 8\times\left(1 - (-\frac{1}{2})\times 2\right)}{\left(1 - (-\frac{1}{2})\times 2\right)(1-2)} = -4$,

$$c_3 = \frac{44 + 6 - 8\left(1 - (-\frac{1}{2})\times 3\right) + 4\left(1 - (-\frac{1}{2})\times 3\right)(1-3)}{\left(1 - (-\frac{1}{2})\times 3\right)(1-3)(1-\frac{3}{2})} = 4.$$

Hence $X(z) = -6 + 8(1 + \frac{1}{2}z^{-1}) - 4(1 + \frac{1}{2}z^{-1})(1 - z^{-1}) + 4(1 + \frac{1}{2}z^{-1})(1 - z^{-1})(1 - \frac{1}{2}z^{-1})$
$= 2 + 2z^{-1} + z^{-2} + z^{-3}$.

3.75 **(a)** $y[n] = \alpha g[n] + \beta h[n]$. Hence, $Y_{NDFT}[k] = \sum_{n=0}^{N-1} y[n]z_k^{-n} = \alpha\sum_{n=0}^{N-1} g[n]z_k^{-n} + \beta\sum_{n=0}^{N-1} h[n]z_k^{-n}$
$= \alpha G_{NDFT}[k] + \beta H_{NDFT}[k]$.

(b) $G_{NDFT}[k] = \sum_{n=0}^{N-1} g[n]z_k^{-n} \Rightarrow G_{NDFT}{}^*[k] = \sum_{n=0}^{N-1} g^*[n](z_k{}^*)^{-n}$. If $z_k{}^*$ also belongs to the samples then $\Rightarrow G_{NDFT}{}^*[k^*] = G^*(z_k{}^*) = \sum_{n=0}^{N-1} g^*[n]z_k{}^{-n}$.

(c) $\text{Re}\{g[n]\} = \frac{1}{2}(g[n] + g^*[n])$. Let $y[n] = \text{Re}\{g[n]\}$. Using Properties (a) and (b) we get
$Y(z_k) = \frac{1}{2}\left(G(z_k) + G^*(z_k{}^*)\right)$.

(d) Let $y[n] = j\,\text{Im}\{g[n]\} = \frac{1}{2}(g[n] - g^*[n])$. Again using Properties (a) and (b)
$Y(z_k) = \frac{1}{2}\left(G(z_k) - G^*(z_k{}^*)\right)$.

(e) Assume $g[n]$ is real, then $\text{Im}\{g[n]\} = 0$. From Property (d) this implies $G(z_k) = G^*(z_k{}^*)$, or equivalently $\text{Re}\{G(z_k)\} = \text{Re}\{G^*(z_k{}^*)\} = \text{Re}\{G(z_k{}^*)\}$. Also

$|G(z_k)| = |G^*(z_k^*)| = |G(z_k^*)|$ and $\arg\{G(z_k)\} = \arg\{G^*(z_k^*)\} = -\arg\{G(z_k^*)\}$.

3.76 $r[n] = \sum_{\ell=-\infty}^{\infty} x[\ell]x[\ell+n]$. Thus , $R(z) = \sum_{n=-\infty}^{\infty} r[n]z^{-n} = \sum_{n=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} x[\ell]x[\ell+n]$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] \sum_{n=-\infty}^{\infty} x[\ell+n]z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell]z^{\ell}\, X(z) = X(z)X(z^{-1}).$$

The ROC of X(z) is $R_{X^-} < |z| < R_{X^+}$ and of $X(z^{-1})$ is $1/R_{X^+} < |z| < 1/R_{X^-}$.
Therefore the ROC of $X(z)X(z^{-1})$ is $\max\{R_{X^-}, 1/R_{X^+}\} < |z| < \min\{R_{X^+}, 1/R_{X^-}\}$.

The total energy of $\{r[n]\}$ = $\sum_{n=-\infty}^{\infty} |r[n]|^2 = \sum_{n=-\infty}^{\infty} r[n]r^*[n]$. Using Parseval's relation, we obtain

$$\text{Energy} = \frac{1}{2\pi j}\oint_C R(v)R(1/v^*)v^{-1}dv = \frac{1}{2\pi j}\oint_C X(v)X(v^{-1})X(1/v^*)X(v^*)v^{-1}dv.$$

3.77 $x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & N \le n \le 2N-1. \end{cases}$ $y[n] = x_e[n] + x_e[2N-1-n]$. Therefore,

$$Y[k] = \sum_{n=0}^{2N-1} y[n]W_{2N}^{nk} = \sum_{n=0}^{N-1} x[n]W_{2N}^{nk} + \sum_{n=N}^{2N-1} x[2N-1-n]W_{2N}^{nk}$$

$$= \sum_{n=0}^{N-1} x[n]W_{2N}^{nk} + \sum_{n=0}^{N-1} x[n]W_{2N}^{(2N-1-n)k} = \sum_{n=0}^{N-1} x[n](W_{2N}^{nk} + W_{2N}^{-k}W_{2N}^{-nk}).$$

Thus, $C_x[k] = W_{2N}^{k/2}Y[k] = \sum_{n=0}^{N-1} x[n](W_{2N}^{k(n+1/2)} + W_{2N}^{-k(n+1/2)})$

$$= \sum_{n=0}^{N-1} 2x[n]\cos\left(\frac{\pi(2n+1)k}{2N}\right), \qquad 0 \le k \le N-1.$$

3.78 $Y[k] = \begin{cases} W_{2N}^{-k/2}C_x[k], & 0 \le k \le N-1, \\ 0, & k = N, \\ -W_{2N}^{-k/2}C_x[2N-k], & N+1 \le k \le 2N-1. \end{cases}$ Thus,

$$y[n] = \frac{1}{2N}\sum_{k=0}^{2N-1} Y[k]W_{2N}^{-nk} = \frac{1}{2N}\sum_{k=0}^{N-1} C_x[k]W_{2N}^{-(n+1/2)k} - \frac{1}{2N}\sum_{k=N+1}^{2N-1} C_x[2N-k]W_{2N}^{-(n+1/2)k}$$

$$= \frac{1}{2N}\sum_{k=0}^{N-1} C_x[k]W_{2N}^{-(n+1/2)k} - \frac{1}{2N}\sum_{k=1}^{N-1} C_x[k]W_{2N}^{-(n+1/2)(2N-k)}$$

$$= \frac{1}{2N}\sum_{k=0}^{N-1} C_x[k]W_{2N}^{-(n+1/2)k} + \frac{1}{2N}\sum_{k=1}^{N-1} C_x[k]W_{2N}^{(n+1/2)k}$$

$$= \frac{C_x[0]}{2N} + \frac{1}{N}\sum_{k=1}^{N-1} C_x[k]\cos\left(\frac{\pi k(2n+1)}{2N}\right),$$

Hence, $\frac{C_x[0]}{2N} + \frac{1}{N}\sum_{k=1}^{N-1} C_x[k]\cos\left(\frac{\pi k(2n+1)}{2N}\right)$, where $w[k] = \begin{cases} 1/2, & k = 0, \\ 1, & 1 \le k \le N-1. \end{cases}$

Moreover, $x[n] = \begin{cases} y[n], & 0 \le n \le N-1, \\ 0, & \text{elsewhere}, \end{cases} = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} w[k]C_x[k]\cos\left(\frac{\pi k(2n+1)}{2N}\right), & 0 \le n \le N-1, \\ 0, & \text{elsewhere}. \end{cases}$

(b) From Eq. (3.163), $x[n] = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} w[k]C_x[k]\cos\left(\frac{(2n+1)\pi k}{2N}\right), & 0 \le n \le N-1, \\ 0, & \text{elsewhere}. \end{cases}$ Hence,

$$\sum_{m=0}^{N-1} 2x[n]\cos\left(\frac{(2n+1)\pi m}{2N}\right) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} w[k]C_x[k]\cos\left(\frac{(2n+1)\pi k}{2N}\right)\cos\left(\frac{(2n+1)\pi m}{2N}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} w[k]C_x[k]\sum_{m=0}^{N-1}\cos\left(\frac{(2n+1)\pi k}{2N}\right)\cos\left(\frac{(2n+1)\pi m}{2N}\right). \qquad (10)$$

Now, $\sum_{m=0}^{N-1}\cos\left(\frac{(2n+1)\pi k}{2N}\right)\cos\left(\frac{(2n+1)\pi m}{2N}\right) = \begin{cases} N, & \text{if } k = m = 0, \\ N/2, & \text{if } k = m, \\ 0, & \text{elsewhere}. \end{cases}$

Thus, Eq. (10) reduces to

$$\sum_{m=0}^{N-1} 2x[n]\cos\left(\frac{(2n+1)\pi m}{2N}\right) = \begin{cases} \frac{1}{N}w[0]C_x[0]\cdot N, & m = 0, \\ \frac{1}{N}w[m]C_x[m]\cdot N, & 1 \le m \le N-1, \end{cases}$$

$$= \begin{cases} C_x[0], & m = 0, \\ C_x[m], & 1 \le m \le N-1, \end{cases} = C_x[m], \; 0 \le m \le N-1.$$

3.79 $y[n] = \alpha g[n] + \beta h[n]$. Thus,

$$C_y[k] = \sum_{n=0}^{N-1} y[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) = \sum_{n=0}^{N-1}(\alpha g[n] + \beta h[n])\cos\left(\frac{\pi k(2n+1)}{2N}\right) =$$

$$= \alpha\sum_{n=0}^{N-1} g[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) + \beta\sum_{n=0}^{N-1} h[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) = \alpha C_g[k] + \beta C_h[k].$$

3.80 $C_x[k] = \sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) \Rightarrow C_x^*[k] = \sum_{n=0}^{N-1} x^*[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right).$

Thus the DCT coefficients of $x^*[n]$ are given by $C_x^*[k]$.

3.81 Note that $\sum_{n=0}^{N-1}\cos\left(\frac{\pi k(2n+1)}{2N}\right)\cos\left(\frac{\pi m(2n+1)}{2N}\right) = \begin{cases} N, & \text{if } k = m = 0, \\ N/2, & \text{if } k = m \text{ and } k \ne 0, \\ 0, & \text{otherwise}. \end{cases}$

Now, $x[n]x^*[n] = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\alpha[k]\alpha[m]C_x^*[m]C_x[k]\cos\left(\frac{\pi(2n+1)k}{2N}\right)\cos\left(\frac{\pi(2n+1)m}{2N}\right)$

Thus, $\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\alpha[k]\alpha[m]C_x^*[m]C_x[k]\sum_{n=0}^{N-1}\cos\left(\frac{\pi(2n+1)k}{2N}\right)\cos\left(\frac{\pi(2n+1)m}{2N}\right)$

Now, using the orthogonality property mentioned above $\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{2N}\sum_{k=0}^{N-1}\alpha[k]|C_x[k]|^2$.

3.82 $X_{DHT}[k]=\sum_{n=0}^{N-1}x[n]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right)$. Now,

$$X_{DHT}[k]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi mk}{N}\right)\right)$$
$$=\sum_{n=0}^{N-1}x[n]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right)\left(\cos\left(\frac{2\pi mk}{N}\right)+\sin\left(\frac{2\pi mk}{N}\right)\right).$$ Therefore,

$$\sum_{k=0}^{N-1}X_{DHT}[k]\left(\cos\left(\frac{2\pi mk}{N}\right)+\sin\left(\frac{2\pi mk}{N}\right)\right)$$
$$=\sum_{n=0}^{N-1}x[n]\sum_{k=0}^{N-1}\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right)\left(\cos\left(\frac{2\pi mk}{N}\right)+\sin\left(\frac{2\pi mk}{N}\right)\right)$$

It can be shown that $\sum_{k=0}^{N-1}\cos\left(\frac{2\pi nk}{N}\right)\cos\left(\frac{2\pi mk}{N}\right)=\begin{cases}N, & \text{if } m=n=0,\\ N/2, & \text{if } m=n\neq 0,\\ N/2, & \text{if } m=N-n,\\ 0, & \text{elsewhere,}\end{cases}$

$$\sum_{k=0}^{N-1}\sin\left(\frac{2\pi nk}{N}\right)\sin\left(\frac{2\pi mk}{N}\right)=\begin{cases}N/2, & \text{if } m=n\neq 0,\\ -N/2, & \text{if } m=N-n,\\ 0, & \text{elsewhere,}\end{cases}$$ and

$$\sum_{k=0}^{N-1}\sin\left(\frac{2\pi nk}{N}\right)\cos\left(\frac{2\pi mk}{N}\right)=\sum_{k=0}^{N-1}\cos\left(\frac{2\pi nk}{N}\right)\sin\left(\frac{2\pi mk}{N}\right)=0.$$

Hence, $x[m]=\frac{1}{N}\sum_{k=0}^{N-1}X_{DHT}[k]\left(\cos\left(\frac{2\pi mk}{N}\right)+\sin\left(\frac{2\pi mk}{N}\right)\right)$.

3.83 **(a)** $y[n]=x(\langle n-n_0\rangle_N)=\begin{cases}x[n-n_0+N], & 0\le n\le n_0-1,\\ x[n-n_0], & n_0\le n\le N-1.\end{cases}$

$$Y_{DHT}[k]=\sum_{n=0}^{N-1}y[n]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right)=\sum_{n=0}^{n_0-1}x[n-n_0+N]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right)$$
$$+\sum_{n=n_0}^{N-1}x[n-n_0]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right).$$

Replacing $n-n_0+N$ by n in the first sum and $n-n_0$ by n in the second sum we get

$$Y_{DHT}[k]=\sum_{n=N-n_0}^{N-1}x[n]\left(\cos\left(\frac{2\pi(n+n_0)k}{N}\right)+\sin\left(\frac{2\pi(n+n_0)k}{N}\right)\right)$$
$$+\sum_{n=0}^{n_0-1}x[n]\left(\cos\left(\frac{2\pi(n+n_0)k}{N}\right)+\sin\left(\frac{2\pi(n+n_0)k}{N}\right)\right)$$
$$=\sum_{n=0}^{N-1}x[n]\left(\cos\left(\frac{2\pi(n+n_0)k}{N}\right)+\sin\left(\frac{2\pi(n+n_0)k}{N}\right)\right)$$
$$=\cos\left(\frac{2\pi n_0k}{N}\right)\sum_{n=0}^{N-1}x[n]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right)$$

$$+\sin\left(\frac{2\pi n_0 k}{N}\right)\sum_{n=0}^{n_0-1} x[n]\left(\cos\left(\frac{2\pi nk}{N}\right)-\sin\left(\frac{2\pi nk}{N}\right)\right)$$

$$= \cos\left(\frac{2\pi n_0 k}{N}\right)X_{DHT}[k]+\sin\left(\frac{2\pi n_0 k}{N}\right)X_{DHT}[-k].$$

(b) The N-point DHT of $x[<-n>_N]$ is $X_{DHT}[-k]$.

(c) $$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} X_{DHT}[k]X_{DHT}[\ell]\times$$

$$\left(\sum_{n=0}^{N-1}\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right)\left(\cos\left(\frac{2\pi n\ell}{N}\right)+\sin\left(\frac{2\pi n\ell}{N}\right)\right)\right).$$

Using the orthogonality property, the product is non-zero if $k = \ell$ and is equal to N. Thus $\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_{DHT}^2[k]$.

3.84 $\cos\left(\frac{2\pi nk}{N}\right)=\frac{1}{2}\left(W_N^{nk}+W_N^{-nk}\right)$, and $\sin\left(\frac{2\pi nk}{N}\right)=\frac{1}{2j}\left(W_N^{nk}-W_N^{-nk}\right)$.

$$X_{DHT}[k]=\sum_{n=0}^{N-1} x[n]\left(\frac{e^{j2\pi nk/N}+e^{-j2\pi nk/N}}{2}+\frac{e^{j2\pi nk/N}-e^{-j2\pi nk/N}}{2j}\right)$$

Therefore $X_{DHT}[k]=\frac{1}{2}(X[N-k]+X[k]-jX[N-k]+jX[k])$.

3.85 $y[n] = x[n]\,Ⓝ\,g[n]$. Thus, $Y_{DHT}[k]=\sum_{n=0}^{N-1} y[n]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right)$

$$= \sum_{r=0}^{N-1} x[r]\sum_{n=0}^{N-1} g[<n-r>_N]\left(\cos\left(\frac{2\pi nk}{N}\right)+\sin\left(\frac{2\pi nk}{N}\right)\right).$$

Fro m results of Problem 3.83

$$Y_{DHT}[k]=\sum_{\ell=0}^{N-1} x[\ell]\left(G_{DHT}[k]\cos\left(\frac{2\pi\ell k}{N}\right)+G_{DHT}[<-k>_N]\sin\left(\frac{2\pi\ell k}{N}\right)\right)$$

$$= G_{DHT}[k]\sum_{\ell=0}^{N-1} x[\ell]\cos\left(\frac{2\pi\ell k}{N}\right)+G_{DHT}[<-k>_N]\sum_{\ell=0}^{N-1} x[\ell]\sin\left(\frac{2\pi\ell k}{N}\right)$$

$$= \frac{1}{2}G_{DHT}[k]\left(X_{DHT}[k]+X_{DHT}[<-k>_N]\right)$$

$$+ \frac{1}{2}G_{DHT}[<-k>_N]\left(X_{DHT}[k]-X_{DHT}[<-k>_N]\right).$$

or $Y_{DHT}[k]=\frac{1}{2}X_{DHT}[k]\left(G_{DHT}[k]+G_{DHT}[<-k>_N]\right)$

$$+\frac{1}{2}X_{DHT}[<-k>_N]\left(G_{DHT}[k]-G_{DHT}[<-k>_N]\right).$$

3.86 (a) $\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, and $\mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$.

(b) From the structure of $\mathbf{H}_2$, $\mathbf{H}_4$ and $\mathbf{H}_8$ it can be seen that

$$\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix}, \text{ and } \mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4 & -\mathbf{H}_4 \end{bmatrix}.$$

(c) $\mathbf{X}_{HT} = \mathbf{H}_N \mathbf{x}$. Therefore $\mathbf{x} = \mathbf{H}_N^{-1}\mathbf{X}_{HT} = N\mathbf{H}_N^T \mathbf{X}_{HT} = N\mathbf{H}_N^* \mathbf{X}_{HT}$. Hence,

$$x[n] = \sum_{k=0}^{N-1} X_{HT}[k](-1)^{\sum_{i=0}^{\ell-1} b_i(n) b_i(k)}.$$

where $b_i(r)$ is the i^{th} bit in the binary representation of r.

3.87 $X(z_\ell) = \sum_{n=0}^{N-1} x[n] z_\ell^{-n} = \sum_{n=0}^{N-1} x[n] A^{-n} V^{\ell n} = \sum_{n=0}^{N-1} x[n] A^{-n} V^{\ell^2/2} V^{n^2/2} V^{-(\ell-n)^2/2}$

$= V^{\ell^2/2} \sum_{n=0}^{N-1} g[n] h[\ell - n]$, where $g[n] = x[n] A^{-n} V^{n^2/2}$ and $h[n] = V^{-n^2/2}$.

A block-diagram representation for the computation of $X(z_\ell)$ using the above scheme is thus precisely Figure P3.5.

3.88 $z_\ell = \alpha^\ell$. Hence, $A_0 V_0^{-\ell} e^{j\theta_0} e^{-j\ell\phi_0} = \alpha^\ell$. Since α is real, we have $A_0 = 1$, $V_0 = 1/\alpha$, $\theta_0 = 0$ and $\phi_0 = 0$.

3.89 (i) N = 3.

$X(z) = x[0] + x[1] z^{-1} + x[2] z^{-2}$, and $H(z) = h[0] + h[1] z^{-1} + h[2] z^{-2}$

$Y_L(z) = h[0]x[0] + (h[1]x[0] + x[1]h[0]) z^{-1} + (h[2]x[0] + h[1]x[1] + h[0]x[2]) z^{-2}$

$\qquad + (h[1]x[2] + h[2]x[1]) z^{-3} + h[2]x[2] z^{-4}$.

On the other hand,

$Y_c(z) = (h[0]x[0] + h[1]x[2] + h[2]x[0]) + (h[0]x[1] + h[1]x[0] + h[2]x[2]) z^{-1}$

$\qquad + (h[0]x[2] + h[1]x[1] + h[2]x[0]) z^{-2}$.

It is easy to see that in this case $Y_c(z) = Y_L(z) \bmod (1 - z^{-3})$.

(ii) N = 4.

$X(z) = x[0] + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3}$ and $H(z) = h[0] + h[1] z^{-1} + h[2] z^{-2} + h[3] z^{-3}$.

$Y_L(z) = h[0]x[0] + (h[1]x[0] + h[0]x[1]) z^{-1} + (h[0]x[2] + h[1]x[1] + h[2]x[0]) z^{-2}$

$\qquad + (h[0]x[3] + h[1]x[2] + h[2]x[1] + h[3]x[0]) z^{-3} + (h[1]x[3] + h[2]x[2] + h[3]x[1]) z^{-4}$

$$+\big(h[2]x[3]+h[3]x[2]\big)z^{-5}+h[3]x[3]z^{-6},$$

whereas, $Y_c(z)=\big(h[0]x[0]+h[1]x[3]+h[2]x[2]+h[3]x[1]\big)$

$$+\big(h[0]x[1]+h[1]x[0]+h[2]x[3]+h[3]x[2]\big)z^{-1}$$
$$+\big(h[0]x[2]+h[1]x[1]+h[2]x[0]+h[3]x[3]\big)z^{-2}$$
$$+\big(h[0]x[3]+h[1]x[2]+h[2]x[1]+h[3]x[0]\big)z^{-3}.$$

Again it can be seen that $Y_c(z)=Y_L(z)\text{mod}(1-z^{-4})$.

(ii) $N = 5$.

$X(z)=x[0]+x[1]z^{-1}+x[2]z^{-2}+x[3]z^{-3}+x[4]z^{-4}$ and $H(z)=h[0]+h[1]z^{-1}+h[2]z^{-2}+h[3]z^{-3}+h[4]z^{-4}$.

$$Y_L(z)=h[0]x[0]+\big(h[1]x[0]+h[0]x[1]\big)z^{-1}+\big(h[0]x[2]+h[1]x[1]+h[2]x[0]\big)z^{-2}$$
$$+\big(h[0]x[3]+h[1]x[2]+h[2]x[1]+h[3]x[0]\big)z^{-3}+\big(h[0]x[4]+h[1]x[3]+h[2]x[2]+h[3]x[1]+h[4]x[0]\big)z^{-4}$$
$$+\big(h[1]x[4]+h[2]x[3]+h[3]x[2]+h[4]x[1]\big)z^{-5}+\big(h[2]x[4]+h[3]x[3]+h[4]x[2]\big)z^{-6}$$
$$+\big(h[4]x[3]+h[3]x[4]\big)z^{-7}+h[4]x[4]z^{-8},$$

whereas, $Y_c(z)=\big(h[0]x[0]+h[4]x[1]+h[3]x[2]+h[2]x[3]+h[1]x[4]\big)$

$$+\big(h[0]x[1]+h[1]x[0]+h[2]x[4]+h[3]x[3]+h[4]x[2]\big)z^{-1}$$
$$+\big(h[0]x[2]+h[1]x[1]+h[2]x[0]+h[3]x[4]+h[4]x[3]\big)z^{-2}$$
$$+\big(h[0]x[3]+h[1]x[2]+h[2]x[1]+h[3]x[0]+h[4]x[4]\big)z^{-3}$$
$$+\big(h[0]x[4]+h[1]x[3]+h[2]x[2]+h[3]x[1]+h[4]x[0]\big)z^{-4}.$$

Again it can be seen that $Y_c(z)=Y_L(z)\text{mod}(1-z^{-5})$

3.90 The dc value of a sequece x[n] is given by $X(e^{j0})$. Now for a finite length sequence $X(e^{j\omega})=\sum_{n=0}^{N-1}x[n]e^{-j\omega n}$. Hence, $X(e^{j0})=\sum_{n=0}^{N-1}x[n]=X[0]$. Hence, for a finite length sequence, the dc value is given by the first coefficient of its DFT.

3.91 (a) $X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$. Let $\hat{X}(z)=\log(X(z)) \Rightarrow X(z)=e^{\hat{X}(z)}$. Thus, $X(e^{j\omega})=e^{\hat{X}(e^{j\omega})}$

(b) $\hat{x}[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\big(X(e^{j\omega})\big)e^{j\omega n}d\omega$. If x[n] is real, then $X(e^{j\omega})=X^*(e^{-j\omega})$. Therefore,

$\log\big(X(e^{j\omega})\big)=\log\big(X^*(e^{-j\omega})\big)$..

$$\hat{x}^*[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\big(X^*(e^{j\omega})\big)e^{-j\omega n}d\omega \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\big(X(e^{-j\omega})\big)e^{-j\omega n}d\omega$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\big(X(e^{j\omega})\big)e^{j\omega n}d\omega \;=\; \hat{x}[n].$$

(c) $\hat{x}_{ev}[n] = \dfrac{\hat{x}[n]+\hat{x}[-n]}{2} = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(X(e^{j\omega})\right)\left(\dfrac{e^{j\omega n}+e^{-j\omega n}}{2}\right)d\omega$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(X(e^{j\omega})\right)\cos(\omega n)\,d\omega,$$

and similarly, $\hat{x}_{od}[n] = \dfrac{\hat{x}[n]-\hat{x}[-n]}{2} = \dfrac{j}{2\pi}\int_{-\pi}^{\pi} \log\left(X(e^{j\omega})\right)\left(\dfrac{e^{j\omega n}-e^{-j\omega n}}{2j}\right)d\omega$

$$= \frac{j}{2\pi}\int_{-\pi}^{\pi} \log\left(X(e^{j\omega})\right)\sin(\omega n)\,d\omega.$$

3.92 $x[n] = a\,\delta[n] + b\,\delta[n-1]$ and $X(z) = a + b\,z^{-1}$. Also,

$\hat{X}(z) = \log(a+b\,z^{-1}) = \log(a) + \log(1+b/a z^{-1}) = \log(a) + \sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{(b/a)^n}{n}z^{-n}$. Therefore,

$$\hat{x}[n] = \begin{cases} \log(a), & \text{if } n=0, \\ (-1)^{n-1}\dfrac{(b/a)^n}{n}, & \text{for } n>0, \\ 0, & \text{elsewhere.} \end{cases}$$

3.93 (a) $\hat{X}(z) = \log(K) + \sum_{k=1}^{N_\alpha}\log\left(1-\alpha_k z^{-1}\right) + \sum_{k=1}^{N_\gamma}\log\left(1-\gamma_k z\right) - \sum_{k=1}^{N_\beta}\log\left(1-\beta_k z^{-1}\right) - \sum_{k=1}^{N_\delta}\log\left(1-\delta_k z\right).$

$$\hat{X}(z) = \log(K) - \sum_{k=1}^{N_\alpha}\sum_{n=1}^{\infty}\frac{\alpha_k^n}{n}z^{-n} - \sum_{k=1}^{N_\gamma}\sum_{n=1}^{\infty}\frac{\gamma_k^n}{n}z^{n} + \sum_{k=1}^{N_\beta}\sum_{n=1}^{\infty}\frac{\beta_k^n}{n}z^{-n} + \sum_{k=1}^{N_\delta}\sum_{n=1}^{\infty}\frac{\delta_k^n}{n}z^{n}.$$

Thus, $$\hat{x}[n] = \begin{cases} \log(K), & n=0, \\ \sum_{n=1}^{N_\beta}\dfrac{\beta_k^n}{n} - \sum_{n=1}^{N_\alpha}\dfrac{\alpha_k^n}{n}, & n>0, \\ \sum_{n=1}^{N_\gamma}\dfrac{\gamma_k^{-n}}{n} - \sum_{n=1}^{N_\delta}\dfrac{\delta_k^{-n}}{n}, & n<0. \end{cases}$$

(b) $|\hat{x}[n]| < N\dfrac{|r|^n}{|n|}$ as $n \to \infty$, where r is the maximum value of α_k, β_k, γ_k and δ_k for all values of k, and N is a constant. Thus $\hat{x}[n]$ is a decaying bounded sequence as $n \to \infty$.

(c) From Part (a) if $\alpha_k = \beta_k = 0$ then $\hat{x}[n] = 0$ for all $n > 0$, and is thus anti-causal.

(d) If $\gamma_k = \delta_k = 0$ then $\hat{x}[n] = 0$ for all $n < 0$ and is thus a causal sequence.

3.94 If X(z) has no poles and zeros on the unit circle then from Problem 3.95, $\gamma_k = \delta_k = 0$ and $\hat{x}[n] = 0$ for all $n < 0$.

$\hat{X}(z) = \log(X(z))$ therefore $\dfrac{d\hat{X}(z)}{dz} = \dfrac{1}{X(z)}\dfrac{dX(z)}{dz}$. Thus , $z\dfrac{dX(z)}{dz} = zX(z)\dfrac{d\hat{X}(z)}{dz}$.

Taking the inverse z-transform we get $n\,x[n] = \sum_{k=-\infty}^{\infty} k\,\hat{x}[k]x[n-k],\ n \neq 0.$

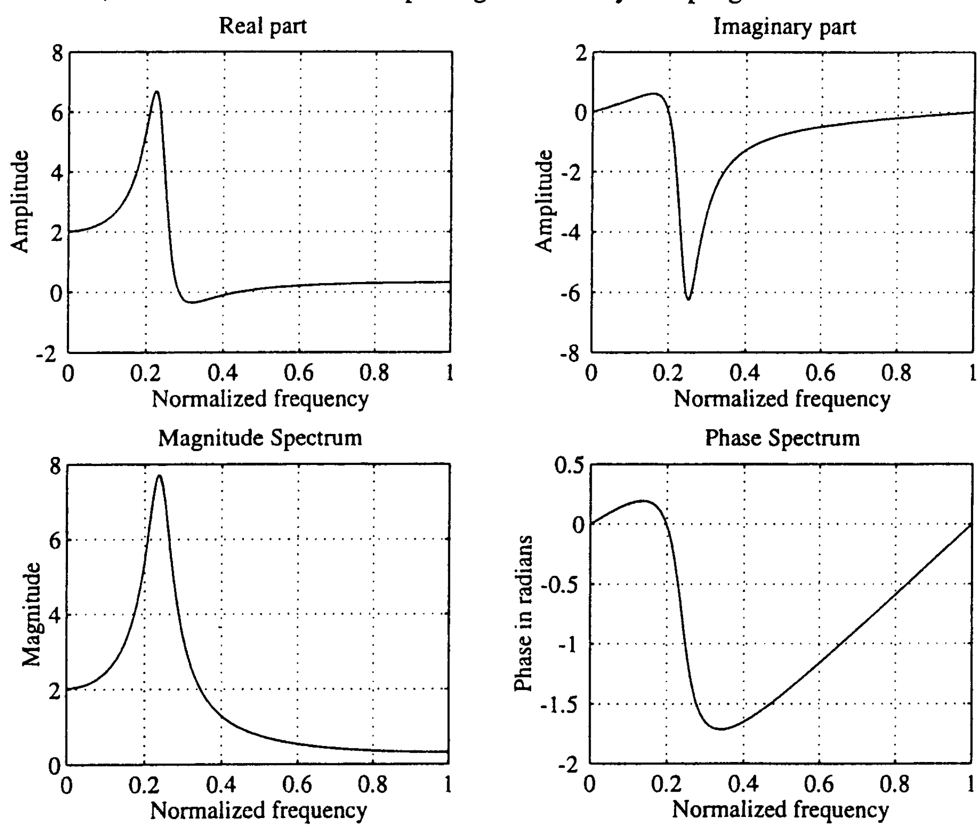
Since x[n] = 0 and $\hat{x}[n] = 0$ for n < 0, thus $x[n] = \sum_{k=0}^{n} \frac{k}{n}\hat{x}[k]x[n-k],\ n \neq 0.$

Or, $x[n] = \sum_{k=0}^{n-1} \frac{k}{n}\hat{x}[k]x[n-k] + \hat{x}[n]x[0]$. Hence, $\hat{x}[n] = \dfrac{x[n]}{x[0]} - \sum_{k=0}^{n-1}\left(\dfrac{k}{n}\right)\dfrac{\hat{x}[k]x[n-k]}{x[0]},\ n \neq 0.$

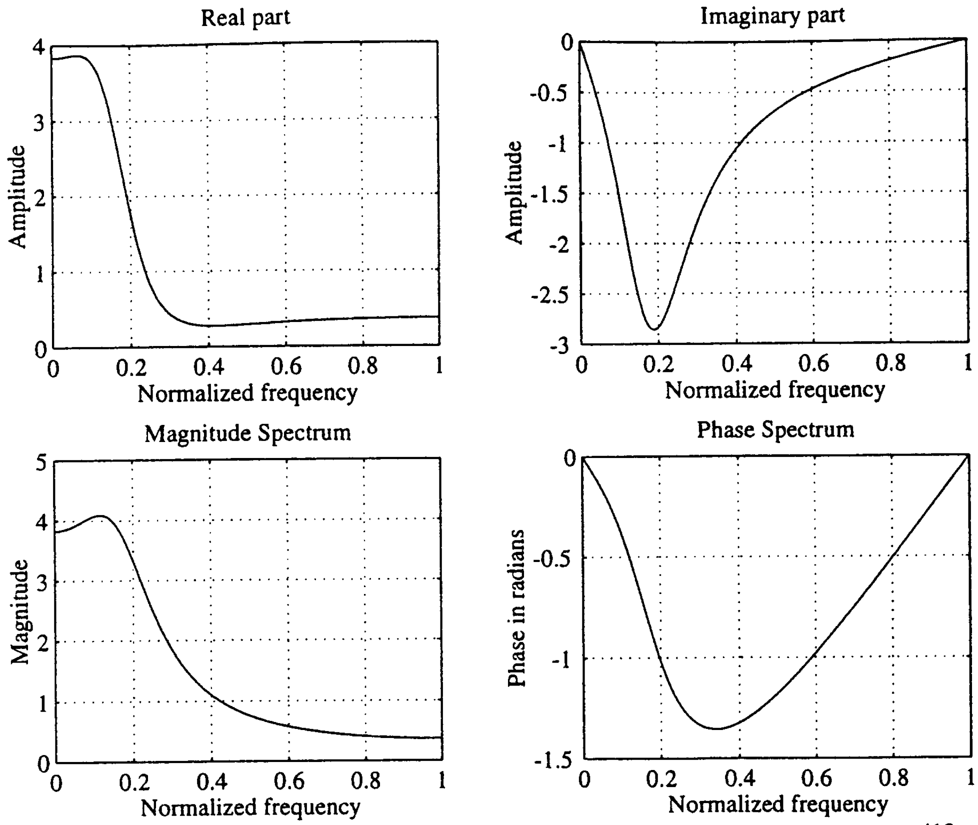
For n = 0, $\hat{x}[0] = \hat{X}(z)\big|_{z=\infty} = X(z)\big|_{z=\infty} = \log(x[0])$. Thus,

$$\hat{x}[n] = \begin{cases} 0, & n < 0, \\ \log(x[0]), & n = 0, \\ \dfrac{x[n]}{x[0]} - \sum_{k=0}^{n-1}\left(\dfrac{k}{n}\right)\dfrac{\hat{x}[k]x[n-k]}{x[0]}, & n > 0. \end{cases}$$

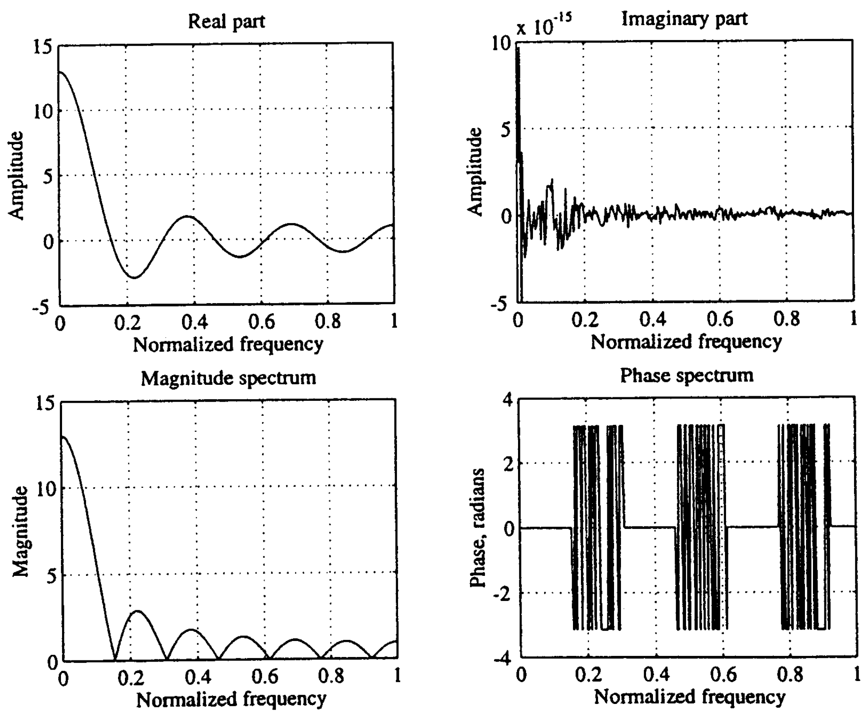
M3.1 **(a)** r = 0.9, θ = 0.75. The various plots generated by the program are shown below:



(b) r = 0.7, θ = 0.5. The various plots generated by the program are shown below:

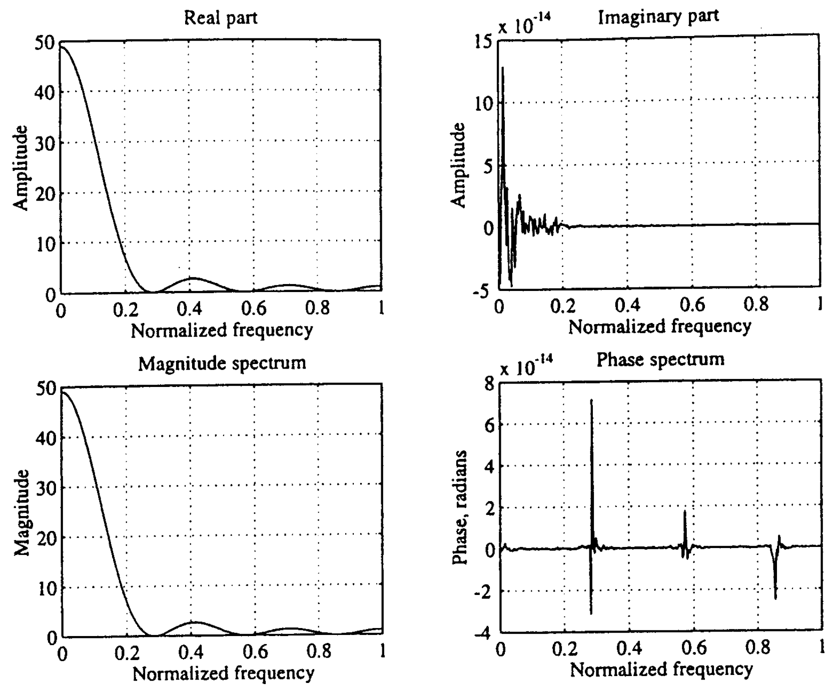


M3.2 (a) $Y_1(e^{j\omega}) = \dfrac{1-e^{-j(2N+1)\omega}}{e^{-jN\omega} - e^{-j(N+1)\omega}}$. For example, for N = 6, $Y_1(e^{j\omega}) = \dfrac{1-e^{-j13\omega}}{e^{-j6\omega} - e^{-j7\omega}}$.



(b) $Y_2(e^{j\omega}) = \dfrac{1-2e^{-j(N+1)\omega} + e^{-j2(N+1)\omega}}{e^{-jN\omega} - 2e^{-j(N+1)\omega} + e^{-j(N+2)\omega}}$. For example, for N = 6,

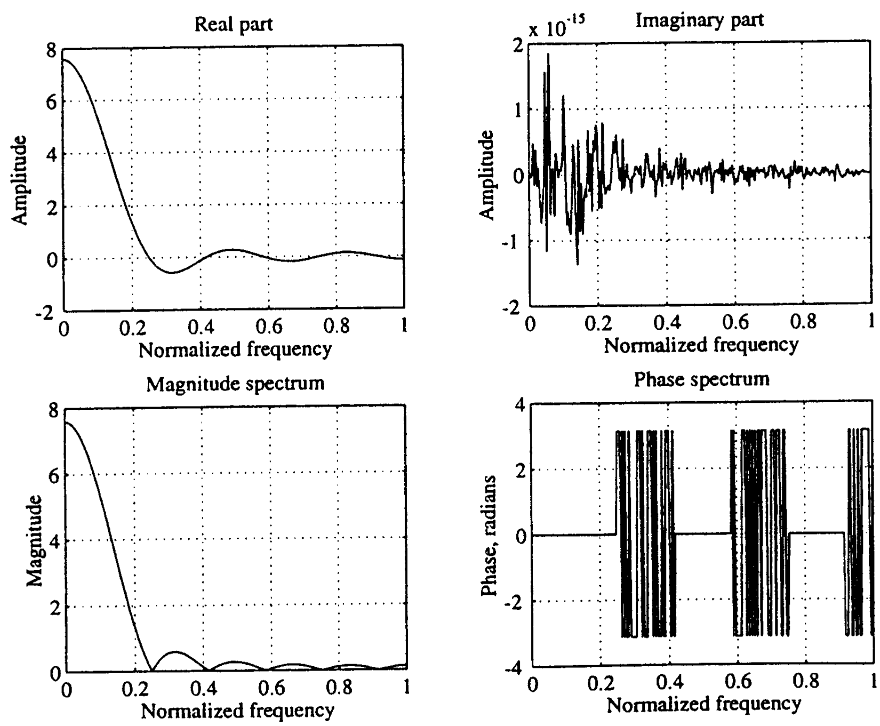
$$Y_2(e^{j\omega}) = \frac{1-2e^{-j7\omega} + e^{-j14\omega}}{e^{-j6\omega} - 2e^{-j7\omega} + e^{-j8\omega}}.$$



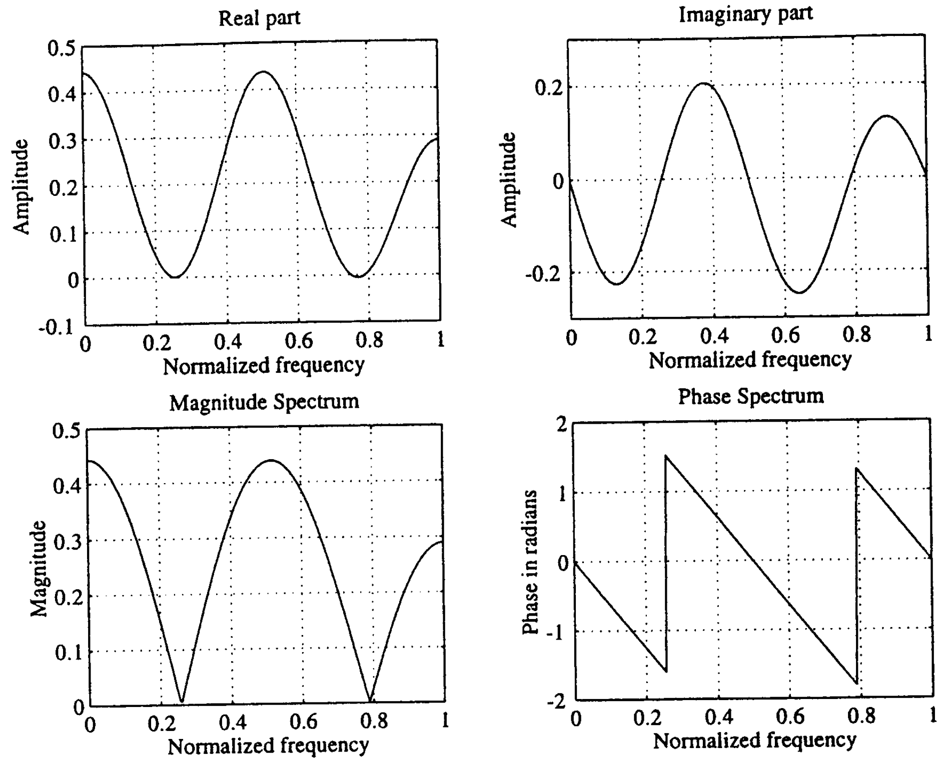
(c) $Y_3(e^{j\omega}) = \frac{1}{2}\sum_{n=-N}^{N}\left(e^{-j\pi n/2N} + e^{j\pi n/2N}\right)e^{-j\omega n} = \sum_{n=-N}^{N}\cos\left(\frac{\pi n}{2N}\right)e^{-j\omega n}$

$$= e^{jN\omega}\sum_{n=-N}^{N}\cos\left(\frac{\pi n}{2N}\right)e^{-j(N+n)\omega} = \frac{\sum_{n=-N}^{N}\cos\left(\frac{\pi n}{2N}\right)e^{-j(N+n)\omega}}{e^{-jN\omega}}.$$

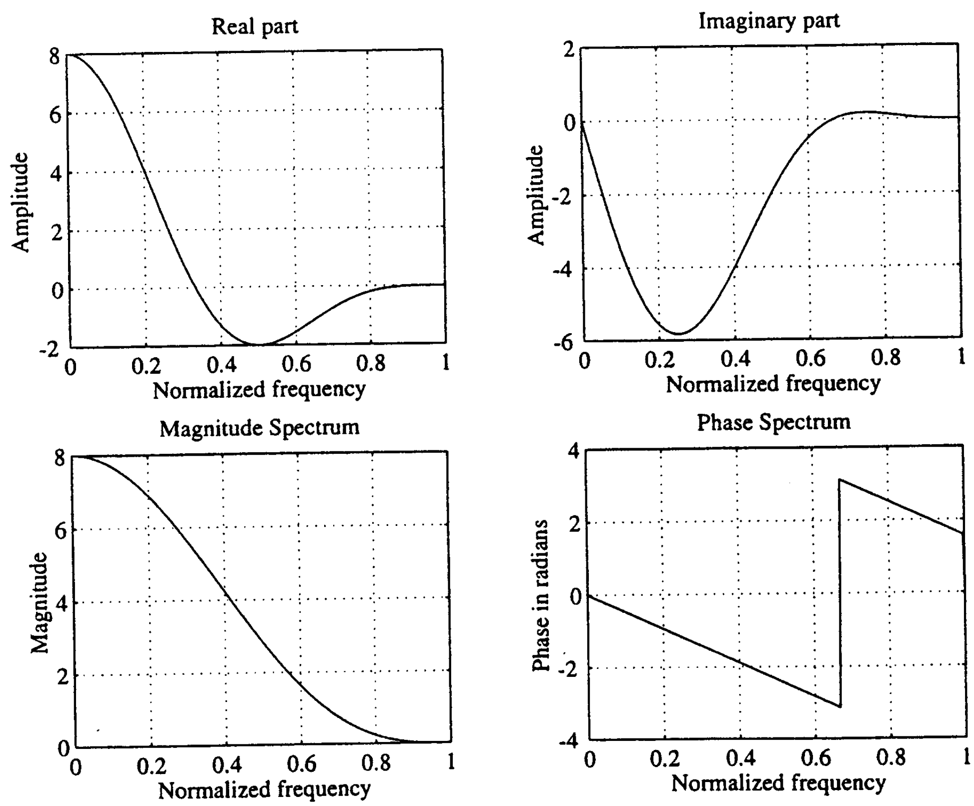
For example, for N = 6, $Y_3(e^{j\omega}) = \frac{\sum_{n=-6}^{6}\cos\left(\frac{\pi n}{2N}\right)e^{-j(6+n)\omega}}{e^{-j6\omega}}$.



M3.3 (a)



(b)



M3.4
```
N = input('The length of the sequence = ');
k = 0:N-1;
gamma = -0.5;
g = exp(gamma*k);
```

```
% g is an exponential sequence
h = sin(2*pi*k/(N/2));
% h is a sinusoidal sequence with period = N/2
[G,w] = freqz(g,1,512);
[H,w] = freqz(h,1,512);

% Property 1
alpha = 0.5;
beta = 0.25;
y = alpha*g+beta*h;
[Y,w] = freqz(y,1,512);
% Plot Y and alpha*G+beta*H to verify that they are equal

% Property 2
n0 = 12; % S equence shifted by 12 samples
y2 = [zeros([1,n0]) g];
[Y2,w] = freqz(y2,1,512);
G0 = exp(-j*w*n0).*G;
% Plot G0 and Y2 to verify they are equal

% Property 3
w0 = pi/2;  % the value of omega0 = pi/2
r=256;  %the value of omega0 in terms of number of samples
k = 0:N-1;
y3 = g.*exp(j*w0*k);
[Y3,w] = freqz(y3,1,512);
k = 0:511;
w = -w0+pi*k/512; % creating G(exp(w-w0))
G1 = freqz(g,1,w');
% Compare G1 and Y3

% Property 4
k = 0:N-1;
y4 = k.*g;
[Y4,w] = freqz(y4,1,512);
% To compute derivative we need sample at pi
y0 = ((-1).^k).*g;
G2 = [G(2:512)' sum(y0)]';
delG = (G2-G)*512/pi;
% Compare Y4, delG

% Property 5
y5 = conv(g,h);
[Y5,w] = freqz(y5,1,512);
% Compare Y5 and G.*H

% Property 6
y6 = g.*h;
[Y6,w] = freqz(y6,1,512,'whole');
[G0,w] = freqz(g,1,512,'whole');
[H0,w] = freqz(h,1,512,'whole');
% Evaluate the sample value at w = pi/2
% and verify with Y6 at pi/2
H1 = [fliplr(H0(1:129)') fliplr(H0(130:512)')]';
val = 1/(512)*sum(G0.*H1);
% Compare val with Y6(129) i.e sample at pi/2
% Can extend this to other points similarly

% Parsevals theorem
val1 = sum(g.*conj(h));
val2 = sum(G0.*conj(H0))/512;
```

```
% Comapre val1 with val2
```

M3.5

```
N = 8;   % Number of samples in sequence
gamma = 0.5;
k = 0:N-1;
x = exp(-j*gamma*k);
y = exp(-j*gamma*fliplr(k));
% r = x[-n] then y = r[n-(N-1)]
% so if X1(exp(jw)) is DTFT of x[-n], then
% X1(exp(jw)) = R(exp(jw)) = exp(jw(N-1))Y(exp(jw))
[Y,w] = freqz(y,1,512);
X1 = exp(j*w*(N-1)).*Y;
m = 0:511;
w = -pi*m/512;
X = freqz(x,1,w');
% Verify X = X1

% Property 2
k = 0:N-1;
y = exp(j*gamma*fliplr(k));
[Y,w] = freqz(y,1,512);
X1 = exp(j*w*(N-1)).*Y;
[X,w] = freqz(x,1,512);
% Verify X1 = conj(X)

% Property 3
y = real(x);
[Y3,w] = freqz(y,1,512);
m = 0:511;
w0 = -pi*m/512;
X1 = freqz(x,1,w0');
[X,w] = freqz(x,1,512);
% Verify Y3 = 0.5*(X+conj(X1))

% Property 4
y = j*imag(x);
[Y4,w] = freqz(y,1,512);
% Verify Y4 = 0.5*(X-conj(X1))

% Property 5
k = 0:N-1;
y = exp(-j*gamma*fliplr(k));
xcs = 0.5*[zeros([1,N-1]) x]+0.5*[conj(y) zeros([1,N-1])];
xacs = 0.5*[zeros([1,N-1]) x]-0.5*[conj(y) zeros([1,N-1])];
[Y5,w] = freqz(xcs,1,512);
[Y6,w] = freqz(xacs,1,512);
Y5 = Y5.*exp(j*w*(N-1));
Y6 = Y6.*exp(j*w*(N-1));
% Verify Y5 = real(X) and Y6 = j*imag(X)
```

M3.6

```
N = 8;
k = 0:N-1;
gamma = 0.5;
x = exp(gamma*k);
y = e xp(gamma*fliplr(k));
xev =0.5*([zeros([1,N-1]) x]+[y zeros([1,N-1])]);
xod = 0.5*([zeros([1,N-1]) x]-[y zeros([1,N-1])]);
[X,w] = freqz(x,1,512);
[Xev,w] = freqz(xev,1,512);
[Xod,w] = freqz(xod,1,512);
Xev = exp(j*w*(N-1)).*Xev;
```

```
Xod = exp(j*w*(N-1)).*Xod;
% Verify real(X)= Xev, and imag(X)= Xod

r = 0:511;
w0 = -pi*r/512;
X1 = freqz(x,1,w0');
% Verify X = conj(X1)
% real(X)= real(X1) and imag(X)= -imag(X1)
% abs(X)= abs(X1) and angle(X)= -angle(X1)
```

M3.7

```
N = input('The size of DFT to be computed =');
for k = 1:N
    for m = 1:N
     D(k,m) = exp(-j*2*pi*(k-1)*(m-1)/N);
    end
end
diff = inv(D)-1/N*conj(D);
% Verify diff is N*N matrix with all elements zero
```

M3.8

```
N = input('The value of N = ');
% Part a
k = -N:N;
y1 = ones([1,2*N+1]);
y2 = y1 - abs(k)/N;
y3 = cos(pi*k/(2*N));
[Y1,w] = freqz(y1,1,512);
[Y2,w] = freqz(y2,1,512);
[Y3,w] = freqz(y3,1,512);
Y1 = exp(j*w*N).*Y1;
Y2 = exp(j*w*N).*Y2;
Y3 = exp(j*w*N).*Y3;
```

M3.9

```
g = [1 -3 4 2 0 -2];
h = [3 0 1 -1 2 1];
x = [3+j*2 -2+j 0+j*3 1+j*4 -3+j*3];
v = [1-j*3 4+j*2 2-j*2 -3+j*5 2+j];
k = 0:5;
z = cos(pi*k/2);
y = 3.^k;
G = fft(g); H = fft(h); X = fft(x);
V = fft(v); Z = fft(z); Y = fft(y);
y1 = ifft(G.*H); y2 = ifft(X.*V); y3 = ifft(Z.*Y);
```

M3.10

```
N = 8; % N is length of the sequence(s)
gamma = 0.5;
k = 0:N-1;
g = exp(-gamma*k); h = cos(pi*k/N);
G = fft(g); H=fft(h);

% Property 1
alpha=0.5; beta=0.25;
x1 = alpha*g+beta*h;
X1 = fft(x1);
% Verify X1=alpha*G+beta*H

% Property 2
n0 = N/2; % n0 is the amount of shift
x2 = [g(n0+1:N) g(1:n0)];
X2 = fft(x2);
% Verify X2(k)= exp(-j*k*n0)G(k)
```

```
% Property 3
k0 = N/2;
x3 = exp(-j*2*pi*k0*k/N).*g;
X3 = fft(x3);
G3 = [G(k0+1:N) G(1:k0)];
% Verify X3=G3

% Property 4
x4 = G;
X4 = fft(G);
G4 = N*[g(1) g(8:-1:2)]; % This forms N*(g mod(-k))
% Verify X4 = G4;

% Property 5
% To calculate circular convolution between
% g and h use eqn (3.67)
h1 = [h(1) h(N:-1:2)];
T = toeplitz(h',h1);
x5 = T*g';
X5 = fft(x5');
% Verify X5 = G.*H

% Property 6
x6 = g.*h;
X6 = fft(x6);
H1 = [H(1) H(N:-1:2)];
T = toeplitz(H.', H1); % .' is the nonconjugate transpose
G6 = (1/N)*T*G.';
% Verify G6 = X6.'
```

M3.11

```
N = 8; % sequence length
gamma = 0.5;
k = 0:N-1;
x = exp(-gamma*k);
X = fft(x);

% Property 1
X1 = fft(conj(x));
G1 = conj([X(1) X(N:-1:2)]);
% Verify X1 = G1

% Property 2
x2 = conj([x(1) x(N:-1:2)]);
X2 = fft(x2);
% Verify X2 = conj(X)

% Property 3
x3 = real(x);
X3 = fft(x3);
G3 = 0.5*(X+conj([X(1) X(N:-1:2)]));
% Verify X3 = G3

% Property 4
x4 = j*imag(x);
X4 = fft(x4);
G4 = 0.5*(X-conj([X(1) X(N:-1:2)]));
% Verify X4 = G4

% Property 5
x5 = 0.5*(x+conj([x(1) x(N:-1:2)]));
X5 = fft(x5);
```

```
% Verify X5 = real(X)

% Property 6
x6 = 0.5*(x-conj([x(1) x(N:-1:2)]));
X6 = fft(x6);
% Verify X6 = j*imag(X)
```

M3.12
```
N = 8;
k = 0:N-1;
gamma = 0.5;
x = exp(-gamma*k);
X = fft(x);

% Property 1
xpe = 0.5*(x+[x(1) x(N:-1:2)]);
xpo = 0.5*(x-[x(1) x(N:-1:2)]);
Xpe = fft(xpe);
Xpo = fft(xpo);
% Verify Xpe = real(X) and Xpo = j*imag(X)

% Property 2
X2 = [X(1) X(N:-1:2)];
% Verify X = conj(X2);
% real(X) = real(X2) and imag(X) = -imag(X2)
% abs(X)= abs(X2) and angle(X) = -angle(X2)
```

M3.13
```
x = [2 1 1 0 3 2 0 3 4 6];
X = fft(x);
X(1)
X(6)
k = 0:9;
sum(exp(-j*4*pi*k/5).*X)
sum(X.*conj(X))
```

M3.14
```
X = [10 -5-j*4 3-j*2 1+j*3 2+j*5 6-j*2 12];
% Since x is real thus X = conj(X(N-k)).
% Use this fact to determine the remaining samples
X = [X conj(X(6:-1:2))];
x = ifft(X);
x(1)
x(7)
sum(x)
n = 0:11;
sum(exp(j*2*pi*n/3).*x)
sum(x.*conj(x))
```

M3.15
```
num = input('The numerator = ');
den = input('The denominator =');
p = roots(den);
z = roots(num);
zplane(z,p);
z = cplxpair(z);
p = cplxpair(p);
lenz = 0; lenp =0;
for k = 1:length(z)
     if (imag(z(k)) ~= 0) lenz = lenz+1;
     end
end
for k = 1:length(p)
      if (imag(p(k)) ~= 0) lenp = lenp+1;
      end
```

```
end
factor = num(1)/den(1)
for k = 1:2:lenz
    factor = [1 -2*real(z(k)) abs(z(k))^2]
end
for k = lenz+1:length(z)
    factor = [1 -z(k)]
end
pause
for k = 1:2:lenp
        factor = [1 -2*real(p(k)) abs(p(k))^2]
end
for k = lenp+1:length(p)
    factor = [1 -p(k)]
end
```

M3.16 **(a)** Aplying Program 3_9 to the rational function of Part (a) we obtain

```
Residues
0.1667    -0.6667

Poles
-4     -1

Constants
0.5000
```

Hence its partial-fraction expansion is given by $\frac{0.1667}{1+4z^{-1}}-\frac{0.6667}{1+z^{-1}}+\frac{1}{2}$. As a result,

$X_a(z)=z^2+3z+\frac{11}{2}+\frac{\frac{2}{3}}{1+4z^{-1}}-\frac{\frac{1}{6}}{1+z^{-1}}$. Hence,

$x_a[n]=\delta[n+2]+3\delta[n+1]+\frac{11}{2}\delta[n]+\frac{1}{6}(-4)^n\mu[n]-\frac{2}{3}(-1)^n\mu[n].$

(b) Aplying Program 3_9 to the rational function of Part (b) we obtain

```
Residues
4     -3

Poles
0.5000      0.2500

Constants
0
```

As a result its partial-fraction expansion is given by $X_b(z)=\frac{4}{1-0.5z^{-1}}-\frac{3}{1-0.25z^{-1}}$, $|z|>0.5$.

Hence, $x_b[n]=4(0.5)^n\mu[n]-3(0.25)^n\mu[n]$.

(c) Aplying Program 3_9 to the rational function of Part (c) we obtain

```
Residues
4     -8

Poles
-1.0000     -0.5000

Constants
4
```

As a result its partial-fraction expansion is given by $X_c(z) = 4 + \frac{4}{1+z^{-1}} - \frac{8}{1+0.5z^{-1}}$,

$0.5 < |z| < 1$. Hence, $x_c[n] = 4\delta[n] - 4(-1)^n \mu[-n-1] - 8(-0.5)^n \mu[n]$.

(d) Aplying Program 3_9 to the rational function of Part (c) we obtain

```
Residues
0.2231    -0.0918     0.7576     0.1111

Poles
-0.6000    0.5000     0.5000     0.2000

Constants
0
```

As a result its partial-fraction expansion is given by

$$X_d(z) = \frac{0.2231}{1+0.6z^{-1}} - \frac{0.0918}{1-0.5z^{-1}} + \frac{0.7576}{(1-0.5z^{-1})^2} + \frac{0.1111}{1-0.2z^{-1}}, \quad |z| > 0.6.$$ Hence,

$$x_d[n] = 0.2231(-0.6)^n \mu[n] - 0.0918(0.5)^n \mu[n] + 0.1111(0.2)^n \mu[n] + 0.7576(n+1)(0.5)^n \mu[n+1].$$

M3.17 (a) (i) Applying Program 3_10 we arrive at

```
Numerator polynomial coefficients
6.0000    -2.0000     0.3750

Denominator polynomial coefficients
1.0000    -0.7500     0.1250
```

Hence, $X_1(z) = \frac{6 - 2z^{-1} + 0.375z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$.

(ii) Applying Program 3_10 we arrive at

```
Numerator polynomial coefficients
1.5000     0.2500     1.8750     0.3125

Denominator polynomial coefficients
1.0000     0.5000     0.2500     0.1250
```

Hence, $X_2(z) = \frac{1.5 + 0.25z^{-1} + 1.875z^{-2} + 0.3125z^{-3}}{1 + 0.5z^{-1} + 0.25z^{-2} + 0.125z^{-3}}$.

(iii) Applying Program 3_10 we arrive at

```
Numerator polynomial coefficients
3.0000     2.4000    -0.2600    -0.7200          0

Denominator polynomial coefficients
1.0000     0.8000     1.0600     0.7200     0.1440
```

Hence, $X_3(z) = \frac{3 + 2.4z^{-1} - 0.26z^{-2} - 0.72z^{-3}}{1 + 0.8z^{-1} + 1.06z^{-2} + 0.72z^{-3} + 0.144z^{-4}}$.

(iv) Applying Program 3_10 we arrive at

```
Numerator polynomial coefficients
6.0000    6.7667    2.4000    0.2667

Denominator polynomial coefficients
1.0000    1.2333    0.5000    0.0667
```

Hence, $X_4(z) = \dfrac{6 + 6.7667 z^{-1} + 2.4 z^{-2} + 0.2667 z^{-3}}{1 + 1.2333 z^{-1} + 0.5 z^{-2} + 0.0667 z^{-3}}$.

(b) (i) $X_1(z) = 3 + \dfrac{14}{2 - z^{-1}} - \dfrac{16}{4 - z^{-1}} = 3 + \dfrac{7}{1 - 0.5 z^{-1}} - \dfrac{4}{1 - 0.25 z^{-1}}$.

Hence, $x_1[n] = 3\delta[n] + 7(0.5)^n \mu[n] - 4(0.25)^n \mu[n]$.

(ii) $X_2(z) = 2.5 + \dfrac{3}{1 + 0.5 z^{-1}} - \dfrac{4 - z^{-1}}{1 + 0.25 z^{-2}} = 2.5 + \dfrac{3}{1 + 0.5 z^{-1}} - \dfrac{4}{1 + 0.25 z^{-2}} + \dfrac{z^{-1}}{1 + 0.25 z^{-2}}$.

Hence, $x_2[n] = 2.5\delta[n] + 3(-0.5)^n \mu[n] - 4(0.5)^n \cos\left(\frac{\pi n}{2}\right)\mu[n] + 2(0.5)^n \sin\left(\frac{\pi n}{2}\right)\mu[n]$.

(iii) $X_3(z) = \dfrac{25}{(5 + 2z^{-1})^2} - \dfrac{10}{5 + 2z^{-1}} + \dfrac{4}{1 + 0.9\, z^{-1}} = \dfrac{1}{(1 + 0.4 z^{-1})^2} - \dfrac{1}{1 + 0.4 z^{-1}} + \dfrac{4}{1 + 0.9\, z^{-1}}$.

Hence, $x_3[n] = (-0.4)^n (n+1)\mu[n+1] - 2(-0.4)^n \mu[n] + 4(\sqrt{0.9})^n \cos\left(\frac{\pi n}{2}\right)\mu[n]$.

(iv) $X_4(z) = 4 + \dfrac{10}{5 + 2z^{-1}} + \dfrac{z^{-1}}{6 + 5z^{-1} + z^{-2}} = 4 + \dfrac{2}{1 + 0.4 z^{-1}} + \dfrac{-1}{1 + 0.5 z^{-1}} + \dfrac{1}{1 + \frac{1}{3} z^{-1}}$.

Hence, $x_4[n] = 4\,\delta[n] + 2(-0.4)^n \mu[n] - (-0.5)^n \mu[n] + \left(-\frac{1}{3}\right)^n \mu[n]$

M3.18 (i) Coefficients of the power series expansion

```
Columns 1 through 7
6.0000    2.5000    1.5000    0.8125    0.4219    0.2148
0.1084

Columns 8 through 14
0.0544    0.0273    0.0137    0.0068    0.0034    0.0017
0.0009

Columns 15 through 21
0.0004    0.0002    0.0001    0.0001    0.0000    0.0000
0.0000

Columns 22 through 28
0.0000    0.0000    0.0000    0.0000    0.0000    Q.0000
0.0000

Columns 29 through 30
0.0000    0.0000
```

(ii) Coefficients of the power series expansion

```
Columns 1 through 7
1.5000   -0.5000    1.7500   -0.6250   -0.0625   -0.0312
0.1094

Columns 8 through 14
-0.0391   -0.0039   -0.0020    0.0068   -0.0024   -0.0002
-0.0001
```

```
Columns 15 through 21
0.0004   -0.0002   -0.0000   -0.0000    0.0000   -0.0000
-0.0000

Columns 22 through 28
-0.0000    0.0000   -0.0000   -0.0000   -0.0000    0.0000
-0.0000

Columns 29 through 30
-0.0000   -0.0000
```

(iii) Coefficients of the power series expansion

```
Columns 1 through 7
3.0000   -0.0000   -3.4400   -0.1280    3.3168   -0.0410
-2.8955

Columns 8 through 14
-0.0098    2.6290   -0.0021   -2.3610   -0.0004    2.1259
-0.0001

Columns 15 through 21
-1.9132   -0.0000    1.7219   -0.0000   -1.5497   -0.0000
1.3947

Columns 22 through 28
-0.0000   -1.2552   -0.0000    1.1297   -0.0000   -1.0167
-0.0000

Columns 29 through 30
0.9151   -0.0000
```

(iv) Coefficients of the power series expansion

```
Columns 1 through 7
6.0000   -0.6331    0.1808   -0.0399    0.0011    0.0066
-0.0060

Columns 8 through 14
0.0040   -0.0024    0.0014   -0.0007    0.0004   -0.0002
0.0001

Columns 15 through 21
-0.0001    0.0000   -0.0000    0.0000   -0.0000    0.0000
-0.0000

Columns 22 through 28
0.0000   -0.0000    0.0000   -0.0000    0.0000   -0.0000
0.0000

Columns 29 through 30
-0.0000    0.0000
```

M3.19

```
% as an example try a sequence x = 0:24;
% calculate the actual uniform dft
% and then use those uniform samples
% with this ndft program to get the
% the original sequence bac
% [X,w] = freqz(x,1,25,'whole');
% use freq = X and  points = exp(j*w)

freq = input('The sample values = ');
```

```
points = input('The frequencies at which samples are taken =
');
L = 1;
len = length(points);
val = zeros(size(1,len));
L = poly(points);
for k = 1:len
    if(freq(k) ~= 0)
        xx = [1 -points(k)];
        [yy, rr] = deconv(L,xx);
        F(k,:) = yy;
        down = polyval(yy,points(k))*(points(k)^(-len+1));
        F(k,:) = freq(k)/down*yy;
        val = val+F(k,:);
    end
end
coeff = val;
```

M3.20

```
x = 0:24;
[X,w] = freqz(x,1,25,'whole');
val = X;
freq = exp(j*w);
len = length(freq);
for k = 1:len
    L = 1;
    for m = 1:k-1
    factor = [1 -freq(m)];
    L = conv(L,factor);
    end
    F(k,:) = [L zeros([1,len-length(L)])];
end
for k = 1:len
    if(k==1) coeff(k) = val(k);
    else
        var = val(k);
        for m = 1:k-1
         var = var-coeff(m)*polyval(F(m,:),freq(k))*(freq(k)^(-
len+1));
        end
         var = var/(polyval(F(k,:),freq(k))*(freq(k)^(1-len)));
        coeff(k) = var;
    end
end
c = zeros([1,len]);
for k = 1:len
    c = c+coeff(k)*F(k,:);
end
```

Chapter 4

4.1 If $u[n] = z^n$ is the input to the LTI discrete-time system, then its output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = z^n H(z),$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$. Hence $u[n] = z^n$ is an eigenfunction of the LTI discrete-time system. If $v[n] = z^n \mu[n]$ is the input to the system, then its output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]v[n-k] = z^n \sum_{k=-\infty}^{\infty} h[k]\mu[n-k]z^{-k} = z^n \sum_{k=-\infty}^{n} h[k]z^{-k}.$$

Since in this case the summation depends upon n, $v[n] = z^n \mu[n]$ is not an eigenfunction of an LTI discrete-time system.

4.2 $h[n] = \delta[n] + \alpha\delta[n-R]$. Taking the DTFT of both sides we get $H(e^{j\omega}) = 1 + \alpha e^{-j\omega R}$. Let $\alpha = |\alpha| e^{j\phi}$, then the maximum value of $|H(e^{j\omega})|$ is $1 + |\alpha|$, and the minimum value of $|H(e^{j\omega})|$ is $1 - |\alpha|$.

4.3 $G(e^{j\omega}) = \left(H(e^{j\omega})\right)^3 = (1 + \alpha e^{-j\omega R})^3$.

4.4 $G(e^{j\omega}) = \sum_{n=0}^{M-1} \alpha^n e^{-j\omega n} = \dfrac{1 - \alpha^M e^{-j\omega M}}{1 - \alpha e^{-j\omega}}$. Note that $G(e^{j\omega}) = H(e^{j\omega})$, for $\alpha = 1$. In order to have the dc value of the magnitude response equal to unity the impulse response should be multiplied by a factor of K, where $K = |(1-\alpha)/(1-\alpha^M)|$.

4.5 The group delay $\tau(\omega)$ of an LTI discrete-time system with a frequency response $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\phi(\omega)}$, is given by $\tau(\omega) = -\dfrac{d(\phi(\omega))}{d\omega}$. Now,

$$\frac{dH(e^{j\omega})}{d\omega} = e^{j\phi(\omega)} \frac{d|H(e^{j\omega})|}{d\omega} + j|H(e^{j\omega})| e^{j\phi(\omega)} \frac{d\phi(\omega)}{d\omega}.$$

Hence, $-j|H(e^{j\omega})| e^{j\phi(\omega)} \dfrac{d\phi(\omega)}{d\omega} = e^{j\phi(\omega)} \dfrac{d|H(e^{j\omega})|}{d\omega} - \dfrac{dH(e^{j\omega})}{d\omega}$. Equivalently,

$$-\frac{d\phi(\omega)}{d\omega} = \frac{e^{j\phi(\omega)}}{j|H(e^{j\omega})| e^{j\phi(\omega)}} \frac{d|H(e^{j\omega})|}{d\omega} - \frac{\frac{dH(e^{j\omega})}{d\omega}}{jH(e^{j\omega})} = \frac{1}{j|H(e^{j\omega})|} \frac{d|H(e^{j\omega})|}{d\omega} + j\frac{\frac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})}.$$

The first term on the right hand side is purely imaginary. Hence,

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega} = \mathrm{Re}\left(\frac{j\dfrac{d(H(e^{j\omega}))}{d\omega}}{H(e^{j\omega})} \right).$$

4.6 Now, $H(e^{j\omega}) = \left((a_1 + a_3)\cos\omega + a_2\right) + j(a_1 - a_3)\sin\omega$. Hence, the frequency response will have zero phase for $a_1 = a_3$.

4.7 Now, $H(e^{j\omega}) = \left((a_1 + a_5)\cos 2\omega + (a_2 + a_4)\cos\omega + a_3\right) + j\left((a_1 - a_5)\sin\omega + (a_2 - a_4)\sin\omega\right)$. Hence, the frequency response will have zero phase for $a_1 = a_5$, and $a_2 = a_4$.

4.8 The frequency response of the LTI discrete-time system is given by

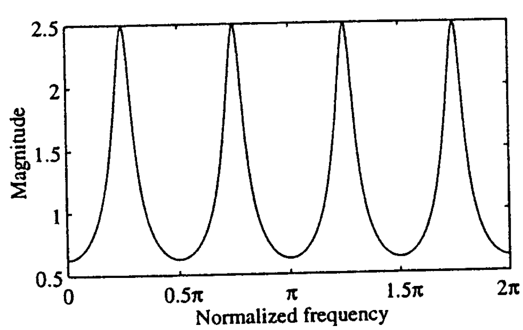
$$\begin{aligned} H(e^{j\omega}) &= a_1 e^{j\omega k} + a_2 e^{j\omega(k-1)} + a_3 e^{j\omega(k-2)} + a_2 e^{j\omega(k-3)} + a_1 e^{j\omega(k-4)} \\ &= e^{j\omega(k-2)}\left(a_1 e^{j2\omega} + a_2 e^{j\omega} + a_3 + a_2 e^{-j\omega} + a_1 e^{-j2\omega}\right) \\ &= e^{j\omega(k-2)}\left(2a_1\cos(2\omega) + 2a_2\cos(\omega) + a_3\right) \end{aligned}$$

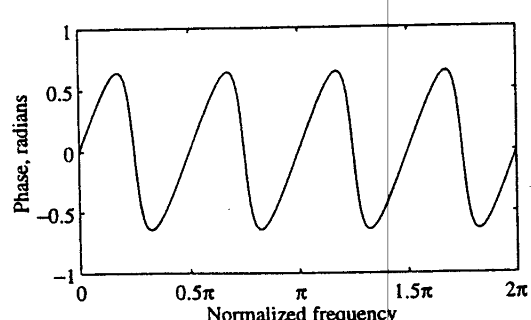
Hence for $H(e^{j\omega})$ will be real for k = 2.

4.9 $Y(e^{j\omega}) = \left|X(e^{j\omega})\right|^{\alpha} e^{j\arg X(e^{j\omega})}$. Hence $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \left|X(e^{j\omega})\right|^{(\alpha-1)}$. Since $H(e^{j\omega})$ is real, it has zero-phase.

4.10 $y[n] = x[n] - \alpha y[n-R]$. $Y(e^{j\omega}) = X(e^{j\omega}) - \alpha e^{-j\omega R} Y(e^{j\omega})$. Hence, $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{1}{1 + \alpha e^{-j\omega R}}$. Maximum value of $\left|H(e^{j\omega})\right|$ is $1/(1-|\alpha|)$, and the minimum value is $1/(1+|\alpha|)$. There are R peaks and dips in the range $0 \le \omega < 2\pi$. The locations of peaks and dips are given by $1 + \alpha e^{-j\omega R} = 1 \pm |\alpha|$, or $e^{-j\omega R} = \pm\dfrac{|\alpha|}{\alpha}$. Let $\alpha = |\alpha| e^{j\phi}$. Then the locations of peaks are given by $\omega = \dfrac{\pi + \phi}{R} + \dfrac{2k\pi}{R}$ and the locations of minima are given by $\omega = \dfrac{2k\pi + \phi}{R}$ where $k = 0, 1, 2, \ldots, R-1$.

Plots of the magnitude and phase responses of $H(e^{j\omega})$ for $\alpha = 0.6$ are shown below:

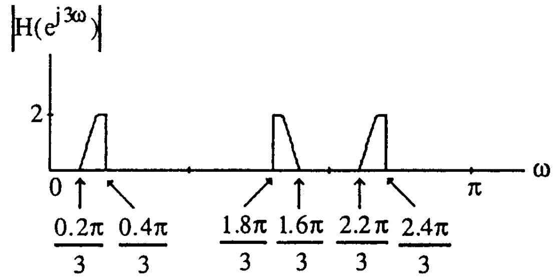




As can be seen from the plot of the magnitude response, the peaks are at $\omega = (2k+1)\pi/4$ and the dips are at $\omega = k\pi/2, k = 0, 1, 2, 3$.

4.11 From the difference equation we get $H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega}) = \left(\gamma + \delta e^{-j\omega}\right)/\left(\alpha + \beta e^{-j\omega}\right)$. The frequency response will be constant if $\gamma = k\alpha$ and $\delta = k\beta$.

4.12



4.13 $H(z) = \sum_{n=-\infty}^{\infty} [0.5]^n z^{-n} \mu[n] = \sum_{n=0}^{\infty} [0.5z^{-1}]^n = \frac{1}{1-0.5z^{-1}}$. The output y[n] of the system to an input $x[n] = \cos(\frac{\pi n}{4})\mu[n]$ is therefore given by

$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = [0.5]^n \sum_{k=0}^{n} [0.5]^{-k} \cos(\frac{\pi k}{4})$. Let $\omega = \pi/4$. Then ,

$$y[n] = (0.5)^{n+1}\left\{\frac{1-(0.5e^{-j\omega})^{n+1}}{1-0.5e^{-j\omega}} + \frac{1-(0.5e^{j\omega})^{n+1}}{1-0.5e^{j\omega}}\right\}$$

$$= (0.5)^{n+1}\left\{\frac{1-2^{n+1}e^{j\omega(n+1)}}{1-2e^{j\omega}} + \frac{1-2^{n+1}e^{-j\omega(n+1)}}{1-2e^{-j\omega}}\right\}$$

$$= (0.5)^{n+1}\left\{\frac{1-2e^{-j\omega}-2^{n+1}e^{j\omega(n+1)}+2^{n+2}e^{j\omega n}+1-2e^{j\omega}-2^{n+1}e^{-j\omega(n+1)}+2^{n+2}e^{-j\omega(n+1)}}{1-2(e^{j\omega}+e^{-j\omega})+4}\right\}$$

$$= (0.5)^{n+1}\left\{\frac{2-4\cos\omega-2^{n+2}\cos(\omega(n+1))+2^{n+3}\cos\omega n}{5-4\cos\omega}\right\}$$

$$= \frac{(0.5)^n-(0.5)^{n+1}\cos\omega-2\cos(\omega(n+1))+4\cos(\omega n)}{5-4\cos\omega}.$$

4.14 $H(e^{j\omega}) = h[0]\left(1+e^{-j2\omega}\right) + h[1]e^{-j\omega} = e^{-j\omega}(2h[0]\cos\omega + h[1])$. Therefore, we require, $H(e^{j0.1}) = 2h[0]\cos(0.1) + h[1] = 1$, and $H(e^{j0.4}) = 2h[0]\cos(0.4) + h[1] = 0$. Solving these two equations we get h[0] = 6.7619 and h[1] = –12.4563.

4.15 (a) $H(e^{j\omega}) = h[0]\left(1+e^{-j4\omega}\right) + h[1]\left(1+e^{-j3\omega}\right) + e^{-j2\omega} = e^{-j2\omega}(2h[0]\cos 2\omega + 2h[1]\cos\omega + h[2])$.

We require $\left|H(e^{j0})\right| = 2h[0] + 2h[1] + h[2] = 1,$

$\left|H(e^{j0.5\pi})\right| = 2h[0]\cos(\pi) + 2h[1]\cos(0.5\pi) + h[2] = -2h[0] + h[2] = 0.5,$

$\left|H(e^{j\pi})\right| = 2h[0]\cos(2\pi) + 2h[1]\cos(\pi) + h[2] = 2h[0] - 2h[1] + h[2] = 0.$

Solving these three equations we get h[0] = 0, h[1] = 0.25, and h[2] = 0.5.

Note: The 4-point DFT H[k] of {h[n]} is given by H[k] = {1 −0.5 0 −0.5}. A 4-point IDFT of H[k] yields h[0] = {0 0.25 0.5 0.25}.

(b) The frequency response of the filter is given by $H(e^{j\omega}) = 0.5(1+\cos\omega)e^{-j2\omega}$.

4.16 (a) As in the previous problem, the conditions on the magnitude response results in the following three eqautions $2h[0]+2h[1]+h[2]=0$, $-2h[0]+h[2]=1$, and $2h[0]-2h[1]+h[2]=0$, whose solution yields h[0] = −0.25, h[1] = 0, h[2] = 0.5.

(b) The frequency response of the filter is given by $H(e^{j\omega}) = 0.5(1-\cos 2\omega)e^{-j2\omega}$.

4.17 (a) $\left|H(e^{j\omega})\right| = 2h[0]\cos 2\omega + 2h[1]\cos\omega + h[2]$. Thus, $\left|H(e^{j0})\right| = 2h[0]+2h[1]+h[2]=1$.

Next, Hence, $\left.\dfrac{d\left|H(e^{j\omega})\right|}{d\omega}\right|_{\omega=0} = 0$. Finally,

$\dfrac{d\left|H(e^{j\omega})\right|}{d\omega} = -4h[0]\sin 2\omega - 2h[1]\sin\omega = 0$, $\dfrac{d^2\left|H(e^{j\omega})\right|}{d\omega^2} = -8h[0]\cos 2\omega - 2h[1]\cos\omega = 0$, and hence,

$$\left.\frac{d^2\left|H(e^{j\omega})\right|}{d\omega^2}\right|_{\omega=0} = -8h[0]-2h[1]=0.$$

Since there are two equations and three unknowns, there is no unique solution. The general form of the solution in terms of h[0] is given by h[1] = −4 h[0] and h[2] = 1 + 6 h[0].

(b) The frequency response is given by $H(e^{j\omega}) = e^{-j2\omega}(2h[0]\cos 2\omega - 8h[0]\cos\omega + 1 + 6h[0])$.

4.18 $\left|H(e^{j\omega})\right| = 2h[0]\cos 2\omega + 2h[1]\cos\omega + h[2]$. Desired specifications are:

$$\left|H(e^{j0.1})\right| = 2h[0]\cos(0.2) + 2h[1]\cos(0.1) + h[2] = 0,$$
$$\left|H(e^{j0.4})\right| = 2h[0]\cos(0.8) + 2h[1]\cos(0.4) + h[2] = 1,$$
$$\left|H(e^{j0.7})\right| = 2h[0]\cos(1.4) + 2h[1]\cos(0.7) + h[2] = 0.$$

Solving these three equations we get h[0] = h[4] = − 21.6426, h[1] = h[3] = 76.1752 , and h[2] = − 109.1669.

4.19 Since the impulse response is real, it follows that $H(e^{j3\pi/2}) = H^*(e^{j\pi/2}) = -2 - j2$. Thus, the 4-point DFT of {h[n]} is given by {H[k]} = {10 −2 + j2 −2 −2 − j2}. Its 4-point IDFT is then {h[n]} = {1 2 3 4}. Therefore $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$.

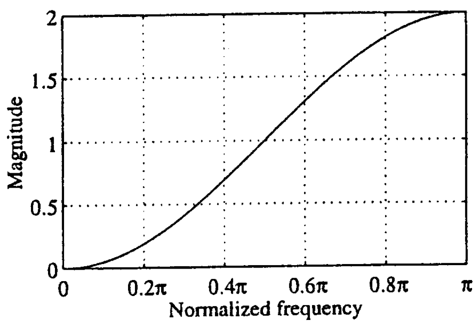
4.20 $H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[1]e^{-j2\omega} + h[0]e^{-j3\omega} = e^{-j3\omega/2}(2h[0]\cos(3\omega/2) + 2h[1]\cos(\omega/2))$.

Thus, $H(e^{j0}) = 2h[0] + 2h[1] = 6$, and

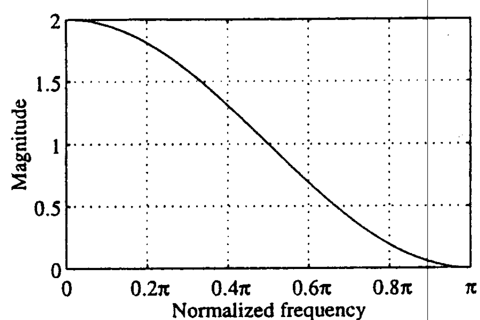
$H(e^{j\pi/2}) = e^{-j3\pi/4}(2h[0]\cos(3\pi/4) + 2h[1]\cos(\pi/4)) = -1 - j$. Thus, $h[0] + h[1] = 3$, and $h[0] - h[1] = -1$. Hence $h[0] = 1$, $h[1] = 2$. Therefore $H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$.

Alternately, note that $H(e^{j\pi}) = e^{-j3\pi/2}(2h[0]\cos(3\pi/2) + 2h[1]\cos(\pi/2)) = 0$, and $H(e^{j3\pi/2}) = H^*(e^{j\pi/2}) = -1 + j$. Hence, the 4-point DFT of the length-4 sequence $\{h[n]\} = \{h[0] \;\; h[1] \;\; h[0] \;\; h[0]\}$ is given by $\{H[k]\} = \{6 \;\; -1-j \;\; 0 \;\; -1+j\}$. Its inverse DFT thus yields $\{h[n]\} = \{1 \;\; 2 \;\; 2 \;\; 1\}$.

4.21 (a) $H_A(e^{j\omega}) = 0.5 - e^{-j\omega} + 0.5e^{-2j\omega}$, $H_B(e^{j\omega}) = 0.5 + e^{-j\omega} + 0.5e^{-2j\omega}$.



(a) $|H_A(e^{j\omega})|$



(b) $|H_B(e^{j\omega})|$

$H_A(e^{j\omega})$ is a highpass filter, whereas, $H_B(e^{j\omega})$ a lowpass filter.

(b) $H_C(e^{j\omega}) = H_B(e^{j\omega}) = H_A(e^{j(\omega+\pi)})$.

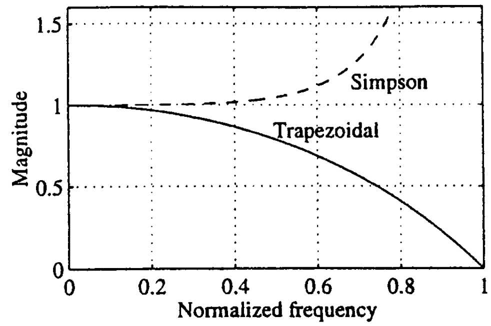
4.22 $y[n] = y[n-1] + 0.5\{x[n] + x[n-1]\}$. Hence,

$Y(e^{j\omega}) = e^{-j\omega}Y(e^{j\omega}) + 0.5\{X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega})\}$, or, $H_{tr}(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = 0.5\left(\dfrac{1+e^{-j\omega}}{1-e^{-j\omega}}\right)$.

4.23 $y[n] = y[n-2] + \frac{1}{3}\{x[n] + 4x[n-1] + x[n-2]\}$. Hence, $H_{sim}(e^{j\omega}) = \dfrac{1}{3}\left(\dfrac{1+4e^{-j\omega}+e^{-2j\omega}}{1-e^{-2j\omega}}\right)$.

Note: To compare the performances of the Trapezoidal numerical integration formula with that of the Simpson's formula, we first observe that if the input is $x_a(t) = e^{j\omega t}$, then the result of integration is $y_a(t) = y_a(t) = \frac{1}{j\omega}e^{j\omega t}$. Thus, the ideal frequency response $H(e^{j\omega})$ is $1/j\omega$.

Hence, we take the ratio of the frequency responses of the approximation to the ideal, and plot the two curves as indicated below.



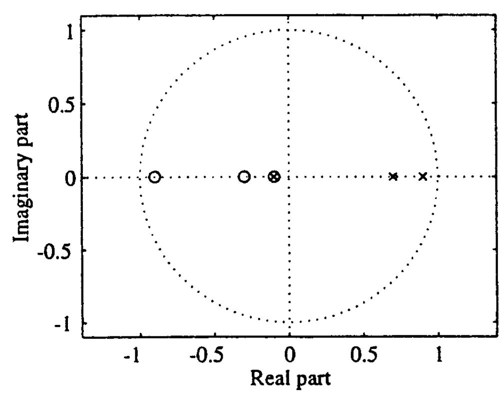
From the above plot it is eveident that the Simpson's formula amplifies high frequencies, whereas, the Trapzoidal formula attenuates them. In the very low frequency range, both formulae yield results close to the ideal. However, Simpson's formula is reasonably accurate for frequencies close to the midband range.

4.24 The transfer function of the composite filter is given by $H(z) = H_3(z)\big(H_1(z) + H_2(z)\big)$ $= (2 + 3z^{-1} + z^{-2})\big(1 + 2z^{-1} + 3z^{-2}\big) = 2 + 7z^{-1} + 13z^{-2} + 11z^{-3} + 3z^{-4}$.

4.25 Taking the z-transform of both sides we get $Y(z) = 2X(z) + 2.6z^{-1}X(z) + 0.78z^{-2}X(z)$ $+ 0.054z^{-3}X(z) + 1.5z^{-1}Y(z) - 0.47z^{-2}Y(z) - 0.063z^{-3}Y(z)$. The transfer function is thus

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + 2.6z^{-1} + 0.78z^{-2} + 0.054z^{-3}}{1 - 1.5z^{-1} + 0.47z^{-2} + 0.063z^{-3}} = \frac{2(1 + 0.9z^{-1})(1 + 0.3z^{-1})}{(1 - 0.9z^{-1})(1 - 0.7z^{-1})}.$$

The system is BIBO stable since both the poles of the transfer functions are inside the unit circle. The pole-zero plot of H(z) is shown below:

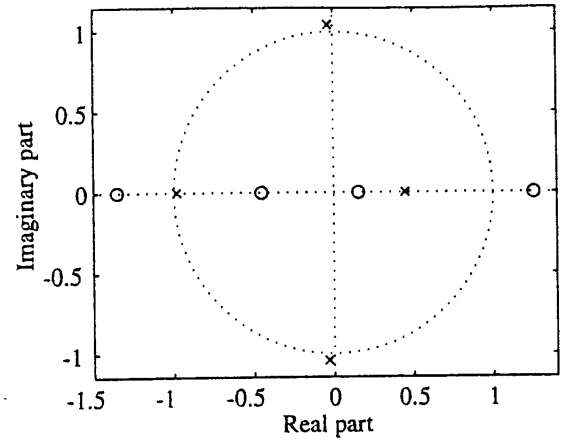


Note the cancellation of a real pole with a real zero in the above plot.

4.26 $$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 1.2z^{-1} - 5.22z^{-2} - 1.548z^{-3} + 0.359z^{-4}}{1 + 0.6z^{-1} + 0.68z^{-2} + 0.55z^{-3} - 0.477z^{-4}}$$

$$= \frac{3(1 - 1.2544z^{-1})(1 + 1.355z^{-1})(1 + 0.4544z^{-1})(1 - 0.155z^{-1})}{(1 + 0.9819z^{-1})(1 - 0.4479z^{-1})(1 - 0.066z^{-1} + 1.0845z^{-2})}.$$

Since there are poles outside the unit circle, hence the system is not BIBO stable. The pole-zero plot of H(z) is shown below:



It can be seen that there is a pair of complex conjugate poles outside the unit circle.

4.27 $H(z) = \dfrac{(2+0.4z^{-1})(0.5+2z^{-1})}{(1-2z^{-1}+2z^{-2})(1-0.5z^{-1})}$. A partial fraction expansion of H(z) is of the form:

$$H(z) = \frac{A}{1-0.5z^{-1}} + \frac{B}{1-(1+j)z^{-1}} + \frac{B^*}{1-(1-j)z^{-1}}.$$

where, $A = (1-0.5z^{-1})H(z)\Big|_{z^{-1}=2} = \dfrac{(2+0.4z^{-1})(0.5+2z^{-1})}{(1-2z^{-1}+2z^{-2})}\Bigg|_{z^{-1}=2}$

$$= \frac{(2+0.4\times 2)(0.5+2\times 2)}{(1-2\times 2+2\times 4)} = \frac{2.8\times 4.5}{5} = 2.52, \text{ and}$$

$B = \left(1-(1+j)z^{-1}\right)H(z)\Big|_{z^{-1}=0.5-j0.5} = \dfrac{(2+0.4z^{-1})(0.5+2z^{-1})}{\left(1-(1-j)z^{-1}\right)(1-0.5z^{-1})}\Bigg|_{z^{-1}=0.5-j0.5} = -0.76 - j0.3.48,$

Hence, $h[n] = 2.52[0.5]^n\mu[n] + \{(-0.76+j3.48)[1+j]^n + (-0.76-j3.48)[1-j]^n\}$

$$= 2.52[0.5]^n\mu[n] + 3.562(\sqrt{2})^n\cos(\frac{\pi}{4}n + 0.568\pi)\mu[n].$$

4.28 (a) $H(z) = \dfrac{2-4z^{-1}}{2+z^{-1}} + \dfrac{1}{3-z^{-1}} = -4 + \dfrac{10}{2+z^{-1}} + \dfrac{1}{3-z^{-1}} = -4 + \dfrac{5}{1-(-0.5)z^{-1}} + \dfrac{1}{3}\dfrac{1}{1-\frac{1}{3}z^{-1}}.$

Hence the impulse response is given by $h[n] = -4\delta[n] + 5(-0.5)^n\mu[n] + (1/3)^{n+1}\mu[n]$.

(b) The z-transform of the input is $X(z) = \dfrac{1}{1-0.4z^{-1}} - \dfrac{5}{1-0.2z^{-1}} = \dfrac{-4+1.8z^{-1}}{(1-0.4z^{-1})(1-0.2z^{-1})}.$

$H(z) = \dfrac{8-13z^{-1}+4z^{-2}}{(2+z^{-1})(3-z^{-1})}$. Now,

$$Y(z) = H(z)X(z) = \frac{(8-13z^{-1}+4z^{-2})(-4+1.8z^{-1})}{(2+z^{-1})(3-z^{-1})(1-0.4z^{-1})(1-0.2z^{-1})}$$

$$= \frac{-15.0794}{1+0.5z^{-1}} + \frac{0.2222}{1-0.4z^{-1}} - \frac{5.8333}{1-0.333z^{-1}} + \frac{15.3571}{1-0.2z^{-1}}.$$

The inverse z-transform of the output is given by

$$y[n] = -15.0794(-0.5)^n\mu[n] + 0.2222(0.4)^n\mu[n] -5.8333(0.333)^n\mu[n] - 15.3571(0.2)^n\mu[-n-1]..$$

4.29 $Y(z) = [H_0(z)F_0(z) - H_0(-z)F_0(-z)]X(z)$. Since the output is a delayed replica of the input, we must have $H_0(z)F_0(z) - H_0(-z)F_0(-z) = z^{-r}$. But $H_0(z) = 1 + z^{-1}$, hence $(1+z^{-1})F_0(z) - (1-z^{-1})F_0(-z) = z^{-r}$. Let $F_0(z) = a_0 + a_1 z^{-1}$. This impiles, $2(a_0 + a_1)z^{-1} = z^{-r}$.

One possible solution is therefore, $r = 1$ and $a_0 = a_1 = 0.25$. Hence $F_0(z) = 0.25(1+z^{-1})$.

4.30 (a) $H_0(z)F_0(z) = \left(E_0(z^2) + z^{-1}E_1(z^2)\right)\left(R_0(z^2) + z^{-1}R_1(z^2)\right)$

$= E_0(z^2)R_0(z^2) + z^{-1}\left((E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2)\right) + z^{-2}E_1(z^2)R_1(z^2)$. Thus,

$H_0(-z)F_0(-z) = \left(E_0(z^2) - z^{-1}E_1(z^2)\right)\left(R_0(z^2) - z^{-1}R_1(z^2)\right)$

$= E_0(z^2)R_0(z^2) - z^{-1}\left(E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2)\right) + z^{-2}E_1(z^2)R_1(z^2).$

As a result, $T(z) = H_0(z)F_0(z) - H_0(-z)F_0(-z) = 2z^{-1}\left(E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2)\right)$.

Hence the condition to be satisfied by $E_0(z)$, $E_1(z)$, $R_0(z)$, $R_1(z)$ for the output to be a delayed replica for input is $E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2) = 0.5z^{-r}$. (11)

(b) In Problem 4.29, $E_0(z) = 1$, $E_1(z) = 1$, $R_0(z) = 0.25$, $R_1(z) = 0.25$. Thus,

$$E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2) = 0.5.$$

Hence the condition of Eq. (11) is satisfied with $r = 0$.

(c) $E_0(z^2) = E_1(z^2) = z^{-2}$. Let $R_0(z^2) = R_1(z^2) = 0.25z^{-2}$. $r = 4$ in this case. Hence $H(z) = z^{-2}(1+z^{-1})$ and $F(z) = \frac{1}{4}z^{-2}(1-z^{-1})$.

4.31 $h[n] = h[N-n]$.

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{N-1} h[N-n]z^{-n} = \sum_{k=0}^{N-1} h[k]z^{-(N-k)} = z^{-N}H(z^{-1}).$$

So if $z = z_0$ is a root then so is $z = 1/z_0$. If $G(z) = 1/H(z)$ then $G(z)$ will have poles both inside and outside the unit circle, and will hence be unstable.

4.32 Since $H(z)$ has a pole at $z = 3$, the given transfer function is unstable. To construct a stable transfer function having the same magnitude response consider another transfer function $G(z)$ defined by $G(z) = \dfrac{2z^3 - 4z^2 + 9}{(1-3z)(z^2+z+0.5)} = H(z)A(z)$ where $A(z) = \dfrac{1-3z}{z-3}$ is an allpass transfer function. Thus $|G(e^{j\omega})| = |H(e^{j\omega})||A(e^{j\omega})|$ but since A(z) is an allpass function, $|A(e^{j\omega})| = 1$. Hence $|G(e^{j\omega})| = |H(e^{j\omega})|$.

Yes, there are infinitely many transfer functions of the form of G(z) = H(z)A(z) having the same magnitude as that of H(z).

4.33 H(z) has poles at $z = 1.5,\ 1,\ \frac{1}{4} \pm j\frac{\sqrt{3}}{4}$. Since H(z) has a pole outside the unit circle it is unstable. To construct a stable transfer function having the same magnitude response consider another transfer function $G(z) = \dfrac{(z^2 - 2z + 1)(z^2 + 3z + 4)}{(1.5z^2 - 2.5z + 1)(z^2 - 0.5z + 0.25)} = H(z)A(z)$, where

$A(z) = \dfrac{z^2 - 2.5z + 1.5}{1.5z^2 - 2.5z + 1}$ is an allpass function. Thus, $|G(e^{j\omega})| = |H(e^{j\omega})||A(e^{j\omega})| = |H(e^{j\omega})|$.

Since there are 4 poles and 4 zeros hence there are $2^8 = 256$ transfer functions having the same magnitude response.

4.34 $H(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$. Substituting the values of **A**, **B**, **C**, and D we get

$$H(z) = \begin{bmatrix} -2 & -1 \end{bmatrix} \begin{bmatrix} z+0.4 & -0.2 \\ -1 & z+0.3 \end{bmatrix}^{-1} \begin{bmatrix} 1.5 \\ 0.4 \end{bmatrix} + 2 = \frac{2z^{-2} - 2z - 2.88}{z^{-2} + 0.7z - 0.08}.$$

4.35 $H(z) = \dfrac{2z^3 - 4z^2 + 9}{(z-3)(z^2 + z + 0.5)} = \dfrac{2 - 4z^{-1} + 9z^{-3}}{1 - 2z^{-1} - 2.5z^{-2} - 1.5z^{-3}}$. The corresponding difference equation is y[n] = 2x[n] – 4x[n – 1] + 9x[n – 3] + 2y[n – 1] + 2.5y[n – 2] + 1.5y[n – 3]. Following the procedure given in Section 2.6.1 we arrive at the following values for the state space parameters $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1.5 & 2.5 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 12 & 5 & 0 \end{bmatrix}$, and D = 2.

4.36 $H(z) = \dfrac{(z^2 - 2z + 1)(z^2 + 3z + 4)}{(z^2 - 2.5z + 1.5)(z^2 - 0.5z + 0.25)} = \dfrac{1 + z^{-1} - z^{-2} - 5z^{-3} + 4z^{-4}}{1 - 3z^{-1} + 3z^{-2} - 1.375z^{-3} + 0.375z^{-4}}$.

The corresponding difference equation is given by y[n] = x[n] + x[n – 1] – x[n – 2] – 5x[n – 3] + 4x[n – 4] + 3y[n – 1] – 3y[n – 2] + 1.375y[n – 3] – 0.375y[n – 4].

Using Eq. (2.96) we arrive at the following values for the state space parameters

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.375 & 1.375 & -3 & 3 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 3.625 & -3.625 & -4 & 4 \end{bmatrix},\ D = 1.$$

4.37 $$h_{HP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{HP}(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \left[\int_{-\pi}^{-\omega_c} e^{j\omega n} d\omega + \int_{\omega_c}^{\pi} e^{j\omega n} d\omega \right]$$

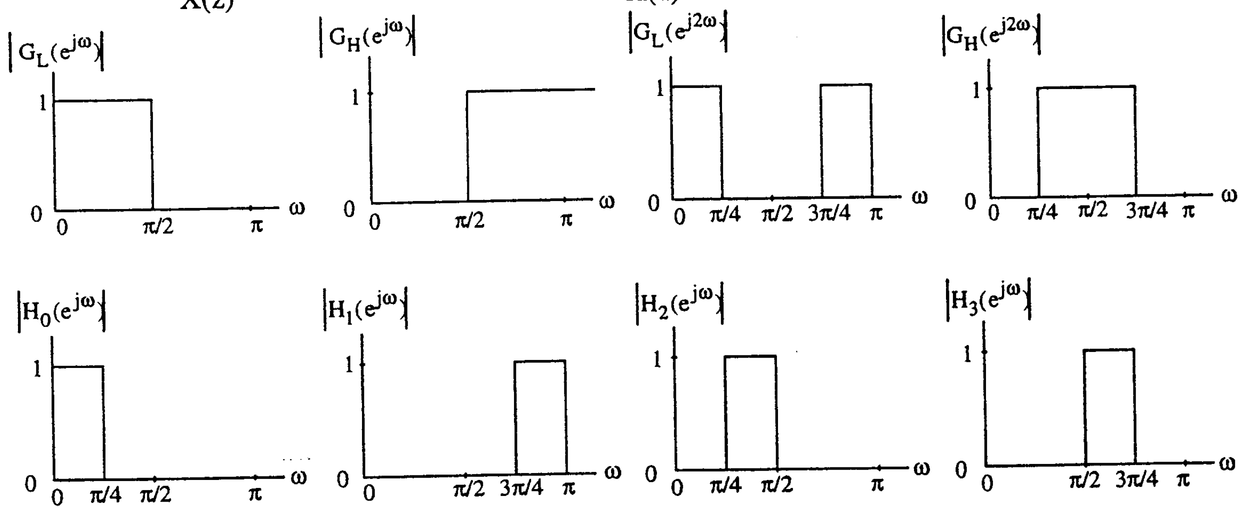
$$= \frac{1}{2\pi}\left[\frac{e^{j\pi n} - e^{j\omega_c n} + e^{-j\omega_c n} - e^{-j\pi n}}{jn}\right] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & \text{for } n = 0, \\ -\frac{\sin(\omega_c n)}{\pi n}, & \text{for } n \neq 0. \end{cases}$$

4.38 The bandpass filter of Figure 4.8(c) can be written as a difference of two lowpass filters one with a cutoff frequency ω_{c2} and another with a cutoff frequency ω_{c1} Hence the impulse response of the bandpass filter is $h_{BP}[n] = h_{LP2}[n] - h_{LP1}[n] = \left[\frac{\sin(\omega_{c2}n) - \sin(\omega_{c1}n)}{\pi n}\right]$.

4.39 Since $H_{BS}(e^{j\omega}) = 1 - H_{BP}(e^{j\omega})$, therefore $h_{BS}[n] = \begin{cases} 1 - \frac{\omega_{c2} - \omega_{c1}}{\pi}, & \text{for } n = 0, \\ \frac{\sin(\omega_{c1}n) - \sin(\omega_{c2}n)}{\pi n}, & \text{for } n \neq 0. \end{cases}$

4.40 From the figure $H_0(z) = \frac{Y_0(z)}{X(z)} = G_L(z)G_L(z^2)$, $H_1(z) = \frac{Y_1(z)}{X(z)} = G_H(z)G_L(z^2)$,

$H_2(z) = \frac{Y_2(z)}{X(z)} = G_L(z)G_H(z^2)$, and $H_3(z) = \frac{Y_3(z)}{X(z)} = G_H(z)G_H(z^2)$,



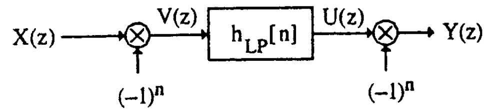
4.41 $G(e^{j\omega}) = H_{LP}(e^{j(\pi-\omega)})$. Since, $H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega < \omega_c, \\ 0, & \omega_c \le \omega < \pi, \end{cases}$ $G(e^{j\omega}) = \begin{cases} 0, & 0 \le \omega < \pi - \omega_c, \\ 1, & \pi - \omega_c \le \omega < \pi. \end{cases}$

Hence $G(e^{j\omega})$ is a highpass filter with a cutoff frequency given by $\omega_0 = \pi - \omega_c$. Also, since $G(z) = H_{LP}(-z)$, hence $g[n] = (-1)^n h_{LP}[n]$.

4.42 $G(z) = H_{LP}(e^{j\omega_o}z) + H_{LP}(e^{-j\omega_o}z)$. Hence, $g[n] = h_{LP}[n]e^{-j\omega_o n} + h_{LP}[n]e^{j\omega_o n} = 2\,h_{LP}[n]\cos(\omega_o n)$. Thus, $G(z)$ is a real coefficient bandpass filter with a center frequency at ω_o and a passband width of $2\omega_p$.

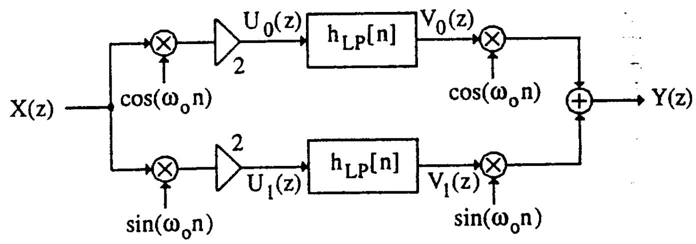
4.43 $F(z) = H_{LP}(e^{j\omega_o}z) + H_{LP}(e^{-j\omega_o}z) + H_{LP}(z)$. Hence, $f[n] = \left(1 + 2\cos(\omega_o n)\right)h_{LP}[n]$. Thus, F(z) is a real coefficient bandstop filter with a center frequency at ω_o and a stopband width of $\left(\pi - 3\omega_p\right)/2$.

4.44



From the figure, $V(z) = X(-z)$, $U(z) = H_{LP}(z)X(-z)$, and $Y(z) = U(-z) = H_{LP}(-z)X(z)$. Hence $H_{eq}(z) = Y(z)/X(z) = H_{LP}(-z)$ which is the highpass filter of Problem 4.41.

4.45



$u_0[n] = 2x[n]\cos(\omega_o n) = x[n]e^{j\omega_o n} + x[n]e^{-j\omega_o n}$ or $U_0(e^{j\omega}) = X(e^{j(\omega+\omega_o)}) + X(e^{j(\omega-\omega_o)})$.

Likewise, $U_1(e^{j\omega}) = jX(e^{j(\omega+\omega_o)}) - jX(e^{j(\omega-\omega_o)})$.

$V_0(z) = H_{LP}(z)X(ze^{j\omega_o}) + H_{LP}(z)X(ze^{-j\omega_o})$,

$V_1(z) = jH_{LP}(z)X(ze^{j\omega_o}) - jH_{LP}(z)X(ze^{-j\omega_o})$,

$Y(z) = \frac{1}{2}\left(V_0(ze^{j\omega_o}) + V_0(ze^{-j\omega_o})\right) + \frac{j}{2}\left(V_1(ze^{j\omega_o}) - V_1(ze^{-j\omega_o})\right)$ which after simplification yields

$$Y(z) = \frac{1}{4}\left\{H_{LP}(ze^{j\omega_o})X(ze^{2j\omega_o}) + H_{LP}(ze^{j\omega_o})X(z) + H_{LP}(ze^{-j\omega_o})X(z) + H_{LP}(ze^{-j\omega_o})X(ze^{-2j\omega_o})\right\}$$
$$-\frac{1}{4}\left\{H_{LP}(ze^{j\omega_o})X(ze^{2j\omega_o}) - H_{LP}(ze^{j\omega_o})X(z) - H_{LP}(ze^{-j\omega_o})X(z) + H_{LP}(ze^{-j\omega_o})X(ze^{-2j\omega_o})\right\}$$

Hence $Y(z) = \frac{1}{2}\left\{H_{LP}(ze^{j\omega_o}) + H_{LP}(ze^{-j\omega_o})\right\}X(z)$, Therefore

$$H_{eq}(z) = \frac{Y(z)}{X(z)} = \frac{1}{2}\left\{H_{LP}(ze^{j\omega_o}) + H_{LP}(ze^{-j\omega_o})\right\}.$$

Thus the structure shown in Figure P4.6 implements the bandpass filter of Problem 4.42.

4.46 See Figure 10.36 of Text.

4.47 $H_0(z) = \frac{1}{2}(1 + z^{-1})$. Thus, $\left|H_0(e^{j\omega})\right| = \cos(\omega/2)$. Now, $G(z) = \left(H_0(z)\right)^M$. Hence, $\left|G(e^{j\omega})\right|^2 = \left|H_0(e^{j\omega})\right|^{2M} = (\cos(\omega/2))^{2M}$. The 3-dB cutoff frequency ω_c of G(z) is thus given by $\left(\cos(\omega_c/2)\right)^{2M} = \frac{1}{2}$. Hence, $\omega_c = 2\cos^{-1}(2^{-1/2M})$.

4.48 $H_1(z) = \frac{1}{2}(1 - z^{-1})$. Thus, $\left|H_1(e^{j\omega})\right|^2 = \sin^2(\omega/2)$. Let $F(z) = \left(H_1(z)\right)^M$. Then $\left|F(e^{j\omega})\right|^2 = (\sin(\omega/2))^{2M}$. The 3-dB cutoff frequency ω_c of F(z) is thus given $\left(\sin(\omega_c/2)\right)^{2M} = \frac{1}{2}$, which yields $\omega_c = 2\sin^{-1}(2^{-1/2M})$.

4.49 $H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$. Note that $H_{LP}(z)$ is stable if $|\alpha| < 1$. Now,

$$\alpha = \frac{1-\sin(\omega_c)}{\cos(\omega_c)} = \frac{\cos^2(\omega_c/2) + \sin^2(\omega_c/2) - 2\sin(\omega_c/2)\cos(\omega_c/2)}{\cos^2(\omega_c/2) - \sin^2(\omega_c/2)}$$

$$= \frac{\cos(\omega_c/2) - \sin(\omega_c/2)}{\cos(\omega_c/2) + \sin(\omega_c/2)} = \frac{1-\tan(\omega_c/2)}{1+\tan(\omega_c/2)}. \qquad (12)$$

If $0 \le \omega < \pi$ then $\tan(\omega_c/2) \ge 0$ hence $|\alpha| < 1$.

4.50 From Eq. (12), $\alpha = \frac{1-\tan(\omega_c/2)}{1+\tan(\omega_c/2)}$, hence $\tan(\omega_c/2) = \frac{1-\alpha}{1+\alpha}$.

4.51 Since $\omega_c = 0.2$, $\alpha = \frac{1-\tan(\omega_c/2)}{1+\tan(\omega_c/2)} = \frac{1-\tan(0.1)}{1+\tan(0.1)} = 0.8176$. Hence,

$$H_{LP}(z) = 0.0912\left(\frac{1+z^{-1}}{1-0.8176z^{-1}}\right)$$

4.52 $H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$. Thus, $H_{HP}(e^{j\omega}) = \frac{1+\alpha}{2}\left(\frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}}\right)$.

$\left|H_{HP}(e^{j\omega})\right|^2 = \left[\frac{1+\alpha}{2}\right]^2\left[\frac{2-2\cos(\omega)}{1+\alpha^2-2\alpha\cos(\omega)}\right]$. At 3-dB cutoff frequency ω_c, $\left|H_{HP}(e^{j\omega_c})\right|^2 = \frac{1}{2}$.

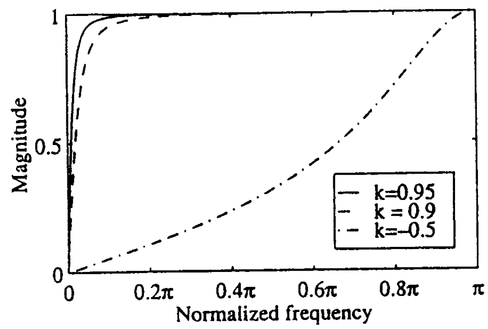
Hence, $\left[\frac{1+\alpha}{2}\right]^2\left[\frac{2-2\cos(\omega_c)}{1+\alpha^2-2\alpha\cos(\omega_c)}\right] = \frac{1}{2}$ which yields $\cos(\omega_c) = \frac{2\alpha}{1+\alpha^2}$.

4.53 $H(z) = \frac{1-z^{-1}}{1-kz^{-1}}$. Hence, $\left|H(e^{j\omega})\right|^2 = \frac{(1-\cos\omega)^2 + \sin^2\omega}{(1-k\cos\omega)^2 + k^2\sin^2\omega}$.

$$\left|H(e^{j\omega})\right|^2 = \frac{(1-\cos(\omega))^2 + \sin^2(\omega)}{(1-k\cos(\omega))^2 + k^2\sin^2(\omega)} = \frac{2-2\cos\omega}{1+k^2-2k\cos\omega}.$$

Now $\left|H(e^{j\pi})\right|^2 = \frac{4}{(1+k)^2}$. Thus, the scaled transfer function is given by

$H(z) = \frac{1+k}{2}\left(\frac{1-z^{-1}}{1-kz^{-1}}\right)$. A plot of the magnitude responses of the scaled transfer function for $k =$ 0.95, 0.9 and –0.5 are given below.



4.54 $H_{BP}(z)=\frac{1-\alpha}{2}\left(\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$. Thus, $H_{BP}(e^{j\omega})=\frac{1-\alpha}{2}\left(\frac{1-e^{-2j\omega}}{1-\beta(1+\alpha)e^{-j\omega}+\alpha e^{-2j\omega}}\right)$,

$\left|H_{BP}(e^{j\omega})\right|^2=\left(\frac{1-\alpha}{2}\right)^2\frac{2(1-\cos 2\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos 2\omega-2\beta(1+\alpha)^2\cos\omega}$, which can be simplified as $\left|H_{BP}(e^{j\omega})\right|^2=\frac{(1-\alpha)^2\sin^2\omega}{(1+\alpha)^2(\cos\omega-\beta)^2+(1-\alpha)^2\sin^2\omega}$.

At the center frequency ω_o, $\left|H_{BP}(e^{j\omega_o})\right|^2=1$. Hence, $(\cos\omega_o-\beta)^2=0$ or $\cos\omega_o=\beta$.

At the 3-dB bandedges ω_1 and ω_2, $\left|H_{BP}(e^{j\omega_i})\right|^2=\frac{1}{2}$, i = 1, 2. This implies

$$(1+\alpha)^2(\cos\omega_i-\beta)^2=(1-\alpha)^2\sin^2\omega_i, \qquad (13)$$

or $\sin\omega_i=\pm\left(\frac{1+\alpha}{1-\alpha}\right)(\cos\omega_i-\beta)$, i = 1, 2. Since $\omega_1<\omega_o<\omega_2$, $\sin\omega_1$ must have positive sign whereas, $\sin\omega_2$ must have negative sign because otherwise, $\sin\omega_2<0$ for ω_2 in $(0,\pi)$.

Now, Eq. (13) can be rewritten as $2(1+\alpha^2)\cos^2\omega_i-2\beta(1+\alpha)^2\cos\omega_i+\beta^2(1+\alpha)^2-(1-\alpha)^2=0$.

Hence, $\cos\omega_1+\cos\omega_2=\beta\frac{(1+\alpha)^2}{1+\alpha^2}$, and $(\cos\omega_1)(\cos\omega_2)=\frac{\beta^2(1+\alpha)^2-(1-\alpha)^2}{2(1+\alpha^2)}$.

Denote $\omega_{3dB}=\omega_2-\omega_1$. Then $\cos(\omega_{3dB})=\cos\omega_2\cos\omega_1+\sin\omega_2\sin\omega_1$

$$=\cos\omega_2\cos\omega_1-\left(\frac{1+\alpha}{1-\alpha}\right)^2\left(\cos\omega_2\cos\omega_1+\beta^2-\beta(\cos\omega_2+\cos\omega_1)\right)=\frac{2\alpha}{1+\alpha^2}.$$

4.55 $H_{BS}(z)=\left(\frac{1+\alpha}{2}\right)\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$.

$\left|H_{BS}(e^{j\omega})\right|^2=\left(\frac{1+\alpha}{2}\right)^2\frac{2+4\beta^2-8\beta\cos\omega+2\cos 2\omega}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos 2\omega-2\beta(1+\alpha)^2\cos\omega}$, which can be simplified as $\left|H_{BS}(e^{j\omega})\right|^2=\frac{(1+\alpha)^2(\cos\omega-\beta)^2}{(1+\alpha)^2(\cos\omega-\beta)^2+(1-\alpha)^2\sin^2\omega}$.

At the center frequency ω_o, $\left|H_{BS}(e^{j\omega_o})\right|^2=0$. Hence, $(\cos\omega_o-\beta)^2=0$ or $\cos\omega_o=\beta$.

At the 3-dB bandedges ω_1 and ω_2, $\left|H_{BS}(e^{j\omega_i})\right|^2 = \frac{1}{2}$, i = 1, 2. This leads to Eq. (13) given on the previous page. Hence, as in the aolution of the previous problem $\omega_{3dB} = \frac{2\alpha}{1+\alpha^2}$.

4.56 $\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)} = 2^{-1/K}$. Let $B = 2^{(K-1)/K}$. Simplifying this equation we get

$$\alpha^2(\cos\omega_c + 1 - B) - 2\alpha(1+\cos\omega_c - B\cos\omega_c) + 1 + \cos\omega_c - B = 0$$

Solving the above quadratic equation for α we obtain

$$\alpha = \frac{2(1+(1-B)\cos\omega_c) \pm \sqrt{4(1+(1-B)\cos\omega_c)^2 - 4(1+\cos\omega_c - B)^2}}{2(1+\cos\omega_c - B)}$$

$$= \frac{\left(1+(1-B)\cos\omega_c\right) \pm \sqrt{(2+2\cos\omega_c - B - B\cos\omega_c)\left(B(1-\cos\omega_c)\right)}}{(1+\cos\omega_c - B)}$$

$$= \frac{1+(1-B)\cos\omega_c \pm \sin\omega_c\sqrt{2B-B^2}}{(1+\cos\omega_c - B)}.$$

For stability we require $|\alpha| < 1$, hence the desired solution is $\alpha = \frac{1+(1-B)\cos\omega_c - \sin\omega_c\sqrt{2B-B^2}}{(1+\cos\omega_c - B)}$.

4.57 $H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$. $\left|H_{HP}(e^{j\omega})\right|^2 = \left[\frac{1+\alpha}{2}\right]^2 \left|\frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}}\right|^2$

$$\left|H_{HP}(e^{j\omega})\right|^{2K} = \left[\frac{1+\alpha}{2}\right]^{2K} \left|\frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}}\right|^{2K} = \left[\frac{1+\alpha}{2}\right]^{2K} \frac{2^K(1-\cos\omega)^K}{\left(1+\alpha^2-2\alpha\cos\omega\right)^K}$$

At the 3-dB cut off frequency ω_c, $\left|H_{HP}(e^{j\omega_c})\right|^{2K} = \frac{1}{2}$. Let $B = 2^{(K-1)/K}$. Simplifying the above equation we get $\alpha^2(1-\cos\omega_c - B) + 2\alpha(1-\cos\omega_c + B\cos\omega_c) + 1 - \cos\omega_c - B = 0$.

Hence, $\alpha = \frac{-2(1-\cos\omega_c + B\cos\omega_c) \pm 2\sqrt{(1-\cos\omega_c + B\cos\omega_c)^2 - (1-\cos\omega_c - B)^2}}{2(1-\cos\omega_c - B)}$

For $H_{HP}(z)$ to be stable, we require $|\alpha| < 1$, hence the desired solution is

$$\alpha = \frac{\sin\omega_c\sqrt{2B-B^2} - (1-\cos\omega_c + B\cos\omega_c)}{(1-\cos\omega_c - B)}.$$

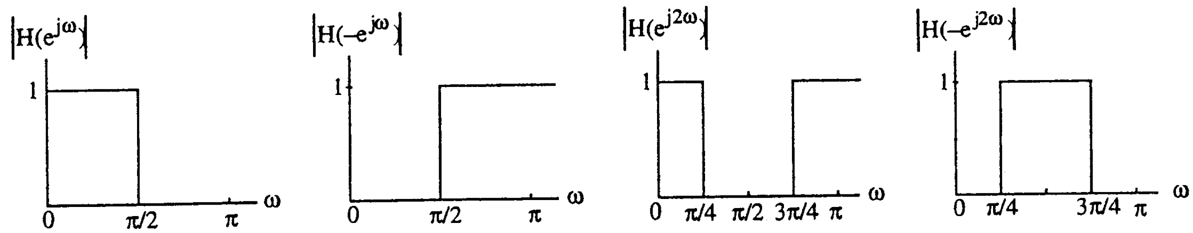
4.58 $H_1(z) = H(-z)$. $H_1(e^{j\omega}) = H(e^{(\pi-\omega)})$. If $H(e^{j\omega})$ is a lowpass filter with a passband edge at ω_p, then $H_1(e^{j\omega}) \approx \begin{cases} 1, & \text{if } \pi - \omega_p \le \omega < \pi, \\ 0, & \text{if } 0 \le \omega < \pi - \omega_p, \end{cases}$ hence $H_1(e^{j\omega})$ is a highpass filter.

Now, $h_1[n] = (-1)^n h[n]$. The passband edge of the highpass filter is given by $\omega_{p,HP} = \pi - \omega_{p,LP}$, and the stopband edge is given by $\omega_{s,HP} = \pi - \omega_{s,LP}$.

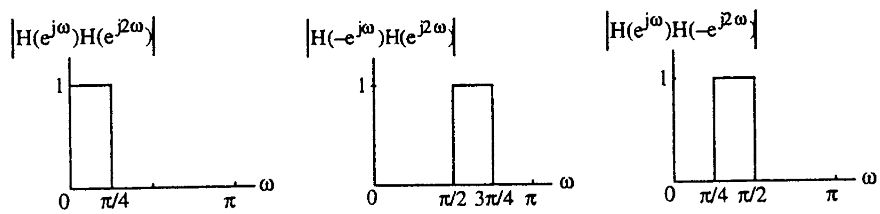
4.59 $H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$, $G_{HP}(z) = \frac{1-\alpha}{2}\left(\frac{1-z^{-1}}{1+\alpha z^{-1}}\right)$. Let $\beta = -\alpha$, hence

$$G_{HP}(z) = \frac{1+\beta}{2}\left(\frac{1-z^{-1}}{1-\beta z^{-1}}\right).$$ Therefore $\omega_c = \cos^{-1}(\beta) = \cos^{-1}(-\alpha)$.

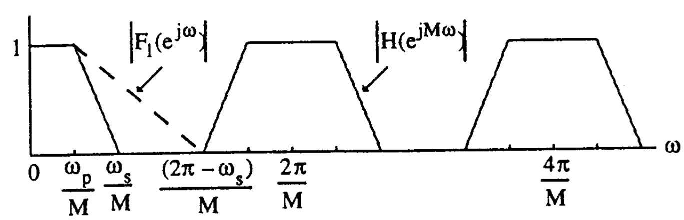
4.60 Tha magnitude responses of $H(z)$, $H(-z)$, $H(z^2)$ and $H(-z^2)$ are shown below.



The magnitude responses of $H(-z)H(z^2)$, $H(-z)H(z^2)$, and $H(z)H(-z^2)$ are shown below.

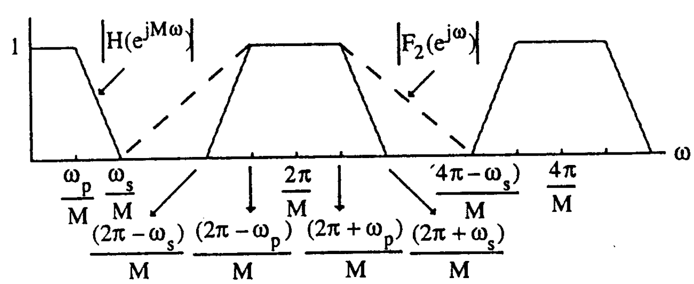


4.61 The magnitude responses of $H(z^M)$ (solid line) and $F_1(z)$ (dashed line) are shown below:



Hence $G_1(z) = H(z^M)F_1(z)$ is a lowpass filter with unity passband magnitude and passband edge at ω_p/M, and a stopband edge at ω_s/M.

The magnitude responses of $H(z^M)$ (solid line) and $F_2(z)$ (dashed line) are shown below:



Hence $G_2(z) = H(z^M)F_2(z)$ is a bandpass filter with unity passband magnitude and passband edges at $(2\pi-\omega_p)/M$ and $(2\pi+\omega_p)/M$, and a stopband edges at $(2\pi-\omega_s)/M$ and $(2\pi+\omega_s)/M$.

4.62 $v[n] = x[-n] \circledast h[n]$, and $u[n] = v[-n] = x[n] \circledast h[-n]$. Hence $y[n] = (h[n] + h[-n]) \circledast x[n]$. Therefore $G(e^{j\omega}) = H(e^{j\omega}) + H^*(e^{j\omega})$. Hence the equivalent transfer function $G(e^{j\omega})$ is real and has zero phase.

4.63 Type I: $\breve{H}(\omega) = \sum_{n=0}^{N/2} a[n]\cos\omega n$. Thus in this case, $P(\omega) = 1$, $Q(\omega) = \sum_{n=0}^{N/2} a[n]\cos\omega n$.

Type 2: $\breve{H}(\omega) = \sum_{n=1}^{(N+1)/2} b[n]\cos\left(\omega(n-\frac{1}{2})\right) = \cos(\omega/2)\sum_{n=1}^{(N+1)/2} b[n]\frac{\cos\left(\omega(n-\frac{1}{2})\right)}{\cos(\omega/2)}$. Now,

$\frac{\cos\left(\omega(n-\frac{1}{2})\right)}{\cos(\omega/2)} = 2\sum_{k=1}^{n-1}(-1)^k\cos\omega k + 1$. Hence we can express $\breve{H}(\omega)$ as

$\breve{H}(\omega) = \cos(\omega/2)\sum_{n=0}^{(N-1)/2}\tilde{b}[n]\cos\omega n$, where $\tilde{b}[n]$ can be obtained by collecting all coefficients.

Here, $P(\omega) = \cos(\omega/2)$ and $Q(\omega) = \sum_{n=0}^{(N-1)/2}\tilde{b}[n]\cos\omega n$.

Type 3: $\breve{H}(\omega) = \sum_{n=1}^{N/2} c[n]\sin\omega n = \sin\omega\sum_{n=1}^{N/2} c[n]\frac{\sin\omega n}{\sin\omega}$. Now, $\frac{\sin\omega n}{\sin\omega} = 1 + 2\sum_{k=1}^{n}\cos\omega k$. Hence we can express $\breve{H}(\omega)$ as $\breve{H}(\omega) = \sin\omega\sum_{n=0}^{(N/2)-1}\tilde{c}[n]\cos\omega n$. Here $P(\omega) = \sin\omega$ and

$Q(\omega) = \sum_{n=0}^{(N/2)-1}\tilde{c}[n]\cos\omega n$.

Type 4: $\breve{H}(\omega) = \sum_{n=1}^{(N+1)/2} d[n]\sin\left(\omega(n-\frac{1}{2})\right) = \sin(\omega/2)\sum_{n=1}^{N/2} c[n]\frac{\sin\left(\omega(n-\frac{1}{2})\right)}{\sin(\omega/2)}$. Now,

$\frac{\sin\left(\omega(n-\frac{1}{2})\right)}{\sin(\omega/2)} = 1 + 2\sum_{k=1}^{n-1}\cos\omega k$. Hence we can express $\breve{H}(\omega)$ as

$\breve{H}(\omega) = \sin(\omega/2)\sum_{n=0}^{(N-1)/2}\tilde{d}[n]\cos\omega n$. As a result, here $P(\omega) = \sin(\omega/2)$ and

$Q(\omega) = \sum_{n=0}^{(N-1)/2}\tilde{d}[n]\cos\omega n$.

4.64 **(a)** Yes the filter no. 2 has a linear-phase response.

(b) None of the filters shown have minimum-phase response.

4.65 $G_1(z) = (6 - z^{-1} - 12z^{-2})(2 + 5z^{-1}) = 30(1 - \frac{3}{2}z^{-1})(1 + \frac{4}{3}z^{-1})(\frac{2}{5} + z^{-1})$.

(a) The other transfer functions having the same magnitude response are:

(i) $H_1(z) = 30(-\frac{3}{2} + z^{-1})(1 + \frac{4}{3}z^{-1})(\frac{2}{5} + z^{-1})$, (ii) $H_2(z) = 30(1 - \frac{3}{2}z^{-1})(\frac{4}{3} + z^{-1})(\frac{2}{5} + z^{-1})$,

(iii) $H_3(z) = 30(1 - \frac{3}{2}z^{-1})(\frac{4}{3} + z^{-1})(1 + \frac{2}{5}z^{-1})$, (iv) $H_4(z) = 30(-\frac{3}{2} + z^{-1})(\frac{4}{3} + z^{-1})(1 + \frac{2}{5}z^{-1})$,

(v) $H_5(z) = 30(-\frac{3}{2} + z^{-1})(1 + \frac{4}{3}z^{-1})(1 + \frac{2}{5}z^{-1})$, (vi) $H_6(z) = 30(1 - \frac{3}{2}z^{-1})(1 + \frac{4}{3}z^{-1})(1 + \frac{2}{5}z^{-1})$,

(vii) $H_7(z) = 30(-\frac{3}{2} + z^{-1})(\frac{4}{3} + z^{-1})(1 + \frac{2}{5}z^{-1})$.

(b) $H_7(z)$ has a minimum phase response, and $G_1(z)$ has a maximum phase response.

(c) The partial energy of the impulse responses of each of the above transfer functions for different values of k are given by

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$G_1(z)$	144	928	1769	5369	5369
$H_1(z)$	324	3573	3769	5369	5369
$H_2(z)$	256	1040	3344	5369	5369
$H_3(z)$	900	949	4793	5369	5369
$H_4(z)$	576	4420	4469	5369	5369
$H_5(z)$	2025	4329	5113	5369	5369
$H_6(z)$	1600	1796	5045	5369	5369
$H_7(z)$	3600	4441	5225	5369	5369

The partial energy remains the same for values of $k > 2$. From the table given above it can be seen that $\sum_{m=0}^{k} |h_i[m]|^2 \le \sum_{m=0}^{k} |h_7[m]|^2$, and $\sum_{m=0}^{\infty} |h_i[m]|^2 = \sum_{m=0}^{\infty} |h_7[m]|^2$, $i = 1, 2, \ldots, 7$.

4.66 A maximum-phase FIR transfer function has all zeros outside the unit circle, and hence, the product of the roots is necessarily greater than 1. This implies that only those FIR transfer functions which have the coefficient of the highest power in z^{-1} (z^{-6} in the present case) greater than 1 can have maximum phase. Thus only $H_1(z)$ and $H_3(z)$ can be maximum-phase transfer functions. Also, maximum-phase transfer functions will have minimum partial-energy (as indicated in the solution of Problem 4.65). Hence, $H_1(z)$ is a maximum-phase transfer function since it has the smallest constant term in comparison with that of $H_3(z)$.

Likewise, a minimum-phase FIR transfer function is characterized by: (1) largest constant term, and (2) the value of the coefficient of the highest power of z^{-1} being less than 1. In the present problem, it can be seen that $H_2(z)$ satisfies these two conditions and, is hence, a minimum-phase transfer function.

Total no. of length-7 sequences having the same magnitude response is 27. Thus, there exist 123 other sequences with the same magnitude responses as those given here.

4.67 (a) $\breve{G}(\omega) = \breve{H}(\omega) + \delta$. $\breve{H}(\omega) = \sum_{n=0}^{N/2} a(n)\cos(\omega n)$ where $a[0] = h[N/2]$. Hence

$$g[n] = \begin{cases} h[n], & \forall n \text{ except } n = N/2, \\ h[N/2] + \delta, & n = N/2. \end{cases}$$ Thus $m = N/2$ and $\alpha = \delta$.

(b) Since $\breve{G}(\omega)$ is real and positive hence it can be expressed as $\breve{G}(\omega) = F(e^{j\omega})F^*(e^{j\omega})$. As $H(z)$ is a linear phase filter so is $G(z)$. Therefore $G(z)$ will have roots at $z = z_o$ and at $z = 1/z_o$. This means that $G(z)$ will have roots inside the unit circle with reciprocal roots outside the unit circle. Hence $F(z)$ can be obtained by assigning to it all the roots that are inside the unit circle. Then $F(z^{-1})$ is automatically assigned the reciprocal roots outside the unit circle.

(c) No, $\breve{H}(\omega)$ can not be expressed as the square magnitude of a minimum-phase FIR filter because $\breve{H}(\omega)$ takes on negative values too.

4.68 $\sum_{n=0}^{K} |h[n]|^2 = 0.95 \sum_{n=0}^{\infty} |h[n]|^2$. Since $H(z) = 1/(1+\alpha z^{-1})$, hence $h[n] = (-\alpha)^n \mu[n]$.. Thus

$\frac{1-|\alpha|^{2K}}{1-|\alpha|^2} = \frac{0.95}{1-|\alpha|^2}$. Solving this equation for K we get $K = 0.5\frac{\log(0.05)}{\log(|\alpha|)}$.

4.69 (a) $F_1(z) = 1 + 2z^{-1} + 3z^{-2}$. $F_1(z)$ has roots at $z = -1 \pm j\sqrt{2}$. Since $H(z)$ is a linear phase FIR transfer function, its roots must exist in reciprocal pairs. Hence if $H(z)$ has roots at $z = -1 \pm j\sqrt{2}$, then it should also have roots at $z = \frac{1}{-1 \pm j\sqrt{2}} = -\frac{1}{3} \pm j\frac{\sqrt{2}}{3}$. Therefore $H(z)$ should atleast have another factor with roots at $z = -\frac{1}{3} \pm j\frac{\sqrt{2}}{3}$. Hence $F_2(z) = 3 + 2z^{-1} + z^{-2}$, which is the mirrorimage polynomial to $F_1(z)$ and $H(z) = F_1(z)F_2(z)$

$= \left(1 + 2z^{-1} + 3z^{-2}\right)\left(3 + 2z^{-1} + z^{-2}\right) = 3 + 8z^{-1} + 14z^{-2} + 8z^{-3} + 3z^{-4}$.

(b) $F_1(z) = 3 + 5z^{-1} - 4z^{-2} - 2z^{-3}$. Its mirror-image olynomial is given by $F_2(z) = -2 - 4z^{-1} + 5z^{-2} + 3z^{-3}$. Therefore, $H(z) = F_1(z)F_2(z)$

$= \left(3 + 5z^{-1} - 4z^{-2} - 2z^{-3}\right)\left(-2 - 4z^{-1} + 5z^{-2} + 3z^{-3}\right)$

$= -6 - 22z^{-1} + 3z^{-2} + 54z^{-3} + 3z^{-4} - 22z^{-5} - 6z^{-6}$.

4.70 $A_1(z) = \frac{1 - d_1^* z}{z - d_1}$. $|A_1(z)|^2 = \frac{\left(1 - d_1^* z\right)\left(1 - d_1 z^*\right)}{(z - d_1)\left(z^* - d_1^*\right)}$. Thus,

$$1-|A_1(z)|^2 = \frac{(z-d_1)(z^*-d_1^*)-(1-d_1^*z)(1-d_1z^*)}{(z-d_1)(z^*-d_1^*)}$$

$$= \frac{|z|^2+|d_1|^2-d_1z^*-zd_1^*-1-|d_1|^2|z|^2+d_1z^*+d_1^*z}{(z-d_1)(z^*-d_1^*)} = \frac{(|z|^2-1)(1-|d_1|^2)}{|z-d_1|^2}.$$

Hence, $1-|A_1(z)|^2 \begin{cases} >0, & \text{if } |z|>1, \\ =0, & \text{if } |z|=1, \\ <0, & \text{if } |z|<1. \end{cases}$ Thus, $|A_1(z)|^2 \begin{cases} <1, & \text{if } |z|>1, \\ =1, & \text{if } |z|=1, \\ >1, & \text{if } |z|<1. \end{cases}$

Thus Eq. (4.129) holds for any first order allpass function. If the allpass is of higher order it can be factored into a product of first order allpass functions. Since Eq. (4.129) holds true for each of these factors individually hence it also holds true for the product.

4.71 An m-th order stable, real allpass transfer function A(z) can be expressed as a product of first-order allpass transfer functions of the form $A_i(z) = \frac{1-d_i^*z}{z-d_i}$. If d_i is complex, then A(z) has another factor of the form $A_i'(z) = \frac{1-d_iz}{z-d_i^*}$. Now,

$A_i(e^{j\omega}) = \frac{1-d_i^*e^{j\omega}}{e^{j\omega}-d_i} = e^{-j\omega}\frac{(1-d_i^*e^{j\omega})(1-d_i^*e^{j\omega})}{(1-d_ie^{-j\omega})(1-d_i^*e^{j\omega})}$. Let $d_i = |d_i|e^{j\theta} = \alpha e^{j\theta}$. Then,

$A_i(e^{j\omega}) = e^{-j\omega}\frac{(1-\alpha e^{-j\theta}e^{j\omega})^2}{\left(1+\alpha^2-2\alpha\cos(\theta-\omega)\right)}$. Therefore,

$$\arg\{A_i(e^{j\omega})\} = -\omega + \arg\left\{\left(1-\alpha e^{-j\theta}e^{j\omega}\right)^2\right\} = -\omega + 2\arg\left\{\left(1-\alpha e^{-j\theta}e^{j\omega}\right)\right\}$$

$$= -\omega + 2\tan^{-1}\left[\frac{\alpha\sin(\theta-\omega)}{1-\alpha\cos(\theta-\omega)}\right].$$

Similarly, $\arg\{A_i'(e^{j\omega})\} = -\omega + 2\tan^{-1}\left[\frac{-\alpha\sin(\theta+\omega)}{1-\alpha\cos(\theta+\omega)}\right]$.

If $d_i = \alpha$ is real, then $\arg\{A_i(e^{j\omega})\} = -\omega + 2\tan^{-1}\left[\frac{-\alpha\sin\omega}{1-\alpha\cos\omega}\right]$.

Now, for real d_i, $\arg\{A_i(e^{j0})\} - \arg\{A_i(e^{j\pi})\} = -0 + 2\tan^{-1}(-0) - \{-\pi + 2\tan^{-1}(-0)\} = \pi.$

For complex d_i, $\arg\{A_i(e^{j0})\} + \arg\{A_i'(e^{j0})\} - \arg\{A_i(e^{j\pi})\} - \arg\{A_i'(e^{j\pi})\}$

$$= -0 + 2\tan^{-1}\left[\frac{\alpha\sin\theta}{1-\alpha\cos\theta}\right] - 0 + 2\tan^{-1}\left[\frac{-\alpha\sin\theta}{1-\alpha\cos\theta}\right]$$

$$+ \pi - 2\tan^{-1}\left[\frac{-\alpha\sin\theta}{1+\alpha\cos\theta}\right] + \pi - 2\tan^{-1}\left[\frac{\alpha\sin\theta}{1+\alpha\cos\theta}\right] = 2\pi.$$

Now, $\tau(\omega) = -\frac{d}{d\omega}\left(\arg\{A(e^{j\omega})\}\right)$.

Therefore, $\int_0^\pi \tau(\omega)d\omega = -\int_0^\pi d\left[\arg\{A(e^{j\omega})\}\right] = \arg\{A(e^{j0})\} - \arg\{A(e^{j\pi})\}$.

Since $\arg\{A(e^{j\omega})\} = \sum_{i=1}^{m} \arg\{A_i(e^{j\omega})\}$, it follows that $\int_0^{\pi} \tau(\omega)d\omega = m\pi$.

4.72 Since G(z) is non-minimum phase but causal, it will have some zeros outside the unit circle. Let $z = \alpha$ be one such zero. We can then write $G(z) = P(z)(1-\alpha z^{-1}) = P(z)(-\alpha^* + z^{-1})\left(\frac{1-\alpha z^{-1}}{-\alpha^* + z^{-1}}\right)$.

Note that $\left(\frac{1-\alpha z^{-1}}{-\alpha^* + z^{-1}}\right)$ is a stable first order allpass function. If we carry out this operation for all zeros of G(z) that are outside the unit circle we can write G(z) = H(z)A(z) where H(z) will have all zeros inside the unit circle and will thus be a minimum phase function and A(z) will be a product of first order allpass functions, and hence an allpass function.

4.73 $H(z) = \frac{(z-0.3)(z^2-2z+2)}{(z-0.5)(z-0.9)}$. In order to correct for magnitude distortion we require the transfer function G(z) to satisfy the following property $|G(e^{j\omega})| = \frac{1}{|H(e^{j\omega})|}$. Hence, one possible solution is $G(z) = \frac{1}{H(z)} = \frac{(z-0.5)(z-0.9)}{(z-0.3)(z^2-2z+2)}$. Since G(z) has poles outside the unit circle. it is not stable. Therefore we require a stable transfer function with magnitude response same as G(z). Using the technique of the previous problem we thus get:

$$G(z) = \frac{(z-0.5)(z-0.9)}{(z-0.3)(z^2-2z+2)}\frac{(2z^2-2z+1)}{(2z^2-2z+1)} = \frac{(z-0.5)(z-0.9)}{(z-0.3)(2z^2-2z+1)}\frac{(2z^2-2z+1)}{(z^2-2z+2)} = P(z)\ A(z)$$

where $P(z) = \frac{(z-0.5)(z-0.9)}{(z-0.3)(2z^2-2z+1)}$ is the desired stable solution such that $|P(e^{j\omega})||H(e^{j\omega})| = 1$..

4.74 **(a)** G(z) = H(z) A(z) where A(z) is an allpass function. Then, $g[0] = \lim_{z\to\infty} G(z)$.

$$|g[0]| = \left|\lim_{z\to\infty} G(z)\right| = \left|\lim_{z\to\infty} H(z)A(z)\right| = \left|\lim_{z\to\infty} H(z)A(z)\right| = \left|\lim_{z\to\infty} H(z)\right|\left|\lim_{z\to\infty} A(z)\right|$$

$$\le \left|\lim_{z\to\infty} H(z)\right| \quad \text{because} \left|\lim_{z\to\infty} A(z)\right| < 1 \quad \text{(see Problem 4.71)}$$

$$\le |h[0]|$$

(b) If λ_ℓ is a zero of H(z), then $|\lambda_\ell| < 1$, since H(z) is a minimum-phase causal stable transfer function which has all zeros inside the unit circle. We can express $H(z) = B(z)(1-\lambda_\ell z^{-1})$. It follows that B(z) is also a minimum-phase causal transfer function.

Now consider the transfer function $F(z) = B(z)(\lambda_\ell^* - z^{-1}) = H(z)\frac{(\lambda_\ell^* - z^{-1})}{(1-\lambda_\ell z^{-1})}$. If h[n], b[n], and f[n] denote, respectively, the inverse z-transforms of H(z), B(z), and F(z), then we get

$$h[n] = \begin{cases} b[0], & n = 0 \\ b[n] - \lambda_\ell b[n-1], & n \ge 1, \end{cases}$$

and $f[n]=\begin{cases}\lambda_\ell^* b[0], & n=0,\\ \lambda_\ell^* b[n]-b[n-1], & n\geq 1.\end{cases}$

Consider $\varepsilon=\sum_{n=0}^{m}|h[n]|^2-\sum_{n=0}^{m}|f[n]|^2=|b[0]|^2-|\lambda_\ell^*|^2|b[0]|^2+\sum_{n=1}^{m}|h[n]|^2-\sum_{n=1}^{m}|f[n]|^2.$

Now $|h[n]|^2=|b[n]|^2+|\lambda_\ell|^2|b[n-1]|^2-\lambda_\ell b[n-1]b^*[n]-\lambda_\ell^* b^*[n-1]b[n]$, and

$|f[n]|^2=|\lambda_\ell|^2|b[n]|^2+|b[n-1]|^2-\lambda_\ell b[n-1]b^*[n]-\lambda_\ell^* b^*[n-1]b[n]$,

Hence, $\varepsilon=|b[0]|^2-|\lambda_\ell|^2|b[0]|^2+\sum_{n=1}^{m}\left(|b[n]|^2+|\lambda_\ell|^2|b[n-1]|^2\right)-\sum_{n=1}^{m}\left(|\lambda_\ell|^2|b[n]|^2-|b[n-1]|^2\right)$

$=\left(1-|\lambda_\ell|^2\right)|b[m]|^2.$

Since $|\lambda_\ell|<1$, $\varepsilon>0$, i.e, $\sum_{n=0}^{m}|h[n]|^2>\sum_{n=0}^{m}|f[n]|^2$. Hence, $\sum_{n=0}^{m}|h[n]|^2\geq\sum_{n=0}^{m}|g[n]|^2$.

4.75 $H(z)=\dfrac{(z+3)(z-2)}{(z-0.25)(z+0.5)}=G(z)A(z)=\dfrac{(3z+1)(1-2z)}{(z-0.25)(z+0.5)}\dfrac{(z+3)(z-2)}{(3z+1)(1-2z)}$

Therefore $G(z)=\dfrac{(3z+1)(1-2z)}{(z-0.25)(z+0.5)}$. The inverse z-transforms of these causal transfer functions are given by $h[n]=\{1,\ 0.75,\ -6.0625,\ 1.6093,\ -1.16015,\ \cdots\}$, and $g[n]=\{-6,\ 2.5,\ -0.375,\ 0.40625,\ -0.1484,\ 0.0879,\ \cdots\}$, respectively. It can be seen that $\sum_{n=0}^{m}|g[n]|^2$ is bigger than $\sum_{n=0}^{m}|h[n]|^2$ for all values of m.

4.76 See Example 4.17

(a) $H_{BS}(z)=\frac{1}{4}(1+z^{-2})^2$. Thus, $H_{BP}(z)=z^{-2}-\frac{1}{4}(1+z^{-2})^2=-\frac{1}{4}(1-z^{-2})^2$.

(b) $H_{BS}(z)=\frac{1}{16}(1+z^{-2})(-1+6z^{-2}-z^{-4})$. Thus, $H_{BP}(z)=z^{-4}-\frac{1}{16}(1+z^{-2})(-1+6z^{-2}-z^{-4})$

$=\frac{1}{16}\left\{1-4z^{-2}+6z^{-4}-4z^{-6}+z^{-8}\right\}=\frac{1}{16}(1-z^{-2})^4.$

(c) $H_{BS}(z)=\frac{1}{32}(1+z^{-2})^2(-3+14z^{-2}-3z^{-4})$. Thus,

$H_{BP}(z)=z^{-4}-\frac{1}{32}(1+z^{-2})^2(-3+14z^{-2}-3z^{-4})=\frac{1}{32}\left\{3-8z^{-2}+10z^{-4}-8z^{-6}+3z^{-8}\right\}$

$=\frac{1}{32}\left\{3-8z^{-2}+10z^{-4}-8z^{-6}+3z^{-8}\right\}.$

4.77 $H_0(z)=A_0(z)+A_1(z)$, and $H_1(z)=A_0(z)-A_1(z)$, where $A_0(z)$ and $A_1(z)$ are allpass functions of orders M and N, respectively, with no common factors. Hence, the orders of $H_0(z)$

and $H_1(z)$ are M+N. Now, we can write $A_0(z) = \frac{z^{-M}D_0(z^{-1})}{D_0(z)}$, and $A_1(z) = \frac{z^{-N}D_1(z^{-1})}{D_1(z)}$.

Then, $H_0(z) = \frac{P(z)}{D(z)} = \frac{z^{-M}D_0(z^{-1})D_1(z) + z^{-N}D_0(z)D_1(z^{-1})}{D_0(z)D_1(z)}$, and

$$H_1(z) = \frac{Q(z)}{D(z)} = \frac{z^{-M}D_0(z^{-1})D_1(z) - z^{-N}D_0(z)D_1(z^{-1})}{D_0(z)D_1(z)}.$$

Since P(z) is of degree M+N and $z^{-(M+N)}P(z^{-1}) = z^{-(M+N)}\left(z^{M}D_0(z)D_1(z^{-1}) + z^{N}D_0(z^{-1})D_1(z)\right)$ $= z^{-N}D_0(z)D_1(z^{-1}) + z^{-M}D_0(z^{-1})D_1(z) = P(z)$. Hence P(z) is symmetric. Similarly one can prove that Q(z) is anti-symmetric.

4.78 $H_0(z) = \frac{1}{2}\left[A_0(z) + A_1(z)\right]$, $H_1(z) = \frac{1}{2}\left[A_0(z) - A_1(z)\right]$. Thus, $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1})$

$$= \frac{1}{4}\left[A_0(z) + A_1(z)\right]\left[A_0(z^{-1}) + A_1(z^{-1})\right] + \frac{1}{4}\left[A_0(z) - A_1(z)\right]\left[A_0(z^{-1}) - A_1(z^{-1})\right]$$

$$= \frac{1}{4}\left[A_0(z)A_0(z^{-1}) + A_0(z)A_1(z^{-1}) + A_1(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})\right]$$

$$+ \frac{1}{4}\left[A_0(z)A_0(z^{-1}) - A_0(z)A_1(z^{-1}) - A_1(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})\right]$$

$= \frac{1}{2}\left[A_0(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})\right] = 1$. Thus, $\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1$ implying that $H_0(z)$ and $H_1(z)$ form a power complementary pair.

4.79 $\left|H_0(e^{j\omega})\right|^2 = \frac{1}{4}\left\{A_0(e^{j\omega})A_0^*(e^{j\omega}) + A_1(e^{j\omega})A_0^*(e^{j\omega}) + A_0(e^{j\omega})A_1^*(e^{j\omega}) + A_1(e^{j\omega})A_1^*(e^{j\omega})\right\}$.

Since $A_0(z)$ and $A_1(z)$ are allpass functions, $A_0(e^{j\omega}) = e^{j\phi_0(\omega)}$ and $A_1(e^{j\omega}) = e^{j\phi_1(\omega)}$.

Therefore $\left|H_0(e^{j\omega})\right|^2 = \frac{1}{4}\left\{2 + e^{j(\phi_0(\omega)-\phi_1(\omega))} + e^{-j(\phi_0(\omega)-\phi_1(\omega))}\right\} \le 1$ as maximum values of $e^{j(\phi_0(\omega)-\phi_1(\omega))}$ and $e^{-j(\phi_0(\omega)-\phi_1(\omega))}$ are 1. $H_0(z)$ is stable since $A_0(z)$ and $A_1(z)$ are stable transfer functions. Hence, $H_0(z)$ is BR.

4.80 $H(z) = \frac{1}{M}\sum_{k=0}^{M-1} A_k(z)$. Thus, $H(z)H(z^{-1}) = \frac{1}{M^2}\sum_{r=0}^{M-1}\sum_{k=0}^{M-1} A_k(z)A_r(z^{-1})$. Hence,

$\left|H(e^{j\omega})\right|^2 = \frac{1}{M^2}\sum_{r=0}^{M-1}\sum_{k=0}^{M-1} e^{j(\phi_k(\omega)-\phi_r(\omega))} \le 1$. Again H(z) is stable since $\{A_i(z)\}$ are stable transfer functions. Hence, H(z) is BR.

4.81 $H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}\right)$ and $H_{BS}(z) = \frac{1-\alpha}{2}\frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$.

$$H_{BS}(z) + H_{BP}(z) = \frac{1 - \alpha - z^{-2} + \alpha z^{-2} + 1 - 2\beta z^{-1} + z^{-2} + \alpha - 2\alpha\beta z^{-1} + \alpha z^{-2}}{2(1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2})} = 1.$$

Hence $H_{BP}(z)$ and $H_{BS}(z)$ are allpass complementary.

4.82 (a) $H_1(z) = \frac{1}{1+\alpha}(1+\alpha z^{-1})$.

$\left|H_1(e^{j\omega})\right|^2 = \frac{1}{(1+\alpha)^2}\left\{(1+\alpha\cos\omega)^2 + (\alpha\sin\omega)^2\right\} = \frac{1+\alpha^2+2\alpha\cos\omega}{(1+\alpha)^2}$. Thus,

$\frac{d\left|H_1(e^{j\omega})\right|^2}{d\omega} = \frac{-2\alpha\sin\omega}{(1+\alpha)^2} < 0$, for $\alpha > 0$. The maximum value of $\left|H_1(e^{j\omega})\right| = 1$ at $\omega = 0$, and the minimum value is at $\omega = \pi$. On the other hand, if $\alpha < 0$, then $\frac{d\left|H_1(e^{j\omega})\right|^2}{d\omega} > 0$, In this case the maximum value of $\left|H_1(e^{j\omega})\right| = (1-\alpha)^2/(1+\alpha)^2 > 1$ at $\omega = \pi$, and the minimum value is at $\omega = 0$. Hence, $H_1(z)$ is BR only for $\alpha > 0$.

(b) $H_2(z) = \frac{1}{1+\beta}(1-\beta z^{-1})$. $\left|H_2(e^{j\omega})\right|^2 = \frac{1+\beta^2-2\beta\cos\omega}{(1+\beta)^2}$. Thus,

$\frac{d\left|H_2(e^{j\omega})\right|^2}{d\omega} = \frac{2\beta\sin\omega}{(1+\beta)^2} > 0$, for $\beta > 0$. The maximum value of $\left|H_2(e^{j\omega})\right| = 1$ at $\omega = \pi$, and the minimum value is at $\omega = 0$. On the other hand, if $\beta < 0$, then $\frac{d\left|H_2(e^{j\omega})\right|^2}{d\omega} < 0$, In this case the maximum value of $\left|H_2(e^{j\omega})\right| = (1-\beta)^2/(1+\beta)^2 > 1$ at $\omega = 0$, and the minimum value is at $\omega = \pi$. Hence, $H_2(z)$ is BR only for $\beta > 0$.

(c) $H_3(z) = \frac{(1+\alpha z^{-1})(1-\beta z^{-1})}{(1+\alpha)(1+\beta)}$. From the results of Parts (a) and (b) it follows that $H_3(z)$ is BR only for $\alpha > 0$ and $\beta > 0$.

(d) $H_4(z) = \frac{(1+0.4z^{-1})(1+0.5z^{-1})(1+0.6z^{-1})}{3.36} = \left(\frac{1+0.4z^{-1}}{1.4}\right)\left(\frac{1+0.5z^{-1}}{1.5}\right)\left(\frac{1+0.6z^{-1}}{1.6}\right)$. Since each individual factors on the right-hand side is BR, $H_4(z)$ is BR.

4.83 (a) $H_1(z) = \frac{2+2z^{-1}}{3+z^{-1}} = \frac{1}{2}\left(1+\frac{1+3z^{-1}}{3+z^{-1}}\right) = \frac{1}{2}\left(A_0(z)+A_1(z)\right)$, where $A_0(z) = 1$ and $A_1(z) = \frac{1+3z^{-1}}{3+z^{-1}}$ are stable allpass functions. In view of Problem 4.80, $H_1(z)$ is BR.

(b) $H_2(z) = \frac{1-z^{-1}}{4+2z^{-1}} = \frac{1}{2}\left(1-\frac{2+4z^{-1}}{4+2z^{-1}}\right) = \frac{1}{2}\left(A_0(z)-A_1(z)\right)$, where $A_0(z) = 1$ and $A_1(z) = \frac{2+4z^{-1}}{4+2z^{-1}}$ are stable allpass functions. In view of Problem 4.79, $H_1(z)$ is BR.

(c) $H_3(z)=\dfrac{1-z^{-2}}{4+2z^{-1}+2z^{-2}}=\dfrac{1}{2}\left(1-\dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}\right)$, where $A_1(z)=\dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$ is a stable allpass function. Hence, $H_3(z)$ is BR.

(d) $H_4(z)=\dfrac{3+6z^{-1}+3z^{-2}}{6+5z^{-1}+z^{-2}}=\left[\dfrac{1}{2}\left(1+\dfrac{1+3z^{-1}}{3+z^{-1}}\right)\right]\left[\dfrac{1}{2}\left(1+\dfrac{1+2z^{-1}}{2+z^{-1}}\right)\right]$ which is seen to be a product of two BR functions. Hence, $H_4(z)$ is BR.

(e) $H_5(z)=\dfrac{3+2z^{-1}+3z^{-2}}{4+2z^{-1}+2z^{-2}}=\dfrac{1}{2}\left(1+\dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}\right)=\dfrac{1}{2}\left(1+A_1(z)\right)$ where $A_1(z)=\dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$ is a stable allpass funcion. Hence, $H_5(z)$ is BR.

(f) $H_6(z)=\dfrac{3+9z^{-1}+9z^{-2}+3z^{-3}}{12+10z^{-1}+2z^{-2}}=\left[\dfrac{1}{2}\left(1+\dfrac{1+3z^{-1}}{3+z^{-1}}\right)\right]\left[\dfrac{1}{2}\left(1+\dfrac{1+2z^{-1}}{2+z^{-1}}\right)\right]\left(\dfrac{1+z^{-1}}{2}\right)$ which is seen to be a product of three BR functions. Hence, $H_6(z)$ is BR.

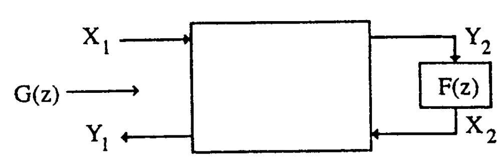
4.84 Since $A_1(z)$ and $A_2(z)$ are LBR, aa $\left|A_1(e^{j\omega})\right|=1$ and $\left|A_2(e^{j\omega})\right|=1$. Thus, $A_1(e^{j\omega})=e^{j\phi_1(\omega)}$, and $A_2(e^{j\omega})=e^{j\phi_2(\omega)}$. Now, $A_1\left(\dfrac{1}{A_2(e^{j\omega})}\right)=A_1\left(e^{-j\phi_2(\omega)}\right)$. Thus, $\left|A_1(e^{-j\phi_2(\omega)})\right|=1$. Hence, $A_1\left(\dfrac{1}{A_2(z)}\right)$ is LBR.

4.85 $F(z)=z\left(\dfrac{G(z)+\alpha}{1+\alpha G(z)}\right)$. Thus, $F(e^{j\omega})=e^{j\omega}\left(\dfrac{G(e^{j\omega})+\alpha}{1+\alpha G(e^{j\omega})}\right)=e^{j\omega}\left(\dfrac{e^{j\phi(\omega)}+\alpha}{1+\alpha e^{j\phi(\omega)}}\right)$ since $G(z)$ is LBR.

$$\left|F(e^{j\omega})\right|^2=\left|\frac{e^{j\phi(\omega)}+\alpha}{1+\alpha e^{j\phi(\omega)}}\right|^2=\frac{(\cos\phi(\omega)+\alpha)^2+(\sin\phi(\omega))^2}{(1+\alpha\cos\phi(\omega))^2+(\alpha\sin\phi(\omega))^2}=\frac{1+2\alpha\cos\phi(\omega)+\alpha^2}{1+2\alpha\cos\phi(\omega)+\alpha^2}=1.$$

Let $z=\lambda$ be a pole of $F(z)$. This implies, $G(\lambda)=-1/\alpha$ or $|G(\lambda)|=|1/\alpha|$. If $|\alpha|<1$, then $|G(\lambda)|>1$, which is satisfied by the LBR $G(z)$ if $|\lambda|<1$. Hence, $F(z)$ is LBR. The order of $F(z)$ is same as that of $G(z)$.

$G(z)$ can be realized in the form of a two-pair constrained by the transfer function $F(z)$.



To this end, we express G(z) in terms of F(z) arriving at $G(z) = \frac{-\alpha + z^{-1}F(z)}{1-\alpha z^{-1}F(z)} = \frac{C + D\,F(z)}{A + B\,F(z)}$,

where A, B, C, and D, are the chain parameters of the two-pair. Comparing the two expressions on the right-hand side we get $A = 1$, $B = -\alpha z^{-1}$, $C = -\alpha$, and $D = z^{-1}$. The corresponding transfer parameters are given by $t_{11} = -\alpha$, $t_{21} = 1$, $t_{12} = (1-\alpha^2)z^{-1}$, and $t_{22} = \alpha z^{-1}$.

4.86 Let $F(z) = G\left(\frac{1}{A(z)}\right)$. Now A(z) being LBR, $A(e^{j\omega}) = e^{j\phi(\omega)}$. Thus,

$F(e^{j\omega}) = G\left(\frac{1}{A(e^{j\omega})}\right) = G(e^{-j\phi(\omega)})$. Since G(z) is a BR function, $\left|G(e^{-j\phi(\omega)})\right| \le 1$. Hence,

$\left|F(e^{j\omega})\right| = \left|G\left(\frac{1}{A(e^{j\omega})}\right)\right| \le 1.$

Let $z = \xi$ be a pole of F(z). Hence, F(z) will be a BR function if $|\xi| < 1$. Let $z = \lambda$ be a pole of G(z). Then this pole is mapped to the location $z = \xi$ of F(z) by the relation

$\left.\frac{1}{A(z)}\right|_{z=\xi} = \lambda$, or $A(\xi) = \frac{1}{\lambda}$. Hence, $|A(\xi)| = \frac{1}{|\lambda|} > 1$ because of Eq. (4.129). This implies, $|\lambda| < 1$. Thus $G\left(\frac{1}{A(z)}\right)$ is a BR function.

4.87 **(a)** $H(z) = \frac{2+2z^{-1}}{3+z^{-1}}$, $G(z) = \frac{1-z^{-1}}{3+z^{-1}}$. Now, $H(z) + G(z) = \frac{2+2z^{-1}}{3+z^{-1}} + \frac{1-z^{-1}}{3+z^{-1}} = \frac{3+z^{-1}}{3+z^{-1}} = 1.$

Next, $H(z)H(z^{-1}) + G(z)G(z^{-1}) = \frac{2+2z^{-1}}{3+z^{-1}}\frac{2+2z}{3+z} + \frac{1-z^{-1}}{3+z^{-1}}\frac{1+z}{3+z} =$

$\frac{4+4z^{-1}+4z+4+1+1-z^{-1}-z}{(3+z^{-1})(3+z)} = 1$. Thus, $\left|H(e^{j\omega})\right|^2 + \left|G(e^{j\omega})\right|^2 = 1$

Hence H(z) and G(z) are both allpass-complementary and power complementary. As a result, they are doubly complementary.

(b) $H(z) = \frac{-1+z^{-2}}{4+2z^{-1}+2z^{-2}}$, $G(z) = \frac{3+2z^{-1}+3z^{-2}}{4+2z^{-1}+2z^{-2}}$. Note $H(z) + G(z) = \frac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$ implying that H(z) and G(z) are allpass complementary. Next, $H(z)H(z^{-1}) + G(z)G(z^{-1})$

$$= \frac{-1+z^{-2}}{4+2z^{-1}+2z^{-2}}\,\frac{-1+z^{2}}{4+2z+2z^{2}} + \frac{3+2z^{-1}+3z^{-2}}{4+2z^{-1}+2z^{-2}}\,\frac{3+2z+3z^{2}}{4+2z+2z^{2}}$$

$$= \frac{1+1-z^{-2}-z^{2}+9+6z+9z^{2}+6z^{-1}+4+6z+9z^{-2}+6z^{-1}+9}{(4+2z^{-1}+2z^{-2})(4+2z+2z^{2})} = 1.$$ Hence, H(z)

and G(z) are also power complementary. As a result, they are doubly-complementary.

4.88 (a) $H_a(z) = \frac{2(1+z^{-1}+z^{-2})}{3+2z^{-1}+z^{-2}} = \frac{1}{2}\left(1+\frac{1+2z^{-1}+3z^{-2}}{3+2z^{-1}+z^{-2}}\right)$. Its power-complementary transfer function therefore is given by $G_a(z) = \frac{1}{2}\left(1-\frac{1+2z^{-1}+3z^{-2}}{3+2z^{-1}+z^{-2}}\right) = \frac{1-z^{-2}}{3+2z^{-1}+z^{-2}}$.

(b) $H_b(z) = \frac{3(1.5+6.5z^{-1}+6.5z^{-2}+1.5z^{-3})}{18+21z^{-1}+8z^{-2}+z^{-3}} = \frac{3(1.5+6.5z^{-1}+6.5z^{-2}+1.5z^{-3})}{(2+z^{-1})(3+z^{-1})(3+z^{-1})}$

$= \frac{1}{2}\left[\frac{(1+2z^{-1})(1+3z^{-1})}{(2+z^{-1})(3+z^{-1})} + \frac{1+3z^{-1}}{3+z^{-1}}\right]$. Its power-complementary transfer function therefore is given by

$$G_b(z) = \frac{1}{2}\left[\frac{(1+2z^{-1})(1+3z^{-1})}{(2+z^{-1})(3+z^{-1})} - \frac{1+3z^{-1}}{3+z^{-1}}\right] = \frac{-1.5-3.5z^{-1}+3.5z^{-2}+1.5z^{-3}}{18+21z^{-1}+8z^{-2}+z^{-3}}.$$

4.89 $\left|G_0(e^{j\omega})\right|^2 + \left|G_1(e^{j\omega})\right|^2 = \frac{1}{4}\left\{\left|A_0(e^{j\omega})+A_1(e^{j\omega})\right|^2 + \left|A_0(e^{j\omega})-A_1(e^{j\omega})\right|^2\right\}$

$= \frac{1}{4}\left\{\left(A_0(e^{j\omega})+A_1(e^{j\omega})\right)\left(A_0^*(e^{j\omega})+A_1^*(e^{j\omega})\right)+\left(A_0(e^{j\omega})-A_1(e^{j\omega})\right)\left(A_0^*(e^{j\omega})-A_1^*(e^{j\omega})\right)\right\}$

$= \frac{1}{4}\{A_0(e^{j\omega})A_0^*(e^{j\omega}) + A_1(e^{j\omega})A_1^*(e^{j\omega}) + A_0(e^{j\omega})A_1^*(e^{j\omega}) + A_1(e^{j\omega})A_0^*(e^{j\omega})$

$+A_0(e^{j\omega})A_0^*(e^{j\omega}) + A_1(e^{j\omega})A_1^*(e^{j\omega}) - A_0(e^{j\omega})A_1^*(e^{j\omega}) - A_1(e^{j\omega})A_0^*(e^{j\omega})\}$

$= \frac{1}{4}\left\{2\left|A_0(e^{j\omega})\right|^2 + 2\left|A_1(e^{j\omega})\right|^2\right\} = 1.$

4.90 $X_1 = AY_2 + BX_2$, $Y_1 = CY_2 + DX_2$. From the first equation, $Y_2 = \frac{1}{A}X_1 - \frac{B}{A}X_2$. Substituting this in the second equation we get $Y_1 = C\left(\frac{1}{A}X_1 - \frac{B}{A}X_2\right) + DX_2 = \frac{C}{A}X_1 + \frac{AD-BC}{A}X_2$, Comparing the last two equations with Eq. (4.146) we arrive at $t_{11} = \frac{C}{A}$, $t_{12} = \frac{AD-BC}{A}$, $t_{21} = \frac{1}{A}$, $t_{22} = -\frac{B}{A}$.

Next, $Y_1 = t_{22}X_1 + t_{12}X_2$, $Y_2 = t_{21}X_1 + t_{22}X_2$. From the second equation we get $X_1 = -\frac{t_{22}}{t_{21}}X_2 + \frac{1}{t_{21}}Y_2$. Substituting this expression in the first equation we get,

$Y_1 = t_{11}\left(-\frac{t_{22}}{t_{21}}X_2 + \frac{1}{t_{21}}Y_2\right) + t_{12}X_2 = \frac{t_{11}}{t_{21}}Y_2 + \frac{t_{12}t_{21}-t_{11}t_{22}}{t_{21}}X_2$. Comparing the last two equations with Eq. (4.149) we arrive at $A = \frac{1}{t_{21}}$, $B = -\frac{t_{22}}{t_{21}}$, $C = \frac{t_{11}}{t_{21}}$, $D = \frac{t_{12}t_{21}-t_{11}t_{22}}{t_{21}}$.

4.91 $\begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix} = \begin{bmatrix} k_1 & (1-k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix}\begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$, $\begin{bmatrix} Y_1'' \\ Y_2'' \end{bmatrix} = \begin{bmatrix} k_2 & (1-k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix}\begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix}$, where $\begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix} = \begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix}$.

Thus, the transfer matrices of the two two-pairs are given by $\boldsymbol{\tau}_1 = \begin{bmatrix} k_1 & (1-k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix}$, and

$\boldsymbol{\tau}_2 = \begin{bmatrix} k_2 & (1-k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix}$. The corresponding chain matrices are obtained using Eq. (4.151a) and are given by $\boldsymbol{\Gamma}_1 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix}$, and $\boldsymbol{\Gamma}_2 = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix}$. Therefore, the chain matrix of the Γ-cascade is given by $\boldsymbol{\Gamma} = {}_1\boldsymbol{\Gamma}_2 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix}\begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1+k_1k_2z^{-1} & k_2z^{-1}+k_1z^{-2} \\ k_1+k_2z^{-1} & k_1k_2z^{-1}+z^{-2} \end{bmatrix}$.

Hence using Eq. (4.151b) we arrive at the transfer matrix of the Γ-cascade as

$$\boldsymbol{\tau} = \begin{bmatrix} \dfrac{k_1+k_2z^{-1}}{1+k_1k_2z^{-1}} & \dfrac{z^{-2}(1-k_1^2)(1-k_1^2)}{1+k_1k_2z^{-1}} \\ \dfrac{1}{1+k_1k_2z^{-1}} & -\dfrac{z^{-1}(k_2+k_1z^{-1})}{1+k_1k_2z^{-1}} \end{bmatrix}.$$

4.92 $\begin{bmatrix} X_1' \\ Y_1' \end{bmatrix} = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix}\begin{bmatrix} Y_2' \\ X_2' \end{bmatrix}$, $\begin{bmatrix} X_1'' \\ Y_1'' \end{bmatrix} = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix}\begin{bmatrix} Y_2'' \\ X_2'' \end{bmatrix}$, where $\begin{bmatrix} Y_2' \\ X_2' \end{bmatrix} = \begin{bmatrix} X_1'' \\ Y_1'' \end{bmatrix}$. Therefore, the chain matrices of the two two-pairs are given by $\boldsymbol{\Gamma}_1 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix}$, and $\boldsymbol{\Gamma}_2 = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix}$. The corresponding transfer matrices are obtained using Eq. (4.151a) and are given by $\boldsymbol{\tau}_1 = \begin{bmatrix} k_1 & (1-k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix}$, and $\boldsymbol{\tau}_2 = \begin{bmatrix} k_2 & (1-k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix}$. The transfer matrix of the τ-cascade is therefore given by $\boldsymbol{\tau} = \boldsymbol{\tau}_2\boldsymbol{\tau}_1 = \begin{bmatrix} k_2 & (1-k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix}\begin{bmatrix} k_1 & (1-k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix}$

$$= \begin{bmatrix} k_1k_2 + z^{-1}(1-k_2^2) & z^{-1}k_2(1-k_1^2) - z^{-2}k_1(1-k_2^2) \\ k_1 - k_2z^{-1} & z^{-1}(1-k_1^2) + k_1k2z^{-2} \end{bmatrix}.$$

Using Eq. (4.151b) we thus arrive at the chain matrix of the τ-cascade:

$$\boldsymbol{\Gamma} = \begin{bmatrix} \dfrac{1}{k_1 - k_2z^{-1}} & \dfrac{-(k_1k_2z^{-1} + 1 - k_1^2)z^{-1}}{k_1 - k_2z^{-1}} \\ \dfrac{k_1k_2 + z^{-1}(1-k_2^2)}{k_1 - k_2z^{-1}} & \dfrac{z^{-2}}{k_1 - k_2z^{-1}} \end{bmatrix}.$$

4.93 For the constrained two pair $H(z) = \dfrac{C + DG(z)}{A + BG(z)}$. Hence, $C = k_m$, $D = z^{-1}$, $A = 1$, $B = k_m z^{-1}$. Substituting these values of the chain parameters in Eq. (4.151a) we get $t_{11} = \dfrac{C}{A} = k_m$, $t_{12} = z^{-1}(1-k_m^2)$, $t_{21} = 1$, $t_{22} = -k_m z^{-1}$.

4.94 Let $D(z) = 1 + d_1z^{-1} + d_2z^{-2} = (1-\lambda_1z^{-1})(1-\lambda_2z^{-1})$. Thus, $d_2 = \lambda_1\lambda_2$ and $d_1 = -(\lambda_1 + \lambda_2)$.

For stability, $|\lambda_i| < 1$, i = 1, 2. As a result, $|d_2| = |\lambda_1 \lambda_2| < 1$.

Case 1: Complex poles: $d_2 > 0$. In this case, $\lambda_2 = \lambda_1^*$. Now,

$\lambda_1, \lambda_2 = \dfrac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2}$. Hence, λ_1 and λ_2 will be complex, if $d_1^2 < 4d_2$. In this case, $\lambda_1 = -\dfrac{d_1}{2} + \dfrac{j}{2}\sqrt{4d_2 - d_1^2}$. Thus, $|\lambda_1|^2 = \dfrac{1}{4}\left(d_1^2 + 4d_2 - d_1^2\right) = d_2 < 1$. Consequently, if the poles are complex and $d_2 < 1$, then they are inside the unit circle.

Case 2: Real poles. In this case we get $-1 < \lambda_i < 1$, i = 1, 2. Since, $|\lambda_i| < 1$, it follows then $|d_1| < |\lambda_1| + |\lambda_2| < 2$. Now, $-1 < \dfrac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} < 1$, or $\pm\sqrt{d_1^2 - 4d_2} < 2 + d_1$.
It is not possible to satisfy the inequality on the right hand side with a minus sign in front of the square root as it would imply then $d_1 < -2$. Therefore,

$$\sqrt{d_1^2 - 4d_2} < 2 + d_1, \text{ or } d_1^2 - 4d_2 < 4 + d_1^2 + 4d_1, \text{ or } -d_1 < 1 + d_2. \qquad (14)$$

Similarly, $\dfrac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} < -1$, or $\pm\sqrt{d_1^2 - 4d_2} > -2 + d_1$. Again it is not possible to satisfy the inequality on the right hand side with a plus sign in front of the square root as it would imply then $d_1 > 2$. Therefore, $-\sqrt{d_1^2 - 4d_2} > -2 + d_1$, or $\sqrt{d_1^2 - 4d_2} < 2 - d_1$, or

$$d_1^2 - 4d_2 < 4 + d_1^2 - 4d_1, \text{ or equivalently, } d_1 < 1 + d_2. \qquad (15)$$

Combining Eqs. (14) and (15) we get $|d_1| < 1 + d_2$.

4.95 **(a)** $|d_1| = 0.92$ and $1 + d_2 = 1.1995$. Since $|d_1| < 1 + d_2$ and $|d_2| < 1$, both roots are inside the unit circle.

(b) $|d_1| = 0.2$ and $1 + d_2 = -0.43$. Since $|d_2| > 1$ and $|d_1| > 1 + d_2$, all roots are not inside the unit circle.

(c) $|d_1| = 1.4562$ and $1 + d_2 = 1.81$. Since $|d_1| < 1 + d_2$ and $|d_2| < 1$, both roots are inside the unit circle.

(d) $|d_1| = 2.1843$ and $1 + d_2 = 1.81$. Since $|d_1| < 1 + d_2$ and $|d_2| < 1$, both roots are inside the unit circle.

4.96 **(a)** $A_3(z) = \dfrac{\frac{1}{12} - \frac{1}{4}z^{-1} - \frac{1}{2}z^{-2} + z^{-3}}{1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{12}z^{-3}}$. Note, $|k_3| = \dfrac{1}{12} < 1$. Using Eq. (4.177) we arrive at

$A_2(z) = \dfrac{-0.2098 - 0.4825z^{-1} + z^{-2}}{1 - 0.4825z^{-1} - 0.2098z^{-2}}$. Here, $|k_2| = 0.2098 < 1$. Continuing this process, we get,

$A_1(z) = \dfrac{-0.6106 + z^{-1}}{1 - 0.6106z^{-1}}$. Finally, $|k_1| = 0.6106 < 1$. Since $|k_i| < 1$, for i = 3, 2, 1, $H_a(z)$ is stable.

(b) $A_3(z) = \dfrac{-\frac{1}{3} + \frac{1}{6}z^{-1} + \frac{13}{6}z^{-2} + z^{-3}}{1 + \frac{13}{6}z^{-1} + \frac{1}{6}z^{-2} - \frac{1}{3}z^{-3}}$. Note, $|k_3| = \frac{1}{3} < 1$. $A_2(z) = \dfrac{1 + 2.5z^{-1} + z^{-2}}{1 + 2.5z^{-1} + z^{-2}}$. Since $|k_2| = 1$, $H_b(z)$ is unstable.

(c) $A_4(z) = \dfrac{\frac{1}{36} + \frac{5}{18}z^{-1} + \frac{18.5}{18}z^{-2} + \frac{5}{3}z^{-3} + z^{-4}}{1 + \frac{5}{3}z^{-1} + \frac{18.5}{18}z^{-2} + \frac{5}{18}z^{-3} + \frac{1}{36}z^{-4}}$. Note, $|k_4| = \frac{1}{36} < 1$.

$A_3(z) = \dfrac{0.2317 + z^{-1} + 1.6602z^{-2} + z^{-3}}{1 + 1.6602z^{-1} + z^{-2} + 0.2317z^{-3}}$. Thus, $|k_3| = 0.2317 < 1$.

$A_2(z) = \dfrac{0.6503 + 1.5096z^{-1} + z^{-2}}{1 + 1.5096z^{-1} + 0.6503z^{-2}}$. Here, $|k_2| = 0.6503 < 1$. Finally, $A_1(z) = \dfrac{0.9147 + z^{-1}}{1 + 0.9147z^{-1}}$. Thus, $|k_1| = 0.9147 < 1$. Since $|k_i| < 1$, for $i = 4, 3, 2, 1$, $H_c(z)$ is stable.

(d) $A_5(z) = \dfrac{\frac{1}{32} + \frac{5}{16}z^{-1} + \frac{5}{4}z^{-2} + \frac{5}{2}z^{-3} + \frac{5}{2}z^{-4} + z^{-5}}{1 + \frac{5}{2}z^{-1} + \frac{5}{2}z^{-2} + \frac{5}{4}z^{-3} + \frac{5}{16}z^{-4} + \frac{1}{32}z^{-5}}$. This implies, $|k_5| = \frac{1}{32} < 1$.

$A_4(z) = \dfrac{0.2346 + 1.173z^{-1} + 2.4633z^{-2} + 2.4927z^{-3} + z^{-4}}{1 + 2.4927z^{-1} + 2.4633z^{-2} + 1.173z^{-3} + 0.2346z^{-4}}$. Thus, $|k_4| = 0.2346 < 1$.

$A_3(z) = \dfrac{0.6225 + 1.9952z^{-1} + 2.3466z^{-2} + z^{-3}}{1 + 2.3466z^{-1} + 1.9952z^{-2} + 0.6225z^{-3}}$. Implying $|k_3| = 0.6225 < 1$.

$A_2(z) = \dfrac{0.8726 + 1.8034z^{-1} + z^{-2}}{1 + 1.8034z^{-1} + 0.8726z^{-2}}$. Hence, $|k_2| = 0.8726 < 1$. Finally, $A_1(z) = \dfrac{0.9630 + z^{-1}}{1 + 0.9630z^{-1}}$. Thus, $|k_1| = 0.9630 < 1$. Since $|k_i| < 1$, for $i = 5, 4, 3, 2, 1$, $H_d(z)$ is stable.

(e) $A_5(z) = \dfrac{\frac{1}{6} + \frac{1}{3}z^{-1} + \frac{1}{2}z^{-2} + \frac{2}{3}z^{-3} + \frac{5}{3}z^{-4} + z^{-5}}{1 + \frac{5}{6}z^{-1} + \frac{2}{3}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{6}z^{-4} + \frac{1}{6}z^{-5}}$. This implies, $|k_5| = \frac{1}{6} < 1$.

$A_4(z) = \dfrac{0.2 + 0.4z^{-1} + 0.6z^{-2} + 0.8z^{-3} + z^{-4}}{1 + 0.8z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4}}$. Thus, $|k_4| = 0.2 < 1$.

$A_3(z) = \dfrac{0.25 + 0.5z^{-1} + 0.75z^{-2} + z^{-3}}{1 + 0.75z^{-1} + 0.5z^{-2} + 0.25z^{-3}}$. Hence, $|k_3| = 0.25 < 1$.

$A_2(z) = \dfrac{0.3333 + 0.6667z^{-1} + z^{-2}}{1 + 0.6667z^{-1} + 0.3333z^{-2}}$. Here, $|k_2| = 0.3333 < 1$. Finally, $A_1(z) = \dfrac{0.5 + z^{-1}}{1 + 0.5z^{-1}}$. As a result, $|k_1| = 0.5 < 1$. Since $|k_i| < 1$, for $i = 5, 4, 3, 2, 1$, $H_e(z)$ is stable.

4.97 **(a)** $A_4(z)=\dfrac{\frac{1}{5}+\frac{2}{5}z^{-1}+\frac{3}{5}z^{-2}+\frac{4}{5}z^{-3}+z^{-4}}{1+\frac{4}{5}z^{-1}+\frac{3}{5}z^{-2}+\frac{2}{5}z^{-3}+\frac{1}{5}z^{-4}}$. Thus, $|k_4|=\frac{1}{5}<1$..

$A_3(z)=\dfrac{0.25+0.5z^{-1}+0.75z^{-2}+z^{-3}}{1+0.75z^{-1}+0.5z^{-2}+0.25z^{-3}}$. Thus, $|k_3|$ = 0.25 < 1. Repeating the p rocess, we arrive at $A_2(z)=\dfrac{0.3333+0.6667z^{-1}+z^{-2}}{1+0.6667z^{-1}+0.3333z^{-2}}$. Thus, $|k_2|$ = 0.3333 < 1. Finally, we get $A_1(z)=\dfrac{0.5+z^{-1}}{1+0.5z^{-1}}$. Thus, $|k_1|$ = 0.5 < 1. Since $|k_i|<1$, for $i = 5, 4, 3, 2, 1$, $D_a(z)$ has all roots inside the unit circle.

(b) $A_3(z)=\dfrac{0.4+0.3z^{-1}+0.2z^{-2}+z^{-3}}{1+0.2z^{-1}+0.3z^{-2}+0.4z^{-3}}$. Thus, $|k_3|=0.4<1$. $A_2(z)=\dfrac{0.2619+0.0952z^{-1}+z^{-2}}{1+0.0952z^{-1}+0.2619z^{-2}}$.

This implies, $|k_2|=0.2619<1$. Next, $A_1(z)=\dfrac{0.0755+z^{-1}}{1+0.0755z^{-1}}$. Hence, $|k_1|=0.0755<1$. Since $|k_i|<1$, for $i = 3, 2, 1$, $D_b(z)$ has all roots inside the unit circle.

4.98 $z=\dfrac{1+s}{1-s}$. Hence, $s=\dfrac{z-1}{z+1}$. Thus, the k-th root in the s-domain is given by

$s_k=\dfrac{(z_k-1)(z_k^*+1)}{|z_k+1|^2}=\dfrac{|z_k|^2-1+z_k-z_k^*}{|z_k+1|^2}$ where z_k is the k-th root in the z-domain. Hence,

$\mathrm{Re}\{s_k\}=\dfrac{|z_k|^2-1}{|z_k+1|^2}$. Since D(z) is a minimum phase polynomial, $|z_k|$ < 1. Therefore, $\mathrm{Re}\{s_k\}$ < 0. Hence B(s) is a strictly Hurwitz polynomial.

4.99 **(a)** $$P_{yy}(\omega)=\left|A(e^{j\omega})\right|^2 P_{xx}(\omega)=\frac{\left|A(e^{j\omega})\right|^2}{1+d\cos(\omega)}=\frac{1}{1+d\cos(\omega)}. \qquad (16)$$

(b) No, The answer does not depend upon the choice of α. However Eq. (16) holds if the filter is stable for which we require $|\alpha|$ < 1.

4.100 **(a)** $\phi_{xy}[\ell]=E\{x[n+\ell]y[n]\}=E\{x[n+\ell]x[n]\}\circledast h[n]=\phi_{xx}[n]\circledast h[n]$. Taking the discrete-time Fourier transform of both sides we get $P_{xy}(\omega)=P_{xx}(\omega)H(e^{j\omega})$. Since $H(e^{j\omega})$ in general is not real, $P_{xy}(\omega)$ in general is not real.

(b) $\phi_{xu}[\ell]=E\{x[n+\ell]u[n]\}=E\{x[n+\ell]h[-n]\circledast y[n]\}=\phi_{xx}[n]\circledast h[n]\circledast h[-n]$. Taking the discrete-time Fourier transform of both sides we get $P_{xu}(\omega)=P_{xx}(\omega)\left|H(e^{j\omega})\right|^2$. Thus, $P_{xu}(\omega)$ is a real function of ω.

M4.1 From Table 3.2, the DTFT of {n h[n]} is $j\frac{dH(e^{j\omega})}{d\omega}$. Hence, the group delay $\tau(\omega)$ using Eq. (4.203) can be computed at a set of N discrete frequency points $\omega_k = 2\pi k/N$, k = 0, 1, . . . , N–1, as follows: $\{\tau(\omega_k)\}$ = Re(DFT{n h[n]}/DFT{h[n]}), where all DFTs are N-points in length with N greater than or equal to length of {h[n]}.

M4.2

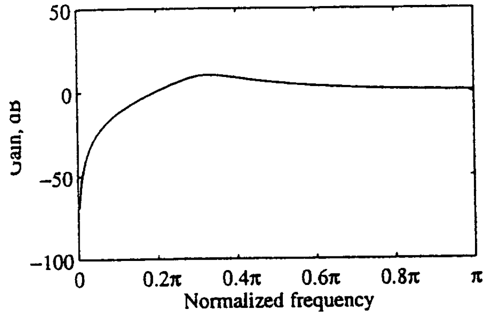
```
L = input('Input sequence length = ');
k = 1:L; y = zeros(1,L);
xa = cos(0.1*k); xb = cos(0.4*k);
x = xa + xb; x0 = 0; x1 = 0;
for n = 1:L;
y(n) = 6.7619*(x(n) + x1) - 12.4563*x0;
x1 = x0; x0 = x(n);
end
subplot(2,2,1);
plot(k-1,xa);axis([0 L-1 -1.2 1.2]);
title('Low frequency component of input');
xlabel('Time index');ylabel('Amplitude');
subplot(2,2,2);
plot(k-1,xb);axis([0 L-1 -1.2 1.2]);
title(High frequency component of input');
xlabel('Time index');ylabel('Amplitude');
subplot(2,2,3);
plot(k-1,x);axis([0 L-1 -1.2 1.2]);
title('Input signal');
xlabel('Time index');ylabel('Amplitude');
subplot(2,2,4);
plot(k-1,y);axis([0 L-1 -1.2 1.2]);
title(Output signal');
xlabel('Time index');ylabel('Amplitude');
```

M4.3

```
L = input('Input sequence length = ');
k = 1:L; y = zeros(1,L);
xa = cos(0.1*k); xb = cos(0.4*k);xc = cos(0.7*k);
x = xa + xb + xc; x0 = 0; x1 = 0; x2 = 0; x3 = 0;
for n = 1:L;
y(n) = -21.6426*(x(n)+x3)+76.1752*(x0+x2)-109.1669*x1;
x3 = x2; x2 = x1; x1 = x0; x0 = x(n);
end
subplot(3,1,1);
plot(k-1,x);axis([0 L-1 -3 3]);
xlabel('Time index');ylabel('Amplitude');
subplot(3,1,2);
plot(k-1,y);axis([0 L-1 -1.2 1.2]);
xlabel('Time index');ylabel('Amplitude');
subplot(3,1,3);
plot(k-1,xb);axis([0 L-1 -1.2 1.2]);
xlabel('Time index');ylabel('Amplitude');
```

M4.4 $H(z) = \frac{2 - 7z^{-1} + 21z^{-2} - 40z^{-3}}{1 - 3z^{-1} + 9z^{-2} - 19z^{-3}}$.

M4.5



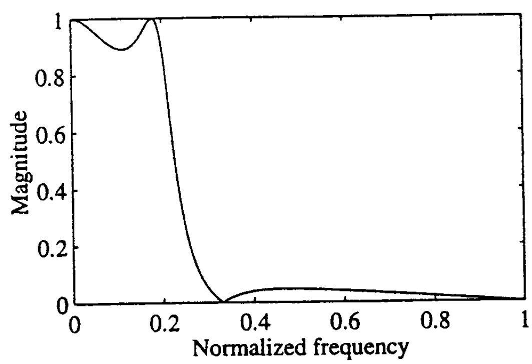
M4.6 (a)

```
num1 = [0  0.75  2   2  1   0.75];
num2 = [0 -0.75  2  -2  1  -0.75];
den = [3  0  3.5  0  1];
w = 0:pi/255:pi;
h1 = freqz(num1, den, w);
h2 = freqz(num2, den, w);
plot(w/pi, abs(h1).*abs(h1)+abs(h2).*abs(h2));
```

(b) Replace the first three lines in the above MATLAB program with the following:

```
num1 = [1   1.5   3.75   2.75   2.75   3.75   1.5   1];
num2 = [1  -1.5   3.75  -2.75   2.75  -3.75   1.5  -1];
den = [6  0  6.5  0  4.75  0  1];
```

M4.7 The magnitude response of H(z) is as shown below from which we observe that H(z) is a lowpass filter.



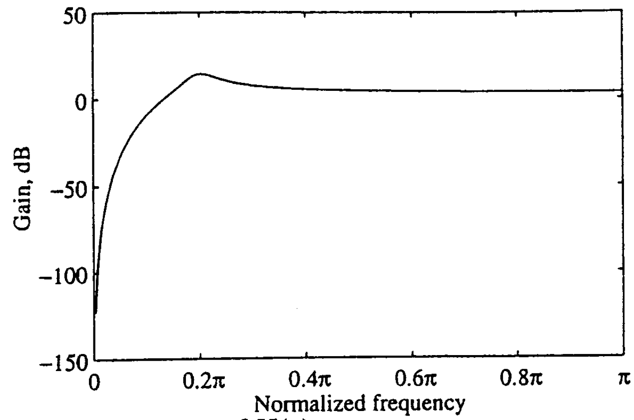
By multiplying out the factors of H(z) we get

$$H(z) = \frac{0.05634 - 0.000935z^{-1} + 0.000935z^{-2} + 0.05634z^{-3}}{1 - 2.1291z^{-1} + 1.783386z^{-2} + 0.543463z^{-3}},$$ The corresponding difference

equation representation is therefore given by

$$y[n] = 0.05634x[n] - 0.000935x[n-1] + 0.000935x[n-2] - 0.05634x[n-3]$$
$$-2.1291y[n-1] + 1.783386y[n-2] + 0.543463y[n-3].$$

M4.8 The magnitude response of H(z) is as shown below from which we observe that H(z) is a highpass filter.



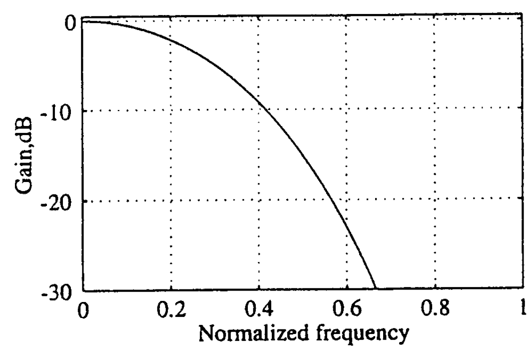
By multiplying the factors of H(z) we get

$$H(z) = \frac{1 - 4z^{-1} + 6z^{-2} - 4z^{-3} + z^{-4}}{1 - 3.0538z^{-1} + 3.8227z^{-2} - 2.2837z^{-3} + 0.5472z^{-4}}$$

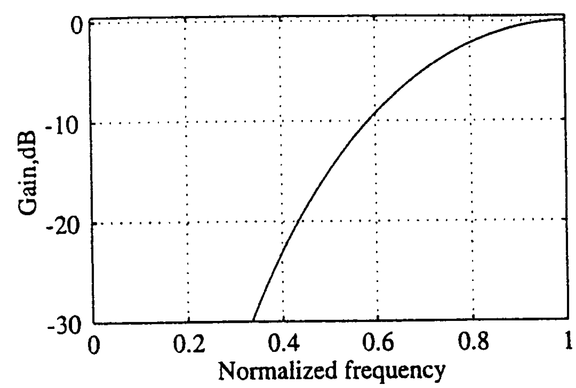
The correspondong difference equation is given by

$$y[n] - 3.0538\,y[n-1] + 3.8227\,y[n-2] - 2.2837\,y[n-3] + 0.5472\,y[n-4]$$
$$= x[n] - 4\,x[n-1] + 6\,x[n-2] - 4\,x[n-3] + x[n-4].$$

M4.9 From Eq. (4.66), we obtain $M = -\frac{1}{2\log_2(\cos(0.12\pi))} = 4.7599$. We choose M = 5. A cascade of 5 first-order lowpass FIR filter has a transfer function given by $G(z) = \frac{1}{32}(1 + z^{-1})^5$, whose gain response is plotted below:

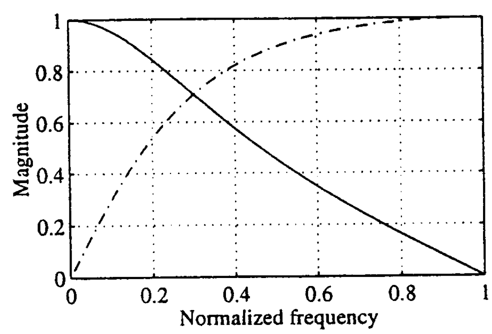


M4.10 For a cascade of M sections of the first-order highpass FIR filter of Eq. (4.67), the 3-dB cutoff frequency wc is given by $\omega_c = 2\sin^{-1}\left(2^{-1/2M}\right)$. Hence, $M = -\frac{1}{2\log_2(\sin(\omega_c/2))}$. Substituting the value of ω_c we then obtain M = 4.76. Choose M = 5. A cascade of 5 first-order highpass FIR filter has a transfer function given by $G(z) = \frac{1}{32}(1 - z^{-1})^5$, whose gain response is plotted below:

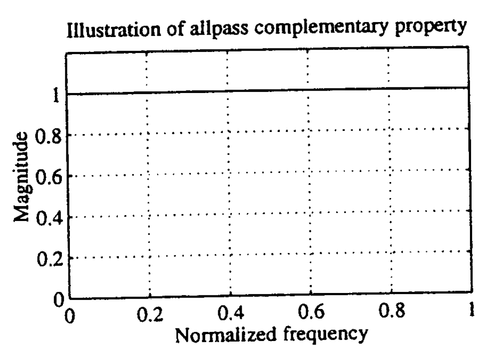


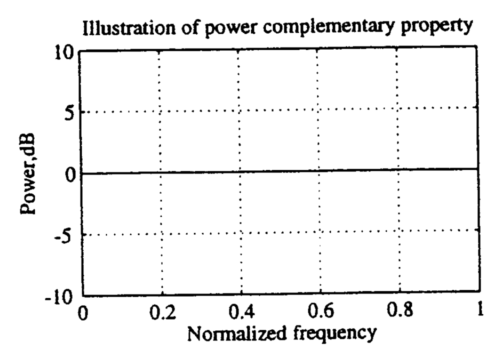
M4.11 From Eq. (4.72b), we obtain $\alpha = 0.32492$. Hence, from Eq. (4.70) we get $H_{LP}(z) = \dfrac{0.33754(1+z^{-1})}{1-0.32492\,z^{-1}}$. Likewise, from Eq. (4.73) we get $H_{HP}(z) = \dfrac{0.66246(1-z^{-1})}{1-0.32492\,z^{-1}}$.

The magnitude responses of $H_{LP}(z)$ and $H_{HP}(z)$ are show below:



The magnitude response of $H_{LP}(z) + H_{HP}(z)$ is shown below on the left-hand side, while the gain response of $|H_{LP}(e^{j\omega})|^2 + |H_{HP}(e^{j\omega})|^2$ is shown below on the right-hand side:

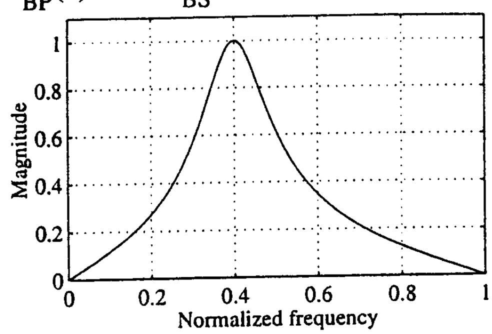


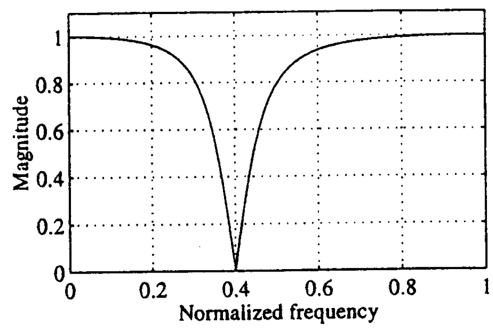


M4.12 From Eq. (4.76) we first obtain $\beta = 0.309017$. From Eq. (4.77) we arrive at two possible values of α: 0.6128 and 1.63185 leading to two posible solutions:

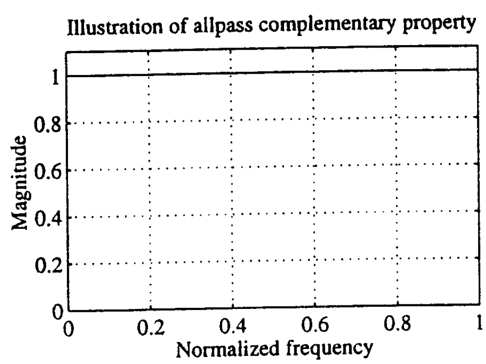
$H'_{BP}(z) = \dfrac{0.1935(1-z^{-2})}{1-0.4984\,z^{-1}+0.6128\,z^{-2}}$, and $H''_{BP}(z) = \dfrac{-0.3159(1-z^{-2})}{1-0.813287\,z^{-1}+1.63185\,z^{-2}}$. It can be seen that $H''_{BP}(z)$ is unstable. From Eq. (4.79), the bandstop transfer function corresponding to

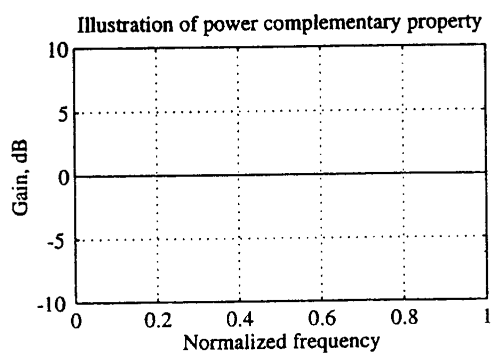
$H'_{BP}(z)$ is given by $H'_{BS}(z) = \dfrac{0.8064(1-0.6180z^{-1}+z^{-2})}{1-0.498383z^{-1}+0.6128z^{-2}}$. A plot of the magnitude responses of $H'_{BP}(z)$ and $H'_{BS}(z)$ are given below:





The magnitude response of $H_{BP}(z) + H_{BS}(z)$ is shown below on the left-hand side, while the gain response of $|H_{BP}(e^{j\omega})|^2 + |H_{BS}(e^{j\omega})|^2$ is shown below on the right-hand side:

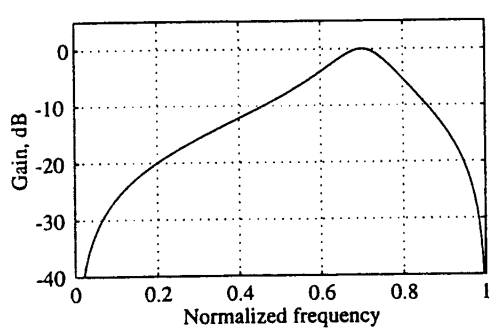




M4.13 From Eq. (4.76), we first obtain $\beta = 0.309017$. From Eq. (4.77) we arrive at two possible values of α: 1.376381 and 0.72654253 leading to two posible solutions:

$$H'_{BP}(z) = \frac{0.1935996(1-z^{-2})}{1+0.94798z^{-1}+0.6128z^{-2}}, \text{ and } H''_{BP}(z) = \frac{-0.3159258(1-z^{-2})}{1+1.5469636z^{-1}+1.63185168z^{-2}}.$$

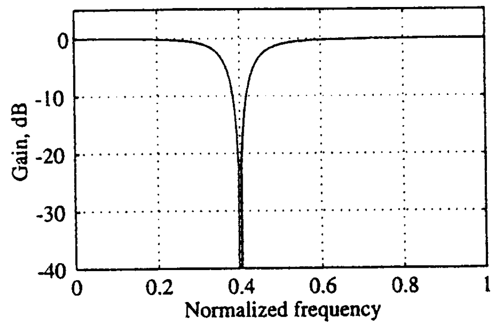
It can be seen that $H''_{BP}(z)$ is unstable. A plot of the gain response of $H'_{BP}(z)$ is shown below:



M4.14 From Eq. (4.76), we first obtain $\beta = 0.309016$. From Eq. (4.77) we arrive at two possible values of α: 0.72654253 and 1.376382 leading to two posible solutions:

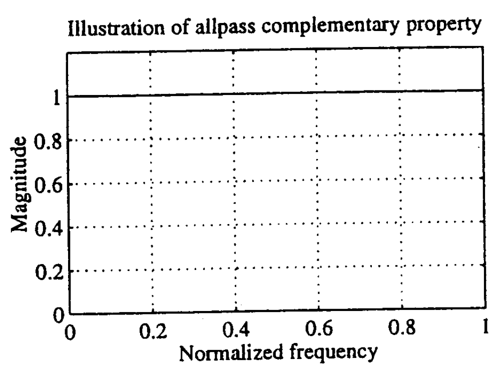
$H'_{BS}(z) = \dfrac{0.86327126(1+0.618033z^{-1}+z^{-2})}{1-0.533531z^{-1}+0.72654253z^{-2}}$, and $H''_{BS}(z) = \dfrac{1.188191(1+0.618033z^{-1}+z^{-2})}{1-0.7343424z^{-1}+1.376382z^{-2}}$.

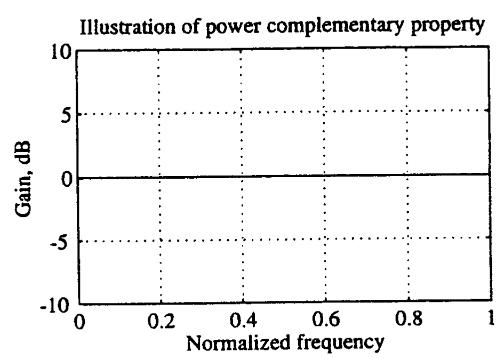
It can be seen that $H''_{BS}(z)$ is unstable. A plot of the gain response of $H''_{BS}(z)$ is shown below:



M4.15 **(a)** Using the following program we arrive at the two plots shown below:

```
b1 = [2 2]; b2 = [1 -1]; den = [3 1];
w = 0:pi/255:pi;
h1 = freqz(b1, den, w); h2 = freqz(b2, den, w);
sum = abs(h1).*abs(h1)+abs(h2).*abs(h2);
subplot(2,1,1);
plot(w/pi,abs(h1+h2));grid
axis([0 1 0 1.2]);
xlabel('Normalized frequency');ylabel('Magnitude');
title('Illustration of allpass complementary property');
subplot(2,1,2);
plot(w/pi,20*log10(sum));grid
axis([0 1 -10 10]);
xlabel('Normalized frequency');ylabel('Gain, dB');
title('Illustration of power complementary property');
```

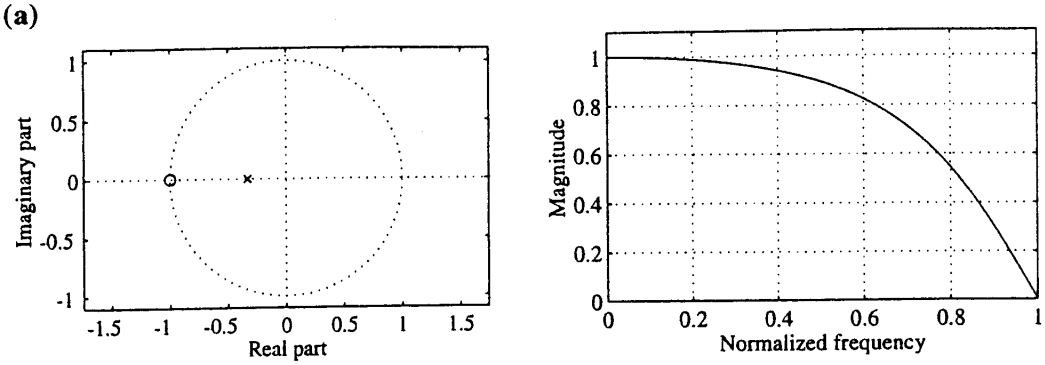




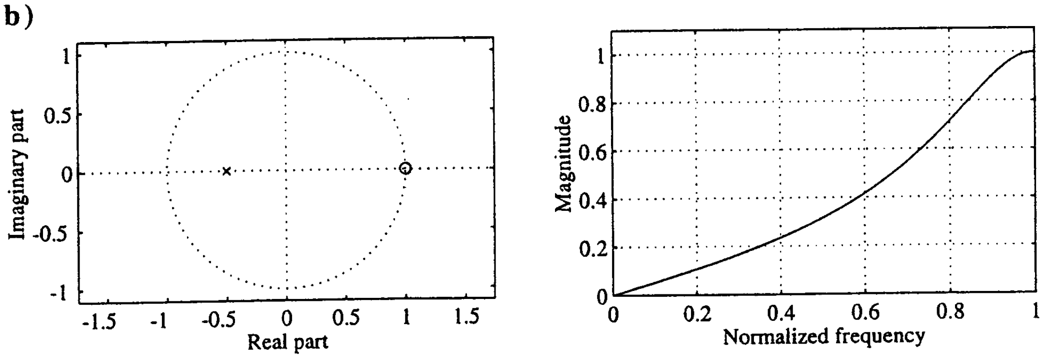
(b) For this part, replace the first line in the above program with the following:

```
b1 = [-1 0 1]; b2 = [3 2 3]; den = [4 2 2];
```

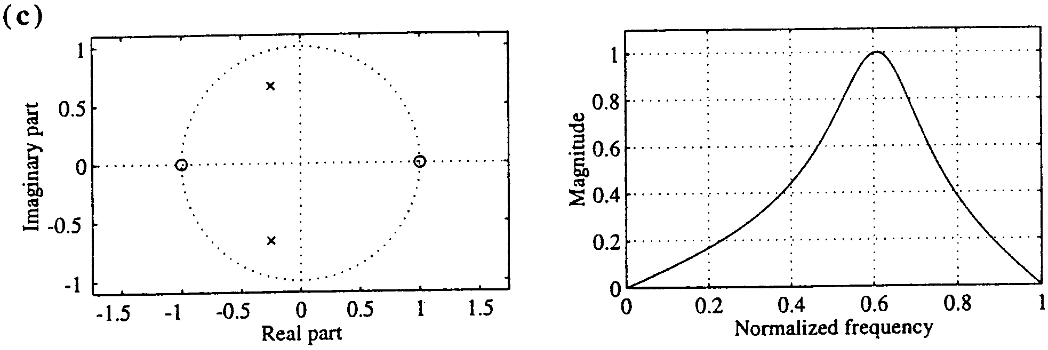
M4.16 **(a)**


Imaginary part
Real part
1
0.5
0
-0.5
-1
-1.5 -1 -0.5 0 0.5 1 1.5
Magnitude
Normalized frequency
1
0.8
0.6
0.4
0.2
0
0 0.2 0.4 0.6 0.8 1

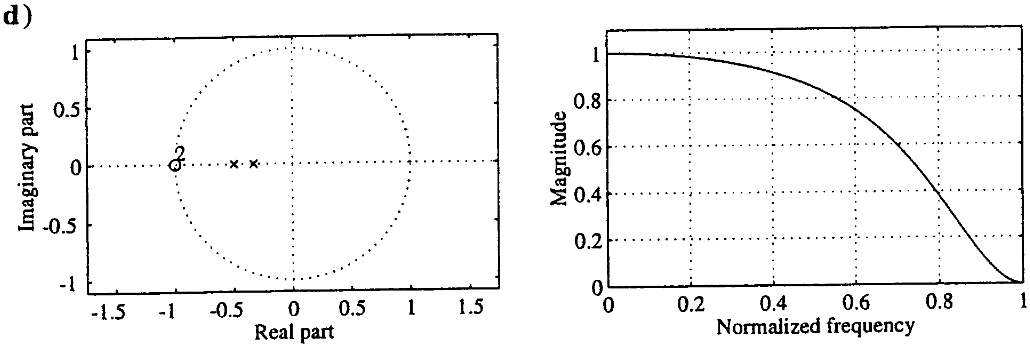
(b)


Imaginary part
Real part
1
0.5
0
-0.5
-1
-1.5 -1 -0.5 0 0.5 1 1.5
Magnitude
Normalized frequency
1
0.8
0.6
0.4
0.2
0
0 0.2 0.4 0.6 0.8 1

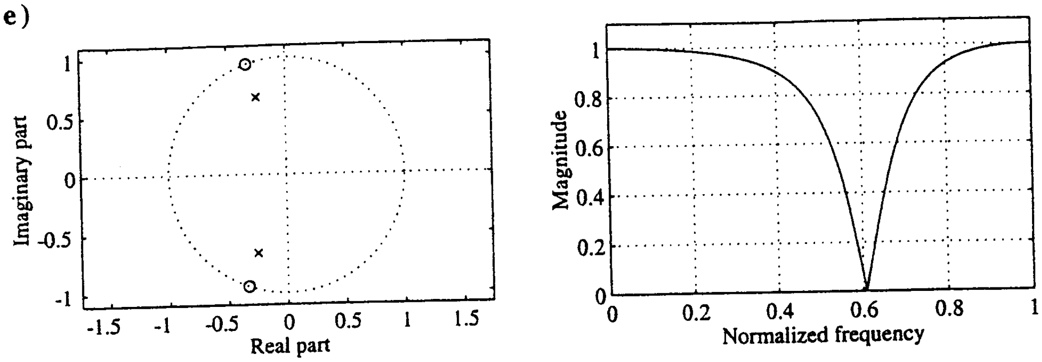
(c)


Imaginary part
Real part
1
0.5
0
-0.5
-1
-1.5 -1 -0.5 0 0.5 1 1.5
Magnitude
Normalized frequency
1
0.8
0.6
0.4
0.2
0
0 0.2 0.4 0.6 0.8 1

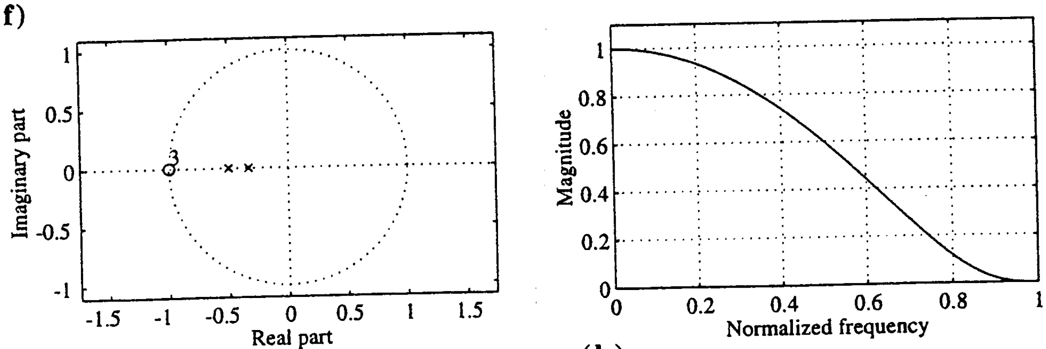
(d)


Imaginary part
Real part
2
1
0.5
0
-0.5
-1
-1.5 -1 -0.5 0 0.5 1 1.5
Magnitude
Normalized frequency
1
0.8
0.6
0.4
0.2
0
0 0.2 0.4 0.6 0.8 1

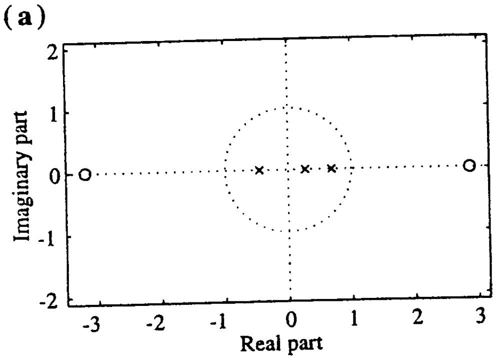
(e)


Imaginary part
Real part
Magnitude
Normalized frequency

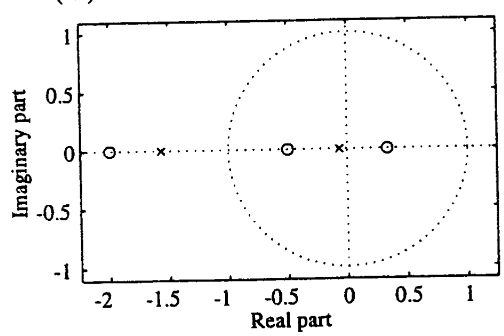
(f)


3
Imaginary part
Real part
Magnitude
Normalized frequency

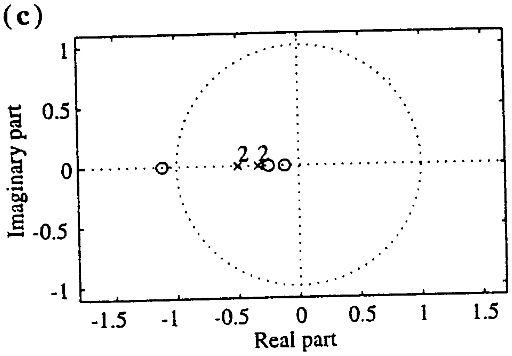
M4.17 (a)


Imaginary part
Real part

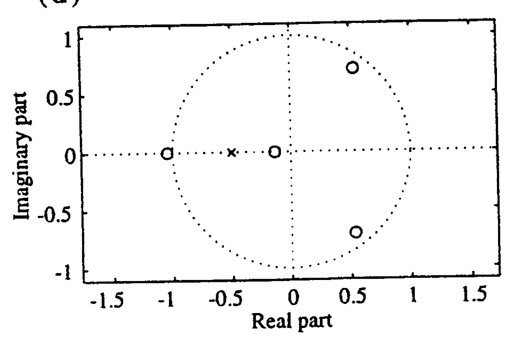
(b)


Imaginary part
Real part

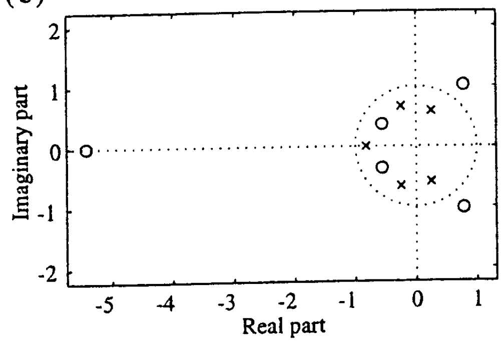
(c)


2 2
Imaginary part
Real part

(d)


Imaginary part
Real part

(e)



M4.18 (a)
```
The stability test parameters are
0.0833    -0.2098    -0.6106
stable = 1
```

(b)
```
The stability test parameters are
-0.3333    1.0000    2.0000
stable = 0
```

(c)
```
The stability test parameters are
0.0278    0.2317    0.6503    0.9147
stable = 1
```

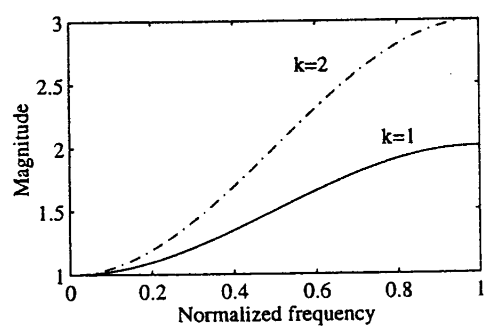
(d)
```
The stability test parameters are
0.0312    0.2346    0.6225    0.8726    0.9630
stable = 1
```

(e)
```
The stability test parameters are
0.1667    0.0286    0.3766    0.3208    0.5277
stable =  1
```

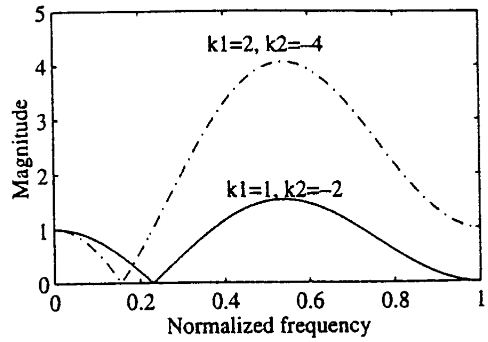
M4.19 (a)
```
The stability test parameters are
0.2000    0.2500    0.3333    0.5000
stable =  1
```

(b)
```
The stability test parameters are
0.4000    0.2619    0.0755
stable = 1
```

M4.20 $H(z) = -\frac{k}{4} + \left(1 + \frac{k}{2}\right)z^{-1} - \frac{k}{4}z^{-2}.$



M4.21 $H(z) = -\frac{k_2}{4} + \frac{k_1}{4}z^{-1} + \left(1 - \frac{k_1}{2} + \frac{k_2}{2}\right)z^{-2} + \frac{k_1}{4}z^{-3} - \frac{k_2}{4}z^{-4}.$



August 16, 1997

Chapter 5

5.1 $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$. Since p(t) is periodic function of time t with a period T, it can be represented as a Fouruer series: $p(t) = \sum_{n=-\infty}^{\infty} c_n e^{j(2\pi nt/T)}$, where $c_n = \int_{-T/2}^{T/2} \delta(t) e^{-j(2\pi nt/T)} dt = \frac{1}{T}$.

Hence $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(2\pi nt/T)}$.

5.2 Since the signal $x_a(t)$ is being sampled at 2 kHz rate, there will be multiple copies of the spectrum at frequencies given by $F_i \pm 2000k$, where F_i is the frequency of the i-th sinusoidal component in $x_a(t)$. Hence,

$F_1 = 300$ Hz,	$F_{1m} = 300, 1700, 2300, \ldots,$ Hz
$F_2 = 400$ Hz,	$F_{2m} = 400, 1600, 2400, \ldots,$ Hz
$F_3 = 1300$ Hz,	$F_{3m} = 700, 1300, 3300, \ldots,$ Hz
$F_4 = 3600$ Hz,	$F_{4m} = 400, 1600, 3600, \ldots,$ Hz
$F_5 = 4300$ Hz,	$F_{5m} = 300, 2300, 4300, \ldots,$ Hz

So after filtering by a lowpass filter with a cutoff at 900 Hz, the frequencies of the sinusoidal components present in $y_a(t)$ are 300, 400, 700 Hz.

5.3 One possible set of values are $F_1 = 350$ Hz, $F_2 = 425$ Hz, $F_3 = 918$ Hz and $F_4 = 3350$ Hz. Another possible set of values are $F_1 = 350$ Hz, $F_2 = 425$ Hz , $F_3 = 918$ Hz, $F_4 = 3425$ Hz. Hence the solution is not unique.

5.4 $t = nT = \frac{n}{3000}$. Therefore,

$$x[n] = 2\cos\left(\frac{600\pi n}{3000}\right) + 4\sin\left(\frac{1400\pi n}{3000}\right) + 3\cos\left(\frac{4400\pi n}{3000}\right) + 7\sin\left(\frac{5400\pi n}{3000}\right)$$

$$= 2\cos\left(\frac{\pi n}{5}\right) + 4\sin\left(\frac{7\pi n}{15}\right) + 3\cos\left(\frac{22\pi n}{15}\right) + 7\sin\left(\frac{27\pi n}{15}\right)$$

$$= 2\cos\left(\frac{\pi n}{5}\right) + 4\sin\left(\frac{7\pi n}{15}\right) + 3\cos\left(\frac{(30-8)\pi n}{15}\right) + 7\sin\left(\frac{(30-3)\pi n}{15}\right)$$

$$= 2\cos\left(\frac{\pi n}{5}\right) + 4\sin\left(\frac{7\pi n}{15}\right) + 3\cos\left(\frac{8\pi n}{15}\right) + 7\sin\left(\frac{\pi n}{5}\right).$$

5.5 $g_a(t) = \cos(10\pi t) + \cos(20\pi t) + \cos(40\pi t)$.

(a) $F_T = 45$ Hz. Hence, $g[n] = \cos\left(\frac{2\pi n}{9}\right) + \cos\left(\frac{4\pi n}{9}\right) + \cos\left(\frac{8\pi n}{9}\right)$.

(b) $F_T = 40$ Hz. Hence, $g[n] = \cos\left(\frac{\pi n}{4}\right) + \cos\left(\frac{\pi n}{2}\right) + \cos(\pi n)$.

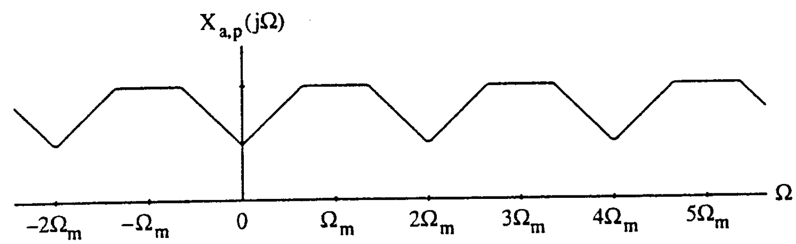
(c) $F_T = 25$ Hz. Hence, $g[n] = \cos\left(\frac{2\pi n}{5}\right) + \cos\left(\frac{4\pi n}{5}\right) + \cos\left(\frac{8\pi n}{5}\right)$.

5.6 Both the channels are sampled at 44.1 kHz. Therefore there are a total of $2\times 44100 = 88200$ samples. Each sample is quantized using 16 bits. Hence the total bit rate of the two channels after sampling and digitization $= 16\times 88200 = 1.412$ Mb/s.

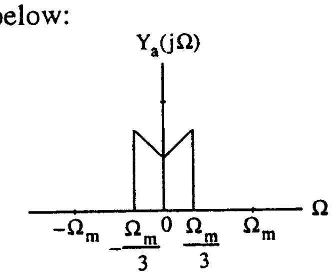
5.7 $h_r(t) = \frac{\sin(\Omega_c t)}{(\Omega_T t/2)}$. Thus, $h_r(nT) = \frac{\sin(\Omega_c nT)}{(\Omega_T nT/2)}$, Since $T = \frac{2\pi}{\Omega_T}$, hence $h_r(nT) = \frac{\sin\left(\frac{2\pi\Omega_c n}{\Omega_T}\right)}{\pi n}$.

If $\Omega_c = \Omega_T/2$, then, $h_r(nT) = \frac{\sin(\pi n)}{\pi n} = \delta[n]$.

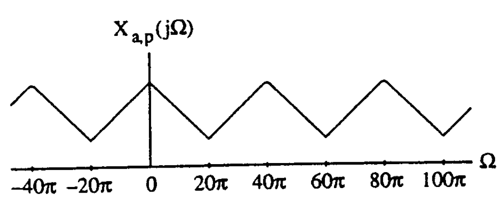
5.8 After sampling the spectrum of the signal is as indicated below:



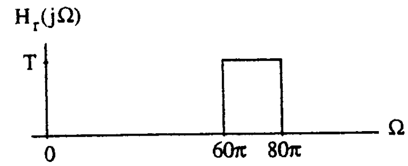
$T = \frac{2\pi}{2\Omega_m} = \frac{\pi}{\Omega_m}$. As a result, $\omega_c = \frac{\Omega_m \pi}{3\Omega_m} = \frac{\pi}{3}$. Hence after the low pass filtering the spectrum of the signal will be as shown below:



5.9 The smallest frquency at which we need to sample $x_a(t)$ is 20 Hz. The frequency response of the sampled version is then given by



Hence for reconstruction we require an ideal bandpass filter with passband from 30 Hz to 40 Hz.



5.10 $A_p = -20\log(1-\delta_p)$ and $A_s = -20\log(\delta_s)$, Therefore $1-\delta_p = 10^{-A_p/20}$ hence $\delta_p = 1-10^{-A_p/20}$ and $\delta_s = 1-10^{-A_s/20}$.

(a) $\delta_p = 0.0057$ and $\delta_s = 5.6234\times10^{-4}$. (b) $\delta_p = 0.0023$ and $\delta_s = 0.0025$.

(c) $\delta_p = 0.0559$ and $\delta_s = 8.9125\times10^{-5}$.

5.11 $H_1(s) = \frac{a}{s+a}$. Thus, $H_1(j\Omega) = \frac{a}{j\Omega+a}$, and hence, i.e. $|H_1(j\Omega)|^2 = \frac{a^2}{a^2+\Omega^2}$.

$|H_1(j\Omega)|^2$ is a monotonically decreasing function with $|H_1(j0)| = 1$ and $|H_1(j\infty)| = 0$. Let the 3-dB cutoff frequency be given by Ω_c. Then, $|H_1(j\Omega_c)|^2 = \frac{a^2}{a^2+\Omega_c^2} = \frac{1}{2}$, and hence, $\Omega_c = a$..

5.12 $H_2(s) = \frac{s}{s+a}$. Thus, $H_2(j\Omega) = \frac{j\Omega}{j\Omega+a}$, and hence, $|H_2(j\Omega)|^2 = \frac{\Omega^2}{a^2+\Omega^2}$.

$|H_2(j\Omega)|^2$ is a monotonically increasing function with $|H_2(j0)| = 0$ and $|H_2(j\infty)| = 1$. The 3-dB cutoff frequency is given by $\frac{\Omega_c^2}{a^2+\Omega_c^2} = \frac{1}{2}$, and hence, $\Omega_c = a$..

5.13 $H_1(s) = \frac{a}{s+a} = \frac{1}{2}\left(1-\frac{s-a}{s+a}\right)$, and $H_2(s) = \frac{s}{s+a} = \frac{1}{2}\left(1+\frac{s-a}{s+a}\right)$. Thus, $A_1(s) = 1$ and $A_2(s) = \frac{s-a}{s+a}$. Since $|A_1(j\Omega)| = 1$ and $|A_2(j\Omega)| = 1$ $\forall\Omega$ hence $A_1(s)$ and $A_2(s)$ are both allpass functions.

5.14 $H_1(s) = \frac{bs}{s^2+bs+\Omega_0^2}$. Thus, $H_1(j\Omega) = \frac{jb\Omega}{jb\Omega+\Omega_0^2-\Omega^2}$, hence $|H_1(j\Omega)|^2 = \frac{b^2\Omega^2}{b^2\Omega^2+(\Omega_0^2-\Omega^2)^2}$.

Now at $\Omega = 0$, $|H_1(j0)| = 0$ and at $\Omega = \infty$, $|H_1(j\infty)| = 0$ and at $\Omega = \Omega_o$, $|H_1(j\Omega_o)| = 1$. Hence $H_1(s)$ has a bandpass response. The 3-dB frequencies are given by $\frac{b^2\Omega_c^2}{b^2\Omega_c^2+(\Omega_0^2-\Omega^2)^2} = \frac{1}{2}$.

Thus, $(\Omega_o^2-\Omega_c^2)^2 = b^2\Omega_c^2$ or $\Omega_c^4-(b^2+2\Omega_o^2)\Omega_c^2+\Omega_o^4 = 0$. Hence if Ω_1 and Ω_2 are the roots of this equation, then so are $-\Omega_1$, $-\Omega_2$, and the product of the roots is Ω_o^4. This implies $\Omega_1\Omega_2 = \Omega_o^2$. Also $\Omega_1^2+\Omega_2^2 = b^2+2\Omega_o^2$. Hence $(\Omega_2-\Omega_1)^2 = b^2$ which gives the desired result $\Omega_2-\Omega_1 = b$.

5.15 $H_2(s) = \dfrac{s^2+\Omega_o^2}{s^2+bs+\Omega_o^2}$. Thus, $H_2(j\Omega) = \dfrac{\Omega_o^2-\Omega^2}{\Omega_o^2-\Omega^2+jb\Omega}$, hence, $|H_2(j\Omega)|^2 = \dfrac{(\Omega^2-\Omega_o^2)^2}{(\Omega^2-\Omega_o^2)^2+b^2\Omega^2}$.

Now, $|H_2(j0)| = 1$, $|H_2(j\infty)| = 1$ and $|H_2(j\Omega_o)| = 1$. Hence, $H_2(s)$ has a bandstop response. As in the earlier problem one can show that $\Omega_1\Omega_2 = \Omega_o^2$ and also $\Omega_2 - \Omega_1 = b$.

5.16 $H_1(s) = \dfrac{1}{2}\left(1 - \dfrac{s^2-bs+\Omega_o^2}{s^2+bs+\Omega_o^2}\right) = \dfrac{1}{2}\{A_1(s) - A_2(s)\}$ and $H_2(s) = \dfrac{1}{2}\left(1 + \dfrac{s^2-bs+\Omega_o^2}{s^2+bs+\Omega_o^2}\right) =$

$\dfrac{1}{2}\{A_1(s) + A_2(s)\}$. Thus, $A_1(s) = 1$ and $A_2(s) = \dfrac{s^2-bs+\Omega_o^2}{s^2+bs+\Omega_o^2}$. Now

$|A_2(j\Omega)|^2 = \dfrac{(\Omega_o^2-\Omega^2)^2+b^2\Omega^2}{(\Omega_o^2-\Omega^2)^2+b^2\Omega^2} = 1$, and is hence $A_2(s)$ is an allpass function. $A_1(s)$ is seen to be an allpass function.

5.17 $|H_a(j\Omega)|^2 = \dfrac{1}{1+(\Omega/\Omega_c)^{2N}}$. $\dfrac{d^k(1/|H_a(j\Omega)|^2)}{d\Omega^k} = 2N(2N-1)\cdots(2N-k+1)\dfrac{\Omega^{2N-k}}{\Omega_c^{2N}}$. Therefore

$\left.\dfrac{d^k(1/|H_a(j\Omega)|^2)}{d\Omega^k}\right|_{\Omega=0} = 0$ for k = 1, 2, . . . , 2N–1. or, equivalently, $\left.\dfrac{d^k(|H_a(j\Omega)|^2)}{d\Omega^k}\right|_{\Omega=0} = 0$ for k = 1, 2, . . . , 2N–1.

5.18 $10\log_{10}\left(\dfrac{1}{1+\varepsilon^2}\right) = -0.5$,, which yields $\varepsilon = 0.3493$. $10\log_{10}\left(\dfrac{1}{A^2}\right) = -60$, which yields A = 1000. Now, $\dfrac{1}{k} = \dfrac{\Omega_s}{\Omega_p} = \dfrac{6.1}{1.5} = 4.0667$ and $\dfrac{1}{k_1} = \dfrac{\sqrt{A^2-1}}{\varepsilon} = 2862.9$. Then, from Eq. (5.31) we get $N = \dfrac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \dfrac{3.4568}{0.6092} = 5.6743$. Hence choose N = 6 as the order.

5.19 The poles are given by $p_\ell = \Omega_c e^{j\frac{\pi(N+2\ell-1)}{2N}}, \ell = 1, 2, \ldots, N$. Here, N = 6 and $\Omega_c = 1$. Hence $p_1 = e^{j7\pi/12} = -0.2588 + j0.9659$, $p_2 = e^{j9\pi/12} = -0.7071 + j0.7071$, $p_3 = e^{j11\pi/12} = -0.9659 + j0.2588$, $p_4 = e^{j13\pi/12} = p_3^*$, $p_5 = e^{j15\pi/12} = p_2^*$, $p_6 = e^{j17\pi/12} = p_1^*$.

5.20 $10\log_{10}\left(\dfrac{1}{1+\varepsilon^2}\right) = -0.5$ which yields $\varepsilon = 0.3493$. Also $A = 10^3$. Hence from Eq. (5.39),

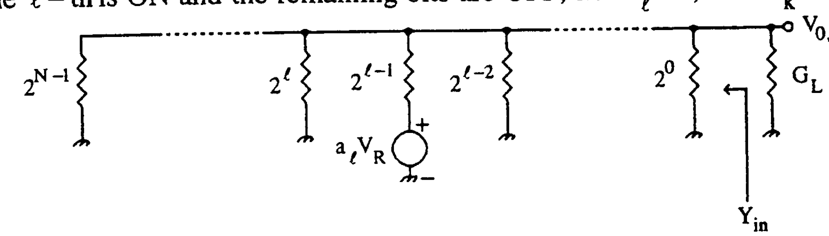
$N = \dfrac{\cosh^{-1}(10^3/0.3493)}{\cosh^{-1}(6.1/1.5)} = 4.159$. We thus choose N = 5 as the order.

5.21 From Problem 5.18 solution, we observe $\frac{1}{k} = 4.0667$ or $k = 0.2459$, and $\frac{1}{k_1} = 2862.9$ or $k_1 = 0.0003493$. Substituting the value of k in Eq. (5.50a) we get $k' = 0.9693$. Then, from Eq. (5.50b) we arrive at $\rho_o = 0.0039$. Substituting the value of in Eq. (5.50c) we get $\rho = 0.0039$. Finally from Eq. (5.49) we arrive at $N = 3.3696$. We choose $N = 4$.

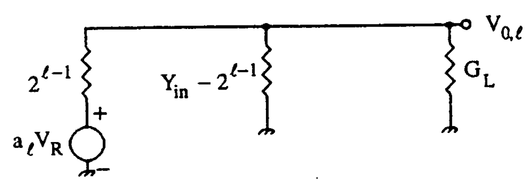
5.22 From Eq. (5.53) $B_N(s) = (2N-1)B_{N-1}(s) + s^2 B_{N-2}(s)$, where $B_1(s) = s + 1$ and $B_2(s) = s^2 + 3s + 3$. Thus, $B_3(s) = 5B_2(s) + s^2 B_1(s) = 5(s^2 + 3s + 3) + s^2(s+1) = s^3 + 6s^2 + 15s + 15$, and
$B_4(s) = 7B_3(s) + s^2 B_2(s) = 7(s^3 + 6s^2 + 15s + 15) + s^2(s^2 + 3s + 3) = s^4 + 10s^3 + 45s^2 + 105s + 105.$

5.23 From Eq. (5.59), the difference in dB in the attenuation levels at Ω_p and Ω_o is given by $20N\log_{10}\left(\frac{\Omega_o}{\Omega_p}\right)$. Hence for $\Omega_o = 2\Omega_p$, attenuation difference in dB is equal to $20N\log_{10} 2 = 6.0206\ N$. Likewise, for $\Omega_o = 3\Omega_p$, attenuation difference in dB is equal to $20N\log_{10} 3 = 9.5424\ N$. Finally, for $\Omega_o = 4\Omega_p$, attenuation difference in dB is equal to $20N\log_{10} 4 = 12.0412\ N$.

5.24 The equivalent representation of the D/A converter of Figure 5.39 reduces to the circuit shown below if the ℓ-th is ON and the remaining bits are OFF, i.e. $a_\ell = 1$, and $a_k = 0$, $k \neq \ell$.



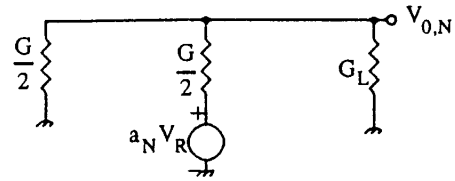
In the above circuit, Y_{in} is the total conductance seen by the load conductance G_L which is given by $Y_{in} = \sum_{i=0}^{N-1} 2^i = 2^N - 1$. The above circuit can be redrawn as indicated below:



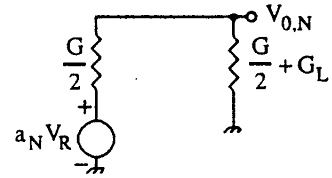
Using the voltage divider relation we then get $V_{0,\ell} = \frac{2^{\ell-1}}{Y_i + G_L} a_\ell V_R$. Using the superposition theorem, the general expression for the output voltage V_0 is thus given by

$$V_0 = \sum_{\ell=1}^{N} \frac{2^{\ell-1}}{Y_i + G_L} a_\ell V_R = \sum_{\ell=1}^{N} 2^{\ell-1} a_\ell \left(\frac{R_L}{1 + (2^N - 1)R_L} \right) V_R.$$

5.25 The equivalent representation of the D/A converter of Figure 5.40 reduces to the circuit shown below if the N-th bit is ON and the remaining bits are OFF, i.e. $a_N = 1$, and $a_k = 0$, $k \neq N$.

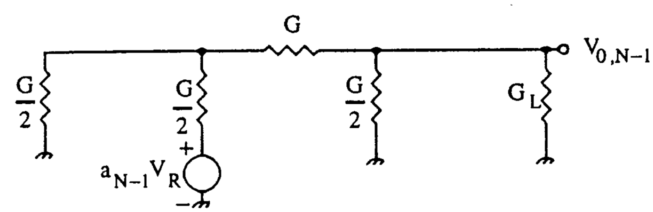


which simplifies to the one shown below

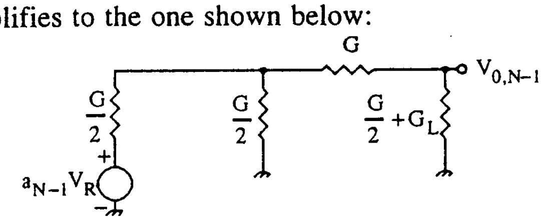


Using the voltage-divider relation we then get $V_{0,N} = \dfrac{\frac{G}{2}}{\frac{G}{2}+G_L+\frac{G}{2}} a_N V_R = \dfrac{R_L}{2(R+R_L)} a_N V_R.$

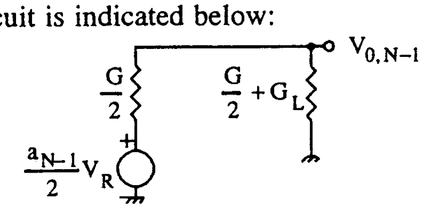
The equivalent representation of the D/A converter of Figure 5.40 reduces to the circuit shown below if the (N–1)-th bit is ON and the remaining bits are OFF, i.e. $a_{N-1} = 1$, and $a_k = 0$, $k \neq N-1$.



The above circuit simplifies to the one shown below:



Its Thevenin equivalent circuit is indicated below:



from which we readily obtain

$$V_{0,N-1} = \frac{\frac{G}{2}}{G+G_L}\,\frac{a_{N-1}}{2} V_R = \frac{R_L}{2(R_L+R)}\,\frac{a_{N-1}}{2} V_R.$$

Following the same procedure we can show that if the ℓ-th bit is ON and the remaining bits are OFF, i.e. $a_\ell = 1$, and $a_k = 0$, $k \neq \ell$, then

$$V_{0,\ell} = \frac{R_L}{2(R_L+R)}\,\frac{a_\ell}{2^{N-\ell}} V_R.$$

Hence, in general we have

$$V_0 = \sum_{\ell=1}^{N} \frac{R_L}{2(R_L + R)} \frac{a_\ell}{2^{N-\ell}} V_R.$$

M5.1 Using the MATLAB statement

```
[N, Wn] = buttord(2*pi*1500, 2*pi*6100, 0.5, 60,'s')
```

we arrive at N = 6 and the 3-dB passband edge frequency Wn = 12120 rad/sec.

M5.2 Using the MATLAB statement

```
[N, Wn] = cheb1ord(2*pi*1500, 2*pi*6100, 0.5, 60,'s')
```

we arrive at N = 5 and the passband edge frequency Wn = 9424 rad/sec.

M5.3 Using the MATLAB statement

```
[N, Wn] = ellipord(2*pi*1500, 2*pi*6100, 0.5, 60,'s')
```

we arrive at N = 4 and the passband edge frequency Wn = 9424 rad/sec.

M5.4 Program 5_2 was modified as indicated below:

```
[N, Wn] = buttord(2*pi*1500, 2*pi*6100, 0.5, 60,'s')
[num,den] = butter(N,2*pi*Wn,'s')
omega = [0: 200: 16000*pi];
h = freqs(num,den,omega);
gain = 20*log10(abs(h));
plot (omega/(2*pi),gain);axis([0 8000 -80 5]);grid
xlabel('Frequency, Hz'); ylabel('Gain, dB')
```

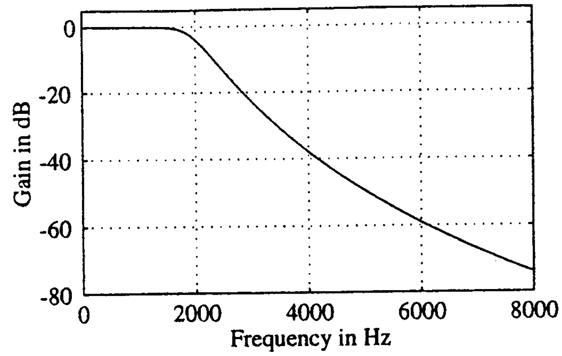
The transfer function of the normalized 6-th order Butterworth lowpass filter is given by

$$H_{an}(s) = \frac{1}{s^6 + 3.8637s^5 + 7.4641s^4 + 9.1416s^3 + 7.4641s^2 + 3.8637s + 1}$$

which has a 3-dB cutoff frequency at 1 rad/sec. To move the 3-dB cutoff frequency to 1929 rad/sec, we denormalize $H_{an}(s)$ to arrive at the desired transfer function:

$$H_a(s) = \frac{1}{\left(\frac{s}{12120}\right)^6 + 3.8637\left(\frac{s}{12120}\right)^5 + 7.4641\left(\frac{s}{12120}\right)^4 + 9.1416\left(\frac{s}{12120}\right)^3 + 7.4641\left(\frac{s}{12120}\right)^2 + 3.8637\left(\frac{s}{12120}\right) + 1}$$

whose gain response is plotted below:



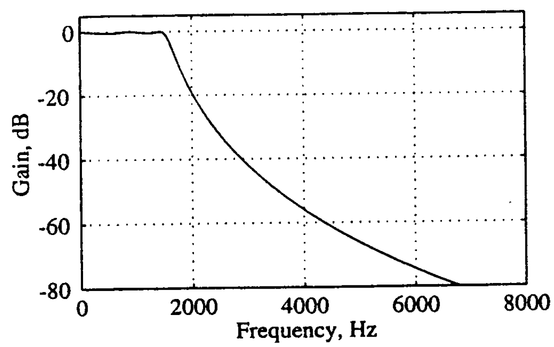
M5.5 The transfer function of a 5-th order Type 1 lowpass Chebyshev filter with a passband edge at 1 rad/sec is given by

$$H_{an}(s) = \frac{1}{s^5 + 1.1725 s^4 + 1.9374 s^3 + 1.3096 s^2 + 0.7525 s + 0.1789}$$

Denormalizing the above transfer function to move the passband edge to 9425 rad/sec we obtain the desired transfer function

$$H_a(s) = \frac{1}{\left(\frac{s}{9.425}\right)^5 + 1.1725\left(\frac{s}{9.425}\right)^4 + 1.9374\left(\frac{s}{9.425}\right)^3 + 1.3096\left(\frac{s}{9.425}\right)^2 + 0.7525\left(\frac{s}{9.425}\right) + 0.1789}$$

whose gain response is plotted below:



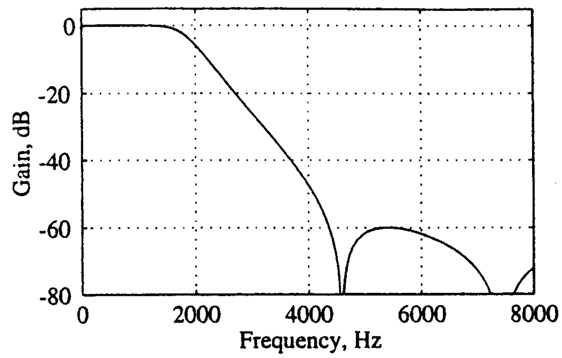
M5.6 The transfer function of a 5-th order Type 2 lowpass Chebyshev filter with a stopband edge at 1 rad/sec is given by

$$H_{an}(s) = \frac{0.005 s^4 + 0.02 s^2 + 0.016}{s^5 + 1.3904 s^4 + 0.9665 s^3 + 0.4171 s^2 + 0.1127 s + 0.016}$$

Denormalizing the above transfer function to move the passband edge to 27430 rad/sec we obtain the desired transfer function

$$H_{an}(s) = \frac{0.005\left(\frac{s}{27430}\right)^4 + 0.02\left(\frac{s}{27430}\right)^2 + 0.016}{\left(\frac{s}{27430}\right)^5 + 1.3904\left(\frac{s}{27430}\right)^4 + 0.9665\left(\frac{s}{27430}\right)^3 + 0.4171\left(\frac{s}{27430}\right)^2 + 0.1127\left(\frac{s}{27430}\right) + 0.016}$$

whose gain response is plotted below:



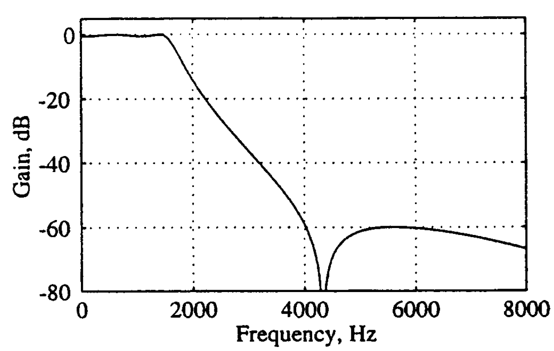
M5.7 The transfer function of a 4-th order elliptic lowpass filter with a passband edge at 1 rad/sec is given by

$$H_{an}(s) = \frac{0.001s^4 + 0.0544s^2 + 0.3852}{s^4 + 1.1911s^3 + 1.7281s^2 + 1.046s + 0.408}$$

Denormalizing the above transfer function to move the passband edge to 9425 rad/sec we obtain the desired transfer function

$$H_{an}(s) = \frac{0.001\left(\frac{s}{9425}\right)^4 + 0.0544\left(\frac{s}{9425}\right)^2 + 0.3852}{\left(\frac{s}{9425}\right)^4 + 1.1911\left(\frac{s}{9425}\right)^3 + 1.7281\left(\frac{s}{9425}\right)^2 + 1.046\left(\frac{s}{9425}\right) + 0.408}$$

whose gain response is plotted below:

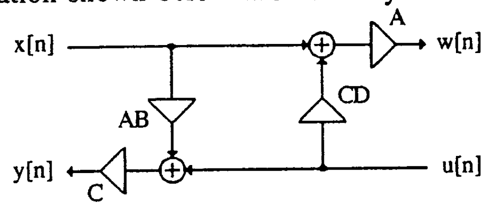


M5.8 The MATLAB program to generate the plots of Figure 5.48 is given below:

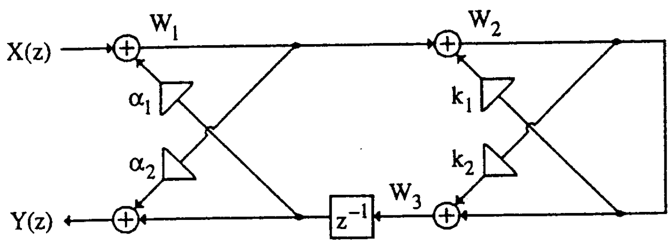
```
w = 0:pi/100:pi; w1 = 0; h2 = freqz(9, [8 1], w);
h1 = freqz([-1/16 9/8 -1/16], 1, w);
for n = 1:101
h3(n) = sin(w1/2)/(w1/2);
w1 = w1 + pi/100;
end
m1 = 20*log10(abs(h1)); m2 = 20*log10(abs(h2));
m3 = 20*log10(abs(h3));
plot(w/pi, m3, 'r-',w/pi, m1+m3, 'b--',w/pi, m2+m3,'r-
');grid
xlabel('Normalized frequency');ylabel('Gain, dB');
text(0.43,-0.58,'DAC','sc');text(0.85,-0.58,'IIR','sc');
text(0.85,-1.67,'FIR','sc');
```

Chapter 6

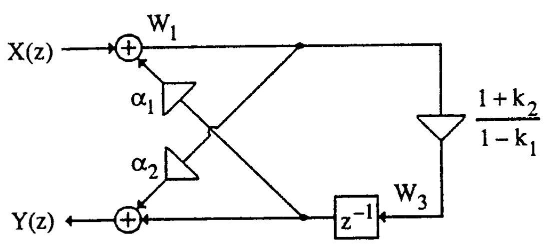
6.1 From Figure P6.1, we get $w[n] = A(x[n] + CDu[n])$, and $y[n] = C(ABx[n] + u[n])$. These two equations can be rewritten as $w[n] = Ax[n] + ACDu[n]$, and $y[n] = CABx[n] + Cu[n]$. The corresponding realization shown below has no delay-free loop:



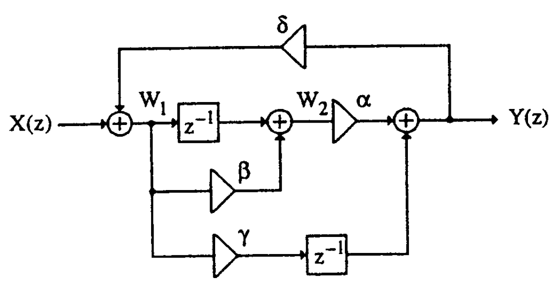
6.2 **(a)**



The only delay-free loop is formed by the loop containing the multiplier k_1. The corresponding equation is given by $W_2 = W_1 + k_1 W_2$, which is equivalent to $W_2 = [1/(1-k_1)]W_1$. Next we observe $W_3 = (1+k_2)W_2 = [(1+k_2)/(1-k_1)]W_2$. Hence, an equivalent delay-free loop realization of Figure P6.2(a) is as indicated below:



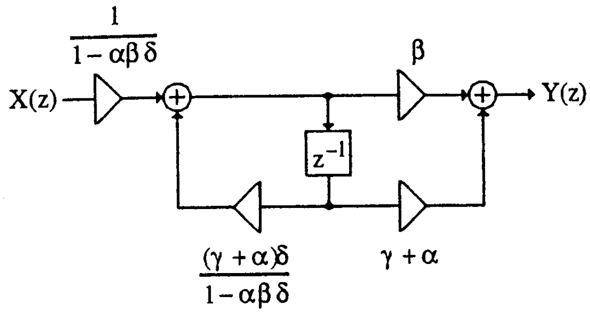
(b)



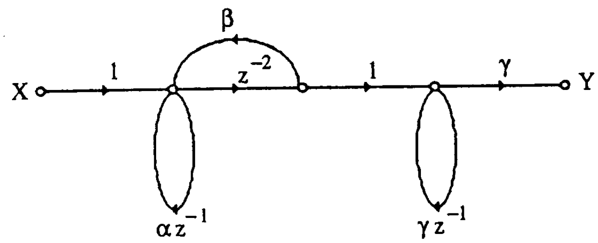
In the above figure, the delay-free loop is formed by the loop containing the multipliers δ, β, and α. Analysis yields $Y(z) = \left(\alpha(z^{-1}+\beta) + \gamma z^{-1}\right)X(z) + \delta\left(\alpha(z^{-1}+\beta)+\gamma z^{-1}\right)Y(z)$ or $\left((1-\alpha\beta\delta) - (\alpha+\gamma)\delta z^{-1}\right)Y(z) = \left(\alpha(z^{-1}+\beta)+\gamma z^{-1}\right)X(z)$. Hence,

$$Y(z) = \frac{1}{1-\alpha\beta\delta}\cdot\frac{\beta + (\alpha+\gamma)z^{-1}}{1-\dfrac{(\alpha+\gamma)\delta}{1-\alpha\beta\delta}z^{-1}}\cdot X(z).$$

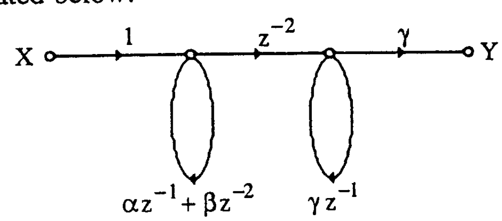
A realization of this equation without a delay-free loop is shown below



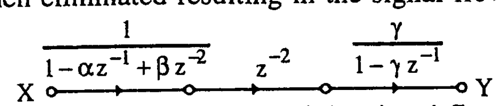
6.3 (a) By splitting the nodes we arive at an equivalent signal-flow graph with two self-loops as shown below:



Another node-splitting results in a third self-loop which can be absorbed into the self-loop with loop gain αz^{-1} as indicated below:



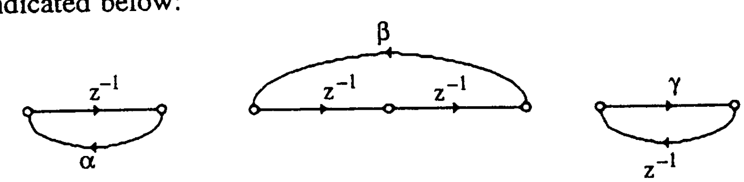
The two self-loops are then eliminated resulting in the signal-flow graph shown below:



From the above graph we observe that the gain of the signal-flow graph is given by

$$H = \frac{Y}{X} = \frac{\gamma z^{-2}}{(1-\alpha z^{-1} - \beta z^{-2})(1-\gamma z^{-1})}.$$

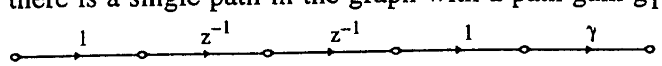
To determine the gain using Mason's gain formula we first observe that there are three loops in the graph as indicated below:



with loop gains αz^{-1}, βz^{-2} and γz^{-1}. Moreover, the last loop is non-touching with the first two loops. Hence, from Eq. (6.??),

$$\Delta = 1 - (\alpha z^{-1} + \beta z^{-2} + \gamma z^{-1}) + (\alpha\gamma z^{-2} + \beta\gamma z^{-3}).$$

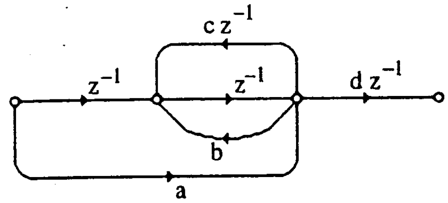
Next, we note that there is a single path in the graph with a path gain $g_1 = \gamma z^{-2}$.



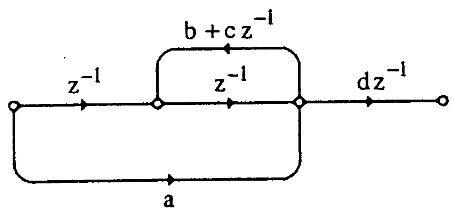
For this path the co-factor is $\Delta_1 = 1$. Therefore, the gain of the signal-flow graph is given by

$$H = \frac{g_1 \Delta_1}{\Delta} = \frac{\gamma z^{-2}}{1 - (\alpha z^{-1} + \beta z^{-2} + \gamma z^{-1}) + (\alpha\gamma z^{-2} + \beta\gamma z^{-3})}.$$

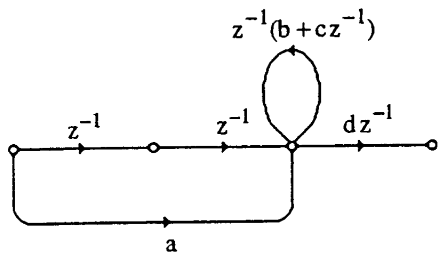
(b) The equivalent graph obtained by node-splitting is as indicated below:



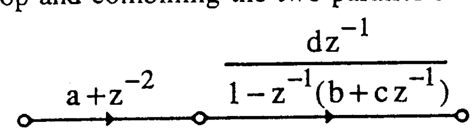
which simplifies to



A second node-splitting results in



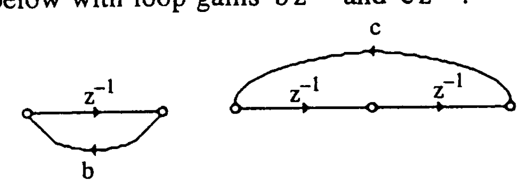
By eliminating the self-loop and combining the two parallel brances we arrive at



from which we obtain thegain of the graph as

$$H = \frac{Y}{X} = \frac{d(a + z^{-2})z^{-1}}{1 - z^{-1}(b + cz^{-1})}.$$

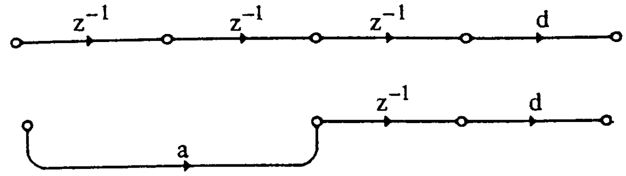
To determine the gain using Mason's gain formula we first observe that there are two loops in the graph as indicated below with loop gains bz^{-1} and cz^{-2}:



There no non-touching loops. Hence, from Eq. (6.??) we get

$$\Delta = 1 - (bz^{-1} + cz^{-2}) = 1 - z^{-1}(b + cz^{-1}).$$

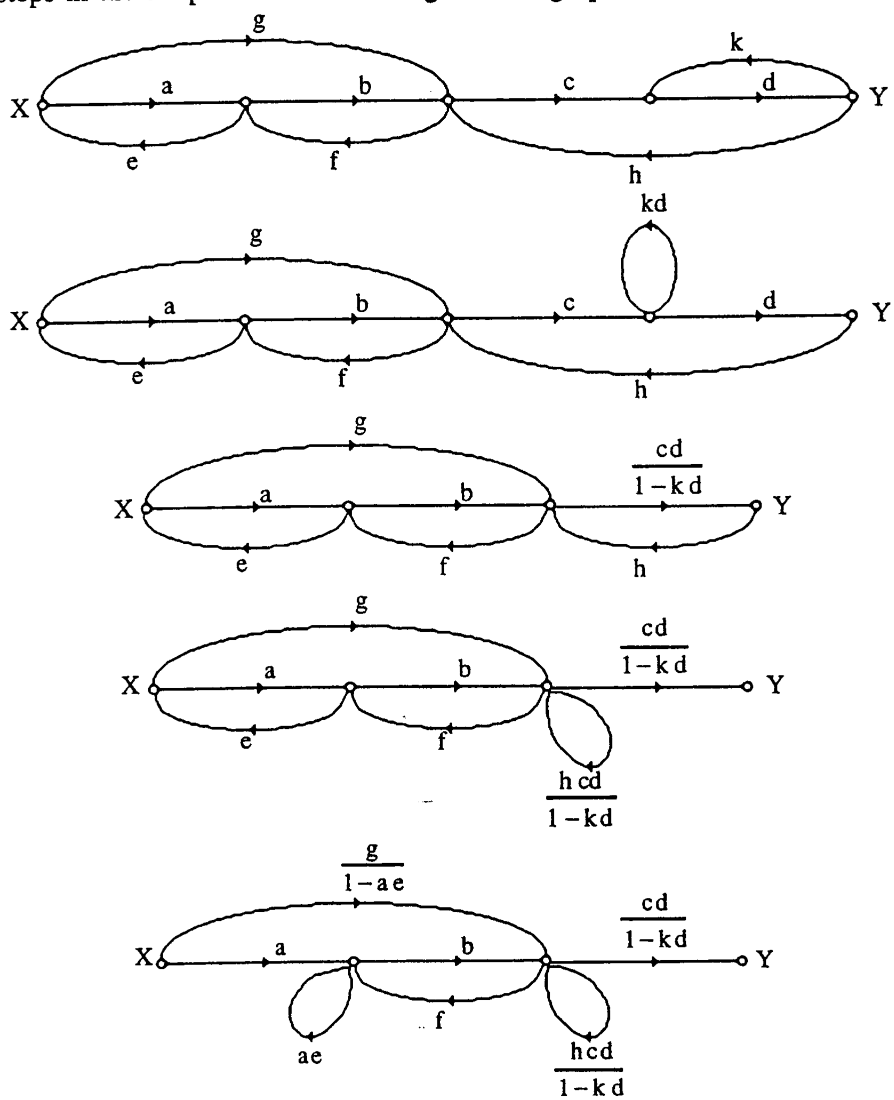
There are two paths as shown below with path gains $g_1 = dz^{-3}$ and $g_2 = adz^{-1}$.

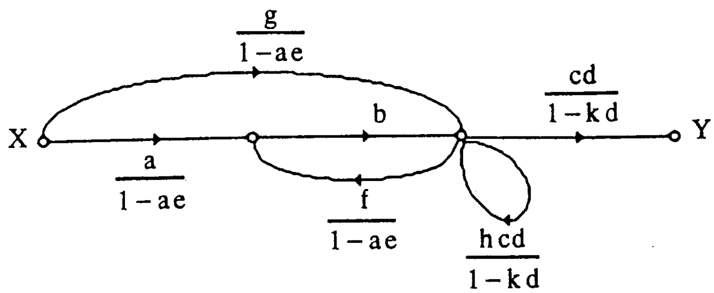


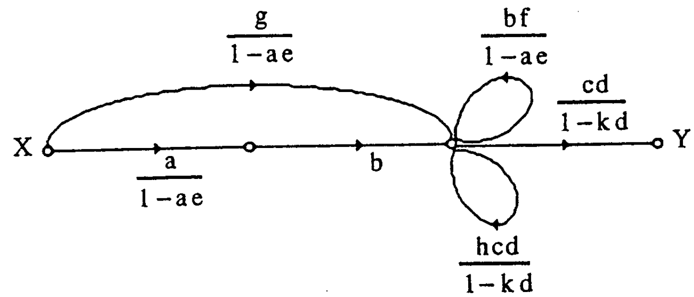
Corresponding co-factors are $\Delta_1 = 1$ and $\Delta_2 = 1$. Therefore, the gain of the graph is given by

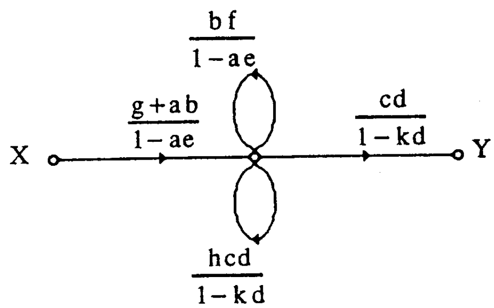
$$H = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta} = \frac{dz^{-3} + adz^{-1}}{1 - z^{-1}(b + cz^{-1})}.$$

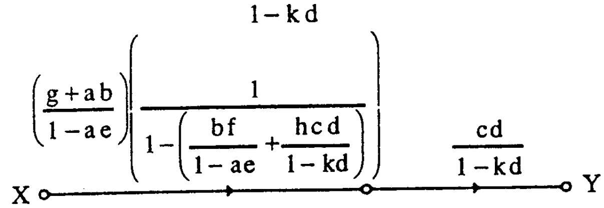
(c) The steps in the simplification of the signal flow graph are shown below











Hence the gain H of the signal flow graph is given by

$$H = \frac{(g+ab)cd}{(1-ae)(1-kd) - bf(1-kd) - hcd(1-ae)}$$

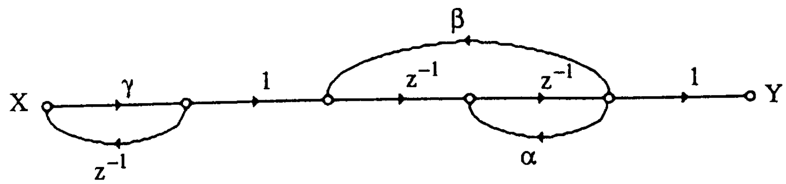
$$= \frac{(g+ab)cd}{1-(ae+kd+bf+hcd)+(aekd+acdhe+bfkd)}.$$

To determine the gain using Mason's gain formula we first observe that there are four loops in the graph with loop gains $L_1 = ae$, $L_2 = bf$, $L_3 = cdh$, and $L_4 = kd$. There are also twopaths with path gains $P_1 = abcd$, and $P_2 = gcd$.

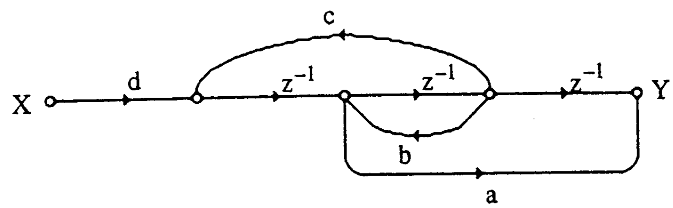
Loops L_1 and L_3, L_1 and L_4, and L_2 and L_4 are non-touching. Hence, $\Delta = 1-(ae+bf+cdh+kd)+(aecdh+aekd+bfkd)$, $\Delta_1 = 1$, and $\Delta_2 = 1$. Therefore,

$$H = \frac{abcd+gcd}{1-(ae+bf+cdh+kd)+(acdhe+adke+bfkd)}.$$

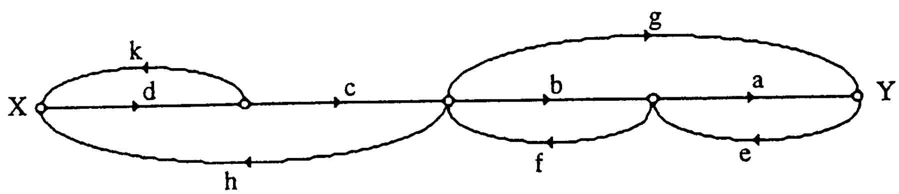
6.4 (a)



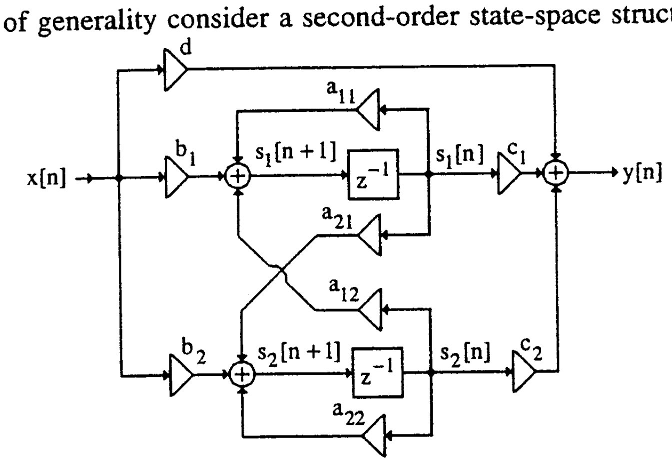
(b)



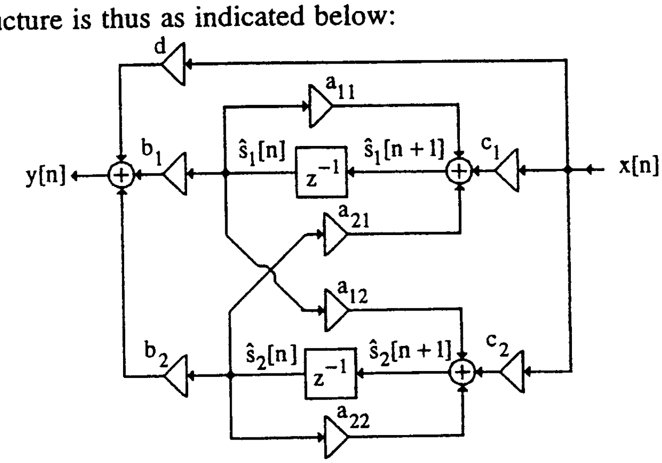
(c)



6.5 Without any loss of generality consider a second-order state-space structure shown below:



Its transposed structure is thus as indicated below:



From the transposed structure, we arrive at the state-space description given by

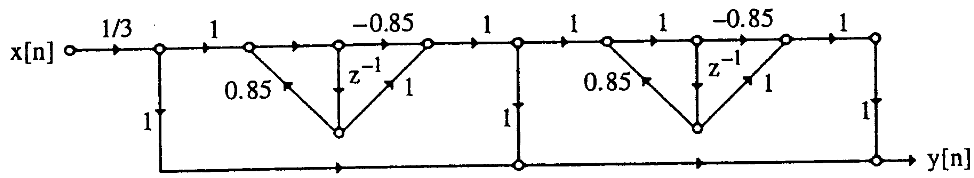
$$\begin{bmatrix}\hat{s}_1[n+1]\\ \hat{s}_2[n+1]\end{bmatrix}=\begin{bmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{bmatrix}\begin{bmatrix}\hat{s}_1[n]\\ \hat{s}_2[n]\end{bmatrix}+\begin{bmatrix}c_1\\ c_2\end{bmatrix}x[n],\quad y[n]=\begin{bmatrix}b_1 & b_2\end{bmatrix}\begin{bmatrix}\hat{s}_1[n]\\ \hat{s}_2[n]\end{bmatrix}+d\,x[n].$$

Thus, the state-space parameters of the transposed structure are given by

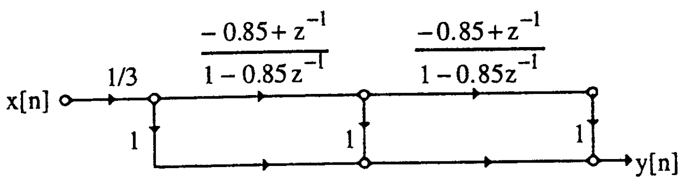
$$\hat{\mathbf{A}}=\begin{bmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{bmatrix}=\mathbf{A}^T,\quad \hat{\mathbf{B}}=\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\mathbf{C}^T,\quad \hat{\mathbf{C}}=\begin{bmatrix}b_1 & b_2\end{bmatrix}=\mathbf{B}^T,\ \text{and}\ \hat{D}=d=D.$$

The transfer function of the transposed structure is given by
$\hat{H}(z)=\hat{\mathbf{C}}[z\mathbf{I}-\hat{\mathbf{A}}]^{-1}\hat{\mathbf{B}}+\hat{D}=\mathbf{B}^T[z\mathbf{I}-\mathbf{A}^T]^{-1}\mathbf{C}^T+D$. Now, the second term on the right-hand side $\mathbf{B}^T[z\mathbf{I}-\mathbf{A}^T]^{-1}\mathbf{C}^T$ is a scalar. If we take the transpose of these product of matrices we get back the same expression. Therefore, $\left(\mathbf{B}^T[z\mathbf{I}-\mathbf{A}^T]^{-1}\mathbf{C}^T\right)^T=\mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}$. Hence, $\hat{H}(z)=H(z)$.

6.6



An equivalent signal flow graph representation of the above graph is shown below



Its transfer function is $H(z)=\dfrac{Y(z)}{X(z)}=\dfrac{1}{3}\left\{1+\dfrac{-0.85+z^{-1}}{1-0.85z^{-1}}+\left(\dfrac{-0.85+z^{-1}}{1-0.85z^{-1}}\right)^2\right\}$ and hence,

the frequency response is $H(e^{j\omega})=\frac{1}{3}\left\{1+e^{j\phi(\omega)}+e^{j2\phi(\omega)}\right\}$, where

$\phi(\omega)=-\omega+2\tan^{-1}\left(\dfrac{-0.85\sin\omega}{1+0.85\cos\omega}\right)$. The magnitude response is

$$\left|H(e^{j\omega})\right|=\frac{1}{3}\left\{(1+\cos\phi+\cos 2\phi)^2+(\sin\phi+\sin 2\phi)^2\right\}^{1/2}.$$

Thus, $\left\{3+2\cos\phi+2\cos 2\phi+2\cos\phi\cos 2\phi+2\sin\phi\sin 2\phi\right\}^{1/2}=2.7$

or, $3+4\cos\phi+2(2\cos^2\phi-1)=(2.7)^2$ or, $4\cos^2\phi+4\cos\phi-6.29=0$. Hence,

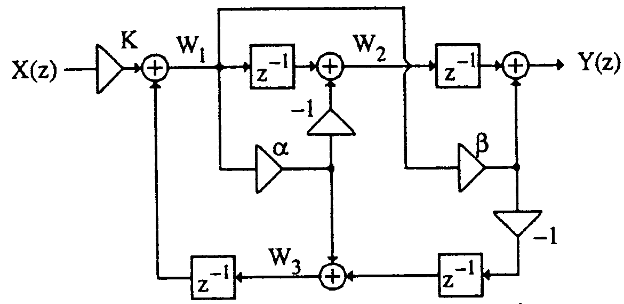
$\cos\phi(\omega)=\dfrac{-4\pm\sqrt{16+(4\times 4\times 6.29)}}{2\times 4}$ or $\cos\phi(\omega)=0.85$. This implies

$$\phi(\omega_o)=-\omega_o+2\tan^{-1}\left(\frac{-0.85\sin\omega_o}{1+0.85\cos\omega_o}\right)=\pm 0.5548\pi,\text{or}$$

$\omega_o\mp 0.5548\pi=2\tan^{-1}\left(\dfrac{-0.85\sin\omega_o}{1+0.85\cos\omega_o}\right)$, whose leads to $\omega_o=\pm 0.3815\pi$. Hence,

$|H(e^{j\omega_o})| = 0.9$ at $\omega_o = \pm 0.3815\pi$.

6.7



From the above figure, we get $W_1 = KX + z^{-1}W_3$, $W_2 = (z^{-1} - \alpha)W_1$, $W_3 = \alpha W_1 - \beta z^{-1}W_1 = (\alpha - \beta z^{-1})W_1$, and $Y = z^{-1}W_2 + \beta W_1$. Substituting the third equation in the first we get $W_1 = KX + z^{-1}(\alpha - \beta z^{-1})W_1$, or $[1 - \alpha z^{-1} + \beta z^{-2}]W_1 = KX$. Next, substituting the second equation in the last one we get $Y = [z^{-1}(z^{-1} - \alpha) + \beta]W_1$. From the last two equations we finally arrive at $H(z) = \frac{Y}{X} = K\left(\frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}}\right)$.

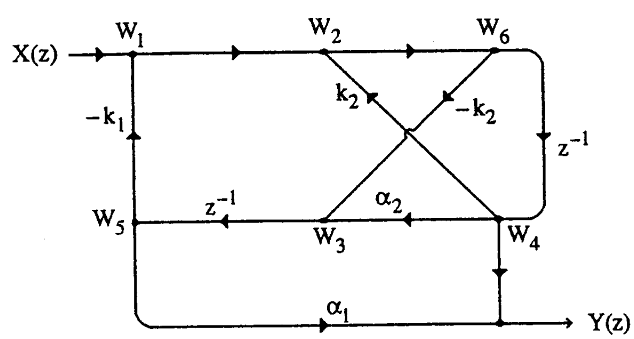
(a) Since the structure employs 4 unit delays to implement a second-order transfer function, it is noncanonic.

(b) and **(c)** We next form $H(z)H(z^{-1}) = K^2\left(\frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}}\right)\left(\frac{\beta - \alpha z + z^2}{1 - \alpha z + \beta z^2}\right)$

$$= K^2\left(\frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}}\right)\left(\frac{\beta z^{-2} - \alpha z^{-1} + 1}{z^{-2} - \alpha z^{-1} + \beta z}\right) = K^2.$$ Therefore, $|H(e^{j\omega})| = K$, for all values of ω. Hence $|H(e^{j\omega})| = 1$ if $K = 1$.

(d) Note H(z) is an allpass transfer function with a constant magnitude at all values of ω.

6.8 The signal-flow graph representation of Figure P6.6 is as shown below:



Loop #1: $\{W_2, W_6, W_4\}$ - loop gain $g_1 = k_2z^{-1}$.
Loop #2: $\{W_1, W_2, W_6, W_4, W_3, W_5\}$ - loop gain $g_2 = -k_1z^{-2}$.
Loop #3: $\{W_1, W_2, W_6, W_3, W_5\}$ - loop gain $g_3 = k_1k_2z^{-1}$.
Therefore $\Delta = 1 - k_2z^{-1} - k_1k_2z^{-1} + k_1z^{-2} = 1 - k_2(1+k_1)z^{-1} + k_1z^{-2}$.

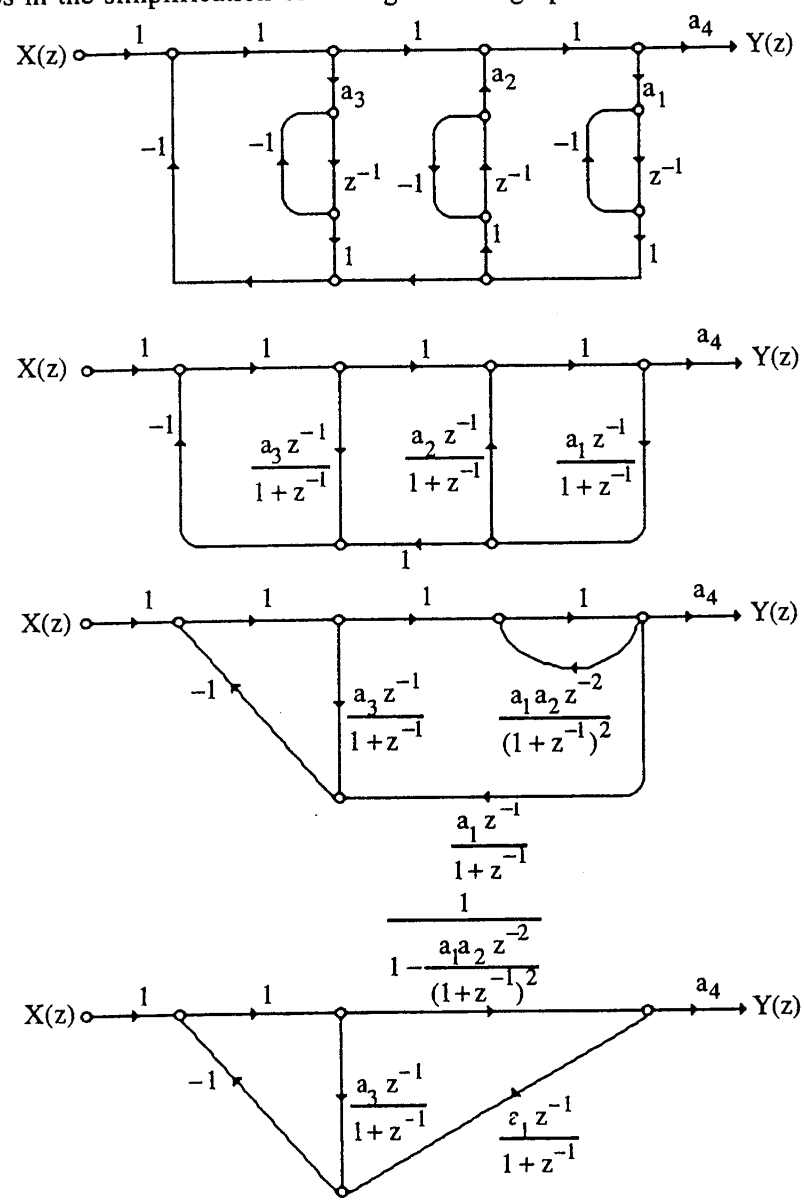
Path #1: $\{X, W_1, W_2, W_6, W_4, Y\}$ - path gain $g_1 = \alpha_2 z^{-1}$ and $\Delta_1 = 1$.
Path #2: $\{X, W_1, W_2, W_6, W_4, W_3, W_5, Y\}$ - path gain $g_2 = \alpha_1 z^{-2}$ and $\Delta_2 = 1$.
Path #3: $\{X, W_1, W_2, W_6, W_5, Y\}$ - path gain $g_3 = -\alpha_1 k_2 z^{-1}$ and $\Delta_3 = 1$.

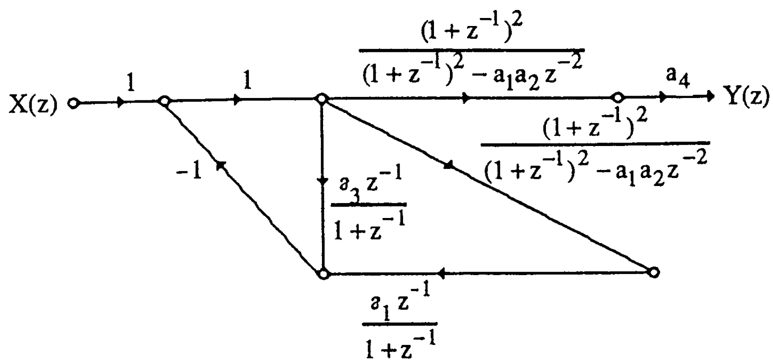
Therefore, $\Delta_1 g_1 + \Delta_2 g_2 + \Delta_3 g_3 = g_1 + g_2 + g_3 = \alpha_2 z^{-1} - \alpha_1 k_2 z^{-1} + \alpha_1 z^{-2}$. Hence,

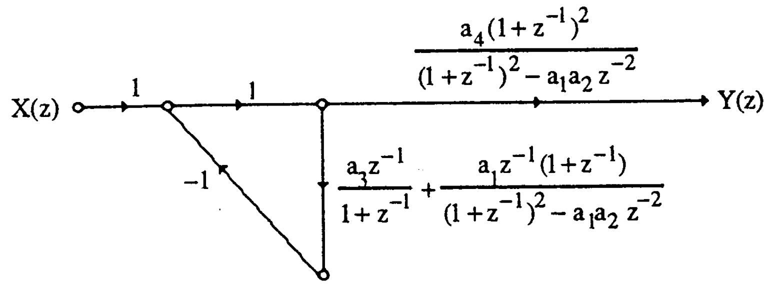
$H = \dfrac{\Delta_1 g_1 + \Delta_2 g_2 + \Delta_3 g_3}{\Delta} = \dfrac{(\alpha_2 - \alpha_1 k_2) z^{-1} + \alpha_1 z^{-2}}{1 - k_2(1+k_1) z^{-1} + k_1 z^{-2}}$. For stability we must have $|k_1| < 1$, and

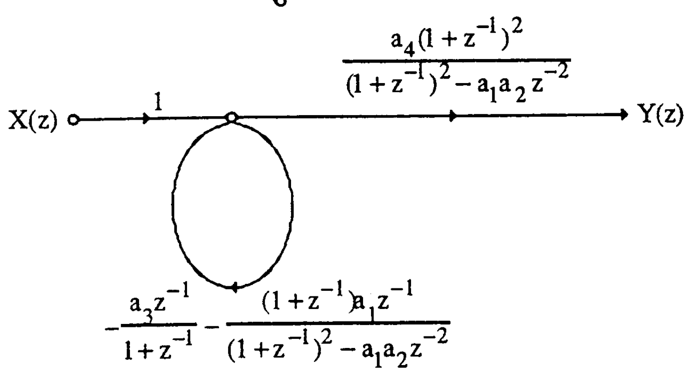
$|k_2(1+k_1)| < 1 + k_1$, or $|k_2| < 1$.

6.9 (a) The steps in the simplification of the signal flow graph are shown below:









Hence the transfer function is given by

$$H(z) = \frac{\dfrac{a_4(1+z^{-1})^2}{(1+z^{-1})^2 - a_1a_2z^{-2}}}{1+\dfrac{a_3z^{-1}}{1+z^{-1}}+\dfrac{a_1(1+z^{-1})^2}{(1+z^{-1})^2-a_1a_2z^{-2}}}$$

$$= \frac{a_4(1+z^{-1})^3}{1+(a_1+a_2+3)z^{-1}+(3-a_1a_2+2a_3+2a_1)z^{-2}+(1+a_1+a_3-a_1a_2-a_1a_2a_3)z^{-3}}.$$

(b) To use Mason's gain formula, we observe that the original signal flow graph has a single path with a pth gain $P_1 = a_4$, and 6 loops with loop gains $L_1 = -a_3z^{-1}$, $L_2 = -z^{-1}$, $L_3 = -a_1z^{-1}$, $L_4 = -z^{-1}$, $L_5 = -z^{-1}$, and $L_6 = a_1a_2z^{-1}$. Hence,

$\Delta_1 = 1-(-3z^{-1})+(z^{-2}+z^{-2}+z^{-2})-(-z^{-3}) = (1+z^{-1})^3$. and

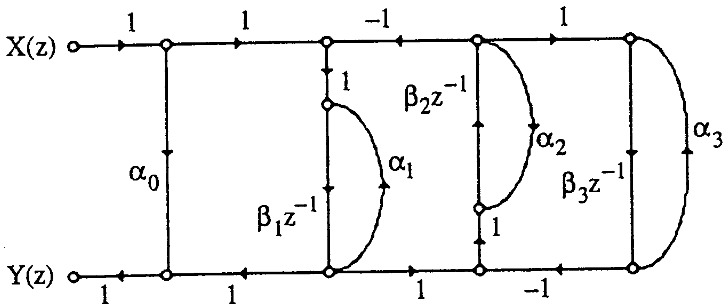
$$\Delta = 1-(-a_3z^{-1}-z^{-1}-a_1z^{-1}-z^{-1}-z^{-1}+a_1a_2z^{-1})$$
$$+(a_3z^{-2}+a_3z^{-2}-a_1a_2a_3z^{-3}+a_1z^{-2}+z^{-2}+z^{-2}+z^{-2}-a_1a_2z^{-3}+a_1z^{-2})$$

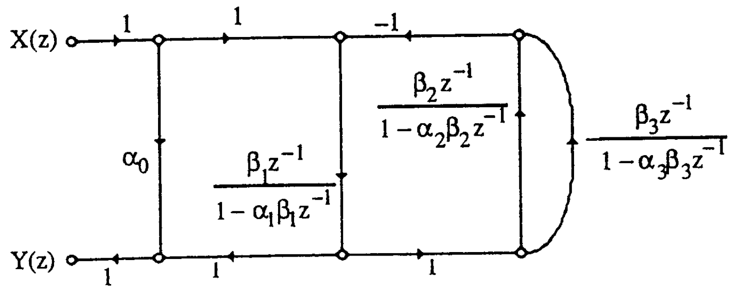
$$- (-z^{-3} - a_3 z^{-3} - a_1 z^{-3}).$$

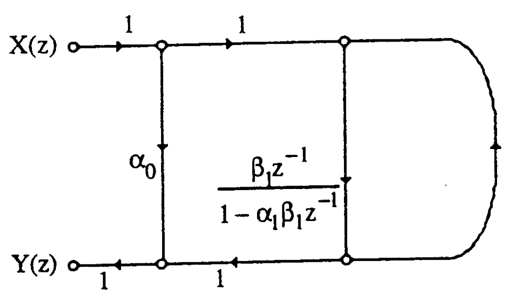
Therefore,

$$H(z) = \frac{a_4(1+z^{-1})^3}{1+(a_1+a_2+3)z^{-1}+(3+2a_1+2a_3-a_1a_2)z^{-2}+(1+a_1+a_3-a_1a_2-a_1a_2a_3)z^{-3}}.$$

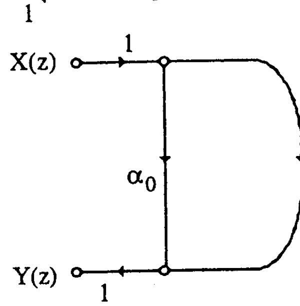
6.10 (a) The steps in the simplification of the signal flow graph are shown below:







$$a(z) = -\frac{\beta_2 z^{-1}(1-\alpha_3\beta_3 z^{-1})}{(1-\alpha_3\beta_3 z^{-1})(1-\alpha_2\beta_2 z^{-1})+\beta_2\beta_3 z^{-2}}$$



$$b(z) = \frac{\beta_1 z^{-1}}{1-\alpha_1\beta_1 z^{-1}+a(z)\beta_1 z^{-1}}$$

Hence the gain $H(z) = \dfrac{Y(z)}{X(z)} = \alpha_0 + b(z) = \alpha_0 + \dfrac{\beta_1 z^{-1}}{1-\alpha_1\beta_1 z^{-1}+a(z)\beta_1 z^{-1}}$

$$= \alpha_0 + \frac{\beta_1 z^{-1}(1-\alpha_3\beta_3 z^{-1})(1-\alpha_2\beta_2 z^{-1})+\beta_1\beta_2\beta_3 z^{-3}}{(1-\alpha_1\beta_1 z^{-1})(1-\alpha_2\beta_2 z^{-1})(1-\alpha_3\beta_3 z^{-1})+(1-\alpha_1\beta_1 z^{-1})\beta_2\beta_3 z^{-2}+(1-\alpha_3\beta_3 z^{-1})\beta_1\beta_2 z^{-2}}$$

(b) To use Mason's gain formula we observe that there are two paths with path gains

$P_1 = \alpha_0$, and $P_2 = \beta_1 z^{-1}$. In addition there are 5 loops with loop gains $L_1 = \alpha_1\beta_1 z^{-1}$, $L_2 = \alpha_2\beta_2 z^{-1}$, $L_3 = -\beta_1\beta_2 z^{-1}$, $L_4 = \alpha_3\beta_3 z^{-1}$, and $L_5 = -\beta_2\beta_3 z^{-1}$. The two co-factors are

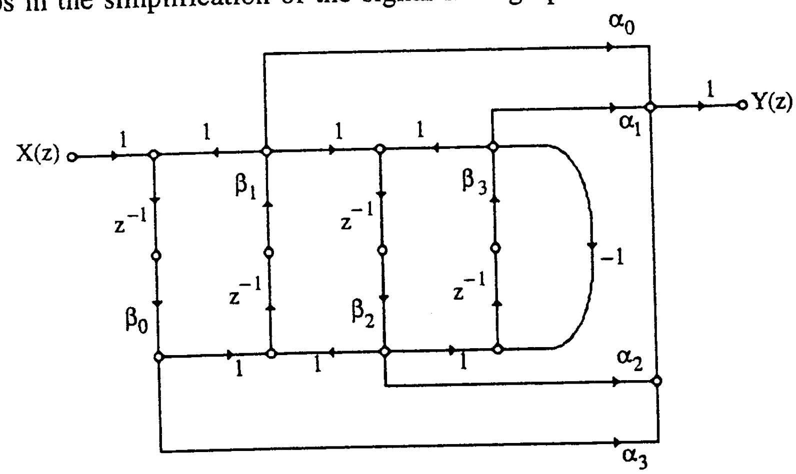
$\Delta_1 = \Delta$, and $\Delta_2 = 1-(\alpha_2\beta_2 z^{-1} + \alpha_3\beta_3 z^{-1} - \beta_2\beta_3 z^{-2}) + (\alpha_2\beta_2\alpha_3\beta_3 z^{-2})$, where

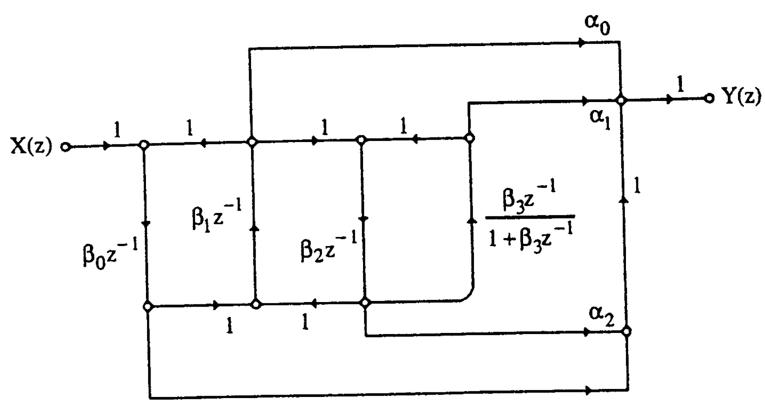
$$\begin{aligned}\Delta = 1 &- (\alpha_1\beta_1 z^{-1} + \alpha_2\beta_2 z^{-1} - \beta_1\beta_2 z^{-2} + \alpha_3\beta_3 z^{-1} - \beta_3\beta_2 z^{-2}) \\ &+(\alpha_1\beta_1\alpha_2\beta_2 z^{-2} + \alpha_1\beta_1\alpha_3\beta_3 z^{-2} - \alpha_1\beta_1\beta_2\beta_3 z^{-3} + \alpha_2\beta_2\alpha_3\beta_3 z^{-2} - \alpha_3\beta_1\beta_2\beta_3 z^{-3}) \\ &- (\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 z^{-3}) \\ = 1 &- (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) z^{-1} + (\alpha_1\beta_1\alpha_2\beta_2 + \alpha_1\beta_1\alpha_3\beta_3 + \alpha_2\beta_2\alpha_3\beta_3 + \beta_1\beta_2 + \beta_2\beta_3) z^{-2} \\ &- (\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 + \alpha_1\beta_1\beta_2\beta_3 + \alpha_3\beta_1\beta_2\beta_3) z^{-3}.\end{aligned}$$

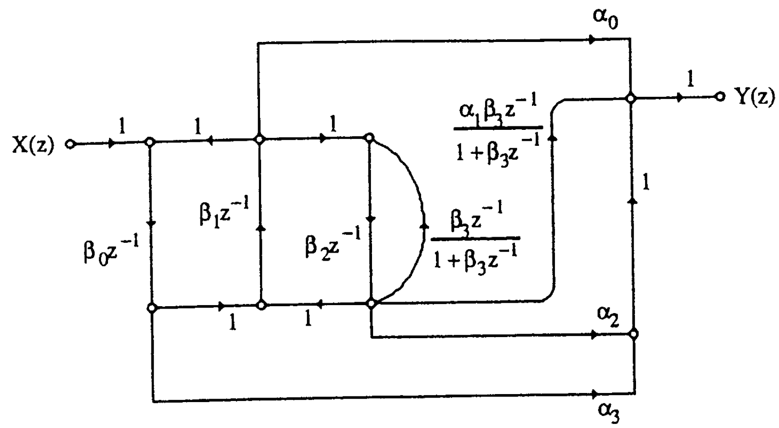
Hence, $H(z) = \dfrac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = P_1 + \dfrac{P_2\Delta_2}{\Delta}$

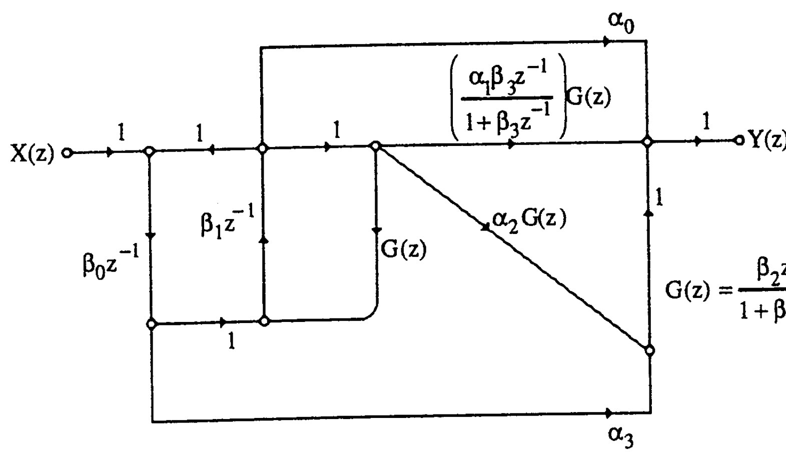
$$= \alpha_0 + \frac{\beta_1 z^{-1}(1-\alpha_3\beta_3 z^{-1})(1-\alpha_2\beta_2 z^{-1}) + \beta_1\beta_2\beta_3 z^{-3}}{(1-\alpha_1\beta_1 z^{-1})(1-\alpha_2\beta_2 z^{-1})(1-\alpha_3\beta_3 z^{-1}) + (1-\alpha_1\beta_1 z^{-1})\beta_2\beta_3 z^{-2} + (1-\alpha_3\beta_3 z^{-1})\beta_1\beta_2 z^{-2}}$$

6.11 (a) The steps in the simplification of the signal flow graph are shown below:

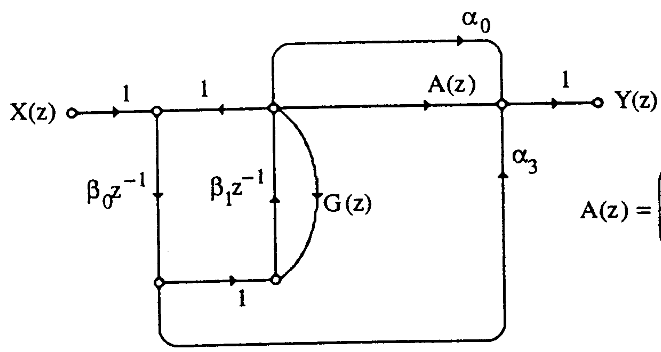




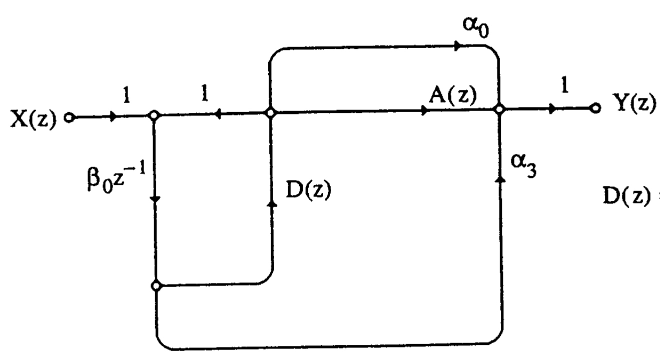




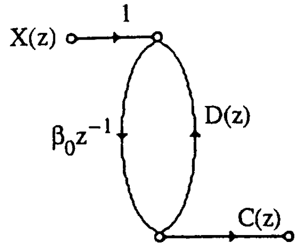
$$G(z) = \frac{\beta_2 z^{-1}(1+\beta_3 z^{-1})}{1+\beta_3 z^{-1}-\beta_2\beta_3 z^{-2}}$$



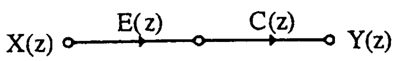
$$A(z) = \left(\frac{\alpha_1\beta_3 z^{-1}}{1+\beta_3 z^{-1}} + \alpha_2\right)G(z)$$



$$D(z) = \frac{\beta_1 z^{-1}}{1-\beta_1 z^{-1}G(z)}$$



$C(z) = (\alpha_0 + A(z))D(z) + \alpha_3$



$$E(z) = \frac{\beta_0 z^{-1}}{1 - D(z)\beta_0 z^{-1}}$$

Now, $H(z) = \dfrac{Y(z)}{X(z)} = C(z)E(z)$, where

$$C(z) = \left[\alpha_0 + \left(\left(\frac{\alpha_1\beta_3 z^{-1}}{1+\beta_3 z^{-1}} + \alpha^2\right)\left(\frac{\beta_2 z^{-1}(1+\beta_3 z^{-1})}{1+\beta_3 z^{-1} - \beta_2\beta_3 z^{-2}}\right)\right)\right]\left[\frac{\beta_1 z^{-1}(1+\beta_3 z^{-1} - \beta_2\beta_3 z^{-2})}{\left(1+\beta_3 z^{-1} - \beta_2\beta_3 z^{-2} - \beta_1\beta_2 z^{-2}(1+\beta_3 z^{-1})\right)}\right] + \alpha_3$$

and

$$E(z) = \frac{\beta_0 z^{-1}}{1 - D(z)\beta_0 z^{-1}} = \frac{\beta_0 z^{-1}(1+\beta_3 z^{-1} - \beta_2\beta_3 z^{-2} - \beta_1\beta_2 z^{-2} - \beta_1\beta_2\beta_3 z^{-3})}{1+\beta_3 z^{-1} - (\beta_2\beta_3 + \beta_1\beta_2 + \beta_1\beta_0)z^{-2} + (\beta_1\beta_2\beta_3 + \beta_1\beta_0\beta_3)z^{-3} + \beta_0\beta_1\beta_2\beta_3 z^{-4}}$$

Hence,

$$H(z) = \frac{\begin{matrix}\alpha_3\beta_0 z^{-1} + (\alpha_0\beta_1\beta_0 + \alpha_3\beta_3\beta_0)z^{-2} + (\alpha_0\beta_1\beta_3\beta_0 + \alpha_2\beta_1\beta_2\beta_0 - \alpha_3\beta_0\beta_2\beta_3 - \alpha_3\beta_0\beta_1\beta_2)z^{-3} \\ + (-\alpha_0 + \alpha_1 + \alpha_2 - \alpha_3)\beta_0\beta_1\beta_2\beta_3 z^{-4}\end{matrix}}{1+\beta_3 z^{-1} - (\beta_2\beta_3 + \beta_1\beta_2 + \beta_1\beta_0)z^{-2} + (\beta_1\beta_2\beta_3 + \beta_1\beta_0\beta_3)z^{-3} + \beta_0\beta_1\beta_2\beta_3 z^{-4}}.$$

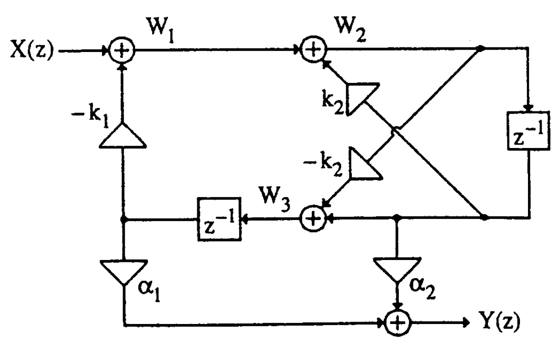
(b) To use Mason's gain formula we observe that there are 4 paths with path gains $P_1 = \alpha_3\beta_0 z^{-1}$, $P_2 = \alpha_0\beta_0\beta_1 z^{-2}$, $P_3 = \alpha_2\beta_0\beta_1\beta_2 z^{-3}$, $P_4 = \alpha_1\beta_0\beta_1\beta_2\beta_3 z^{-4}$. In addition, there are 4 loops with loop gains $L_1 = \beta_0\beta_1 z^{-2}$, $L_2 = \beta_1\beta_2 z^{-2}$, $L_3 = \beta_2\beta_3 z^{-2}$, and $L_4 = -\beta_3 z^{-1}$.

The co-factors are given by

$\Delta_1 = 1 - (\beta_1\beta_2 z^{-2} + \beta_2\beta_3 z^{-2} - \beta_3 z^{-1}) + (-\beta_1\beta_2\beta_3 z^{-3})$, $\Delta_2 = 1 - (\beta_2\beta_3 z^{-2} - \beta_3 z^{-1})$,
$\Delta_3 = 1 + \beta_3 z^{-1}$, and $\Delta_4 = 1$. The determinant is given by

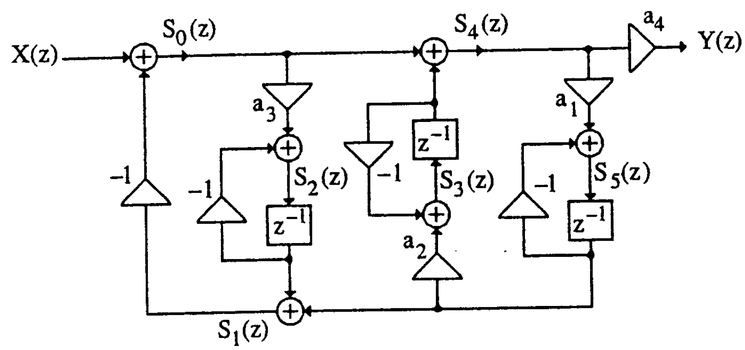
$$\Delta = 1 - (\beta_0\beta_1 z^{-2} + \beta_1\beta_2 z^{-2} + \beta_2\beta_3 z^{-2} - \beta_3 z^{-1}) + (\beta_0\beta_1\beta_2\beta_3 z^{-4} - \beta_0\beta_1\beta_3 z^{-3} - \beta_1\beta_2\beta_3 z^{-3})$$
$$= 1 + \beta_3 z^{-1} - (\beta_0\beta_1 + \beta_1\beta_2 + \beta_2\beta_3)z^{-2} - (\beta_0\beta_1\beta_3 + \beta_1\beta_2\beta_3)z^{-3} + \beta_0\beta_1\beta_2\beta_3 z^{-4}$$

6.12



From the above figure we get $W_1 = X - k_1 z^{-1} W_3$, $W_2 = W_1 + k_2 z^{-1} W_2$, $W_3 = (-k_2 + z^{-1}) W_2$, $Y = \alpha_1 z^{-1} W_3 + \alpha_2 z^{-1} W_2$. From second equation we get $W_1 = W_2 - k_2 z^{-1} W_2 = (1 - k_2 z^{-1}) W_2$. Substituting this and the third equation in the first we then obtain $X - k_1 z^{-1}(z^{-1} - k_2) W_2 = (1 - k_2 z^{-1}) W_2$, or $X = [k_1 z^{-1}(z^{-1} - k_2) + 1 - k_2 z^{-1}] W_2$. Next, from the third and fourth equations we get $Y = [\alpha_1 z^{-1}(z^{-1} - k_2) + \alpha_2 z^{-1}] W_2$. Finally, from the last two equations we arrive at $H = \dfrac{Y}{X} = \dfrac{(\alpha_2 - \alpha_1 k_2) z^{-1} + \alpha_1 z^{-2}}{1 - k_2(1 + k_1) z^{-1} + k_1 z^{-2}}$.

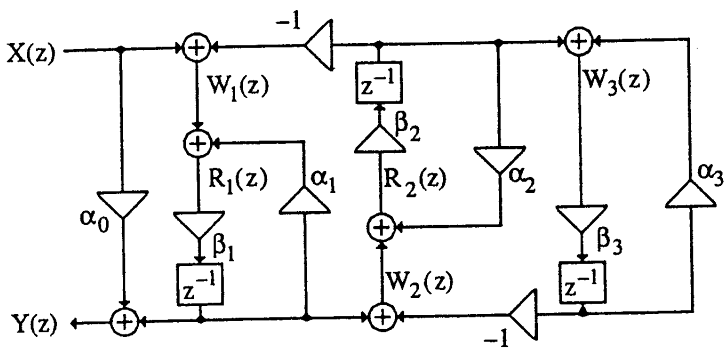
6.13



Analysis yields $S_0(z) = X(z) - S_1(z)$, $S_1(z) = z^{-1} S_2(z) + z^{-1} S_5(z)$, $S_2(z) = a_3 S_0(z) - z^{-1} S_2(z)$, $S_3(z) = a_3 z^{-1} S_5(z) - z^{-1} S_3(z)$, $S_4(z) = S_0(z) + z^{-1} S_3(z)$, $S_5(z) = a_1 S_4(z) - z^{-1} S_5(z)$, and $Y(z) = a_4 S_4(z)$. Eliminating $S_0(z)$, $S_1(z)$, $S_2(z)$, $S_3(z)$, $S_4(z)$ and $S_5(z)$ from these equations we get after some algebra

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_4(1 + z^{-1})^3}{1 + (3 + a_1 + a_3) z^{-1} + (3 + 2a_1 + 2a_3 - a_1 a_2) z^{-2} + (1 + a_1 + a_3 - a_1 a_2 - a_1 a_2 a_3) z^{-3}}.$$

6.14



Analysis yields $Y(z) = \alpha_0 X(z) + \beta_1 z^{-1} R_1(z)$, $W_1(z) = X(z) - \beta_2 z^{-1} R_2(z)$,
$R_1(z) = W_1(z) + \alpha_1 \beta_1 z^{-1} R_1(z)$, $W_2(z) = \beta_1 z^{-1} R_1(z) - \beta_3 z^{-1} W_3(z)$,
$R_2(z) = W_2(z) + \alpha_2 \beta_2 z^{-1} R_2(z)$, $W_3(z) = \beta_2 z^{-1} R_2(z) + \alpha_3 \beta_3 z^{-1} W_3(z)$.

From the third equation we get $W_1(z) = (1 - \alpha_1 \beta_1 z^{-1}) R_1(z)$. From the sixth equation we get

$W_3(z) = \dfrac{\beta_2 z^{-1} R_2(z)}{1-\alpha_3\beta_3 z^{-1}}$. From the fifth equation we get $R_2(z) = \dfrac{W_2(z)}{1-\alpha_2\beta_2 z^{-1}}$.

Rewriting the fourth equation we get $W_2(z) - \beta_3 z^{-1} W_3(z) = \beta_1 z^{-1} R_1(z)$, in which we substitute the expressions for $W_3(z)$ and $W_1(z)$ resulting in

$$R_2(z)\left\{(1-\alpha_2\beta_2 z^{-1}) + \frac{\beta_2\beta_3 z^{-1}}{(1-\alpha_3\beta_3 z^{-1})}\right\} = \beta_1 z^{-1} R_1(z), \text{ or}$$

$$R_2(z) = \frac{\beta_1 z^{-1}(1-\alpha_3\beta_3 z^{-1})}{\beta_2\beta_3 z^{-1} + (1-\alpha_2\beta_2 z^{-1})(1-\alpha_3\beta_3 z^{-1})} R_1(z).$$

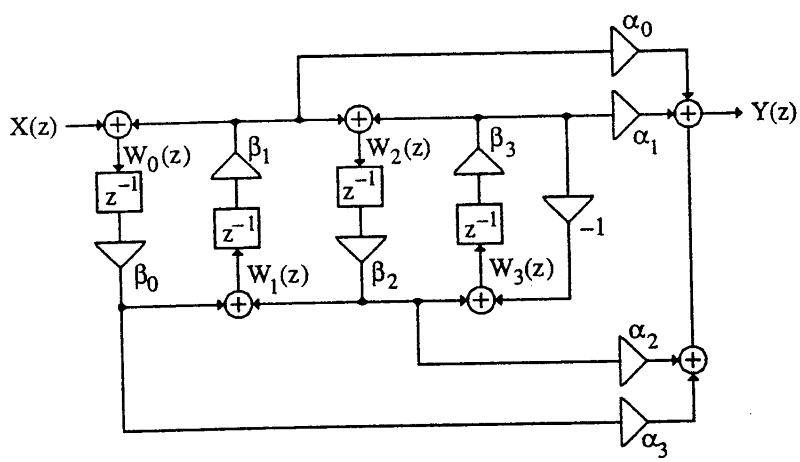
Combining $W_1(z) = X(z) - \beta_2 z^{-1} R_2(z)$, $W_1(z) = (1-\alpha_1\beta_1 z^{-1}) R_1(z)$. and making use of the expression for $R_2(z)$ we arrive at

$$R_1(z) = \frac{\beta_2\beta_3 z^{-2} + (1-\alpha_2\beta_2 z^{-1})(1-\alpha_3\beta_3 z^{-1})}{\beta_2\beta_3 z^{-2}(1-\alpha_1\beta_1 z^{-1}) + (1-\alpha_1\beta_1 z^{-1})(1-\alpha_2\beta_2 z^{-1})(1-\alpha_3\beta_3 z^{-1}) + \beta_1\beta_2 z^{-2}(1-\alpha_3\beta_3 z^{-1})} X(z).$$

Substituting the above in $Y(z) = \alpha_0 X(z) + \beta_1 z^{-1} R_1(z)$, we finally get

$$H(z) = \frac{Y(z)}{X(z)} = \alpha_0 + \frac{\beta_1\beta_2\beta_3 z^{-2} + \beta_1 z^{-1}(1-\alpha_2\beta_2 z^{-1})(1-\alpha_3\beta_3 z^{-1})}{(1-\alpha_1\beta_1 z^{-1})(1-\alpha_2\beta_2 z^{-1})(1-\alpha_3\beta_3 z^{-1}) + \beta_2\beta_3 z^{-2}(1-\alpha_1\beta_1 z^{-1}) + \beta_1\beta_2 z^{-2}(1-\alpha_3\beta_3 z^{-1})}.$$

6.15



Analysis yields $W_0(z) = X(z) + \beta_1 z^{-1} W_1(z)$, $W_1(z) = \beta_0 z^{-1} W_0(z) + \beta_2 z^{-1} W_2(z)$,
$W_2(z) = \beta_1 z^{-1} W_1(z) + \beta_3 z^{-1} W_3(z)$, $W_3(z) = \beta_2 z^{-1} W_2(z) - \beta_3 z^{-1} W_3(z)$.

From these equations we get $W_3(z) = \dfrac{\beta_2 z^{-1}}{1+\beta_3 z^{-1}} W_2(z)$, and

$$\left(\frac{1+\beta_3 z^{-1} - \beta_2\beta_3 z^{-2}}{1+\beta_3 z^{-1}}\right) W_2(z) = \beta_1 z^{-1} W_1(z), \text{ Or, } W_2(z) = \left(\frac{\beta_1 z^{-1}(1+\beta_3 z^{-1})}{1+\beta_3 z^{-1} - \beta_2\beta_3 z^{-2}}\right) W_1(z),$$

In addition, $W_1(z) = \left(\dfrac{\beta_0 z^{-1}(1+\beta_3 z^{-1} - \beta_2\beta_3 z^{-2})}{1+\beta_3 z^{-1} - (\beta_2\beta_3 + \beta_1\beta_2) z^{-2} - \beta_1\beta_2\beta_3 z^{-3}}\right) W_0(z).$

Now, $W_0(z)\left(1-\dfrac{\beta_0\beta_1 z^{-2}(1+\beta_3 z^{-1}-\beta_2\beta_3 z^{-2})}{1+\beta_3 z^{-1}-\beta_2(\beta_1+\beta_3)z^{-2}-\beta_1\beta_2\beta_3 z^{-3}}\right)=X(z)$, Hence,

$$W_0(z)=\left(\frac{1+\beta_3 z^{-1}-\beta_2(\beta_1+\beta_3)z^{-2}-\beta_1\beta_2\beta_3 z^{-3}}{1+\beta_3 z^{-1}-(\beta_1\beta_2+\beta_2\beta_3+\beta_0\beta_1)z^{-2}-\beta_1\beta_3(\beta_2+\beta_0)z^{-3}+\beta_0\beta_1\beta_2\beta_3 z^{-4}}\right)X(z),$$

$$W_1(z)=\left(\frac{\beta_0 z^{-1}(1+\beta_3 z^{-1}-\beta_2\beta_3 z^{-2})}{1+\beta_3 z^{-1}-(\beta_1\beta_2+\beta_2\beta_3+\beta_0\beta_1)z^{-2}-\beta_1\beta_3(\beta_2+\beta_0)z^{-3}+\beta_0\beta_1\beta_2\beta_3 z^{-4}}\right)X(z),$$

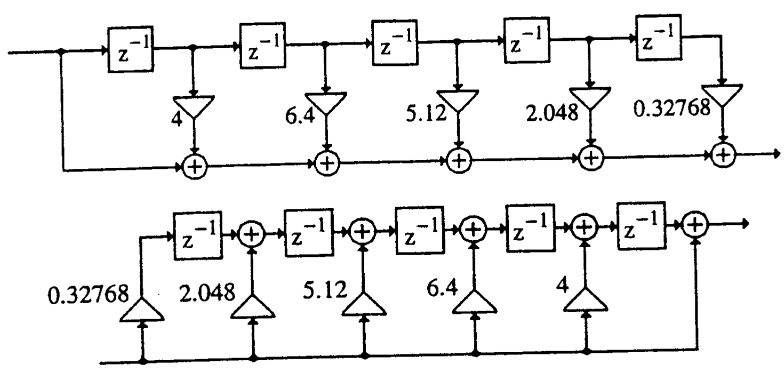
$$W_2(z)=\left(\frac{\beta_1\beta_0 z^{-2}(1+\beta_3 z^{-1})}{1+\beta_3 z^{-1}-(\beta_1\beta_2+\beta_2\beta_3+\beta_0\beta_1)z^{-2}-\beta_1\beta_3(\beta_2+\beta_0)z^{-3}+\beta_0\beta_1\beta_2\beta_3 z^{-4}}\right)X(z),$$

$$W_3(z)=\left(\frac{\beta_0\beta_1\beta_2 z^{-3}}{1+\beta_3 z^{-1}-(\beta_1\beta_2+\beta_2\beta_3+\beta_0\beta_1)z^{-2}-\beta_1\beta_3(\beta_2+\beta_0)z^{-3}+\beta_0\beta_1\beta_2\beta_3 z^{-4}}\right)X(z).$$

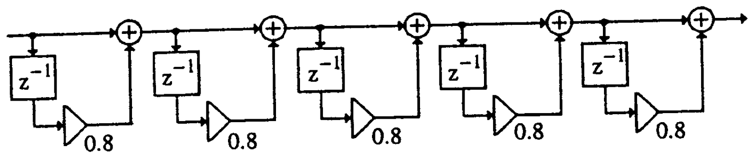
Finally, $Y(z)=\alpha_3\beta_0 z^{-1}W_0(z)+\alpha_0\beta_1 z^{-1}W_1(z)+\alpha_2\beta_2 z^{-1}W_2(z)+\alpha_1\beta_3 z^{-1}W_3(z)$. Substituting the expressions for $W_0(z)$, $W_1(z)$, $W_2(z)$ and $W_3(z)$ in the above we get

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\begin{array}{c}\alpha_3\beta_0 z^{-1}+(\alpha_3\beta_3\beta_0+\alpha_0\beta_1\beta_0)z^{-2}+(\alpha_0\beta_0\beta_1\beta_3+\alpha_2\beta_1\beta_2\beta_0-\alpha_3\beta_2\beta_1\beta_0-\alpha_3\beta_0\beta_2\beta_3)z^{-3}\\+(-\alpha_3+\alpha_2-\alpha_0+\alpha_1)\beta_0\beta_1\beta_2\beta_3 z^{-3}\end{array}}{1+\beta_3 z^{-1}-(\beta_1\beta_2+\beta_2\beta_3+\beta_0\beta_1)z^{-2}-\beta_1\beta_3(\beta_2+\beta_0)z^{-3}+\beta_0\beta_1\beta_2\beta_3 z^{-4}}$$

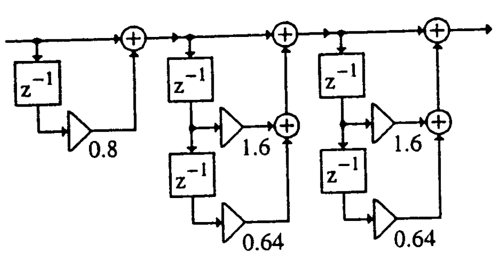
6.16 (a)



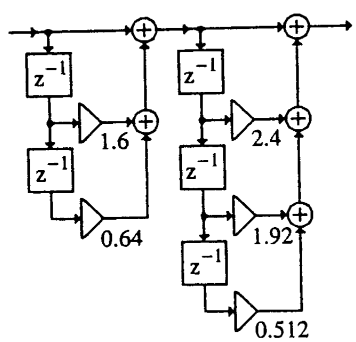
(b)



(c)

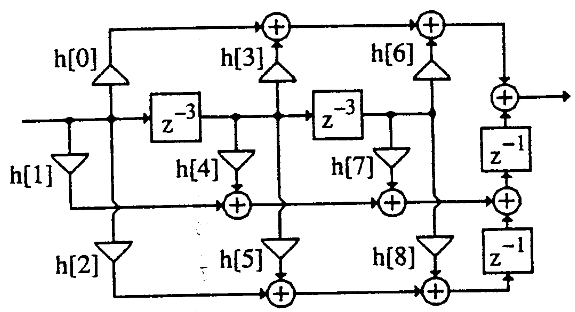


(d)

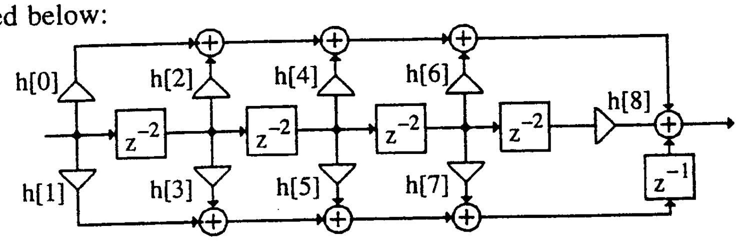


Note: All structures use 5 delays, 5 two-input adders, and 5 multipliers.

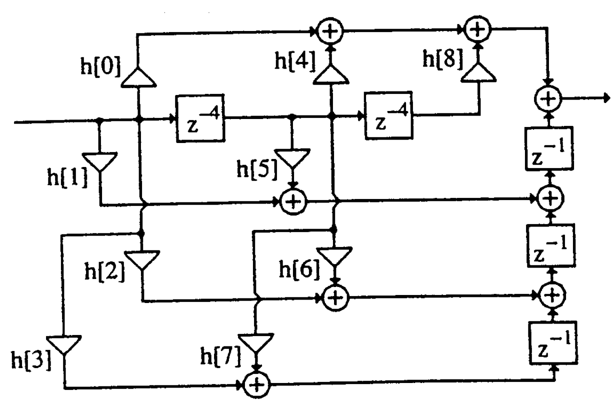
6.17 $H(z)=\left(h[0]+h[3]z^{-3}+h[6]z^{-6}\right)+z^{-1}\left(h[1]+h[4]z^{-3}+h[7]z^{-6}\right)+z^{-2}\left(h[2]+h[5]z^{-3}+h[8]z^{-6}\right)$
$=H_0(z^3)+z^{-1}H_1(z^3)+z^{-2}H_2(z^3)$. Hence, $H_0(z)=h[0]+h[3]z^{-1}+h[6]z^{-2}$,
$H_1(z)=h[1]+h[4]z^{-1}+h[7]z^{-2}$, and $H_2(z)=h[2]+h[5]z^{-1}+h[8]z^{-2}$. A canonic 3-branch polyphase realization of H(z) is thus as indicated below:



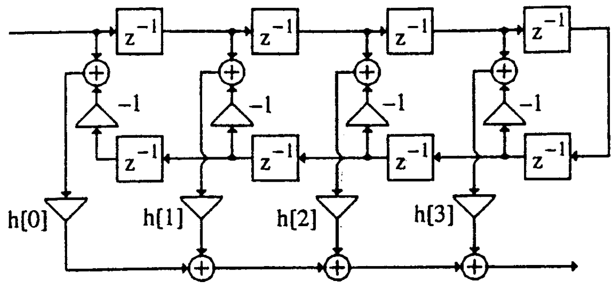
6.18 $H(z)=\left(h[0]+h[2]z^{-2}+h[4]z^{-4}+h[6]z^{-6}+h[8]z^{-8}\right)+z^{-1}\left(h[1]+h[3]z^{-2}+h[5]z^{-4}+h[7]z^{-6}\right)$
$=H_0(z^2)+z^{-1}H_1(z^2)$, where $H_0(z)=h[0]+h[2]z^{-1}+h[4]z^{-2}+h[6]z^{-3}+h[8]z^{-4}$, and
$H_1(z)=h[1]+h[3]z^{-1}+h[5]z^{-2}+h[7]z^{-3}$. A canonic 2-branch polyphase realization of H(z) is thus as indicated below:



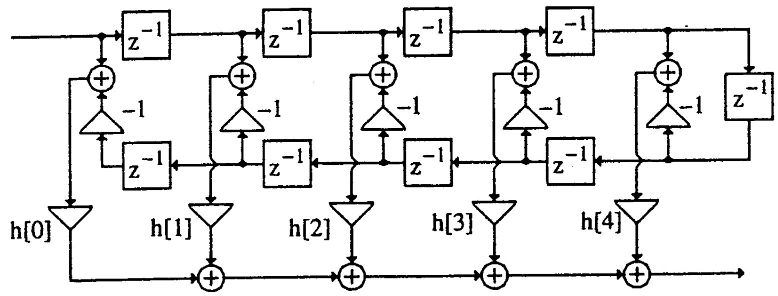
6.19 $H(z)=\left(h[0]+h[4]z^{-4}+h[8]z^{-8}\right)+z^{-1}\left(h[1]+h[5]z^{-4}\right)+z^{-2}\left(h[2]+h[6]z^{-4}\right)+z^{-3}\left(h[3]+h[7]z^{-4}\right)$
$=H_0(z^4)+z^{-1}H_1(z^4)+z^{-2}H_2(z^4)+z^{-3}H_3(z^4)$, where $H_0(z)=h[0]+h[4]z^{-1}+h[8]z^{-2}$,
$H_1(z)=h[1]+h[5]z^{-1}$, $H_2(z)=h[2]+h[6]z^{-1}$, and $H_3(z)=h[3]+h[7]z^{-1}$. A canonic 4-branch polyphase realization of H(z) is thus as indicated below:



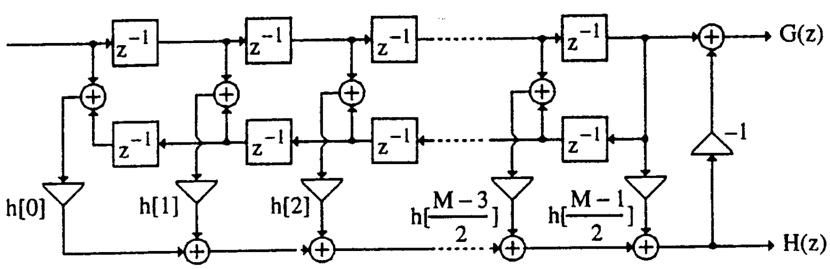
6.20



6.21



6.22



The total number of unit delays in the realization is equal to (M–1) which is same as that required in the realization of H(z). Likewise, the total number of multipliers in the overall structure is equal to (M+1)/2 which is also same as that required in the realization of H(z).

6.23 Without any loss of generality, assume M = 5 which means N = 11. In this case the transfer function is given by $H(z) =$

$$z^{-5}\left[h[5]+h[4](z+z^{-1})+h[3](z^2+z^{-2})+h[2](z^3+z^{-3})+h[1](z^4+z^{-4})+h[0](z^5+z^{-5})\right].$$

Now, the recursion relation for the Chebyshev polynomial is given by

$T_r(x)=2xT_{r-1}(x)-T_{r-2}(x)$, for $r \geq 2$ with $T_0(x)=1$ and $T_1(x)=x$. Hence,

$T_2(x)=2xT_1(x)-T_0(x)=2x^2-1,$

$T_3(x)=2xT_2(x)-T_1(x)=2x(2x^2-1)-x=4x^3-3x,$

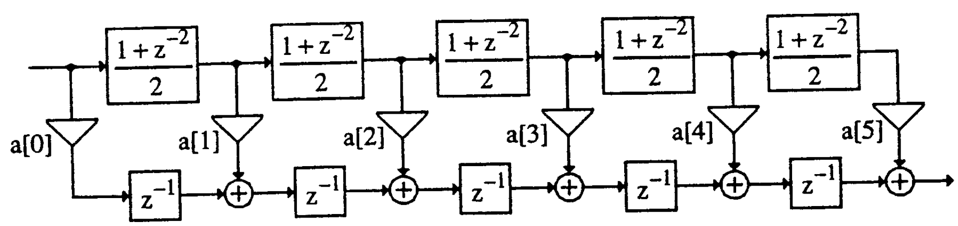
$T_4(x)=2xT_3(x)-T_2(x)=2x(4x^3-3x)-(2x^2-1)=8x^4-8x^2+1,$

$T_5(x)=2xT_4(x)-T_3(x)=2x(8x^4-8x^2+1)-(4x^3-3x)=16x^5-20x^3+5x.$

We can thus rewrite the expression inside the square brackets given above as

$$h[5]+2h[4]T_1\left(\frac{z+z^{-1}}{2}\right)+2h[3]T_2\left(\frac{z+z^{-1}}{2}\right)+2h[2]T_3\left(\frac{z+z^{-1}}{2}\right)$$
$$+2h[1]T_4\left(\frac{z+z^{-1}}{2}\right)+h[0]T_5\left(\frac{z+z^{-1}}{2}\right)$$
$$= h[5]+2h[4]\left(\frac{z+z^{-1}}{2}\right)+2h[3]\left[2\left(\frac{z+z^{-1}}{2}\right)^2-1\right]+2h[2]\left[4\left(\frac{z+z^{-1}}{2}\right)^3-3\left(\frac{z+z^{-1}}{2}\right)\right]$$
$$+2h[1]\left[8\left(\frac{z+z^{-1}}{2}\right)^4-8\left(\frac{z+z^{-1}}{2}\right)^2+1\right]+2h[0]\left[16\left(\frac{z+z^{-1}}{2}\right)^5-20\left(\frac{z+z^{-1}}{2}\right)^3+5\left(\frac{z+z^{-1}}{2}\right)\right]$$
$= \sum_{n=0}^{5} a[n]\left(\frac{z+z^{-1}}{2}\right)^n$, where $a[0]=h[5]-2h[3]+2h[1]$, $a[1]=2h[4]-6h[2]+10h[0]$, $a[2]=4h[3]-16h[1]$, $a[3]=8h[2]-40h[1]$, $a[4]=16h[1]$, and $a[5]=32h[0]$.

A realization of $H(z)=z^{-5}\left[\sum_{n=0}^{5} a[n]\left(\frac{z+z^{-1}}{2}\right)^n\right] = a[0]z^{-5}+a[1]z^{-4}\left(\frac{1+z^{-2}}{2}\right)$
$$+a[2]z^{-3}\left(\frac{1+z^{-2}}{2}\right)^2+a[3]z^{-2}\left(\frac{1+z^{-2}}{2}\right)^3+a[4]z^{-1}\left(\frac{1+z^{-2}}{2}\right)^4+a[5]\left(\frac{1+z^{-2}}{2}\right)^5$$
is thus as shown below:



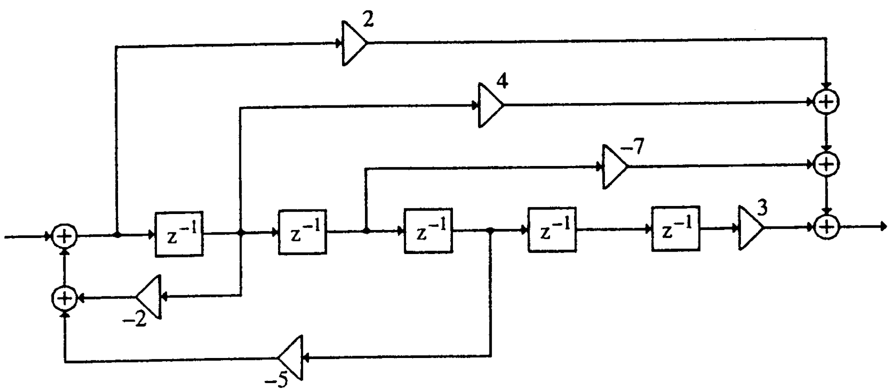
6.24 Consider $H(z)=\frac{P(z)}{D(z)}=\frac{P_1(z)}{D_1(z)}\cdot\frac{P_2(z)}{D_2(z)}\cdot\frac{P_3(z)}{D_3(z)}$. Assume all zeros of P(z) and D(z) are complex. Note that the numerator of the first stage can be one of the 3 factors, $P_1(z)$, $P_2(z)$, and $P_3(z)$, the numerator of the second stage can be one of the remaining 2 factors, and the numerator of the third stage is the remaining factor. Likewise, the denominator of the first stage can be one of the 3 factors, $D_1(z)$, $D_2(z)$, and $D_3(z)$, the deniminator of the second stage can be one of the remaining 2 factors, and the denominator of the third stage is the remaining factor. Hence, there are $6\times 6 = (3!)^2$ different types of cascade realizations.

If the zeros of P(z) and D(z) are all real, then P(z) has 6 real zeros and D(z) has 6 real zeros. In this case then there are $=(6!)^2$ different types of cascade realizations.

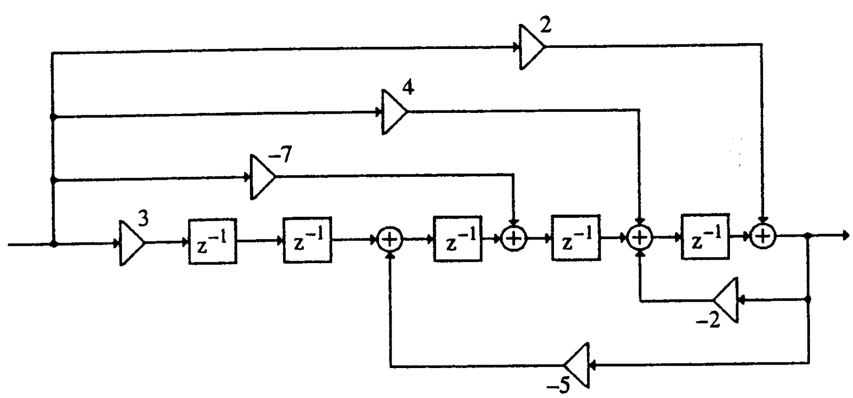
6.25 $H(z) = \prod_{i=1}^{K} \frac{P_i(z)}{D_i(z)}$. Here the numerator of the first stage can be chosen in $\binom{K}{1}$ ways, the numerator of the second stage can be chosen in $\binom{K-1}{1}$ ways, and until the there is only possible choice for the numerator of the K-th stage. Likewise, the denominator of the first stage can be chosen in $\binom{K}{1}$ ways, the denominator of the second stage can be chosen in $\binom{K-1}{1}$ ways, and until the there is only possible choice for the denominator of the K-th stage. Hence the total number of possible cascade realizations are equal to

$$\binom{K}{1}^2\binom{K-1}{1}^2\binom{K-2}{1}^2\cdots\binom{2}{1}^2\binom{1}{1}^2 = (K!)^2$$

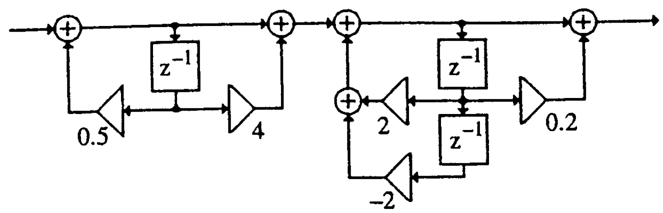
6.26 A canonic direct form II realization of $H(z) = \frac{2+4z^{-1}-7z^{-2}+3z^{-5}}{1+2z^{-1}+5z^{-2}}$ is shown below:



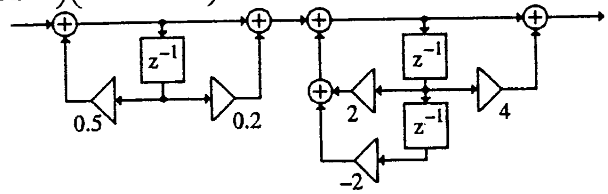
Its transposed realization is indicated below:



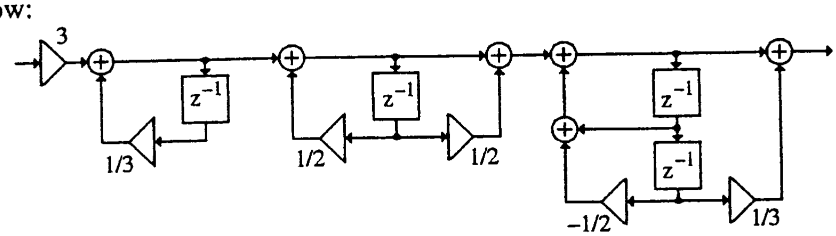
6.27 **(a)** A cascade canonic realization of $H_1(z) = \left(\frac{1+0.2z^{-1}}{1-2z^{-1}+2z^{-2}}\right)\left(\frac{1+4z^{-1}}{1-0.5z^{-1}}\right)$ is shown below:



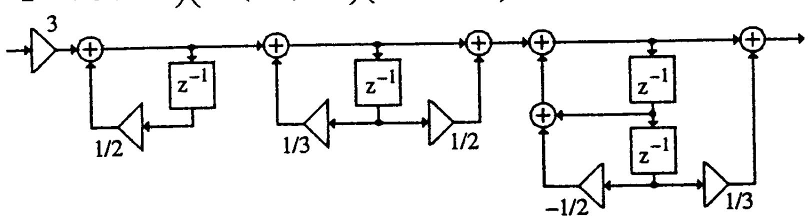
Another cascade canonic realization is obtained by a different pole-zero pairing given by $H_1(z) = \left(\dfrac{1+4z^{-1}}{1-2z^{-1}+2z^{-2}}\right)\left(\dfrac{1+0.2z^{-1}}{1-0.5z^{-1}}\right)$ resulting in



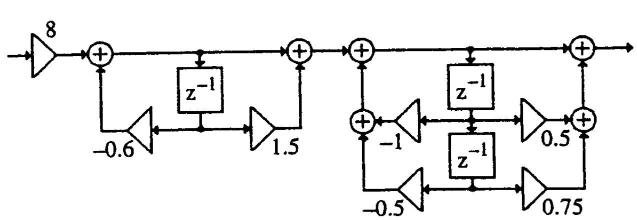
(b) A cascade canonic realization of $H_2(z) = 3\left(\dfrac{1+(1/3)z^{-2}}{1-z^{-1}+0.5z^{-2}}\right)\left(\dfrac{1+0.5z^{-1}}{1-0.5z^{-1}}\right)\left(\dfrac{1}{1-(1/3)z^{-1}}\right)$ is shown below:



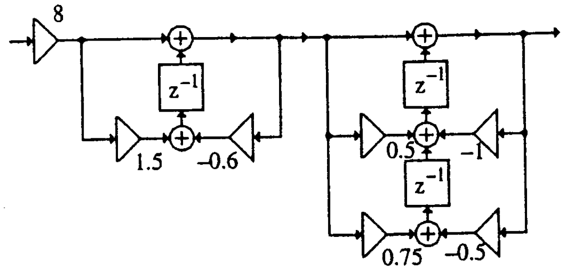
Another cascade canonic realization is obtained by a different pole-zero pairing given by $H_2(z) = 3\left(\dfrac{1+(1/3)z^{-2}}{1-z^{-1}+0.5z^{-2}}\right)\left(\dfrac{1+0.5z^{-1}}{1-(1/3)z^{-1}}\right)\left(\dfrac{1}{1-0.5z^{-1}}\right)$ resulting in



(c) A cascade canonic realization of $H_3(z) = 8\left(\dfrac{1+0.5z^{-1}+0.75z^{-2}}{1+z^{-1}+0.5z^{-2}}\right)\left(\dfrac{1+1.5z^{-1}}{1+0.6z^{-1}}\right)$ is shown below:

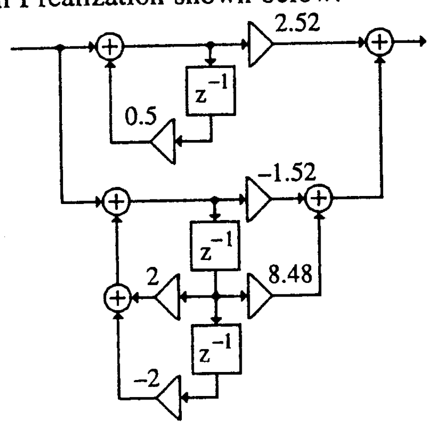


Transpose of the above structure yields a second cascade canonic realization shown below:



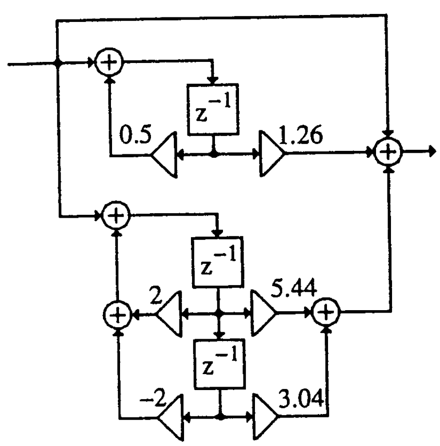
6.28 **(a)** $H_1(z)=\frac{-0.76-j3.48}{1-(1+j)z^{-1}}+\frac{-0.76+j3.48}{1-(1-j)z^{-1}}+\frac{2.52}{1-0.5z^{-1}}=\frac{-1.52+8.48z^{-1}}{1-2z^{-1}+2z^{-2}}+\frac{2.52}{1-0.5z^{-1}}$,

which leads to Parallel Form I realization shown below:



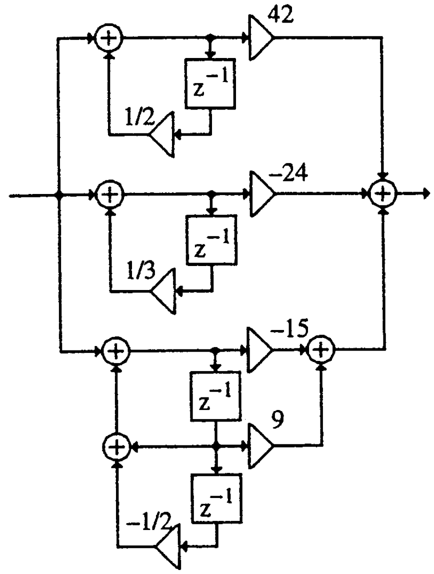
$H_1(z)=1+\frac{2.72-j\,4.24}{1-(1+j)z^{-1}}+\frac{2.72+j\,4.24}{1-(1-j)z^{-1}}+\frac{1.26}{1-0.5z^{-1}}=1+\frac{5.44z^{-1}+3.04z^{-2}}{1-2z^{-1}+2z^{-2}}+\frac{1.26z^{-1}}{1-0.5z^{-1}}$, which

leads to Parallel Form II realization shown below:



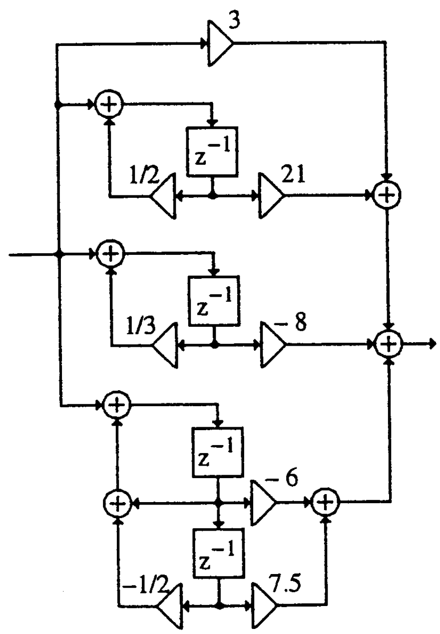
(b) $H_2(z)=\frac{-7.5-j1.5}{1-(\frac{1}{2}+j\frac{1}{2})z^{-1}}+\frac{-7.5+j1.5}{1-(\frac{1}{2}-j\frac{1}{2})z^{-1}}+\frac{42}{1-\frac{1}{2}z^{-1}}-\frac{24}{1-\frac{1}{3}z^{-1}}$

$$= \frac{-15+9z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} + \frac{42}{1-\frac{1}{2}z^{-1}} - \frac{24}{1-\frac{1}{3}z^{-1}},$$ which leads to Parallel Form I realization shown below:



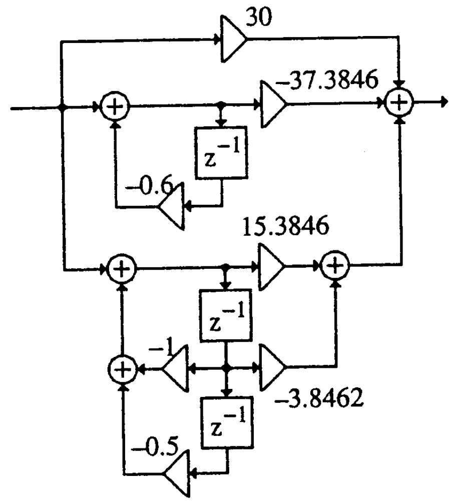
$$H_2(z) = 3 + \frac{(-3-j4.5)z^{-1}}{1-(\frac{1}{2}+j\frac{1}{2})z^{-1}} + \frac{(-3+j4.5)z^{-1}}{1-(\frac{1}{2}-j\frac{1}{2})z^{-1}} + \frac{21z^{-1}}{1-\frac{1}{2}z^{-1}} - \frac{8z^{-1}}{1-\frac{1}{3}z^{-1}}$$

$$= 3 + \frac{-6z^{-1}+7.5z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}} + \frac{21z^{-1}}{1-\frac{1}{2}z^{-1}} - \frac{8z^{-1}}{1-\frac{1}{3}z^{-1}},$$ which leads to Parallel Form II realization shown below:



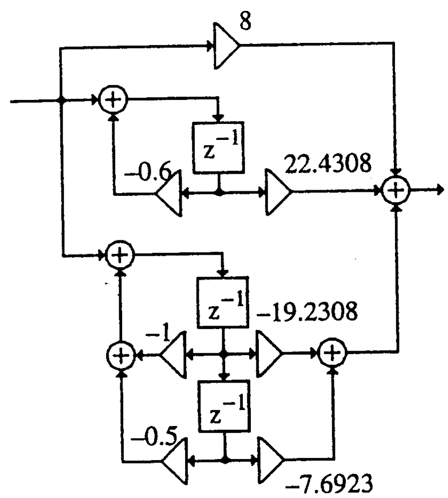
(c) $H_3(z) = 30 + \dfrac{7.6923 - j11.5358}{1+(0.5-j0.5)z^{-1}} + \dfrac{7.6923 + j11.5358}{1+(0.5+j0.5)z^{-1}} - \dfrac{37.3846}{1+0.6z^{-1}}$

$= 30 + \dfrac{15.3846 - 3.8462z^{-1}}{1+z^{-1}+0.5z^{-2}} - \dfrac{37.3846}{1+0.6z^{-1}}$ which leads to Parallel Form I realization shown below:



$H_3(z) = 8 + \dfrac{(-9.6154 - j1.9231)z^{-1}}{1+(0.5-j0.5)z^{-1}} + \dfrac{(-9.6154 + j1.9231)z^{-1}}{1+(0.5+j0.5)z^{-1}} + \dfrac{22.4308z^{-1}}{1+0.6z^{-1}}$

$= 8 + \dfrac{-19.2308z^{-1} - 7.6923z^{-2}}{1+z^{-1}+0.5z^{-2}} + \dfrac{22.4308z^{-1}}{1+0.6z^{-1}}$ which leads to Parallel Form II realization shown below:



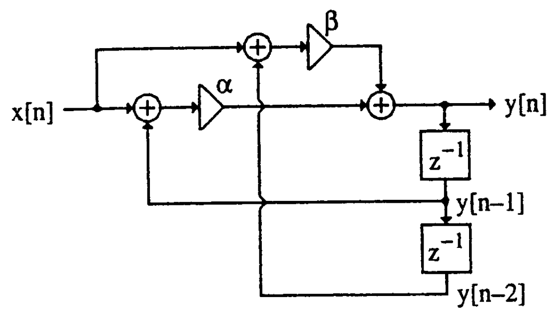
6.29 The structure of Figure P.611 is seen to be a Parallel Form II realization. A direct partial fraction expansion of H(z) as a ratio of polynomials in z results in

$H(z) = 3 + \dfrac{2}{z+0.5} - \dfrac{2}{z+2} = 3 + \dfrac{2z^{-1}}{1+0.5z^{-1}} - \dfrac{2z^{-1}}{1+2z^{-1}}$. Comparing this expansion with the structure of Figure P6.11, we observe that both multiplier coefficients in the top path are incorrect. The multiplier coefficient of value 5 should have a value of -2, and the multiplier coefficient of value –2 should have a value of 2.

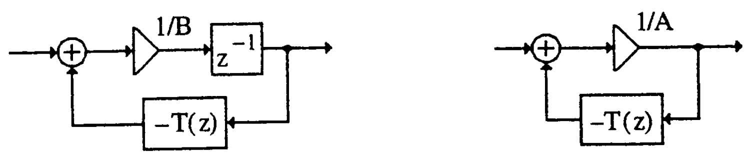
6.30 The structure of Figure P.612 is seen to be a Parallel Form I realization. A direct partial fraction expansion of H(z) as a ratio of polynomials in z^{-1} results in

$H(z) = \frac{3}{1+0.5z^{-1}} + \frac{2}{1+2z^{-1}}$. Comparing this expansion with the structure of Figure P6.12, we observe that A = 3 and B = −1/3.

6.31 The difference equation corresponding to the transfer function $H(z) = \frac{Y(z)}{X(z)} = \frac{1+\alpha+\beta}{1+\alpha z^{-1}+\beta z^{-2}}$ is given by $y[n]+\alpha y[n-1]+\beta y[n-2] = (1+\alpha+\beta)x[n]$, which can be rewritten in the form $y[n] = x[n]+\alpha(x[n]-y[n-1])+\beta(x[n]-y[n-2])$. A realization of H(z) based on this equation is thus as shown below:



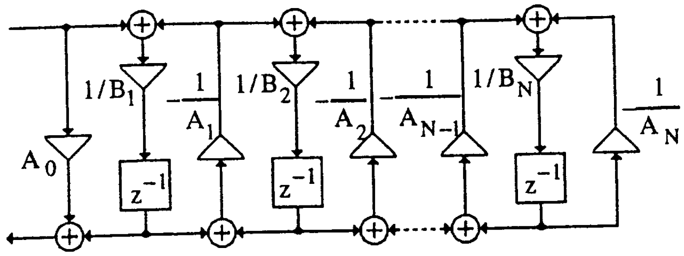
6.32 In order to implement H(z) in the form of Eq. (6.145), we need building blocks that realize the two functions: $G_1(z) = \frac{1}{Bz+T(z)}$, and $G_2(z) = \frac{1}{A+T(z)}$, where A and B are real. Figures below show possible realization of $G_1(z)$ and $G_2(z)$.



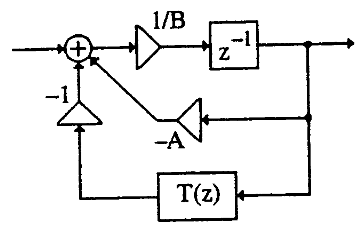
To develop a realization of H(z) of Eq. (6.145), we first rewrite it as $H(z) = A_0 + \frac{1}{B_1 z + T_1(z)}$, where

$$T_1(z) = \cfrac{1}{A_1 + \cfrac{1}{B_1 z + \cfrac{1}{A_2 + \cfrac{1}{\ddots + \cfrac{1}{B_N z + \cfrac{1}{A_N}}}}}}.$$

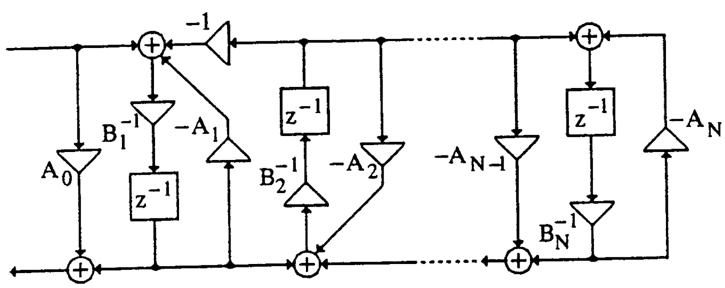
Note that the second term in the expression for H(z) is similar to that of $G_1(z)$. Next, T1(z) is expressed in the form of $G_2(z)$ and realized accordingly. This process is repeated leading to the complete realization shown below:



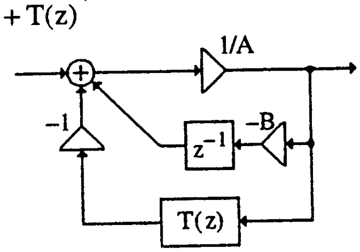
6.33 The basic buiding block in the implementation of H(z) of Eq. (6.146) has a transfer function of the form $G_3(z) = \dfrac{1}{Bz + A + T(z)}$, whose realization is sketched below:



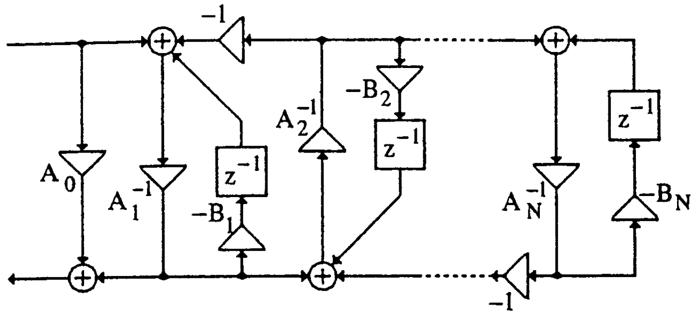
The complete realization of H(z) is as indicated below for N odd.



6.34 The basic buiding block in the implementation of H(z) of Eq. (6.147) has a transfer function of the form $G_4(z) = \dfrac{1}{Bz^{-1} + A + T(z)}$, whose realization is sketched below:



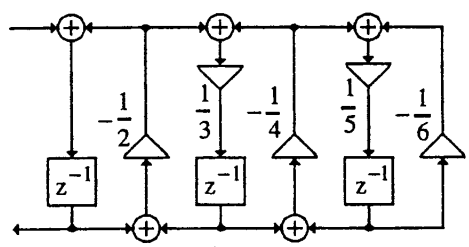
The complete realization of H(z) is as indicated below for N odd. It can be seen that the structure shown has delay-free-loops (for example, the loop containing the multipliers $1/A_1$ and $1/A_2$) and is therefore not ralizable.



6.35 A continued fraction expansion of $H(z) = \dfrac{720z^2 + 240z + 12}{720z^3 + 600z^2 + 72z + 1}$ in the form of Eq. (6.145) is given by

$$H(z) = \cfrac{1}{z + \cfrac{1}{2 + \cfrac{1}{3z + \cfrac{1}{4 + \cfrac{1}{5z + \cfrac{1}{6}}}}}},$$

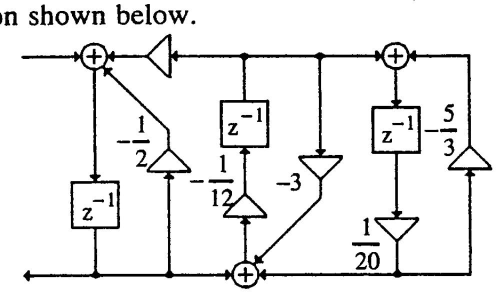
resulting in the realization shown below.



Likewise, a continued fraction expansion of H(z) in the form of Eq. (6.146) is given by

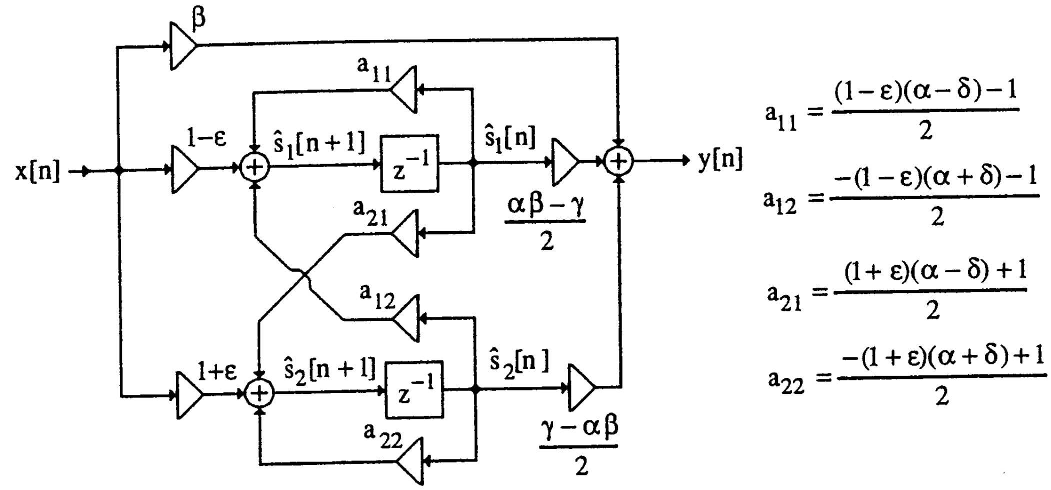
$$H(z) = \cfrac{1}{z + \frac{1}{2} + \cfrac{1}{-12z + 3 + \cfrac{1}{20z + \cfrac{5}{3}}}},$$

resulting in the realization shown below.



6.36 The new state-space parameters are $\hat{\mathbf{A}} = \frac{1}{2}\begin{bmatrix} (1-\varepsilon)(\alpha-\delta)-1 & -(1-\varepsilon)(\alpha+\delta)-1 \\ (1+\varepsilon)(\alpha-\delta)+1 & -(1+\varepsilon)(\alpha+\delta)+1 \end{bmatrix},$

$\hat{\mathbf{B}} = \begin{bmatrix} 1-\varepsilon \\ 1+\varepsilon \end{bmatrix}$, $\hat{C} = \frac{1}{2}[\alpha\beta-\gamma \quad \gamma-\alpha\beta]$, and $\hat{D} = \beta$. The corresponding state-space structure is shown below:



6.37 **(a)** From the state-space description, $\mathbf{A} = \begin{bmatrix} -0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{C} = [2 \quad 3]$, and D = 2, we arrive at $H(z) = \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + D = [2 \quad 3]\begin{bmatrix} z+0.5 & -0.5 \\ -1 & z \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2$

$$= \frac{1}{z^2+0.5z-0.5}[2 \quad 3]\begin{bmatrix} z & 0.5 \\ 1 & z+0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 = \frac{2z^2+3z+2}{z^2+0.5z-0.5}.$$

(b) $\hat{\mathbf{A}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -0.5 & 0.5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & -1 \end{bmatrix}$, $\hat{\mathbf{B}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$\hat{\mathbf{C}} = [2 \quad 3]\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [5 \quad -1]$, and $\hat{D} = 2$.

6.38 **(a)** From the state-space description, $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{C} = [1 \quad 3]$, and D = 3, we arrive at $H(z) = \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + D = [1 \quad 3]\begin{bmatrix} z-2 & 1 \\ -3 & z-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3$

$$= \frac{1}{z^2-3z+5}[1 \quad 3]\begin{bmatrix} z-1 & -1 \\ 3 & z-2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 = \frac{3z^2-2z+9}{z^2-3z+5}.$$

(b) $\hat{\mathbf{A}} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1.25 & 2.25 \\ -1.25 & 1.75 \end{bmatrix}$, $\hat{\mathbf{B}} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$,

$\hat{\mathbf{C}} = [1 \quad 3]\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} = [8 \quad -2]$, and $\hat{D} = 3$.

6.39 From Figure P6.13 we get $s_1[n+1] = \alpha s_1[n] - \alpha\beta s_2[n] + (1-\alpha\beta)x[n]$, $s_2[n+1] = s_1[n]$, and $y[n] = -\alpha s_1[n] + (1+\alpha\beta)s_2[n] + \alpha\beta x[n]$. Hence, the statespace parameters are given by

$\mathbf{A}=\begin{bmatrix}\alpha & -\alpha\beta\\ 1 & 0\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}1-\alpha\beta\\ 0\end{bmatrix}$, $\mathbf{C}=[-\alpha \quad 1+\alpha\beta]$, and $\mathbf{D}=\alpha\beta$. Therefore,

$$H(z)=[-\alpha \quad 1+\alpha\beta]\begin{bmatrix}z-\alpha & \alpha\beta\\ -1 & z\end{bmatrix}^{-1}\begin{bmatrix}1-\alpha\beta\\ 0\end{bmatrix}+\alpha\beta=\frac{\beta z^2-\alpha z+1}{z^2-\alpha z+\beta}.$$

6.40 **(a)** From the structure of Figure P.6/14 it follows that $H_N(z)=\dfrac{Y_1}{X_1}=\dfrac{C+DH_{N-1}(z)}{A+BH_{N-1}(z)}$, from which we get $H_{N-1}(z)=\dfrac{C-AH_N(z)}{BH_N(z)-D}$. Substituting the expression for $H_N(z)$ we then arrive at

$$H_{N-1}(z)=\frac{C\left(1+\sum_{i=1}^{N}d_iz^{-i}\right)-A\left(\sum_{i=0}^{N}p_iz^{-i}\right)}{B\left(\sum_{i=0}^{N}p_iz^{-i}\right)-D\left(1+\sum_{i=1}^{N}d_iz^{-i}\right)}$$

$$=\frac{(C-Ap_0)+(Cd_1-Ap_1)z^{-1}+\cdots++(Cd_{N-1}-Ap_{N-1})z^{-N+1}++(Cd_N-Ap_N)z^{-N}}{(Bp_0-D)+(Bp_1-Dd_1)z^{-1}+\cdots+(Bp_{N-1}-Dd_{N-1})z^{-N+1}+(Bp_N-Dd_N)z^{-N}}.$$

Substituting the values $A = 1$, $B = d_Nz^{-1}$, $C = p_0$, and $D = p_Nz^{-1}$, we get $H_{N-1}(z)$

$$=\frac{(p_0-p_0)+(p_0d_1-p_1)z^{-1}+\cdots+(p_0d_{N-1}-p_{N-1})z^{-N+1}+(p_0d_N-p_N)z^{-N}}{(d_Np_0-p_N)z^{-1}+(d_Np_1-p_Nd_1)z^{-2}+\cdots+(d_Np_{N-1}-p_Nd_{N-1})z^{-N}+(d_Np_N-p_Nd_N)z^{-N-1}}$$

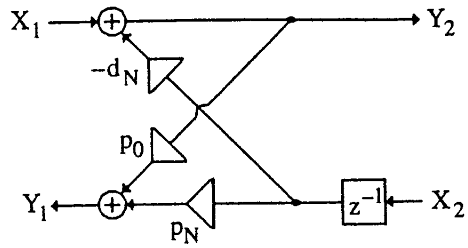
$$=\frac{(p_0d_1-p_1)+(p_0d_2-p_2)z^{-1}+\cdots+(p_0d_{N-1}-p_{N-1})z^{-N+2}+(p_0d_N-p_N)z^{-N+1}}{(p_0d_N-p_N)+(p_1d_N-p_Nd_1)z^{-1}+\cdots+(p_{N-1}d_N-p_Nd_{N-1})z^{-N+1}}$$

$$=\frac{p_0'+p_1'z^{-1}+\cdots+p_{N-2}'z^{-N+2}+p_{N-1}'z^{-N+1}}{1+d_1'z^{-1}+\cdots+d_{N-1}'z^{-N+1}}$$ where

$$p_k'=\frac{p_kd_{k+1}-p_{k+1}}{p_0d_N-p_N},\quad k=0,1,\ldots,N-1,\text{ and } d_k'=\frac{p_kd_N-p_Nd_k}{p_0d_N-p_N},\quad k=1,2,\ldots,N-\cdot.$$

(b) From the chain parameters, we obtain for the first stage $t_{11}=\dfrac{C}{A}=p_0$, $t_{12}=\dfrac{AD-BC}{A}=(p_N-p_0d_N)z^{-1}$, $t_{21}=\dfrac{1}{A}=1$, and $t_{22}=-\dfrac{B}{A}=-d_Nz^{-1}$. The corresponding input-output relations are then given by

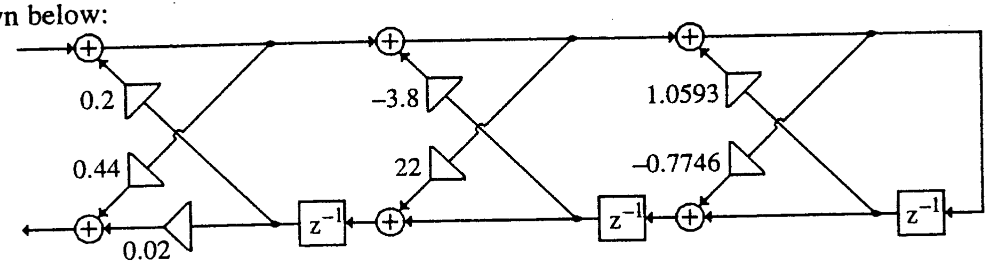
$Y_1=p_0X_1+(p_N-p_0d_N)z^{-1}=p_0(X_1-d_N\,z^{-1}X_2)+p_Nz^{-1}X_2$,
$Y_2=X_1-d_N\,z^{-1}X_2$. Substituting the second equation into the first we rewrite it as
$Y_1=p_0Y_2+p_Nz^{-1}X_2$. A realization of the two-pair based on the last two equations is therefore as indicated below:



Except for the first stage, all other stages require 2 multipliers. Hence the total number of multipliers needed to implement an N-th order transfer function $H_N(z)$ is 2N+1. The total number of two-input adders required is 2N while the overall realization is canonic requiring N delays.

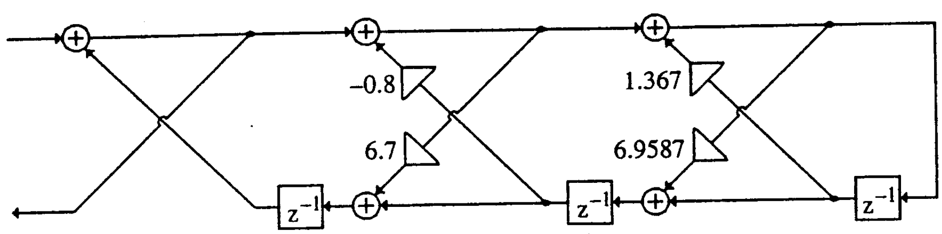
6.41 From $H_3(z)=\dfrac{0.44z^{-1}+0.362z^{-2}+0.002z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$, using Eq. (6.152) we arrive at $H_2(z)=\dfrac{22+18.1z^{-1}+z^{-2}}{1+4.8z^{-1}+8.8z^{-2}}$. Repeating the procedure we obtain $H_1(z)=\dfrac{1.0593+z^{-1}}{1+0.7746z^{-1}}$.

From $H_3(z)$, $H_2(z)$ and $H_1(z)$ we then arrive at the cascaded lattice realization of $H_3(z)$ as shown below:



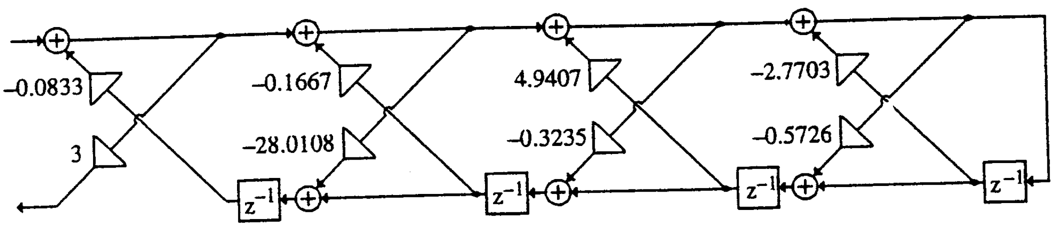
6.42 **(a)** From $H_3(z)=\dfrac{1+4.2z^{-1}+0.8z^{-2}}{1-2.5z^{-1}+3z^{-2}-z^{-3}}$, using Eq. (6.152) we arrive at $H_2(z)=\dfrac{6.7-2.2z^{-1}+z^{-2}}{1+4.2z^{-1}+0.8z^{-2}}$. Repeating the procedure we obtain $H_1(z)=\dfrac{6.9587+z^{-1}}{1-1.367z^{-1}}$.

From $H_3(z)$, $H_2(z)$ and $H_1(z)$ we then arrive at the cascaded lattice realization of $H_3(z)$ as shown below:

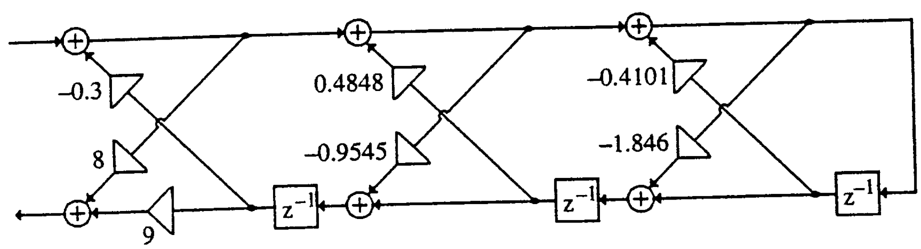


(b) From $H_4(z)=\dfrac{3+1.5z^{-1}+z^{-2}+0.5z^{-3}}{1-1.8333z^{-1}+1.5z^{-2}-0.5833z^{-3}+0.0833z^{-4}}$, using Eq. (6.152) we arrive at $H_3(z)=\dfrac{-28.0108+14.0056z^{-1}-9.0032z^{-2}+z^{-3}}{1+0.5z^{-1}+0.3333z^{-2}+0.1667z^{-3}}$, Repeating the procedure we obtain

$H_2(z) = \frac{4.0407 + 0.0587z^{-1} + z^{-2}}{1 - 0.3236z^{-1} + 0.3235z^{-2}}$, and $H_1(z) = \frac{-2.7703 + z^{-1}}{1 + 0.5726z^{-1}}$. From $H_4(z)$, $H_3(z)$, $H_2(z)$ and $H_1(z)$ we then arrive at the cascaded lattice realization of $H_4(z)$ as shown below:



(c) From $H_3(z) = \frac{8 + 16z^{-1} + 12z^{-2} + 9z^{-3}}{1 + 1.6z^{-1} + 1.1z^{-2} + 0.3z^{-3}}$, using Eq. (6.152) we arrive at $H_2(z) = \frac{0.4848 + 0.4848z^{-1} + z^{-2}}{1 + 1.45.45z^{-1} + 0.9545z^{-2}}$. Repeating the procedure we obtain $H_1(z) = \frac{-0.4101 + z^{-1}}{1 + 1.846z^{-1}}$. From $H_3(z)$, $H_2(z)$ and $H_1(z)$ we then arrive at the cascaded lattice realization of $H_3(z)$ as shown below:



6.43 When $H_N(z)$ is an allpass transfer function of the form

$H_N(z) = A_N(z) = \frac{d_N + d_{N-1}z^{-1} + \cdots + z^{-N}}{1 + d_1 z^{-1} + \cdots + d_N z^{-N}}$, then from Eq. (6.152a), the numerator coefficients of $H_{N-1}(z)$ are given by $p_k' = \frac{p_0 d_{k+1} - p_{k+1}}{p_0 d_N - p_N} = \frac{d_N d_{k+1} - d_{N-k-1}}{d_N^2 - 1}$, and

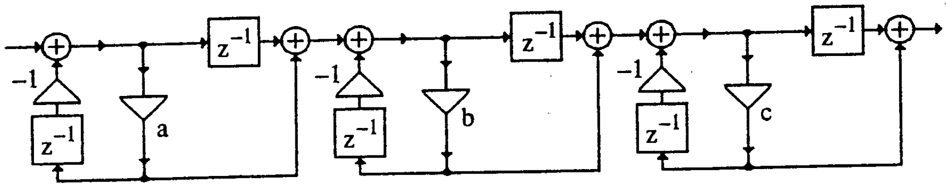
$d'_{N-k-1} = \frac{p_{N-k-1} d_N - d_{N-k-1}}{p_0 d_N - p_N} = \frac{d_{k+1} d_N - d_{N-k-1}}{d_N^2 - 1} = p_k'$, implying $H_{N-1}(z)$ is an allpass transfer function of order N–1. Since here $p_N = 1$ and $p_0 = d_N$, the lattice structure of Problem 6.40 then reduces to the lattice structure employed in the Gray-Markel realization procedure.

6.44 (a) Consider the realization of Type 1B allpass transfer function. From its transfer parameters given in Eq. (6.62b) we arrive at $Y_1 = z^{-1}X_1 + (1 + z^{-1})X_2 = z^{-1}(X_1 + X_2) + X_2$, and $Y_2 = (1 - z^{-1})X_1 - z^{-1}X_2 = X_1 - z^{-1}(X_1 + X_2)$. A realization of the two-pair based on these two equations is as shown below which leads to the structure of Figure 6.36(b).

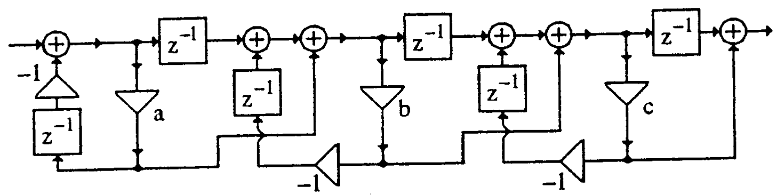
(b) From the transfer parameters of Type $1A_t$ allpass given in Eq. (6.62c) we obtain $Y_1 = z^{-1}X_1 + X_2$, and $Y_2 = (1 - z^{-2})X_1 - z^{-1}X_2 = X_1 - z^{-1}(z^{-1}X_1 + X_2) = X_1 - z^{-1}Y_1$. A realization of the two-pair based on these two equations is as shown below which leads to the structure of Figure 6.36(c).

(c) From the transfer parameters of Type $1B_t$ allpass given in Eq. (6.62d) we obtain $Y_1 = z^{-1}X_1 + (1 - z^{-1})X_2 = z^{-1}(X_1 - X_2) + X_2$, and $Y_2 = (1 + z^{-1})X_1 - z^{-1}X_2 = X_1 + z^{-1}(X_1 - X_2)$. A realization of the two-pair based on these two equations is as shown below which leads to the structure of Figure 6.36(d).

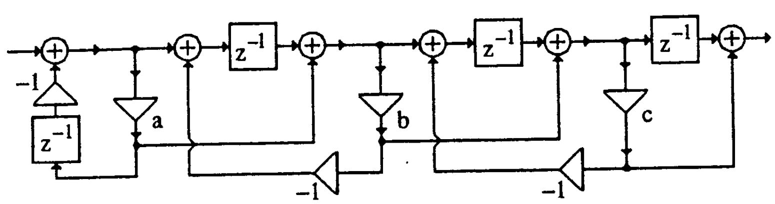
6.45 **(a)** A cascade connection of three Type 1A first-order allpass networks is a shown below which is seen to require 6 delays:



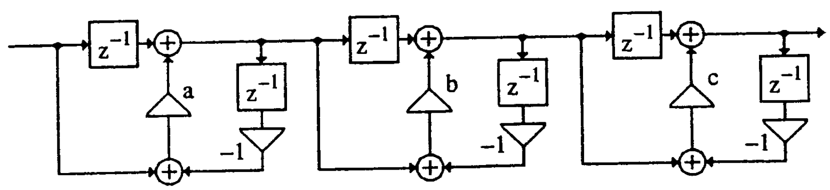
Simple block-diagram manipulations results in the structure shown below:



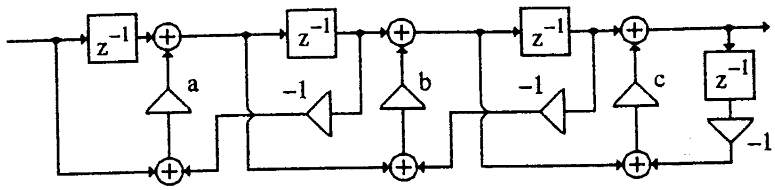
By delay-sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 4 delays.



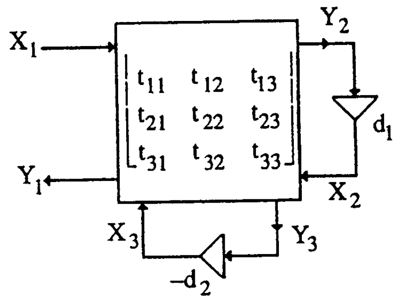
(b) A cascade connection of three Type $1A_t$ first-order allpass networks is a shown below which is seen to require 6 delays:



By delay-sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 4 delays.



6.46 We realize $A_2(z)=\dfrac{d_1d_2+d_1z^{-1}+z^{-2}}{1+d_1z^{-1}+d_1d_2z^{-2}}$ in the form of a constrained three-pair as indicated below:



From the above figure, we have $\begin{bmatrix}Y_1\\Y_2\\Y_3\end{bmatrix}=\begin{bmatrix}t_{11} & t_{12} & t_{13}\\t_{21} & t_{22} & t_{23}\\t_{31} & t_{32} & t_{33}\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}$, and $X_2=d_1Y_2$, $X_3=-d_2Y_3$.

From these equations, after some algebra we get $A_2(z)=\dfrac{Y_1}{X_1}=\dfrac{N(z)}{D(z)}$, where

$D(z)=1-d_1t_{22}+d_2t_{33}+d_1d_2(t_{23}t_{32}-t_{22}t_{33})$, and

$$N(z)=t_{11}-d_1(t_{11}t_{22}-t_{12}t_{21})+d_2(t_{11}t_{33}-t_{13}t_{31})$$
$$+d_1d_2\{t_{21}(t_{12}t_{33}-t_{13}t_{32})+t_{31}(t_{22}t_{13}-t_{12}t_{23})+t_{11}(t_{23}t_{32}-t_{22}t_{33})\}.$$

Comparing trhe denominator of the desired allpass transfer function with D(z) we get $t_{22}=z^{-1}$, $t_{33}=0$, $t_{23}t_{32}=z^{-2}$. Next, comparing trhe numerator of the desired allpass transfer function with N(z) we get $t_{11}=z^{-1}$, $t_{12}t_{21}=z^{-1}(z^{-2}-1)$, $t_{13}t_{31}=0$, and $t_{32}(t_{11}t_{23}-t_{21}t_{13})+t_{31}(t_{22}t_{13}-t_{12}t_{23})=1$. Substituting the appropriate transfer parameters from the previous equations into the last equation we simplify it to $t_{13}t_{21}t_{32}+t_{31}t_{12}t_{23}=z^{-4}-1$. Since $t_{13}t_{31}=0$, either $t_{13}=0$, or $t_{31}=0$. (Both cannot be simultaneously equal to zero, as this will violate the condition $t_{13}t_{21}t_{32}+t_{31}t_{12}t_{23}=z^{-4}-1$.

Consider the case $t_{13}=0$. Then the equation above reduces to $t_{31}t_{12}t_{23}=z^{-4}-1$. From this equation and $t_{23}t_{32}=z^{-2}$. it follows that

$t_{32}=z^{-2}$, $t_{23}=1$, $t_{31}t_{12}=z^{-4}-1=(z^{-1}-1)(z^{-1}+1)(z^{-2}+1)$.

There are four possible realizable sets of values of t_{21} and t_{31} satisfying the last equation and $t_{12}t_{21} = z^{-1}(z^{-2}-1)$. These lead to four different realizable transfer matrices for the three-pair:

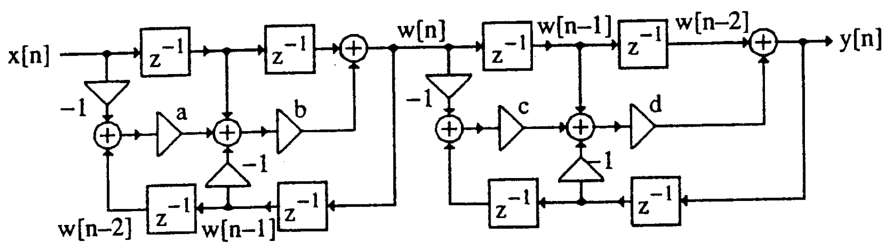
Type 2A: $\begin{bmatrix} z^{-2} & z^{-2}-1 & 0 \\ z^{-1} & z^{-1} & 1 \\ z^{-2}+1 & z^{-2} & 0 \end{bmatrix}$, Type 2B: $\begin{bmatrix} z^{-2} & z^{-1}+1 & 0 \\ z^{-1}(z^{-1}-1) & z^{-1} & 1 \\ (z^{-2}+1)(z^{-1}-1) & z^{-2} & 0 \end{bmatrix}$,

Type 2C: $\begin{bmatrix} z^{-2} & z^{-1}-1 & 0 \\ z^{-1}(z^{-1}+1) & z^{-1} & 1 \\ (z^{-2}+1)(z^{-1}+1) & z^{-2} & 0 \end{bmatrix}$, Type 2D: $\begin{bmatrix} z^{-2} & 1 & 0 \\ z^{-1}(z^{-2}-1) & z^{-1} & 1 \\ z^{-4}-1 & z^{-2} & 0 \end{bmatrix}$.

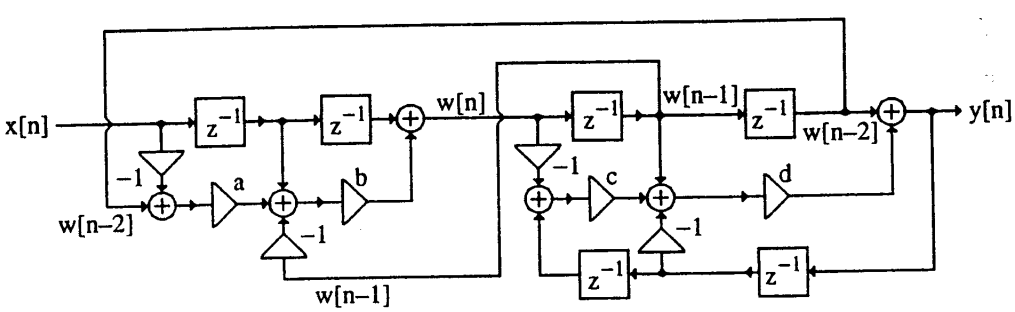
A realization of each Type 2 allpass structures is obtained by implementing its respective transfer matrix, and then constraining the Y_2 and X_2 variables through the multiplier d_1 and constraining the Y_3 and X_3 variables through the multiplier $-d_2$ resulting in the four structures shown in Figure 6.38 of text.

It can be easily shown that the allpass structures obtained for the case $t_{31} = 0$ are precisely the transpose of the structures of Figure 6.38.

6.47 A cascade connection of two Type 2D second-order allpass networks is a shown below which is seen to require 8 delays:



By delay-sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 6 delays.



The minimum number of multipliers needed to implement a cascade of M Type 2D second-order allpass sections is thus 4 + 2(M–1) = 2(M + 1).

6.48 We realize $A_2(z) = \dfrac{d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}}$ in the form of a constrained three-pair as indicated in the figure in the solution of Problem 6.46. Comparing the numerator and the denominator of the Type 3 allpass transfer function with N(z) and D(z) given in the solution of Problem 6.46 we arrive at $t_{11} = z^{-2}$, $t_{22} = z^{-1}$, $t_{33} = z^{-2}$, $t_{23}t_{32} = z^{-2}$, $t_{23}t_{32} = z^{-1}(z^{-2} - 1)$, $t_{13}t_{31} = z^{-4} - 1$, $t_{13}t_{21}t_{32} + t_{13}t_{21}t_{32} = z^{-3}(z^{-2} - 1) + z^{-1}(z^{-4} - 1)$. To solve the last four equations, we preselect t_{23} and t_{32} satiasfying $t_{23}t_{32} = z^{-2}$,and then determine realizable values for t_{12}, t_{21}, t_{13}, and t_{31},

Choice #1: $t_{23} = z^{-1}$, $t_{32} = z^{-2}$. This leads to four possible realizable sets of values of t_{12}, t_{21}, t_{13}, and t_{31}, satisfying the constraint equations given earlier and resulting in the transfer matrices given below:

Type 3A: $\begin{bmatrix} z^{-2} & z^{-2}-1 & z^{-2}-1 \\ z^{-1} & z^{-1} & z^{-1} \\ z^{-2}+1 & z^{-2} & z^{-2} \end{bmatrix}$, Type 3B: $\begin{bmatrix} z^{-2} & z^{-1}+1 & z^{-1}+1 \\ z^{-1}(z^{-1}-1) & z^{-1} & z^{-1} \\ (z^{-1}-1)(z^{-2}+1) & z^{-2} & z^{-2} \end{bmatrix}$,

Type 3C: $\begin{bmatrix} z^{-2} & z^{-1}-1 & z^{-1}-1 \\ z^{-1}(z^{-1}+1) & z^{-1} & z^{-1} \\ (z^{-1}+1)(z^{-2}+1) & z^{-2} & z^{-2} \end{bmatrix}$, Type 3D: $\begin{bmatrix} z^{-2} & 1 & 1 \\ z^{-1}(z^{-2}-1) & z^{-1} & z^{-1} \\ z^{-4}-1 & z^{-2} & z^{-2} \end{bmatrix}$.

A realization of each Type 3 allpass structures is obtained by implementing its respective transfer matrix, and then constraining the Y_2 and X_2 variables through the multiplier d_1 and constraining the Y_3 and X_3 variables through the multiplier $-d_2$. Realizations of Types 3A, 3C and 3D allpass are shown in Figure 6.38 of text. The realization of Type 3B allpass is shown below:

Choice #2: $t_{23} = 1$, $t_{32} = z^{-3}$. This leads to four possible realizable sets of values of t_{12}, t_{21}, t_{13}, and t_{31}, satisfying the constraint equations given earlier and resulting in the transfer matrices given below:

Type 3E: $\begin{bmatrix} z^{-2} & z^{-1}(z^{-2}-1) & z^{-2}-1 \\ 1 & z^{-1} & 1 \\ z^{-2}+1 & z^{-3} & z^{-2} \end{bmatrix}$, Type 3F: $\begin{bmatrix} z^{-2} & z^{-1}(z^{-1}+1) & z^{-1}+1 \\ z^{-1}-1 & z^{-1} & 1 \\ (z^{-1}-1)(z^{-2}+1) & z^{-3} & z^{-2} \end{bmatrix}$,

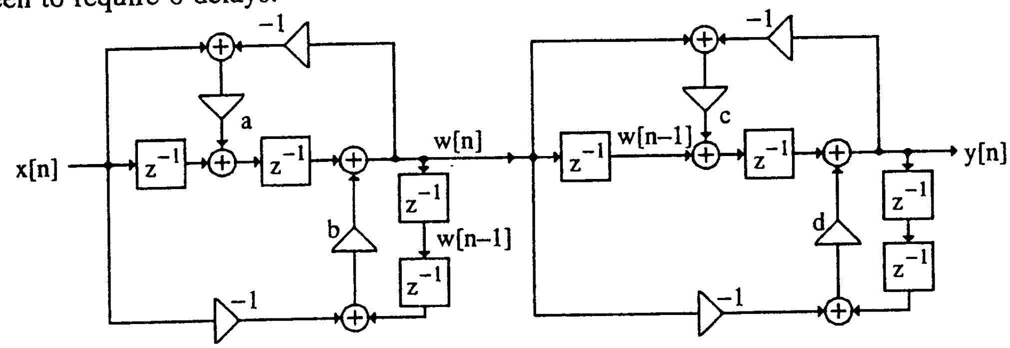
Type 3G: $\begin{bmatrix} z^{-2} & z^{-1}(z^{-1}-1) & z^{-1}-1 \\ z^{-1}+1 & z^{-1} & 1 \\ (z^{-1}+1)(z^{-2}+1) & z^{-3} & z^{-2} \end{bmatrix}$, Type 3H: $\begin{bmatrix} z^{-2} & z^{-1}(z^{-1}-1) & z^{-1}-1 \\ z^{-2}-1 & z^{-1} & 1 \\ z^{-4}-1 & z^{-3} & z^{-2} \end{bmatrix}$.

Realization of Types 3H is shown in Figure 6.38 of text. The realizations of Types 3E, 3F and 3G allpass are shown below:

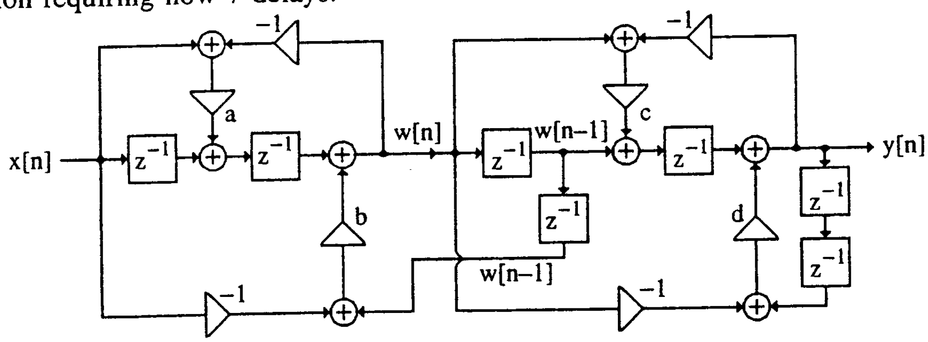
Choice #3: $t_{23} = z^{-2}$, $t_{32} = z^{-1}$. The structures in this case are the transpose of the Types 3A, 3B, 3C and 3D allpass networks of Choice #1 given above.

Choice #4: $t_{23} = z^{-3}$, $t_{32} = 1$. The structures in this case are the transpose of the Types 3E, 3F, 3G and 3H allpass networks of Choice #2 given above.

6.49 A cascade connection of two Type 3H second-order allpass networks is a shown below which is seen to require 8 delays:



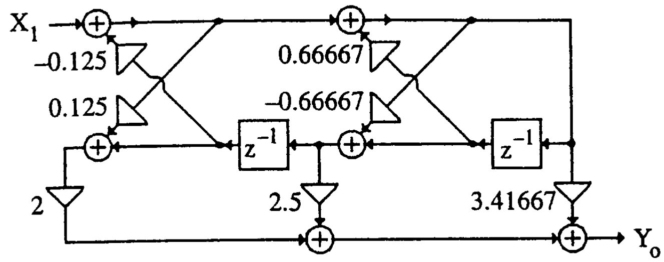
By delay-sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 7 delays.



The minimum number of multipliers needed to implement a cascade of M Type 3H second-order allpass sections is thus 4 + 3(M–1) = 3M + 1.

6.50 **(a)** $H_1(z) = \dfrac{2+z^{-1}+2z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}$. Choose, $A_2(z) = \dfrac{0.125-0.75z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}$. Note $k_2 = A_2(\infty) = d_2 = 0.125 < 1$. Using Eq. (6.71) we next determine the coefficients of $A_1(z)$ arriving at $A_1(z) = \dfrac{0.66667+z^{-1}}{1-0.66667z^{-1}}$. Here, $k_1 = A_1(\infty) = d_1' = -0.66667$, Hence, $|k_1| = 0.66667 < 1$. Therefore, $A_2(z)$ and hence $H_1(z)$ is stable.

To determine the feed-forward coefficients we use $\alpha_1 = p_2 = 2$, $\alpha_2 = p_1 - \alpha_1 d_1 = 2.5$, $\alpha_3 = p_0 - \alpha_1 d_2 - \alpha_2 d_1' = 3.41667$. Final realization of $H_1(z)$ is thus as shown below:

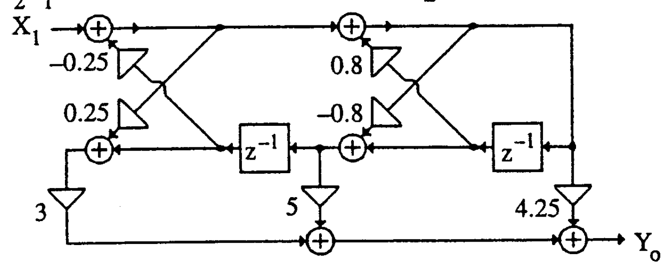


(b) $H_2(z) = \dfrac{1+2z^{-1}+3z^{-2}}{1-z^{-1}+0.25z^{-2}}$. Thus, $A_2(z) = \dfrac{0.25-z^{-1}+z^{-2}}{1-z^{-1}+0.25z^{-2}}$. Note $k_2 = A_2(\infty) = d_2 = 0.25 < 1$.

Using Eq. (6.71) we next determine the coefficients of $A_1(z)$ arriving at $A_1(z) = \dfrac{-0.8+z^{-1}}{1-0.8z^{-1}}$.

Here, $k_1 = A_1(\infty) = -0.8$. Thus, $|k_1| = 0.8 < 1$. Therefore, $A_2(z)$ and hence $H_2(z)$ is stable.

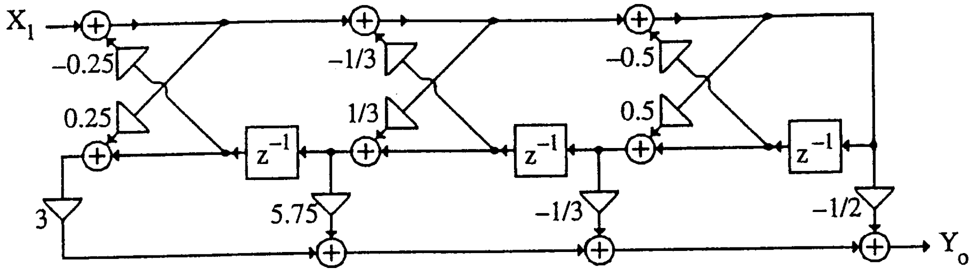
To determine the feed-forward coefficients we use $\alpha_1 = p_2 = 3$, $\alpha_2 = p_1 - \alpha_1 d_1 = 5$, $\alpha_3 = p_0 - \alpha_1 d_2 - \alpha_2 k_1 = 4.25$. Final realization of $H_2(z)$ is thus as shown below:



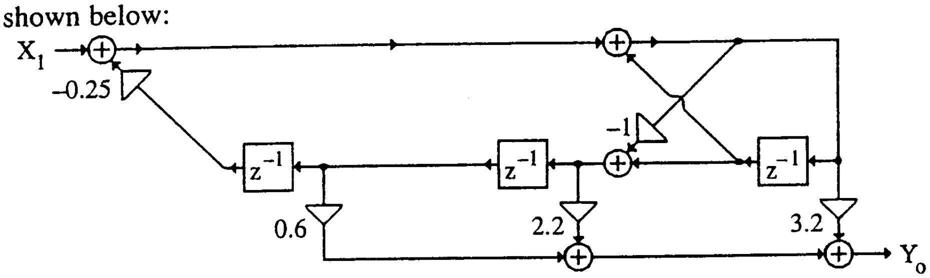
(c) $H_3(z) = \dfrac{2+5z^{-1}+8z^{-2}+3z^{-3}}{1+0.75z^{-1}+0.5z^{-2}+0.25z^{-3}}$. This implies, $A_3(z) = \dfrac{0.25+0.5z^{-1}+0.75z^{-2}+z^{-3}}{1+0.75z^{-1}+0.5z^{-2}+0.25z^{-3}}$.
Note $k_3 = A_3(\infty) = d_3 = 0.25 < 1$. Using Eq. (6.71) we next determine the coefficients of $A_2(z)$ arriving at $A_2(z) = \dfrac{\frac{1}{3}+\frac{2}{3}z^{-1}+z^{-2}}{1+\frac{2}{3}z^{-1}+\frac{1}{3}z^{-2}}$. Note $k_2 = A_2(\infty) = d_2' = 1/3 < 1$. Using Eq. (6.71) we next determine the coefficients of $A_1(z)$ arriving at $A_1(z) = \dfrac{0.5+z^{-1}}{1+0.5z^{-1}}$. Thus,

$k_1 = A_1(\infty) = d_1'' = 0.5 < 1$. Since $|k_i| < 1$ for i = 3, 2, 1, $A_3(z)$ and hence $H_3(z)$ is stable.

To determine the feed-forward coefficients we make use of Eq. (6.98) and obtain $\alpha_1 = p_3 = 3$, $\alpha_2 = p_2 - \alpha_1 d_1 = 5.75$, $\alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = -\frac{1}{3}$. $\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = -\frac{1}{2}$.
Final realization of $H_3(z)$ is thus as shown below:



(d) $H_4(z)=\dfrac{1+1.6z^{-1}+0.6z^{-2}}{1-z^{-1}-0.25z^{-2}+0.25z^{-3}}$. This implies $A_3(z)=\dfrac{0.25-0.25z^{-1}-z^{-2}+z^{-3}}{1-z^{-1}-0.25z^{-2}+0.25z^{-3}}$. Note $k_3=A_3(\infty)=d_3=0.25<1$. Using Eq. (6.71) we next determine the coefficients of $A_2(z)$ arriving at $A_2(z)=\dfrac{-1+z^{-1}}{1-z^{-1}}$. Note $k_2=A_2(\infty)=d_2'=0$. Using Eq. (6.71) we next determine the coefficients of $A_1(z)$ arriving at $A_1(z)=\dfrac{-1+z^{-1}}{1-z^{-1}}$. $k_1=A_1(\infty)=d_1''=-1$. Since $|k_1|=1, A_3(z)$ and hence $H_4(z)$ is unstable. Feed-forward coefficients are next determined using Eq. (6.98) and are given by $\alpha_1=0, \alpha_2=0.6, \alpha_3=2.2, \alpha_4=3.2$. Final realization of $H_4(z)$ is thus as shown below:



(e) $H_5(z)=\dfrac{3+1.5z^{-1}+z^{-2}+0.5z^{-3}}{1-1.8333z^{-1}+1.5z^{-2}-0.5833z^{-3}+0.0833z^{-4}}$. Hence,

$A_4(z)=\dfrac{0.0833-0.5833z^{-1}+1.5z^{-2}-1.8333z^{-3}+z^{-4}}{1-1.8333z^{-1}+1.5z^{-2}-0.5833z^{-3}+0.0833z^{-4}}$. Note $k_4=A_4(\infty)=d_4=0.0833<1$.

Using Eq. (6.71) we next determine the coefficients of $A_3(z)$ arriving at

$A_3(z)=\dfrac{0.433595+1.38466z^{-1}-1.7972z^{-2}+z^{-3}}{1-1.7972z^{-1}+1.38466z^{-2}-0.433595z^{-3}}$. Thus, $k_3=A_3(\infty)=d_3'=0.433595<1$.

Continuing this process we obtain $A_2(z)=\dfrac{0.74558-1.4739z^{-1}+z^{-2}}{1-1.4739z^{-1}+0.74558z^{-2}}$. Thus,

$k_2=A_2(\infty)=d_2'=0.74558<1$. Finally we arrive at $A_1(z)=\dfrac{-0.84436+z^{-1}}{1-0.84436z^{-1}}$. This implies

$k_1=A_1(\infty)=d_1'''=-0.84436$. Since $|k_i|<1$ for i = 4, 3, 2, 1, $A_4(z)$ and hence $H_5(z)$ is stable.

Feed-forward coefficients are next determined using Eq. (6.98) and are given by $\alpha_1=0, \alpha_2=0.5, \alpha_3=1.89859, \alpha_4=3.606, \alpha_5=4.846$.

6.51 (a) $H(z)=\dfrac{(2+0.4z^{-1})(0.5+2z^{-1})}{(1-2z^{-1}+2z^{-2})(1-0.5z^{-1})}=\dfrac{1+4.2z^{-1}+0.8z^{-2}}{1-2.5z^{-1}+3z^{-2}-z^{-3}}$. We first form

$A_3(z)=\dfrac{-1+3z^{-1}-2.5z^{-2}+z^{-3}}{1-2.5z^{-1}+3z^{-2}-z^{-3}}$. Note that $k_3 = d_3 = -1$. Since $|k_3|=1$, H(z) is unstable. The realization method cannnot be continued any further indicating that Gray-Market realization does not exist for H(z).

(b) $H(z)=\dfrac{3z(z+\frac{1}{2})(z^2+\frac{1}{3})}{(z-\frac{1}{2})(z-\frac{1}{3})(z^2-z+\frac{1}{2})}$. We first form

$A_4(z)=\dfrac{0.08333-0.58333z^{-1}+1.5z^{-2}-1.8333z^{-3}+z^{-4}}{1-1.8333z^{-1}+1.5z^{-2}-0.58333z^{-3}+0.08333z^{-4}}$. Thus, $k_4 = d_4 = 0.08333$. Next we determine $A_3(z)=z\left[\dfrac{A_4(z)-k_4}{1-k_4A_4(z)}\right]=\dfrac{-0.433564+1.384615z^{-1}-1.7972z^{-2}+z^{-3}}{1-1.7972z^{-1}+1.384615z^{-2}-0.433564z^{-3}}$. Hence $k_3=d_3'=-0.43356$. Next form $A_2(z)=z\left[\dfrac{A_3(z)-k_3}{1-k_3A_3(z)}\right]=\dfrac{0.7456-1.14739z^{-1}+z^{-2}}{1-1.14739z^{-1}+0.7456z^{-2}}$. Hence $k_2=d_2''=0.7456$. Finally, $A_1(z)=z\left[\dfrac{A_2(z)-k_2}{1-k_2A_2(z)}\right]=\dfrac{-0.844398+z^{-1}}{1-0.844398z^{-1}}$. Thus, $k_1 = -0.844398$.

Since, $|k_i|<1$, for i = 4, 3, 2, 1, H(z) is stable.
The feedfoward multipliers in the Gray-Markel realizations are given by
$\alpha_1=3,\ \alpha_2=7,\ \alpha_3=9.08037,\ \alpha_4=5.94172$, and $\alpha_5=1.03218$.

(c) $H(z)=\dfrac{(2+3z^{-1})(4+2z^{-1}+3z^{-2})}{(1+0.6z^{-1})(1+z^{-1}+0.5z^{-2})}=\dfrac{8+16z^{-1}+12z^{-2}+9z^{-3}}{1+1.6z^{-1}+1.1z^{-2}+0.3z^{-3}}$. Thus,

$A_3(z)=\dfrac{0.3+1.1z^{-1}+1.6z^{-2}+z^{-3}}{1+1.6z^{-1}+1.1z^{-2}+0.3z^{-3}}$. Hence, $k_3 = 0.3$.

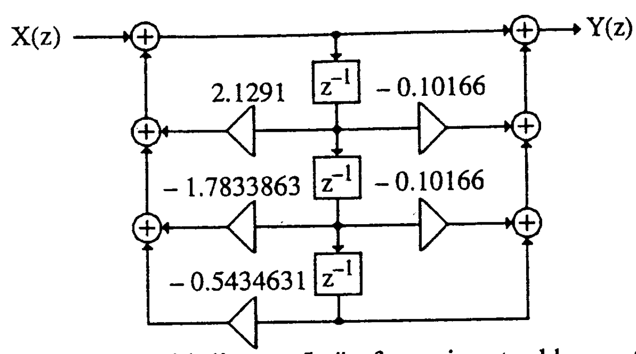
$A_2(z)=z\left[\dfrac{A_3(z)-k_3}{1-k_3A_3(z)}\right]=\dfrac{0.68132+1.3956z^{-1}+z^{-2}}{1+1.3956z^{-1}+0.68132z^{-2}}$. Hence, $k_2 = 0.68132$.

$A_1(z)=z\left[\dfrac{A_2(z)-k_2}{1-k_2A_2(z)}\right]=\dfrac{0.83+z^{-1}}{1+0.83z^{-1}}$. Hence, $k_1 = 0.83$. Since, $|k_i|<1$, for i = 4, 3, 2, 1, H(z) is stable.

The feedfoward multipliers in the Gray-Markel realizations are given by
$\alpha_1=9,\ \alpha_2=-2.4,\ \alpha_3=-0.5505,\ \alpha_4=7.3921$.

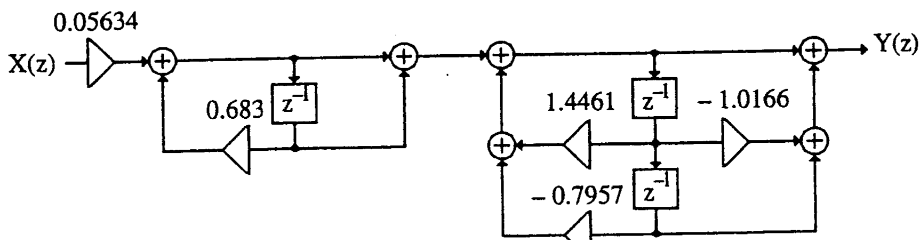
6.52 $H(z)=\dfrac{0.5634(1+z^{-1})(1-1.10166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$.

(a) Direct canonic form - $H(z)=\dfrac{0.05634(1-0.10166z^{-1}-0.10166z^{-2}+z^{-3})}{1-2.1291z^{-1}+1.78339z^{-2}-0.54346z^{-3}}$.



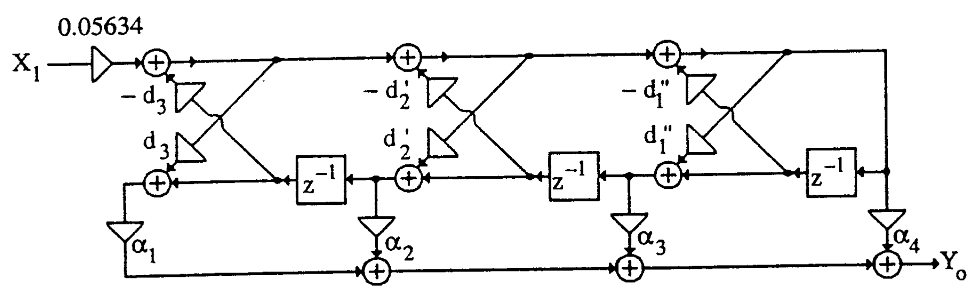
Hardware requirements: # of multipliers = 5, # of two-input adders = 6, # of delays = 3.

(b) Cascade Form



Hardware requirements: # of multipliers = 5, # of two-input adders = 6, # of delays = 3.

(c) Gray-Markel Form -



$d_3 = -0.5434631,\ \ d_2' = 0.8881135,\ \ d_1'' = -0.8714813.$

$\alpha_1 = p_3 = 1,\ \ \alpha_2 = p_2 - \alpha_1 d_1 = 2.02744,\ \ \alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = 1.45224,$

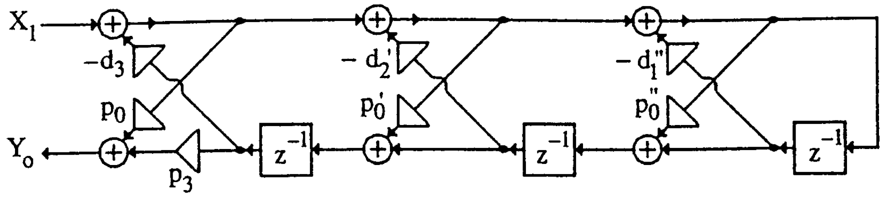
$\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = 1.00702.$

Hardware requirements: # of multipliers = 9, # of two-input adders = 6, # of delays = 3.

(d) Cascaded Lattice Structure -

Using Eqn. (6.152) we obtain $H_2(z) = \dfrac{p_0' + p_1' z^{-1} + p_2' z^{-2}}{1 + d_1' z^{-1} + d_2' z^{-2}} = \dfrac{1.3136 - 1.2213 z^{-1} + z^{-2}}{1 - 1.4152 z^{-1} + 1.1197 z^{-2}}.$

and $H_1(z) = \dfrac{p_0'' + p_1'' z^{-1}}{1 + d_1'' z^{-1}} = \dfrac{-1.3546 + z^{-1}}{1 - 0.1015 z^{-1}}.$ The final structure is as shown below:

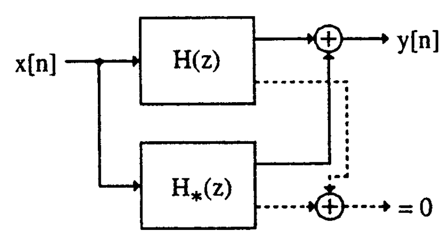


where $p_0 = 0.5634$, $d_3 = -0.54346$, $p_3 = 0.5634$, $p_0' = 1.3136$, $d_2' = 1.1197$, $p_0'' = -1.3546$, and $d_1'' = -0.1015$.

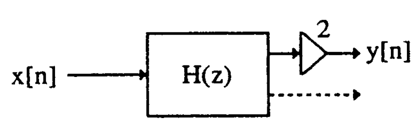
6.53 (a) A partial fraction expansion of G(z) is of the form $G(z) = d + \sum_{i=1}^{N/2} \frac{\upsilon_i}{z-\lambda_i} + \sum_{i=1}^{N/2} \frac{\upsilon_i^*}{z-\lambda_i^*}$. If we define $H(z) = \frac{d}{2} + \sum_{i=1}^{N/2} \frac{\upsilon_i}{z-\lambda_i}$, then we can write G(z) = H(z) + H∗(z), where H∗(z) represents the transfer function obtained from H(z) by conjugating its coefficients.

(b) In this case, the partial fraction expansion of G(z) is of the form $G(z) = d + \sum_{i=1}^{N_r} \frac{\rho_i}{z-\xi_i} + \sum_{i=1}^{N_c/2} \frac{\upsilon_i}{z-\lambda_i} + \sum_{i=1}^{N_c/2} \frac{\upsilon_i^*}{z-\lambda_i^*}$, where N_r and N_c are the number of real poles ξ_i's and complex poles λ_i's, respectively, with residues ρ_i's and υ_i's. We can thus decompose G(z) as G(z) = H(z) + H∗(z), where $H(z) = \frac{d}{2} + \sum_{i=1}^{N_r} \frac{\rho_i/2}{z-\xi_i} + \sum_{i=1}^{N_c/2} \frac{\upsilon_i}{z-\lambda_i}$.

(c) An implementation of real coefficient G(z) is thus simply a parallel connection of two complex filters characterized by transfer functions H(z) and H∗(z) as indicated in the figure below:



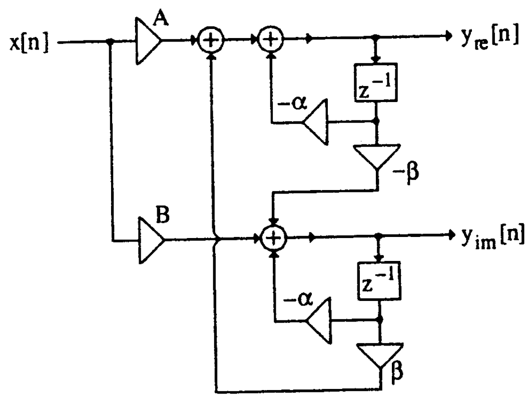
However, for a real valued input x[n], the output of H(z) is the complex conjugate of H∗(z). As a result, two times the real part of the output of H(z) is the desired real-valued sequence y[n] indicating that single complex filter H(z) is sufficient to realize G(z) as indicated below:



6.54 From $H(z) = \frac{Y(z)}{X(z)} = \frac{A + jB}{1 + (\alpha + \beta) z^{-1}}$, we arrive at the difference equation representation $y_{re}[n] + j y_{im}[n] = -(\alpha + j\beta)\left(y_{re}[n-1] + j y_{im}[n-1]\right) + A\,x[n] + jB\,x[n]$, which is equivalent to a set of two difference equations involving all real variables and real multiplier coefficients:

$y_{re}[n] = -\alpha y_{re}[n-1] + \beta y_{im}[n-1] + A x[n]$, and $y_{im}[n] = -\beta y_{re}[n-1] - \alpha y_{im}[n-1] + B x[n]$.

A realization of H(z) based on the last two equations is shown below:



To determine the transfer function $Y_{re}(z)/X(z)$, we take the z-transforms of the last two difference equations and arrive at $(1+\alpha z^{-1})Y_{re}(z) - \beta z^{-1} Y_{im}(z) = A X(z)$, and $\beta z^{-1} Y_{re}(z) + (1+\alpha z^{-1}) Y_{im}(z) = B X(z)$. Solving these two equations we get $\dfrac{Y_{re}(z)}{X(z)} = \dfrac{A + (A\alpha + B\beta)z^{-1}}{1 + 2\alpha z^{-1} + (\alpha^2 + \beta^2)z^{-2}}$, which is seen to be a second-order transfer function.

6.55 An m-th order complex allpass function is given by

$$A_m(z) = \frac{\alpha_m^* + \alpha_{m-1}^* z^{-1} + \cdots + \alpha_1^* z^{-(m-1)} + \alpha_m^* z^{-m}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_{m-1} z^{-(m-1)} + \alpha_m z^{-m}}.$$

To generate an (m–1)-th order allpass we use the recursion

$$A_{m-1}(z) = \frac{P_{m-1}(z)}{D_{m-1}(z)} = z\left[\frac{A_m(z) - k_m^*}{1 - k_m A_m(z)}\right].$$

Substituting the expression for $A_m(z)$ in the above we obtain after some algebra

$$P_{m-1}(z) = z[\alpha_m^* + \alpha_{m-1}^* z^{-1} + \alpha_{m-2}^* z^{-2} + \cdots + \alpha_1^* z^{-(m-1)} + z^{-m}$$
$$- \alpha_m^* (1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_{m-1} z^{-(m-1)} + \alpha_m z^{-m})]$$
$$= (\alpha_{m-1}^* - \alpha_m^* \alpha_1) + (\alpha_{m-2}^* - \alpha_m^* \alpha_2) z^{-1} + \cdots + (\alpha_1^* - \alpha_m^* \alpha_{m-1}) z^{-(m-2)} + (1 - |\alpha_m|^2) z^{-(m-1)},$$

$$D_{m-1}(z) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + + \alpha_{m-1} z^{-(m-1)} + \alpha_m z^{-m}$$
$$- \alpha_m (\alpha_m^* + \alpha_{m-1}^* z^{-1} + \alpha_{m-2}^* z^{-2} + \cdots + + \alpha_1^* z^{-(m-1)} + z^{-m})$$
$$= (1 - |\alpha_m|^2) + (\alpha_1 - \alpha_m \alpha_{m-1}^*) z^{-1} + (\alpha_2 - \alpha_m \alpha_{m-2}^*) z^{-2} + \cdots + (\alpha_{m-1} - \alpha_m \alpha_1^*) z^{-(m-1)}.$$

Hence, $A_{m-1}(z)$ is a complex allpass function of order m–1 given by

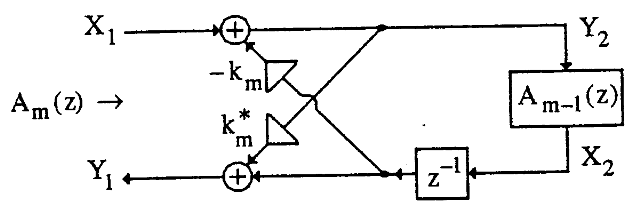
$$A_{m-1}(z) = \frac{\beta_{m-1}^* + \beta_{m-2}^* z^{-1} + \cdots + \beta_1^* z^{-(m-2)} + z^{-(m-1)}}{1 + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_{m-2} z^{-(m-2)} + \beta_{m-1} z^{-(m-1)}},$$

where $\beta_k = \dfrac{\alpha_k - \alpha_m \alpha^*_{m-k}}{1-|\alpha_m|^2}$, $k = 1, 2, \ldots, m-1$.

To develop a realization of $A_m(z)$ we express $A_m(z)$ in terms of $A_{m-1}(z)$:

$$A_m(z) = \frac{Y_1}{X_1} = \frac{k^*_m - z^{-1}A_{m-1}(z)}{1 + k_m z^{-1} A_{m-1}(z)},$$

and compare it with Eq. (6.73) resulting in the following expressions for the transfer parameters of the two-pair: $t_{11} = k^*_m$, $t_{22} = -k_m z^{-1}$, and $t_{12}t_{21} = (1 - k_m k^*_m)z^{-1}$. As in the case of the realization of a real allpass function, there are many possible choices for t_{12} and t_{21}. We choose $t_{12} = (1 - k_m k^*_m)z^{-1}$, $t_{21} = 1$. The corresponding input-output relations of the two-pair $Y_1 = k^*_m X_1 + (1 - k^*_m k_m)z^{-1}X_2 = k^*_m(X_1 - k_m z^{-1}X_2) + z^{-1}X_2$, and $Y_2 = X_1 - k_m z^{-1}X_2$. A realization of $A_m(z)$ based on the above two-pair relations is indicated below:



By continuing this process, we arrive at a cascaded lattice realization of a complex allpass transfer function.

6.56 (a) $H_1(z) = \dfrac{2+2z^{-1}}{3+z^{-1}} = \dfrac{1}{2}\left(1 + \dfrac{1+3z^{-1}}{3+z^{-1}}\right) = \dfrac{1}{2}\left(A_0(z) + A_1(z)\right)$, where $A_0(z) = 1$ and

$A_1(z) = \dfrac{1+3z^{-1}}{3+z^{-1}}$.

(b) $H_2(z) = \dfrac{1-z^{-1}}{4+2z^{-1}} = \dfrac{1}{2}\left(1 - \dfrac{2+4z^{-1}}{4+2z^{-1}}\right) = \dfrac{1}{2}\left(A_0(z) - A_1(z)\right)$, where $A_0(z) = 1$ and

$A_1(z) = \dfrac{2+4z^{-1}}{4+2z^{-1}}$.

(c) $H_3(z) = \dfrac{1-z^{-2}}{4+2z^{-1}+2z^{-2}} = \dfrac{1}{2}\left(1 - \dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}\right) = \dfrac{1}{2}\left(A_0(z) - A_1(z)\right)$, where $A_0(z) = 1$ and

$A_1(z) = \dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$.

(d) $H_4(z) = \dfrac{3+9z^{-1}+9z^{-2}+3z^{-3}}{12+10z^{-1}+2z^{-2}} = \dfrac{1}{2}\left(\dfrac{3+9z^{-1}+9z^{-2}+3z^{-3}}{(3+z^{-1})(2+z^{-1})}\right) = \dfrac{1}{2}\left[z^{-1}\left(\dfrac{1+3z^{-1}}{3+z^{-1}}\right) + \left(\dfrac{1+2z^{-1}}{2+z^{-1}}\right)\right]$

$= \frac{1}{2}\left(A_0(z) + A_1(z)\right)$, where $A_0(z) = z^{-1}\left(\frac{1+3z^{-1}}{3+z^{-1}}\right)$ and $A_1(z) = \left(\frac{1+2z^{-1}}{2+z^{-1}}\right)$.

6.57 (a) From the equation given we get

$y[2\ell] = h[0]x[2\ell] + h[1]x[2\ell-1] + h[2]x[2\ell-2] + h[3]x[2\ell-3] + h[4]x[2\ell-4] + h[5]x[2\ell-5]$, and
$y[2\ell+1] = h[0]x[2\ell+1] + h[1]x[2\ell] + h[2]x[2\ell-1] + h[3]x[2\ell-2] + h[4]x[2\ell-3] + h[5]x[2\ell-4]$.

Rewriting the above two equations in a matrix form we arrive at

$$\begin{bmatrix} y[2\ell] \\ y[2\ell+1] \end{bmatrix} = \begin{bmatrix} h[0] & 0 \\ h[1] & h[0] \end{bmatrix}\begin{bmatrix} x[2\ell] \\ x[2\ell+1] \end{bmatrix} + \begin{bmatrix} h[2] & h[1] \\ h[3] & h[3] \end{bmatrix}\begin{bmatrix} x[2\ell-2] \\ x[2\ell-1] \end{bmatrix}$$
$$+ \begin{bmatrix} h[4] & h[3] \\ h[5] & h[4] \end{bmatrix}\begin{bmatrix} x[2\ell-4] \\ x[2\ell-3] \end{bmatrix} + \begin{bmatrix} 0 & h[5] \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x[2\ell-6] \\ x[2\ell-5] \end{bmatrix},$$

which can be alternately expressed as

$$\mathbf{Y}_\ell = \mathbf{H}_0\mathbf{X}_\ell + \mathbf{H}_1\mathbf{X}_{\ell-1} + \mathbf{H}_2\mathbf{X}_{\ell-2} + \mathbf{H}_3\mathbf{X}_{\ell-3},$$

where $\mathbf{Y}_\ell = \begin{bmatrix} y[2\ell] \\ y[2\ell+1] \end{bmatrix}$, $\mathbf{X}_\ell = \begin{bmatrix} x[2\ell] \\ x[2\ell+1] \end{bmatrix}$, $\mathbf{H}_0 = \begin{bmatrix} h[0] & 0 \\ h[1] & h[0] \end{bmatrix}$, $\mathbf{H}_1 = \begin{bmatrix} h[2] & h[1] \\ h[3] & h[3] \end{bmatrix}$,
$\mathbf{H}_2 = \begin{bmatrix} h[4] & h[3] \\ h[5] & h[4] \end{bmatrix}$, and $\mathbf{H}_3 = \begin{bmatrix} 0 & h[5] \\ 0 & 0 \end{bmatrix}$.

(b) Here $y[3\ell] = h[0]x[3\ell] + h[1]x[3\ell-1] + h[2]x[3\ell-2] + h[3]x[3\ell-3] + h[4]x[3\ell-4] + h[5]x[3\ell-5]$,
$y[3\ell+1] = h[0]x[3\ell+1] + h[1]x[3\ell] + h[2]x[3\ell-1] + h[3]x[3\ell-2] + h[4]x[3\ell-3] + h[5]x[3\ell-4]$, and
$y[3\ell+2] = h[0]x[3\ell+2] + h[1]x[3\ell+1] + h[2]x[3\ell] + h[3]x[3\ell-1] + h[4]x[3\ell-2] + h[5]x[3\ell-3]$.

Rewriting the above two equations in a matrix form we arrive at

$$\begin{bmatrix} y[3\ell] \\ y[3\ell+1] \\ y[3\ell+2] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix}\begin{bmatrix} x[3\ell] \\ x[3\ell+1] \\ x[3\ell+2] \end{bmatrix} + \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix}\begin{bmatrix} x[3\ell-3] \\ x[3\ell-2] \\ x[3\ell-1] \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & h[2] & h[1] \\ 0 & 0 & h[2] \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x[3\ell-6] \\ x[3\ell-5] \\ x[3\ell-4] \end{bmatrix},$$

which can be alternately expressed as $\mathbf{Y}_\ell = \mathbf{H}_0\mathbf{X}_\ell + \mathbf{H}_1\mathbf{X}_{\ell-1} + \mathbf{H}_2\mathbf{X}_{\ell-2}$, where
$\mathbf{Y}_\ell = \begin{bmatrix} y[3\ell] \\ y[3\ell+1] \\ y[3\ell+2] \end{bmatrix}$, $\mathbf{X}_\ell = \begin{bmatrix} x[3\ell] \\ x[3\ell+1] \\ x[3\ell+2] \end{bmatrix}$, $\mathbf{H}_0 = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix}$, $\mathbf{H}_1 = \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix}$, and
$\mathbf{H}_2 = \begin{bmatrix} 0 & h[2] & h[1] \\ 0 & 0 & h[2] \\ 0 & 0 & 0 \end{bmatrix}$.

(c) Following a procedure similar to that outlined in Parts (a) and (b) above, we can shw that here
$\mathbf{Y}_\ell = \mathbf{H}_0\mathbf{X}_\ell + \mathbf{H}_1\mathbf{X}_{\ell-1} + \mathbf{H}_2\mathbf{X}_{\ell-2}$, where

$$\mathbf{Y}_\ell = \begin{bmatrix} y[4\ell] \\ y[4\ell+1] \\ y[4\ell+2] \\ y[4\ell+3] \end{bmatrix}, \quad \mathbf{X}_\ell = \begin{bmatrix} x[4\ell] \\ x[4\ell+1] \\ x[4\ell+2] \\ x[4\ell+3] \end{bmatrix}, \quad \mathbf{H}_0 = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} h[4] & h[3] & h[2] & h[1] \\ h[5] & h[4] & h[3] & h[2] \\ 0 & h[5] & h[4] & h[3] \\ 0 & 0 & h[5] & h[4] \end{bmatrix}, \text{ and } \mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0 & h[5] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

6.58 (a) $d_0\, y[2\ell] + d_1\, y[2\ell-1] + d_2\, y[2\ell-2] + d_3\, y[2\ell-3] + d_4\, y[2\ell-4]$

$$= p_0\, x[2\ell] + p_1\, x[2\ell-1] + p_2\, x[2\ell-2] + p_3\, x[2\ell-3] + p_4\, x[2\ell-4],$$

$d_0\, y[2\ell+1] + d_1\, y[2\ell] + d_2\, y[2\ell-1] + d_3\, y[2\ell-2] + d_4\, y[2\ell-3]$

$$= p_0\, x[2\ell+1] + p_1\, x[2\ell] + p_2\, x[2\ell-1] + p_3\, x[2\ell-2] + p_4\, x[2\ell-3],$$

Rewriting the above two equations in a matrix form we arrive at

$$\begin{bmatrix} d_0 & 0 \\ d_1 & d_0 \end{bmatrix}\begin{bmatrix} y[2\ell] \\ y[2\ell+1] \end{bmatrix} + \begin{bmatrix} d_2 & d_1 \\ d_3 & d_2 \end{bmatrix}\begin{bmatrix} y[2\ell-2] \\ y[2\ell-1] \end{bmatrix} + \begin{bmatrix} d_4 & d_3 \\ 0 & d_4 \end{bmatrix}\begin{bmatrix} y[2\ell-4] \\ y[2\ell-3] \end{bmatrix}$$

$$= \begin{bmatrix} p_0 & 0 \\ p_1 & p_0 \end{bmatrix}\begin{bmatrix} x[2\ell] \\ x[2\ell+1] \end{bmatrix} + \begin{bmatrix} p_2 & p_1 \\ p_3 & p_2 \end{bmatrix}\begin{bmatrix} x[2\ell-2] \\ x[2\ell-1] \end{bmatrix} + \begin{bmatrix} p_4 & p_3 \\ 0 & p_4 \end{bmatrix}\begin{bmatrix} x[2\ell-4] \\ x[2\ell-3] \end{bmatrix},$$

which can be alternately expressed as $\mathbf{D}_0\mathbf{Y}_\ell + \mathbf{D}_1\mathbf{Y}_{\ell-1} + \mathbf{D}_2\mathbf{Y}_{\ell-2} = \mathbf{P}_0\mathbf{X}_\ell + \mathbf{P}_1\mathbf{X}_{\ell-1} + \mathbf{P}_2\mathbf{X}_{\ell-1}$,

where $\mathbf{Y}_\ell = \begin{bmatrix} y[2\ell] \\ y[2\ell+1] \end{bmatrix}$, $\mathbf{X}_\ell = \begin{bmatrix} x[2\ell] \\ x[2\ell+1] \end{bmatrix}$, $\mathbf{D}_0 = \begin{bmatrix} d_0 & 0 \\ d_1 & d_0 \end{bmatrix}$, $\mathbf{D}_1 = \begin{bmatrix} d_2 & d_1 \\ d_3 & d_2 \end{bmatrix}$, $\mathbf{D}_2 = \begin{bmatrix} d_4 & d_3 \\ 0 & d_4 \end{bmatrix}$,

$\mathbf{P}_0 = \begin{bmatrix} p_0 & 0 \\ p_1 & p_0 \end{bmatrix}$, $\mathbf{P}_1 = \begin{bmatrix} p_2 & p_1 \\ p_3 & p_2 \end{bmatrix}$, and $\mathbf{P}_2 = \begin{bmatrix} p_4 & p_3 \\ 0 & p_4 \end{bmatrix}$.

(b)
$$\begin{bmatrix} d_0 & 0 & 0 \\ d_1 & d_0 & 0 \\ d_2 & d_1 & d_0 \end{bmatrix}\begin{bmatrix} y[3\ell] \\ y[3\ell+1] \\ y[3\ell+2] \end{bmatrix} + \begin{bmatrix} d_3 & d_2 & d_1 \\ d_4 & d_3 & d_2 \\ 0 & d_4 & d_3 \end{bmatrix}\begin{bmatrix} y[3\ell-3] \\ y[3\ell-2] \\ y[3\ell-1] \end{bmatrix} + \begin{bmatrix} 0 & 0 & d_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y[3\ell-6] \\ y[3\ell-5] \\ y[3\ell-4] \end{bmatrix}$$

$$= \begin{bmatrix} p_0 & 0 & 0 \\ p_1 & p_0 & 0 \\ p_2 & p_1 & p_0 \end{bmatrix}\begin{bmatrix} x[3\ell] \\ x[3\ell+1] \\ x[3\ell+2] \end{bmatrix} + \begin{bmatrix} p_3 & p_2 & p_1 \\ p_4 & p_3 & p_2 \\ 0 & p_4 & p_3 \end{bmatrix}\begin{bmatrix} x[3\ell-3] \\ x[3\ell-2] \\ x[3\ell-1] \end{bmatrix} + \begin{bmatrix} 0 & 0 & p_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x[3\ell-6] \\ x[3\ell-5] \\ x[3\ell-4] \end{bmatrix},$$

which can be alternately expressed as $\mathbf{D}_0\mathbf{Y}_\ell + \mathbf{D}_1\mathbf{Y}_{\ell-1} + \mathbf{D}_2\mathbf{Y}_{\ell-2} = \mathbf{P}_0\mathbf{X}_\ell + \mathbf{P}_1\mathbf{X}_{\ell-1} + \mathbf{P}_2\mathbf{X}_{\ell-1}$,

where $\mathbf{Y}_\ell = \begin{bmatrix} y[3\ell] \\ y[3\ell+1] \\ y[3\ell+2] \end{bmatrix}$, $\mathbf{X}_\ell = \begin{bmatrix} x[3\ell] \\ x[3\ell+1] \\ x[3\ell+2] \end{bmatrix}$, $\mathbf{D}_0 = \begin{bmatrix} d_0 & 0 & 0 \\ d_1 & d_0 & 0 \\ d_2 & d_1 & d_0 \end{bmatrix}$, $\mathbf{D}_1 = \begin{bmatrix} d_3 & d_2 & d_1 \\ d_4 & d_3 & d_2 \\ 0 & d_4 & d_3 \end{bmatrix}$,

$\mathbf{D}_2 = \begin{bmatrix} 0 & 0 & d_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{P}_0 = \begin{bmatrix} p_0 & 0 & 0 \\ p_1 & p_0 & 0 \\ p_2 & p_1 & p_0 \end{bmatrix}$, $\mathbf{P}_1 = \begin{bmatrix} p_3 & p_2 & p_1 \\ p_4 & p_3 & p_2 \\ 0 & p_4 & p_3 \end{bmatrix}$, and $\mathbf{P}_2 = \begin{bmatrix} 0 & 0 & p_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(c)
$$\begin{bmatrix} d_0 & 0 & 0 & 0 \\ d_1 & d_0 & 0 & 0 \\ d_2 & d_1 & d_0 & 0 \\ d_3 & d_2 & d_1 & d_0 \end{bmatrix}\begin{bmatrix} y[4\ell] \\ y[4\ell+1] \\ y[4\ell+2] \\ y[4\ell+3] \end{bmatrix} + \begin{bmatrix} d_4 & d_3 & d_2 & d_1 \\ 0 & d_4 & d_3 & d_2 \\ 0 & 0 & d_4 & d_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}\begin{bmatrix} y[4\ell-4] \\ y[4\ell-3] \\ y[4\ell-2] \\ y[4\ell-1] \end{bmatrix}$$

$$= \begin{bmatrix} p_0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} x[4\ell] \\ x[4\ell+1] \\ x[4\ell+2] \\ x[4\ell+3] \end{bmatrix} + \begin{bmatrix} p_4 & p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & p_4 & p_3 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \begin{bmatrix} x[4\ell-4] \\ x[4\ell-3] \\ x[4\ell-2] \\ x[4\ell-1] \end{bmatrix},$$

which can be alternately expressed as $\mathbf{D}_0\mathbf{Y}_\ell + \mathbf{D}_1\mathbf{Y}_{\ell-1} = \mathbf{P}_0\mathbf{X}_\ell + \mathbf{P}_1\mathbf{X}_{\ell-1}$, where

$$\mathbf{Y}_\ell = \begin{bmatrix} y[4\ell] \\ y[4\ell+1] \\ y[4\ell+2] \\ y[4\ell+3] \end{bmatrix}, \quad \mathbf{X}_\ell = \begin{bmatrix} x[4\ell] \\ x[4\ell+1] \\ x[4\ell+2] \\ x[4\ell+3] \end{bmatrix}, \quad \mathbf{D}_0 = \begin{bmatrix} d_0 & 0 & 0 & 0 \\ d_1 & d_0 & 0 & 0 \\ d_2 & d_1 & d_0 & 0 \\ d_3 & d_2 & d_1 & d_0 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} d_4 & d_3 & d_2 & d_1 \\ 0 & d_4 & d_3 & d_2 \\ 0 & 0 & d_4 & d_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix},$$

$$\mathbf{P}_0 = \begin{bmatrix} p_0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 & p_0 \end{bmatrix}, \text{ and } \mathbf{P}_1 = \begin{bmatrix} p_4 & p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & p_4 & p_3 \\ 0 & 0 & 0 & p_4 \end{bmatrix}.$$

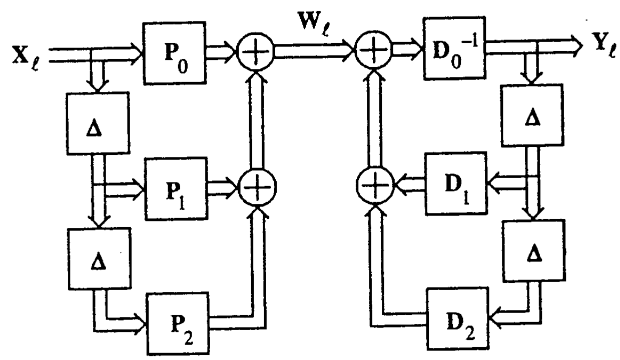
6.59 We first rewrite the second-order block-difference equation

$$\mathbf{D}_0\mathbf{Y}_\ell + \mathbf{D}_1\mathbf{Y}_{\ell-1} + \mathbf{D}_2\mathbf{Y}_{\ell-2} = \mathbf{P}_0\mathbf{X}_\ell + \mathbf{P}_1\mathbf{X}_{\ell-1} + \mathbf{P}_2\mathbf{X}_{\ell-1},$$

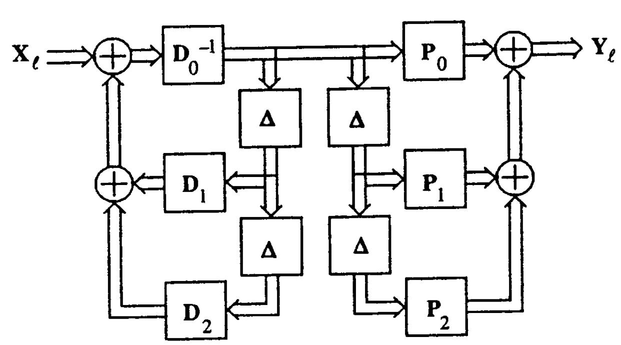
as two separate equations: $\mathbf{W}_\ell = \mathbf{P}_0\mathbf{X}_\ell + \mathbf{P}_1\mathbf{X}_{\ell-1} + \mathbf{P}_2\mathbf{X}_{\ell-1}$, and

$$\mathbf{Y}_\ell = -\mathbf{D}_0^{-1}\mathbf{D}_1\mathbf{Y}_{\ell-1} - \mathbf{D}_0^{-1}\mathbf{D}_2\mathbf{Y}_{\ell-2} + \mathbf{D}_0^{-1}\mathbf{W}_\ell.$$

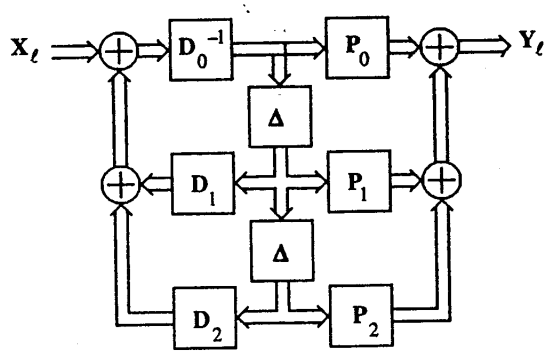
A cascade realization of the IIR block digital filter based on the above two equations is thus as shown below:



By interchanging the locations of the two block sections in the above structure we get an equivalent realization as indicated below:



Finally, by delay-sharing the above structure reduces to a canonic realization as shown below:



6.60 From the given state equations we get

$$s[n+1] = \mathbf{A}s[n] + \mathbf{B}x[n],$$
$$s[n+2] = \mathbf{A}s[n+1] + \mathbf{B}x[n+1] = \mathbf{A}(\mathbf{A}s[n] + \mathbf{B}x[n]) + \mathbf{B}x[n+1] = \mathbf{A}^2 s[n] + \mathbf{AB}x[n] + \mathbf{B}x[n+1],$$
$$\vdots$$
$$s[n+L] = \mathbf{A}^L s[n] + \mathbf{A}^{L-1}\mathbf{B}x[n] + \mathbf{A}^{L-2}\mathbf{B}x[n+1] + \cdots + \mathbf{AB}x[n+L-2] + \mathbf{B}\,x[n+L-1]. \qquad (17)$$

Likewise, from the output equation we get

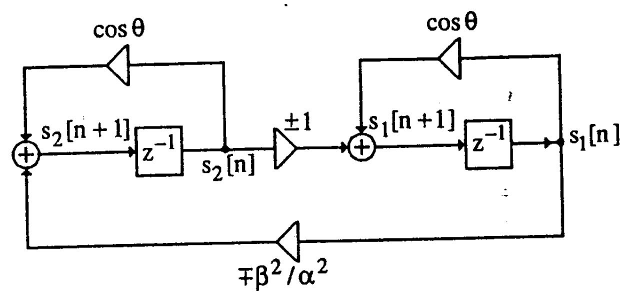
$$y[n] = \mathbf{C}s[n] + \mathbf{D}x[n],$$
$$y[n+1] = \mathbf{C}s[n+1] + \mathbf{D}x[n+1] = \mathbf{C}(\mathbf{A}s[n] + \mathbf{B}x[n]) + \mathbf{D}x[n+1] = \mathbf{CA}s[n] + \mathbf{CB}x[n] + \mathbf{D}x[n+1],$$
$$\vdots$$
$$y[n+L-1] = \mathbf{C}s[n+L-1] + \mathbf{D}x[n+L-1]$$
$$= \mathbf{C}\left(\mathbf{A}^{L-1}s[n] + \mathbf{A}^{L-2}\mathbf{B}x[n] + \cdots + \mathbf{AB}x[n+L-3] + \mathbf{B}x[n+L-2]\right) + \mathbf{D}x[n+L-1]$$

By setting n = kL in Eq. (17) and the last L output equations and using the notations $\mathbf{Y}_k = [y[kL] \quad y[kL+1] \quad \cdots \quad y[kL+L-1]]^T$, $\mathbf{X}_k = [x[kL] \quad x[kL+1] \quad \cdots \quad x[kL+L-1]]^T$, $\mathbf{S}_k = s[kL]$, $\boldsymbol{\mathcal{A}} = \mathbf{A}^L$, $\boldsymbol{\mathcal{B}} = [\mathbf{A}^{L-1}\mathbf{B} \quad \mathbf{A}^{L-2}\mathbf{B} \quad \cdots \quad \mathbf{AB} \quad \mathbf{B}]$,

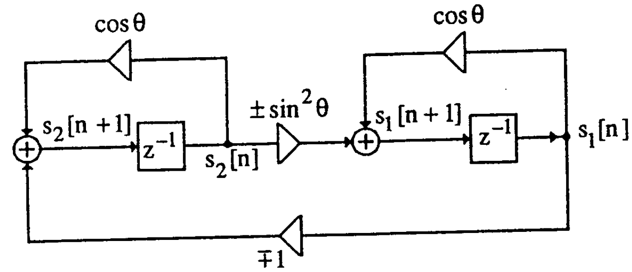
$\boldsymbol{\mathcal{C}} = [\mathbf{C} \quad \mathbf{CA} \quad \cdots \quad \mathbf{CA}^{L-2} \quad \mathbf{CA}^{L-1}]^T$, and $\boldsymbol{\mathcal{D}} = \begin{bmatrix} \mathbf{D} & 0 & \cdots & 0 \\ \mathbf{CB} & \mathbf{D} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{CA}^{L-2}\mathbf{B} & \mathbf{CA}^{L-1}\mathbf{B} & \cdots & \mathbf{D} \end{bmatrix}$, we arrive at the block-state-space representation of the form

$$\mathbf{S}_{k+1} = \boldsymbol{\mathcal{A}}\,\mathbf{S}_k + \boldsymbol{\mathcal{B}}\,\mathbf{X}_k,$$
$$\mathbf{Y}_k = \boldsymbol{\mathcal{C}}\,\mathbf{S}_k + \boldsymbol{\mathcal{D}}\,\mathbf{X}_k.$$

6.61 By setting $\alpha \sin\theta = \pm\beta$ in Eq. (6.138), the state-space description of the sine-cosine generator reduces to $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \cos\theta & \pm 1 \\ \mp\frac{\beta^2}{\alpha^2} & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$, which leads to the three-multiplier structure shown below:



6.62 By setting $\alpha = \pm\beta\sin\theta$ in Eq. (6.138), the state-space description of the sine-cosine generator reduces to $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \cos\theta & \pm\sin^2\theta \\ \mp 1 & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$, which leads to the three-multiplier structure shown below:

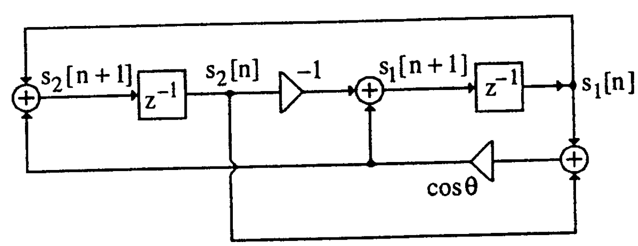


6.63 Let $\frac{\beta}{\alpha} = -\frac{1+\cos\theta}{\sin\theta}$. Then $\frac{\alpha}{\beta}\sin\theta = \frac{-\sin^2\theta}{\cos\theta+1} = \frac{\cos^2\theta - 1}{\cos\theta+1} = \cos\theta - 1$, and, $-\frac{\beta}{\alpha}\sin\theta = \cos\theta + 1$.

Substituting these values in Eq. (6.138) we arrive at $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos\theta - 1 \\ \cos\theta + 1 & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$,

These equations can be alternately rewritten as

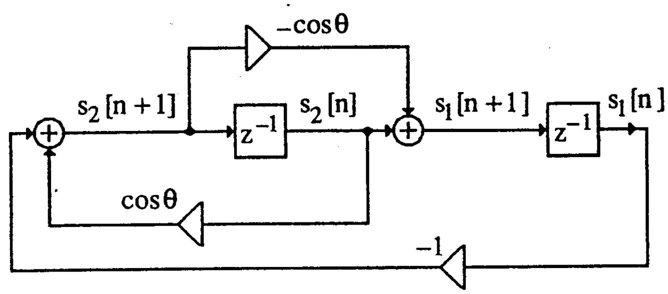
$s_1[n+1] = \cos\theta\big(s_1[n] + s_2[n]\big) - s_2[n]$, and $s_2[n+1] = \cos\theta\big(s_1[n] + s_2[n]\big) + s_1[n]$. A realization based on the last two equations results in a single-multiplier structure as indicated below:



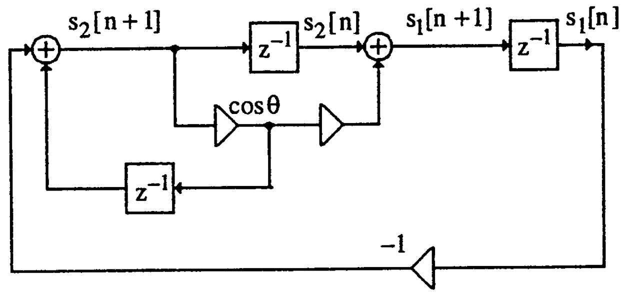
6.64 $$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & \frac{\alpha(C - \cos\theta)}{\beta\sin\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} C & \frac{\alpha(1 - C\cos\theta)}{\beta\sin\theta} \\ -\frac{\beta}{\alpha}\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}.$$

If C = 0, choose $\alpha = \beta\sin\theta$. Then

$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & -\cos\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$, which can be realized with two multipliers as shown below:



The above structure can be modified to yield a single multiplier realization as indicated below:

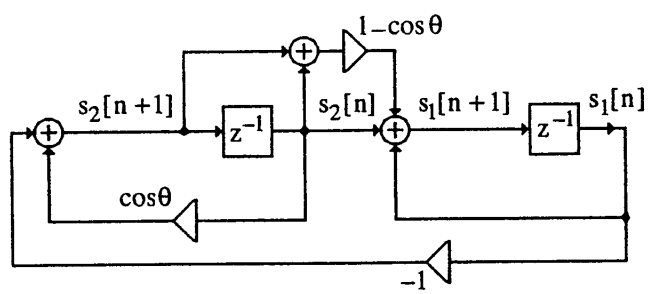


6.65 If C = 1, then $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\alpha(1-\cos\theta)}{\beta\sin\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} 1 & \dfrac{\alpha(1-\cos\theta)}{\beta\sin\theta} \\ -\dfrac{\beta}{\alpha}\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}.$

Choose $\alpha = \beta\sin\theta$. This leads to

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1-\cos\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} 1 & 1-\cos\theta \\ -1 & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}.$$

A two-multiplier realization of the above equation is shown below:



To arrive at an one-multiplier realization we observe that the two equations describing the sine-cosine generator are given by $s_1[n+1] = (1-\cos\theta)s_2[n+1] + s_1[n] + (1-\cos\theta)s_2[n]$, and $s_2[n+1] = -s_1[n] + \cos\theta\, s_2[n]$. Substituting the second equation in the first equation we arrive at an alternate description in the form

$$s_1[n+1] = -\cos\theta\, s_2[n+1] + s_2[n],$$
$$s_2[n+1] = -s_1[n] + \cos\theta\, s_2[n].$$

A realization of above is identical to the single-multiplier structure of Problem 6.64,

6.66 From Figure P6.17(a), the output-input relation of the channel is given by

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} 1 & H_{12}(z) \\ H_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}.$$

Likewise, the output-input relation of the channel separation circuit of Figure P6.17(b) is given by $\begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 & -G_{12}(z) \\ -G_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix}$. Hence, the overall system is characterized by

$$\begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 & -G_{12}(z) \\ -G_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & H_{12}(z) \\ H_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} =$$

$$\begin{bmatrix} 1-H_{21}(z)G_{12}(z) & H_{12}(z)-G_{12}(z) \\ H_{21}(z)-G_{21}(z) & 1-H_{12}(z)G_{21}(z) \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}.$$

The cross-talk is eliminated if $V_1(z)$ is a function of either $X_1(z)$ or $X_2(z)$, and similarly, if $V_2(z)$ is a function of either $X_1(z)$ or $X_2(z)$. From the above equation it follows that if $H_{12}(z) = G_{12}(z)$, and $H_{21}(z) = G_{21}(z)$, then $V_1(z) = \left(1 - H_{21}(z)G_{12}(z)\right)X_1(z)$, and $V_2(z) = \left(1 - H_{12}(z)G_{21}(z)\right)X_2(z)$. Alternately, if $G_{12}(z) = H_{21}^{-1}(z)$, and $G_{21}(z) = H_{12}^{-1}(z)$, then

$$V_1(z) = \left(\frac{H_{12}(z)H_{21}(z)-1}{H_{21}(z)}\right)X_2(z), \text{ and } V_2(z) = \left(\frac{H_{12}(z)H_{21}(z)-1}{H_{12}(z)}\right)X_1(z).$$

6.67 (a) The difference equation corresponding to the transfer function

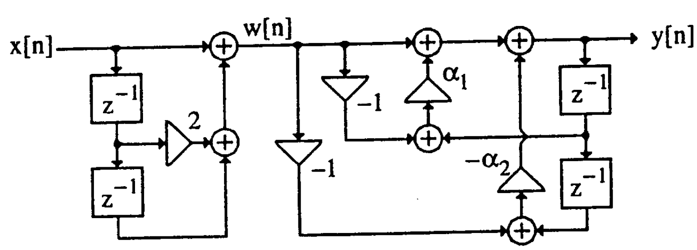
$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1+\alpha_1+\alpha_2)(1+2z^{-1}+z^{-2})}{1-\alpha_1 z^{-1}+\alpha_2 z^{-2}}$$ is given by

$y[n] + \alpha_1 y[n-1] - \alpha_2 y[n-2] = (1+\alpha_1+\alpha_2)(x[n] + 2x[n-1] + x[n-2])$, which can be rewritten as

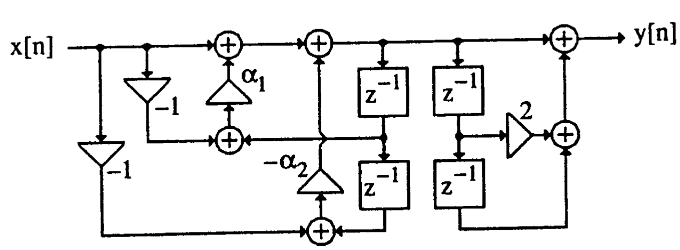
$y[n] = (x[n] + 2x[n-1] + x[n-2]) + \alpha_1(y[n-1] - x[n] - 2x[n-1] - x[n-2])$

$\quad -\alpha_2(y[n-2] - x[n] - 2x[n-1] - x[n-2])$. Denoting $w[n] = x[n] + 2x[n-1] + x[n-2]$, the difference equation representation becomes

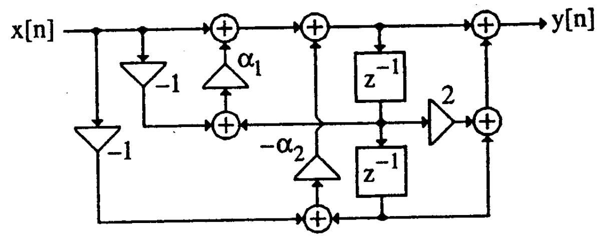
$y[n] = w[n] + \alpha_1(y[n-1] - w[n]) - \alpha_2(y[n-2] - w[n])$. A realization of H(z) based on the last two equations is as indicated below where the first stage realizes w[n] while the second stage realizes y[n].



An interchange of the two stages leads to an equivalent realization shown below:



Finally, by delay sharing the above structure reduces to a canonic realization as shown below:



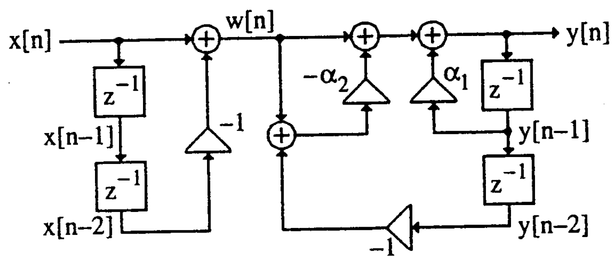
(b) The difference equation corresponding to the transfer function

$H(z) = \frac{Y(z)}{X(z)} = \frac{(1-\alpha_2)(1-z^{-2})}{1-\alpha_1 z^{-1} + \alpha_2 z^{-2}}$ is given by

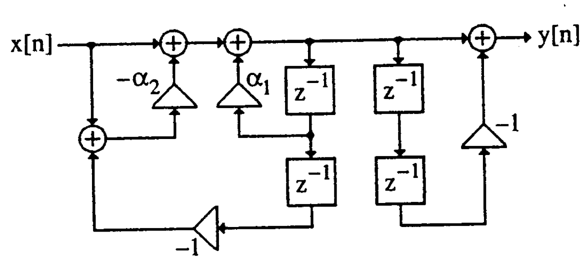
$y[n] - \alpha_1 y[n-1] + \alpha_2 y[n-2] = (1-\alpha_2)(x[n] - x[n-2])$ which can be rewritten as

$y[n] = \alpha_1 y[n-1] - \alpha_2 y[n-2] + x[n] - \alpha_2 x[n] - x[n-2] - \alpha_2 x[n-2]$

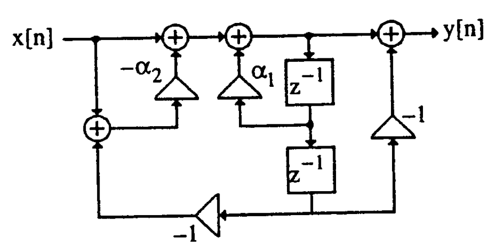
$= \alpha_1 y[n-1] - \alpha_2 (x[n] - x[n-2] + y[n-2]) + (x[n] - x[n-2])$. Denoting $w[n] = x[n] - x[n-2]$, we can rewrite the last equation as $y[n] = \alpha_1 y[n-1] - \alpha_2 (w[n] + y[n-2]) + w[n]$. A realization of H(z) based on the last two equations is as shown below:



An interchange of the two stages leads to an equivalent realization shown below:



Finally, by delay sharing the above structure reduces to a canonic realization as shown below:



M6.1 **(a)** $H_1(z) = 0.01551933(1 - 10.37288 z^{-1})(1 - 1.845984\, z^{-1})(1 + z^{-1})$
$\times (1 - 0.541716 z^{-1})(1 - 0.096405 z^{-1})(1 + 0.26456 z^{-1} + z^{-2})(1 + 1.432115 z^{-1} + z^{-2})$.

(b) $H_2(z) = 0.0636786(1 - 3.5834665z^{-1} + 4.975546z^{-2})(1 + 1.89771z^{-1} + z^{-2})$
$\times(1 + 1.320477z^{-1} + z^{-2})(1 - 0.720216z^{-1} + 0.200983z^{-2})$.

M6.2 (a) Using the command `[z, p, k] = tf2zp(num, den)` we first obtain the zeros, poles, and the gain constant of G(z). Next using `sos = zp2sos(z, p, k)` we arrive at the second-order factors for the numerator and the denominator of G(z) as given below:

$$G(z) = \frac{(0.04954 - 0.10063z^{-1} + 0.05111z^{-2})(0.16881 - 0.33235z^{-1} + 0.16361z^{-2})}{(1 + 1.31014z^{-1} + 0.51507z^{-2})(1 + 1.06398z^{-1} + 0.79662z^{-2})}$$

(b) From the above factored form various cascade realizations can be easily obtained.

(c) To develop the parallel form I realization we make use of the command `[r, p, k] = residuez(num,den)` resulting in the expansion

$$G(z) = 0.0204 + \frac{-0.0959 + 0.1055z^{-1}}{1 + 1.064z^{-1} + 0.7966z^{-2}} + \frac{0.0839 - 0.1509z^{-1}}{1 + 1.3101z^{-1} + 0.5151z^{-2}}.$$

To develop the parallel form II realization we make use of the command `[r, p, k] = residue(num,den)` resulting in the expansion

$$G(z) = 0.0084 + \frac{0.2075z^{-1} + 0.0764z^{-2}}{1 + 1.064z^{-1} + 0.7966z^{-2}} + \frac{-0.2608z^{-1} - 0.0432z^{-2}}{1 + 1.3101z^{-1} + 0.5151z^{-2}}.$$

M6.3 (a) Using the command `[z, p, k] = tf2zp(num, den)` we first obtain the zeros, poles, and the gain constant of G(z). Next using `sos = zp2sos(z, p, k)` we arrive at the second-order factors for the numerator and the denominator of G(z) as given below:

$$H(z) = \frac{(1 - 1.5484z^{-1} + z^{-2})(1 - 0.54281z^{-1} + z^{-2})}{(1 + 0.721695z^{-1} + 0.35088z^{-2})(1 + 0.174477z^{-1} + 0.837424z^{-2})}.$$

(b) From the above factored form various cascade realizations can be easily obtained.

(c) To develop the parallel form I realization we make use of the command `[r, p, k] = residuez(num,den)` resulting in the expansion

$$H(z) = 3.403277 + \frac{-1.718146 - 5.81889z^{-1}}{1 + 0.721695z^{-1} + 0.35088z^{-2}} + \frac{-0.68513125 + 1.47198z^{-1}}{1 + 0.174477z^{-1} + 0.837424z^{-2}}.$$

To develop the parallel form II realization we make use of the command `[r, p, k] = residue(num,den)` resulting in the expansion

$$H(z) = 1 + \frac{-4.578917z^{-1} + 0.60286z^{-2}}{1 + 0.721695z^{-1} + 0.35088z^{-2}} + \frac{1.59152z^{-1} + 0.57374535z^{-2}}{1 + 0.174477z^{-1} + 0.837424z^{-2}}.$$

M6.4 (a) $H_1(z) = \dfrac{1 + 4.2z^{-1} + 0.8z^{-2}}{1 - 2.5z^{-1} + 3z^{-2} - z^{-3}}$. The state-space representation of $H_1(z)$ is obtained using the command `[A, B, C, D] = tf2ss(num, den)` resulting in

$$\mathbf{A}=\begin{bmatrix}2.5 & -3 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{C}=[6.7\ \ -2.2\ \ 1],\ \mathrm{D}=0$$

$H_2(z)=\dfrac{3z^4+1.5z^3+z^2+0.5z}{z^4-\frac{11}{6}z^3+\frac{3}{2}z^2-\frac{7}{12}z+\frac{1}{12}}$. The state-space representation of $H_2(z)$ is obtained using the command `[A, B, C, D] = tf2ss(num, den)` resulting in

$$\mathbf{A}=\begin{bmatrix}\frac{11}{6} & -\frac{3}{2} & \frac{7}{12} & -\frac{1}{12}\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\ \mathbf{C}=[7\ \ -3.5\ \ 2.25\ \ -0.25],\ \mathrm{D}=3.$$

$H_3(z)=\dfrac{8+16z^{-1}+12z^{-2}+9z^{-3}}{1+1.6z^{-1}+1.1z^{-2}+0.3z^{-3}}$. The state-space representation of $H_3(z)$ is obtained using the command `[A, B, C, D] = tf2ss(num, den)` resulting in

$$\mathbf{A}=\begin{bmatrix}-1.6 & -1.1 & -0.3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{C}=[3.2\ \ 3.2\ \ 6.6],\ \mathrm{D}=8.$$

(b) As discussed in Section 6.6.3, equivalent state-space structures can be generated by applying a non-singular linear transform **Q** to the original state-vector.

For $H_1(z)$ in part (a), we choose $\mathbf{Q}=\begin{bmatrix}1 & -2 & 1\\ 0 & -1 & 2\\ 1 & 0 & 3\end{bmatrix}$ which leads to

$$\hat{\mathbf{A}}=\begin{bmatrix}\frac{3}{4} & \frac{1}{2} & -\frac{1}{4}\\ -\frac{7}{6} & \frac{1}{3} & \frac{1}{6}\\ \frac{13}{12} & \frac{13}{6} & \frac{17}{12}\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}1\\0\\1\end{bmatrix},\ \hat{\mathbf{C}}=\begin{bmatrix}\frac{47}{12} & -\frac{169}{30} & \frac{167}{60}\end{bmatrix},\ \hat{\mathrm{D}}=1.$$

For $H_2(z)$ in Part (a) choose $\mathbf{Q}=\begin{bmatrix}1 & 1 & 2 & 2\\ 0 & 2 & 1 & 1\\ 0 & 0 & 3 & 1\\ 0 & 0 & 0 & 2\end{bmatrix}$. This results in

$$\hat{\mathbf{A}}=\begin{bmatrix}\frac{17}{6} & -\frac{7}{6} & -\frac{23}{36} & 2\\ 2 & -\frac{1}{2} & -\frac{5}{6} & -\frac{4}{3}\\ 0 & \frac{3}{2} & -\frac{1}{6} & -\frac{2}{3}\\ 0 & 0 & \frac{2}{3} & \frac{1}{3}\end{bmatrix},\ \hat{\mathbf{B}}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\ \hat{\mathbf{C}}=\begin{bmatrix}7 & -\frac{21}{4} & -\frac{13}{6} & -\frac{41}{12}\end{bmatrix},\ \hat{\mathrm{D}}=3.$$

For $H_3(z)$ in Part (a) choose $\mathbf{Q}=\begin{bmatrix}0 & 0 & 2\\ 1 & 1 & 3\\ 2 & 1 & 2\end{bmatrix}$. This leads to

$$\hat{\mathbf{A}} = \begin{bmatrix} -4 & 4 & -2 \\ -4.25 & 4.4 & -2.5 \\ -1 & 1.8 & -2 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \hat{\mathbf{C}} = [-1.5 \quad 3.2 \quad 0], \quad \hat{\mathrm{D}} = 8.$$

M6.5
```
Lattice parameters are
0.77656938900011    0.85103309181315    0.74259237362294
0.41031508000000

Feedforward multipliers are
Columns 1 through 4

0.00836323900000   -0.05330831748615    0.13786902252618
-0.14267464184046

Column 5

0.03798368487069
```

M6.6
```
Lattice parameters are
0.21531549398220    0.84121556913458    0.44027115051364
0.29383441000000

Feedforward multipliers are
Columns 1 through 4

1.00000000000000   -2.98739663550000    3.81703303348362
-1.23557167409275

Column 5

-0.92347974655755
```

M6.7 Lattice realizations for both the FIR transfer functions of Problem M6.1 do not exist as some of the the zeros of the transfer functions are on the unit circle. This fact can be verified by determining the zero locations using the command `roots`. For the transfer function $H_1(z)$, the zeros on the unit circle are at `-0.1323 + 0.9912i`, `-0.1323 - 0.9912i`, `-0.7161 + 0.6980i`, and `- 0.7161 - 0.6980i`, and `-1.0`. For the transfer function $H_2(z)$, the zeros on the unit circle are at `0.9489 + 0.3157i`, `0.9489 - 0.3157i`, `0.6602 + 0.7511i`, and `0.6602 - 0.7511i`.

M6.8 Using `roots` we first determine the poles of the denominator of G(z) which are given by

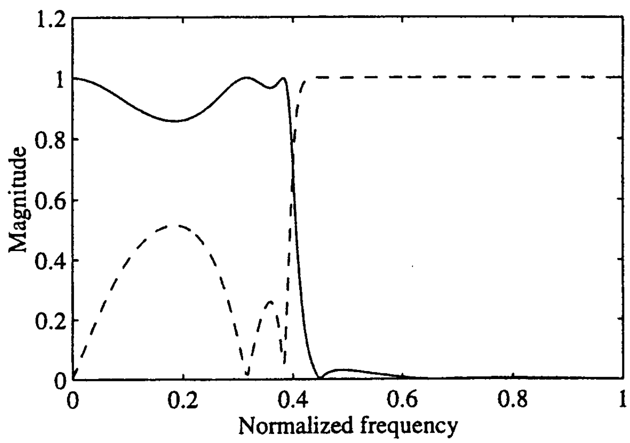
```
0.30473968579505 + 0.89710028813368i
0.30473968579505 - 0.89710028813368i
0.39886375978027 + 0.67325319068160i
0.39886375978027 - 0.67325319068160i
0.58469310884937
```

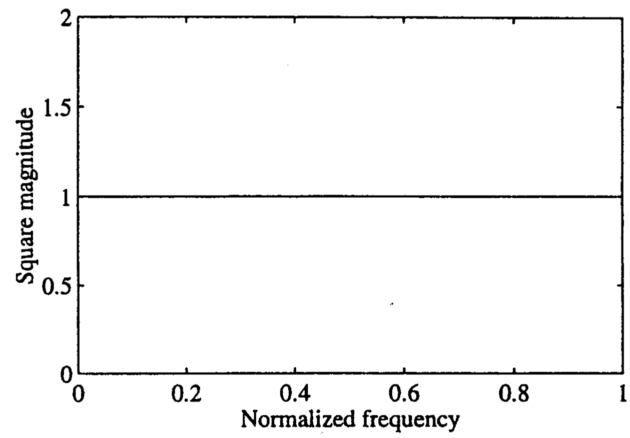
Pairing the poles according to the pole-interlacing rule we then obtain

$$G(z) = \frac{1}{2}\left[A_0(z) + A_1(z)\right], \text{ where } A_0(z) = \frac{-0.524852 + 1.2540135z^{-1} - 1.19417248z^{-2} + z^{-3}}{1 - 1.19417248z^{-1} + 1.2540135z^{-2} - 0.524852z^{-3}}$$

and $A_1(z) = \dfrac{0.61236215 - 0.79772751 z^{-1} + z^{-2}}{1 - 0.79772751 z^{-1} + 0.61236215 z^{-2}}$. The power-complementary transfer function

is therefore given by $H(z) = \frac{1}{2}[A_0(z) - A_1(z)]$. A plot of the magnitude responses of G(z) and H(z) is shown below along with sum of their square magnitude responses.

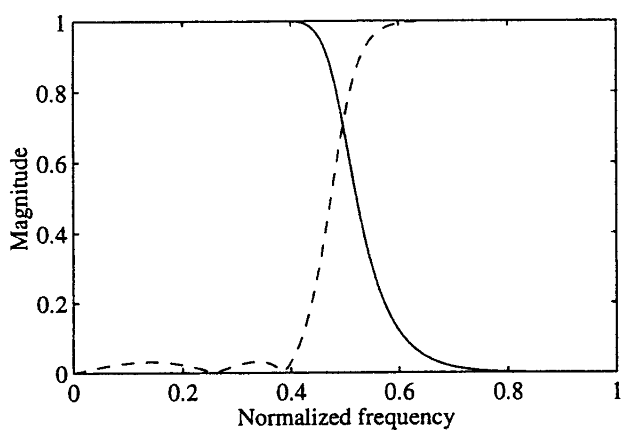


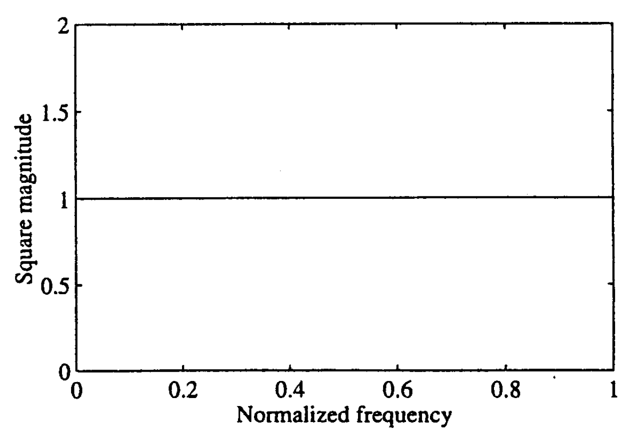


M6.9 Here $G(z) = \frac{1}{2}[A_0(z) + A_1(z)]$, where $A_0(z) = \dfrac{-0.125605 + 0.65652317 z^{-1} - 0.25278 z^{-2} + z^{-3}}{1 - 0.25278 z^{-1} + 0.65652317 z^{-2} - 0.125605 z^{-3}}$

and $A_1(z) = \dfrac{0.20023118 - 0.26642 z^{-1} + z^{-2}}{1 - 0.26642 z^{-1} + 0.20023118 z^{-2}}$. The power-complementary transfer function

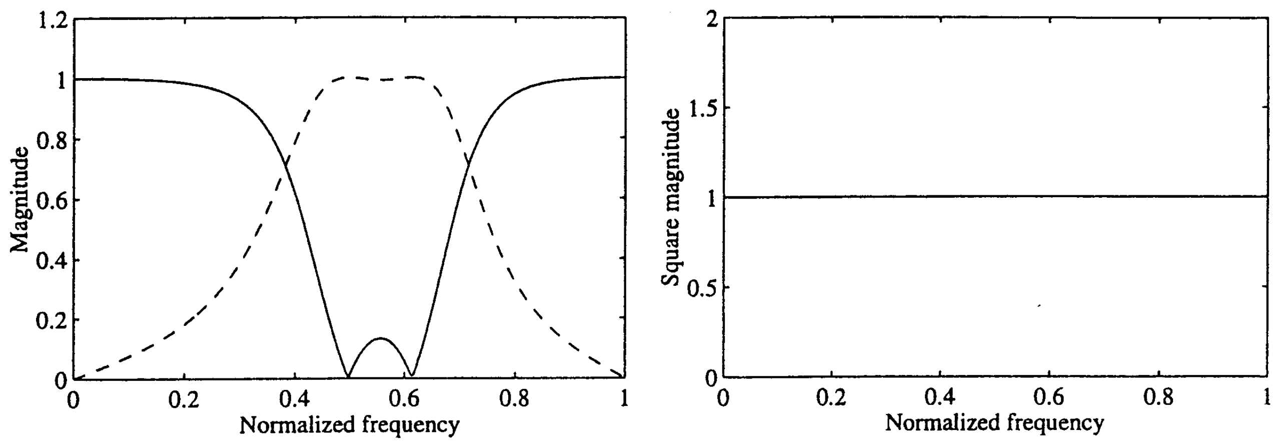
is therefore given by $H(z) = \frac{1}{2}[A_0(z) - A_1(z)]$. A plot of the magnitude responses of G(z) and H(z) is shown below along with sum of their square magnitude responses.





M6.10 Here $G(z) = \frac{1}{2}[A_0(z) + A_1(z)]$, where $A_0(z) = \dfrac{0.499081516 - 0.299801 z^{-1} + z^{-2}}{1 - 0.299801 z^{-1} + 0.499081516 z^{-2}}$, and

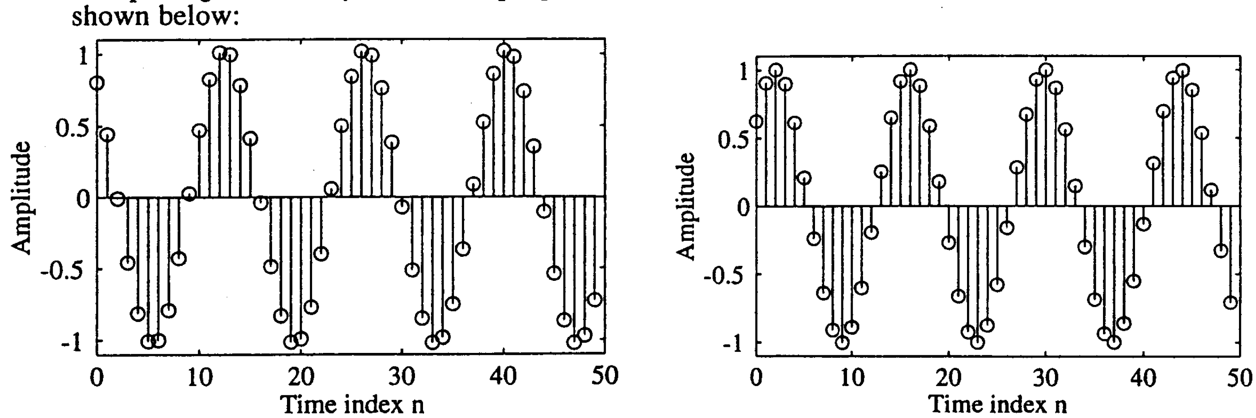
$A_0(z) = \frac{0.545432 + 0.782231z^{-1} + z^{-2}}{1 + 0.782231z^{-1} + 0.545432z^{-2}}$, The power-complementary transfer function is therefore given by $H(z) = \frac{1}{2}[A_0(z) - A_1(z)]$. A plot of the magnitude responses of G(z) and H(z) is shown below along with sum of their square magnitude responses.



M6.11 The MATLAB program for the simulation of the sine-cosine generator of Problem 6.63 is given below:

```
s10 = 0.1; s20 = 0.1; a = 0.9;
y1 = zeros(1,50);y2 = y1;
for n = 1:50;
y1(n) = a*(s10 + s20) - s20;y2(n) = a*(s10 + s20) + s10;
s10 = y1(n);s20 = y2(n);
end
k = 1:1:50;
stem(k-1,y1/abs(y1(7)));axis([0 50 -1.1 1.1]);
xlabel('Time index n');ylabel('Amplitude');
pause
stem(k-1,y2/y2(3));axis([0 50 -1.1 1.1]);
xlabel('Time index n');ylabel('Amplitude');
```

The plots generated by the above program for initial conditions $s1[-1] = s2[-1] = 0.1$ are shown below:

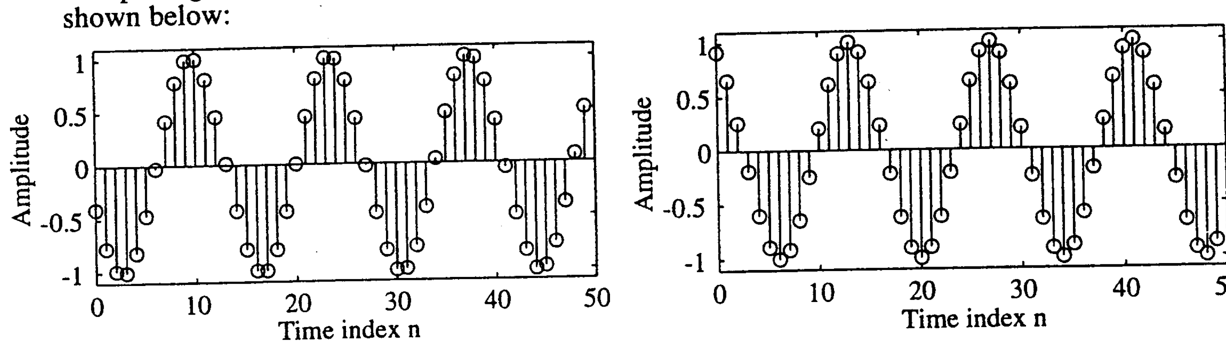


The outputs are zero for zero initial conditions. Non-zero initial conditions of equal values appear to have no effects on the outputs. However, unequal initial conditions have effects on the amplitudes and phases of the two output sequences.

M6.12 The MATLAB program for the simulation of the sine-cosine generator of Problem 6.63 is given below:

```
s10 = .1; s20 = 1; a = 0.9;
y1 = zeros(1,50);y2 = y1;
for n = 1:50;
y1(n) = -s20 + a*s10;y2(n) = -a*y1(n) + s10;
s10 = y1(n);s20 = y2(n);
end
k = 1:1:50;
stem(k-1,y1/y1(11));axis([0 50 -1.1 1.1]);
xlabel('Time index n');ylabel('Amplitude');
pause
stem(k-1,y2/y2(14));axis([0 50 -1.1 1.1]);
xlabel('Time index n');ylabel('Amplitude');
```

The plots generated by the above program for initial conditions s1[–1] = s2[–1] = 0.1 are shown below:



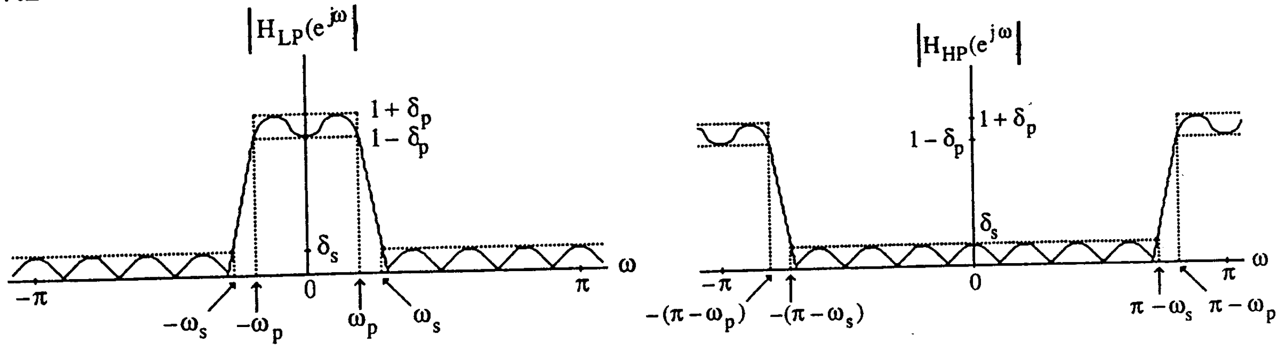
The outputs are zero for zero initial conditions. Non-zero initial conditions of equal values appear to have no effects on the outputs. However, unequal initial conditions have effects on the amplitudes and phases of the two output sequences.

M6.13 Since the single multiplier sine-cosine generator of Problem 6.65 is identical to the single-multiplier structure of Problem 6.64, the simulation program given above in the solution of Problem M6.12 also holds here.

Chapter 7

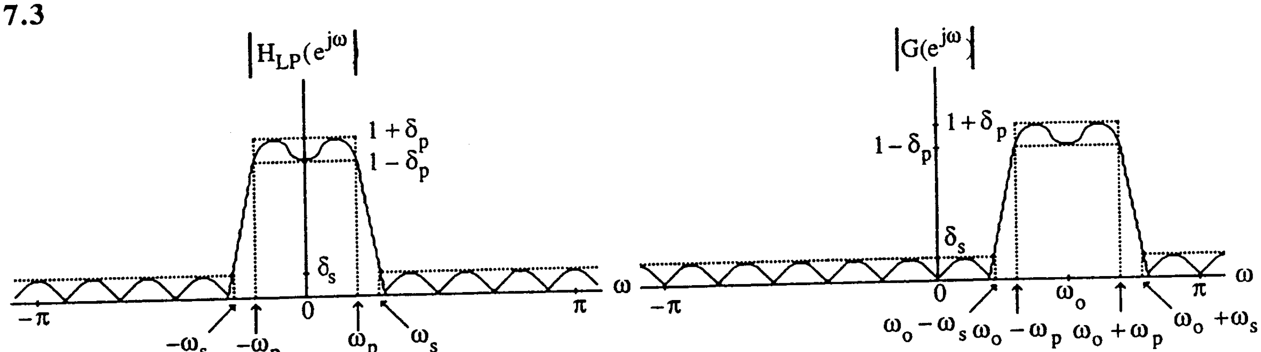
7.1 $G(z) = H^2(z)$ or equivalently, $G(e^{j\omega}) = H^2(e^{j\omega})$. $\left|G(e^{j\omega})\right| = \left|H^2(e^{j\omega})\right| = \left|H(e^{j\omega})\right|^2$. Let δ_p and δ_s denote the passband and stopband ripples of $H(e^{j\omega})$, respectively. Also, let $\delta_{p,2} = 2\delta_p$, and $\delta_{s,2}$ denote the passband and stopband ripples of $G(e^{j\omega})$, respectively. Then $\delta_{p,2} = 1-(1-\delta_p)^2$, and $\delta_{s,2} = (\delta_s)^2$. For a cascade of M sections, $\delta_{p,M} = 1-(1-\delta_p)^M$, and $\delta_{s,M} = (\delta_s)^M$.

7.2



Therefore, the passband edge and the stopband edge of the highpass filter are given by $\omega_{p,HP} = \pi - \omega_p$, and $\omega_{s,HP} = \pi - \omega_s$, respectively.

7.3



Note that G(z) is a complex bandpass filter with a passband in the range $0 \le \omega \le \pi$. Its passband edges are at $\omega_{p,BP} = \omega_o \pm \omega_p$, and stopband edges at $\omega_{s,BP} = \omega_o \pm \omega_s$. A real coefficient bandpass transfer function can be generated according to $G_{BP}(z) = H_{LP}(e^{j\omega_o}z) + H_{LP}(e^{-j\omega_o}z)$ which will have a passband in the range $0 \le \omega \le \pi$ and another passband in the range $-\pi \le \omega \le 0$. However because of the overlap of the two spectra a simple formula for the bandedges cannot be derived.

7.4 From Eqs. (7.29) and (7.32) if $H_a(s) = \dfrac{A}{s+\alpha}$ then $G(z) = \dfrac{A}{1-e^{-\alpha T}z^{-1}}$. We can express

$$H_a(s)=\frac{\lambda}{\lambda^2+(s+\beta)^2}=\frac{1}{2j}\left(\frac{1}{s+\beta-j\lambda}\right)-\frac{1}{2j}\left(\frac{1}{s+\beta+j\lambda}\right).$$ Hence,

$$G(z)=\frac{1}{2j}\left(\frac{1}{1-e^{-(\beta-j\lambda)T}z^{-1}}-\frac{1}{1-e^{-(\beta+j\lambda)T}z^{-1}}\right)=\frac{1}{2j}\left(\frac{-e^{-\beta T}e^{-j\lambda T}z^{-1}+e^{-\beta T}e^{j\lambda T}z^{-1}}{(1-e^{-(\beta-j\lambda)T}z^{-1})(1-e^{-(\beta+j\lambda)T}z^{-1})}\right)$$

$$=\frac{1}{2j}\left(\frac{z^{-1}e^{-\beta T}(e^{j\lambda T}-e^{-j\lambda T})}{1-2z^{-1}e^{-\beta T}\cos(\lambda T)+e^{-2\beta T}z^{-2}}\right)=\frac{z^{-1}e^{-\beta T}\sin(\lambda T)}{1-2z^{-1}e^{-\beta T}\cos(\lambda T)+e^{-2\beta T}z^{-2}}$$

$$=\frac{ze^{-\beta T}\sin(\lambda T)}{z^2-2ze^{-\beta T}\cos(\lambda T)+e^{-2\beta T}}.$$

7.5 $$H_a(s)=\frac{s+\beta}{\lambda^2+(s+\beta)^2}=\frac{1}{2}\left(\frac{1}{s+\beta+j\lambda}\right)+\frac{1}{2}\left(\frac{1}{s+\beta-j\lambda}\right).$$ Hence

$$G(z)=\frac{1}{2}\left(\frac{1}{1-e^{-(\beta+j\lambda)T}z^{-1}}+\frac{1}{1-e^{-(\beta-j\lambda)T}z^{-1}}\right)=\frac{1}{2}\left(\frac{1-e^{-\beta T}e^{j\lambda T}z^{-1}+1-e^{-\beta T}e^{-j\lambda T}z^{-1}}{1-2z^{-1}e^{-\beta T}\cos(\lambda T)+e^{-2\beta T}z^{-2}}\right)$$

$$=\frac{1-z^{-1}e^{-\beta T}\cos(\lambda T)}{1-2z^{-1}e^{-\beta T}\cos(\lambda T)+e^{-2\beta T}z^{-2}}=\frac{z^2-ze^{-\beta T}\cos(\lambda T)}{z^2-2ze^{-\beta T}\cos(\lambda T)+e^{-2\beta T}}.$$

7.6 Assume $h_a(t)$ is causal. Now, $h_a(t)=\oint H_a(s)e^{st}ds$. Hence $g[n]=h_a(nT)==\oint H_a(s)e^{snT}ds$. Therefore,

$$G(z)=\sum_{n=0}^{\infty}g[n]z^{-n}=\sum_{n=0}^{\infty}\oint H_a(s)e^{snT}z^{-n}ds=\oint H_a(s)\sum_{n=0}^{\infty}z^{-n}e^{snT}ds=\oint\frac{H_a(s)}{1-e^{sT}z^{-1}}ds.$$

Hence $$G(z)=\sum_{\text{all poles of } H_a(s)}\text{Residues}\left[\frac{H_a(s)}{1-e^{sT}z^{-1}}\right].$$

7.7 $H_a(s)=\dfrac{A}{s+\alpha}$. The transfer function has a pole at $s=-\alpha$. Now

$$G(z)=\underset{\text{at } s=-\alpha}{\text{Residue}}\left[\frac{A}{(s+\alpha)(1-e^{sT}z^{-1})}\right]=\left.\frac{A}{1-e^{sT}z^{-1}}\right|_{s=-\alpha}=\frac{A}{1-e^{-\alpha T}z^{-1}}.$$

7.8 $H_a(s)=\dfrac{3s^3+7s^2+10s+7}{(s^2+s+1)(s^2+2s+3)}=\dfrac{s+2}{s^2+s+1}+\dfrac{2s+1}{s^2+2s+3}$. Let $H_1(s)=\dfrac{s+2}{s^2+s+1}$ and $H_2(s)=\dfrac{2s+1}{s^2+2s+3}$. We can express $H_1(s)=\dfrac{s+2}{s^2+s+1}=\dfrac{s+2}{(s+\frac{1}{2})^2+\frac{3}{4}}$. Comparing it with Eq. (7.37), we get $C=1$, $D=2$, $\beta=\frac{1}{2}$, $\lambda=\frac{\sqrt{3}}{2}$, and $A=\sqrt{3}$. Hence, from Eq. (7.40) we arrive at $$G_1(z)=\frac{z^2+ze^{-T/2}\left(\sqrt{3}\sin(\frac{\sqrt{3}}{2}T)-\cos(\frac{\sqrt{3}}{2}T)\right)}{z^2-2ze^{-T/2}\cos(\frac{\sqrt{3}}{2}T)+e^{-T}}.$$

Similarly, we can show $G_2(z) = \dfrac{2z^2 + ze^{-T}\left(-\frac{1}{\sqrt{2}}\sin(\sqrt{2}T) - 2\cos(\sqrt{2}T)\right)}{z^2 - 2ze^{-T}\cos(\sqrt{2}T) + e^{-2T}}$.

Hence, $G(z) = G_1(z) + G_2(z)$

$$= \frac{z^2 + ze^{-T/2}\left(\sqrt{3}\sin(\frac{\sqrt{3}}{2}T) - \cos(\frac{\sqrt{3}}{2}T)\right)}{z^2 - 2ze^{-T/2}\cos(\frac{\sqrt{3}}{2}T) + e^{-T}} + \frac{2z^2 + ze^{-T}\left(-\frac{1}{\sqrt{2}}\sin(\sqrt{2}T) - 2\cos(\sqrt{2}T)\right)}{z^2 - 2ze^{-T}\cos(\sqrt{2}T) + e^{-2T}}.$$

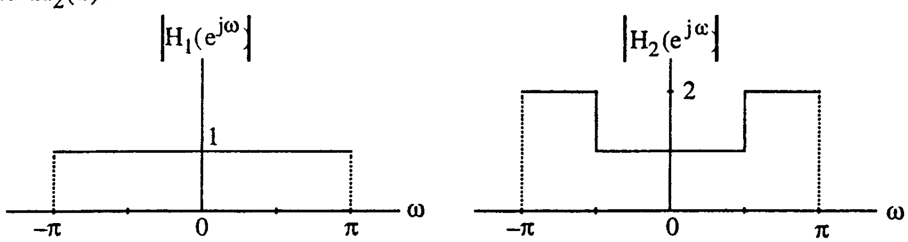
For T = 0.1 sec, we then get $G(z) = \dfrac{z^2 - 0.9487z}{z^2 - 1.90245z + 0.904837} + \dfrac{2z^2 - 1.8818z}{z^2 - 1.791608z + 0.81873}$.

7.9 $H_a(s) = \dfrac{4s^2 + 10s + 8}{(s^2 + 2s + 3)(s+1)} = \dfrac{1}{s+1} + \dfrac{3s+5}{s^2 + 2s + 3} = \dfrac{1}{s+1} + \dfrac{3s+5}{(s+1)^2 + 2}$.

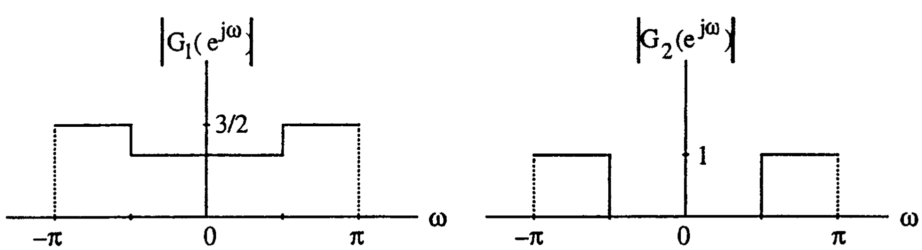
Hence, $G(z) = \dfrac{1}{1 - e^{-T}z^{-1}} + \dfrac{3z^2 + ze^{-T}\left(\sqrt{2}\sin(\sqrt{2}T) - 3\cos(\sqrt{2}T)\right)}{z^2 - 2ze^{-T}\cos(\sqrt{2}T) + e^{-2T}}$.

For T = 0.2, we thus get $G(z) = \dfrac{z}{z - 0.8187} + \dfrac{3z^2 - 2.03545z}{z^2 - 1.57299z + 0.67032}$.

7.10 For no aliasing $T \le \dfrac{\pi}{\Omega_c}$. Figure below shows the magnitide responses of the digital filters $H_1(z)$ and $H_2(z)$.



(a) The magnitude responses of the digital filters $G_1(z)$ and $G_2(z)$ are shown below:



(b) As can be seen from the above $G_1(z)$ is a multi-band filter, whereas, $G_2(z)$ is a highpass filter.

7.11 (a) $H_a(s) = \displaystyle\sum_{k=0}^{R} \frac{A_k}{s - \alpha_k}$. Hence $h_a(t) = \mathcal{L}^{-1}\left\{\dfrac{H_a(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\displaystyle\sum_{k=0}^{R} \frac{A_k}{s(s - \alpha_k)}\right\}$.

Thus $h_a(t) = \mathcal{L}^{-1}\left\{\sum_{k=0}^{R}\frac{A_k}{\alpha_k}\left(\frac{1}{s-\alpha_k}-\frac{1}{s}\right)\right\} = \sum_{k=0}^{R}\frac{A_k}{\alpha_k}\left(e^{\alpha_k t}-1\right)\mu(t).$

Hence $g_\mu[n] = h_a(nT) = \sum_{k=0}^{R}\frac{A_k}{\alpha_k}\left(e^{\alpha_k nT}-1\right)\mu(nT).$ As a result,

$$G_\mu(z) = \sum_{n=-\infty}^{\infty} g_\mu[n]z^{-n} = \sum_{k=0}^{R}\frac{A_k}{\alpha_k}\left(\frac{1}{1-e^{\alpha_k T}z^{-1}}-\frac{1}{1-z^{-1}}\right) = \sum_{k=0}^{R}\frac{A_k}{\alpha_k}\left(\frac{z^{-1}(e^{\alpha_k T}-1)}{(1-e^{\alpha_k T}z^{-1})(1-z^{-1})}\right).$$

Now the transfer function G(z), which is the z-transform of the impulse response g[n], is related to the z-transform of the step response $g_\mu[n]$ by $G(z) = (1-z^{-1})G_\mu(z)$. Hence

$$G(z) = \sum_{k=0}^{R}\frac{A_k}{\alpha_k}\left(\frac{z^{-1}(e^{\alpha_k T}-1)}{1-e^{\alpha_k T}z^{-1}}\right).$$

(b) Now, $\frac{H_a(j\omega)}{j\omega} \approx 0$ if $|\omega| \ge \frac{1}{2T}$. Consider the digital transfer function obtained by sampling $F(s) = \frac{H_a(s)}{s}$. Now this digital transfer function would correspond to the z-transform of the sampled step response of $H_a(s)$. Thus

$$G_\mu(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty}F(j\frac{\omega}{T}+j\frac{2\pi k}{T}) = \frac{1}{T}F(j\frac{\omega}{T}) = \frac{H_a(j\omega/T)}{j\omega}..$$

Since $G(z) = (1-z^{-1})G_\mu(z)$, hence $G(e^{j\omega T}) = (1-e^{-j\omega T})G_\mu(e^{j\omega T}) = (1-e^{-j\omega T})\frac{H_a(j\omega)}{j\omega T}$.

Since $\omega T << 1$, therefore $1-e^{-j\omega T} \approx j\omega T$. Thus, $G(e^{j\omega T}) \cong H_a(j\omega)$.

7.12 $H_a(s)$ is causal and stable and $|H_a(s)| \le 1\ \forall s$, Now, $G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$. Thus, G(z) is causal and stable. Now,

$G(e^{j\omega}) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right)} = H_a(s)\Big|_{s=j\frac{2}{T}\tan(\omega/2)} = H_a\left(j\frac{2}{T}\tan(\omega/2)\right).$ Therefore,

$\left|G(e^{j\omega})\right| = \left|H_a\left(j\frac{2}{T}\tan(\omega/2)\right)\right| \le 1$ for all values of ω. Hence, G(z) is a BR function.

7.13 The ideal L-band digital filter $H_{ML}(z)$ with an ideal frequency response given by $H_{ML}(e^{j\omega})$ $= A_k$ for $\omega_{k-1} \le \omega \le \omega_k$, $k = 1,2,\ldots,L$, can be considered as sum of L ideal bandpass filters with cutoff frequencies at $\omega_{c1}^k = \omega_{k-1}$ and $\omega_{c2}^k = \omega_k$, where $\omega_{c1}^0 = 0$ and $\omega_{c2}^L = \pi$. Now from Eq. (7.90) the impulse response of a bandpass filter is given by $h_{BP}[n] =$ $\frac{\sin(\omega_{c2}n)}{\pi n} - \frac{\sin(\omega_{c1}n)}{\pi n}$. Therefore, $h_{BP}^k[n] = A_k\left(\frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1}n)}{\pi n}\right)$. Hence,

$$h_{ML}[n] = \sum_{k=1}^{L} h_{BP}^{k}[n] = \sum_{k=1}^{L} A_k\left(\frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1} n)}{\pi n}\right)$$

$$= A_1\left(\frac{\sin(\omega_1 n)}{\pi n} - \frac{\sin(0n)}{\pi n}\right) + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k\left(\frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1} n)}{\pi n}\right)$$

$$+ A_L\left(\frac{\sin(\omega_L n)}{\pi n} - \frac{\sin(\omega_{L-1} n)}{\pi n}\right)$$

$$= A_1 \frac{\sin(\omega_1 n)}{\pi n} + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_{k-1} n)}{\pi n} - A_L \frac{\sin(\omega_{L-1} n)}{\pi n}$$

$$= \sum_{k=1}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L} A_k \frac{\sin(\omega_{k-1} n)}{\pi n}.$$

Since $\omega_L = \pi$, $\sin(\omega_L n) = 0$. We add a term $A_L \frac{\sin(\omega_L n)}{\pi n}$ to the first sum in the above expression and change the index range of the second sum, resulting in

$$h_{ML}[n] = \sum_{k=1}^{L} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^{L-1} A_{k+1} \frac{\sin(\omega_k n)}{\pi n}.$$

Finally, since $A_{L+1} = 0$, we can add a term $A_{L+1} \frac{\sin(\omega_L n)}{\pi n}$ to the second sum. This leads to

$$h_{ML}[n] = \sum_{k=1}^{L} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^{L} A_{k+1} \frac{\sin(\omega_k n)}{\pi n} = \sum_{k=1}^{L} (A_k - A_{k+1}) \frac{\sin(\omega_k n)}{\pi n}.$$

7.14 $H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases}$ Therefore,

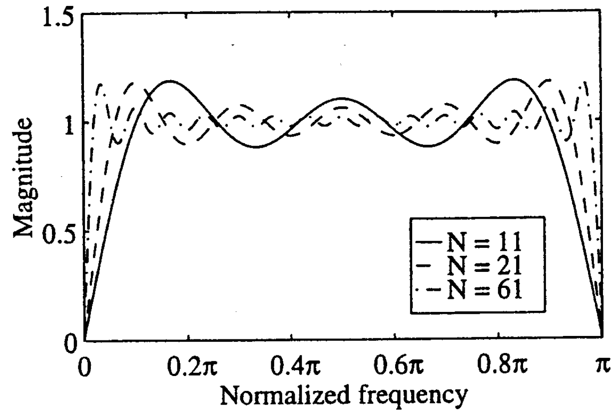
$$h_{HT}[n] = \frac{1}{2\pi}\int_{-\pi}^{0} H_{HT}(e^{j\omega}) e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{0}^{\pi} H_{HT}(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} j e^{j\omega n} d\omega - \frac{1}{2\pi}\int_{0}^{\pi} j e^{j\omega n} d\omega = \frac{2}{2\pi n}(1 - \cos(\pi n)) = \frac{2\sin^2(\pi n/2)}{\pi n} \quad \text{if } n \neq 0.$$

For $n = 0$, $h_{HT}[0] = \frac{1}{2\pi}\int_{-\pi}^{0} j d\omega - \frac{1}{2\pi}\int_{0}^{\pi} j d\omega = 0.$

Hence, $h_{HT}[n] = \begin{cases} 0, & \text{if } n = 0, \\ \frac{2\sin^2(\pi n/2)}{\pi n}, & \text{if } n \neq 0. \end{cases}$

Since $h_{HT}[n] = -h_{HT}[-n]$ and the length of the truncated impulse response is odd, it is a Type 3 linear-phase FIR filter.



From the frequency response plots given above, we observe the presence of ripples at the bandedges due to the Gibbs phenonomenon caused by the truncation of the impulse response.

7.15 $\mathcal{H}\{x[n]\} = \sum_{k=-\infty}^{\infty} h_{HT}[n-k]x[k].$

Hence $\mathcal{F}\{\mathcal{H}\{x[n]\}\} = H_{HT}(e^{j\omega})X(e^{j\omega}) = \begin{cases} jX(e^{j\omega}), & -\pi < \omega < 0, \\ -jX(e^{j\omega}), & 0 < \omega < \pi. \end{cases}$

(a) Let $y[n] = \mathcal{H}\{\mathcal{H}\{\mathcal{H}\{\mathcal{H}\{x[n]\}\}\}\}$. Hence $Y(e^{j\omega}) = \begin{cases} j^4 X(e^{j\omega}), & -\pi < \omega < 0, \\ (-j)^4 X(e^{j\omega}), & 0 < \omega < \pi, \end{cases} = X(e^{j\omega}).$

Therefore, $y[n] = x[n]$.

(b) Define $g[n] = \mathcal{H}\{x[n]\}$, and $h^*[n] = x[n]$. Then $\sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = \sum_{\ell=-\infty}^{\infty} g[\ell]h^*[\ell].$

But from the Parsevals' relation in Table 3.2, $\sum_{\ell=-\infty}^{\infty} g[\ell]h^*[\ell] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})G(e^{j\omega})\,d\omega.$

Therefore, $\sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})\,d\omega$ where

$H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases}$ Since the integrand $H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})$ is an odd function of ω, $\int_{-\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})\,d\omega = 0$. A a result, $\sum_{\ell=-\infty}^{\infty} \mathcal{H}\{x[\ell]\}x[\ell] = 0.$

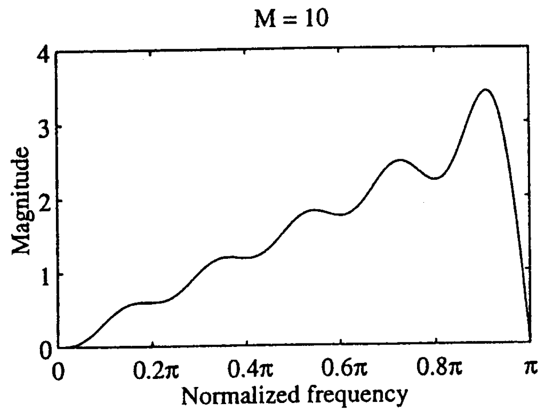
7.16 $H_{DIF}(e^{j\omega}) = j\omega$. Hence,

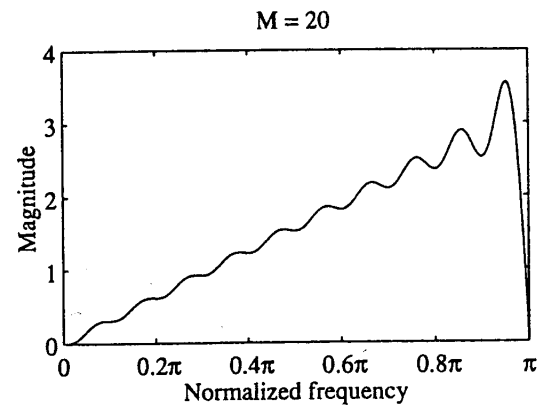
$$h_{DIF}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{j\omega n}\,d\omega = \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega e^{j\omega n}\,d\omega = \frac{j}{2\pi}\left(\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right)\Bigg|_{-\pi}^{\pi}.$$ Therefore,

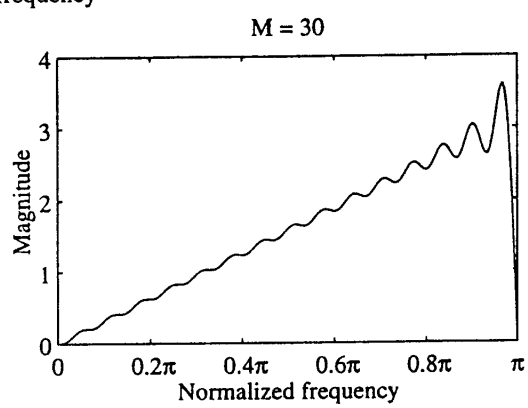
$$h_{DIF}[n] = \frac{\cos(\pi n)}{n} - \frac{\sin(\pi n)}{\pi n^2} = \frac{\cos(\pi n)}{n},$$ if $n \neq 0$. For $n = 0$, $h_{DIF}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega\,d\omega = 0.$

Hence, $h_{DIF}[n] = \begin{cases} 0, & n = 0, \\ \dfrac{\cos(\pi n)}{n}, & |n| > 0. \end{cases}$ Since $h_{DIF}[n] = -h_{DIF}[-n]$, the truncated impulse response is a Type 3 linear-phase FIR filter.

The magnitude responses of the above differentiator for several values of M are given below:







7.17 $N = 2M + 1$. $\hat{h}_{HP}[n] = \begin{cases} 1 - \dfrac{\omega_c}{\pi}, & \text{for } n = M, \\ -\dfrac{\sin(\omega_c(n-m))}{\pi(n-m)}, & \text{if } n \neq M,\ 0 \leq n < N, \\ 0. & \text{otherwise.} \end{cases}$

Now $\hat{H}_{HP}(z) + \hat{H}_{LP}(z) = \sum_{n=-\infty}^{\infty} \hat{h}_{HP}[n] z^{-n} + \sum_{n=-\infty}^{\infty} \hat{h}_{LP}[n] z^{-n} = \sum_{n=0}^{N-1} \hat{h}_{HP}[n] z^{-n} + \sum_{n=0}^{N-1} \hat{h}_{LP}[n] z^{-n}$

$$= \sum_{n=0}^{N-1} \left(\hat{h}_{HP}[n] + \hat{h}_{LP}[n] \right) z^{-n}.$$

But $\hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \hat{h}_{HP}[n] + \hat{h}_{LP}[n] = \begin{cases} 0, & 0 \leq n \leq N-1,\ n \neq M, \\ 1, & n = M, \end{cases}$,

Hence, $\hat{H}_{HP}(z) + \hat{H}_{LP}(z) = z^{-M}$, i.e. the two filters are delay-complementary.

7.18 $H_{LLP}(e^{j\omega}) = \begin{cases} |\omega|, & |\omega| < \omega_c, \\ 0, & \text{otherwise.} \end{cases}$ Therefore,

$$h_{LLP}[n] = \frac{1}{2\pi}\left(-\int_{-\omega_c}^{0} \omega e^{j\omega n} d\omega + \int_{0}^{\omega_c} \omega e^{j\omega n} d\omega\right) = \frac{1}{2\pi}\left(-\left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right]\Bigg|_{-\omega_c}^{0} + \left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right]\Bigg|_{0}^{\omega_c}\right)$$

$$= \frac{1}{2\pi}\left(\frac{\omega_c e^{j\omega_c n} - \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} + e^{-j\omega_c n} - 2}{n^2}\right) = \frac{\omega_c}{\pi n}\sin(\omega_c n) + \frac{\cos(\omega_c n) - 1}{\pi n^2}.$$

7.19 $H_{BLDIF}(e^{j\omega}) = \begin{cases} \omega, & |\omega| < \omega_c, \\ 0, & \text{otherwise.} \end{cases}$ Hence,

$$h_{BLDIF}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \omega e^{j\omega n} d\omega = \frac{1}{2\pi}\left(\left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right]\Bigg|_{-\omega_c}^{\omega_c}\right)$$

$$= \frac{1}{2\pi}\left(\frac{\omega_c e^{j\omega_c n} + \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{n^2}\right) = -j\frac{\omega_c}{\pi n}\cos(\omega_c n) + j\frac{1}{\pi n^2}\sin(\omega_c n).$$

7.20 $$H_{LP}(e^{j\omega}) = \begin{cases} 1, & -\omega_p \le \omega \le \omega_p, \\ 1 - \left(\dfrac{\omega - \omega_p}{\omega_s - \omega_p}\right), & \omega_p < \omega \le \omega_s, \\ 1 + \left(\dfrac{\omega + \omega_p}{\omega_s - \omega_p}\right), & -\omega_s < \omega \le \omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

Now, for $n \ne 0$, $h_{LP}[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega$

$$= \frac{1}{2\pi}\left[\int_{-\omega_p}^{\omega_p} e^{j\omega n} d\omega + \int_{\omega_p}^{\omega_s}\left(1 - \frac{\omega - \omega_p}{\Delta\omega}\right) e^{j\omega n} d\omega + \int_{-\omega_s}^{-\omega_p}\left(1 + \frac{\omega + \omega_p}{\Delta\omega}\right) e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\omega_s}^{\omega_s} e^{j\omega n} d\omega - \int_{\omega_p}^{\omega_s} \frac{\omega - \omega_p}{\Delta\omega} e^{j\omega n} d\omega + \int_{-\omega_s}^{-\omega_p} \frac{\omega + \omega_p}{\Delta\omega} e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left\{\frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega}\left[\frac{(\omega - \omega_p) e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right]\Bigg|_{\omega_p}^{\omega_s} + \frac{1}{\Delta\omega}\left[\frac{(\omega + \omega_p) e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right]\Bigg|_{-\omega_s}^{-\omega_p}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega}\left[\frac{\Delta\omega\, e^{j\omega_s n}}{jn} + \frac{e^{j\omega_s n} - e^{j\omega_p n}}{n^2} + \frac{-\Delta\omega\, e^{-j\omega_s n}}{jn} - \frac{e^{-j\omega_p n} - e^{-j\omega_s n}}{n^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{2\sin(\omega_s n)}{\pi n} - \frac{2}{\Delta\omega}\left[\frac{\Delta\omega\sin(\omega_s n)}{n} + \frac{\cos(\omega_s n) - \cos(\omega_p n)}{n^2}\right]\right\}$$

$$= \frac{1}{\Delta\omega}\left\{\frac{\cos(\omega_p n)}{\pi n^2} - \frac{\cos(\omega_s n)}{\pi n^2}\right\} = \frac{1}{\Delta\omega}\left\{\frac{\cos((\omega_c - \Delta\omega/2)n)}{\pi n^2} - \frac{\cos((\omega_c + \Delta\omega/2)n)}{\pi n^2}\right\}$$

$$= \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n}\frac{\sin(\omega_c n)}{\pi n}.$$

Now for n = 0, $h_{LP}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})d\omega = \frac{1}{2\pi}$(area under the curve)

$$= \frac{1}{2\pi}\frac{2(\omega_s + \omega_p)}{2} = \frac{\omega_c}{\pi}.$$

Hence, $h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n}\frac{\sin(\omega_c n)}{\pi n}, & \text{if } n \neq 0. \end{cases}$

An alternate approach to solving this problem is as follows. Consider the frequency response $G(e^{j\omega}) = \frac{d H_{LP}(e^{j\omega})}{d\omega} = \begin{cases} 0, & -\omega_p \le \omega \le \omega_p, \\ -\frac{1}{\Delta\omega}, & \omega_p < \omega < \omega_s, \\ \frac{1}{\Delta\omega}, & -\omega_s < \omega < -\omega_p, \\ 0, & \text{elsewhere.} \end{cases}$ Its inverse DTFT is given by

$$g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_s}^{-\omega_p}\frac{1}{\Delta\omega}e^{j\omega n}\,d\omega - \frac{1}{2\pi}\int_{\omega_p}^{\omega_s}\frac{1}{\Delta\omega}e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi\Delta\omega}\left(\frac{e^{j\omega n}}{jn}\Bigg|_{-\omega_s}^{-\omega_p} - \frac{e^{j\omega n}}{jn}\Bigg|_{\omega_p}^{\omega_s}\right) = \frac{1}{j\pi n\Delta\omega}\left(\cos(\omega_p n) - \cos(\omega_s n)\right).$$

Thus, $h_{LP}[n] = \frac{j}{n}g[n] = \frac{1}{\pi n^2\Delta\omega}\left(\cos(\omega_p n) - \cos(\omega_s n)\right)$

$$= \frac{1}{\pi n^2\Delta\omega}\left[\cos\left(\left(\omega_c - \frac{\Delta\omega}{2}\right)n\right) - \cos\left(\left(\omega_s + \frac{\Delta\omega}{2}\right)n\right)\right]$$

$$= \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n}\cdot\frac{\sin(\omega_c n)}{\pi n}, \text{ for } n \neq 0.$$

For n = 0, $h_{LP}[n] = \frac{\omega_c}{\pi}$.

7.21 Consider the case when the transition region is approximated by a second order spline. In this case the ideal frequency response can be constructed by convolving an ideal, no-transition-band frequency response with a triangular pulse of width $\Delta\omega = \omega_s - \omega_p$, which in turn can be obtained by convolving two rectangular pulses of width $\Delta\omega/2$. In the time domain this implies that the impulse response of a filter with transition band approximated by a second order spline is given by the product of the impulse response of an ideal low pass filter with no transition region and square of the impulse response of a rectangular pulse. Now,

$H_{LP(ideal)}[n] = \dfrac{\sin(\omega_c n)}{\pi n}$ and $H_{rec}[n] = \dfrac{\sin(\Delta\omega n/4)}{\Delta\omega n/4}$. Hence, $H_{LP}[n] = H_{LP(ideal)}[n]\left(H_{rec}[n]\right)^2$.

Thus for a lowpass filter with a transition region approximated by a second order spline

$$h_{LP}[n] = \begin{cases} \dfrac{\omega_c}{\pi}, & \text{if } n = 0, \\ \left(\dfrac{\sin(\Delta\omega n/4)}{\Delta\omega n/4}\right)^2 \dfrac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$$

Similarly the frequency response of a lowpass filter with the transition region specified by a P-th order spline can be obtained by convolving in the frequency domain an ideal filter with no transition region with P rectangular pulses of width $\Delta\omega/P$. Hence,

$H_{LP}[n] = H_{LP(ideal)}[n]\left(H_{rec}[n]\right)^P$, where the rectangular pulse is of width $\Delta\omega/P$. Thus

$$h_{LP}[n] = \begin{cases} \dfrac{\omega_c}{\pi}, & \text{if } n = 0, \\ \left(\dfrac{\sin(\Delta\omega n/2P)}{\Delta\omega n/2P}\right)^P \dfrac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$$

7.22 Consider another filter with a frequency response $G(e^{j\omega})$ given by

$$G(e^{j\omega}) = \begin{cases} 0, & 0 \le \omega \le \omega_p, \\ \dfrac{-\pi}{2\Delta\omega}\sin\left(\dfrac{\pi(\omega-\omega_p)}{\Delta\omega}\right), & \omega_p < \omega \le \omega_s, \\ \dfrac{-\pi}{2\Delta\omega}\sin\left(\dfrac{\pi(\omega+\omega_p)}{\Delta\omega}\right), & -\omega_s \le \omega \le -\omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly $G(e^{j\omega}) = \dfrac{dH(e^{j\omega})}{d\omega}$. Now,

$$g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})e^{j\omega n}d\omega = \frac{-\pi}{8\pi\Delta\omega j}\left(\int_{\omega_p}^{\omega_s} e^{j\left(\frac{\pi(\omega-\omega_p)}{\Delta\omega}\right)}e^{j\omega n}d\omega - \int_{\omega_p}^{\omega_s} e^{-j\left(\frac{\pi(\omega-\omega_p)}{\Delta\omega}\right)}e^{j\omega n}d\omega \right.$$
$$\left. + \int_{-\omega_s}^{-\omega_p} e^{j\left(\frac{\pi(\omega+\omega_p)}{\Delta\omega}\right)}e^{j\omega n}d\omega - \int_{-\omega_s}^{-\omega_p} e^{-j\left(\frac{\pi(\omega+\omega_p)}{\Delta\omega}\right)}e^{j\omega n}d\omega\right)$$

$$= \frac{-1}{8\Delta\omega j}\left(e^{(-j\pi\omega_p/\Delta\omega)}\left[\frac{e^{j\omega_s(n+\pi/\Delta\omega)} - e^{j\omega_p(n+\pi/\Delta\omega)}}{j(n+\pi/\Delta\omega)}\right] - e^{(j\pi\omega_p/\Delta\omega)}\left[\frac{e^{j\omega_s(n-\pi/\Delta\omega)} - e^{j\omega_p(n-\pi/\Delta\omega)}}{j(n-\pi/\Delta\omega)}\right]\right.$$

$$\left. + e^{(j\pi\omega_p/\Delta\omega)}\left[\frac{e^{-j\omega_p(n+\pi/\Delta\omega)} - e^{-j\omega_s(n+\pi/\Delta\omega)}}{j(n+\pi/\Delta\omega)}\right] - e^{(-j\pi\omega_p/\Delta\omega)}\left[\frac{e^{-j\omega_p(n-\pi/\Delta\omega)} - e^{-j\omega_s(n-\pi/\Delta\omega)}}{j(n-\pi/\Delta\omega)}\right]\right)$$

$$= \frac{-1}{4\Delta\omega j}\left(\frac{-\sin(\omega_s n) - \sin(\omega_p n)}{(n+\pi/\Delta\omega)} - \frac{-\sin(\omega_s n) - \sin(\omega_p n)}{(n-\pi/\Delta\omega)}\right) = \frac{\left(\sin(\omega_s n) + \sin(\omega_p n)\right)}{4\Delta\omega j}\left(\frac{-2\pi/\Delta\omega}{n^2 - \pi^2/\Delta\omega^2}\right)$$

$$= \frac{\sin(\omega_c n)\cos(\Delta\omega n/2)}{\Delta\omega j}\left(\frac{1}{\pi\left(1-(\Delta\omega/\pi)^2 n^2\right)}\right).$$

Now $h[n] = \frac{j}{n} g[n]$. Therefore, $h[n] = \left(\frac{\cos(\Delta\omega n/2)}{1-(\Delta\omega/\pi)^2 n^2}\right)\left(\frac{\sin(\omega_c n)}{\pi n}\right)$.

7.23 Let $w_R[n] = \frac{1}{k}$, Since the convolution of two length N sequences produces a sequence of length 2N– 1, therefore 2N –1 = 2M + 1 which gives N = M + 1. Therefore, $w_R[n] = \frac{1}{k}$, $-\frac{M}{2} \le n \le \frac{M}{2}$. Now, $w[n] = w_R[n] \circledast w_R[n] = \begin{cases} \frac{1}{k^2}(M+1-|n|), & -M \le n \le M, \\ 0, & \text{elsewhere,} \end{cases}$ or

$$w[n] = \begin{cases} \frac{M+1}{k^2}\left(1 - \frac{|n|}{M+1}\right), & -M \le n \le M, \\ 0, & \text{elsewhere.} \end{cases}$$

Now $\frac{M+1}{k^2} = 1$ which yields $k = \sqrt{M+1}$. Hence a Bartlett window of length 2M + 1 is obtained by the linear convolution of 2 length M + 1 rectangular windows scaled by a factor of $\frac{1}{\sqrt{M+1}}$ each. The DTFT $\Psi_{BART}(e^{j\omega})$ of the Bartlett window is thus given by

$$\Psi_{BART}(e^{j\omega}) = \left(\Psi_R(e^{j\omega})\right)^2 = \frac{1}{M+1}\frac{\sin^2\left(\frac{\omega(M+1)}{2}\right)}{\sin^2\left(\frac{\omega}{2}\right)},$$ where $\Psi_R(e^{j\omega})$ is the DTFT of the rectangular window. Hence $\Psi_{BART}(e^{j\omega}) = 0$ at $\omega = \pm\frac{2\pi}{M+1}, \pm\frac{4\pi}{M+1}, \ldots\ldots$

Therefore $\Delta_{ML,BART} = \frac{4\pi}{M+1}$. It should be noted that the main lobe width given here is for a Bartlett window of length 2M + 1 = 2N – 1 as compared to that of a rectangular window of length N = M + 1. The maximas of the DTFT $\Psi_{BART}(e^{j\omega})$ of the Bartlett window occur at the same location as the DTFT $\Psi_R(e^{j\omega})$ of the rectangular window. Since $\Psi_{BART}(e^{j\omega}) = \left(\Psi_R(e^{j\omega})\right)^2$, it follows then $A_{s\ell,BART} = 2 \times A_{s\ell,R} = 2 \times 13.3 = 26.6$ dB.

7.24 $w_{GC}[n]=\left[\alpha+\beta\cos\left(\frac{2\pi n}{2M+1}\right)+\gamma\cos\left(\frac{4\pi n}{2M+1}\right)\right]w_R[n]$

$$=\left[\alpha+2\beta\left(e^{j\left(\frac{2\pi n}{2M+1}\right)}+e^{-j\left(\frac{2\pi n}{2M+1}\right)}\right)+2\gamma\left(e^{j\left(\frac{4\pi n}{2M+1}\right)}+e^{-j\left(\frac{4\pi n}{2M+1}\right)}\right)\right]w_R[n]$$

Hence, $\Psi_{GC}(e^{j\omega})=\alpha\Psi_R(e^{j\omega})\ +2\beta\Psi_R\left(e^{j\left(\omega-\frac{2\pi}{2M+1}\right)}\right)+2\beta\Psi_R\left(e^{j\left(\omega+\frac{2\pi}{2M+1}\right)}\right)$

$$+2\gamma\Psi_R\left(e^{j\left(\omega-\frac{4\pi}{2M+1}\right)}\right)+2\gamma\Psi_R\left(e^{j\left(\omega+\frac{4\pi}{2M+1}\right)}\right).$$

For the Hann window : $\alpha = 0.5$, $\beta = 0.5$ and $\gamma = 0$. Hence

$$\Psi_{HANN}(e^{j\omega})=0.5\Psi_R(e^{j\omega})+\Psi_R\left(e^{j\left(\omega-\frac{2\pi}{2M+1}\right)}\right)+\Psi_R\left(e^{j\left(\omega+\frac{2\pi}{2M+1}\right)}\right)$$

$$=0.5\frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)}+\frac{\sin\left((2M+1)\left(\frac{\omega}{2}-\frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2}-\frac{\pi}{2M+1}\right)}+\frac{\sin\left((2M+1)\left(\frac{\omega}{2}+\frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2}+\frac{\pi}{2M+1}\right)}.$$

For the Hamming window $\alpha = 0.54$, $\beta = 0.46$, and $\gamma = 0$. Hence

$$\Psi_{HAMMING}(e^{j\omega})=0.54\Psi_R(e^{j\omega})\ +0.92\Psi_R\left(e^{j\left(\omega-\frac{2\pi}{2M+1}\right)}\right)+0.92\Psi_R\left(e^{j\left(\omega+\frac{2\pi}{2M+1}\right)}\right).$$

$$=0.54\frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)}+0.92\frac{\sin\left((2M+1)\left(\frac{\omega}{2}-\frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2}-\frac{\pi}{2M+1}\right)}+0.92\frac{\sin\left((2M+1)\left(\frac{\omega}{2}+\frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2}+\frac{\pi}{2M+1}\right)}..$$

For the Blackmann window $\alpha = 0.42$, $\beta = 0.5$ and $\gamma = 0.08$

$$\Psi_{BLACK}(e^{j\omega})=0.54\Psi_R(e^{j\omega})\ +\Psi_R\left(e^{j\left(\omega-\frac{2\pi}{2M+1}\right)}\right)+W_R\left(e^{j\left(\omega+\frac{2\pi}{2M+1}\right)}\right)$$

$$+0.16\Psi_R\left(e^{j\left(\omega-\frac{4\pi}{2M+1}\right)}\right)+0.16\Psi_R\left(e^{j\left(\omega-\frac{4\pi}{2M+1}\right)}\right)$$

$$=0.42\frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)}+\frac{\sin\left((2M+1)\left(\frac{\omega}{2}-\frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2}-\frac{\pi}{2M+1}\right)}+\frac{\sin\left((2M+1)\left(\frac{\omega}{2}+\frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2}+\frac{\pi}{2M+1}\right)}$$

$$+0.16\frac{\sin\left((2M+1)\left(\frac{\omega}{2}-\frac{2\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2}-\frac{2\pi}{2M+1}\right)}+0.16\frac{\sin\left((2M+1)\left(\frac{\omega}{2}-\frac{2\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2}-\frac{2\pi}{2M+1}\right)}.$$

7.25 **(a)** N = 1 and hence, $x_a(t)=\alpha_0+\alpha_1 t$. Without any loss of generality, for L = 5, we first fit the data set $\{x[k]\}$, $-5\le k\le 5$, by the polynomial $x_a(t)=\alpha_0+\alpha_1 t$ with a minimum mean-square error at t = –5, –4, . . . , 0, 1, . . , 5, and then replace x[0] with a new value $\bar{x}[0] = x_a(0) = \alpha_0$.

Now, the mean-square error is given by $\varepsilon(\alpha_0,\alpha_1)=\sum_{k=-5}^{5}\left\{x[k]-\alpha_0-\alpha_1 k\right\}^2$. We set

$\frac{\partial\varepsilon(\alpha_0,\alpha_1)}{\partial\alpha_0}=0$ and $\frac{\partial\varepsilon(\alpha_0,\alpha_1)}{\partial\alpha_1}=0$ which yields $11\alpha_0+\alpha_1\sum_{k=-5}^{5}k=\sum_{k=-5}^{5}x[k]$, and $\alpha_0\sum_{k=-5}^{5}k+\alpha_1\sum_{k=-5}^{5}k^2=\sum_{k=-5}^{5}k\,x[k]$.

From the first equation we get $\bar{x}[0]=\alpha_0=\frac{1}{11}\sum_{k=-5}^{5}x[k]$. In the general case we thus have

$\bar{x}[n]=\alpha_0=\frac{1}{11}\sum_{k=-5}^{5}x[n-k]$, which is a moving average filter of length 11.

(b) N = 2 and hence, $x_a(t)=\alpha_0+\alpha_1 t+\alpha_2 t^2$. Here, we fit the data set $\{x[k]\}$, $-5\le k\le 5$, by the polynomial $x_a(t)=\alpha_0+\alpha_1 t+\alpha_2 t^2$ with a minimum mean-square error at t = –5, –4, . . . , 0, 1, . . , 5, and then replace x[0] with a new value $\bar{x}[0] = x_a(0) = \alpha_0$. The mean-square error is now given by $\varepsilon(\alpha_0,\alpha_1,\alpha_2)=\sum_{k=-5}^{5}\left(x[k]-\alpha_0-\alpha_1 k-\alpha_2 k^2\right)^2$. We set

$\frac{\partial\varepsilon(\alpha_0,\alpha_1,\alpha_2)}{\partial\alpha_0}=0$, $\frac{\partial\varepsilon(\alpha_0,\alpha_1,\alpha_2)}{\partial\alpha_1}=0$, and $\frac{\partial\varepsilon(\alpha_0,\alpha_1,\alpha_2)}{\partial\alpha_2}=0$, which yields

$11\alpha_0+110\alpha_2=\sum_{k=-5}^{5}x[k]$, $110\alpha_1=\sum_{k=-5}^{5}k\,x[k]$, $110\alpha_0+1958\alpha_2=\sum_{k=-5}^{5}k^2x[k]$. From the first and the third equations we then get

$$\alpha_0=\frac{1958\sum_{k=-5}^{5}x[k]-110\sum_{k=-5}^{5}k^2x[k]}{(1958\times 11)-(110)^2}=\frac{1}{429}\sum_{k=-5}^{5}\left(89-5k^2\right)x[k].$$

Hence, here we replace x[n] with a new value $\bar{x}[n]=\alpha_0$ which is a weighted combination of the original data set $\{x[k]\}$, $-5\le k\le 5$:

$$\bar{x}[n]=\frac{1}{429}\sum_{k=-5}^{5}\left(89-5k^2\right)x[n-k]$$
$$=\frac{1}{429}(-36\,x[n+5]+9\,x[n+4]+44\,x[n+3]+69\,x[n+2]+84\,x[n+1]+89\,x[n]$$

$$+84\,x[n-1]+69\,x[n-2]+44\,x[n-3]+9\,x[n-4]-36\,x[n-5]).$$

(c) The impulse response of the FIR filter of Part (a) is given by

$$h_1[n]=\frac{1}{11}\{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\},$$

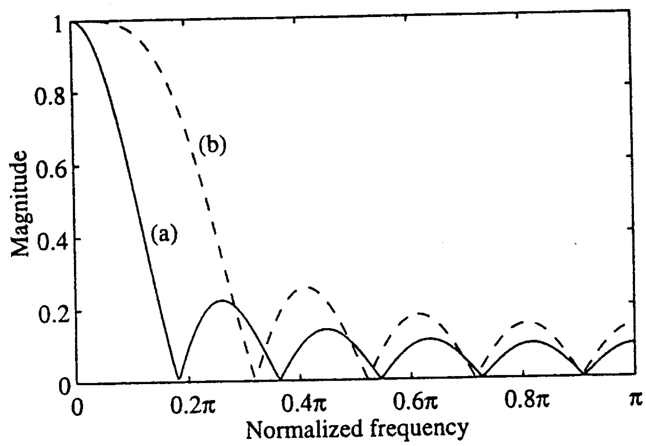
whereas, the impulse response of the FIR filter of Part (b) is given by

$$h_1[n]=\frac{1}{429}\{-36 \quad 9 \quad 44 \quad 69 \quad 84 \quad 89 \quad 84 \quad 69 \quad 44 \quad 9 \quad -36\}.$$

The corresponding frequency responses are given by

$$H_1(e^{j\omega})=\frac{1}{11}\sum_{k=-5}^{5}e^{-j\omega k}, \text{ and } H_2(e^{j\omega})=\frac{1}{429}\sum_{k=-5}^{5}\left(89-5k^2\right)e^{-j\omega k}.$$

A plot of the magnitude responses of these two filters are shown below from which it can be seen that the filter of Part (b) has a wider passband and thus provides smoothing over a larger frequency range than the filter of Part (a).

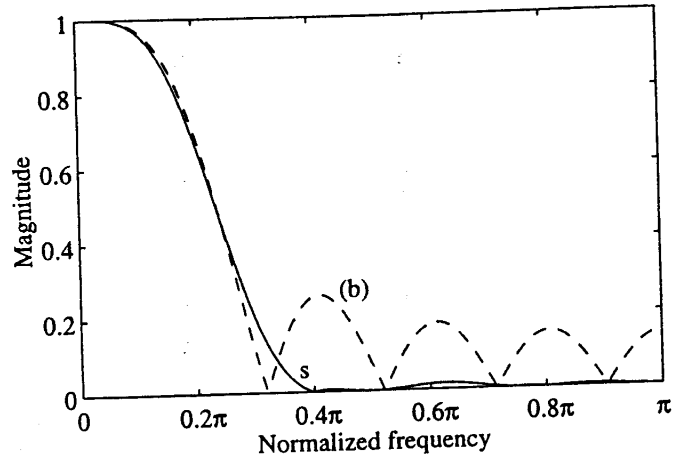


7.26 $y[n] = \frac{1}{320}\{-3x[n-7]-6x[n-6]-5x[n-5]+3x[n-4]+21x[n-3]+46x[n-2]$
$+67x[n-1]+74x[n]+67x[n+1]+46x[n+2]+21x[n+3]$
$+3x[n+4]5x[n+5]-6x[n+6]-3x[n+7]\}.$

Hence, $$\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{320}\left\{-3e^{-j7\omega}-6e^{-j6\omega}-5e^{-j5\omega}+3e^{-j4\omega}+21e^{-j3\omega}+46e^{-j2\omega}+67e^{-j\omega}+74\right.$$
$$\left.+67e^{j\omega}+46e^{j2\omega}+21e^{j3\omega}+3e^{j4\omega}-5e^{j5\omega}-6e^{j6\omega}-3e^{j7\omega}\right\}.$$

Thus, $$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{160}\{74+67\cos\omega+46\cos(2\omega)+21\cos(3\omega)+3\cos(4\omega)$$
$$-5\cos(5\omega)-6\cos(6\omega)-3\cos(7\omega)\}.$$

The magnitude response of the above FIR filter is plotted below (solid line marked 's') along with that of the FIR filter of Part (b) of Problem 7.25 (dashed line marked '(b)'). Note that both filters have roughly the same passband but the Spencer's filter has very large attenuation in the stopband and hence it provides better smoothing than the filter of Part (b).



7.27 (a) L = 3. $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$. Now P(0) = 0 is satisfied by the way P(x) has been defined. Also to ensure P(1) = 1 we require $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Choose m = 1 and n = 1, Since $\left.\frac{dP(x)}{dx}\right|_{x=0} = 0$, hence $\left.\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2\right|_{x=0} = 0$, implying $\alpha_1 = 0$. Also since $\left.\frac{dP(x)}{dx}\right|_{x=1} = 0$, hence $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$. Thus solving the three equations:

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \qquad \alpha_1 = 0, \qquad \text{and} \qquad \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$$

we arrive at $\alpha_1 = 0$, $\alpha_2 = 3$, and $\alpha_3 = -2$. Therefore, $P(x) = 3x^2 - 2x^3$.

(b) L = 4. Hence, $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$. Choose m = 2 and n = 1 (alternatively one can choose m = 1 and n = 2 for better stopband performance). Then,

P(1) = 1 $\Rightarrow$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$,

$$\left.\frac{dP(x)}{dx}\right|_{x=0} = 0 \Rightarrow \left.\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3\right|_{x=0} = 0,$$

$$\left.\frac{d^2P(x)}{dx^2}\right|_{x=0} = 0 \Rightarrow \left.2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2\right|_{x=0} = 0,$$

$$\left.\frac{dP(x)}{dx}\right|_{x=1} = 0 \Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 0.$$

Solving the above simultaneous equations we get $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 4$, and $\alpha_4 = -3$. Therefore, $P(x) = 4x^3 - 3x^4$.

(c) L = 5. Hence $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5$. Choose m = 2 and n = 2. Following a procedure similar to that in parts (a) and (b) we get $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 10$, $\alpha_4 = -15$, and $\alpha_5 = 6$.

7.28 From the given specifications $\frac{\omega_s}{M} = 0.2\pi$. We also require $\frac{\omega_s}{M} \le \frac{2\pi - \omega_s}{M}$, for no overlap to occur in $|H(e^{jM\omega})|$ (see Figure P7.3(b)). Hence $0.2\pi \le \frac{2\pi}{M} - 0.2\pi$ which implies $M \le 5$.

We choose M = 4. Hence specifications for H(z) are $\omega_p = 0.6\pi$, $\omega_s = 0.8\pi$, $\delta_p = 0.001$ and $\delta_s = 0.001$. Substituting these values in Eq. (7.13), we arrive at the estimated value of the FIR filter length as N = 33.

Specifications for F(z) are $\omega_p = 0.15\pi$, $\omega_s = 0.3\pi$, $\delta_p = 0.001$ and $\delta_s = 0.001$. Hence, from Eq. (7.13), the esimated length of F(z) is N = 43.

Therefore the total number of multiplications required per output sample is $\left(\frac{33}{2}+43\right)\times\frac{1}{2} =$ 30. (The division by 2 is due to the fact that H(z) and F(z) are linear phase filters.)

On the other hand for a direct single stage implementation, we note that the specifications of G(z) are: $\omega_p = 0.15\pi$, $\omega_s = 0.2\pi$, $\delta_p = 0.002$, $\delta_s = 0.001$. Hence, the filter length of G(z) from Eq. (7.13) is 121. Therefore the total number of multiplications required per output sample is 121/2 = 61.

7.29 **(a)** For Type 1 FIR filter, $H(e^{j\omega}) = e^{-j\frac{\omega(N-1)}{2}}\left|H(e^{j\omega})\right|$. Since in frequency sampling approach we sample the DTFT $H(e^{j\omega})$ at N points given by $\omega = \frac{2\pi k}{N}$, $k = 0,1,\ldots,N-1$, therefore $H[k] = H(e^{j2\pi k/N}) = \left|H_d(e^{j2\pi k/N})\right|e^{-j2\pi k(N-1)/2N}$, $k = 0,1,\ldots,N-1$. Since the filter is of Type 1, (N – 1) is even, thus $e^{j2\pi(N-1)/2} = 1$. Moreover, h[n] being real, $H(e^{j\omega}) = H^*(e^{j\omega})$. Thus, $H(e^{j\omega}) = e^{j\omega(N-1)/2}\left|H(e^{j\omega})\right|$, $\pi \le \omega < 2\pi$. Hence,

$$H[k] = \begin{cases} \left|H_d(e^{j2\pi k/N})\right|e^{-j2\pi k(N-1)/2N}, & k = 0,1,2,\ldots,\frac{N-1}{2}, \\ \left|H_d(e^{j2\pi k/N})\right|e^{j2\pi(N-k)(N-1)/2N}, & k = \frac{N+1}{2},\frac{N+3}{2},\ldots,N-1. \end{cases}$$

(b) For Type 2 FIR filter

$$H[k] = \begin{cases} \left|H_d(e^{j2\pi k/N})\right|e^{-j2\pi k(N-1)/2N}, & k = 0,1,2,\ldots,\frac{N}{2}-1, \\ 0, & k = \frac{N}{2}, \\ \left|H_d(e^{j2\pi k/N})\right|e^{j2\pi k(N-k)(N-1)/2N} & k = \frac{N}{2}+1,\ldots,N-1. \end{cases}$$

(c) In Example 7.16, N = 37, and $\left|H_d(e^{j\omega})\right| = \begin{cases} 1, & 0 < \omega \le 0.3\pi, \\ 0, & 0.3\pi < \omega \le 1.7\pi, \\ 1, & 1.7\pi < \omega \le 2\pi. \end{cases}$

Therefore, $H[k] = \begin{cases} e^{-j(2\pi k/37)18}, & 0 \le k < 6, \\ 0, & 6 \le k < 32, \\ e^{j(2\pi(37-k)/37)18}, & 32 \le k \le 36. \end{cases}$

Hence, $H[k] = \begin{cases} e^{-j(2\pi k/37)18}, & k = 0,1,\ldots,5,32,33,\ldots,36, \\ 0, & k = 6,7,\ldots,31. \end{cases}$

7.30 By expressing $\cos(\omega n) = T_n(\cos\omega)$, where $T_n(x)$ is the n-th order Chebyshev polynomial in x, we first rewrite Eq. (7.142) in the form:

$$\breve{H}(\omega) = \sum_{n=0}^{M} a[n]\cos(\omega n) = \sum_{n=0}^{M} \alpha_n \cos^n(\omega).$$

Therefore, we can rewrite Eq. (7.145) repeated below for convenience

$$P(\omega_i)\left[\breve{H}(\omega_i) - D(\omega_i)\right] = (-1)^i \varepsilon, \quad 1 \le i \le M+2,$$

in a matrix form as

$$\begin{bmatrix} 1 & \cos(\omega_1) & \cdots & \cos^M(\omega_1) & 1/P(\omega_1) \\ 1 & \cos(\omega_2) & \cdots & \cos^M(\omega_2) & -1/P(\omega_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{M+1}) & \cdots & \cos^M(\omega_{M+1}) & (-1)^M/P(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cdots & \cos^M(\omega_{M+2}) & (-1)^{M+1}/P(\omega_{M+2}) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_M \\ \varepsilon \end{bmatrix} = \begin{bmatrix} D(\omega_1) \\ D(\omega_2) \\ \vdots \\ D(\omega_{M+1}) \\ D(\omega_{M+2}) \end{bmatrix}.$$

Note that the coefficients $\{\alpha_i\}$ are different from the coefficients $\{a[i]\}$ of Eq. (7.142). To determine the expression of ε we use Cramer's rule arriving at $\varepsilon = \frac{\Delta_\varepsilon}{\Delta}$, where

$$\Delta = \det\begin{bmatrix} 1 & \cos(\omega_1) & \cdots & \cos^M(\omega_1) & 1/P(\omega_1) \\ 1 & \cos(\omega_2) & \cdots & \cos^M(\omega_2) & -1/P(\omega_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{M+1}) & \cdots & \cos^M(\omega_{M+1}) & (-1)^M/P(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cdots & \cos^M(\omega_{M+2}) & (-1)^{M+1}/P(\omega_{M+2}) \end{bmatrix}, \text{ and}$$

$$\Delta_\varepsilon = \det\begin{bmatrix} 1 & \cos(\omega_1) & \cdots & \cos^M(\omega_1) & D(\omega_1) \\ 1 & \cos(\omega_2) & \cdots & \cos^M(\omega_2) & D(\omega_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{M+1}) & \cdots & \cos^M(\omega_{M+1}) & D(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \cdots & \cos^M(\omega_{M+2}) & D(\omega_{M+2}) \end{bmatrix}.$$

Expanding both determinants using the last column we get $\Delta_\varepsilon = \sum_{i=1}^{M+2} b_i D(\omega_{i+1})$, and

$\Delta = \sum_{i=1}^{M+2} b_i \frac{(-1)^{i-1}}{P(\omega_i)}$, where

$$b_i = \det\begin{bmatrix} 1 & \cos(\omega_1) & \cos^2(\omega_1) & \cdots & \cos^M(\omega_1) \\ 1 & \cos(\omega_2) & \cos^2(\omega_2) & \cdots & \cos^M(\omega_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos(\omega_{i-1}) & \cos^2(\omega_{i-1}) & \cdots & \cos^M(\omega_{i-1}) \\ 1 & \cos(\omega_{i+1}) & \cos^2(\omega_{i+1}) & \cdots & \cos^M(\omega_{i+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos(\omega_{M+2}) & \cos^2(\omega_{M+2}) & \cdots & \cos^M(\omega_{M+2}) \end{bmatrix}.$$

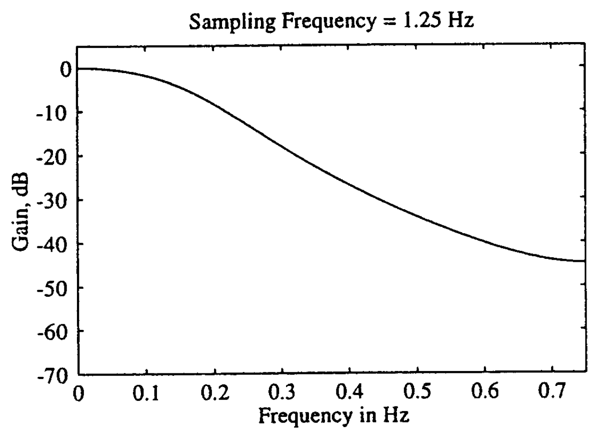
The above matrix is seen to be a Vandermonde matrix and is determinant is given by

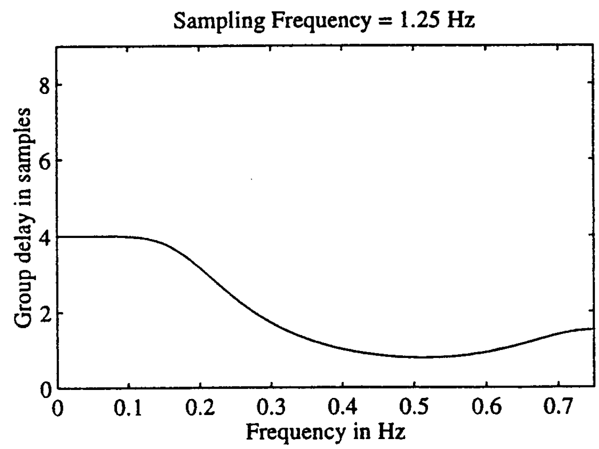
$$b_i = \prod_{\substack{k \neq \ell,\, k > \ell \\ k, \ell \neq i}} (\cos\omega_k - \cos\omega_\ell).$$

Define $c_i = \dfrac{b_i}{\prod_{\substack{r=1 \\ r \neq i}}^{M+2} b_r}$. It can be shown by induction that $c_i = \prod_{\substack{n=1 \\ n \neq i}}^{M+2} \dfrac{1}{\cos\omega_i - \cos\omega_n}$. Therefore,

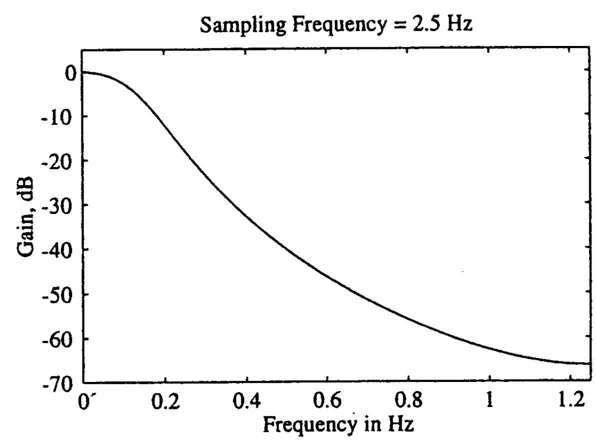
$$\varepsilon = \frac{\sum_{i=1}^{M+2} b_i D(\omega_i)}{\sum_{i=1}^{M+2} b_i \dfrac{(-1)^i}{P(\omega_i)}} = \frac{c_1 D(\omega_1) + c_2 D(\omega_2) + \cdots + c_{M+2} D(\omega_{M+2})}{\dfrac{c_1}{P(\omega_1)} - \dfrac{c_2}{P(\omega_2)} + \cdots + \dfrac{c_{M+2}(-1)^{M+1}}{P(\omega_{M+2})}}$$

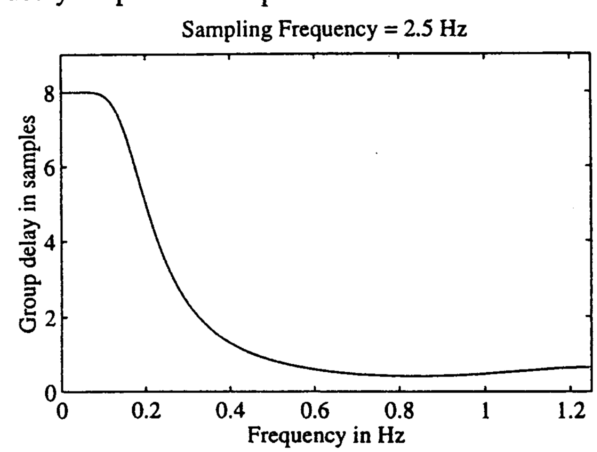
M7.1 The plots of the gain response and the group delay response for $F_T = 1.25$ are shown below:



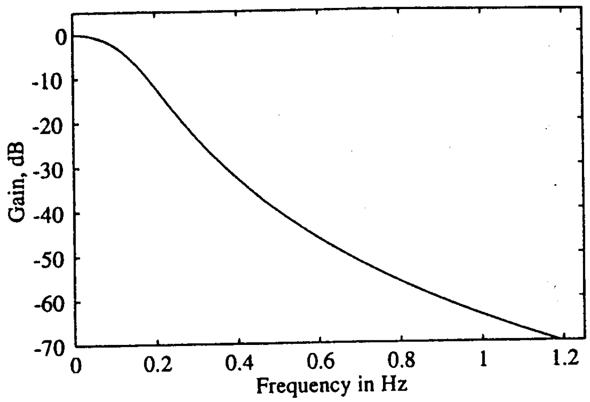


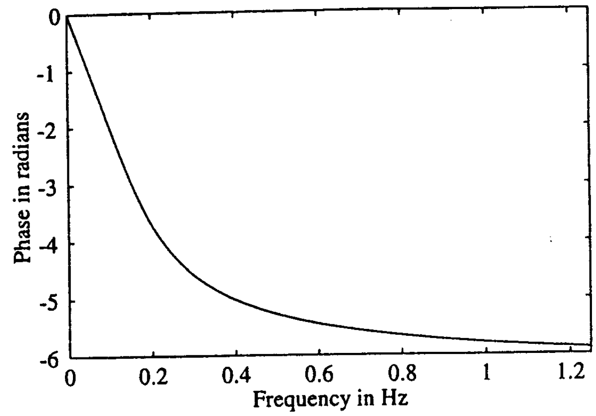
The plots of the gain response and the group delay response for $F_T = 2.5$ are shown below:





The plots of the gain response and the phase response of the Bessel filter are shown below:





M7.2 Given specifications: $\omega_p = 0.0907\pi$, $\omega_s = 0.453515\pi$,

$$20\log_{10}\left|G(e^{j0.0907\pi})\right| \geq -0.5, \text{ and } 20\log_{10}\left|G(e^{j0.453515\pi})\right| \leq -50.$$

(a) Impulse Invariance Method: Let T = 1. Assume no aliasing. Then the specifications of $H_a(s)$ is same as that of G(z), i.e.

$\Omega_p = 0.0907\pi$, $\Omega_s = 0.453515\pi$, $20\log_{10}\left|H_a(j0.0907\pi)\right| \geq -0.5$, and $20\log_{10}\left|H_a(j0.453515\pi)\right| \leq -50$. Now,

$20\log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right) = -0.5$ which yields $\varepsilon^2 = 0.12202$. Similarly, $20\log_{10}\left(\frac{1}{A}\right) = -50$ which yields A^2 = 100,000. Therefore, the inverse discrimination ratio is given by $\frac{1}{k_1} = \sqrt{\frac{A^2-1}{\varepsilon^2}} = \sqrt{\frac{99999}{0.12202}} = 905.2787$, and the inverse transition ratio is given by $\frac{1}{k} = \frac{0.453515\pi}{0.0907\pi} = 5$. Hence, from Eq. (5.31), the order of the lowpass Butterworth filter is given by $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{\log_{10}(905.2787)}{\log_{10}(5)} = 4.23$. We choose N = 5 as the order. Next we determine the 3-dB cutoff frequency Ω_c by meeting exactly the passband edge specifications. From Eq. (5.30a), we have $\left(\Omega_p/\Omega_c\right)^{10} = \varepsilon^2$ or $\Omega_c = 1.23412\,\Omega_p = 0.351653$.

Using `buttap` we determine the normalized analog Butterworth transfer function of 5-th order with a 3-dB cutoff frequency at $\Omega_c = 1$ which is

$$H_{an}(s) = \frac{1}{(s+1)(s^2+0.618034s+1)(s^2+1.618034s+1)}$$
$$= \frac{1.89442}{s+1} + \frac{-0.2763932s - 0.8944272}{s^2+0.618034s+1} + \frac{-1.618034s}{s^2+1.618034s+1}..$$

We denormalize $H_{an}(s)$ to move Ω_c to 0.351653 leading to

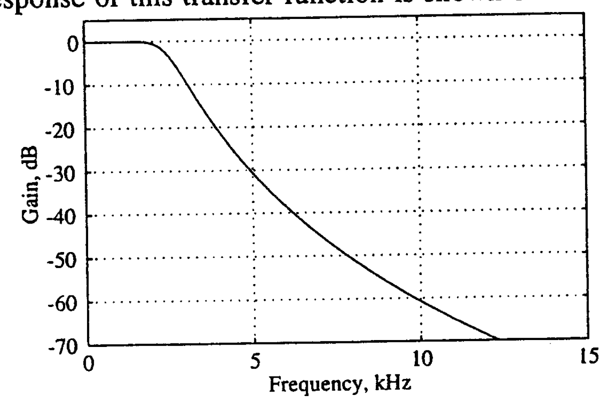
$$H_a(s) = H_{an}\left(\frac{s}{0.351653}\right) = \frac{0.666178}{s+0.351653} + \frac{-0.097197s - 0.110605}{s^2 + 0.2173335s + 0.12366} + \frac{-0.5689865s}{s^2 + 0.5689865s + 0.12366}.$$

$$= \frac{0.666178}{s+0.351653} + \frac{-0.097197s - 0.110605}{(s+0.10866675)^2 + 0.33444^2} + \frac{-0.5689865s}{(s+0.28449325)^2 + 0.20669685^2}.$$

Comparing each term in the last expression with Eq. (7.37) and applying the transformation of Eq. (7.40) we arrive at

$$G(z) = \frac{0.666178z}{z-0.703524} + \frac{-0.097197z^2 - 0.00572z}{z^2 - 1.6946575z + 0.80466} + \frac{-0.5689865z^2 + 0.5399173z}{z^2 - 1.47276z + 0.5661}.$$

A plot of the gain response of this transfer function is shown below.



(b) Bilinear Transformation Method: Let T = 2. From Eq. (7.52), the bandedges of the analog prototype are $\Omega_p = 0.143443$, and $\Omega_s = 0.863675$. The inverse transition ratio is now $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 6.02103$. The inverse discrimination ratio remains the same as in Part (a).

Substituting these values in Eq (5.31) we obtain $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{\log_{10}(905.2787)}{\log_{10}(6.02103)} = 3.79235.$

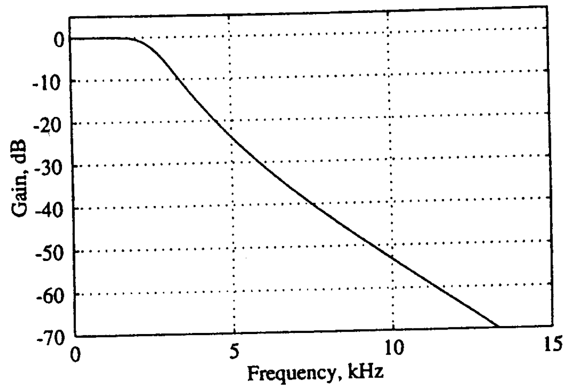
We choose the filter order as N = 4. From Eq. (5.30a), we have $\left(\Omega_p / \Omega_c\right)^8 = \varepsilon^2$ or $\Omega_c = 1.300757\,\Omega_p = 0.1865845$. Since the order is same as that in the example of Section 7.5.1, we note that the normalized analog Butterworth transfer function of 4-th order with a 3-dB cutoff frequency at $\Omega_c = 1$ is $H_{an}(s) = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}$. We denormalize $H_{an}(s)$ to move Ω_c to 0.1865845 leading to

$$H_a(s) = H_{an}\left(\frac{s}{0.1865845}\right) = \frac{0.001212}{(s^2 + 0.14281175s + 0.03481376)(s^2 + 0.34477077s + 0.03481376)}.$$

Applying bilinear transformation to $H_a(s)$ we finally arrive at

$$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.000746(1+z^{-1})^4}{(1-1.639207z^{-1} + 0.757458z^{-2})(1-1.399242z^{-1} + 0.500182z^{-2})}.$$

A plot of the gain response of this transfer function is shown below.



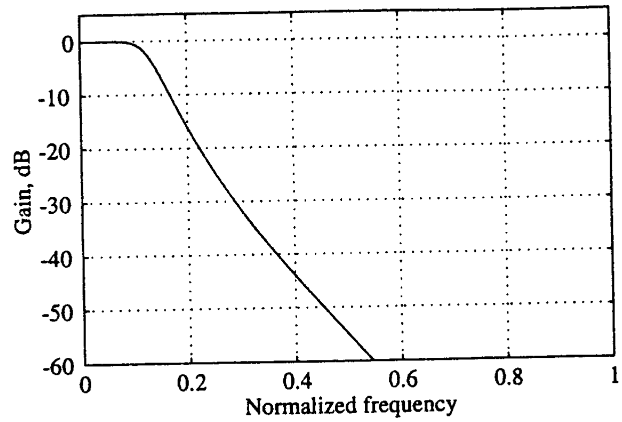
M7.3 The modified program is as given below:

```
format long
Wp = input('Passband edge frequency = ');
Ws = input('Stopband edge frequency = ');
Rp = input('Maximum passband deivation in dB = ');
Rs = input('Minimum stopband deivation in dB = ');
[N, Wn] = buttord(Wp, Ws, Rp, Rs);
[b,a] = butter(N,Wn);
disp('Numerator polynomial'); disp(b);
disp('Denominator polynomial'); disp(a);
w = 0:pi/255:pi;
h = freqz(b,a,w);
gain = 20*log10(abs(h));
plot (w/pi,gain);grid;axis([0 1 -60 5]);
xlabel('Normalized frequency');ylabel('Gain, dB')
```

The data generated by running this program for the filter specifications: Wp = 0.0907, Ws = 0.453515, Rp = 0.5 dB and Rs = 50 dB are as follows:

```
Numerator polynomial
  Columns 1 through 4
   0.00103344    0.004133765    0.0062006479    0.004133765

  Column 5
   0.00103344

Denominator polynomial
  Columns 1 through 4
   1.0   -2.94748748    3.3619516   -1.742887

  Column 5
   0.34495788
```

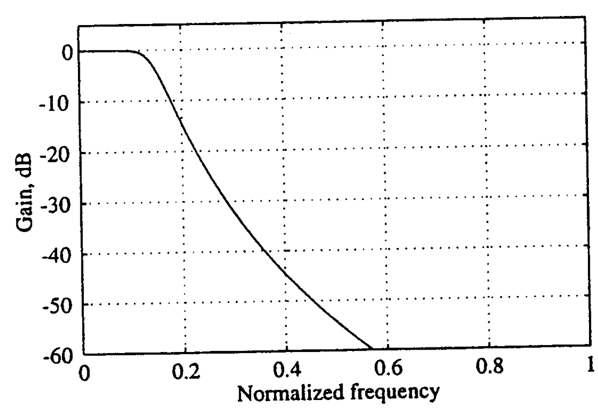
and the gain response of the above filter is as indicated below:



M7.4 The modified program is as given below:

```
Wp = input('Passband edge = ');
Ws = input('Stopband edge = ');
Rp = input('Maximum passband deivation = ');
Rs = input('Minimum stopband attenuation = ');
[N, Wn] = buttord(Wp, Ws, Rp, Rs, 's');
[num,den] = butter(N, Wn, 's');
FT = input('Type in sampling frequency = ');
[b,a] = impinvar(num,den,FT);
[h,omega] = freqz(b,a,512);
mag = 20*log10(abs(h));
plot(omega/pi,mag); axis([0 1 -60 5]); grid
xlabel('Normalized frequency');ylabel('Gain, dB')
```

The gain response of the filter generated for the input data: Wp = 0.0907π, Ws = 0.453515π, Rp = 0.5 and Rs = 50 is as indicated below:



M7.5 Given specifications: $\omega_p = 0.0907\pi$, $\omega_s = 0.453515\pi$,

$$20\log_{10}\left|G(e^{j0.0907\pi})\right| \geq -0.5, \text{ and } 20\log_{10}\left|G(e^{j0.453515\pi})\right| \leq -50.$$

(a) Impulse Invariance Method: Let T = 1. Assume no aliasing. Then the specifications of $H_a(s)$ is same as that of G(z), i.e.

$\Omega_p = 0.0907\pi$, $\Omega_s = 0.453515\pi$, $20\log_{10}\left|H_a(j0.0907\pi)\right| \geq -0.5$, and

$20\log_{10}\left|H_a(j0.453515\pi)\right| \leq -50$. Now,

$20\log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right) = -0.5$ which yields $\varepsilon^2 = 0.12202$. Similarly, $20\log_{10}\left(\frac{1}{A}\right) = -50$ which yields A^2 = 100,000. Therefore, the inverse discrimination ratio is given by $\frac{1}{k_1} = \sqrt{\frac{A^2-1}{\varepsilon^2}} = \sqrt{\frac{99999}{0.12202}} = 905.2787$, and the inverse transition ratio is given by $\frac{1}{k} = \frac{0.453515\pi}{0.0907\pi} = 5$. Hence, from Eq. (5.39), the order of the lowpass Type 1 Chebyshev filter is given by $N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 3.2722$. We choose the filter order as N = 4. Using `cheb1ap` for N = 4 and Rp = 0.5 we arrive at the transfer function of a normalized 4-th order Type 1 Chebyshev lowpass filter given by

$$H_{an}(s) = \frac{0.357847}{(s^2+0.84668s+0.356412)(s^2+0.350706s+1.06352)}$$
$$= \frac{0.200634s+0.455915}{s^2+0.84667s+0.356412} + \frac{-0.200634s-0.356406}{s^2+0.350706s+1.06352}.$$

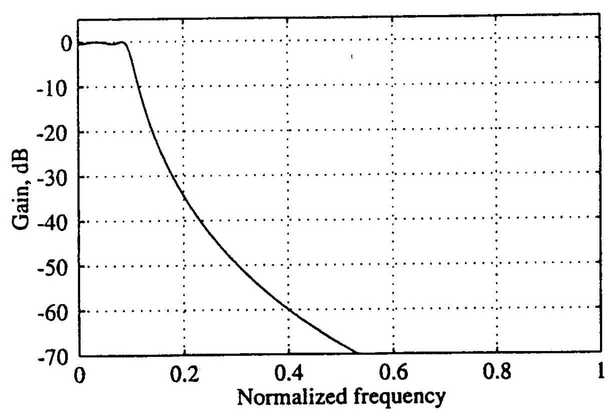
We denormalize $H_{an}(s)$ to move the passband edge from $\Omega_p = 1$ to $\Omega_p = 0.0907\pi = 0.284942$ leading to

$$H_a(s) = H_{an}\left(\frac{s}{0.284942}\right) = \frac{0.05717s+0.037017}{s^2+0.241252s+0.0289378} + \frac{-0.05717s-0.0289373}{s^2+0.099931s+0.08634926}$$
$$= \frac{0.05717s+0.037017}{(s+0.120626)^2+(0.1199465)^2} + \frac{-0.05717s-0.0289373}{(s+0.0499654)^2+(0.2895733)^2}.$$

Comparing each term in the last expression with Eq. (7.37) and applying the transformation of Eq. (7.40) we arrive at

$$G(z) = \frac{0.05717z^2-0.02367535z}{z^2-1.7599938z+0.7856436} + \frac{-0.05717z^2+0.027655z}{z^2-1.8233146z+0.9049}.$$

A plot of the gain response of this transfer function is shown below.



(b) Bilinear Transformation Method: Let T = 2. From Eq. (7.52), the bandedges of the analog prototype are $\Omega_p = 0.143443$, and $\Omega_s = 0.863675$. The inverse transition ratio is now

$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 6.02103$. The inverse discrimination ratio remains the same as in Part (a). Hence, from Eq. (5.39), the order of the lowpass Type 1 Chebyshev filter is given by

$N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 3.023$. We choose the filter order as N = 4. From Part (a), the transfer function of a normalized 4-th order Type 1 Chebyshev lowpass filter is given by

$$H_{an}(s) = \frac{0.357847}{(s^2 + 0.84668s + 0.356412)(s^2 + 0.350706s + 1.06352)}.$$

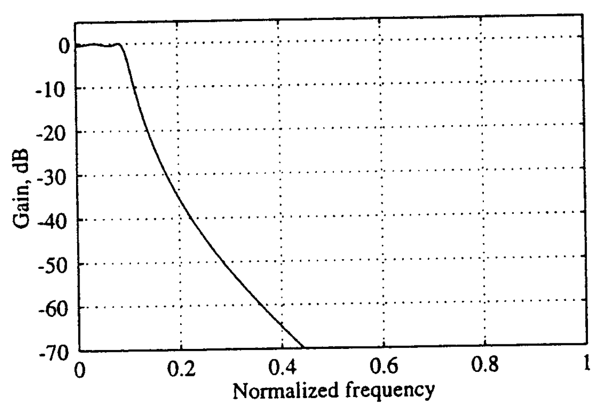
We denormalize $H_{an}(s)$ to move Ω_c to 0.143443 leading to

$$H_a(s) = H_{an}\left(\frac{s}{0.143443}\right) = \frac{0.0001515}{(s^2 + 0.12145s + 0.0073335)(s^2 + 0.05030632s + 0.021883)}.$$

A bilinear transformation of the above with T = 2 results in

$$G(z) = \frac{0.00012518(1 - z^{-1})^4}{(1 - 1.758825z^{-1} + 0.784812z^{-2})(1 - 1.8245234z^{-1} + 0.9061615z^{-2})}.$$

A plot of the gain response of this transfer function is shown below.



M7.6 The modified program is as given below:

```
Wp = input('Passband edge frequency = ');
Ws = input('Stopband edge frequency = ');
Rp = input('Maximum passband deviation = ');
Rs = input('Minimum stopband attenuation = ');
[N, Wn] = cheb1ord(Wp, Ws, Rp, Rs);
[b,a] = cheby1(N,Rp,Wn);
disp('Numerator polynomial'); disp(b);
disp('Denominator polynomial'); disp(a);
w = 0:pi/255:pi;
h = freqz(b,a,w);
plot (w/pi,20*log10(abs(h)));axis([0 1 -60 5]); grid
xlabel('Normalized frequency'); ylabel('Gain, dB');
```

The data generated by running this program for the filter specifications: Wp = 0.0907, Ws = 0.453515, Rp = 0.5 dB and Rs = 50 dB are as follows:

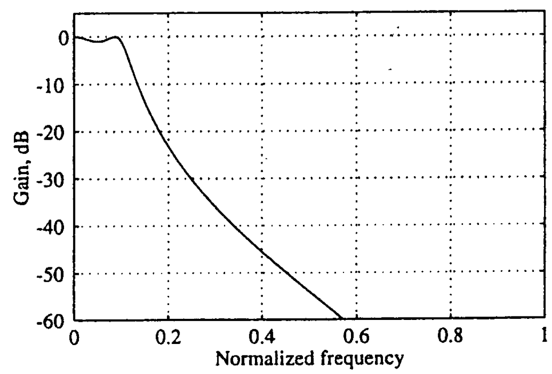
```
Numerator polynomial
   1.0e-03 *
```

```
Columns 1 through 4
 0.12517957    0.5007183     0.751077     0.5007182767

Column 5
 0.1251796
```

Denominator polynomial

```
Columns 1 through 4
 1.0      -3.583348     4.89999    -3.025687

Column 5
 0.71116642
```

and the gain response of the above filter is as indicated below:



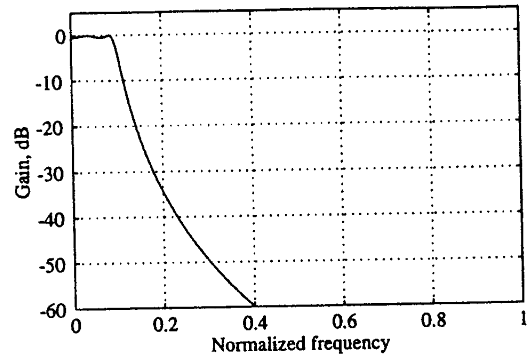
M7.7 The modified program is as given below:

```
format long
Wp = input('Passband edge = ');
Ws = input('Stopband edge = ');
Rp = input('Maximum passband deivation = ');
Rs = input('Minimum stopband attenuation = ');
[N, Wn] = cheb1ord(Wp, Ws, Rp, Rs, 's');
[num,den] = cheby1(N, Rp, Wn, 's');
FT = input('Type in sampling frequency = ');
[b,a] = impinvar(num,den,FT);
disp('Numerator polynomial');disp(real(b));
disp('Denominator polynomial');disp(real(a));
[h,omega] = freqz(b,a,512);
mag = 20*log10(abs(h));
plot(omega/pi,mag); axis([0 1 -60 5]); grid
xlabel('Normalized frequency');ylabel('Gain, dB')
```

The data generated by running this program for the filter specifications: Wp = 0.0907π = 0.28494245368, Ws = 0.453515π = 1.4247593, Rp = 0.5 dB and Rs = 50 dB are as follows:

```
Numerator coefficients
 0.0        0.00035928       0.0013133667     0.00030296

Denominator coefficients
  1.0       -3.58330565      4.8995585 -3.0250868
     0.71092666
```

and the gain response of the above filter is as indicated below:



M7.8 Given specifications: $\omega_p = 0.0907\pi$, $\omega_s = 0.453515\pi$,

$$20\log_{10}\left|G(e^{j0.0907\pi})\right| \ge -0.5, \text{ and } 20\log_{10}\left|G(e^{j0.453515\pi})\right| \le -50.$$

(a) Impulse Invariance Method: Let T = 1. Assume no aliasing. Then the specifications of $H_a(s)$ is same as that of G(z), i.e.

$\Omega_p = 0.0907\pi$, $\Omega_s = 0.453515\pi$, $20\log_{10}\left|H_a(j0.0907\pi)\right| \ge -0.5$, and

$20\log_{10}\left|H_a(j0.453515\pi)\right| \le -50$. Now,

$20\log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right) = -0.5$ which yields $\varepsilon^2 = 0.12202$. Similarly, $20\log_{10}\left(\frac{1}{A}\right) = -50$ which yields $A^2 = 100{,}000$. Therefore, the inverse discrimination ratio is given by

$\frac{1}{k_1} = \sqrt{\frac{A^2-1}{\varepsilon^2}} = \sqrt{\frac{99999}{0.12202}} = 905.2787$, and the inverse transition ratio is given by

$\frac{1}{k} = \frac{0.453515\pi}{0.0907\pi} = 5$. Hence, from Eq. (5.49), the order of the lowpass elliptic filter is given by

$N \cong \frac{2\log_{10}(4/k_1)}{\log_{10}(1/\rho)} = 2.7447$. We choose the filter order as N = 3. Using `ellipap` for N = 3, Rp = 0.5 and Rs = 50 we arrive at the transfer function of a normalized 3-rd order elliptic lowpass filter given by

$$H_{an}(s) = \frac{0.6414285}{s+0.6414285} + \frac{-0.6049954\,s}{s^2+0.6049954\,s+1.144384}.$$

We denormalize $H_{an}(s)$ to move the passband edge from $\Omega_p = 1$ to $\Omega_p = 0.0907\pi = 0.284942$

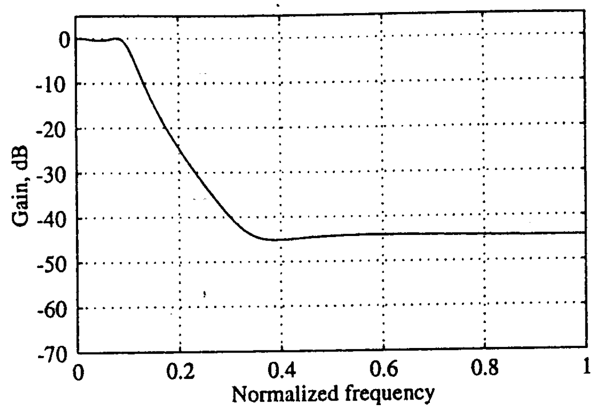
leading to $H_a(s) = H_{an}\left(\frac{s}{0.2894942}\right) = \frac{0.18276992}{s+0.18276992} + \frac{-0.1723886\,s}{s^2+0.1723886\,s+0.0929147}$

$$= \frac{0.18276992}{s+0.18276992} + \frac{-0.1723886\,s}{(s+0.0861943)^2+(0.2923787)^2}.$$

Applying the impulse invariance transformation we get

$$G(z) = \frac{0.18276992\,z}{z-0.83295978} + \frac{-0.1723886\,z^2+0.1648786\,z}{z^2-1.75696\,z+0.84165}.$$

A plot of the gain response of the above transfer function is given below. Observe the effect of aliasing in the gain response. To minimize the aliasing effect, one should choose N = 4.



(b) Bilinear Transformation Method: Let T = 2. From Eq. (7.52), the bandedges of the analog prototype are $\Omega_p = 0.143443$, and $\Omega_s = 0.863675$. The inverse transition ratio is now $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 6.02103$. The inverse discrimination ratio remains the same as in Part (a). Hence, from Eq. (5.49), the order of the lowpass elliptic filter is given by $N \cong \frac{2\log_{10}(4/k_1)}{\log_{10}(1/\rho)} = 2.5813$.

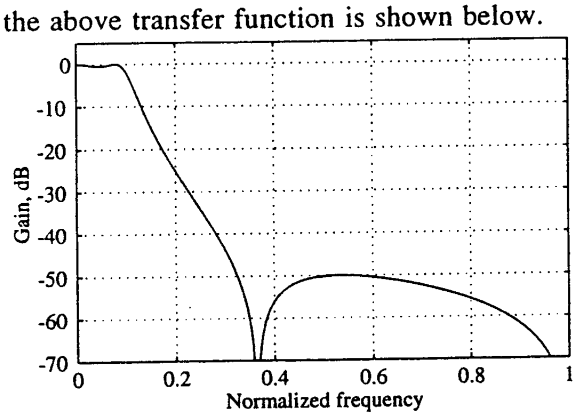
We choose N = 3. Then, $H_{an}(s) = \frac{0.036433s^2 + 00.73404045}{s^3 + 1.246424s^2 + 1.532445s + 0.73404045}$. We denormalize $H_{an}(s)$ to move the passband edge from $\Omega_p = 1$ to $\Omega_p = 0.143443$ leading to

$$H_a(s) = H_{an}\left(\frac{s}{0.143443}\right) = \frac{0.005226s^2 + 0.0021665}{s^3 + 0.17879s^2 + 0.03153143s + 0.0021665}.$$

Applying bilinear transformation to the above with T = 2 we get

$$G(z) = \frac{0.006097 - 2.5903415z^{-1} + 2.5903415z^{-2} - 0.006097z^{-3}}{1 - 2.59034z^{-1} + 2.306147z^{-2} - 0.701511z^{-3}}$$

The gain response of the above transfer function is shown below.



M7.9 The modified program is as given below:

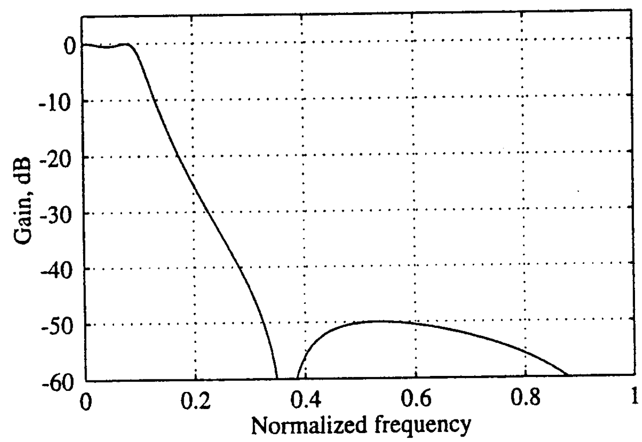
```
Wp = input('Passband edge = ');
Ws = input('Stopband edge = ');
Rp = input('Maximum passband deivation = ');
Rs = input('Minimum stopband attenuation = ');
[N, Wn] = ellipord(Wp, Ws, Rp, Rs);
```

```
[b,a] = ellip(N,Rp,Rs,Wn);
disp('Numerator polynomial'); disp(b);
disp('Denominator polynomial'); disp(a);
w = 0:pi/255:pi;
h = freqz(b,a,w);
gain = 20*log10(abs(h));
plot (w/pi,gain);axis([0 1 -60 5]);grid
xlabel('Normalized frequency'); ylabel('Gain, dB')
```

The data generated by running this program for the filter specifications: Wp = 0.0907, Ws = 0.453515, Rp = 0.5 dB and Rs = 50 dB are as follows:

```
Numerator polynomial
    0.006097    0.001050255    0.001050255    0.006097

Denominator polynomial
    1.0   -2.59034    2.3061467   -0.701511
```

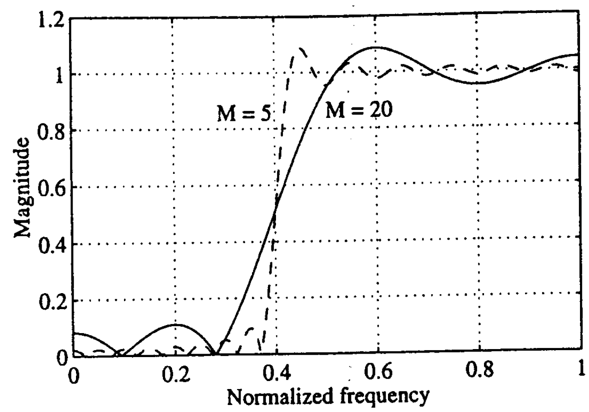
and the gain response of the above filter is as indicated below:



M7.10 The impulse response coefficients of the truncated FIR highpass filter with cutoff frequency at 0.4π can be generated using the following MATLAB statements:

```
n = -M:M;
num = - 0.4*sinc(0.4*n);
num(M+1) = 0.6;
```

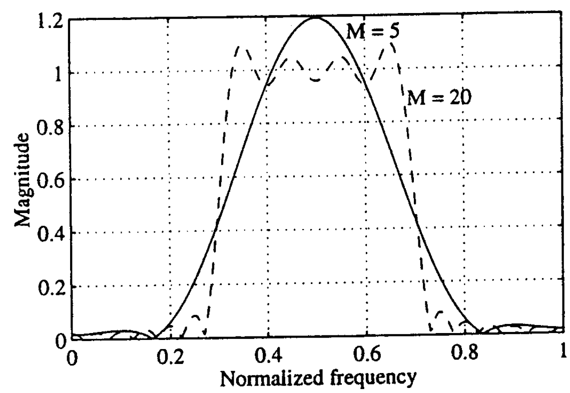
The magnitude responses of the truncated FIR highpass filter for two values of M are shown below:



M7.11 The impulse response coefficients of the truncated FIR bandpass filter with cutoff frequencies at 0.7π and 0.3π can be generated using the following MATLAB statements:

```
n = -M:M;
num = 0.7*sinc(0.7*n) - 0.3*sinc(0.3*n);
```

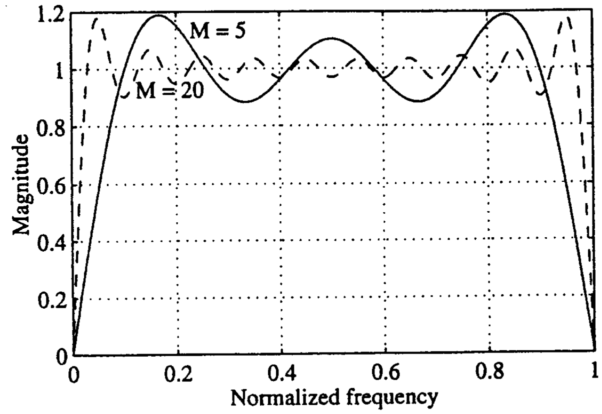
The magnitude responses of the truncated FIR bandpass filter for two values of M are shown below:



M7.12 The impulse response coefficients of the truncated Hilbert transformer can be generated using the following MATLAB statements:

```
n = 1:M;
c = 2*sin(pi*n/2).*sin(pi*n/2);b = c./(pi*n);
num = [-fliplr(b) 0 b];
```

The magnitude responses of the truncated Hilbert transformer for two values of M are shown below:



M7.13 The desired function $D(x) = 1.1x^3 - 9x^2 + 1$ is defined for $-1 \le x \le 6$. We wish to approximate D(x) by a quadratic function $a_2 x^2 + a_1 x + a_0$ by minimizing the peak value of the absolute error $|D(x) - a_2 x^2 - a_1 x - a_0|$. Since there are four unknownsm a_0, a_1, a_2 and ε, we need four extremal points on x in the range $-1 \le x \le 6$, which we arbitrarily choose as $x_1 = -1$, $x_2 = 2$, $x_3 = 4$ and $x_4 = 6$. We then solve the four linear equations

$a_0 + a_1 x_\ell + a_2 x_\ell^2 - (-1)^\ell \varepsilon = D(x_\ell)$, $\ell = 1,2,3,4$, which lead to

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 4 & 16 & 1 \\ 1 & 6 & 36 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} -9.1 \\ -26.2 \\ -72.6 \\ -85.4 \end{bmatrix}.$$

Its solution yields $a_0 = -10.88$, $a_1 = -11.44$, $a_2 = -0.42$, and $\varepsilon = -9.24$. Figure (a) below shows the plot of the error $\mathcal{E}_1(x) = 1.1x^3 - 8.58x^2 + 11.44x + 11.88$ along with the values of the error at the chosen extremal points.

The next set of extremal points are those points where $\mathcal{E}_1(x)$ assumes its maximum absolute values. These extremal points are given by $x_1 = -1$, $x_2 = 0.79$, $x_3 = 4.41$ and $x_4 = 6$. The new values of the unknowns are obtained by solving

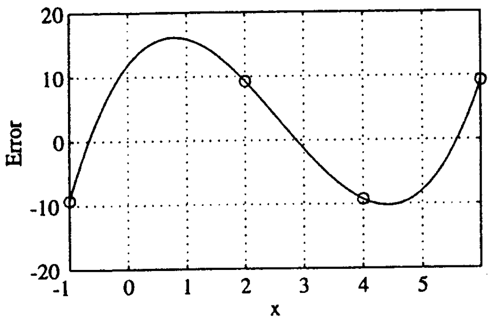
$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0.79 & 0.6241 & -1 \\ 1 & 4.41 & 19.4481 & 1 \\ 1 & 6 & 36 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} -9.1 \\ -4.0746 \\ -79.6902 \\ -85.4 \end{bmatrix}$$

whose solution yields $a_0 = -7.0706$, $a_1 = -10.464$, $a_2 = -0.7578$, and $\varepsilon = -11.7355$. Figure (b) below shows the plot of the error $\mathcal{E}_2(x) = 1.1x^3 - 8.2422x^2 + 10.464x + 8.0706$ along with the values of the error at the chosen extremal points.

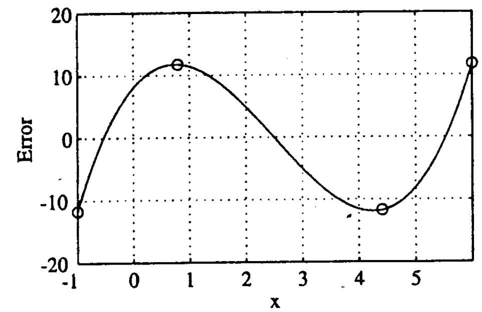
The third set of extremal points are those points where $\mathcal{E}_2(x)$ assumes its maximum absolute values. These extremal points are given by $x_1 = -1$, $x_2 = 0.75$, $x_3 = 4.25$ and $x_4 = 6$. The new values of the unknowns are obtained by solving

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0.75 & 0.5625 & -1 \\ 1 & 4.25 & 18.0625 & 1 \\ 1 & 6 & 36 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} -9.1 \\ -3.5984 \\ -77.1203 \\ -85.4 \end{bmatrix}$$

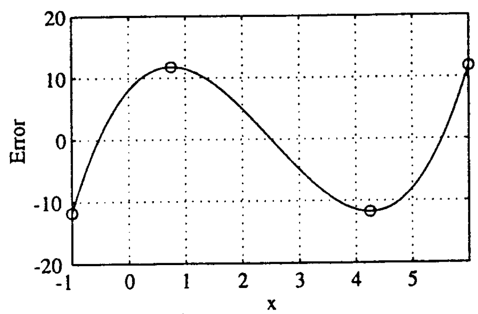
which yields $a_0 = -7.0781$, $a_1 = -10.5187$, $a_2 = -0.75$, and $\varepsilon = -11.7906$. Figure (c) below shows the plot of the error $\mathcal{E}_3(x) = 1.1x^3 - 8.25x^2 + 10.5187x + 8.0781$ along with the values of the error at the chosen extremal points. This time the algorithm has converged as ε is also the maximum value of the absolute error.



(a)



(b)



(c)

M7.14 From Eqs. (7.7) and (7.8), the normalized bandedges are given by $\omega_p = 0.2\pi$ **and** $\omega_s = 0.4\pi$. Therefore, $\omega_c = 0.3\pi$ **and** $\Delta\omega = 0.2\pi$.

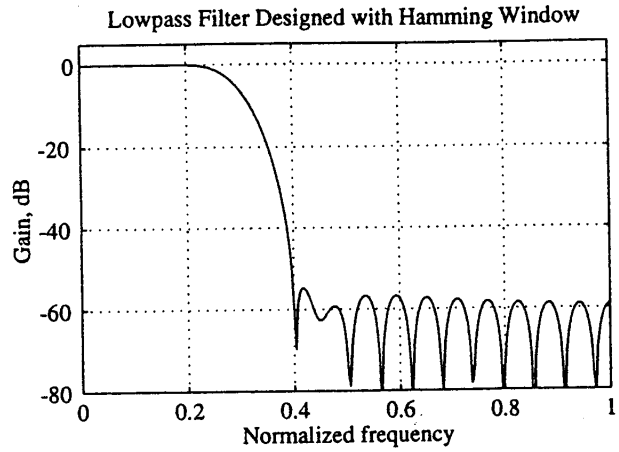
(a) Using Eq. (7.105) and Table 7.3, the estimated length of the FIR filter for designing using the Hamming window is given by N = 2M + 1 where M = 3.32π/ Δω = 16.6. We therefore choose M = 17 or equivalently, N = 35. The MATLAB program used to generate the windowed filter coefficients are

```
n = -M:M;
num = 0.4*sinc(0.4*n);
c = hamming(2*M+1);
b = num.*c';d = b/sum(b);
```

The first 18 filter coefficients generated are given by

h[0] = – 0.00046211, h[1] = 0.00102536, h[2] = 0.002352888, h[3] = 0.00198659,
h[4] = – 0.00151117, h[5] = – 0.0066184, h[6] = – 0.00782857, h[7] = 0,
h[8] = 0.014212866, h[9] = 0.022003622, h[10] = 0.00934131, h[11] = – 0.02319375,
h[12] = – 0.0519384, h[13] = – 0.0410902, h[14] = 0.03047765, h[15] = 0.146418535,
h[16] = 0.25507435, h[17] = 0.299498888.

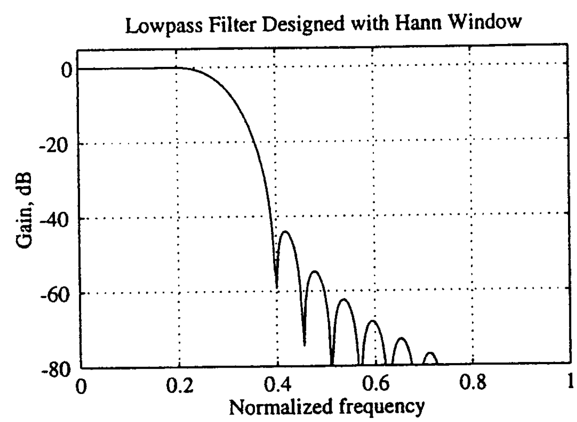
The remaining filter coefficients are obtained using the symmetry condition h[n] = h[35 − n]. The gain response of the filter is indicated below:



(b) For designing the FIR filter using the Hann window, we obtain freom Eq. (7.105) and Table 7.3, $M = 3.11\pi/\Delta\omega = 15.55$. We choose M = 16, and hence N = 33. The first 17 filter coefficients generated are given by

h[0] = 0.00009963, h[1] = 0.00071704, h[2] = 0.001001623, h[3] = − 0.00098813,
h[4] = − 0.00501611, h[5] = − 0.0064928, h[6] = 0, h[7] = 0.0129964375,
h[8] = 0.02068223, h[9] = 0.0089555, h[10] = − 0.0225584, h[11] = − 0.05105241,
h[12] = − 0.0407017, h[13] = 0.0303554, h[14] = 0.14636637, h[15] = 0.255520755,
h[16] = 0.30022913.

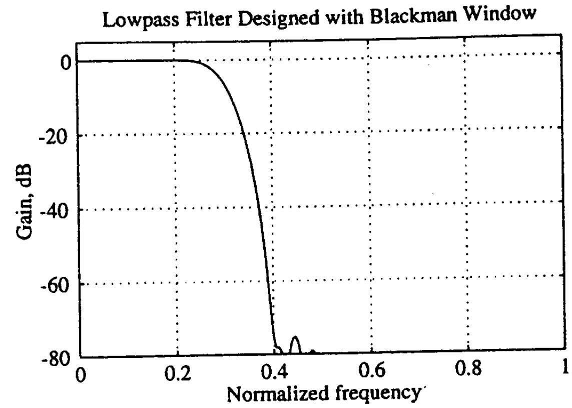
The remaining filter coefficients are obtained using the symmetry condition h[n] = h[33 − n]. The gain response of the above FIR filter is given below:



(c) For designing the FIR filter using the Blackman window, we obtain freom Eq. (7.105) and Table 7.3, $M = 5.56\pi/\Delta\omega = 27.8$. We choose M = 28, and hence N = 57. The first 29 filter coefficients generated are given by

h[0] = 0, h[1] = 0.000004146, h[2] = − 0.0000332, h[3] = − 0.000135,
h[4] = − 0.0001512, h[5] = 0.000134064, h[6] = 0.00064516, h[7] = 0.0008148,
h[8] = 0, h[9] = − 0.00161654, h[10] = − 0.0025762, h[11] = 0.00111272,
h[12] = 0.0027674, h[13] = 0.00606946, h[14] = 0.0045437, h[15] = − 0.00301145,
h[20] = 0.02701594, h[21] = 0.01086963, h[22] = − 0.025841, h[23] = − 0.0558985,
h[24] = − 0.0430485, h[25] = 0.0312948, h[26] = 0.1482652, h[27] = 0.25618604,
h[28] = 0.29999305.

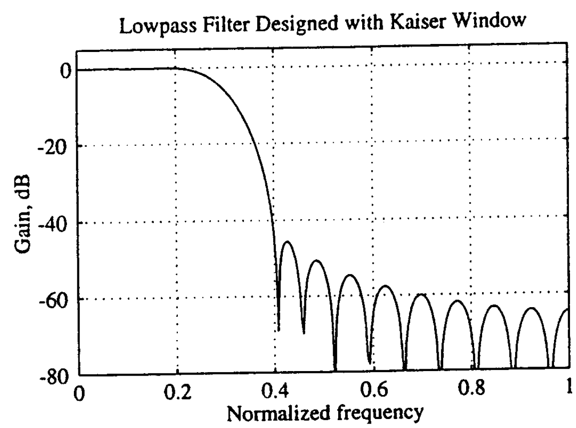
The remaining filter coefficients are obtained using the symmetry condition h[n] = h[57 – n]. The gain response of the above FIR filter is given below:



M7.15 Substituting $\alpha_s = 45$ in Eq. (7.112) we obtain $\beta = 3.975433$. Next, substituting the value of $\alpha_s = 45$ and $N \approx 26.8$. We choose the next higher off integer value of 27 as the filter length N, and hence M = 13. The first 14 filter coefficients are given by

h[0] = – 0.0006856, h[1] = – 0.0038303, h[2] = – 0.0052589, h[3] = 0,
h[4] = 0.011384778, h[5] = 0.01865897, h[6] = 0.00828507, h[7] = – 0.0213207,
h[8] = – 0.04912314, h[9] = – 0.0397382, h[10] = 0.0299733, h[11] = 0.1456924,
h[12] = 0.25557446, h[13] = 0.3007754.

The remaining filter coefficients are obtained using the symmetry condition h[n] = h[27 – n]. The gain response of the above FIR filter is given below:



M7.16
```
N = 39;
for k = 1:N+1
    w = 2*pi*(k-1)/40;
    if(w >= 0.6*pi & w <= 1.4*pi) H(k)= 1;
    else H(k) = 0;
    end
    if(w <= pi) phase(k) = i*exp(-i*w*N/2);
    else phase(k) = -i*exp(i*(2*pi-w)*N/2);
    end
end
H = H.*phase;
```

```
f = ifft(H);
[FF,w] = freqz(f,1,512);
k = 0:N;
subplot(211)
stem(k,real(f));
xlabel('Time index n'); ylabel('Amplitude')
subplot(212)
plot(w/pi,20*log10(abs(FF)));
xlabel('Normalized Frequency'); ylabel('Magnitude, dB');
axis([0 1 -50 5]); grid on
```

M7.17

```
% Length = 45 and bandpass hence Type 3 filter
N = 44; L = N+1;
for k = 1:L
    w = 2*pi*(k-1)/L;
    if (w >= 0.3*pi & w <= 0.5*pi) H(k) = i*exp(-i*w*N/2);
    elseif (w >= 1.5*pi & w <= 1.7*pi) H(k) = -i*exp(i*(2*pi-
w)*N/2);
    else H(k) = 0;
    end
end
f = ifft(H);
[FF,w] = freqz(f,1,512);
k = 0:N;
subplot(211);
stem(k,real(f));
xlabel('Time index n'); ylabel('Amplitude');
subplot(212);
plot(w/pi,20*log10(abs(FF)));
ylabel('Magnitude, dB'); xlabel('Normalized Frequency');
axis([0 1 -50 5]); grid;
```

M7.18

```
N=36; L=N+1;
for k = 0:36
     if(k <= 5 | k>=32) H(k+1) = exp(-i*2*pi*k/L*18);
     elseif (k == 6 | k == 31) H(k+1) = 0.5*exp(-
i*2*pi*k/L*18);
     end
end
f = ifft(H);
[FF,w] = freqz(f,1,512);
k = 0:N;
subplot(211);
stem(k,real(f));
xlabel('Time index n'); ylabel('Amplitude');
subplot(212);
plot(w/pi,20*log10(abs(FF)));
xlabel('Normalized Frequency'); ylabel('Magnitude, dB');
axis([0 1 -70 5]);grid;
```

M7.19

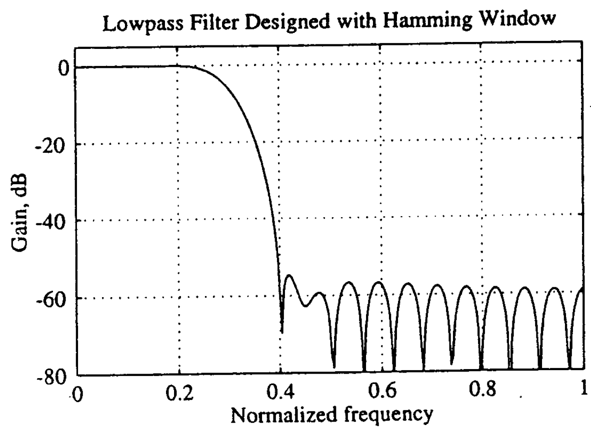
```
N = 36; L=N+1;
for k = 0:36
    if(k <= 5 | k >= 32) H(k+1) = exp(-i*2*pi*k/L*18);
    elseif (k ==6 | k == 31) H(k+1) = 2/3*exp(-i*2*pi*k/L*18);
    elseif (k ==7 | k == 30) H(k+1) = 1/3*exp(-i*2*pi*k/L*18);
     end
end
f = ifft(H);
```

```
[FF,w] = freqz(f,1,512);
k = 0:N;
subplot(211);
stem(k,real(f));
xlabel('Time index n'); ylabel('Amplitude');
subplot(212);
plot(w/pi,20*log10(abs(FF)));
xlabel('Normalized Frequency'); ylabel('Magnitude, dB');
axis([0 1 -70 5]); grid;
```

M7.20 **(a)** From M7.14(a) above, we note $\omega_c = 0.3\pi$ and M = 17. The first 18 filter coefficients generated using `fir1` are given by

h[0] = – 0.000462, h[1] = 0.00102536, h[2] = 0.002352888, h[3] = 0.001986597,
h[4] = – 0.0015112, h[5] = – 0.0066184, h[6] = – 0.00782857, h[7] = 0,
h[8] = 0.014212866, h[9] = 0.022003623, h[10] = 0.0093413, h[11] = – 0.023193746,
h[12] = – 0.0519384, h[13] = – 0.0410902, h[14] = 0.03047765, h[15] = 0.14641853,
h[16] = 0.255074347, h[17] = 0.299498887.

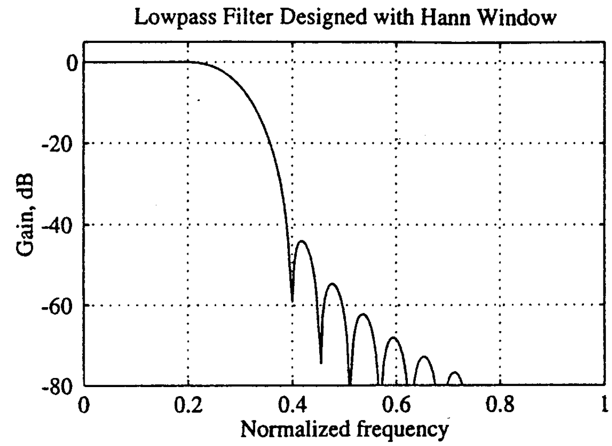
The remaining filter coefficients are obtained using the symmetry condition h[n] = h[35 – n]. The gain response of the filter is indicated below:



(b) From M7.14 (b), M = 16. The first 17 filter coefficients generated using `fir1` are given by

h[0] = 0.00009963, h[1] = 0.00071704, h[2] = 0.00100162, h[3] = – 0.000988135,
h[4] = – 0.0050161, h[5] = – 0.0064928, h[6] = 0, h[7] = 0.0129964375,
h[8] = 0.020682232, h[9] = 0.00895552, h[10] = – 0.0225584, h[11] = – 0.05105241,
h[12] = – 0.0407017, h[13] = 0.0303554, h[14] = 0.14636637, h[15] = 0.25552075,
h[16] = 0.30022913.

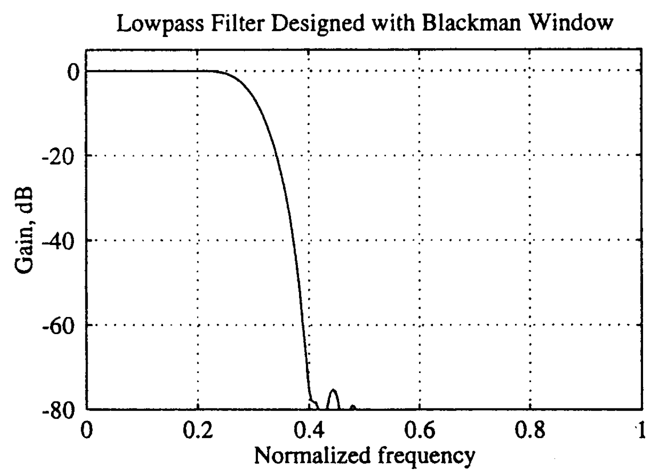
The remaining filter coefficients are obtained using the symmetry condition h[n] = h[33 – n]. The gain response of the above FIR filter is given below:



(c) From M7.14(c) M = 28. The first 29 filter coefficients generated by `fir1` are given by

h[0] = 0, h[1] = 0.000004146, h[2] = – 0.0000332, h[3] = – 0.000135,
h[4] = – 0.0001512, h[5] = 0.000134064, h[6] = 0.00064516, h[7] = 0.0008148,
h[8] = 0, h[9] = – 0.00161654, h[10] = – 0.0025762, h[11] = 0.00111272,
h[12] = 0.0027674, h[13] = 0.00606947, h[14] = 0.0045437, h[15] = – 0.00301126,
h[16] = – 0.0115838, h[17] = – 0.012234, h[18] = 0, h[19] = 0.01863545,
h[20] = 0.02701594, h[21] = 0.01086963, h[22] = – 0.025841, h[23] = – 0.0558985,
h[24] = – 0.0430485, h[25] = 0.0312947, h[26] = 0.1482652, h[27] = 0.25618604,
h[28] = 0.299993047.

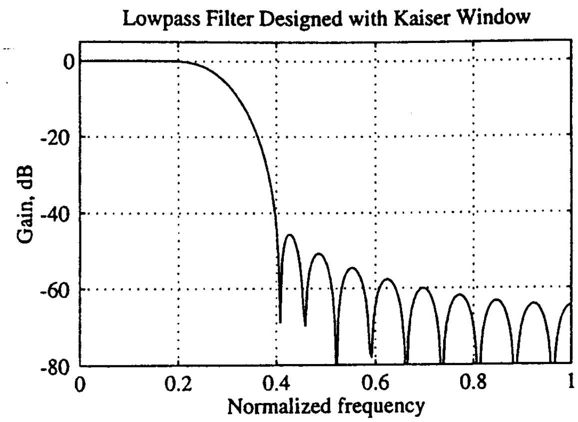
The remaining filter coefficients are obtained using the symmetry condition h[n] = h[57 – n]. The gain response of the above FIR filter is given below:



(d) The first 14 filter coefficients generated using `fir1` are given by

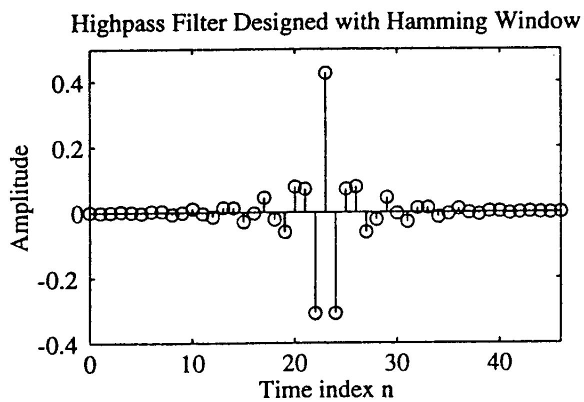
h[0] = – 0.0006856, h[1] = – 0.00383026, h[2] = – 0.0052589, h[3] = 0,
h[4] = 0.011384777, h[5] = 0.018658976, h[6] = 0.0082851, h[7] = – 0.0213207,
h[8] = – 0.04912314, h[9] = – 0.03973819, h[10] = 0.02997333, h[11] = 0.1456924,
h[12] = 0.25557445, h[13] = 0.3007754.

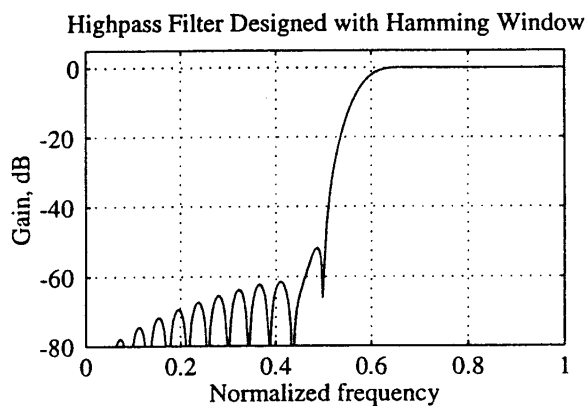
The remaining filter coefficients are obtained using the symmetry condition h[n] = h[27 – n]. The gain response of the above FIR filter is given below:



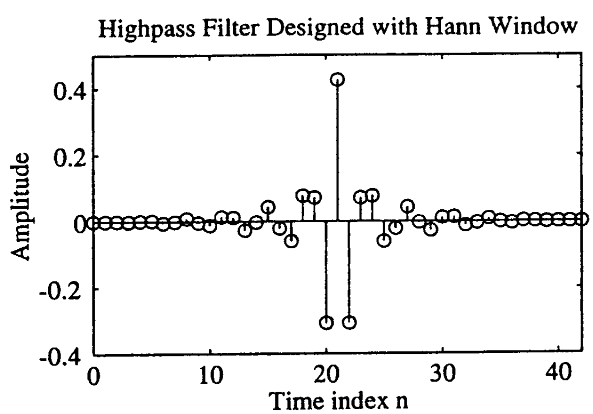
M7.21 From the band edge specifications we obtain $\omega_c = 0.575\pi$ and $\Delta\omega = 0.15\pi$.

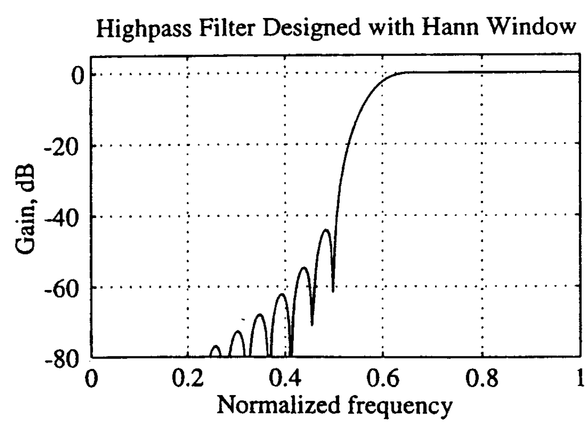
(a) Using Eq. (7.105) and Table 7.3, the estimated length of the FIR filter for designing using the Hamming window is given by N = 2M + 1 where $M = 3.32\pi/\Delta\omega = 22.13$. We therefore choose M = 23 or equivalently, N = 47. The impulse response coefficients and the gain response of the designed FIR filter are shown below:



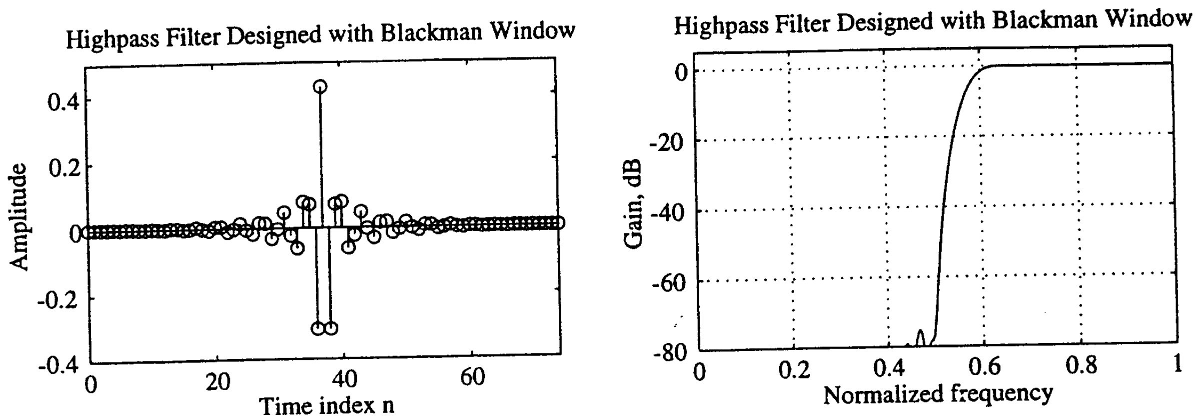


(b) Using Eq. (7.105) and Table 7.3, the estimated length of the FIR filter for designing using the Hann window is given by N = 2M + 1 where $M = 3.11\pi/\Delta\omega = 20.7333$. We therefore choose M = 21 or equivalently, N = 43. The impulse response coefficients and the gain response of the designed FIR filter are shown below:

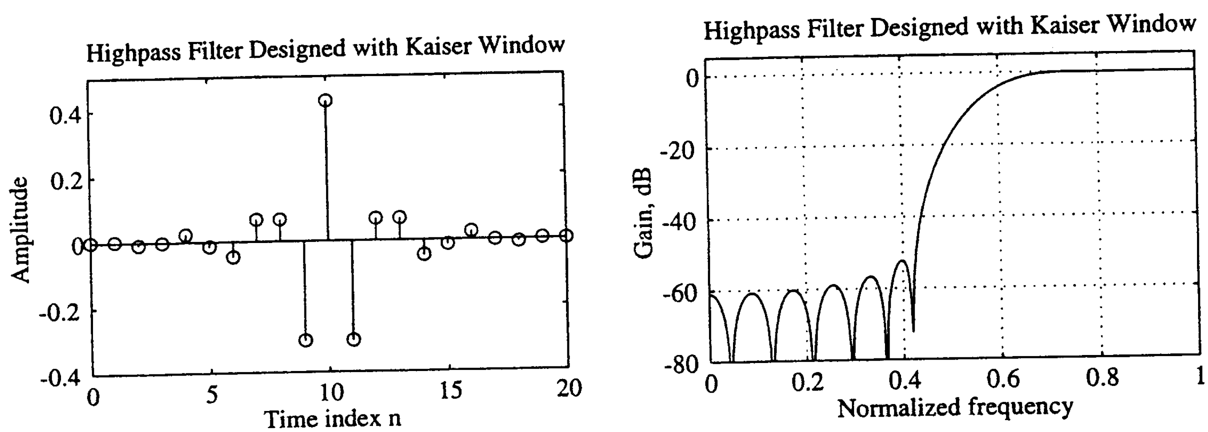




(c) Using Eq. (7.105) and Table 7.3, the estimated length of the FIR filter for designing using the Blackman window is given by N = 2M + 1 where $M = 5.56\pi/\Delta\omega = 37.0667$. We therefore choose M = 37 or equivalently, N = 75. The impulse response coefficients and the gain response of the designed FIR filter are shown below:

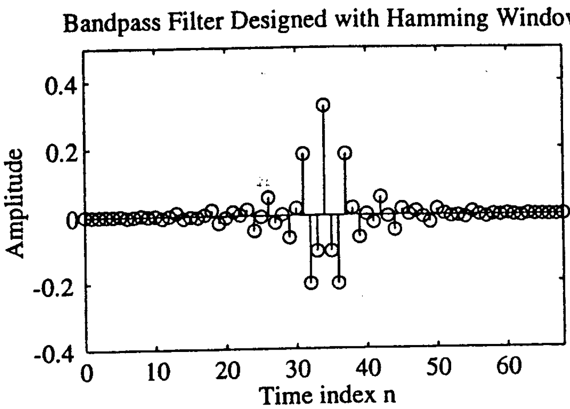


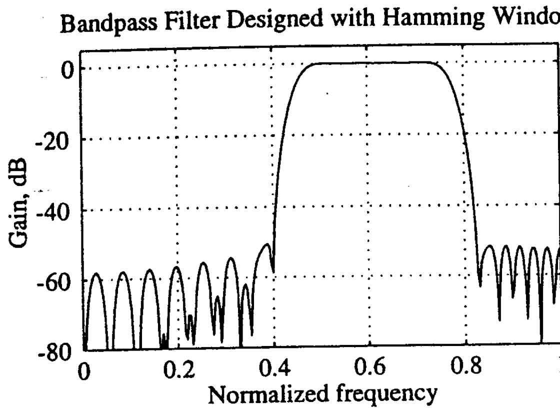
(d) From Eq. (7.112), we get $\beta = 4.5335$ and from Eq. (7.113) we get N = 20.5218. We choose the next higher odd integer 21 for the filter lengt,h and hence M = 10. The impulse response coefficients and the gain response of the designed FIR filter are shown below:



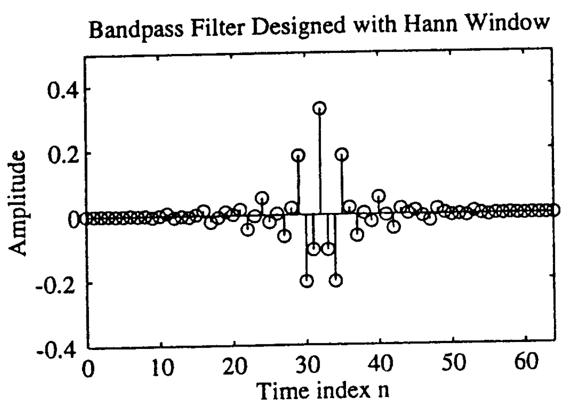
M7.22 From the band edge specifications, $\omega_{p1} = 0.5\pi$, $\omega_{p2} = 0.7\pi$, $\omega_{s1} = 0.4\pi$, and $\omega_{s2} = 0.85\pi$, we get the two cutoff frequencies as $\omega_{c1} = 0.45\pi$, and $\omega_{c2} = 0.775\pi$, and the two transition bandwidths as $\Delta\omega_1 = 0.1\pi$, and $\Delta\omega_2 = 0.15\pi$. From Eq. (7.90) we observe that the FIR bandpass filter coefficients are given by the difference of the filter coefficients of two FIR lowpass filters. Hence, we estimate the filter lengths of the two lowpass filters with cutoff frequencies ω_{c1} and ω_{c2} using Eq. (7.105) and Table 7.3, and use the larger of the two as the filter lengths of both lowpass filters.

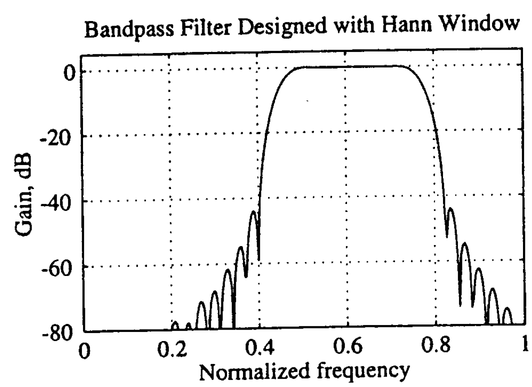
(a) For the bandpass filter designed using the Hamming window we get $M_1 = \frac{3.32}{\omega_{c1}} = \frac{3.32}{0.1} = 33.2$ and $M_2 = \frac{3.32}{\omega_{c2}} = \frac{3.32}{0.15} = 22.1333$. We choose M = 34 and hence, N = 69. The impulse response coefficients and the gain response of the designed FIR filter are shown below:



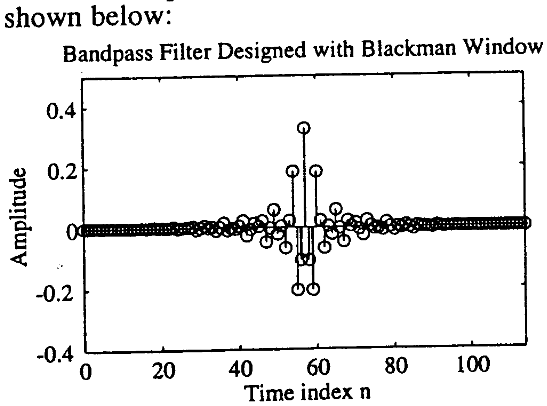


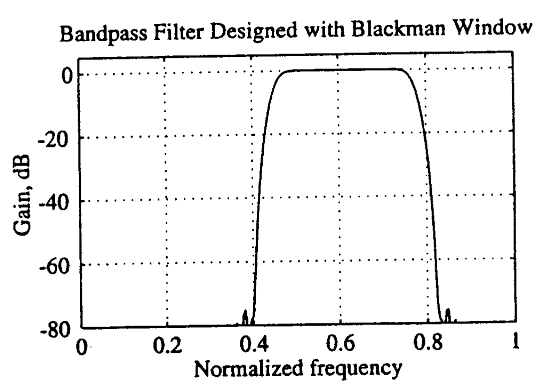
(b) For the bandpass filter designed using the Hann window we get $M_1 = \frac{3.11}{\omega_{c1}} = \frac{3.11}{0.1} = 31.1$ and $M_2 = \frac{3.11}{\omega_{c2}} = \frac{3.11}{0.15} = 20.7333$. We choose M = 32 and hence, N = 65. The impulse response coefficients and the gain response of the designed FIR filter are shown below:



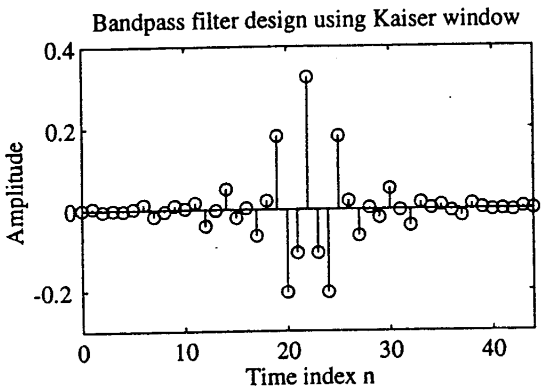


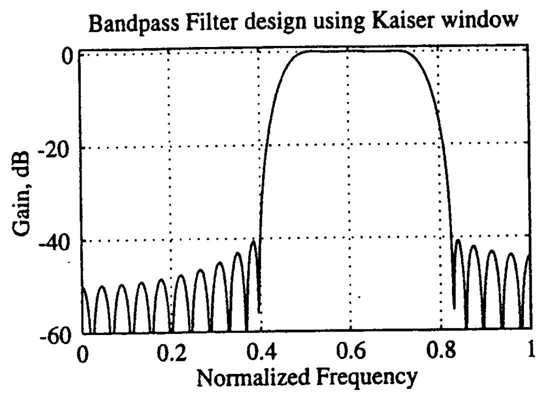
(c) For the bandpass filter designed using the Blackman window we get
$M_1 = \frac{5.56}{\omega_{c1}} = \frac{5.56}{0.1} = 55.6$ and $M_2 = \frac{5.56}{\omega_{c2}} = \frac{5.56}{0.15} = 37.0667$. We choose M = 56 and hence, N = 113. The impulse response coefficients and the gain response of the designed FIR filter are shown below:





(d) Given $\alpha_s = 40, \omega_{s1} = 0.4\pi$, $\omega_{p1} = 0.5\pi$, $\omega_{p2} = 0.7\pi$, and $\omega_{s2} = 0.85\pi$. Using Eq. (7.112) we first compute $\beta = 3.3953$. We compute the filter length using Eq. (7.113) twice, first with $\Delta f = (\omega_{p1} - \omega_{s1})/2\pi = 0.05$ which yields N = 45 and second with $\Delta f = (\omega_{s2} - \omega_{p21})/2\pi = 0.075$ which yields N = 30. We thus choose the larger of these two estimates as the filter length N = 45. The impulse response coefficients and the gain response of the designed FIR filter are shown below:

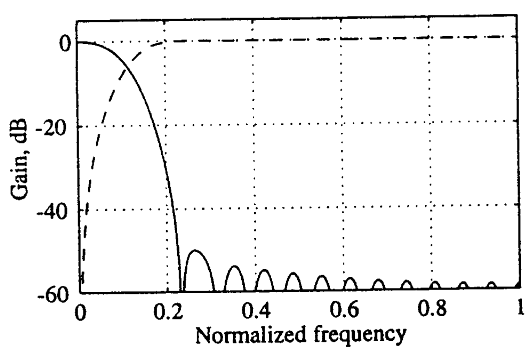




M7.23 From Eq. (7.7), the normalized crossover frequency $\omega_c = \dfrac{2\pi \times 2.5}{44.1} = 0.1134\pi$. The delay-complementary FIR lowpass and highpass filter coefficients of length 31 are then generated using the MATLAB statements:

```
d1  =  fir1(30,0.1134);
d2  =  -d1;d2(16)  =  1  -  d1(16);
```

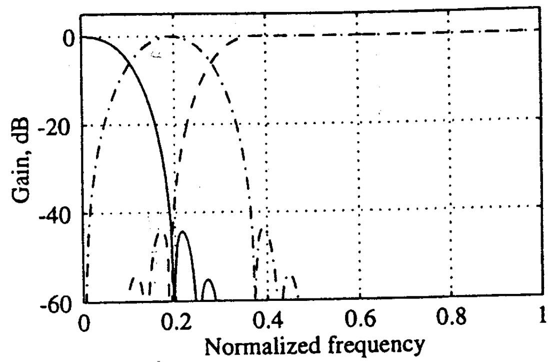
The gain responses of the two filters generated using the above program are as indicated below:



M7.24 The filter coefficients of the FIR lowpass and the highpass, and their delay-complementary FIR bandpass filter are generated using the MATLAB program:

```
c  =  hanning(35);
d1  =  fir1(34,0.1043,c);
d2  =  fir1(34,0.2812,'high',c);
d3  =  -d1-d2;d3(18)  =  1  -  d1(18)  -  d2(18);
```

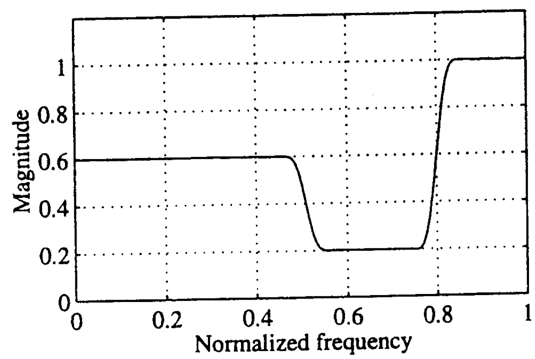
The gain responses of the three FIR filters generated using the above program are as given below:



M7.25 The filter coefficients are generated using the MATLAB program:

```
fpts = [0 0.5 0.52 0.79 0.81 1];
mval = [0.6 0.6 0.2 0.2 1.0 1.0];
b = fir2(90,fpts,mval);
```

The gain response of the multiband FIR filter generated using the above program is as given below:

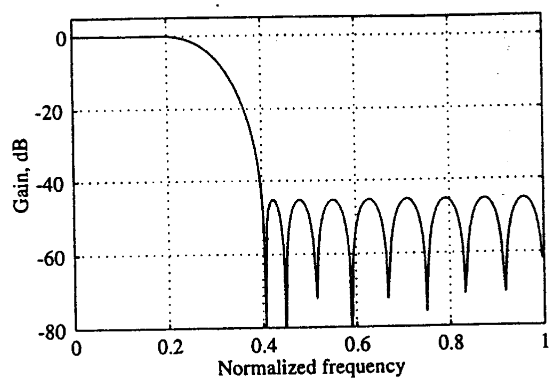


M7.26 From Eq. (7.6) with $\alpha_{max} = 0.1$ we obtain $\delta_p = 0.0057$. Likewise, from Eq. (7.4) with $\alpha_s = 45$ we obtain $\delta_s = 0.0056$. Using these values, and the specified values of the passband and stopband edge frequencies, and the sampling frequency in Program 7.1 we arrive at the filter length N = 22. Next, using Program 7.8 we determine the impulse response coefficients of the FIR lowpass filter and its gain response. The input data used are: filter order = 21, band edge vector = [0 0.2 0.4 1] and the magnitude level vector = [1 1 0 0]. The filter generated did not meet the stopband edge specifications. The specifications were met when the filter order was increased to 23.

The first 12 filter coefficients are given by

h[0] = – 0.00548387, h[1] = – 0.00324835, h[2] = 0.00526428, h[3] = 0.01638097,
h[4] = 0.016687436, h[5] = – 0.00391905, h[6] = – 0.03667825, h[7] = – 0.0510055,
h[8] = – 0.01321462, h[9] = 0.083808425, h[10] = 0.20496879, h[12] 0.289164653.

The remaining 12 filter coefficients are given by the symmetry condition h[n] = h[23 – n]. The gain response of the above filter is given below:

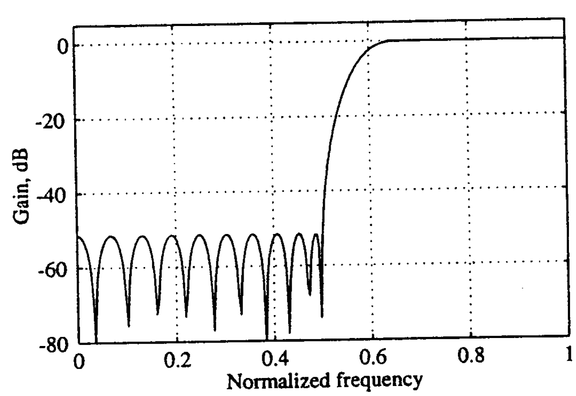


The filter was also designed by using remezord to estimate the filter order and then using `remez` to determine the filter coefficients. To this end the MATLAB program used is given below:

```
Rp = 0.1; Rs = 45; FT = 10;
f = [1  2]; m = [1 0];
dev = [(10^(Rp/20) -1)/(10^(Rp/20) +1) 10^(-Rs/20)];
[N, fo, mo, wo] = remezord(f, m, dev, FT);
b = remez(N,fo,mo,wo);
```

The filter obtained using the above program is of length 23, but its gain response did not meet the specifications. The specifications were met when the filter order was increased to 24.

M7.27 From Eq. (7.6) with $\alpha_{max} = 0.2$ we obtain $\delta_p = 0.01138$. Likewise, from Eq. (7.4) with $\alpha_s = 50$ we obtain $\delta_s = 0.0031623$. Using these values, and the specified values of the passband and stopband edge frequencies, and the sampling frequency in Program 7.1 we arrive at the filter length N = 29. Next, using Program 7.8 we determine the impulse response coefficients of the FIR lowpass filter and its gain response. The input data used are: filter order = 28, band edge vector = [0 0.5 0.65 1] and the magnitude level vector = [0 0 1 1]. The filter generated did not meet the stopband edge specifications. The specifications were met when the filter order was increased to 36.

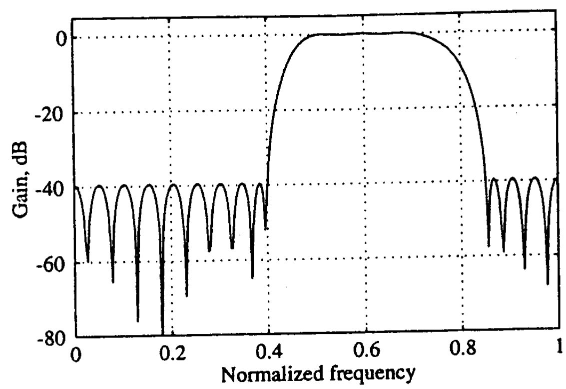


M7.28 The FIR bandpass filter was designed using the following MATLAB program:

```
Rp = 0.15; Rs = 40; FT = 2;
f = [0.4 0.5  0.7 0.85]; m = [0 1 0];
dev = [10^(-Rs/20) (10^(Rp/20) -1)/(10^(Rp/20) +1) 10^(-
Rs/20)];
```

```
[N, fo, mo, wo] = remezord(f, m, dev, FT);
b = remez(N,fo,mo,wo);
```

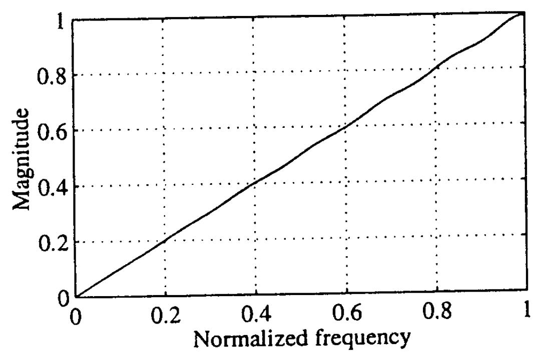
The gain response of the resulting filter is given below:



M7.29 The MATLAB program used to design the 27-th order differentiator is given below.

```
b = remez(27,[0 1],[0 1],'differentiator');
w = 0:pi/255:pi;
h = freqz(b,1,w);
plot(w/pi,abs(h));grid
xlabel('Normalized frequency');ylabel('Magnitude');
```

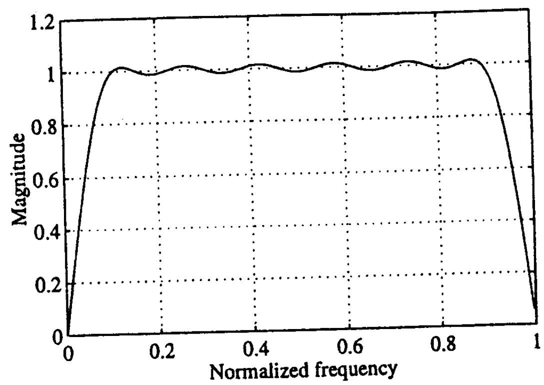
The magnitude response of the differentiator designed is shown below:



M7.30 The MATLAB program used to design the 26-th order Hilbert transformer is given below.

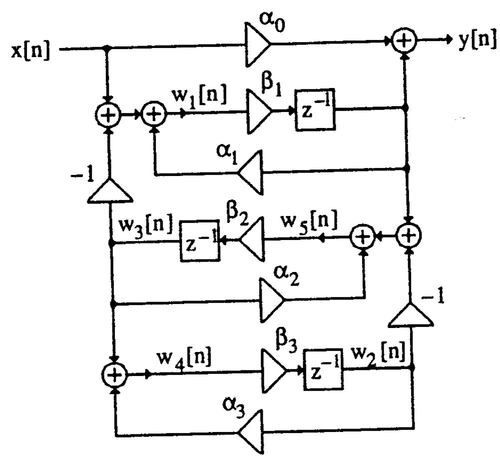
```
f = [0.01 0.08 0.1 0.9 0.92 1];
m = [0 0 1 1 0 0];
wt = [1 60 1];
b = remez(26,f,m,wt,'hilbert');
```

Note that the the passband and the stopbands have been weighted to reduce the ripple in the passband. The magnitude response of the Hilbert transformer designed is shown below:



Chapter 8

8.1



Analysis yields
$$w_1[n] = \alpha_1\beta_1 w_1[n-1] + x[n] - w_3[n],$$
$$w_2[n] = \beta_3 w_4[n-1],$$
$$w_3[n] = \beta_2 w_5[n-1],$$
$$w_4[n] = \alpha_3 w_2[n] + w_3[n],$$
$$w_5[n] = \alpha_2 w_3[n] - w_2[n] + \beta_1 w_1[n-1],$$
$$y[n] = \alpha_0 x[n] + \beta_1 w_1[n-1].$$

In matrix form the above set of equations is given by:

$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_3 & 1 & 0 & 0 & 0 \\ 0 & -1 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ y[n] \end{bmatrix} + \begin{bmatrix} \alpha_1\beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} x[n] \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_0 x[n] \end{bmatrix}$$

Here the **F** matrix is given by

$$\mathbf{F}=\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_3 & 1 & 0 & 0 & 0 \\ 0 & -1 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the **F** matrix contains nonzero entries above the main diagonal, the above set of equations are not computable.

8.2 A computable set of equations of the structure of Figure P8.1 is given by

$$\begin{aligned} w_2[n] &= \beta_3 \, w_4[n-1], \\ w_3[n] &= \beta_2 \, w_5[n-1], \\ w_1[n] &= \alpha_1\beta_1 \, w_1[n-1] - w_3[n] + x[n], \\ w_4[n] &= \alpha_3 w_2[n] + w_3[n], \\ w_5[n] &= \alpha_2 w_3[n] - w_2[n] + \beta_1 \, w_1[n-1], \\ y[n] &= \alpha_0 x[n] + \beta_1 \, w_1[n-1]. \end{aligned}$$

In matrix form the above set of equations is given by:

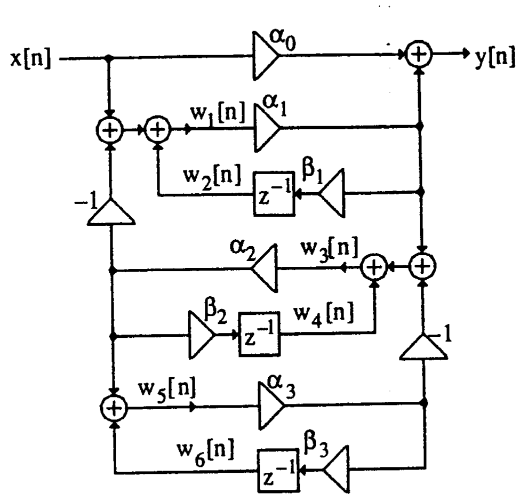
$$\begin{bmatrix} w_2[n] \\ w_3[n] \\ w_1[n] \\ w_4[n] \\ w_5[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \alpha_3 & 1 & 0 & 0 & 0 & 0 \\ -1 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_2[n] \\ w_3[n] \\ w_1[n] \\ w_4[n] \\ w_5[n] \\ y[n] \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & 0 \\ 0 & 0 & \alpha_1\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x[n] \\ 0 \\ 0 \\ \alpha_0 x[n] \end{bmatrix}$$

Here the **F** matrix is given by

$$\mathbf{F}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \alpha_3 & 1 & 0 & 0 & 0 & 0 \\ -1 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the **F** matrix does not contain nonzero entries above the main diagonal, the new set of equations are computable.

8.3



Analysis yields $w_1[n] = x[n] - \alpha_2\, w_3[n] + w_2[n],$

$w_2[n] = \alpha_1\, \beta_1 w_1[n-1],$

$w_3[n] = \alpha_1\, w_1[n] - \alpha_3\, w_5[n] + w_4[n],$

$w_4[n] = \alpha_2\, \beta_2 w_3[n-1],$

$w_5[n] = \alpha_2\, w_3[n] + w_6[n],$

$w_6[n] = \alpha_3\, \beta_3 w_5[n-1],$

$y[n] = \alpha_0 x[n] + \alpha_1 w_1[n].$

In matrix form

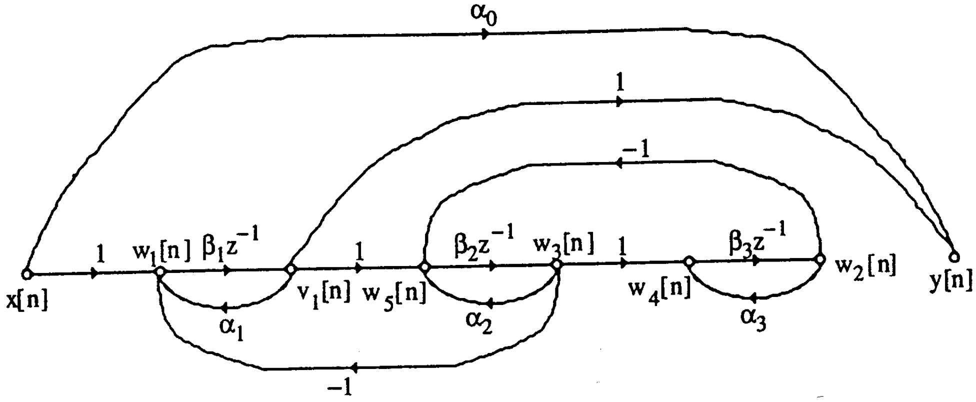
$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ w_6[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 1 & -\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ w_6[n] \\ y[n] \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2\beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3\beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ w_6[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} x[n] \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_0 x[n] \end{bmatrix}$$

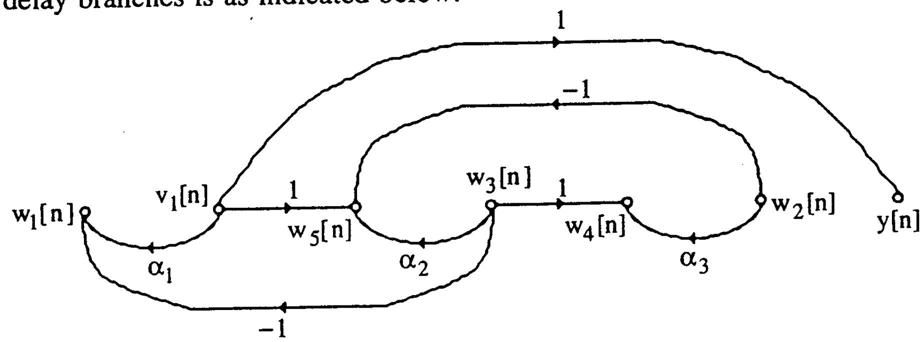
Here the **F** matrix is given by $\mathbf{F} = \begin{bmatrix} 0 & 1 & -\alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 1 & -\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Since the **F** matrix contains nonzero entries above the main diagonal, the above set of equations are not computable.

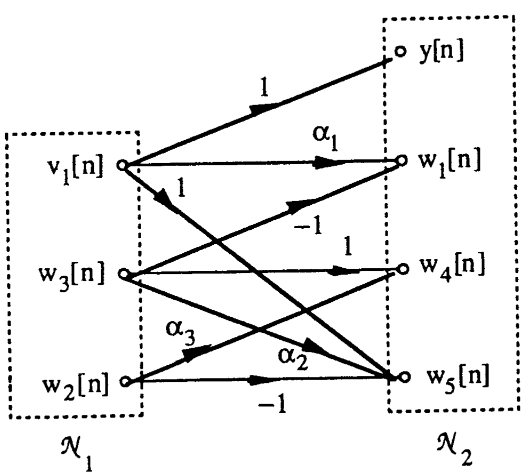
8.4 The signal-flow graph representation of the structure of Figure P8.1 is shown below:



The reduced signal-flow graph obtained by removing the branches going out of the input node and the delay branches is as indicated below:



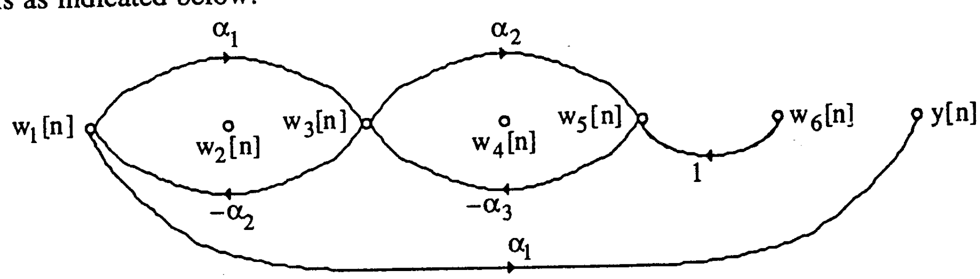
From the above signal-flow graph we arrive at its precedence graph shown below:



In the above precedence graph, the set $\mathcal{N}_1$ contains nodes with only outgoing branches and the final set $\mathcal{N}_2$ contains nodes with only incoming branches. As a result, the structure of Figure P8.1 has no delay-free loops. A valid computational algorithm by computing the node variables in set $\mathcal{N}_1$ first in any order followed by computing the node variables in set $\mathcal{N}_2$ in any order. For example, one valid computational algorithm is given by

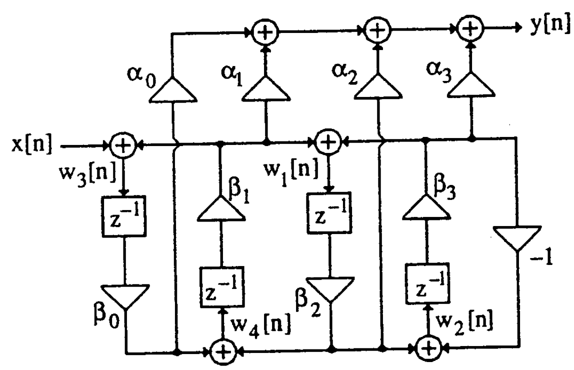
$v_1[n] = \beta_1 w_1[n-1]$,
$w_3[n] = \beta_2 w_5[n-1]$,
$w_2[n] = \beta_3 w_4[n-1]$,
$w_1[n] = \alpha_1 v_1[n] - w_3[n] + x[n]$,
$w_4[n] = \alpha_3 w_2[n] + w_3[n]$,
$w_5[n] = \alpha_2 w_3[n] - w_2[n] + \beta_1 v_1[n]$,
$y[n] = \alpha_0 x[n] + v_1[n]$.

8.5 The reduced signal-flow graph obtained by removing the branches going out of the input node and the delay branches from the signal-flow graph representation of the structure of Figure P8.2 is as indicated below:



The only node with outgoing branch is $w_6[n]$ and hence it is the only member of the set $\mathcal{N}_1$. Since it is not possible to find a set of nodes $\mathcal{N}_2$ with incoming branches from $\mathcal{N}_1$ and all other branches being outgoing, the structure of Figure P8.2 has delay-free loops and is therefore not realizable.

8.6 (a)



Analysis yields $w_1[n] = \beta_1 w_4[n-1] + \beta_3 w_2[n-1]$,
$w_2[n] = \beta_2 w_1[n-1] - \beta_3 w_2[n-1]$,
$w_3[n] = x[n] + \beta_1 w_4[n-1]$,
$w_4[n] = \beta_0 w_3[n-1] + \beta_2 w_1[n-1]$,
$y[n] = \alpha_0 \beta_0 w_3[n-1] + \alpha_1 \beta_1 w_4[n-1] + \alpha_2 \beta_2 w_1[n-1] + \alpha_3 \beta_3 w_2[n-1]$.

In matrix form

$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ y[n] \end{bmatrix} + \begin{bmatrix} 0 & \beta_3 & 0 & \beta_1 & 0 \\ \beta_2 & -\beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1 & 0 \\ \beta_2 & 0 & \beta_0 & 0 & 0 \\ \alpha_2\beta_2 & \alpha_3\beta_3 & \alpha_0\beta_0 & \alpha_1\beta_1 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x[n] \\ 0 \\ 0 \end{bmatrix}$$

Since in the **F** matrix there are non-zero entries above the main diagonal, the above set of equatioms are computable. Note also that the **F** matrix being a null matrix, the variables can be computed in any order.

(b) The reduced signal-flow graph obtained by removing the branches going out of the input node and the delay branches from the signal-flow graph representation of the structure of Figure P8.3 will have no branches. As a result, the structure of Figure P8.3 is realizable and the variables can be computed in any order.

8.7 From Eq. (8.15), $\mathbf{p} = \mathbf{H}_1 \begin{bmatrix} 1 \\ \mathbf{d} \end{bmatrix}$, where $\mathbf{p} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix}$, and

$$\mathbf{H}_1 = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix} = \begin{bmatrix} 3.0 & 0 & 0 \\ 1.72 & 3.0 & 0 \\ 4.43 & 1.72 & 3.0 \end{bmatrix}. \text{ Hence, } \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 3.0 & 0 & 0 \\ 1.72 & 3.0 & 0 \\ 4.43 & 1.72 & 3.0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.1 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 3.0 \\ 2.02 \\ 4.002 \end{bmatrix}.$$

Hence $P(z) = 3.0 + 2.02\,z^{-1} + 4.002\,z^{-2}$.

8.8 Here $\mathbf{H}_1 = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ -1 & 5 & 0 \\ -8 & -1 & 5 \end{bmatrix}$, $\mathbf{H}_2 = \begin{bmatrix} h[2] & h[1] \\ h[3] & h[2] \end{bmatrix} = \begin{bmatrix} -8 & -1 \\ 19 & -8 \end{bmatrix}$, $\mathbf{h} = \begin{bmatrix} h[3] \\ h[4] \end{bmatrix} = \begin{bmatrix} 19 \\ -14 \end{bmatrix}$.

Hence, $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -\begin{bmatrix} -8 & -1 \\ 19 & -8 \end{bmatrix}^{-1} \begin{bmatrix} 19 \\ -14 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \mathbf{H}_1 \begin{bmatrix} 1 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ -1 & 5 & 0 \\ -8 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 5 \end{bmatrix}$.

Thus, $H(z) = \dfrac{5 + 9z^{-1} + 5z^{-2}}{1 + 2z^{-1} + 3z^{-2}}$.

8.9 Here $\mathbf{H}_1 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ -3 & -2 & 3 & 0 \\ 1 & -3 & -2 & 3 \end{bmatrix}$, $\mathbf{H}_2 = \begin{bmatrix} 1 & -3 & -2 \\ 15 & 1 & -3 \\ -21 & 15 & 1 \end{bmatrix}$, $\mathbf{h} = \begin{bmatrix} 15 \\ -21 \\ -7 \end{bmatrix}$.

Hence, $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = -\begin{bmatrix} 1 & -3 & -2 \\ 15 & 1 & -3 \\ -21 & 15 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ -21 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ -3 & -2 & 3 & 0 \\ 1 & -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \end{bmatrix}$.

8.10 Here $\mathbf{H}_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 \\ 14 & -5 & 2 & 0 \\ 14 & 14 & -5 & 2 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$. Hence, $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 \\ 14 & -5 & 2 & 0 \\ 14 & 14 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 54 \end{bmatrix}$. Thus,

$P(z) = 2 + z^{-1} + 3z^{-2} + 54z^{-3}.$

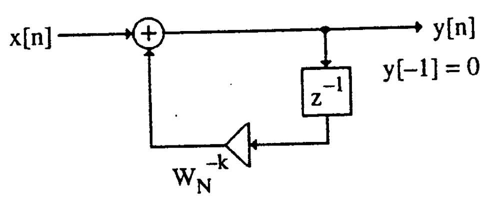
8.11 Here, $\mathbf{H}_1 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ -11 & 5 & 0 & 0 & 0 \\ 15 & -11 & 5 & 0 & 0 \\ -25 & 15 & -11 & 5 & 0 \\ 42 & -25 & 15 & -11 & 5 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 3 \end{bmatrix}$. Hence,

$$\mathbf{p} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ -11 & 5 & 0 & 0 & 0 \\ 15 & -11 & 5 & 0 & 0 \\ -25 & 15 & -11 & 5 & 0 \\ 42 & -25 & 15 & -11 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \\ 4 \end{bmatrix}.$$ Thus, $P(z) = 5 - z^{-1} + 3z^{-2} - 2z^{-3} + 4z^{-4}$.

8.12 The k-th sample of an N-point DFT is given by $X[k] = \sum_{k=0}^{N-1} x[n] W_N^{nk}$. Thus the computation of X[k] requires N complex multiplications and N–1 complex additions. Now each complex multiplication, in turn, requires 4 real multiplications, and 2 real additions. Likewise, each complex addition requires 2 real additions. As a result, the N complex multiplications needed to compute X[k] require a total of 4N real multiplications and a total of 2N real additions. Similarly, the N–1 complex additions needed in the computation of X[k] require a total of 2N–2 real additions. Hence, each sample of the N-point DFT involves 4N real multiplications and 4N–2 real additions. The computation of all N DFT samples thus requires $4N^2$ real multiplications and (4N–2)N real additions.

8.13 Let the two complex numbers be $\alpha = a + jb$ and $\beta = c + jd$. Then, $\alpha\beta = (a + jb)(c + jd)$ $= (ac - bd) + j(ad + bc)$ which requires 4 real multiplications and 2 real additions. Consider the products $(a+b)(c+d)$, ac and bd which require 3 real multiplications and 2 real additions. The imaginary part of $\alpha\beta$ can be formed from $(a+b)(c+d) - ac - bd = ad + bc$ which now requires 2 real additions. Likewise, the real part of $\alpha\beta$ can be formed by forming ac – bd requiring an additional real addition. Hence, the complex multplication $\alpha\beta$ can be computed using 3 real multplications and 5 real additions.

8.14



$H_k(z) = \dfrac{1}{1 - W_N^{-k} z^{-1}}$. Hence, $Y(z) = \dfrac{X(z)}{1 - W_N^{-k} z^{-1}} = \dfrac{1 + z^{-N/2}}{1 - W_N^{-k} z^{-1}}$.

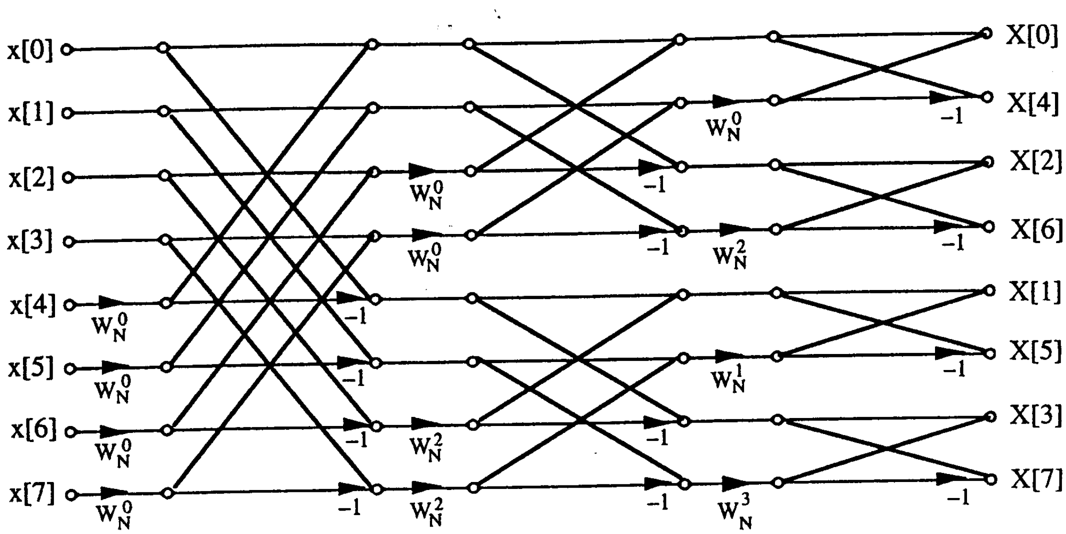
For k = 1, $Y(z) = \dfrac{1 + z^{-N/2}}{1 - W_N^{-1} z^{-1}} = (1 + z^{-N/2})(1 + W_N^{-1} z^{-1} + W_N^{-2} z^{-2} + \cdots + W_N^{-N/2} z^{-N/2} + \cdots$

Hence, $y[n] = \{1, W_N^{-1}, W_N^{-2}, \cdots, W_N^{-(N-2)/2}, 1 + W_N^{-N/2}, \cdots, 1 + W_N^{-(N-1)}, \cdots\}$

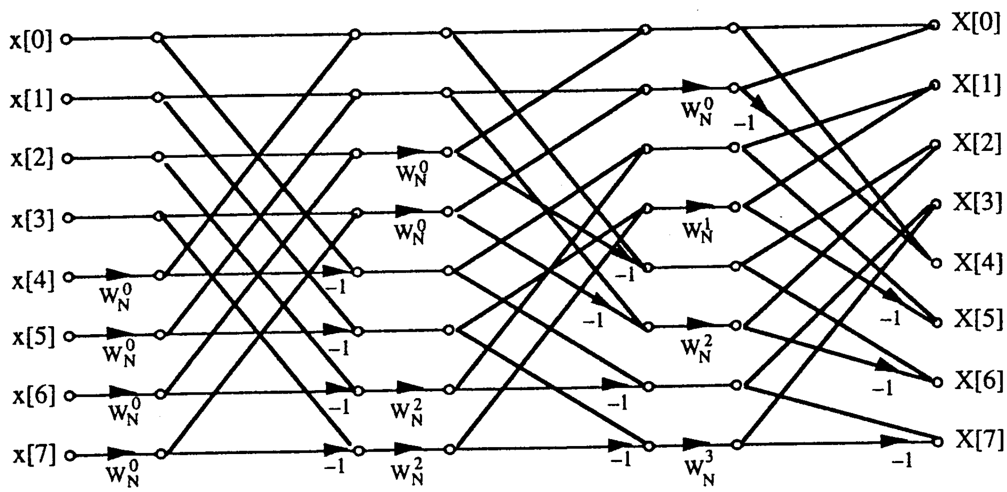
For k = N/2, $Y(z) = \dfrac{1 + z^{-N/2}}{1 - W_N^{-N/2} z^{-1}}$. Hence,

$y[n] = \{1, \quad -1, \quad 1, \quad -1, \quad \ldots \quad -1, \quad 2, \quad 0, \quad 2, \quad 0, \quad \ldots \quad 0\}$

8.15



8.16



8.17 $X[k]=\sum_{n=0}^{N-1} x[n]\,W_N^{nk}$

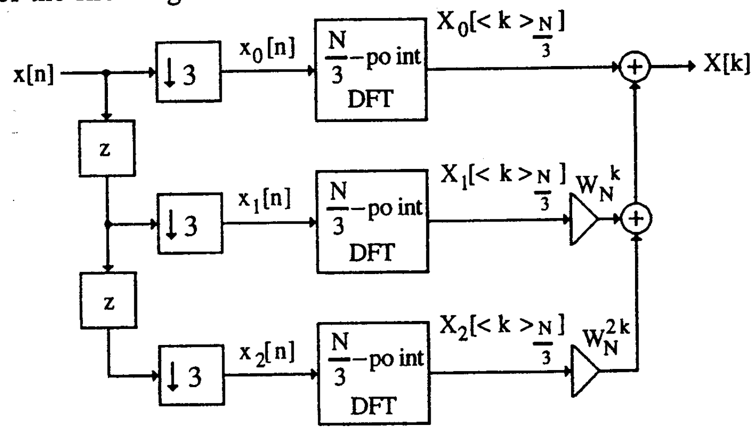
$$=\sum_{n=0}^{\frac{N}{r_1}-1} x[nr_1]\,W_N^{nkr_1}+\sum_{n=0}^{\frac{N}{r_1}-1} x[nr_1+1]\,W_N^{nkr_1+k}+\cdots+\sum_{n=0}^{\frac{N}{r_1}-1} x[nr_1+r_1-1]\,W_N^{nkr_1+(r_1-1)k}$$

$$=\sum_{i=0}^{r_1-1}\ \underbrace{\sum_{n=0}^{\frac{N}{r_1}-1} x[nr_1+i]\,W_N^{nkr_1}}_{(N/r_1)-\text{point DFT}}\cdot W_N^{ki}$$

Thus, if the (N/r_1)-point DFT has been calculated, we need at the first stage an additional (r_1-1) multiplications to compute one sample the N-point DFT X[k] and as a result, additional $(r_1-1)N$ multiplications are required to compute all N samples of the N-point DFT. Decomposing it further, it follows then that additional $(r_2-1)N$ multiplications are needed at the second stage, and so on. Therefore, Total number of multiply (add) operations = $(r_1 - 1)N + (r_2 - 1)N + \ldots + (r_\nu - 1)N$

$$= \left(\sum_{i=1}^{\nu} r_i - \nu\right)N.$$

8.18 $X(z) = X_0(z^3) + z^{-1}X_1(z^3) + z^{-2}X_2(z^3)$. Thus, the N-point DFT can be expressed as $X[k] = X_0[\langle k \rangle_{N/3}] + W_N^k X_1[\langle k \rangle_{N/3}] + W_N^{2k} X_2[\langle k \rangle_{N/3}]$. Hence, the structural interpretation of the first stage of the radix-3 DFT is as indicated below:



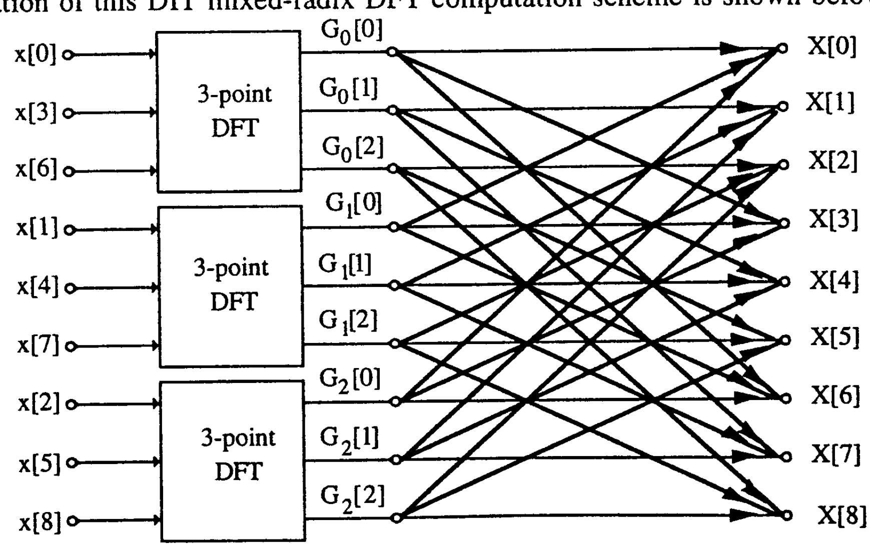
8.19 $X[k] = \sum_{n=0}^{8} x[n] W_9^{nk} = \left(x[0]W_9^{0\cdot k} + x[3]W_9^{3k} + x[6]W_9^{6k}\right)$

$$+\left(x[1]W_9^{k} + x[4]W_9^{4k} + x[7]W_9^{7k}\right) + \left(x[2]W_9^{2k} + x[5]W_9^{5k} + x[8]W_9^{8k}\right)$$

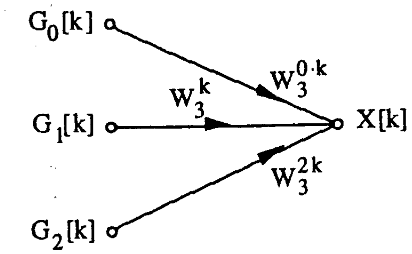
$$= \left(x[0]W_3^{0\cdot k} + x[3]W_3^{k} + x[6]W_3^{2k}\right) + \left(x[1]W_3^{0\cdot k} + x[4]W_3^{k} + x[7]W_3^{2k}\right)W_9^{k}$$

$$+\left(x[2]W_3^{0\cdot k} + x[5]W_3^{k} + x[8]W_3^{2k}\right)W_9^{2k} = G_0[\langle k \rangle_3] + G_1[\langle k \rangle_3]W_9^{k} + G_2[\langle k \rangle_3]W_9^{2k},$$

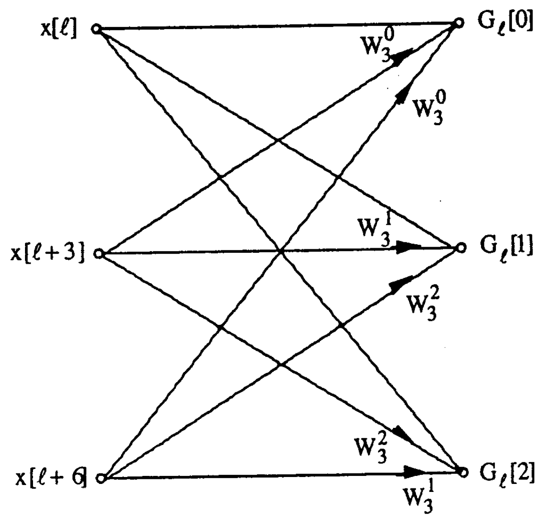
where $G_0[\langle k \rangle_3] = x[0]W_3^{0\cdot k} + x[3]W_3^{k} + x[6]W_3^{2k}$, $G_1[\langle k \rangle_3] = x[1]W_3^{0\cdot k} + x[4]W_3^{k} + x[7]W_3^{2k}$, **and** $G_2[\langle k \rangle_3] = x[2]W_3^{0\cdot k} + x[5]W_3^{k} + x[8]W_3^{2k}$, are three 3-point DFTs. A flow-graph representation of this DIT mixed-radix DFT computation scheme is shown below:



where the twiddle factors for computing the DFT samples are indicated below for a typical DFT sample:



In the above diagram, the 3-point DFT computation is carried out as indicated below:



8.20 $X[k]=\sum_{n=0}^{14} x[n]W_{15}^{nk}=\left(x[0]W_{15}^{0\cdot k}+x[3]W_{15}^{3k}+x[6]W_{15}^{6k}+x[9]W_{15}^{9k}+x[12]W_{15}^{12k}\right)$

$+\left(x[1]W_{15}^{k}+x[4]W_{15}^{4k}+x[7]W_{15}^{7k}+x[10]W_{15}^{10k}+x[13]W_{15}^{13k}\right)$

$+\left(x[2]W_{15}^{2k}+x[5]W_{15}^{5k}+x[8]W_{15}^{8k}+x[11]W_{15}^{11k}+x[14]W_{15}^{14k}\right)$

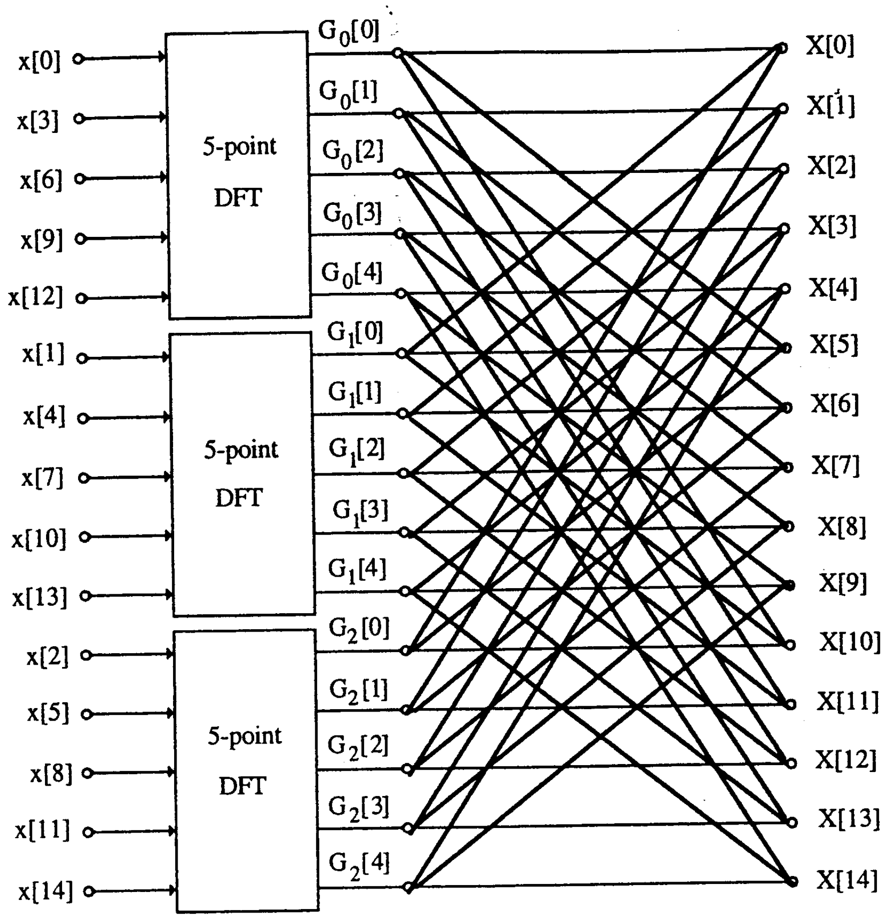
$=G_0[\langle k\rangle_5]+G_1[\langle k\rangle_5]W_{15}^{k}+G_2[\langle k\rangle_5]W_{15}^{2k}$, where

$G_0[\langle k\rangle_5]=x[0]W_5^{0\cdot k}+x[3]W_5^{k}+x[6]W_5^{2k}+x[9]W_5^{3k}+x[12]W_5^{4k}$,

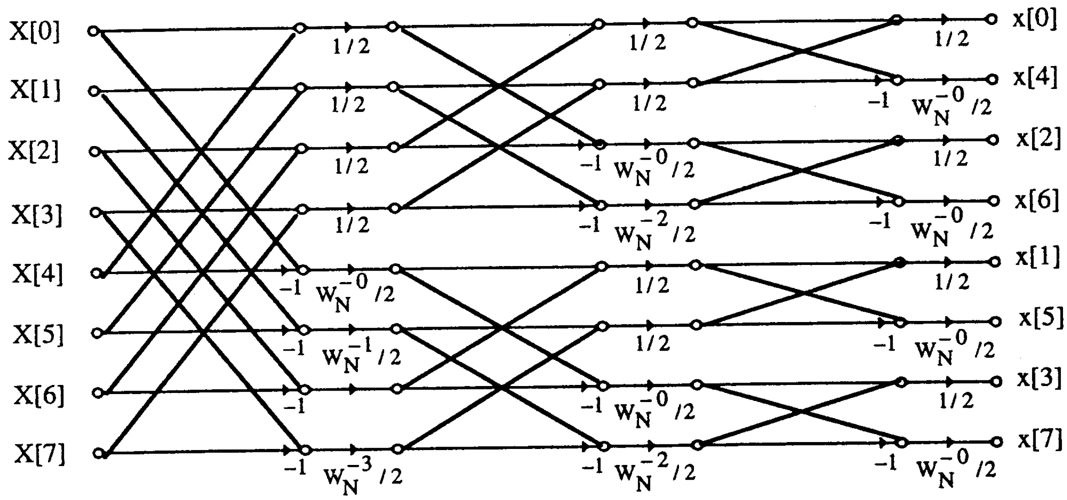
$G_1[\langle k\rangle_5]=x[1]W_5^{0\cdot k}+x[4]W_5^{k}+x[7]W_5^{2k}+x[10]W_5^{3k}+x[13]W_5^{4k}$, and

$G_2[\langle k\rangle_5]=x[2]W_5^{0\cdot k}+x[5]W_5^{k}+x[8]W_5^{2k}+x[11]W_5^{3k}+x[14]W_5^{4k}$.

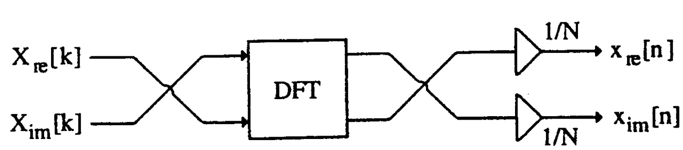
A flow-graph representation of this DIT mixed-radix DFT computation scheme is shown below:


x[0]
x[3]
x[6]
x[9]
x[12]
x[1]
x[4]
x[7]
x[10]
x[13]
x[2]
x[5]
x[8]
x[11]
x[14]
5-point DFT
5-point DFT
5-point DFT
G0[0]
G0[1]
G0[2]
G0[3]
G0[4]
G1[0]
G1[1]
G1[2]
G1[3]
G1[4]
G2[0]
G2[1]
G2[2]
G2[3]
G2[4]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
X[8]
X[9]
X[10]
X[11]
X[12]
X[13]
X[14]

8.21


X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
1/2
–1
$W_N^{-0}/2$
$W_N^{-1}/2$
$W_N^{-2}/2$
$W_N^{-3}/2$
x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]

8.22


$X_{re}[k]$
$X_{im}[k]$
DFT
1/N
1/N
$x_{re}[n]$
$x_{im}[n]$

Now, by definition, $q[n] = \mathrm{Im}\{X[n]\} + j\,\mathrm{Re}\{X[n]\}$. Its N-point DFT is $Q[k] = \sum_{n=0}^{N-1} q[n] W_N^{nk}$.

Thus, $$\mathrm{Re}\{Q[k]\} = \sum_{m=0}^{N-1}\left(\mathrm{Im}\{X[m]\}\cos\left(\frac{2\pi mk}{N}\right) + \mathrm{Re}\{X[m]\}\sin\left(\frac{2\pi mk}{N}\right)\right), \quad (18)$$

$$\mathrm{Im}\{Q[k]\} = \sum_{m=0}^{N-1}\left(-\mathrm{Im}\{X[m]\}\sin\left(\frac{2\pi mk}{N}\right) + \mathrm{Re}\{X[m]\}\cos\left(\frac{2\pi mk}{N}\right)\right), \quad (19)$$

From the definition of the inverse DFT we observe $x[k] = \frac{1}{N}\sum_{m=0}^{N-1} X[m] W_N^{-mk}$. Hence,

$$\mathrm{Re}\{x[k]\} = \frac{1}{N}\sum_{m=0}^{N-1}\left(\mathrm{Re}\{X[m]\}\cos\left(\frac{2\pi mk}{N}\right) - \mathrm{Im}\{X[m]\}\sin\left(\frac{2\pi mk}{N}\right)\right), \quad (20)$$

$$\mathrm{Im}\{x[k]\} = \frac{1}{N}\sum_{m=0}^{N-1}\left(\mathrm{Im}\{X[m]\}\cos\left(\frac{2\pi mk}{N}\right) + \mathrm{Re}\{X[m]\}\sin\left(\frac{2\pi mk}{N}\right)\right).12` \quad (21)$$

Comparing Eqs. (19) and (20), we get $\mathrm{Re}\{x[n]\} = \frac{1}{N}\mathrm{Im}\{Q[k]\}\Big|_{k=n}$,

and comparing Eqs. (18) and (21) we get $\mathrm{Im}\{x[n]\} = \frac{1}{N}\mathrm{Re}\{Q[k]\}\Big|_{k=n}$.

8.23 $r[n] = X[<-n>_N] = \begin{cases} X[0], & \text{if } n = 0, \\ X[N-n], & \text{if } n \neq 0. \end{cases}$ Therefore,

$$R[k] = \sum_{n=0}^{N-1} r[n] W_N^{nk} = r[0] + \sum_{n=1}^{N-1} r[n] W_N^{nk} = X[0] + \sum_{n=1}^{N-1} X[N-n] W_N^{nk}$$

$$= X[0] + \sum_{n=1}^{N-1} X[n] W_N^{(N-n)k} = X[0] + \sum_{n=1}^{N-1} X[n] W_N^{-nk} = \sum_{n=0}^{N-1} X[n] W_N^{-nk} = N\,x[k].$$

Thus, $x[n] = \frac{1}{N}\cdot R[k]\Big|_{k=n}$.

8.24 Method #1: Linear convolution - $y[n] = \sum_{m=0}^{2} x[m]h[n-m]$, n = 0, 1, . . . , 5.

Total # of real multiplications required = 1 + 2 +3 + 3 + 2 + 1 = 12.

Method #2: Here each sequence needs to be padded with zeros to make it a length-6 sequence. Then the total # of real multiplications required = 6^2 = 36.

Method #3: To implement the linear convolution via DFT we need to follow the scheme of Figure 3.12 requiring two forward DFTs and one inverse DFT. For a radix-2 FFT and IFFT, the sequences need to be zero-padded to make them of length that is a power of 2, whose smallest value here is 8.

Now, the first stage of the 8-point radix-2 FFT requires 0 complex multiplications, the second stage requires 0 complex multiplications, and the last stage requires 2 complex multiplications resulting in a total of 2 complex multiplications. The multiplication of the two 8-point DFTs requires 8 complex multiplications and the IFFT of the product requires 2 complex multiplications. Hence the computation of the linear convolution via the FFT approach requires 8 + 2 + 2 + 2 = 14 complex multiplications. A direct implementation of a complex multiplication requires 4 real multiplications resulting in a total of $14 \times 4 = 56$ real multiplications for Method #3. However, if a complex multiply can be implemented using 3 real multiplies (see Problem 8.13), in which case Method #3 requires a total of $14 \times 3 = 42$ real multiplications.

8.25 Method #1: Total # of real multiplications required
$= 1 + 2 + 3 + 4 + 5 + 6 + 6 + 6 + 5 + 4 + 3 + 2 + 1 = 48.$

Method #2: Total # of real multiplications required $= 13^2 = 169$.

Method #3: Here the desired length of the DFT (and the IDFT) is 16. Now, the first stage of the 16-point radix-2 FFT requires 0 complex multiplications, the second stage requires 0 complex multiplications, the third stage requires 4 complex multiplications, and the last stage requires 6 complex multiplications resulting in a total of 10 complex multiplications. The multiplication of the two 16-point DFTs requires 16 complex multiplications and the IFFT of the product requires 10 complex multiplications. Hence the computation of the linear convolution via the FFT approach requires 16 + 10 + 10 + 10 = 46 complex multiplications. Assuming the implementation of a complex multiply using 3 real multiplications (See Problem 8.13), we need here a total of $46 \times 3 = 138$ real multiplications.

8.26 **(a)** Since the impulse response of the filter is of length 65, the transform length N should be greater than 65. If L denotes the number of input samples used for convolution, then L = N − 64. So for every L samples of the input sequence, an N-point DFT is computed and multiplied with an N-point DFT of the impulse response sequence h[n] (which needs to be computed only once), and finally an N-point inverse of the product sequence is evaluated. Hence, the total number $\mathcal{R}_M$ of complex multiplications required (assuming N is a power-of-2) is given by

$$\mathcal{R}_M = \left\lceil \frac{1024}{N-64} \right\rceil \left(N \log_2 N + N \right) + \frac{N}{2} \log_2 N$$

It should be noted that in developing the above expression, multiplications due to twiddle factors of values ± 1 and $\pm j$ have not been excluded. The values of $\mathcal{R}_M$ for different values of N are as follows:

For N = 128, $\mathcal{R}_M = 16{,}832$

for N = 256, $\mathcal{R}_M = 14{,}848$

for N = 512, $\mathcal{R}_M = 17{,}664$

Hence, N = 256 is the appropriate choice for the transform length requiring 14,848 complex multiplications or equivalently, $14{,}848 \times 3 = 44{,}544$ real multiplications.

Since the first stage of the FFT calculation process requires only multiplications by ± 1, the total number of complex multiplications for N = 256 is actually

$$\mathcal{R}_M = \left\lceil \frac{1024}{N-64} \right\rceil \left(N \log_2 N + N \right) + \frac{N}{2} \log_2 N - \frac{N}{2} = 13{,}184$$

or equivalently, $13{,}184 \times 3 = 39{,}552$ real multiplications.

(b) For direct convolution, # of real multiplications $= 2\left(\sum_{n=1}^{64} n \right) + 65(1064 - 64) = 66{,}560.$

8.27 (a)

$$\mathbf{V}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\ 1 & 0 & 0 & 0 & W_8^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^7 \end{bmatrix}, \quad \mathbf{V}_4 = \begin{bmatrix} 1 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_8^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_8^2 \end{bmatrix},$$

$$\mathbf{V}_2 = \begin{bmatrix} 1 & W_8^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & W_8^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & W_8^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & W_8^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & W_8^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & W_8^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^4 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As can be seen from the above, multiplication by each matrix $\mathbf{V}_k$, k = 1, 2, 3, requires at most 8 complex multiplications.

(b) The transpose of the matrices given in part (a) are as follows:

$$\mathbf{V}_8^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ W_8^0 & 0 & 0 & 0 & W_8^4 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 & 0 & W_8^5 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 & 0 & 0 & W_8^6 & 0 \\ 0 & 0 & 0 & W_8^3 & 0 & 0 & 0 & W_8^7 \end{bmatrix}, \quad \mathbf{V}_4^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ W_8^0 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & W_8^2 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & W_8^0 & 0 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_8^2 & 0 & W_8^2 \end{bmatrix},$$

$$\mathbf{V}_2^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ W_8^0 & W_8^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & W_8^0 & W_8^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_8^0 & W_8^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_8^0 & W_8^4 \end{bmatrix}, \quad \mathbf{E}^T = \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to show that the flow-graph representation of $\mathbf{D}_8 = \mathbf{E}^T\mathbf{V}_2^T\mathbf{V}_4^T\mathbf{V}_8^T$ is precisely the 8-point DIF FFT algorithjm of Figure 8.27.

8.28 $X[2\ell]=\sum_{n=0}^{N-1}x[n]W_N^{2\ell n}=\sum_{n=0}^{\frac{N}{2}-1}x[n]W_N^{2\ell n}+\sum_{n=N/2}^{N-1}x[n]W_N^{2\ell n}, \quad \ell=0,1,\ldots,\frac{N}{2}-1.$

Replacing n by $n+\frac{N}{2}$ in the right-most sum we get

$$X[2\ell]=\sum_{n=0}^{\frac{N}{2}-1}x[n]W_N^{2\ell n}+\sum_{n=0}^{\frac{N}{2}-1}x[n+\frac{N}{2}]W_N^{2\ell n}W_N^{N\ell}=\sum_{n=0}^{\frac{N}{2}-1}\left\{x[n]+x[n+\frac{N}{2}]\right\}W_{N/2}^{\ell n},\ 0\le\ell\le\frac{N}{2}-1.$$

$$X[4\ell+1]=\sum_{n=0}^{\frac{N}{4}-1}x[n]W_N^{(4\ell+1)n}+\sum_{n=\frac{N}{4}}^{\frac{N}{2}-1}x[n]W_N^{(4\ell+1)n}+\sum_{n=\frac{N}{2}}^{\frac{3N}{4}-1}x[n]W_N^{(4\ell+1)n}+\sum_{n=\frac{3N}{4}}^{N-1}x[n]W_N^{(4\ell+1)n},$$

where $0\le\ell\le\frac{N}{4}-1$. Replacing n by $n+\frac{N}{4}$ in the second sum, n by $n+\frac{N}{2}$ in the third sum, and n by $n+\frac{3N}{4}$ in the fourth sum, we get

$$X[4\ell+1]=\sum_{n=0}^{\frac{N}{4}-1}x[n]W_N^{4\ell n}W_N^{n}+\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{N}{4}]W_N^{4\ell n}W_N^{n}W_N^{\ell N}W_N^{N/4}$$

$$+\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{N}{2}]W_N^{4\ell n}W_N^{n}W_N^{2\ell N}W_N^{N/2}+\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{3N}{4}]W_N^{4\ell n}W_N^{n}W_N^{3\ell N}W_N^{3N/4}$$

Now, $W_N^{\ell N}=W_N^{2\ell N}=W_N^{3\ell N}=1$, $W_N^{N/4}=-j$, $W_N^{N/2}=-1$, and $W_N^{3N/4}=+j$. Therefore,

$$X[4\ell+1]=\sum_{n=0}^{\frac{N}{4}-1}\left\{\left(x[n]-x[n+\frac{N}{2}]\right)-j\left(x[n+\frac{N}{4}]-x[n+\frac{3N}{4}]\right)\right\}W_N^{n}W_{N/4}^{\ell n},\ 0\le\ell\le\frac{N}{4}-1.$$

Similarly, $X[4\ell+3]=\sum_{n=0}^{\frac{N}{4}-1}x[n]W_N^{(4\ell+3)n}+\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{N}{4}]W_N^{(4\ell+3)n}W_N^{(4\ell+3)N/4}$

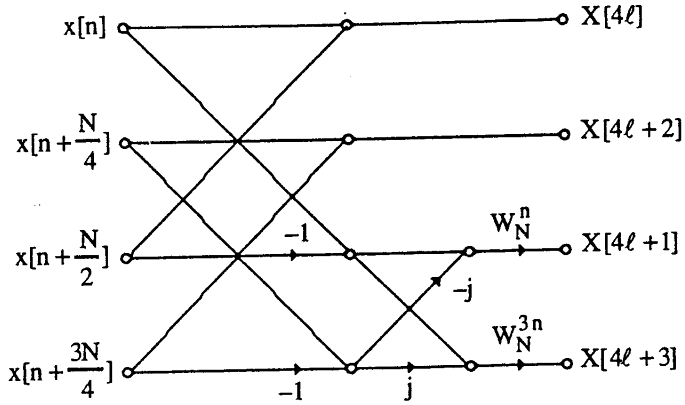
$$+\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{N}{2}]W_N^{(4\ell+3)n}W_N^{(4\ell+3)N/2}+\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{3N}{4}]W_N^{(4\ell+3)n}W_N^{(4\ell+3)3N/4}$$

$$=\sum_{n=0}^{\frac{N}{4}-1}x[n]W_N^{4\ell n}W_N^{3n}+\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{N}{4}]W_N^{4\ell n}W_N^{3n}W_N^{\ell N}W_N^{3N/4}$$

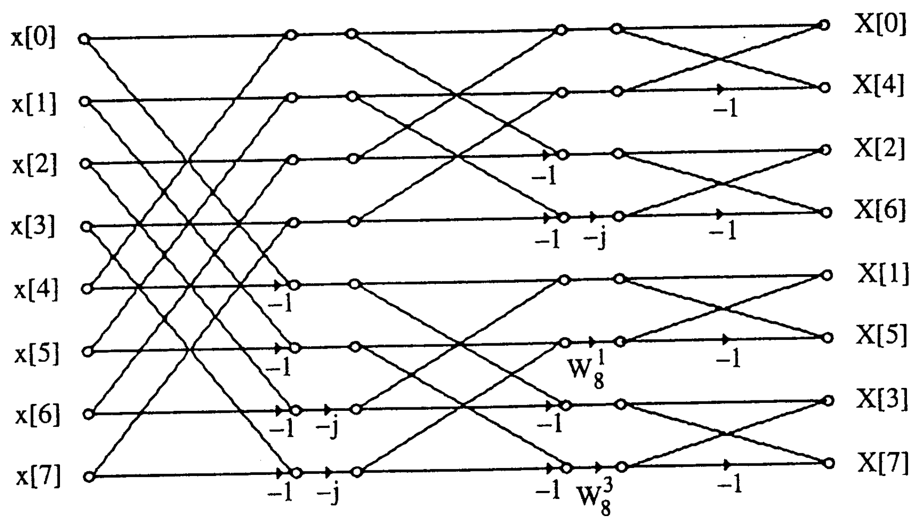
$$=\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{N}{2}]W_N^{4\ell n}W_N^{3n}W_N^{2\ell N}W_N^{6N/4}+\sum_{n=0}^{\frac{N}{4}-1}x[n+\frac{3N}{4}]W_N^{4\ell n}W_N^{3n}W_N^{3\ell N}W_N^{9N/4}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left\{ \left(x[n] - x[n+\frac{N}{2}] \right) + j \left(x[n+\frac{N}{4}] - x[n+\frac{3N}{4}] \right) \right\} W_N^{3n} W_{N/4}^{\ell n}, \; 0 \le \ell \le \frac{N}{4} - 1.$$

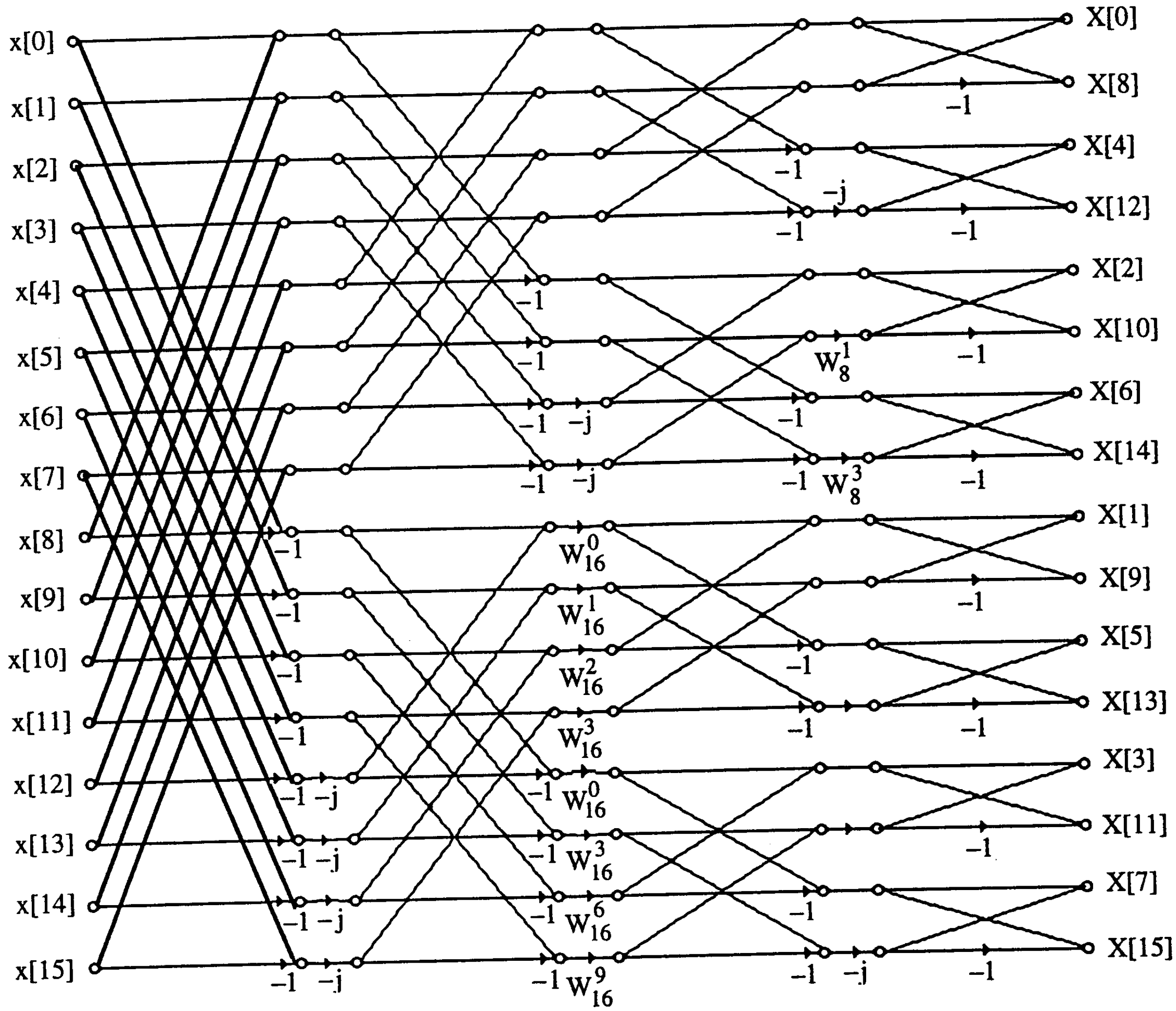
The butterfly here is as shown below which is seen to require two complex multiplications.



8.29 From the flow-graph of the 8-point split-radix FFT algorithm given below it can be seen that the total number of complex multiplications required is 2. On the other hand, the total number of complex multiplications required for a standard DIF FFT algorithm is also 2.



8.30 If multiplications by ±j, ±1 are ignored, the flow-graph shown below requires 8 complex = 24 real multiplications. A radix-2 DIF 16-point FFT algorithm, on the other hand, requires 10 complex multiplications = 30 real multiplications.



8.31 (a) From Eq. (8.98a), $x[n]=x[n_1+N_1n_2]$ where $0 \le n_1 \le N_1-1$, and $0 \le n_2 \le N_2-1$. Now, using Eq. (8.98b) we can rewrite $X[k]=\sum_{n=0}^{N-1} x[n]W_N^{nk}$, $0 \le k \le N-1$, as

$$X[N_2k_1+k_2]=\sum_{n=0}^{N_1N_2-1} x[n]W_N^{n(N_2k_1+k_2)}=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x[n_1+N_1n_2]W_N^{(n_1+N_1n_2)(N_2k_1+k_2)}$$

$$=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x[n_1+N_1n_2]W_N^{n_1N_2k_1}W_N^{N_1n_2N_2k_1}W_N^{N_1n_2k_2}W_N^{n_1k_2}.$$

Since, $W_N^{n_1N_2k_1}=W_{N_1}^{n_1k_1}$, $W_N^{N_1N_2n_1k_1}=1$, and $W_N^{N_1n_2k_2}=W_{N_2}^{n_2k_2}$, we get

$$X[k]=X[N_2k_1+k_2]=\sum_{n_1=0}^{N_1-1}\left[\left(\sum_{n_2=0}^{N_2-1} x[n_1+N_1n_2]W_{N_2}^{n_2k_2}\right)W_N^{n_1k_2}\right]W_{N_1}^{n_1k_1}. \tag{22}$$

(b) For $N_1 = 2$ and $N_2 = N/2$, the above DFT computation scheme leads to

$$X[k] = X[\frac{N}{2}k_1 + k_2] = \sum_{n_1=0}^{1}\left[\left(\sum_{n_2=0}^{\frac{N}{2}-1} x[n_1 + 2n_2] W_{N/2}^{n_2k_2}\right) W_N^{n_1k_2}\right] W_2^{n_1k_1}$$

$$= \sum_{n_2=0}^{\frac{N}{2}-1} x[2n_2] W_{N/2}^{n_2k_2} + W_{N_1}^{k_1} W_N^{k_2} \sum_{n_2=0}^{\frac{N}{2}-1} x[2n_2 + 1] W_{N/2}^{n_2k_2}$$

$$= \sum_{n_2=0}^{\frac{N}{2}-1} x[2n_2] W_{N/2}^{n_2k_2} + W_N^{k} \sum_{n_2=0}^{\frac{N}{2}-1} x[2n_2 + 1] W_{N/2}^{n_2k_2}$$

which is seen to be the first stage in the DIT DFT computation.

On the other hand, for $N_1 = N/2$ and $N_2 = 2$, the above DFT computation scheme leads to

$$X[k] = X[2k_1 + k_2] = \sum_{n_1=0}^{\frac{N}{2}-1}\left[\left(\sum_{n_2=0}^{1} x[n_1 + \frac{N}{2}n_2] W_2^{n_2k_2}\right) W_N^{n_1k_2}\right] W_{N/2}^{n_1k_1}$$

$$= \sum_{n_1=0}^{\frac{N}{2}-1} \left(x[n_1] + x[n_1 + \frac{N}{2}](-1)^{k_2}\right) W_N^{n_1k_2}\, W_{N/2}^{n_1k_1}$$

$$= \sum_{n_1=0}^{\frac{N}{2}-1} \left\{\left(x[n_1] + (-1)^{k_2} x[n_1 + \frac{N}{2}]\right) W_N^{n_1k_2}\right\} W_{N/2}^{n_1k_1}$$

which represents the first stage of the DIF FFT algorithm.

(c) In the DFT computation scheme of Eq. (22), we first compute a total of N_2 N_1-point DFTs, multiply all the $N_1N_2 = N$ computed DFT coefficients by the twiddle factors $W_N^{n_1k_2}$, and finally calculate a total of N_2 N_1-point DFTs. If $\mathcal{R}(N)$ denotes the total number of multiplications needed to compute an N-point DFT, then the total number of multplications required in the DFT computation scheme of Eq. (22) is given by

(i) $N_2 \cdot \mathcal{R}(N_1)$ for the first step,
(ii) $N_2N_1 = N$ for multiplications by the twiddle factors, and
(iii) $N_1 \cdot \mathcal{R}(N_2)$ for the last step.

Therefore, $\mathcal{R}(N) = N_2 \cdot \mathcal{R}(N_1) + N + N_1 \cdot \mathcal{R}(N_2) = N(\frac{1}{N_1}\mathcal{R}(N_1) + \frac{1}{N_2}\mathcal{R}(N_2) + 1)$.

(d) For $N = 2^{\nu}$, choose $N_i = 2$, $i = 1, 2, \ldots \nu$. Now from Figure 8.24 for a 2-point DFT $\mathcal{R}(N_i) = 2$. Hence, $\mathcal{R}(N) = N\left(\frac{\nu}{2}\right) = \frac{N}{2}\log_2 N$.

8.32 **(a)** N = 10. Choose $N_1 = 2$ and $N_2 = 5$.

$n_2 \backslash n_1$	0	1
0	x[0]	x[1]
1	x[2]	x[3]
2	x[4]	x[5]
3	x[6]	x[7]
4	x[8]	x[9]

$k_2 \backslash k_1$	0	1
0	X[0]	X[5]
1	X[1]	X[6]
2	X[2]	X[7]
3	X[3]	X[8]
4	X[4]	X[9]

(b) N = 12. Choose $N_1 = 4$ and $N_2 = 3$.

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[1]	x[2]	x[3]
1	x[4]	x[5]	x[6]	x[7]
2	x[8]	x[9]	x[10]	x[11]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[3]	X[6]	X[9]
1	X[1]	X[4]	X[7]	X[10]
2	X[2]	X[5]	X[8]	X[11]

(c) N = 15. Choose $N_1 = 3$ and $N_2 = 5$.

$n_2 \backslash n_1$	0	1	2
0	x[0]	x[1]	x[2]
1	x[3]	x[4]	x[5]
2	x[6]	x[7]	x[8]
3	x[9]	x[10]	x[11]
4	x[12]	x[13]	x[14]

$k_2 \backslash k_1$	0	1	2
0	X[0]	X[5]	X[10]
1	X[1]	X[6]	X[11]
2	X[2]	X[7]	X[12]
3	X[3]	X[8]	X[13]
4	X[4]	X[9]	X[14]

(d) N = 20. Choose $N_1 = 4$ and $N_2 = 5$.

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[1]	x[2]	x[3]
1	x[4]	x[5]	x[6]	x[7]
2	x[8]	x[9]	x[10]	x[11]
3	x[12]	x[13]	x[14]	x[15]
4	x[16]	x[17]	x[18]	x[19]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[5]	X[10]	X[15]
1	X[1]	X[6]	X[11]	X[16]
2	X[2]	X[7]	X[12]	X[17]
3	X[3]	X[8]	X[13]	X[18]
4	X[4]	X[9]	X[14]	X[19]

8.33 **(a)** $n = \langle An_1 + Bn_2 \rangle_N, \quad k = \langle Ck_1 + Dk_2 \rangle_N,$ where $N = N_1 N_2$.

$$X[k] = X[\langle Ck_1 + Dk_2 \rangle_N] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[\langle An_1 + Bn_2 \rangle_N] W_N^{(An_1+Bn_2)(Ck_1+Dk_2)}$$

$$= \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[\langle An_1 + Bn_2 \rangle_N] W_N^{ACn_1k_1} W_N^{ADn_1k_2} W_N^{BDn_2k_2} W_N^{BCn_2k_1}.$$

To completely eliminate the twiddle factors we require

$$W_N^{ACn_1k_1} W_N^{ADn_1k_2} W_N^{BDn_2k_2} W_N^{BCn_2k_1} = W_{N_1}^{n_1k_1} W_{N_2}^{n_2k_2}.$$

To achieve this we need to choose the constants A, B, C, and D, such that

$$<AD>_N = 0,\ <BC>_N = 0,\ <AC>_N = N_2, \text{ and } <BD>_N = N_1. \tag{23}$$

Then, we can write $X[<Ck_1 + Dk_2>_N] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[<An_1 + Bn_2>_N] W_{N_1}^{n_1 k_1} W_{N_2}^{n_2 k_2}$.

(b) We shall show that the following choice of the constants satisfy the constraints of Eq. (23):

$$A = N_2,\quad B = N_1,\quad C = N_2 <N_2^{-1}>_{N_1}, \text{ and } D = N_1 <N_1^{-1}>_{N_2},$$

where $<N_1^{-1}>_{N_2}$ is the multiplicative inverse of N_1 evaluated modulo N_2, i.e. if $<N_1^{-1}>_{N_2} = \alpha$, then $<N_1\alpha>_{N_2} = 1$. Hence, $N_1\alpha$ must be expressible as $N_2\beta + 1$, where β is any integer. Likewise, $<N_2^{-1}>_{N_1}$ is the multiplicative inverse of N_2 evaluated modulo N_1, and if $<N_2^{-1}>_{N_1} = \gamma$, then $N_2\gamma = N_1\delta + 1$, where δ is any integer. Now, from Eq. (23),
$<AC>_N = <N_2 \cdot N_2 <N_2^{-1}>_{N_1}>_N = <N_2(N_1\delta + 1)>_N = <N_2N_1\delta + N_2)>_N = N_2$. Similarly,
$<BD>_N = <N_1 \cdot N_1 <N_1^{-1}>_{N_2}>_N = <N_1(N_2\beta + 1)>_N = <N_1N_2\delta + N_1)>_N = N_1$. Next, we observe that $<AD>_N = <N_2 \cdot N_1 <N_1^{-1}>_{N_2}>_N = <N <N_1^{-1}>_{N_2}>_N = <N\alpha>_N = 0$, and
$<BC>_N = <N_1 \cdot N_2 <N_2^{-1}>_{N_1}>_N = <N <N_2^{-1}>_{N_1}>_N = <N\gamma>_N = 0$.

Hence, $X[k] = X[<Ck_1 + Dk_2>_N] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[<An_1 + Bn_2>_N] W_N^{N_2 n_1 k_1} W_N^{N_1 n_2 k_2}$

$$= \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[<An_1 + Bn_2>_N] W_{N_1}^{n_1 k_1} W_{N_2}^{n_2 k_2}$$

$$= \sum_{n_2=0}^{N_2-1} \left(\sum_{n_1=0}^{N_1-1} x[<An_1 + Bn_2>_N] W_{N_1}^{n_1 k_1} \right) W_{N_2}^{n_2 k_2}.$$

8.34 (a) N = 10. Choose $N_1 = 2$ and $N_2 = 5$.

$A = 5,\ B = 2,\ C = 5 <5^{-1}>_2 = 5,\ D = 2 <2^{-1}>_5 = 6.$

$n = <5n_1 + 2n_2>_{10}$ $\qquad$ $k = <5k_1 + 6k_2>_{10}$

$n_2 \backslash n_1$	0	1
0	x[0]	x[5]
1	x[2]	x[7]
2	x[4]	x[9]
3	x[6]	x[1]
4	x[8]	x[3]

$k_2 \backslash k_1$	0	1
0	X[0]	X[5]
1	X[6]	X[1]
2	X[2]	X[7]
3	X[8]	X[3]
4	X[4]	X[9]

(b) N = 12. Choose $N_1 = 4$ and $N_2 = 3$.

$A = 3,\ B = 4,\ C = 3 <3^{-1}>_4 = 9,\ D = 4 <4^{-1}>_3 = 4.$

$n = <3n_1 + 4n_2>_{12}$ $\qquad$ $k = <9k_1 + 4k_2>_{12}$

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[3]	x[6]	x[9]
1	x[4]	x[7]	x[10]	x[1]
2	x[8]	x[11]	x[2]	x[5]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[9]	X[6]	X[3]
1	X[4]	X[1]	X[10]	X[7]
2	X[8]	X[5]	X[2]	X[11]

(c) N = 15. Choose $N_1 = 3$ and $N_2 = 5$.

$A = 5,\ B = 3,\ C = 5 < 5^{-1} >_3 = 10,\ D = 3 < 3^{-1} >_5 = 6.$

$n = < 5n_1 + 3n_2 >_{15}$ $\quad k = < 10k_1 + 6k_2 >_{15}$

$n_2 \backslash n_1$	0	1	2
0	x[0]	x[5]	x[10]
1	x[3]	x[8]	x[13]
2	x[6]	x[11]	x[1]
3	x[9]	x[14]	x[4]
4	x[12]	x[2]	x[7]

$k_2 \backslash k_1$	0	1	2
0	X[0]	X[10]	X[5]
1	X[6]	X[1]	X[11]
2	X[12]	X[7]	X[2]
3	X[3]	X[13]	X[8]
4	X[9]	X[4]	X[14]

(d) N = 20. Choose $N_1 = 4$ and $N_2 = 5$.

$A = 5,\ B = 4,\ C = 5 < 5^{-1} >_4 = 5,\ D = 4 < 4^{-1} >_5 = 16.$

$n = < 5n_1 + 4n_2 >_{20}$ $\quad k = < 5k_1 + 16k_2 >_{20}$

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[5]	x[10]	x[15]
1	x[4]	x[9]	x[14]	x[19]
2	x[8]	x[13]	x[18]	x[3]
3	x[12]	x[17]	x[2]	x[7]
4	x[16]	x[1]	x[6]	x[11]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[5]	X[10]	X[15]
1	X[16]	X[1]	X[6]	X[11]
2	X[12]	X[17]	X[2]	X[7]
3	X[8]	X[13]	X[18]	X[3]
4	X[4]	X[9]	X[14]	X[19]

8.35 N = 12. $N_1 = 4$ and $N_2 = 3$.

$A = 3,\ B = 4,\ C = 3 < 3^{-1} >_4 = 9,\ D = 4 < 4^{-1} >_3 = 4.$

$n = < 3n_1 + 4n_2 >_{12}$ $\quad k = < 9k_1 + 4k_2 >_{12}$

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[3]	x[6]	x[9]
1	x[4]	x[7]	x[10]	x[1]
2	x[8]	x[11]	x[2]	x[5]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[9]	X[6]	X[3]
1	X[4]	X[1]	X[10]	X[7]
2	X[8]	X[5]	X[2]	X[11]

$n = < 9n_1 + 4n_2 >_{12}$ $\quad k = < 3k_1 + 4k_2 >_{12}$

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[9]	x[6]	x[3]
1	x[4]	x[1]	x[10]	x[7]
2	x[8]	x[5]	x[2]	x[11]

$k_2 \backslash k_1$	0	1	2	3
0	Y[0]	Y[3]	Y[6]	Y[9]
1	Y[4]	Y[7]	Y[10]	Y[1]
2	Y[8]	Y[11]	Y[2]	Y[5]

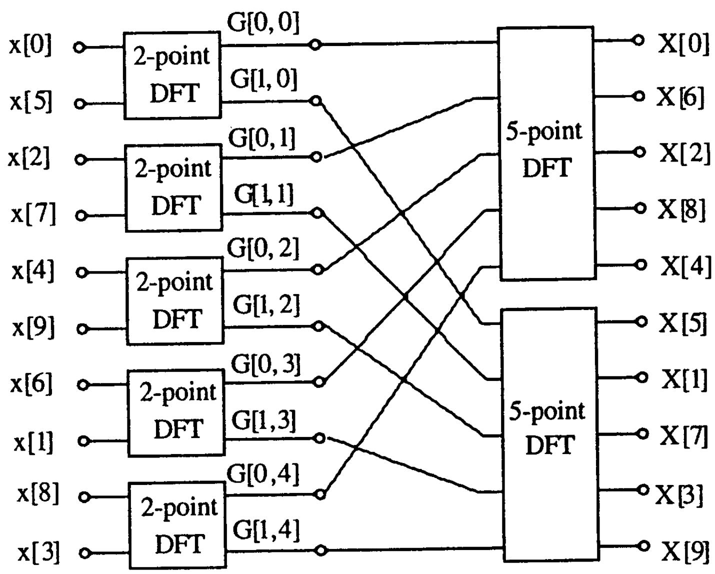
Hence, X[2k] = Y[2k], and X[2k + 1] = $X[2k+1] = Y[< 6 + (2k+1) >_{12}]$, k = 0, 1, . . ., 5,

8.36 $N = 10$, $N_1 = 2$, and $N_2 = 5$. Choose $A = 5$, $B = 2$, $C = 5 < 5^{-1} >_2 = 5$, $D = 2 < 2^{-1} >_5 = 6$.

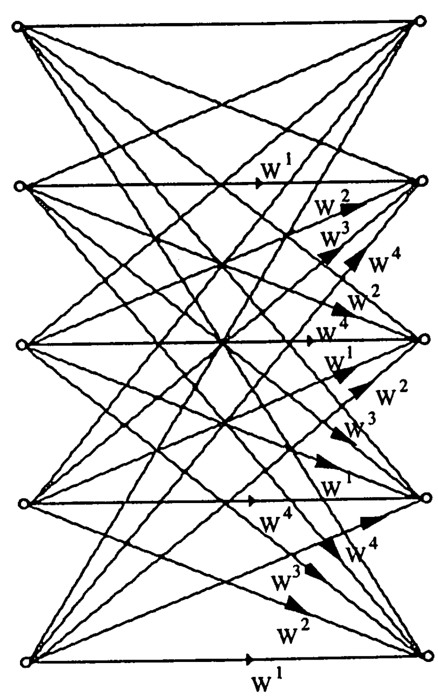
n_2 \ n_1	0	1
0	x[0]	x[5]
1	x[2]	x[7]
2	x[4]	x[9]
3	x[6]	x[1]
4	x[8]	x[3]

n_2 \ k_1	0	1
0	G[0,0]	G[1,0]
1	G[0,1]	G[1,1]
2	G[0,2]	G[1,2]
3	G[0,3]	G[1,3]
4	G[0,4]	G[1,4]

k_2 \ k_1	0	1
0	X[0]	X[5]
1	X[6]	X[1]
2	X[2]	X[7]
3	X[8]	X[3]
4	X[4]	X[9]



The flow-graph of the 2-point DFT is given in Figure 8.21. The flow-graph of the 5-point DFT is shown below

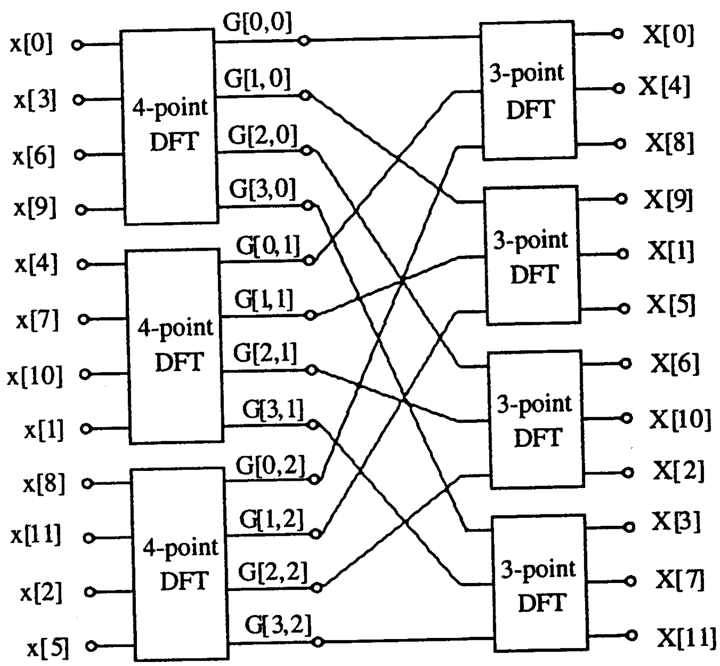


8.37 $N = 12$, $N_1 = 4$, and $N_2 = 3$. Choose $A = 3$, $B = 4$, $C = 3<3^{-1}>_4 = 9$, and $D = 4<4^{-1}>_3 = 4$.

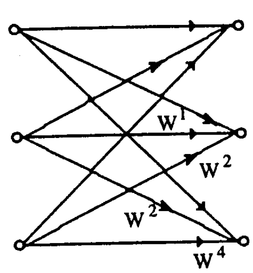
n_2 \ n_1	0	1	2	3
0	x[0]	x[3]	x[6]	x[9]
1	x[4]	x[7]	x[10]	x[1]
2	x[8]	x[11]	x[2]	x[5]

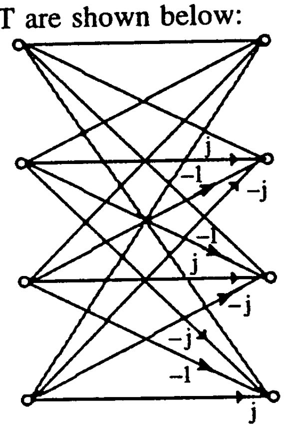
n_2 \ k_1	0	1	2	3
0	G[0, 0]	G[1,0]	G[2,0]	G[3,0]
1	G[0, 1]	G[1,1]	G[2,1]	G[3, 1]
2	G[0, 2]	G[1,2]	G[2,2]	G[3, 2]

k_2 \ k_1	0	1	2	3
0	X[0]	X[9]	X[6]	X[3]
1	X[4]	X[1]	X[10]	X[7]
2	X[8]	X[5]	X[2]	X[11]



The flow-graphs of the 3-point DFT and the 4-point DFT are shown below:





8.38 **(a)** $Y(z) = H(z)X(z)$ or $y[0]+y[1]z^{-1}+y[2]z^{-2} = \left(h[0]+h[1]z^{-1}\right)\left(x[0]+x[1]z^{-1}\right)$.

Now, $Y(z_0) = Y(-1) = y[0]-y[1]+y[2] = H(-1)X(-1) = (h[0]-h[1])(x[0]-x[1])$,

$Y(z_1) = Y(\infty) = y[0] = H(\infty)X(\infty) = h[0]x[0]$,

$$Y(z_2) = Y(1) = y[0] + y[1] + y[2] = H(1)X(1) = (h[0] + h[1])(x[0] + x[1]).$$

From Eqs. (3.158) and (3.159), we can write

$$Y(z) = \frac{I_0(z)}{I_0(z_0)} Y(z_0) + \frac{I_1(z)}{I_1(z_1)} Y(z_1) + \frac{I_2(z)}{I_2(z_2)} Y(z_2),$$

where $I_0(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) = (1 - z_1 z^{-1})(1 - z^{-1})\Big|_{z_1=\infty}$,

$$I_1(z) = (1 - z_0 z^{-1})(1 - z_2 z^{-1}) = (1 + z^{-1})(1 - z^{-1}) = 1 - z^{-2},$$

$$I_2(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1}) = (1 + z^{-1})(1 - z_1 z^{-1})\Big|_{z_1=\infty}.$$

Therefore, $\frac{I_0(z)}{I_0(z_0)} = -\frac{1}{2} z^{-1}(1 - z^{-1})$, $\frac{I_1(z)}{I_1(z_1)} = (1 - z^{-2})$, and $\frac{I_2(z)}{I_2(z_2)} = \frac{1}{2} z^{-1}(1 + z^{-1})$. Hence,

$$\begin{aligned}
Y(z) &= -\frac{1}{2} z^{-1}(1 - z^{-1}) Y(z_0) + (1 - z^{-2}) Y(z_1) + \frac{1}{2} z^{-1}(1 + z^{-1}) Y(z_2) \\
&= Y(z_1) + \left(-\frac{1}{2} Y(z_0) + \frac{1}{2} Y(z_2)\right) z^{-1} + \left(\frac{1}{2} Y(z_0) - Y(z_1) + \frac{1}{2} Y(z_2)\right) z^{-2} \\
&= h[0]x[0] + \left(-\frac{1}{2}(h[0] - h[1])(x[0] - x[1]) + \frac{1}{2}(h[0] + h[1])(x[0] + x[1])\right) z^{-1} \\
&\quad + \left(\frac{1}{2}(h[0] - h[1])(x[0] - x[1]) - h[0]x[0] + \frac{1}{2}(h[0] + h[1])(x[0] + x[1])\right) z^{-2} \\
&= h[0]x[0] + (h[0]x[1] + h[1]x[0]) z^{-1} + h[1]x[1] z^{-2}.
\end{aligned}$$

Ignoring the mu,tiplications by $\frac{1}{2}$, computation of the coefficients of Y(z) require the values of $Y(z_0)$, $Y(z_1)$, and $Y(z_2)$ which can be evaluated using only 3 multiplications.

(b) Y(z) = H(z)X(z) or

$$y[0] + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + y[4]z^{-4} = \left(h[0] + h[1]z^{-1} + h[2]z^{-2}\right)\left(x[0] + x[1]z^{-1} + x[2]z^{-2}\right).$$

Now, $Y(z_0) = Y(-\frac{1}{2}) = (h[0] - 2h[1] + 4h[2])(x[0] - 2x[1] + 4x[2])$,

$$Y(z_1) = Y(-1) = (h[0] - h[1] + h[2])(x[0] - x[1] + x[2]),$$

$$Y(z_2) = Y(\infty) = h[0]x[0],$$

$$Y(z_3) = Y(1) = (h[0] + h[1] + h[2])(x[0] + x[1] + x[2]),$$

$$Y(z_4) = Y(\frac{1}{2}) = (h[0] + 2h[1] + 4h[2])(x[0] + 2x[1] + 4x[2]).$$

From Eqs. (3.158) and (3.159), we can write

$$Y(z) = \frac{I_0(z)}{I_0(z_0)} Y(z_0) + \frac{I_1(z)}{I_1(z_1)} Y(z_1) + \frac{I_2(z)}{I_2(z_2)} Y(z_2) + \frac{I_3(z)}{I_3(z_3)} Y(z_3) + \frac{I_4(z)}{I_4(z_4)} Y(z_4),$$

where $I_0(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1})(1 - z_4 z^{-1}) = (1 - z^{-2})(1 - \frac{1}{2} z^{-1})(1 - z_2 z^{-1})\Big|_{z_2=\infty}$,

$$I_1(z) = (1 - z_0 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1})(1 - z_4 z^{-1}) = (1 - \frac{1}{4} z^{-2})(1 - z^{-1})(1 - z_2 z^{-1})\Big|_{z_2=\infty},$$

$$I_2(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})(1 - z_3 z^{-1})(1 - z_4 z^{-1}) = (1 - \frac{1}{4} z^{-2})(1 - z^{-2}),$$

$$I_3(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_4 z^{-1}) = (1 - \frac{1}{4} z^{-2})(1 + z^{-1})(1 - z_2 z^{-1})\Big|_{z_2=\infty},$$

$$I_4(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1}) = (1 - z^{-2})(1 + \frac{1}{2} z^{-1})(1 - z_2 z^{-1})\Big|_{z_2=\infty}.$$

Therefore, $\frac{I_0(z)}{I_0(z_0)} = \frac{1}{12} z^{-1}(1 - \frac{1}{2} z^{-1})(1 - z^{-2})$, $\frac{I_1(z)}{I_1(z_1)} = -\frac{2}{3} z^{-1}(1 - z^{-1})(1 - \frac{1}{4} z^{-2})$,

$\frac{I_2(z)}{I_2(z_2)} = (1 - z^{-2})(1 - \frac{1}{4} z^{-2})$, $\frac{I_3(z)}{I_3(z_3)} = \frac{2}{3} z^{-1}(1 + z^{-1})(1 - \frac{1}{4} z^{-2})$, and

$\frac{I_4(z)}{I_4(z_4)} = -\frac{1}{12} z^{-1}(1 + \frac{1}{2} z^{-1})(1 - z^{-2})$.

Hence, $$Y(z) = \frac{1}{12}(z^{-1} - \frac{1}{2} z^{-2} - z^{-3} + \frac{1}{2} z^{-4}) Y(z_0) - \frac{2}{3}(z^{-1} - z^{-2} - \frac{1}{4} z^{-3} + \frac{1}{4} z^{-4}) Y(z_1)$$
$$+ \frac{2}{3}(z^{-1} + z^{-2} - \frac{1}{4} z^{-3} - \frac{1}{4} z^{-4}) Y(z_3) - \frac{1}{12}(z^{-1} + \frac{1}{2} z^{-2} - z^{-3} - \frac{1}{2} z^{-4}) Y(z_4)$$
$$= Y(z_2) + \left(\frac{1}{12} Y(z_0) - \frac{2}{3} Y(z_1) + \frac{2}{3} Y(z_3) - \frac{1}{12} Y(z_4)\right) z^{-1}$$
$$+ \left(-\frac{1}{24} Y(z_0) + \frac{2}{3} Y(z_1) - \frac{5}{4} Y(z_2) + \frac{2}{3} Y(z_3) - \frac{1}{24} Y(z_4)\right) z^{-2}$$
$$+ \left(-\frac{1}{12} Y(z_0) + \frac{1}{6} Y(z_1) - \frac{1}{6} Y(z_3) + \frac{1}{12} Y(z_4)\right) z^{-3}$$
$$+ \left(\frac{1}{24} Y(z_0) - \frac{1}{6} Y(z_1) + \frac{1}{4} Y(z_2) - \frac{1}{6} Y(z_3) + \frac{1}{24} Y(z_4)\right) z^{-4}.$$

Substituting the expressions for $Y(z_0)$, $Y(z_1)$, $Y(z_2)$, $Y(z_3)$, and $Y(z_4)$, in the above equation, we then arive at the expressions for the coefficients $\{y[n]\}$ in terms of the coefficients $\{h[n]\}$ and $\{x[n]\}$. For example, $y[0] = Y(z_2) = h[0]x[0]$.

$$y[1] = \frac{1}{12}\left(Y(z_0) - Y(z_4)\right) + \frac{2}{3}\left(Y(z_3) - Y(z_1)\right) =$$
$$= \frac{1}{12}\Big(\big[(h[0] - 2h[1] + 4h[2])(x[0] - 2x[1] + 4x[2])\big] - \big[(h[0] + 2h[1] + 4h[2])(x[0] + 2x[1] + 4x[2])\big]\Big)$$
$$+ \frac{2}{3}\Big(\big[(h[0] + h[1] + h[2])(x[0] + x[1] + x[2])\big] - \big[(h[0] - h[1] + h[2])(x[0] - x[1] + x[2])\big]\Big)$$
$$= h[0]x[1] + h[1]x[0], = h[0]x[1] + h[1]x[0].$$ In a similar manner we can show,

$$y[2] = -\frac{1}{24}\left(Y(z_0) + Y(z_4)\right) + \frac{2}{3}\left(Y(z_1) + Y(z_3)\right) - \frac{5}{4} Y(z_2) = h[0]x[2] + h[1]x[1] + h[2]x[0],$$

$$y[3] = \frac{1}{12}\left(Y(z_4) - Y(z_0)\right) + \frac{1}{2}\left(Y(z_1) - Y(z_3)\right) = h[1]x[2] + h[2]x[1],$$ and

$$y[4] = \frac{1}{24}\left(Y(z_0) + Y(z_4)\right) - \frac{1}{6}\left(Y(z_1) + Y(z_3)\right) + \frac{1}{4} Y(z_2) = h[2]x[2].$$

Hence, ignoring the mu,tiplications by $\frac{1}{12}$, $\frac{2}{3}$, $\frac{5}{4}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{24}$, computation of the coefficients of $Y(z)$ require the values of $Y(z_0)$, $Y(z_1)$, $Y(z_2)$, $Y(z_3)$, and $Y(z_4)$ which can be evaluated using only 5 multiplications.

8.39 $Y(z) = H(z)X(z)$ or $y[0] + y[1]z^{-1} + y[2]z^{-2} = \left(h[0] + h[1]z^{-1}\right)\left(x[0] + x[1]z^{-1}\right)$

$$h[0]x[0]+(h[0]x[1]+h[1]x[0])z^{-1}+h[1]x[1]z^{-2}.$$

Hence, $y[0]=h[0]x[0]$, $y[1]=h[0]x[1]+h[1]x[0]$, and $y[2]=h[1]x[1]$.
Now, $(h[0]+h[1])(x[0]+x[1])-h[0]x[0]-h[1]x[1]=h[0]x[1]+h[1]x[0]=y[1]$. As a result, evaluation of H(z)X(z) requires the computation of 3 products, h[0]x[0], h[1]x[1], and $(h[0]+h[1])(x[0]+x[1])$. In addition, it requires 4 additions, h[0] + h[1], x[0] + x[1], and $(h[0]+h[1])(x[0]+x[1])-h[0]x[0]-h[1]x[1]$.

8.40 Let the two length-N sequencess be denoted by $\{h[n]\}$ and $\{x[n]\}$. Denote the sequence generated by the linear convolution of h[n] and x[n] as y[n], i.e. $y[n]=\sum_{\ell=0}^{2N-1}h[\ell]x[n-\ell]$. Computation of $\{y[n]\}$ thus requires 2N multiplications. Let H(z) and X(z) denote the z-transforms of $\{h[n]\}$ and $\{x[n]\}$, i.e. $H(z)=\sum_{n=0}^{N-1}h[n]z^{-n}$, and $X(z)=\sum_{n=0}^{N-1}x[n]z^{-n}$. Rewrite H(z) and X(z) in the form $H(z)=H_0(z)+z^{-N/2}H_1(z)$, and $X(z)=X_0(z)+z^{-N/2}X_1(z)$, where $H_0(z)=\sum_{n=0}^{(N/2)-1}h[n]z^{-n}$, $H_1(z)=\sum_{n=0}^{(N/2)-1}h[n+\frac{N}{2}]z^{-n}$, $X_0(z)=\sum_{n=0}^{(N/2)-1}x[n]z^{-n}$, and $X_1(z)=\sum_{n=0}^{(N/2)-1}x[n+\frac{N}{2}]z^{-n}$. Therefore, we can write

$$\begin{aligned}Y(z)&=\left(H_0(z)+z^{-N/2}H_1(z)\right)\left(X_0(z)+z^{-N/2}X_1(z)\right)\\&=H_0(z)X_0(z)+z^{-N/2}\left(H_0(z)X_1(z)+H_0(z)X_1(z)\right)+z^{-N}H_1(z)X_1(z)\\&=Y_0(z)+z^{-N/2}Y_1(z)+z^{-N}Y_2(z),\end{aligned}$$

where $Y_0(z)=H_0(z)X_0(z)$, $Y_1(z)=H_0(z)X_1(z)+H_1(z)X_0(z)$, $Y_2(z)=H_1(z)X_1(z)$.

Now $Y_0(z)$ and $Y_1(z)$ are products of two polynomials of degree $\frac{N}{2}$, and hence, require $\left(\frac{N}{2}\right)^2$ multiplications each. Now, we can write $Y_1(z)=\left(H_0(z)+H_1(z)\right)\left(X_0(z)+X_1(z)\right)-Y_0(z)-Y_2(z)$. Since $\left(H_0(z)+H_1(z)\right)\left(X_0(z)+X_1(z)\right)$ is a product of two polynomials of degree $\frac{N}{2}$, and hence, it can be computed using $\left(\frac{N}{2}\right)^2$ multiplications. As a result, Y(z) = H(z)X(z) can be computed using $3\left(\frac{N}{2}\right)^2$ multiplications instead of N^2 multiplications.

If N is a power-of-2, $\frac{N}{2}$ is even, and the same procedure can be applied to compute $Y_0(z)$, $Y_1(z)$, and $Y_2(z)$ reducing further the number of multiplications. This process can be continued until, the sequences being convolved are of length 1 each.

Let $\mathcal{R}(N)$ denote the total number of multiplications required to compute the linear convolution of two length-N sequences. Then, in the method outlined above, we have $\mathcal{R}(N) = 3\cdot\mathcal{R}(N/2)$ with $\mathcal{R}(1) = 1$. A solution of this equation is given by $\mathcal{R}(N) = 3^{\log_2 N}$.

8.41 The dynamic range of a signed B-bit integer η is given by $-(2^{(B-1)}-1)\le\eta<(2^{(B-1)}-1)$ which for B = 32 is given by $-(2^{31}-1)\le\eta<(2^{31}-1)$.

(a) For E = 6 and M = 25, the value of a 32-bit floating point number is given by $\eta = (-1)^s 2^{E-31}(M)$. Hence, the value of the largest number is $\approx 2^{32}$, and the value of the smallest number is $\approx -2^{32}$. The dynamic range is therefore $\approx 2 \times 2^{32}$.

(b) For E = 7 and M = 24, the value of a 32-bit floating point number is given by $\eta = (-1)^s 2^{E-63}(M)$. Hence, the value of the largest number is $\approx 2^{64}$, and the value of the smallest number is $\approx -2^{64}$. The dynamic range is therefore $\approx 2 \times 2^{64}$.

(c) For E = 8 and M = 23, the value of a 32-bit floating point number is given by $\eta = (-1)^s 2^{E-127}(M)$. Hence, the value of the largest number is $\approx 2^{128}$, and the value of the smallest number is $\approx -2^{128}$. The dynamic range is therefore $\approx 2 \times 2^{128}$.

Hence, the dynamic range in a floating-point representation is much larger than that in a fixed-point representation with the same wordlength.

8.42 A 32-bit floating-point number in the IEEE Format has E = 8 and M = 23. Also the exponent E is coded in a biased form as E – 127 with certain conventions for special cases such as E = 0, 255, and M = 0 (See text pages 540 and 541).

Now a positive 32-bit floating point number η represented in the "normalized" form have an exponent in the range $0 < E < 255$, and is of the form $\eta = (-1)^s 2^{E-127}(1_\Delta M)$. Hence, the smallest positive number that can be represented will have E = 1, and M = $\underbrace{0\ 0\ \cdots\ 0\ 0}_{22 \text{ bits}}$, and has therefore a value given by $2^{-126} \cong 1.18 \times 10^{-38}$. For the largest positive number, E = 254, and $M = \underbrace{1\ 1\ \cdots\ 1\ 1}_{22 \text{ bits}}$. Thus here $\eta = (-1)^0 2^{127}(1_\Delta \underbrace{1\ 1\ \cdots\ 1}_{22 \text{ bits}}) \approx 2^{127} \times 2 \cong 3.4 \times 10^{38}$.

Note: For representing numbers less than 2^{-126}, IEEE format uses the "de-normalized" form where E = 0, and $\eta = (-1)^s 2^{-126}(0_\Delta M)$. In this case, the smallest positive number that can be represented is given by $\eta = (-1)^0 2^{-126}(0_\Delta \underbrace{0\ 0\ \cdots\ 0\ 1}_{22 \text{ bits}}) \approx 2^{-149} \cong 1.4013 \times 10^{-45}$.

8.43 For a two's-complement binary fraction $s_\Delta a_{-1}a_{-2}\cdots a_{-b}$ the decimal equivalent for s = 0 is simply $\sum_{i=1}^{b} a_{-i}2^{-i}$. For s = 1, the decimal equivalent is given by $-\left(\sum_{i=1}^{b}(1-a_{-i})2^{-i} + 2^{-b}\right)$

$$= -\sum_{i=1}^{b} 2^{-i} + \sum_{i=1}^{b} a_{-i}2^{-i} - 2^{-b} = -(1-2^{-b}) + \sum_{i=1}^{b} a_{-i}2^{-i} - 2^{-b} = -1 + \sum_{i=1}^{b} a_{-i}2^{-i}.$$

Hence, the decimal equivalent of $s_\Delta a_{-1}a_{-2}\cdots a_{-b}$ is given by $-s + \sum_{i=1}^{b} a_{-i}2^{-i}$.

8.44 For a two's-complement binary fraction $s_\Delta a_{-1}a_{-2}\cdots a_{-b}$ the decimal equivalent for s = 0 is simply $\sum_{i=1}^{b} a_{-i}2^{-i}$. For s = 1, the decimal equivalent is given by $-\sum_{i=1}^{b}(1-a_{-i})2^{-i}$

$$= -\sum_{i=1}^{b} 2^{-i} + \sum_{i=1}^{b} a_{-i} 2^{-i} = -(1-2^{-b}) + \sum_{i=1}^{b} a_{-i} 2^{-i}.$$ Hence, the decimal equivalent of $s_{\Delta} a_{-1} a_{-2} \cdots a_{-b}$ is given by $-s(1-2^{-b}) + \sum_{i=1}^{b} a_{-i} 2^{-i}$.

8.45 **(a)** $\eta = -0.625_{10}$. (i) Signed-magnitude representation = $1_{\Delta}10100000$
(ii) Ones'-complement representation = $1_{\Delta}01011111$
(iii) Two's-complement representation = $1_{\Delta}01100000$

(b) $\eta = -0.7734375_{10}$. (i) Signed-magnitude representation = $1_{\Delta}11000110$
(ii) Ones'-complement representation = $1_{\Delta}00111001$
(iii) Two's-complement representation = $1_{\Delta}00111010$

(c) $\eta = -0.36328125_{10}$. (i) Signed-magnitude representation = $1_{\Delta}01011101$
(ii) Ones'-complement representation = $1_{\Delta}10100010$
(iii) Two's-complement representation = $1_{\Delta}10100011$

(d) $\eta = -0.94921875_{10}$. (i) Signed-magnitude representation = $1_{\Delta}11110011$
(ii) Ones'-complement representation = $1_{\Delta}00001100$
(iii) Two's-complement representation = $1_{\Delta}00001101$

8.46 **(a)** $\eta = 0.625_{10} = 1_{\Delta}101000$, **(b)** $\eta = -0.625_{10} = 0_{\Delta}011000$,
(c) $\eta = 0.359375_{10} = 1_{\Delta}010111$, **(d)** $\eta = -0.359375_{10} = 0_{\Delta}101001$,
(e) $\eta = 0.90625_{10} = 1_{\Delta}111010$, **(f)** $\eta = -0.90625_{10} = 0_{\Delta}000110$.

8.47 **(a)** SD-representation $= 0_{\Delta}00\bar{1}10\bar{1}1\bar{1}$, **(b)** SD-representation $= 0_{\Delta}10000\bar{1}1\bar{1}$,
(c) SD-representation $= 0_{\Delta}\bar{1}1\bar{1}1000\bar{1}$.

8.48 **(a)** $\underbrace{1101}_{D}\underbrace{0110}_{6}\underbrace{1100}_{C}\underbrace{0111}_{7} = D6C7$, **(b)** 0101 1111 1010 1001 = 5FA9,
(c) 1011 0100 0010 1110 = B42E.

8.49 **(a)** The addition of the positive binary fractions $0_{\Delta}10101$ and $0_{\Delta}01111$ is given below:

$$\begin{array}{cccccccl} 1 & & 1 & 1 & 1 & 1 & & \leftarrow \text{carry} \\ 0 & \Delta & 1 & 0 & 1 & 0 & 1 & \\ +\,0 & \Delta & 0 & 1 & 1 & 1 & 1 & \\ \hline 1 & \Delta & 0 & 0 & 1 & 0 & 0 & \end{array}$$

As the sign bit is a 1 there has been an overflow and the sum is not correct.

(b) The addition of the positive binary fractions $0_{\Delta}01011$ and $0_{\Delta}10001$ is given below:

```
          1 1       ← carry
  0 Δ 0 1 0 1 1
+ 0 Δ 1 0 0 0 1
----------------
  0 Δ 1 1 1 0 0
```

8.50 **(a)** The difference of the two positive binary fractions $0_\Delta 10101 - 0_\Delta 01111$ can be carried out as an addition of the positive binary fraction $0_\Delta 10101$ and the two's-complement representation of $-0_\Delta 01111$ which is given by $1_\Delta 10001$. The process is illustrated below:

```
   1            1    ← carry
   0 Δ 1 0 1 0 1
+  1 Δ 1 0 0 0 1
-----------------
  1 0 Δ 0 0 1 1 0
  ↑
 drop
```

The extra bit 1 on the left of the sign bit is dropped resulting in $0_\Delta 00110$ which is the correct difference.

(b) The difference of the two positive binary fractions $0_\Delta 10001 - 0_\Delta 01011$ can be carried out as an addition of the positive binary fraction $0_\Delta 10001$ and the two's-complement representation of $-0_\Delta 01011$ which is given by $1_\Delta 10101$. The process is illustrated below:

```
   1            1    ← carry
   0 Δ 1 0 0 0 1
+  1 Δ 1 0 1 0 1
-----------------
  1 0 Δ 0 0 1 1 0
  ↑
 drop
```

The extra bit 1 on the left of the sign bit is dropped resulting in $0_\Delta 00110$ which is the correct difference.

8.51 **(a)** The difference of the two positive binary fractions $0_\Delta 10101 - 0_\Delta 01111$ can be carried out as an addition of the positive binary fraction $0_\Delta 10101$ and the ones'-complement representation of $-0_\Delta 01111$ which is given by $1_\Delta 10000$. The process is illustrated below:

```
   1            1    ← carry
   0 Δ 1 0 1 0 1
+  1 Δ 1 0 0 0 0
-----------------
  1 0 Δ 0 0 1 0 1
   ·...........→ 1  ←     end around carry
-----------------
   0 Δ 0 0 1 1 0
```

The extra bit 1 on the left side of the sign bit is brought around and added to the LSB resulting in the correct difference given by $0_\Delta 00110$.

(b) The difference of the two positive binary fractions $0_\Delta 10001 - 0_\Delta 01011$ can be carried out as an addition of the positive binary fraction $0_\Delta 10001$ and the ones'-complement representation of $-0_\Delta 01011$ which is given by $1_\Delta 10100$. The process is illustrated below:

$$
\begin{array}{rcccccccl}
 & 1 & & & & & & & \leftarrow \text{carry} \\
 & 0 & \Delta & 1 & 0 & 0 & 0 & 1 & \\
+ & 1 & \Delta & 1 & 0 & 1 & 0 & 0 & \\
\hline
 & 1\,0 & \Delta & 0 & 0 & 1 & 0 & 1 & \\
 & & & & & & & \to 1 & \leftarrow \quad \text{end around carry} \\
\hline
 & 0 & \Delta & 0 & 0 & 1 & 1 & 0 &
\end{array}
$$

The extra bit 1 on the left side of the sign bit is brought around and added to the LSB resulting in the correct difference given by $0_\Delta 00110$.

8.52 $\eta_1 = 0.6875_{10} = 0_\Delta 1011$, $\eta_2 = 0.8125_{10} = 0_\Delta 1101$, $\eta_3 = -0.5625_{10} = 1_\Delta 0111$. Now, $\eta_1 + \eta_2 + \eta_3 = 0.6875_{10} + 0.8125_{10} - 0.5625_{10} = 0.9375_{10}$. We first form the binary addition $\eta_1 + \eta_2 = 0_\Delta 1011 + 0_\Delta 1101 = 1_\Delta 1000$ indicating an overflow. If we ignore the overflow and add to the partial sum η_3 we arrive at $(\eta_1 + \eta_2) + \eta_3 = 1_\Delta 1000 + 1_\Delta 0111 = 10_\Delta 1111$. Dropping the leading 1 from the sum we get $\eta_1 + \eta_2 + \eta_3 = 0_\Delta 1111$ whose decimal equivalent is 0.9375_{10}. As a result, the correct sum is obtained inspite of intermediate overflow that occurred.

8.53 (a) $0_\Delta 1\,1\,1\,0\,1 \times 1_\Delta 1\,0\,1\,1\,1$. In this case the multiplier is a negative number. Now a multiplier $D = d_{s\Delta} d_{-1} d_{-2} \cdots d_{-b}$ has the value $-d_s + \sum_{i=1}^{b} d_{-i} 2^{-i}$. So the product is given by

$$x = A \cdot D = -Ad_s + A \cdot (0_\Delta d_{-1} d_{-2} \cdots d_{-b}).$$

Hence we first ognore the sign bit of the multiplier and in the end if the sign bit is a 1, the value of the multiplicand is subtracted from the partial product.

$$
\begin{array}{lcc l}
 & 0 & \Delta & 1\,1\,1\,0\,1 \\
 & \times\, 0 & \Delta & 1\,0\,1\,1\,1 \\
\hline
P^{(0)} & 0 & \Delta & 0\,0\,0\,0\,0 \\
 & +\,0 & \Delta & 1\,1\,1\,0\,1 \\
\hline
 & 0 & \Delta & 1\,1\,1\,0\,1 \\
\hline
P^{(1)} & 0 & \Delta & 0\,1\,1\,1\,0\,1 \\
 & +\,0 & \Delta & 1\,1\,1\,0\,1 \\
\hline
 & 1 & \Delta & 0\,1\,0\,1\,1\,1 \\
\hline
P^{(2)} & 0 & \Delta & 1\,0\,1\,0\,1\,1\,1 \\
 & +\,0 & \Delta & 1\,1\,1\,0\,1 \\
\hline
 & 1 & \Delta & 1\,0\,0\,1\,0\,1\,1 \\
\hline
P^{(3)} & 0 & \Delta & 1\,1\,0\,0\,1\,0\,1\,1 \\
\hline
P^{(4)} & 0 & \Delta & 0\,1\,1\,0\,0\,1\,0\,1\,1 \\
 & +\,0 & \Delta & 1\,1\,1\,0\,1 \\
\hline
 & 1 & \Delta & 0\,1\,0\,0\,1\,1\,0\,1\,1 \\
\hline
P^{(5)} & 0 & \Delta & 1\,0\,1\,0\,0\,1\,1\,0\,1\,1
\end{array}
$$

As $d_s = 1$, we need to subtract $0_\Delta 1\,1\,1\,0\,1$ from $P^{(5)}$ or equivalently, adding its two's-complement $1_\Delta 0\ 0\ 0\ 1\ 1$ to $P^{(5)}$ as indicated below:

$$\begin{array}{rcl}
 & 0 \ \Delta & 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1 \\
+ & 1 \ \Delta & 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0 \\
\hline
 & 1 \ \Delta & 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1
\end{array}$$

Hence the final product is $1_\Delta 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1$. Its decimal equivalent is -0.254882813_{10} which is the correct result of the product $0.90625_{10} \times (-0.28125_{10})$.

(b) In this case the multiplicand is negative. Hence we follow the same steps as given on Page 547 of Text except the addition is now two's-complement addition.

$$\begin{array}{rrl}
 & 1 \ \Delta & 1\ 0\ 1\ 0\ 1 \\
 & \times\ 0 \ \Delta & 1\ 0\ 1\ 1\ 1 \\
\hline
P^{(0)} & 0 \ \Delta & 0\ 0\ 0\ 0\ 0 \\
 & +1 \ \Delta & 1\ 0\ 1\ 0\ 1 \\
\hline
P^{(1)} & 1 \ \Delta & 1\ 1\ 0\ 1\ 0\ 1 \\
 & +1 \ \Delta & 1\ 0\ 1\ 0\ 1 \\
\hline
 & 1\ 1 \ \Delta & 0\ 1\ 1\ 1\ 1\ 1 \\
\hline
P^{(2)} & 1 \ \Delta & 1\ 0\ 1\ 1\ 1\ 1\ 1 \\
 & +1 \ \Delta & 1\ 0\ 1\ 0\ 1 \\
\hline
 & 1\ 1 \ \Delta & 0\ 1\ 1\ 0\ 0\ 1\ 1 \\
\hline
P^{(3)} & 1 \ \Delta & 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \\
\hline
P^{(4)} & 1 \ \Delta & 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \\
 & +1 \ \Delta & 1\ 0\ 1\ 0\ 1 \\
\hline
 & 1\ 1 \ \Delta & 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \\
\hline
P^{(5)} & 1 \ \Delta & 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1
\end{array}$$

Hence the result of the multiplication $1_\Delta 1\ 0\ 1\ 0\ 1 \times 0_\Delta 1\ 0\ 1\ 1\ 1$ is $1_\Delta 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1$,

The decimal equivalent is $(-0.34375_{10}) \times 0.71875 = -0.247070313_{10}$.

8.54 **(a)** Again in this case the multiplier is negative. Hence we can write

$$x = A \cdot D = A \cdot \left(-d_s(1-2^{-b}) + \sum_{i=1}^{b} d_{-i} 2^{-i} \right) = -Ad_s + Ad_s 2^{-b} + \tilde{x},$$

where $\tilde{x}$ is the product of A and D with its sign bit d_s ignored. Hence, we first form $\tilde{x}$ and if ds = 1, we subtract A from $\tilde{x}$, and add to the result a shifted version (b bits to the right) of A.

Now $\tilde{x}$ is same as the product obtained in Problem 8.52(a) and is given by

$$\tilde{x} = 0_\Delta 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1.$$

Subtracting A from $\tilde{x}$ is equivalent to adding the ones'-complement of A to $\tilde{x}$ as indicated below:

$$\begin{array}{rcl}
 & 0 \ \Delta & 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1 \\
+ & 1 \ \Delta & 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1 \\
\hline
 & 1 \ \Delta & 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0
\end{array}$$

Next we apply the final corection of adding $0_{\Delta}0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1$ to $\tilde{x} - A$:

$$\begin{array}{r}
1\ _{\Delta}\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0 \\
+\,0\ _{\Delta}\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1 \\
\hline
1\ _{\Delta}\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1
\end{array}$$

Hence the result of the product $0_{\Delta}1\ 1\ 1\ 0\ 1 \times 1_{\Delta}1\ 0\ 1\ 1\ 1$ is given by $1_{\Delta}1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1$.

The decimal equivalent is $0.90625_{10} \times (-0.25_{10}) = -0.22265625_{10}$.

(b) In this case the multiplicand is negative. Hence we follow the same steps as on Page 547 of Text except the addition is now ones'-complement addition:

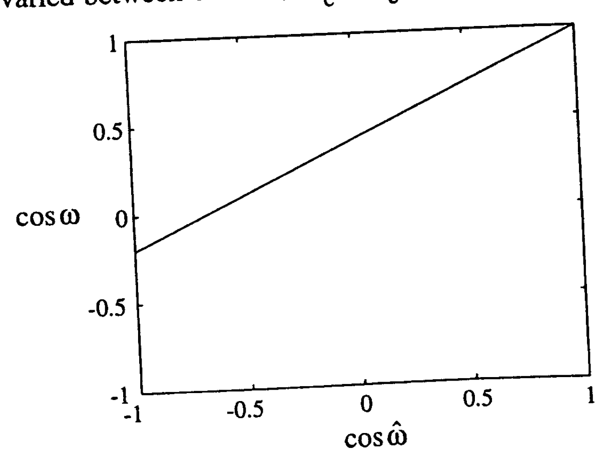
$$\begin{array}{ll}
 & 1\ _{\Delta}\ 1\ 0\ 1\ 0\ 1 \\
 & \times\ 0\ _{\Delta}\ 1\ 0\ 1\ 1\ 1 \\
\hline
P^{(0)} & 0\ _{\Delta}\ 0\ 0\ 0\ 0\ 0 \\
 & +\,1\ _{\Delta}\ 1\ 0\ 1\ 0\ 1 \\
\hline
 & 1\ _{\Delta}\ 1\ 0\ 1\ 0\ 1 \\
\hline
P^{(1)} & 1\ _{\Delta}\ 1\ 1\ 0\ 1\ 0\ 1 \\
 & +\,1\ _{\Delta}\ 1\ 0\ 1\ 0\ 1 \\
\hline
 & 1\ 1\ _{\Delta}\ 0\ 1\ 1\ 1\ 1\ 1 \\
 & \quad\cdots\cdots\cdots\rightarrow 1 \\
\hline
 & 1\ _{\Delta}\ 1\ 0\ 0\ 0\ 0\ 1 \\
\hline
P^{(2)} & 1\ _{\Delta}\ 1\ 1\ 0\ 0\ 0\ 0\ 1 \\
 & +\,1\ _{\Delta}\ 1\ 0\ 1\ 0\ 1 \\
\hline
 & 1\ 1\ _{\Delta}\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\
 & \quad\cdots\cdots\cdots\rightarrow 1 \\
\hline
 & 1\ _{\Delta}\ 0\ 1\ 1\ 1\ 0\ 0\ 1 \\
\hline
P^{(3)} & 1\ _{\Delta}\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1 \\
\hline
P^{(4)} & 1\ _{\Delta}\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1 \\
 & +\,1\ _{\Delta}\ 1\ 0\ 1\ 0\ 1 \\
\hline
 & 1\ 1\ _{\Delta}\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1 \\
 & \quad\cdots\cdots\cdots\rightarrow 1 \\
\hline
 & 1\ _{\Delta}\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \\
\hline
P^{(5)} & 1\ _{\Delta}\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1
\end{array}$$

Hence the result of the product $1_{\Delta}1\ 0\ 1\ 0\ 1 \times 0_{\Delta}1\ 0\ 1\ 1\ 1$ is given by $1_{\Delta}1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1$.

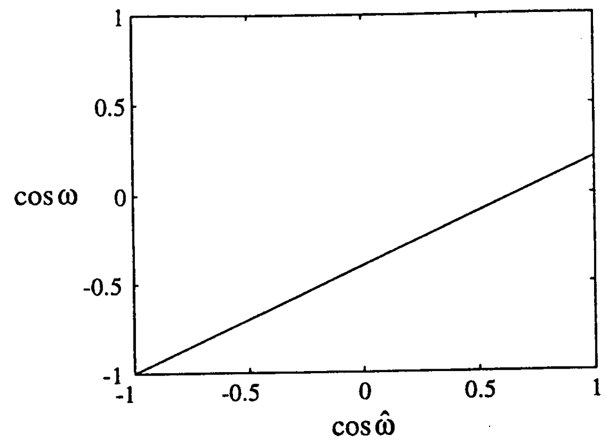
The decimal equivalent is $(-0.3125_{10}) \times 0.71875_{10} = -0.224609375_{10}$.

8.55 The transformation $\cos\omega = \alpha + \beta\cos\hat{\omega}$ is equivalent to $\dfrac{e^{j\omega} + e^{-j\omega}}{2} = \alpha + \beta\left(\dfrac{e^{j\hat{\omega}} + e^{-j\hat{\omega}}}{2}\right)$, which by analytic continuation can be expressed as $\frac{z+z^{-1}}{2} = \alpha + \beta\left(\frac{\hat{z}+\hat{z}^{-1}}{2}\right)$. Now, let H(z) be a Type I linear-phase FIR transfer function of degree 2M. As indicated in Eq. (6.143), H(z) can be expressed as $H(z) = z^{-M}\sum_{n=0}^{M} a[n]\left(\frac{z+z^{-1}}{2}\right)^n$ with a frequency response given by $H(e^{j\omega}) = e^{-jM\omega}\sum_{n=0}^{M} a[n](\cos\omega)^n$, with $\breve{H}(\omega) = \sum_{n=0}^{M} a[n](\cos\omega)^n$ denoting the amplitude function or the zero-phase frequency response. The amplitude function or the zero-phase frequency response of the transformed filter obtained by applying the mapping $\cos\omega = \alpha + \beta\cos\hat{\omega}$ is therefore given by $\breve{H}(\hat{\omega}) = \sum_{n=0}^{M} a[n](\alpha + \beta\cos\hat{\omega})^n$. Or equivalently, the transfer function of the transformed filter is given by $H(\hat{z}) = \hat{z}^{-M}\sum_{n=0}^{M} a[n]\left(\alpha + \beta\frac{\hat{z}+\hat{z}^{-1}}{2}\right)^n$. A convenient way to realize $H(\hat{z})$ is to consider the realization of the parent transfer function H(z) in the form of a Taylor structure as outlined in Problem 6.23 which is obtained by expressing H(z) in the form $H(z) = \sum_{n=0}^{M} a[n] z^{-M+n}\left(\frac{1+z^{-2}}{2}\right)^n$. Similarly, the transfer function of the transformed filter can be expressed as $H(\hat{z}) = \sum_{n=0}^{M} a[n]\hat{z}^{-M+n}\left(\alpha\hat{z}^{-1} + \beta\frac{1+\hat{z}^{-2}}{2}\right)^n$. As a result, the transformed filter can be realized by replacing each block $\frac{1+z^{-2}}{2}$ in the Taylor structure realization of H(z) by the block $\alpha\hat{z}^{-1} + \beta\frac{1+\hat{z}^{-2}}{2}$.

Now, for a lowpass-to-lowpass transformation, we can impose the condition $\breve{H}(\hat{\omega})\big|_{\hat{\omega}=0} = \breve{H}(\omega)\big|_{\omega=0}$. This condition is met if $\alpha + \beta = 1$ and $0 \le \alpha < 1$. In this case, the transformation reduces to $\cos\omega = \alpha + (1-\alpha)\cos\hat{\omega}$. From the plot of the mapping given below it follows that as α is varied between 0 and 1, $\hat{\omega}_c < \omega_c$.

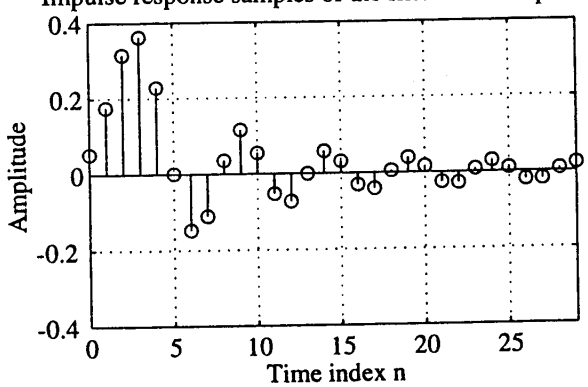


On the other hand if $\hat{\omega}_c > \omega_c$ is desired along with a lowpass-to-lowpass transformation, we can impose the condition $\breve{H}(\hat{\omega})\big|_{\hat{\omega}=\pi} = \breve{H}(\omega)\big|_{\omega=\pi}$. This condition is met if $\beta = 1+\alpha$ and $-1 < \alpha \le 0$. The corresponding transformation is now given by $\cos\omega = \alpha + (1+\alpha)\cos\hat{\omega}$. From the plot of the mapping given below it follows that as α is varied between -1 and 0, $\hat{\omega}_c > \omega_c$.



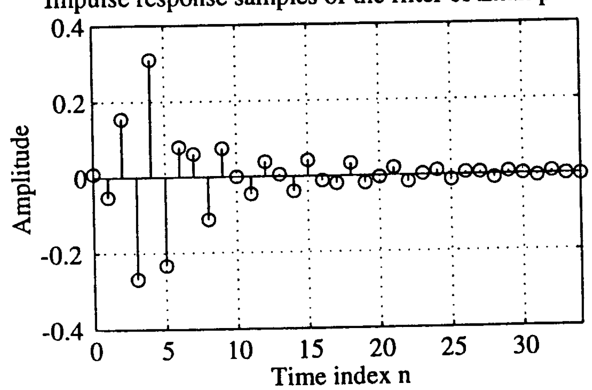
M8.1

Impulse response samples of the filter of Example 7.19

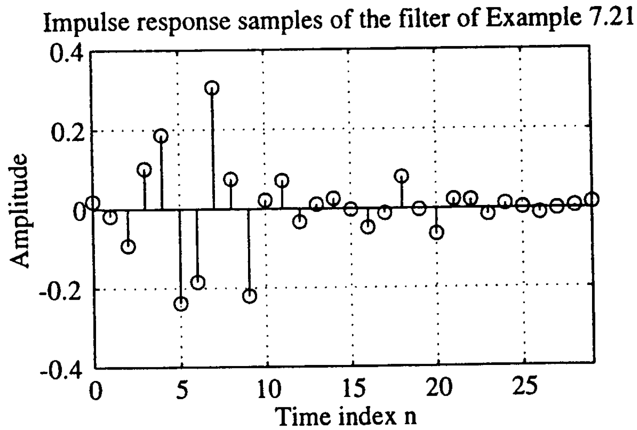


M8.2

Impulse response samples of the filter of Example 7.20



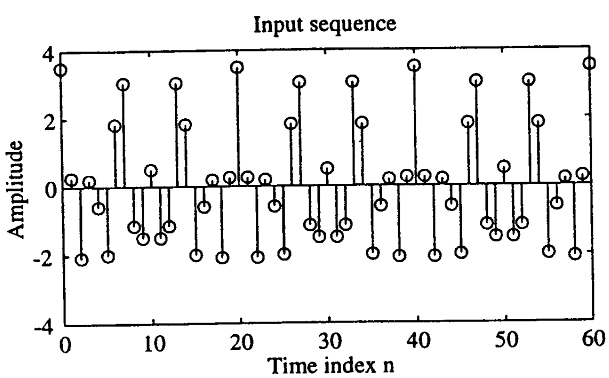
M8.3

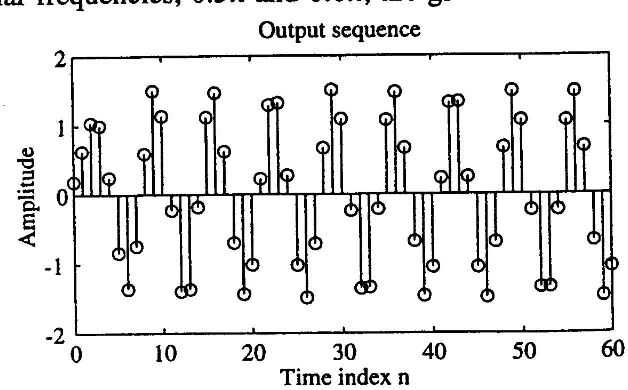


M8.4 Modified Program 8_2 is given below:

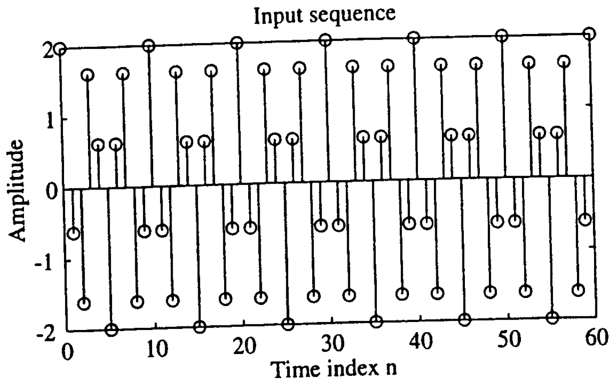
```
k = 0:60;
w = input('Angular frequency vector = ');
num = input('Numerator coefficients = ');
den = input('Denominator coefficients = ');
A = 1.5;B = 2.0;
x1 = A*cos(w(1)*k); x2 = B*cos(w(2)*k);
x = x1+x2;
subplot(2,1,1);
stem(k,x);
title('Input sequence');
xlabel('Time index n');ylabel('Amplitude');
order = max(length(num),length(den));
si = [zeros(1, order-1)];
y = filter(num,den,x,si);
subplot(2,1,2);
stem(k,y);
title('Output sequence');
xlabel('Time index n'); ylabel('Amplitude');
```

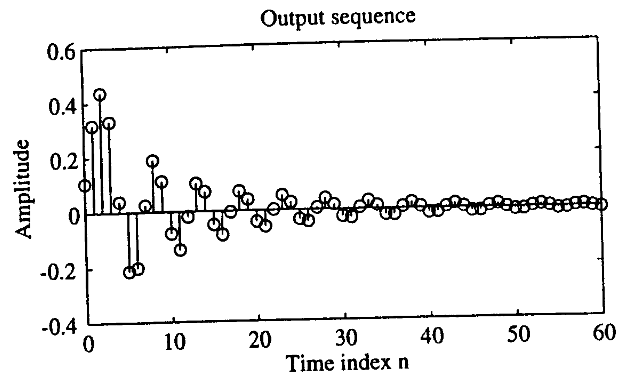
The plots generated by this program for the filter of Example 7.19 for an input composed of a sum of two sinusoidal sequences of angular frequencies, 0.3π and 0.6π, are given below:



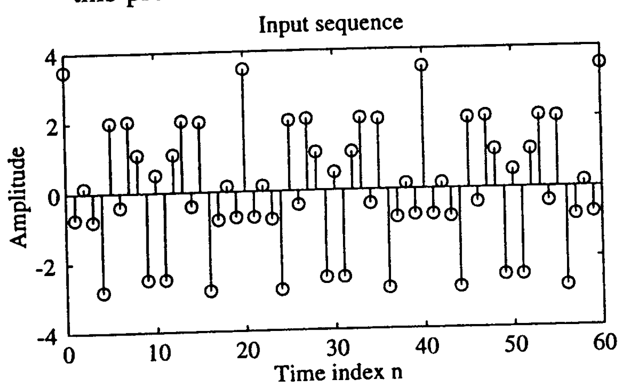


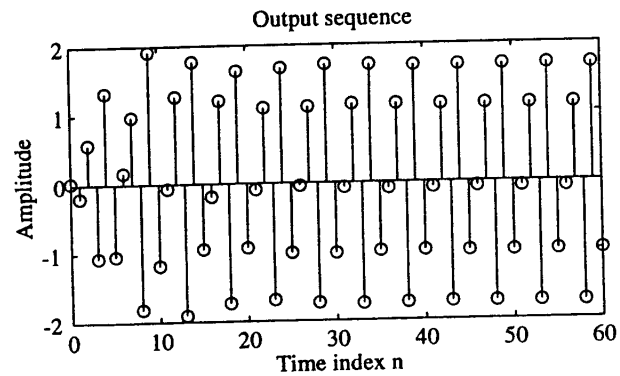
The blocking of the high-frequency signal by the lowpass filter can be demonstrated by replacing the statement `y = filter(num,den,x,si)` in the above program with the following: `y = filter(num,den,x2,si)`. The plots of the input high-frequency signal component and the corresponding output are indicated below:



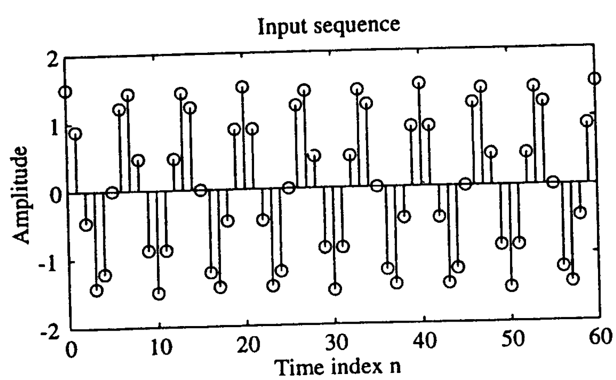


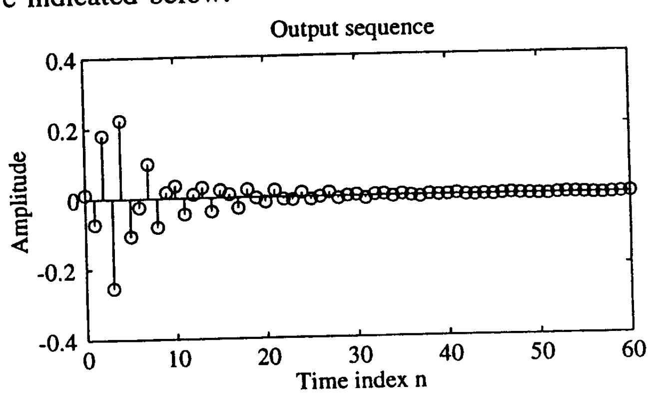
M8.5 The plots generated by using the modified program of Problem M8.4 and using the data of this problem are shown below:





The blocking of the low-frequency signal by the highpass filter can be demonstrated by replacing the statement `y = filter(num,den,x,si)` in the above program with the following: `y = filter(num,den,x1,si)`. The plots of the input high-frequency signal component and the corresponding output are indicated below:





M8.6
```
% The factors for the transfer of example 7.16 are
% Num1 = [0.2546     0.2546         0]
% Den1 = [1.0000    -0.4909         0]
% Num2 = [0.2982     0.1801    0.2982]
% Den2 = [1.0000    -0.7624    0.5390]
% Num3 = [0.6957    -0.0660    0.6957]
% Den3 = [1.0000    -0.5574    0.8828]
N = input('The total number of sections =');
for k = 1:N;
    num(k,:) = input('The numerator =');
```

```
        den(k,:) = input('The denominator =');
    end
    A = 1.5;B = 2.0;
    k = 1:51;
    w1 = 0.3*pi;w2 = 0.6*pi;
    x1 = A*cos(w1*(k-1));x2 = B*cos(w2*(k-1));
    x = x1+x2;
    si = [0 0];
    for k = 1:N
        y(k,:) = filter(num(k,:),den(k,:),x,si);
        x = y(k,:);
    end
    k = 1:51;
    stem(k-1,x);axis([0 50 -4 4]);
    xlabel('Time index n'); ylabel('Amplitude');
    title('Output sequence')
```

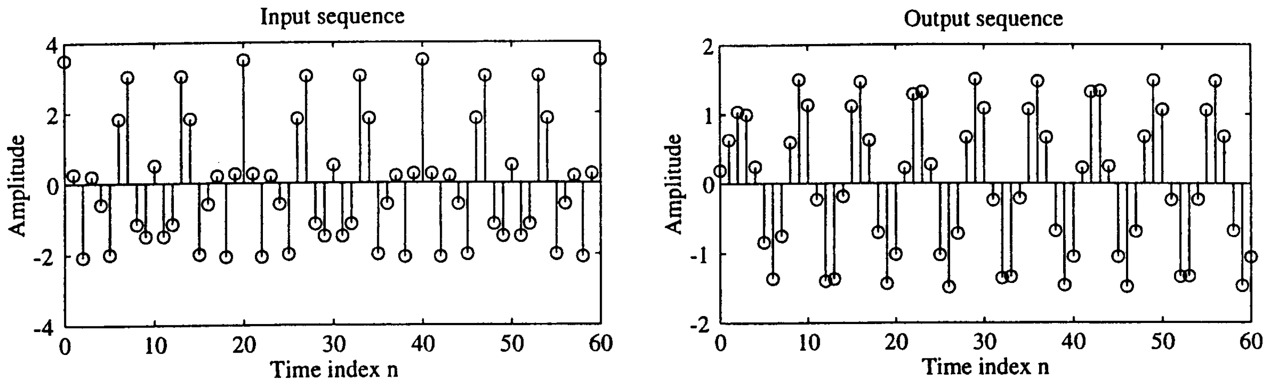
M8.7

```
%The factors for the highpass filter are
 % Num1 = [0.0495   -0.1006    0.0511]
 % Den1 = [1.0000    1.3101    0.5151]
 % Num2 = [0.1688   -0.3323    0.1636]
 % Den2 = [1.0000    1.0640    0.7966]
```

M8.8 The M-file function `filter2` to implement an IIR causal digital filter in direct form II is given below:

```
function [y,sf] = filter2(num,den,x,si);
% Y = FILTER2(P, D, X) filters the data in vector X with the
% filter described by vectors P and D to create the filtered
% data Y.  The filter is a "Direct Form II" implementation
% of the difference equation:
%
% y(n) = p(1)*x(n) + p(2)*x(n-1) + ... + p(np+1)*x(n-np)
%                - d(2)*y(n-1) - ... - d(nd+1)*y(n-nd)
%
% [Y,SF] = FILTER2(P,D,X,SI) gives access to initial and
% final conditions, SI and SF, of the delays.
%
dlen = length(den); nlen = length(num);
N = max(dlen,nlen); M = length(x);
sf = zeros(1,N-1); y = zeros(1,M);
if nargin ~= 3,
     sf = si;
end
if dlen < nlen,
     den = [den zeros(1,nlen - dlen)];
     else
     num = [num zeros(1, dlen - nlen)];
end
num = num/den(1);
den = den/den(1);
for n = 1:M;
wnew = [1 -den(2:N)]*[x(n) sf]';
K = [wnew sf];
y(n) = K*num';
sf = [wnew sf(1:N-2)];
end
```

To apply this function to filter a sum of two sinusoidal sequences, we replace the statement `y = filter(num,den,x,si)` in the MATLAB program given in the solution of Problem M8.4 with the statement `y = filter2(num,den,x,si)`. The plots generated by the modified program for the data given in this problem are given below:



M8.9 The MATLAB program that can be used to compute all DFT samples using the function gfft and the function fft is as follows:

```
clear
N = input('Desired DFT length = ');
x = input('Input sequence = ');
for j = 1:N
    Y(j) = gfft(x,N,j-1);
end
disp('DFT samples computed using gfft are ');
disp(Y);
disp('DFT samples computed using fft are ');
X = fft(x,N); disp(X);
```

Results obtained for the computation of two input sequences {x[n]} of lengths 8, and 12, respectrively, are given below:

```
Desired DFT length = 8
Input sequence = [1 2 3 4 4 3 2 1]

FFT values computed using gfft are
Columns 1 through 4
20.0000     -5.8284 + 2.4142i     0     -0.1716 + 0.4142i
Columns 5 through 8
     0     -0.1716 - 0.4142i     0     -5.8284 - 2.4142i

FFT values using fft are
Columns 1 through 4
20.0000     -5.8284 - 2.4142i     0     -0.1716 - 0.4142i
Columns 5 through 8
     0     -0.1716 + 0.4142i     0     -5.8284 + 2.4142i

Desired DFT length = 12
Input sequence = [2 4  8  12 1 3 5  7  9  6 0 1]

FFT values computed using gfft are
Columns 1 through 4
58.0000     -8.3301 + 5.5000i     -12.5000 +19.9186i
-1.0000 - 7.0000i
Columns 5 through 8
```

```
8.5000 - 7.7942i     0.3301 + 5.5000i     -8.0000
0.3301 - 5.5000i
Columns 9 through 12
8.5000 + 7.7942i     -1.0000 + 7.0000i     -12.5000
-19.9186i   -8.3301 - 5.5000i

FFT values using fft are
Columns 1 through 4
58.0000      -8.3301 - 5.5000i     -12.5000 -19.9186i
-1.0000 + 7.0000i
Columns 5 through 8
8.5000 + 7.7942i     0.3301 - 5.5000i     -8.0000 + 0.0000i
0.3301 + 5.5000i
Columns 9 through 12
8.5000 - 7.7942i     -1.0000 - 7.0000i     -12.5000
+19.9186i   -8.3301 + 5.5000i
```

M8.10 The MATLAB program that can be used to verify the plots of Figure 8.37 is given below:

```
[z,p,k] = ellip(5,0.5,40,0.4);
a = conv([1 -p(1)],[1 -p(2)]);b =[1 -p(5)];
c = conv([1 -p(3)],[1 -p(4)]);
w = 0:pi/255:pi;
alpha = 0;
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;
an2 = a(3) + (a(3) -1)*a(2)*alpha;
g = b(2) - (1 - b(2)*b(2))*alpha;
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;
cn2 = c(3) + (c(3) -1)*c(2)*alpha;
a = [1 an1 an2];b = [1 g]; c = [1 cn1 cn2];
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);
h3 = freqz(fliplr(c),c,w);
ha = 0.5*(h1.*h2 + h3);ma = 20*log10(abs(ha));
alpha = 0.1;
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;
an2 = a(3) + (a(3) -1)*a(2)*alpha;
g = b(2) - (1 - b(2)*b(2))*alpha;
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;
cn2 = c(3) + (c(3) -1)*c(2)*alpha;
a = [1 an1 an2];b = [1 g]; c = [1 cn1 cn2];
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);
h3 = freqz(fliplr(c),c,w);
hb = 0.5*(h1.*h2 + h3);mb = 20*log10(abs(hb));
alpha = -0.25;
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;
an2 = a(3) + (a(3) -1)*a(2)*alpha;
g = b(2) - (1 - b(2)*b(2))*alpha;
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;
cn2 = c(3) + (c(3) -1)*c(2)*alpha;
a = [1 an1 an2];b = [1 g]; c = [1 cn1 cn2];
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);
h3 = freqz(fliplr(c),c,w);
hc = 0.5*(h1.*h2 + h3);mc = 20*log10(abs(hc));
plot(w/pi,ma,'r-',w/pi,mb,'b--',w/pi,mc,'g-.');axis([0 1 -80
5]);
xlabel('Normalized frequency');ylabel('Gain, dB');
legend('b--','alpha = 0.1 ','w','','r-','alpha = 0
','w','','g-.','alpha = -0.25 ');
```

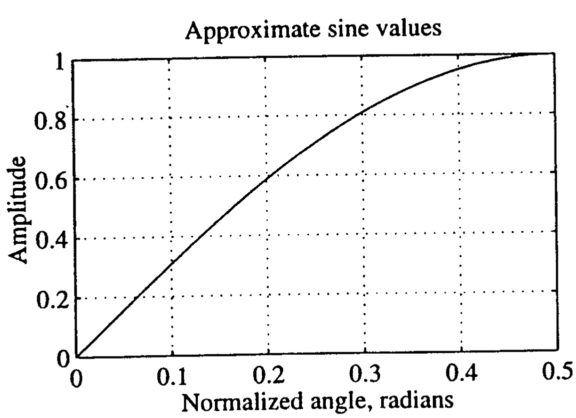
M8.11 The MATLAB program that can be used to verify the plots of Figure 8.39 is given below:

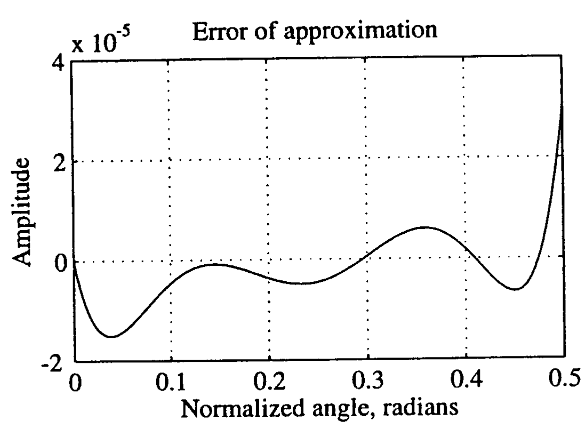
```
w = 0:pi/255:pi;
wc2 = 0.31*pi;
f = [0 0.36 0.46 1];m = [1 1 0 0];
b1 = remez(50, f, m);
h1 = freqz(b1,1,w);
m1 = 20*log10(abs(h1));
n = -25:-1;
c = b1(1:25)./sin(0.41*pi*n);
d = c.*sin(wc2*n);q = (b1(26)*wc2)/(0.4*pi);
b2 = [d q fliplr(d)];
h2 = freqz(b2,1,w);
m2 = 20*log10(abs(h2));
wc3 = 0.51*pi;
d = c.*sin(wc3*n);q = (b1(26)*wc3)/(0.4*pi);
b3 = [d q fliplr(d)];
h3 = freqz(b3,1,w);
m3 = 20*log10(abs(h3));
plot(w/pi,m1,'r-',w/pi,m2,'b--',w/pi,m3,'g-.');
axis([0 1 -80 5]);
xlabel('Normalized frequency');ylabel('Gain, dB');
legend('b--','wc = 0.31π','w','','r-','wc = 0.41π','w','','g-
.','wc = 0.51π')
```

M8.12 The MATLAB program to evaluate Eq. (8.104) is given below:

```
x = 0:0.001:0.5;
y = 3.140625*x + 0.0202636*x.^2 - 5.325196*x.^3 +
0.5446778*x.^4 + 1.800293*x.^5;
x1 = pi*x;
z = sin(x1);
plot(x,y);xlabel('Normalized angle,
radians');ylabel('Amplitude');
title('Approximate sine values');grid;axis([0 0.5 0 1]);
pause
plot(x,y-z);xlabel('Normalized angle,
radians');ylabel('Amplitude');
title('Error of approximation');grid;
```

The plots generated by the above program are as indicated below:

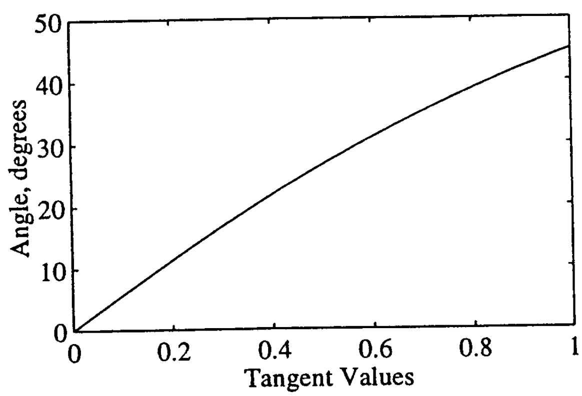


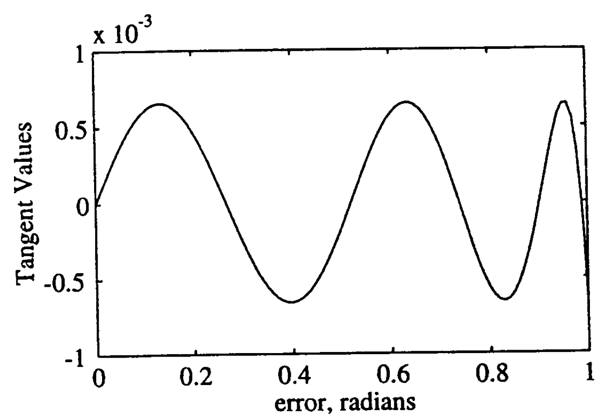


M8.13 The MATLAB program to evaluate Eq. (8.110) is given below:

```
k = 1;
for x = 0:.01:1
     op1 = 0.318253*x+0.00331*x^2-0.130908*x^3+0.068524*x^4-
0.009159*x^5;
     op2 = 0.999866*x-0.3302995*x^3+0.180141*x^5-
0.085133*x^7+0.0208351*x^9;
     arctan1(k) = op1*180/pi;
     arctan2(k) = 180*op2/pi;
     actual(k) = atan(x)*180/pi;
    k = k+1;
end
subplot(211)
x = 0:.01:1;
plot(x,arctan2);
ylabel('Angle, degrees');
xlabel('Tangent Values');
subplot(212)
plot(x,actual-arctan2,'--');
ylabel('Tangent Values');
xlabel('error, radians');
```

The plots generated by the above program are as indicated below:





Note: Expansion for arctan(x) given by Eq. (8.110) gives the result in normalized radians, i.e. the actual value in radians divided by π.

Chapter 9

9.1 Two's Complement Truncation - Assume x > 0. The relative error e_t is given by

$$e_t = \frac{Q(x)-x}{x} = \frac{Q(M)-M}{M} = \frac{\sum_{i=1}^{b} a_{-i}2^{-i} - \sum_{i=1}^{\beta} a_{-i}2^{-i}}{M} = \frac{-\sum_{i=b+1}^{\beta} a_{-i}2^{-i}}{M}.$$

Now e_t will be a minimum if all a_{-i}'s are 1 and will be a maximum if all a_{-i}'s are 0 for $b+1 \le i \le \beta$,

Thus $-\frac{\delta}{M} \le e_t \le \frac{0}{M}$. Since $0.5 \le M \le 1$ hence $-2\delta \le e_t \le 0$.

Now consider x < 0. Here, the relative error e_t is given by

$$e_t = \frac{Q(x)-x}{x} = \frac{Q(M)-M}{M} = \frac{-1+\sum_{i=1}^{b} a_{-i}2^{-i} + 1 - \sum_{i=1}^{\beta} a_{-i}2^{-i}}{M}.$$ As before, $-\frac{\delta}{M} \le e_t \le \frac{0}{M}$. In this case $-1 < M \le -0.5$, and, hence $0 \le e_t \le 2\delta$.

Ones' Complement Truncation - Assume again x > 0. The relative error e_t is given by

$$e_t = \frac{Q(x)-x}{x} = \frac{Q(M)-M}{M} = \frac{\sum_{i=1}^{b} a_{-i}2^{-i} - \sum_{i=1}^{\beta} a_{-i}2^{-i}}{M} = \frac{-\sum_{i=b+1}^{\beta} a_{-i}2^{-i}}{M}.$$

Now e_t will be a minimum if all a_{-i}'s are 1 and will be a maximum if all a_{-i}'s are 0 for $b+1 \le i \le \beta$,

Thus $-\frac{\delta}{M} \le e_t \le \frac{0}{M}$. Since $0.5 \le M \le 1$ hence $-2\delta \le e_t \le 0$.

Now consider x < 0. Here, the relative error e_t is given by

$$e_t = \frac{Q(x)-x}{x} = \frac{Q(M)-M}{M} = \frac{-(1-2^{-b}) + \sum_{i=1}^{b} a_{-i}2^{-i} + (1-2^{-\beta}) - \sum_{i=1}^{\beta} a_{-i}2^{-i}}{M}$$

$$= \frac{(2^{-b}-2^{-\beta}) + \sum_{i=b+1}^{b} a_{-i}2^{-i}}{M}.$$

Now e_t will be a maximum if all a_{-i}'s are 0 and will be a minimum if all a_{-i}'s are 1, In this case,

$\frac{0}{M} < e_t \le \frac{\delta}{M}$. Since, $-0.5 \le M < -1$, hence $-2\delta < e_t \le 0$.

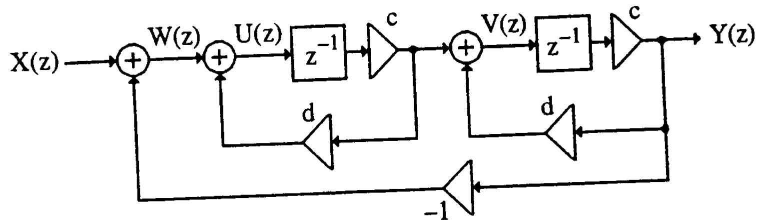
Sign-Magnitude Truncation - Assume x > 0. Here,

$$e_t = \frac{Q(M)-M}{M} = \frac{-\sum_{i=1}^{b} a_{-i}2^{-i} + \sum_{i=1}^{\beta} a_{-i}2^{-i}}{M} = \frac{\sum_{i=b+1}^{\beta} a_{-i}2^{-i}}{M}.$$ Since, $-1 \le M \le -0.5$, hence, $-2\delta \le e_t < 0$.

Rounding - $\varepsilon = Q(M) - M$. Hence $\frac{-\delta}{2} \le \varepsilon \le \frac{\delta}{2}$. This implies $\frac{-\delta}{2M} \le e_r \le \frac{\delta}{2M}$.

Since $-1 < M \le -0.5$, and, hence $-\delta \le e_r \le \delta$.

9.2 Analysis of the digital filter structure yields



$W(z) = X(z) - Y(z)$,

$U(z) = W(z) + cdz^{-1}U(z)$ which implies $U(z) = \dfrac{W(z)}{1-cdz^{-1}} = \dfrac{X(z)-Y(z)}{1-cdz^{-1}}$,

$cz^{-1}V(z) = Y(z)$ implies $V(z) = \dfrac{zY(z)}{c}$. Also $V(z) = cz^{-1}U(z) + dY(z)$.

Substituting for U(z) and V(z) we get $\dfrac{zY(z)}{c} = cz^{-1}\dfrac{X(z)-Y(z)}{1-cdz^{-1}} + dY(z)$. Thus,

$Y(z)\{1 - cdz^{-1} - cdz^{-1}(1 - cdz^{-1}) + c^2z^{-2}\} = c^2z^{-2}X(z)$, hence

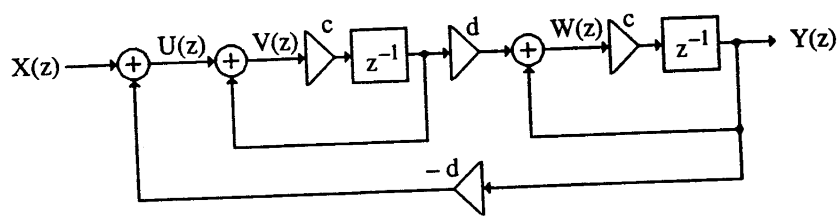
$$H(z) = \frac{Y(z)}{X(z)} = \frac{c^2}{z^2 - 2cdz + c^2(1+d^2)}.$$

Let $c^2(1 + d^2) = r^2$ and $cd = r\cos\theta$, Solving we get $\cos\theta = \dfrac{d}{\sqrt{1+d^2}}$.

Thus, $d = \cot\theta$ and $c = r\sin\theta$. Hence, $\Delta d = -\text{cosec}^2\theta(\Delta\theta)$, and $\Delta c = \sin\theta(\Delta r) + r\cos\theta(\Delta\theta)$.

Or, $\begin{bmatrix}\Delta c\\ \Delta d\end{bmatrix} = \begin{bmatrix}\sin\theta & r\cos\theta\\ 0 & -\text{cosec}^2\theta\end{bmatrix}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix}$, i.e.

$$\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} = -\sin\theta\begin{bmatrix}-\text{cosec}^2\theta & -r\cos\theta\\ 0 & \sin\theta\end{bmatrix}\begin{bmatrix}\Delta c\\ \Delta d\end{bmatrix} = \begin{bmatrix}\frac{1}{\sin\theta} & r\sin\theta\cdot\cos\theta\\ 0 & -\sin^2\theta\end{bmatrix}\begin{bmatrix}\Delta c\\ \Delta d\end{bmatrix}$$



Analysis of the above structure yields, $U(z) = X(z) - dY(z)$

$V(z) = U(z) + cz^{-1}V(z)$ which implies $V(z) = \dfrac{U(z)}{1-cz^{-1}}$.

$W(z) = cdz^{-1}V(z) + cz^{-1}W(z)$ or $W(z) = \dfrac{cdz^{-1}}{1-cz^{-1}}V(z) = \dfrac{cdz^{-1}}{(1-cz^{-1})^2}U(z)$.

$Y(z) = cz^{-1}W(z)$ or $\dfrac{Y(z)}{cz^{-1}} = \dfrac{cdz^{-1}(X(z)-dY(z))}{(1-cz^{-1})^2}$.

Thus $Y(z)\{(1 - cz^{-1})^2 + c^2dz^{-2}\} = c^2dz^{-2}X(z)$. Hence,

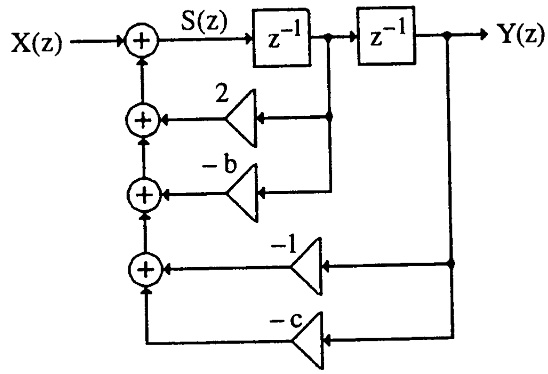
$$H(z) = \frac{Y(z)}{X(z)} = \frac{c^2d}{z^2 - 2cz + c^2(1+d^2)}.$$

Again let $c = r\cos\theta$ and $c^2(1 + d^2) = r^2$.

Therefore, $1+d^2=\sec^2\theta$ which implies $d=\tan\theta$. Taking partials we then get
$\Delta c=\cos\theta(\Delta r)-r\sin\theta(\Delta\theta)$, and $\Delta d=\sec^2\theta(\Delta\theta)$. Thus,

$$\begin{bmatrix}\Delta c\\ \Delta d\end{bmatrix}=\begin{bmatrix}\cos\theta & -r\sin\theta\\ 0 & \sec^2\theta\end{bmatrix}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix},\text{ which yields }\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix}=\begin{bmatrix}\frac{1}{\cos\theta} & r\sin\theta\cdot\cos\theta\\ 0 & \cos^2\theta\end{bmatrix}\begin{bmatrix}\Delta c\\ \Delta d\end{bmatrix}.$$

9.3



Analysis yields $X(z) + 2z^{-1}S(z) - b\,z^{-1}S(z) - Y(z) - cY(z) = S(z)$ and $Y(z) = z^{-2}S(z)$. Thus

$$H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-2}}{1+(b-2)z^{-1}+(1+c)z^{-2}}=\frac{1}{z^2+(b-2)z+(1+c)}.$$

Let $1 + c = r^2$ and $2-b=2r\cos\theta$. Taking partials we get
$\Delta c=2r(\Delta r)$, and $-\Delta b=2\cos\theta(\Delta r)-2r\sin\theta(\Delta\theta)$. Equivalently,

$$\begin{bmatrix}\Delta c\\ \Delta b\end{bmatrix}=\begin{bmatrix}2r & 0\\ 2\cos\theta & -2r\sin\theta\end{bmatrix}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix}\text{ or }\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix}=\begin{bmatrix}\frac{1}{2r} & 0\\ \frac{1}{2r^2\tan\theta} & -\frac{1}{2r\sin\theta}\end{bmatrix}\begin{bmatrix}\Delta c\\ \Delta b\end{bmatrix}.$$

From Example 9.2 the pole sensitivities for Figure 9.9 are given by

$$\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta\beta\end{bmatrix}.$$

9.4 (a) For direct form implementation $B(z)=z^3+b_2z^2+b_1z+b_0=(z-z_1)(z-z_2)(z-z_3)$ where $z_1=r_1e^{j\theta_1}$, $z_2=r_1e^{-j\theta_1}$, and $z_3=r_3e^{j\theta_3}$. Thus, $B(z)=(z^2-2r_1\cos\theta_1 z+r_1^2)(z-r_3)=$ $(z^2-1.4461z+0.7857)(z-0.683)$. Thus, $2r_1\cos\theta_1=1.4461$, $r_1^2=0.7857$, $r_3=0.683$, and $\theta_3=0$.

From the above $\cos\theta_1=\dfrac{1.4461}{2\sqrt{0.7857}}=0.8157$.

$$\frac{1}{B(z)}=\frac{1}{(z^2-1.4461z+0.7857)(z-0.683)}$$
$$=\frac{-1.8903-j0.1477}{z-0.7231-j0.5127}+\frac{-1.8903+j0.1477}{z-0.7231+j0.5127}+\frac{3.7807}{z-0.683}.$$

$\mathbf{P}_1=\left[\cos\theta_1 \quad r_1 \quad r_1^2\cos\theta_1\right]=\left[0.8157 \quad 0.8864 \quad 0.6409\right]$,
$\mathbf{Q}_1=\left[-\sin\theta_1 \quad 0 \quad r_1^2\sin\theta_1\right]=\left[-0.5785 \quad 0 \quad 0.4545\right]$,
$R_1 = -1.8903$, $X_1 = -0.1477$,
$\mathbf{P}_3=\left[\cos\theta_3 \quad r_3 \quad r_3^2\cos\theta_1\right]=\left[1 \quad 0.683 \quad 0.4665\right]$, $\mathbf{Q}_3=\left[-\sin\theta_3 \quad 0 \quad r_3^2\sin\theta_3\right]=\left[0 \quad 0 \quad 0\right]$,
$R_3 = 4.7807$, and $X_3 = 0$.

Thus, $\Delta r_1 = \left(-R_1\mathbf{P}_1 + X_1\mathbf{Q}_1\right)\cdot\Delta\mathbf{B} = 1.6274\,\Delta b_0 + 1.6756\,\Delta b_1 + 1.1444\,\Delta b_2,$

$$\Delta\theta_1 = -\frac{1}{r_1}\left(X_1\mathbf{P}_1 + \mathbf{R}_1\mathbf{Q}_1\right)\cdot\Delta\mathbf{B} = -1.0978\,\Delta b_0 + 0.1477\,\Delta b_1 + 1.0760\,\Delta b_2,$$

$$\Delta r_3 = \left(-R_3\mathbf{P}_3 + X_3\mathbf{Q}_3\right)\cdot\Delta\mathbf{B} = -R_3\mathbf{P}_3\cdot\Delta\mathbf{B} = -4.7807\,\Delta b_0 - 3.2652\,\Delta b_1 - 2.2302\,\Delta b_2,$$

$$\Delta\theta_3 = -\frac{1}{r_3}\left(X_3\mathbf{P}_3 + \mathbf{R}_3\mathbf{Q}_3\right)\cdot\Delta\mathbf{B} = 0.$$

(b) Cascade Form: $B(z) = z^3 + b_2z^2 + b_1z + b_0 = (z^2 + c_1z + c_0)(z + d_0) = B_1(z)B_2(z)$ where $B_1(z) = z^2 + c_1z + c_0 = (z - z_1)(z - z_2) = (z - r_1e^{j\theta_1})(z - r_1e^{-j\theta_1}) = z^2 - 2r_1\cos\theta_1 z + r_1^2$ and $B_2(z) = z + d_0 = z - r_3e^{j\theta_3}$. Comparing with the denominator of the given transfer function we **get** $2r_1\cos\theta_1 = 1.4461$, $r_1^2 = 0.7857$, $r_3 = 0.683$, and $\theta_3 = 0$. Hence, $r_1 = 0.8864$ and $\cos\theta_1 = \mathbf{0.8157}$.

Now, $\dfrac{1}{B_1(z)} = \dfrac{-j0.9752}{z - 0.7230 - j0.5127} + \dfrac{j0.9752}{z - 0.7230 + j0.5127}$. Hence, $R_1 = 0$ and $X_1 = -0.9752$.

$\mathbf{P}_1 = \left[\cos\theta_1 \quad r_1\right] = [0.8157 \quad 0.8864]$, and $\mathbf{Q}_1 = \left[-\sin\theta_1 \quad 0\right] = [-0.5785 \quad 0]$. Next, we observe $\dfrac{1}{B_1(z)} = \dfrac{1}{z - 0.683}$. This implies, $R_3 = 1$ and $X_3 = 0$. $\mathbf{P}_3 = \cos\theta_3 = 1$, $\mathbf{Q}_3 = -\sin\theta_3 = 0$. Thus,

$$\Delta r_1 = (-R_1\mathbf{P}_1 + X_1\mathbf{Q}_1)\cdot\left[\Delta c_0 \quad \Delta c_1\right]^T = X_1\mathbf{Q}_1\cdot\left[\Delta c_0 \quad \Delta c_1\right]^T = 0.5642\,\Delta c_0$$

$$\Delta\theta_1 = -\frac{1}{r_1}\left(X_1\mathbf{P}_1 + \mathbf{R}_1\mathbf{Q}_1\right)\cdot\left[\Delta c_0 \quad \Delta c_1\right]^T = -\frac{1}{r_1}X_1\mathbf{P}_1\cdot\left[\Delta c_0 \quad \Delta c_1\right]^T = 0.8974\,\Delta c_0 + 0.8864\,\Delta c_1$$

$$\Delta r_3 = (-R_3\mathbf{P}_3 + X_3\mathbf{Q}_3)\cdot\Delta d_0 = -R_3\mathbf{P}_3\cdot\Delta d_0 = -\Delta d_0$$

$$\Delta\theta_3 = -\frac{1}{r_3}\left(X_3\mathbf{P}_3 + \mathbf{R}_3\mathbf{Q}_3\right)\cdot\Delta d_0 = -\frac{1}{r_3}\mathbf{R}_3\mathbf{Q}_3\cdot\Delta d_0 = 0$$

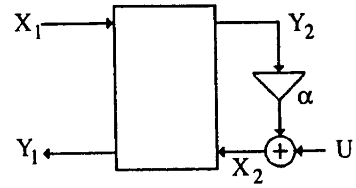
9.5 In terms of transfer parameters, the input-output relation of the two-pair of Figure P9.3 is given by $Y_1 = t_{11}X_1 + t_{12}X_2$, and $Y_2 = t_{21}X_1 + t_{22}X_2$, where $X_2 = \alpha Y_2$. From Eq. (4.156b) we get

$$H(z) = \frac{Y_1(z)}{X_1(z)} = t_{11} + \frac{\alpha t_{12}t_{21}}{1 - \alpha t_{22}}.$$

From the above, $\dfrac{\partial H(z)}{\partial\alpha} = \dfrac{t_{12}t_{21}\left(1 - \alpha t_{22}\right) + t_{22}\alpha t_{12}t_{21}}{\left(1 - \alpha t_{22}\right)^2} = \dfrac{t_{12}t_{21}}{\left(1 - \alpha t_{22}\right)^2}.$

Substituting $X_2 = \alpha Y_2$ in $Y_2 = t_{21}X_1 + t_{22}X_2$ we get the expression for the scaling transfer function

$F_\alpha(z) = \dfrac{Y_2(z)}{X_1(z)} = \dfrac{t_{21}}{1 - \alpha t_{22}}$. Now from the structure given below we observe that the noise transfer function $G_\alpha(z)$ is given by $G_\alpha(z) = \left.\dfrac{Y_1}{U}\right|_{X_1 = 0}$. From the structure we also observe that $X_2 = \alpha Y_2 + U$. Substituting this in the transfer relations we arrive at with $X_1 = 0$, $Y_1 = t_{12}X_2 = t_{12}(\alpha Y_2 + U)$, and $Y_2 = t_{22}X_2 = t_{22}(\alpha Y_2 + U)$. From these two equations we obtain



after some algebra $Y_1 = \frac{\alpha t_{12} t_{22}}{1-\alpha t_{22}} U + t_{12} U = \frac{t_{12}}{1-\alpha t_{22}} U$. Hence the noise transfer function is given by $G_\alpha(z) = \left.\frac{Y_1}{U}\right|_{X_1=0} = \frac{t_{12}}{1-\alpha t_{22}}$. Therefore, $\frac{\partial H(z)}{\partial \alpha} = \frac{t_{21}}{1-\alpha t_{22}} \cdot \frac{t_{12}}{1-\alpha t_{22}} = F_\alpha(z) G_\alpha(z)$.

9.6 (a) $W_M(\omega) = \left[\frac{1}{2M-1}\left(M-1+\frac{\sin(M\omega)}{\sin(\omega)}\right)\right]^{1/2} = \left[\frac{1}{2M-1}\left(1+4\sum_{n=1}^{(M-1)/2}\cos^2(\omega n)\right)\right]^{1/2}$.

Since the maximum value of $\cos^2(\omega n) = 1$ and the minimum value = 0, hence

$\max\{W_M(\omega)\} = \left[\frac{1}{2M-1}\left(1+4\left(\frac{M-1}{2}\right)\right)\right]^{1/2} = 1$, and $\min\{W_M(\omega)\} = \left[\frac{1}{2M-1}\right]^{1/2} > 0$.

Hence $0 < W_M(\omega) \le 1$.

(b) $W_M(0) = \left[\frac{1}{2M-1}\left(1+4\sum_{m=1}^{(M-1)/2} 1\right)\right]^{1/2} = 1$.

$W_M(\pi) = \left[\frac{1}{2M-1}\left(1+4\sum_{m=1}^{(M-1)/2} 1\right)\right]^{1/2} = 1$.

(c) $\lim_{M\to\infty} W_M(\pi) = \left[\lim_{M\to\infty}\frac{M-1}{2M-1} + \lim_{M\to\infty}\frac{1}{2M-1}\frac{\sin(M\omega)}{\sin(\omega)}\right]^{1/2}$.

Since $|\sin(M\omega) \le 1|$, $\lim_{M\to\infty}\frac{1}{2M-1}\frac{\sin(M\omega)}{\sin(\omega)} = 0$, and $\lim_{M\to\infty} W_M(\pi) = \frac{1}{\sqrt{2}}$.

9.7 From Eq. (9.75) $SNR_{A/D} = 6.0206b + 16.81 - 20\log_{10}(K)$ dB. For a constant value of K, an increase in b by 1 bit increases the $SNR_{A/D}$ by 6.0206 dB and an increase in b by 2 bits increases the $SNR_{A/D}$ by 12.0412 dB.

If K = 4 and b = 7 then $SNR_{A/D} = 6.0206\times 7 + 16.81 - 20\log_{10}(4)$ $= 6.0206\times 7 + 16.81 - 20\times 0.6021 = 46.9122$ dB. Therefore for K = 4 and b = 9, $SNR_{A/D}$ = 46.91 + 12.04 = 58.95 dB; for K = 4 and b = 11, $SNR_{A/D}$ = 58.95 + 12.04 = 70.99 dB; for K = 4 and b = 13, $SNR_{A/D}$ = 70.99+ 12.04 = 83.03; for K = 4 and b = 15, $SNR_{A/D}$ = 83.03 + 12.04 = 95.08.

If K = 6 and b = 7 then $SNR_{A/D} = 6.0206\times 7 + 16.81 - 20\log_{10}(6)$ $= 6.0206\times 7 + 16.81 - 20\times 0.7782 = 43.3902$ dB. Therefore for K = 6 and b = 9, $SNR_{A/D}$ = 43.39 + 12.04 = 55.43 dB; for K = 6 and b = 11, $SNR_{A/D}$ = 43.39 + 12.04 = 67.47 dB; for K = 6 and b = 13, $SNR_{A/D}$ = 67.47 + 12.04 = 79.51; for K = 6 and b = 15, $SNR_{A/D}$ = 79.51 + 12.04 = 91.56.

If K = 8 and b = 7 then $SNR_{A/D} = 6.0206\times 7 + 16.81 - 20\log_{10}(8)$

$= 6.0206 \times 7 + 16.81 - 20 \times 0.9031 = 40.8922$ dB. Therefore for K = 6 and b = 9, $SNR_{A/D}$ = 40.89 + 12.04 = 52.93 dB; for K = 6 and b = 11, $SNR_{A/D}$ = 52.93 + 12.04 = 64.97 dB; for K = 6 and b = 13, $SNR_{A/D}$ = 64.97 + 12.04 = 77.01; for K = 6 and b = 15, $SNR_{A/D}$ = 77.01 + 12.04 = 89.05.

9.8 (a) $H_1(z) = \dfrac{(z+1)(z+2)}{(z+\frac{1}{2})(z+\frac{1}{3})} = 1 + \dfrac{-4.5}{z+\frac{1}{2}} + \dfrac{6.667}{z+\frac{1}{3}}$. Making use of Eq. (9.86) and Table 9.4, we get

$$\sigma_{1,n}^2 = 1 + \frac{(4.5)^2}{1-\left(\frac{1}{2}\right)^2} + \frac{(6.667)^2}{1-\left(\frac{1}{3}\right)^2} + \frac{2\times(-4.5)\times 6.667}{1-\frac{1}{6}} = 1 + 27 + 50 - 72 = 6.$$

(b) $H_2(z) = \dfrac{6(5z+2)(0.3z^2+0.6z+1)}{(2z+1)(3z+1)(z^2+0.6z+0.3)} = \dfrac{9.3}{z+\frac{1}{2}} + \dfrac{7.8947}{z+\frac{1}{3}} + \dfrac{-15.6947z - 0.6853}{z^2+0.6z+0.3}$

Again from Eq. (9.86) and Table 9.44, $\sigma_{2,n}^2 = \dfrac{(9.3)^2}{1-\left(\frac{1}{2}\right)^2} + \dfrac{(7.8947)^2}{1-\left(\frac{1}{3}\right)^2} + 2\times\dfrac{9.3\times 7.8947}{1-\frac{1}{6}}$

$$+ \frac{[(15.69)^2+(0.685)^2][1-(0.3)^2] - 2\times(1-0.3)\times 0.6\times 15.69\times 0.685}{[1-(0.3)^2]^2 + 2\times 0.3\times(0.6)^2 - [1+(0.3)^2]\times(0.6)^2}$$

$$+ 2\times\frac{9.3(-15.69 - 0.685\times(-0.5))}{1+0.6\times(-0.5)+0.3\times(-0.5)^2} + 2\times\frac{7.8947(-15.69-0.685\times(-0.333))}{1+0.6\times(-0.333)+0.3\times(-0.333)^2}$$

$= 115.32 + 70.115 + 176.2 + 330.\ 73 - 368.44 - 292.97 = 30.9.$

(c) $H_3(z) = \dfrac{(z+1)^2}{z^2+0.5z+0.8} = 1 + \dfrac{1.5z+0.2}{z^2+0.5z+0.8}$.

$$\sigma_{3,n}^2 = 1 + \frac{[(1.5)^2+(0.2)^2][1-(0.8)^2] - 2\times[0.2\times 1.5 - 1.5\times 0.2\times 0.8]\times 0.5}{[1-(0.8)^2]^2 + 2\times 0.8\times(0.5)^2 - [1+(0.8)^2]\times(0.5)^2} = 1 + 6.3913 = 7.3913.$$

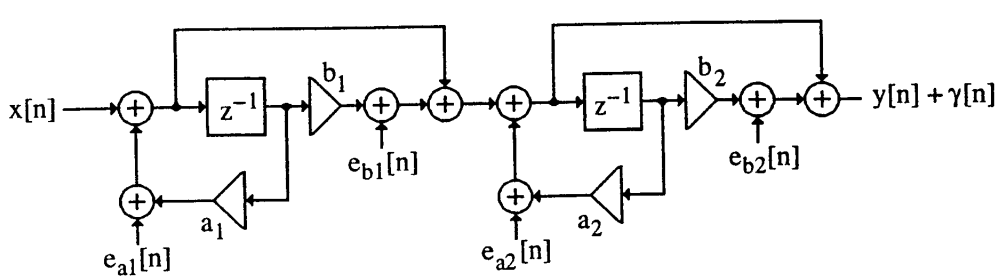
9.9 $H(z) = C + \dfrac{A}{1-\alpha z^{-1}} + \dfrac{B}{1-\beta z^{-1}} = C + \dfrac{Az}{z-\alpha} + \dfrac{Bz}{z-\beta} = C + \dfrac{Az - A\alpha + A\alpha}{z-\alpha} + \dfrac{Bz - B\beta + B\beta}{z-\beta}$

$= (A+B+C) + \dfrac{A\alpha}{z-\alpha} + \dfrac{B\beta}{z-\beta}$. From Eq. (9.86) and using Table 9.4 we obtain

$\sigma_{v,n}^2 = (A+B+C)^2 + \dfrac{(A\alpha)^2}{1-\alpha^2} + \dfrac{(B\beta)^2}{1-\beta^2} + \dfrac{2A\alpha B\beta}{1-\alpha\beta}$. For the values given, we get

$$\sigma_{v,n}^2 = (7)^2 + \frac{(1.8)^2}{1-(0.9)^2} + \frac{(-2.4)^2}{1-(-0.8)^2} + \frac{2(1.8)(-2.4)}{1-(0.9)(-0.8)} = 49.00 + 17.0526 + 16.00 - 5.0233 = 77.0293$$

9.10 For quantization before addition the noise model is



Let $G_1(z)$ denote the noise transfer function due to $e_{a1}[n]$, $G_2(z)$ denote the noise transfer function due to $e_{b1}[n]$ and $e_{a2}[n]$, and $G_3(z)$ denote the noise transfer function due to $e_{b2}[n]$. Let $\sigma_{1,n}^2$ denote the normalized output noise variance due to $e_{a1}[n]$, $\sigma_{2,n}^2$ denote the normalized output noise variance due to $e_{b1}[n]$ and $e_{a2}[n]$, and $\sigma_{3,n}^2$ denote the normalized output noise variance due to $e_{b2}[n]$.

Structure I: $a_1 = -\frac{1}{3}$, $b_1 = 3$, $a_2 = -\frac{1}{2}$, and $b_2 = 5$.

Here, $G_1(z) = \dfrac{(z+3)(z+5)}{(z+\frac{1}{3})(z+\frac{1}{2})} = 1 + \dfrac{74.67}{z+\frac{1}{3}} - \dfrac{67.5}{z+\frac{1}{2}}$. Thus,

Thus, the normalized output noise variance due to $e_{a1}[n]$ is

$$\sigma_{1,n}^2 = 1 + \frac{(74.67)^2}{1-\frac{1}{9}} + \frac{(67.5)^2}{1-\frac{1}{4}} + \frac{2\times 74.67\times(-67.5)}{1-\frac{1}{6}} = 252.$$

Next, $G_2(z) = \dfrac{z+5}{z+\frac{1}{2}} = 1 + \dfrac{4.5}{z+\frac{1}{2}}$. Thus, $\sigma_{2,n}^2 = 1 + \dfrac{(4.5)^2}{1-\frac{1}{4}} = 28.$

Finally, $G_3(z) = 1$, hence $\sigma_{3,n}^2 = 1$.

Therefore total normalized noise variance at the output = $\sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{3,n}^2 = 309.$

Structure II: $a_1 = -\frac{1}{2}$, $b_1 = 3$, $a_2 = -\frac{1}{3}$, and $b_2 = 5$.

Here, $G_1(z) = \dfrac{(z+3)(z+5)}{(z+\frac{1}{3})(z+\frac{1}{2})} = 1 + \dfrac{74.67}{z+\frac{1}{3}} - \dfrac{67.5}{z+\frac{1}{2}}$. Therefore, $\sigma_{1,n}^2 = 252.$

Next $G_2(z) = \dfrac{z+5}{z+1/3} = 1 + \dfrac{14/3}{z+1/3}$. Thus, $\sigma_{2,n}^2 = 1 + \dfrac{(14/3)^2}{1-\frac{1}{9}} = 25.5.$

Finally, $G_3(z) = 1$, hence $\sigma_{3,n}^2 = 1$.

Therefore total normalized noise variance at the output = $\sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{3,n}^2 = 304.$

Structure III: $a_1 = -\frac{1}{2}$, $b_1 = 5$, $a_2 = -\frac{1}{3}$, and $b_2 = 3$.

Here $G_1(z)$ is the same as in the earlier cases. Hence, $\sigma_{1,n}^2 = 252$.

Next, $G_2(z) = \dfrac{z+3}{z+1/3} = 1 + \dfrac{8/3}{z+1/3}$. Thus, $\sigma_{2,n}^2 = 1 + \dfrac{(8/3)^2}{1-\frac{1}{9}} = 9.$ Finally, $G_3(z) = 1$, hence $\sigma_{3,n}^2 = 1$. Therefore total normalized noise variance at the output = $\sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{3,n}^2 = 271.$

Structure IV: $a_1 = -\frac{1}{3}$, $b_1 = 5$, $a_2 = -\frac{1}{2}$, and $b_2 = 3$.

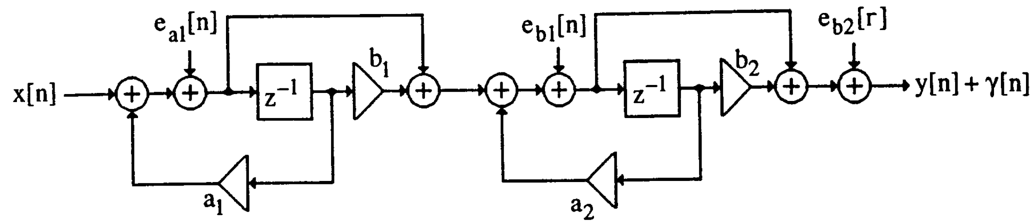
$G_1(z)$ is the same as before, hence $\sigma_{1,n}^2 = 252$.

$G_2(z) = \dfrac{z+3}{z+\frac{1}{2}} = 1 + \dfrac{\frac{5}{2}}{z+\frac{1}{2}}$. Thus, $\sigma_{2,n}^2 = 1 + \dfrac{(5/2)^2}{1-\frac{1}{4}} = 9.333.$

Therefore total normalized noise variance at the output = $\sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{3,n}^2 = 271.666.$

Structure III has the lowest variance amongst all four structures.

For quantization after addition the noise model is



Here the transfer function from noise sources $e_{a1}[n]$, $e_{b1}[n]$ and $e_{b2}[n]$ are the same as that calculated above, the only difference is that in this case there is a single noise source $e_{b1}[n]$ due to the product roundoff caused by the multipliers b_1 and a_2.

Structure I: Total normalized noise variance at the output = $\sigma_{1,n}^2+\sigma_{2,n}^2+\sigma_{3,n}^2=281$.

Structure II: Total normalized noise variance at the output = $\sigma_{1,n}^2+\sigma_{2,n}^2+\sigma_{3,n}^2=278.5$.

Structure III: Total normalized noise variance at the output = $\sigma_{1,n}^2+\sigma_{2,n}^2+\sigma_{3,n}^2=262$.

Structure IV: Total normalized noise variance at the output = $\sigma_{1,n}^2+\sigma_{2,n}^2+\sigma_{3,n}^2=262.33$.

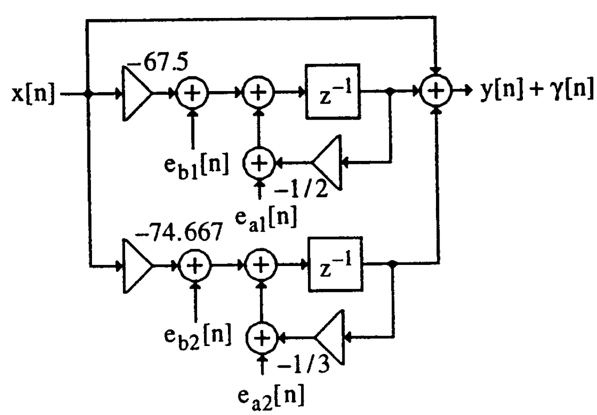
In this case also Structure III has the lowest normalized noise variance.

9.11 Quantization of products before addition

Parallel form II is obtained by performing a partial fraction of H(z) in positive powers of z: (See Section 6.5.3)

$$H(z) = \frac{(z+3)(z+5)}{(z+\frac{1}{3})(z+\frac{1}{2})} = 1+\frac{-67.5}{z+\frac{1}{2}}+\frac{74.667}{z+\frac{1}{3}}$$

The noise model for this structure is shown as below



Now the noise transfer function from the noise sources $e_{a1}[n]$ and $e_{b1}[n]$ to the output is $G_1(z) = \frac{1}{z+\frac{1}{2}}$. Thus the normalized output noise variance from these error sources is given by

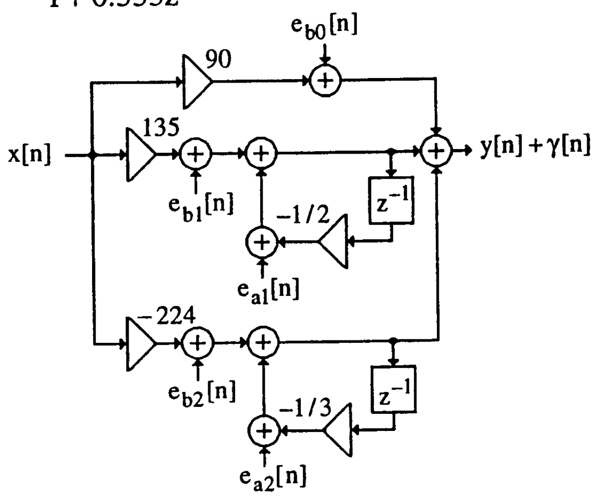
$$\sigma_{\gamma 1,n}^2 = 2\left[\frac{1}{2\pi j}\oint G_1(z)G_1(z^{-1})z^{-1}dz\right] = 2\left(\frac{1}{1-(0.5)^2}\right) = 2.6667.$$

The noise transfer function from the noise sources $e_{a2}[n]$ and $e_{b2}[n]$ to the output is $G_2(z) = \frac{1}{z+\frac{1}{3}}$. Thus the normalized output noise variance due to these noise sources is given by

$\sigma^2_{\gamma 2,n} = 2\left[\frac{1}{2\pi j}\oint G_2(z)G_2(z^{-1})z^{-1}dz\right] = 2\left(\frac{1}{1-(1/3)^2}\right) = 2.25.$ Therefore the total normalized noise variance at the output $= 2.6667 + 2.25 = 4.9167.$

Parallel form I is obtained by performing a partial fraction of H(z) in powers of z^{-1}: (See Section 6.5.3)

$H(z) = 90 + \frac{135}{1+0.5z^{-1}} - \frac{224}{1+0.333z^{-1}}$. The noise model for this structure is



Now the noise transfer function from the noise sources $e_{a1}[n]$ and $e_{b1}[n]$ to the output is $G_3(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = 1 + \frac{\frac{1}{2}}{z-\frac{1}{2}}$. Thus the total normalized quantization noise variance at the output due to these noise sources is given by $\sigma^2_{\gamma 1,n} = 2\left[\frac{1}{2\pi j}\oint G_3(z)G_3(z^{-1})z^{-1}dz\right] = 2\left(1+\frac{(0.5)^2}{1-(0.5)^2}\right) = 2.6667.$

The the noise transfer function from the noise sources $e_{a1}[n]$ and $e_{b1}[n]$ to the output is $G_4(z) = \frac{1}{1-\frac{1}{3}z^{-1}} = 1 + \frac{\frac{1}{3}}{z-\frac{1}{3}}$. The total normalized quantization noise variance at the output due to these noise sources is given by $\sigma^2_{\gamma 2,n} = 2\left[\frac{1}{2\pi j}\oint G_4(z)G_4(z^{-1})z^{-1}dz\right] = 2\left(1+\frac{(1/3)^2}{1-(1/3)^2}\right) = 2.25.$

The normalized noise variance at the output due to the noise source $e_{b0}[n] = 1$.
Thus the total normalized noise variance at the output = 2.6667 + 2.25 + 1 = 5.9167.

<u>Quantization of products after addition</u>
For Parallel Form II total normalized quantization noise variance at the output due to $e_{a1}[n]$ and $e_{b1}[n]$ is given by $\sigma^2_{\gamma 1,n} = \frac{1}{2\pi j}\oint G_1(z)G_1(z^{-1})z^{-1}dz = \frac{1}{1-(0.5)^2} = 1.3333.$

The total normalized quantization noise variance at the output due to $e_{a2}[n]$ and $e_{b2}[n]$ is given by

$$\sigma^2_{\gamma 2,n} = \frac{1}{2\pi j}\oint G_2(z)G_2(z^{-1})z^{-1}dz = \frac{1}{1-(1/3)^2} = 1.125.$$

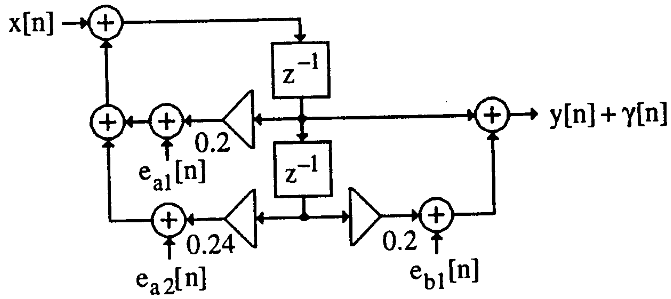
Thus the total noise variance at the output = 1.3333 + 1.125 = 2.4583.

Similarly for Parallel Form I the total normalized quantization noise variance at the output due to $e_{a1}[n]$ and $e_{b1}[n]$ = 1.3333, the total normalized quantization noise variance at the output due to $e_{a1}[n]$ and $e_{b1}[n]$ = 1.125, and the normalized noise variance at the output due to the noise source $e_{b0}[n]$ = 1.

Thus total noise variance at the output = = 1.3333 + 1.125 + 1 = 3.4583.

9.12 $$H(z) = \frac{z^{-1}+0.2z^{-2}}{1-0.2z^{-1}-0.24z^{-2}},$$

Direct Form Realization - The noise model in this case is given by



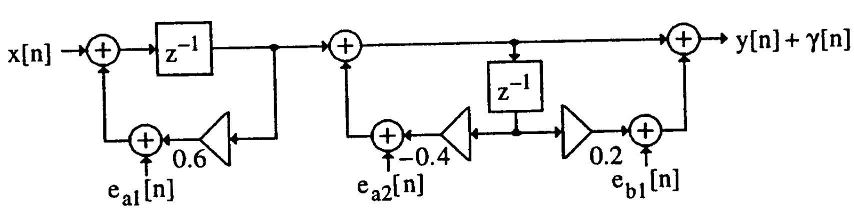
The noise transfer function from noise sources $e_{a1}[n]$, and $e_{a2}[n]$ is $H(z) = \frac{0.8}{z-0.6} + \frac{0.2}{z+0.4}$, while the noise transfer function from the noise source $e_{b1}[n]$ is $H_b(z) = 1$. From Table 9.4, the total normalized noise variance at the output due to $e_{a1}[n]$ and $e_{a2}[n]$ is therefore given by

$$2\left[\frac{1}{2\pi j}\oint_C H(z)H(z^{-1})z^{-1}dz\right] = 2\left[\frac{(0.8)^2}{1-(0.6)^2} + \frac{(0.2)^2}{1-(0.4)^2} + 2\times\frac{0.8\times 0.2}{1+0.24}\right] = 2.6114.$$

The normalized noise variance at the output due to $e_{b1}[n]$ = 1. Therefore the total normalized noise variance at the output is $\sigma^2_{\gamma,n} = 3.6114$.

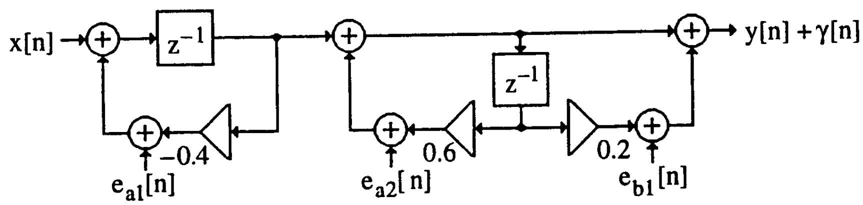
Cascade Form Realization - $H(z) = \frac{z^{-1}(1+0.2z^{-1})}{(1-0.6z^{-1})(1+0.4z^{-1})}$

One possible cascade realization (Cascade Form #1) with the noise sources is shown below.



Now the noise transfer function from $e_{a1}[n]$ to the output is $H(z)$. Thus the normalized noise power at the output due to $e_{a1}[n]$ = 1.3057. The noise transfer function from $e_{a2}[n]$ to the output is $\dfrac{1+0.2z^{-1}}{1+0.4z^{-1}} = 1-\dfrac{0.2}{z+0.4}$. The corresponding normalized noise power at the ouput = $1+\dfrac{(0.2)^2}{1-(0.4)^2}$ = 1.0476. Finally, the noise transfer function from $e_{b1}[n]$ to the output is 1, and hence the corresponding normalized noise power at the output = 1. Therefore the total normalized noise power at the output = 1.3057 + 1.0476 + 1 = 3.3533.

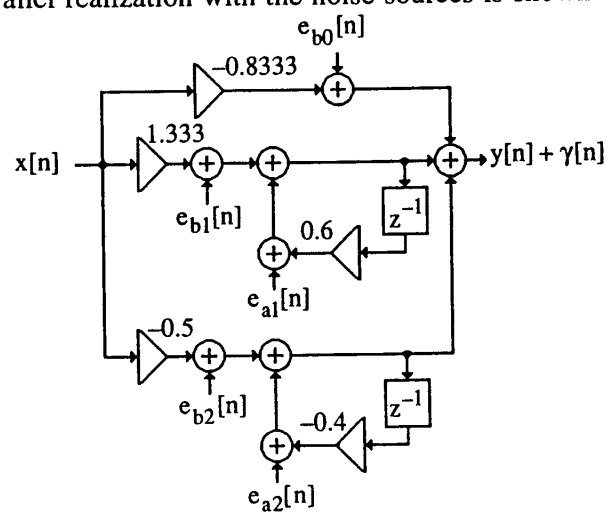
Another possible cascade realization (Cascade Form #2) with the noise sources is shown below.



Now the noise transfer function from $e_{a1}[n]$ to the output is H(z). Thus the normalized noise power at the output due to $e_{a1}[n]$ = 1.3057. The noise transfer function from $e_{a2}[n]$ to the output is $\dfrac{1+0.2z^{-1}}{1-0.6z^{-1}}=1+\dfrac{0.8}{z-0.6}$. The corresponding normalized noise power at the ouput = $1+\dfrac{(0.8)^2}{1-(-0.6)^2}=1+1=2$. Finally, the noise transfer function from $e_{b1}[n]$ to the output is 1, and hence the corresponding normalized noise power at the output = 1. Therefore the total normalized noise power at the output = 1.3057 + 2 + 1 = 4.3057.

Parallel Form I - $H(z)=-0.8333+\dfrac{1.333}{1-0.6z^{-1}}-\dfrac{0.5}{1+0.4z^{-1}}$.

The corresponding parallel realization with the noise sources is shown below.



The noise transfer function from $e_{a1}[n]$ and $e_{b1}[n]$ to the output is $G_a(z) = 1/(1 - 0.6z^{-1})$. Thus the total normalized noise power at the output due to these noise sources is

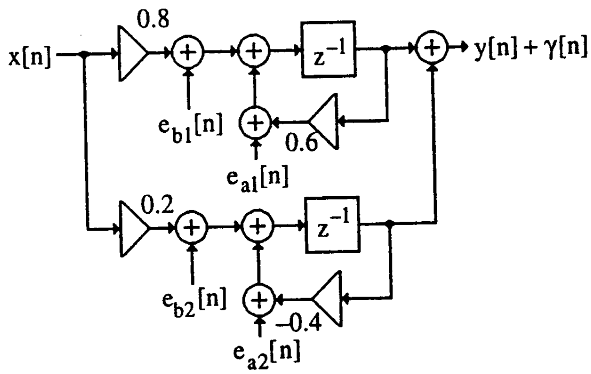
$$2\left[\frac{1}{2\pi j}\oint_C G_a(z)G_a(z^{-1})z^{-1}dz\right]=2\left[1+\frac{(-0.6)^2}{1-(-0.6)^2}\right]=3.125.$$

The noise transfer function from $e_{a1}[n]$ and $e_{b1}[n]$ to the output is $G_b(z) = 1/(1 + 0.4z^{-1})$. Thus the total normalized noise power at the output due to these noise sources is

$$2\left[\frac{1}{2\pi j}\oint_C G_b(z)G_b(z^{-1})z^{-1}dz\right]=2\left[1+\frac{(0.4)^2}{1-(0.4)^2}\right]=2.381.$$

The noise transfer function from the noise sources $e_{b0}[n]$ is 1. The normalized noise power at the output due to this noise source is 1.
Thus total normalized noise variance at the output = 3.125 + 2.381 + 1 = 6.506.

Parallel Form II - $H(z)=\frac{0.8}{z-0.6}+\frac{0.2}{z+0.4}$.
The corresponding parallel realization with the noise sources is shown below.



The noise transfer function from $e_{a1}[n]$ and $e_{b1}[n]$ to the output is $G_c(z) = z^{-1}/(1 - 0.6z^{-1})$ Thus the total normalized noise power at the output due to these noise sources is

$$2\left[\frac{1}{2\pi j}\oint_C G_c(z)G_c(z^{-1})z^{-1}dz\right]=2\left[\frac{1}{1-(-0.6)^2}\right]=3.125.$$

The noise transfer function from $e_{a1}[n]$ and $e_{b1}[n]$ to the output is $G_d(z) = z^{-1}/(1 + 0.4z^{-1})$. Thus the total normalized noise power at the output due to these noise sources is

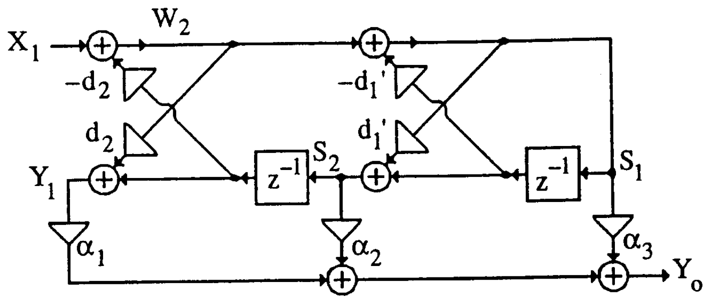
$$2\left[\frac{1}{2\pi j}\oint_C G_d(z)G_d(z^{-1})z^{-1}dz\right]=2\left[\frac{1}{1-(0.4)^2}\right]=2.381.$$

Thus total normalized noise variance at the output = 3.125 + 2.381 = 5.506.

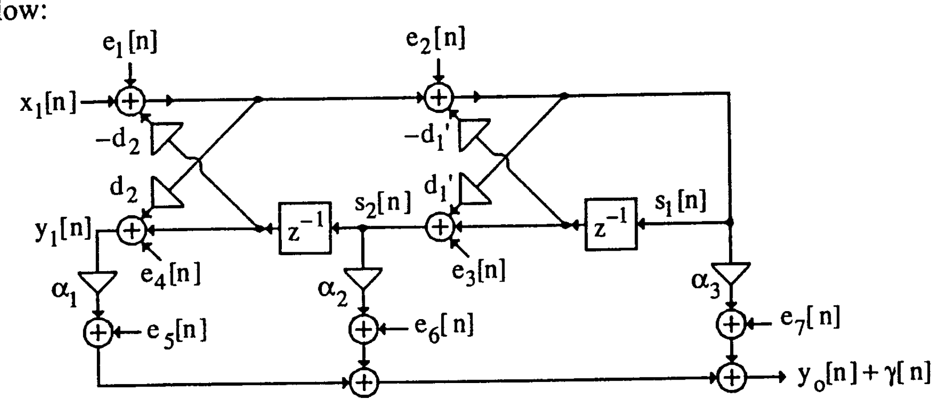
Hence of the five realizations, the Cascade Form #1 has the lowest product roundoff noise.

Note: If the multiplier at the input of the bottom branch of Parallel Form I structure is instead placed at the output, the total normalized noise variance at the output = 5.4226. Likewise, if the multiplier at the input of the two branches of Parallel Form II structure is instead placed at the output, the total normalized noise variance at the output = 3.04761. In this case, the modified Parallel Form II structure will have the lowest product roundoff noise.

9.13 The Gray-Markel realization of the transfer function of Problem 9.12 can be readily obtained using Program 6_3 resulting in the structure shown below where $d_2=-0.24$, $d_1'=-0.2632$, $\alpha_1=0.2$, $\alpha_2=1.04$, and $\alpha_3=0.3217$.



The noise model of the above structure assuming quntization of products before addition is as given below:



It can be seen from the noise model that the noise transfer function from the noise source $e_1[n]$ to the filter output is $G_1(z) = H(z) = \dfrac{z^{-1} + 0.2z^{-2}}{1 - 0.2z^{-1} - 0.24z^{-2}}$. Let $G_2(z)$, $G_3(z)$, and $G_4(z)$, denote the noise transfer functions from the noises sources $e_2[n]$, $e_3[n]$, and $e_4[n]$, respectively. Analyzing the structure we arrive at the expressions for these noise transfer functions:

$$G_2(z) = \frac{(\alpha_2 d_1' + \alpha_3) + \left[\alpha_1(1-d_2^2)d_1' + \alpha_2\right]z^{-1} + \alpha_1(1-d_2^2)z^{-2}}{1 - 0.2z^{-1} - 0.24z^{-2}} = \frac{0.04797 + 0.9904z^{-1} + 0.1885z^{-2}}{1 - 0.2z^{-1} - 0.24z^{-2}},$$

$$G_3(z) = \frac{\alpha_2 + \left[\alpha_1(1-d_2^2) + \alpha_2 d_1' - \alpha_3 d_2\right]z^{-1} + \alpha_1(1-d_2^2)d_1' z^{-2}}{1 - 0.2z^{-1} - 0.24z^{-2}} = \frac{1.04 - 0.008z^{-1} - 0.0496z^{-2}}{1 - 0.2z^{-1} - 0.24z^{-2}},$$

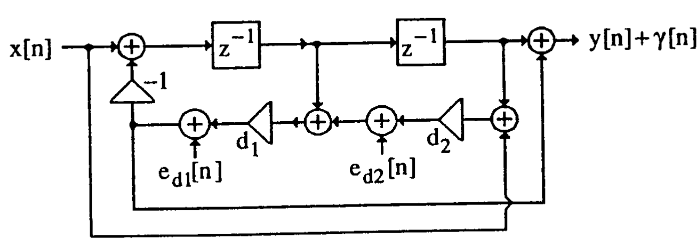
$$G_4(z) = \alpha_1 = 0.2.$$

The noise transfer functions from the remaining three noise sources are all equal to 1. Using Program 9_1 we determine the normalized output noise variances due to $e_1[n]$, $e_2[n]$, and $e_3[n]$, resulting in

$\sigma_1^2 = 1.3057$, $\sigma_2^2 = 1.3080$, $\sigma_3^2 = 1.1968$, and $\sigma_4^2 = 0.04$.

Hence, the total normalized output noise variances = 1.3057 + 1.7169 + 1.3080 + 0.04 + 3 = 6.8505.

9.14 The noise model for the given structure is as shown below.



The noise transfer function from $e_{d1}[n]$ to the output is

$$G_1(z)=\frac{Y(z)}{E_{d1}(z)}=\frac{1-z^{-2}}{1+d_1z^{-1}+d_1d_2z^{-2}}=\frac{z^2-1}{z^2+d_1z+d_1d_2}=1-\frac{d_1z+d_1d_2+1}{z^2+d_1z+d_1d_2}.$$

Using Table 9.4, the normalized noise power at the output due to $e_{d1}[n]$ is thus given by

$$\sigma_{1,n}^2=\frac{1}{2\pi j}\oint_C G_1(z)G_1(z^{-1})z^{-1}dz$$

$$=1+\frac{(d_1^2+(1+d_1d_2)^2)(1-d_1^2d_2^2)-2d_1((1+d_1d_2)d_1-(1+d_1d_2)d_1d_1d_2)}{(1-d_1^2d_2^2)^2+2d_1^3d_2-(1+d_1^2d_2^2)d_1^2}$$

$$=1+\frac{((1+d_1d_2)^2-d_1^2)(1-d_1^2d_2^2)}{(1-d_1^2d_2^2)^2+2d_1^3d_2-(1+d_1^2d_2^2)d_1^2}.$$

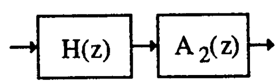
The noise transfer function from $e_{d2}[n]$ to the output is

$$G_2(z)=\frac{Y(z)}{E_{d2}(z)}=\frac{d_1-d_1z^{-2}}{1+d_1z^{-1}+d_1d_2z^{-2}}=d_1+\frac{-d_1^2z-d_1^2d_2-d_1}{1+d_1z^{-1}+d_1d_2z^{-2}}=-d_1G_1(z).$$

Thus $\sigma_{2,n}^2=d_1^2\sigma_{1,n}^2$. Hence total normalized noise power at the output

$$=(1+d_1^2)\,\sigma_{1,n}^2=(1+d_1^2)\left\{1+\frac{(1-d_1^2d_2^2)(1+d_1^2d_2^2+2d_1d_2-d_1^2)}{(1-d_1^2d_2^2)^2+2d_1^3d_2-(1+d_1^2d_2^2)d_1^2}\right\}.$$

9.15



Let e[n] be a noise source inside H(z) due to product roundoff with an associated noise transfer function $G_e(z)$. Then the normalized noise power at the output of the above cascade structure is given by $\sigma_{e,n}^2=\frac{1}{2\pi j}\oint_C G_e(z)A_2(z)G_e(z^{-1})A_2(z^{-1})z^{-1}dz=\frac{1}{2\pi j}\oint_C G_e(z)G_e(z^{-1})z^{-1}dz$ since $A_2(z)A_2(z^{-1})=1$.

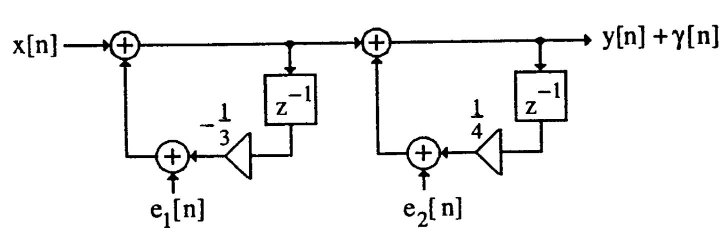
Assume that H(z) is realized in a cascade form with the lowest product round-off noise power $\sigma_{h,n}^2$. Then the noise power at the output of the cascade due to all product round-off in H(z) still remains $\sigma_{h,n}^2$. From Problem 9.10 solution given earlier, we observe $\sigma_{h,n}^2$ = 271 (or 262 if quantization after addition is assumed).

Now $d_1 = -0.4$ and $d_2 = -2.25$. Thus, from Problem 9.14 solution given above

$$\sigma_{a,n}^2=(1+d_1^2)\left\{1+\frac{(1-d_1^2d_2^2)(1+d_1^2d_2^2+2d_1d_2-d_1^2)}{(1-d_1^2d_2^2)^2+2d_1^3d_2-(1+d_1^2d_2^2)d_1^2}\right\}=[1+(0.14)^2][1+19]=23.2.$$

Hence, the total normalized noise variance at the output = 271 + 23.2 = 294.2.

9.16 The noise model for the structure is as shown below



The noise transfer function from $e_1[n]$ to the output is

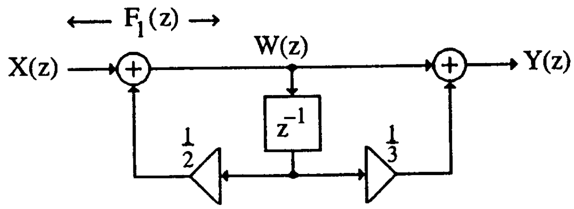
$G_1(z) = \dfrac{1}{(1+\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})} = 1 - \dfrac{\frac{4}{21}}{z+\frac{1}{3}} + \dfrac{\frac{3}{28}}{z-\frac{1}{4}}$. Thus, the normalized noise power at the output due to $e_1[n]$ is $\sigma_1^2 = 1 + \dfrac{\left(\frac{4}{21}\right)^2}{1-\frac{1}{9}} + \dfrac{\left(\frac{3}{28}\right)^2}{1-\frac{1}{16}} + 2\times\dfrac{\left(-\frac{4}{21}\right)\times\left(\frac{3}{28}\right)}{1+\frac{1}{12}} = 1.0154.$

The noise transfer function from $e_2[n]$ to the output is $G_2(z) = \dfrac{z}{z-\frac{1}{4}} = 1 + \dfrac{\frac{1}{4}}{z-\frac{1}{4}}$.

Thus, the normalized noise power at the output due to $e_2[n]$ is $\sigma_2^2 = 1 + \dfrac{(1/4)^2}{1-(1/4)^2} = 1.0667.$

Hence the total normalized noise at the output = 1.0154 + 1.0667 = 2.0821.

9.17 The unscaled structure is shown below.

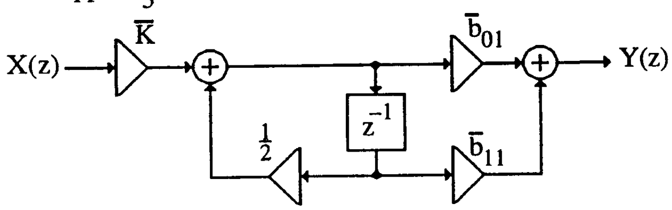


Now $F_1(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} = \dfrac{z}{z-\frac{1}{2}} = 1 + \dfrac{\frac{1}{2}}{z-\frac{1}{2}}$. Using Table 9.4 we obtain $\|F_1\|_2^2 = 1 + \dfrac{\frac{1}{4}}{1-\frac{1}{4}} = \dfrac{4}{3}$.

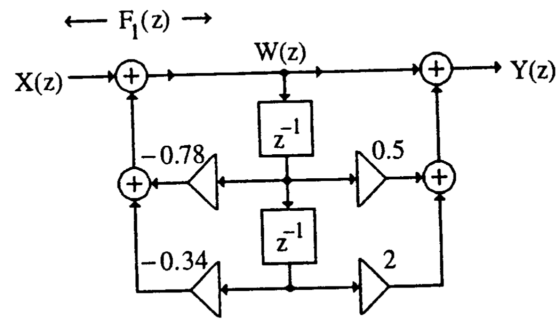
Also, $H(z) = \dfrac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \dfrac{z+\frac{1}{3}}{z-\frac{1}{2}} = 1 + \dfrac{-\frac{1}{6}}{z-\frac{1}{2}}$. Using Table 9.4 we obtain $\|H\|_2^2 = 1 + \dfrac{\frac{1}{36}}{1-\frac{1}{4}} = \dfrac{28}{27}$.

From Eq. (9.129a), $\|F_1\|_2 = \alpha_1 = \frac{2}{\sqrt{3}}$, and from Eq. (9.129b), $\|H\|_2 = \alpha_2 = \sqrt{\frac{28}{27}}$.

The scaled structure is as shown below, where $\bar{b}_{01} = \beta_1$ and $\bar{b}_{11} = \frac{\beta_1}{3}$. From Eq. (9.133a), $\bar{K} = \beta_0 K = \beta_0 = \frac{1}{\alpha_1} = \frac{\sqrt{3}}{2} = 0.866$, and from Eq. (9.133b), $\beta_1 = \frac{\alpha_1}{\alpha_2} = \sqrt{\frac{18}{14}} = 1.1339$. Therefore, $\bar{b}_{01} = \beta_1 = 0.866$, and $\bar{b}_{11} = \frac{\beta_1}{3} = 0.378$.



9.18 The unscaled structure is shown below.



Here $F_1(z) = \frac{1}{1+0.78z^{-1}+0.34z^{-2}} = 1 - \frac{0.78z+0.34}{z^2+0.78z+0.34}$. Using Table 9.4 we get

$$\|F_1\|_2^2 = 1 + \frac{[(0.78)^2+(0.34^2)][1-(0.34)^2] - 2\times(1-0.34)\times 0.78\times 0.34\times 0.78}{[1-(0.34)^2]^2 + 2\times 0.34\times(0.78)^2 - [1+(0.34)^2]\times(0.78)^2} = 1.7102.$$

Next, we note $H(z) = \frac{1+0.5z^{-1}+2z^{-2}}{1+0.78z^{-1}+0.34z^{-2}} = 1 + \frac{-0.28z+1.66}{z^2+0.78z+0.34}$.

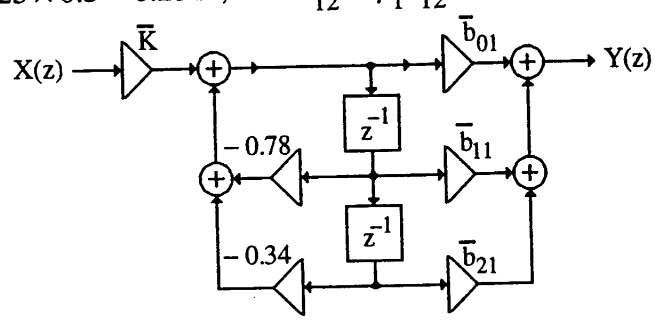
Using Table 9.4 we then get

$$\|H\|_2^2 = 1 + \frac{[(0.28)^2+(0.1.66)^2][1-(0.34)^2] - 2\times(1-0.34)\times 0.78\times 1.66\times(-0.28)}{[1-(0.34)^2]^2 + 2\times 0.34\times(0.78)^2 - [1+(0.34)^2]\times(0.78)^2} = 6.772.$$

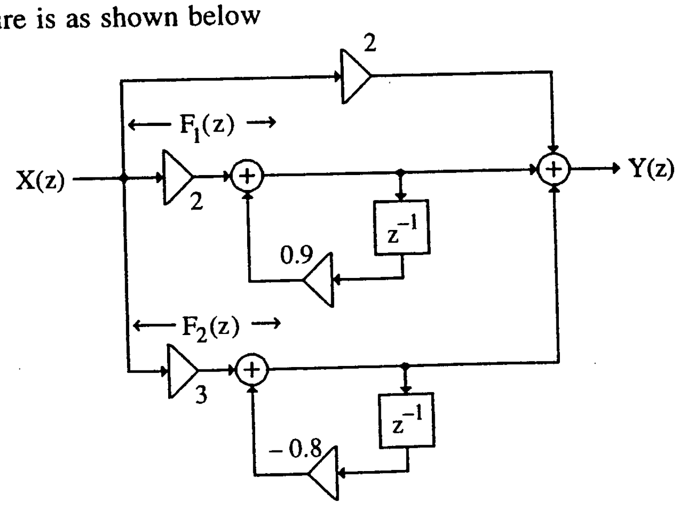
From Eq. (9.129a), $\|F_1\|_2 = \alpha_1 = \sqrt{1.7102}$, and $\|H\|_2 = \alpha_2 = \sqrt{6.772}$. Hence, from Eq. (9.133a),

$\beta_0 = \frac{1}{\alpha_1} = \frac{1}{\sqrt{1.7102}} = 0.7647$, and from Eq. (1.33b), $\beta_1 = \frac{\alpha_1}{\alpha_2} = \frac{\sqrt{1.7102}}{\sqrt{6.772}} = 0.5025.$

The scaled structure is as shown below, where $\bar{K} = \beta_0 K = \beta_0 = 0.7647$, $\bar{b}_{01} = \beta_1 = 0.5025$, $\bar{b}_{11} = \beta_1 b_{11} = 0.5025\times 0.5 = 0.2513$, and $\bar{b}_{12} = \beta_1 b_{12} = 0.5025\times 2 = 1.005$.



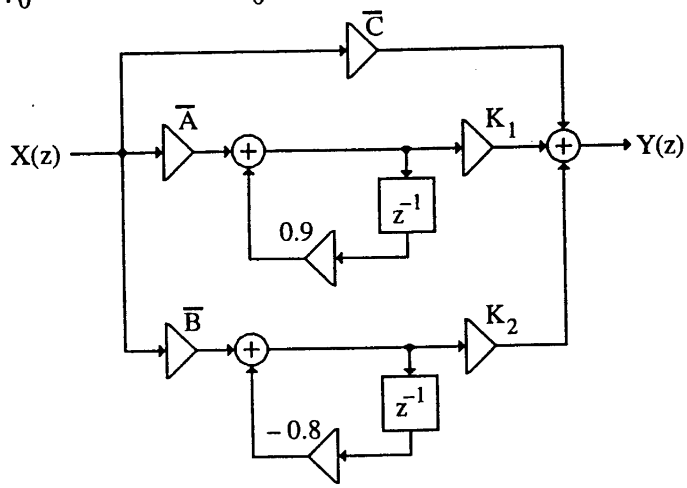
9.19 The unscaled structure is as shown below



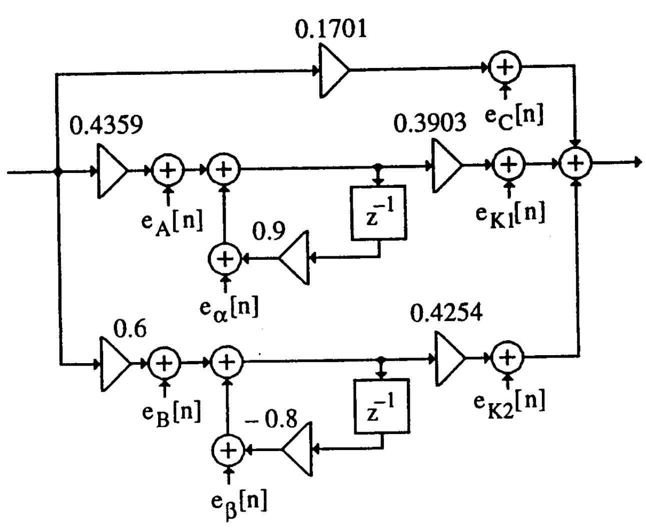
Note $F_1(z) = \dfrac{2}{1-0.9z^{-1}} = \dfrac{2(z-0.9)+1.8}{z-0.9} = 2 + \dfrac{1.8}{z-0.9}$. Hence $\|F_1\|_2^2 = 4 + \dfrac{(1.8)^2}{1-(0.8)^2} = 21.0526$.

Likewise $F_2(z) = \dfrac{3}{1+0.8z^{-1}} = \dfrac{3(z+0.8)-2.4}{z+0.8} = 3 + \dfrac{-2.4}{z+0.8}$. Hence, $\|F_2\|_2^2 = 9 + \dfrac{(-2.4)^2}{1-(0.8)^2} = 25.0$.

Also, $H(z) = 2 + \dfrac{2z}{z-0.9} + \dfrac{3z}{z+0.8} = \dfrac{7z^2-1.3z-1.44}{z^2-0.1z+0.72} = 7 + \dfrac{-0.6z-6.48}{z^2-0.1z+0.72}$. This yields $\|H\|_2^2 = 138.177$. Denote $\gamma_0 = \|H\|_2 = \sqrt{138.177} = 11.7549$, $\gamma_1 = \|F_1\|_2 = \sqrt{21.0526} = 4.5883$, and $\gamma_2 = \|F_2\|_2 = \sqrt{25} = 5.0$. The scaled structure is shown below, where $\bar{A} = \dfrac{2}{\gamma_1} = 0.4359$, $\bar{B} = \dfrac{3}{\gamma_2} = 0.6$, $\bar{C} = \dfrac{2}{\gamma_0} = 0.1701$, $K_1 = \dfrac{\gamma_1}{\gamma_0} = 0.3903$, and $K_2 = \dfrac{\gamma_2}{\gamma_0} = 0.4254$.



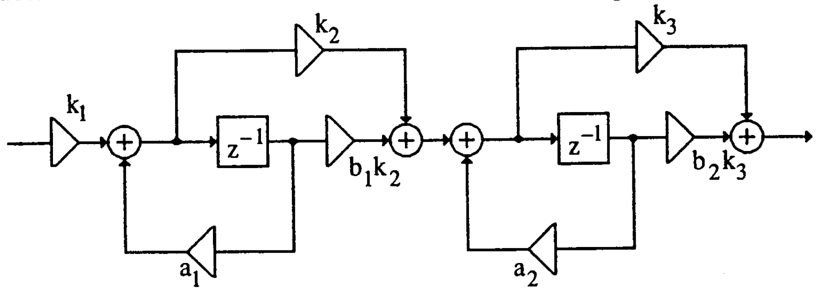
The noise model for the above scaled structure is given below



The noise transfer function from the noise sources $e_A[n]$ and $e_\alpha[n]$ is given by $G_1(z) = \dfrac{0.3903}{1-0.9z^{-1}}$ $= \dfrac{0.3903z}{z-0.9} = 0.3903 + \dfrac{0.2973}{z-0.9}$. Hence, the normalized output noise variance due to these noise sources is given by $\sigma_1^2 = 2\left((0.3903)^2 + \dfrac{(0.2973)}{1-(0.9)^2}\right) = 2 \times 0.8018 = 1.6036$.

Likewise, the noise transfer function from $e_B[n]$ and $e_\beta[n]$ is given by $G_2(z) = \frac{0.4254}{1+0.8z^{-1}}$ $= 0.4254 + \frac{-0.3403}{z+0.8}$. Hence, the normalized output noise variance due to these noise sources is given by $\sigma_2^2 = 2\left((0.4254)^2 + \frac{(-0.3403)}{1-(0.8)^2}\right) = 2 \times 0.5027 = 1.0054$. The noise transfer function from the remsaining three noise sources = 1 resulting in a normalized output noise variance $\sigma_3^2 = 3$. The total normalized output noise variance = $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1.6036 + 1.0054 + 3 = 5.609$.

9.20 The scaled structure is as shown below. The value of the scaling constants is found below.



Structure I: $a_1 = -\frac{1}{3}$, $b_1 = 3$, $a_2 = -\frac{1}{2}$, and $b_2 = 5$.

$F_1(z) = \frac{1}{1+\frac{1}{3}z^{-1}}$, thus $\|F_1\|_2^2 = \frac{1}{1-\frac{1}{9}} = 1.125$. Hence $\gamma_1 = \|F_1\|_2 = 1.0606$.

$F_2(z) = \frac{(1+3z^{-1})}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = \frac{z^2+3z}{z^2+\frac{5}{6}z+\frac{1}{6}}$. Hence $\|F_2\|_2^2 = 1 + 11 = 12$. Hence $\gamma_2 = \|F_2\|_2 = 3.4641$.

$\|H\|_2^2 = 252$. Hence, $\gamma_0 = \|H\|_2 = 15.8745$. The scaling multipliers are therefore given by

$k_1 = \frac{1}{\gamma_1} = 0.9428$, $k_2 = \frac{\gamma_1}{\gamma_2} = 0.30617$, $k_3 = \frac{\gamma_2}{\gamma_0} = 0.2182$. $b_1k_2 = 0.9185$, and $b_2k_3 = 1.091$.

The noise at the output due to the scaling constant k_1 and multiplier a_1 have a variance $\sigma_1^2 = \gamma_1^2 = 1.125$.

Noise at the output due to a_2, $1/k_1$ and b_1/k_1 have variance σ_2^2 which is calculated below.

The noise transfer function for these noise sources is $G_2(z) = \frac{1}{4.5825}\frac{z+5}{z+\frac{1}{2}}$. Therefore σ_2^2 = 1.3333. Hence the total noise power (variance) at the output = $2\times1.125 + 3\times1.3333 + 2 = 8.25$.

In case quantization is carried out after addition, then the total noise power at the output = 1.25 + 1.33333 + 1 = 3.125.

Structure II: $a_1 = -\frac{1}{2}$, $b_1 = 3$, $a_2 = -\frac{1}{3}$, and $b_2 = 5$.

$F_1(z) = \frac{1}{1+\frac{1}{2}z^{-1}}$ thus $\|F_1\|_2^2 = \frac{4}{3}$ or $\gamma_1 = \|F_1\|_2 = 1.1547$.

$F_2(z) = \dfrac{1+3z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$. Thus $\|F_2\|_2^2 = 12$ or $\gamma_2 = \|F_2\|_2 = 3.4641$.

$\|H\|_2^2 = 252$ or $\gamma_0 = \|H\|_2 = 15.8745$.

The scaling multipliers are therefore given by $k_1 = \dfrac{1}{\gamma_1} = 0.866$, $k_2 = \dfrac{\gamma_1}{\gamma_2} = 0.3333$,

$k_3 = \dfrac{\gamma_2}{\gamma_0} = 0.2182$. $b_1k_2 = 1$, and $b_2k_3 = 1.091$.

The noise at the output due to the scaling constant k_1 and multiplier a_1 have a variance $\sigma_1^2 = \gamma_1^2 = 4/3$.

Noise at the output due to a_2, $1/k_1$ and b_1/k_1 have variance σ_2^2 which is calculated below.

The noise transfer function for these noise sources is $G_2(z) = \dfrac{1}{4.5825}\left(\dfrac{z+5}{z+\frac{1}{3}}\right)$. Therefore $\sigma_2^2 = \|G_2\|_2^2 = 1.2143$. Hence the total noise power (variance) at the output $= 2\times\frac{4}{3} + 3\times 1.2143 + 2 = 8.3096$.

In case quantization is carried out after addition, then the total noise power at the output $= \frac{4}{3} + 1.2143 + 1 = 3.7476$.

<u>Structure III</u>: $a_1 = -\frac{1}{2}$, $b_1 = 5$, $a_2 = -\frac{1}{3}$, and $b_2 = 3$.

$F_1(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}$ thus $\|F_1\|_2^2 = \frac{4}{3}$ or $\gamma_1 = \|F_1\|_2 = 1.1547$.

$F_2(z) = \dfrac{1+5z^{-1}}{(1+\frac{1}{3}z^{-1})(1+\frac{1}{2}z^{-1})}$ thus $\|F_2\|_2^2 = 39.6$ or $\gamma_2 = \|F_2\|_2 = 6.2928$.

$\|H\|_2^2 = 252$ or $\gamma_0 = \|H\|_2 = 15.8745$.

The scaling multipliers are therefore given by $k_1 = \dfrac{1}{\gamma_1} = 0.866$, $k_2 = \dfrac{\gamma_1}{\gamma_2} = 0.1835$,

$k_3 = \dfrac{\gamma_2}{\gamma_0} = 0.3964$. $b_1k_2 = 0.9175$, and $b_2k_3 = 1.1892$.

As above $\sigma_1^2 = \frac{4}{3}$. Again $G_2(z) = \dfrac{1}{2.5226}\left(\dfrac{z+3}{z+\frac{1}{3}}\right)$. Thus $\sigma_2^2 = \|G_2\|_2^2 = 1.4143$.

Hence the total noise power (variance) at the output $= 2\times\frac{4}{3} + 3\times 1.4143 + 2 = 8.90956$.

In case quantization is carried out after addition then the total noise power at the output $= \frac{4}{3} + 1.4143 + 1 = 3.7476$.

<u>Structure IV</u> : $a_1 = -\frac{1}{3}$, $b_1 = 5$, $a_2 = -\frac{1}{2}$, and $b_2 = 3$.

$F_1(z) = \dfrac{1}{1+\frac{1}{3}z^{-1}}$ thus $\|F_1\|_2^2 = \frac{9}{8}$ or $\gamma_1 = \|F_1\|_2 = 1.0606.$

$F_2(z) = \dfrac{1+5z^{-1}}{(1+\frac{1}{3}z^{-1})(1+\frac{1}{2}z^{-1})}$ thus $\|F_2\|_2^2 = 39.6$ or $\gamma_2 = \|F_2\|_2 = 6.2928..$

$\|H\|_2^2 = 252$ or $\gamma_0 = \|H\|_2 = 15.8745.$

The scaling multipliers are therefore given by $k_1 = \dfrac{1}{\gamma_1} = 0.9428,\ k_2 = \dfrac{\gamma_1}{\gamma_2} = 0.16854,$

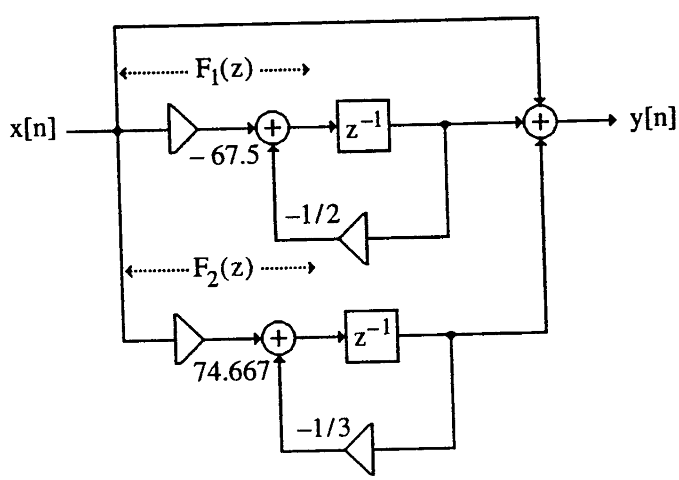
$k_3 = \dfrac{\gamma_2}{\gamma_0} = 0.3964.$ $b_1k_2 = 0.8427$, and $b_2k_3 = 1.1892.$

As above $\sigma_1^2 = 1.125$. Again $G_2(z) = \dfrac{1}{2.5226}\left(\dfrac{z+3}{z+\frac{1}{2}}\right)$. Thus $\sigma_2^2 = \|G_2\|_2^2 = 1.4667.$

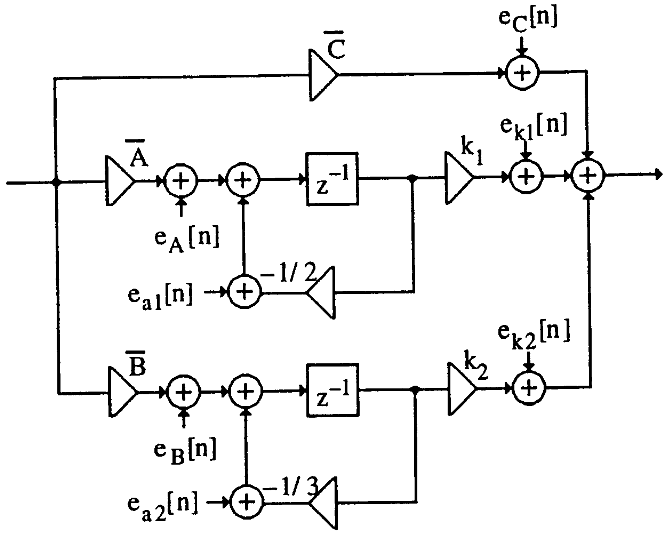
Hence the total noise power (variance) at the output $= 2\times 1.125 + 3\times 1.4667 + 2 = 8.6501.$

In case quantization is carried out after addition then the total noise power at the output $= 1.125 + 1.4667 + 1 = 3.5917.$

9.21 Parallel form II realization: The corresponding unscaled structure is obtained from a partial fraction expansion of H(z) in z in the form: $H(z) = 1 - \dfrac{67.5}{z+\frac{1}{2}} + \dfrac{74.667}{z+\frac{1}{3}}$, and is shown on the next page. Here the scaling transfer function $F_1(z)$ for the middle branch is given by $F_1(z) = -\dfrac{6.75}{z+\frac{1}{2}}$. Hence, $\|F_1\|_2^2 = \dfrac{(6.75)^2}{1-\frac{1}{4}} = 6075.$ This implies $\gamma_1 = \|F_1\|_2 = 77.9423$. Similarly, the scaling transfer function for the bottom branch is given by $F_2(z) = \dfrac{74.667}{z+\frac{1}{3}}$. Hence,



$\|F_2\|_2^2 = \dfrac{(74.667)^2}{1-\frac{1}{9}} = 6272.1.$ Thus, $\gamma_2 = \|F_2\|_2 = 79.1963.$ Finally, $\|H\|_2^2 = 252.$ Hence, $\gamma_0 = \|H\|_2 = 15.8745.$ The scaled structure is shown below:

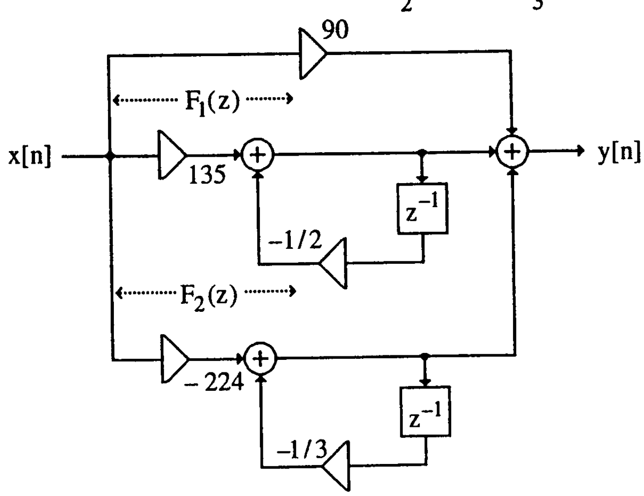


where $\overline{A} = \frac{-67.5}{\gamma_1} = -0.86602$, $\overline{B} = \frac{74.667}{\gamma_2} = 0.94285$, $\overline{C} = \frac{1}{\gamma_0} = 0.06299$, $k_1 = \frac{\gamma_1}{\gamma_0} = 4.9099$, $k_2 = \frac{\gamma_2}{\gamma_0} = 4.9889$.

The output noise variances due to $e_C[n]$, $e_{k1}[n]$, and $e_{k2}[n]$ are each equal to 1. The output noise variances due to $e_A[n]$ and $e_{a1}[n]$ are each equal to $k_1^2\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{4}{3}\times(4.9099)^2 = 32.1429$.

Likewise, the output noise variances due to $e_B[n]$ and $e_{a2}[n]$ are each equal to $k_2^2\left(\frac{1}{1-\frac{1}{9}}\right) = \frac{9}{8}\times(4.9889)^2 = 28.125$. Total noise variance at the output is therefore = $3+(2\times 32.1429)+(2\times 28.125) = 123.536$.

<u>Parallel form I realization</u>: The corresponding structure is obtained from a partial fraction expansion of H(z) in z^{-1} the form: $H(z) = 90 + \frac{135}{1+\frac{1}{2}z^{-1}} - \frac{224}{1+\frac{1}{3}z^{-1}}$, and is shown below:



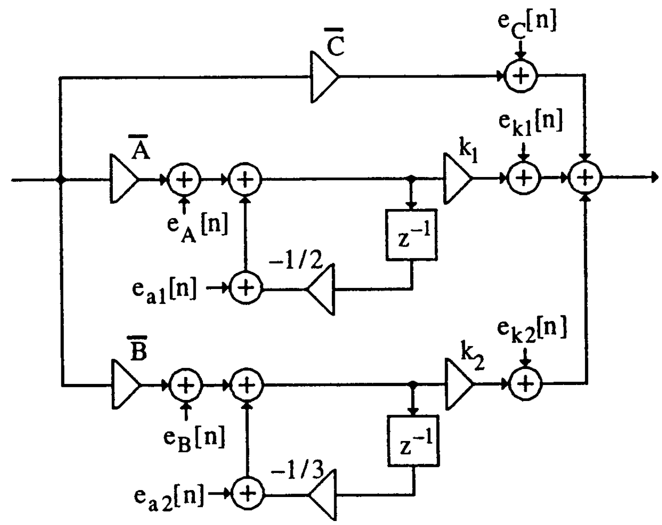
Here the scaling transfer function $F_1(z)$ for the middle branch is given by

$F_1(z) = \frac{135z}{z+\frac{1}{2}} = 135 - \frac{67.5}{z+\frac{1}{2}}$. Hence, $\|F_1\|_2^2 = (135)^2 + \frac{(67.5)^2}{1-\frac{1}{4}} = 24300$. This implies

$\gamma_1 = \|F_1\|_2 = 155.8846$. Similarly, the scaling transfer function for the bottom branch is given by

$F_2(z) = \frac{-224z}{z+\frac{1}{3}} = -224 + \frac{74.6667}{z+\frac{1}{3}}$. Hence, $\|F_2\|_2^2 = (224)^2 + \frac{(74.6667)^2}{1-\frac{1}{9}} = 56448$. This implies

$\gamma_2 = \|F_2\|_2 = 237.5879$. Finally, $\|H\|_2^2 = 252$. Hence, $\gamma_0 = \|H\|_2 = 15.8745$. The scaled structure is shown below:



where $\overline{A} = \frac{135}{\gamma_1} = 0.866$, $\overline{B} = \frac{-224}{\gamma_2} = -0.9428$, $\overline{C} = \frac{90}{\gamma_0} = 5.6695$, $k_1 = \frac{\gamma_1}{\gamma_0} = 9.8198$, and $k_2 = \frac{\gamma_2}{\gamma_0} = 14.9666$.

The output noise variances due to $e_C[n]$, $e_{k1}[n]$, and $e_{k2}[n]$ are each equal to 1. The output noise variances due to $e_A[n]$ and $e_{a1}[n]$ are each equal to $k_1^2\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{4}{3}\times(9.8198)^2 = 128.5841$.

Likewise, the output noise variances due to $e_B[n]$ and $e_{a2}[n]$ are each equal to

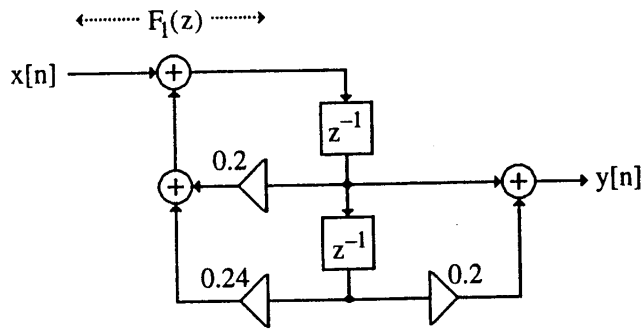
$k_2^2\left(\frac{1}{1-\frac{1}{9}}\right) = \frac{9}{8}\times(14.9666)^2 = 252$. Total noise variance at the output is therefore =

$3 + 2\times128.58 + 2\times252 = 764.164$.

For quantization of products after addition, the total output noise variance in the case of Parallel form II structure (after scaling) is $1 + 32.1429 + 28.125 = 61.27$, and total output noise variance in the case of Parallel form I structure (after scaling) is $1 + 128.58 + 252 = 381.58$.

9.22 $H(z) = \frac{z^{-1} + 0.2z^{-2}}{1 - 0.2z^{-1} - 0.24z^{-2}} = \frac{z^{-1}(1+0.2z^{-1})}{(1-0.6z^{-1})(1+0.4z^{-1})}$.

(a) The unscaled direct form realization is shown below:

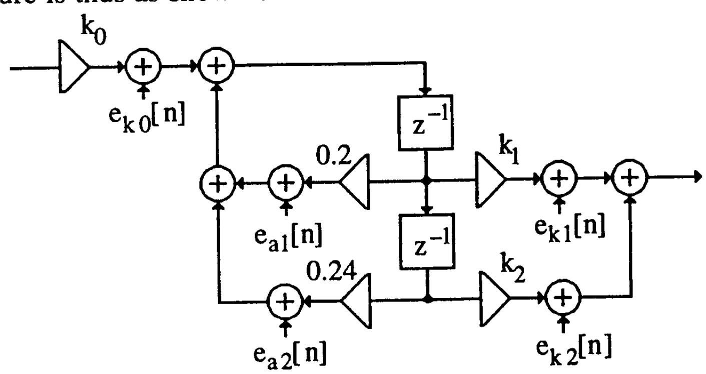


The scaling transfer function $F_1(z)$ is given by $F_1(z)= \dfrac{1}{1-0.2z^{-1}-0.24z^{-2}}$

$= 1+\dfrac{0.2z+0.24}{z^2-0.2z-0.24}$. Hence $\|F_1\|_2^2 = 1 + 0.1401 = 1.1401$. This implies $\gamma_1 = \|F_1\|_2 = 1.0677$.

For scaling the output, we compute $\|H\|_2^2 = 1 + .3057 = 1.3057$. Hence, $\gamma_2 = \|H\|_2 = 1.1427$.

Ther scaled structure is thus as shown below:

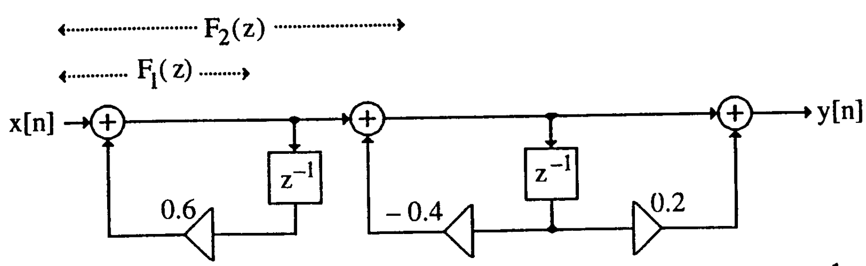


where $k_0 = \dfrac{1}{\gamma_1} = 0.8771$, $k_1 = \dfrac{\gamma_1}{\gamma_2} = 0.9344$, and $k_2 = \dfrac{0.2\gamma_1}{\gamma_2} = 0.1869$.

Thus the noise at the output due to the noise sources $e_{k1}[n]$ and $e_{k2}[n]$ = 1.

Also the noise variance at the output due to the noise sources $e_{k0}[n]$, $e_{a1}[n]$, and $e_{a2}[n]$ = γ_1^2=1.1401. Thus the total noise variance at the output = $2 + 3\times 1.1401 = 5.4203$.

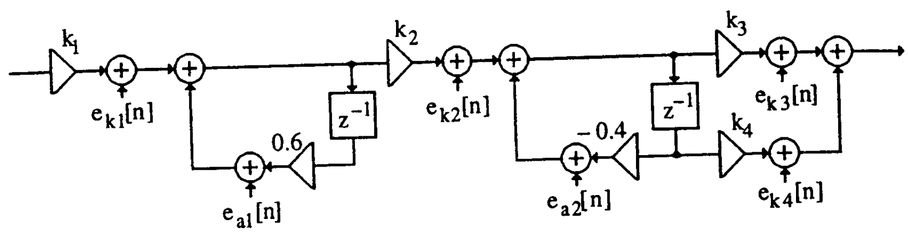
(b) Cascade Form - The unscaled structure is shown below:



In this case the scaling transfer function $F_1(z)$ is given by $F_1(z) = \dfrac{1}{1-0.6z^{-1}}$. Hence, $\|F_1\|_2^2$ = 1.5625 which implies $\gamma_1 = \|F_1\|_2 = 1.25$. The scaling transfer function $F_2(z)$ is given by

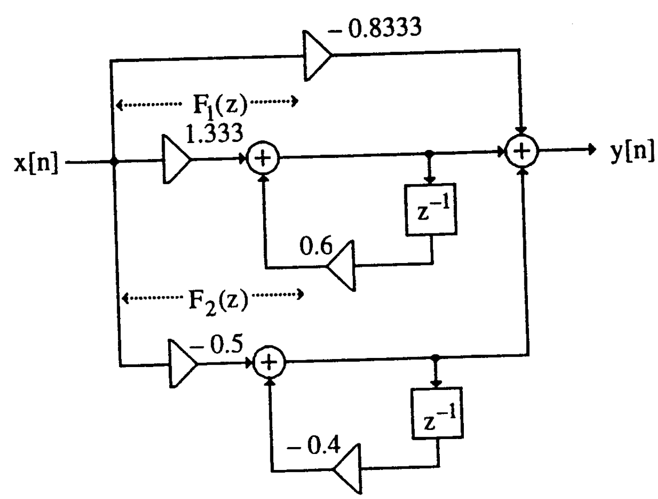
$F_2(z) = \frac{z^{-1}}{(1-0.6z^{-1})(1+0.4z^{-1})} = \frac{z}{z^2-0.2z-0.24}$ which implies $\|F_2\|_2^2 = 1.1401$ or $\gamma_2 = \|F_2\|_2 =$ 1.0678. Finally, $H(z) = \frac{z^{-1}(1+0.2z^{-1})}{(1-0.6z^{-1})(1+0.4z^{-1})} = \frac{z+0.2}{z^2-0.2z-0.24}$ which implies $\|H\|_2^2$ = 1.3057 or $\gamma_0 = \|H\|_2 = 1.1427$.

The scaled structure is as shown below:



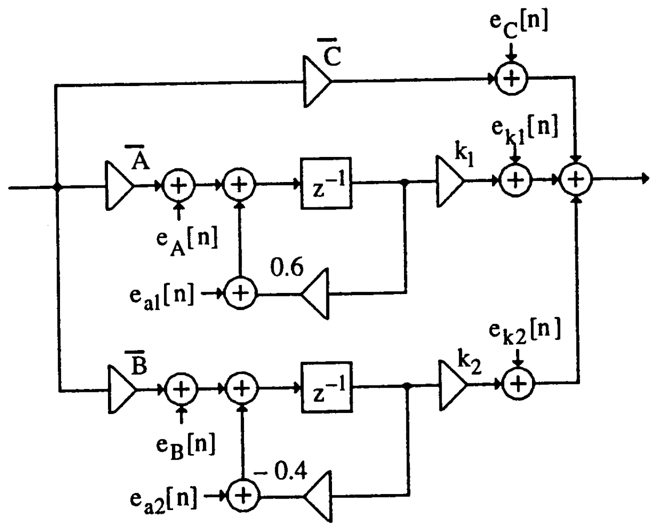
where $k_1 = \frac{1}{\gamma_1} = 0.8$, $k_2 = \frac{\gamma_1}{\gamma_2} = 1.1706$, $k_3 = \frac{\gamma_2}{\gamma_0} = 0.9345$, and $k_4 = 0.2k_3 = 0.1869$.
The noise variance at the output due to the noise sources $e_{k3}[n]$, and $e_{k4}[n]$ are = 1 each. The noise variance at the output due to the noise sources $e_{k2}[n]$ and $e_{a2}[n]$ are each equal to = 0.91487, and the noise variance at the output due to the noise sources $e_{k1}[n]$ and $e_{a1}[n]$ are each equal to = $\gamma_1^2 = 1.5625$. Thus the total noise variance at the output = 2(1 + 1.5625 + 0.91487) = 6.9547.

(c) Parallel Form I Realization: A partial-fraction expansion of H(z) in z^{-1} is given by $H(z) = \frac{z^{-1}+0.2z^{-2}}{1-0.2z^{-1}-0.24z^{-2}} = -0.8333 + \frac{1.3333}{1-0.6z^{-1}} + \frac{-0.5}{1+0.4z^{-1}}$. The correspondsing unscaled realization is thus as shown below:



The scaling transfer function $F_1(z)$ is given by $F_1(z) = \frac{1.3333}{1-0.6z^{-1}}$. This implies $\|F_1\|_2^2 = 2.7776$. Hence $\gamma_1 = \|F_1\|_2 = 1.6666$. The scaling transfer function $F_2(z)$ is given by $F_2(z) = \frac{-0.5}{1+0.4z^{-1}}$.

This implies $\|F_2\|_2^2 = 0.29761$. Hence $\gamma_2 = \|F_2\|_2 = 0.5455$. Finally, $\|H\|_2^2 = 1.30568$. This implies $\gamma_0 = \|H\|_2 = 1.14267$. The scaled structure is thus as shown below:

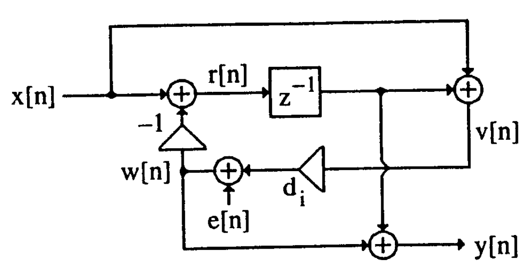


where $\bar{A} = \frac{1.333}{\gamma_1} = 0.8$, $\bar{B} = \frac{-0.5}{\gamma_2} = -0.9166$, $\bar{C} = \frac{0.8333}{\gamma_0} = 0.7293$, $k_1 = \frac{\gamma_1}{\gamma_0} = 1.4585$, and $k_2 = \frac{\gamma_2}{\gamma_0} = 0.4774$.

The noise variance at the output due to the noise sources $e_C[n]$, $e_{k1}[n]$, and $e_{k2}[n]$ are each = 1. The noise variance at the output due to the noise sources $e_{a1}[n]$ and $e_A[n]$ are each = $\frac{k_1^2}{1-(0.6)^2} = 3.3215$. The noise variance at the output due to the noise sources $e_{a2}[n]$ and $e_B[n]$ are each = $\frac{k_2^2}{1-(0.4)^2} = 0.271317$. Hence, the total output noise variance = $3\times1 + 2\times3.3215 + 2\times0.271317$ = 10.1856.

If quantization is after addition of products, then the total output noise variance = 1 + 3.3215 + 0.271317 = 4.592817.

9.23 The noise model for the allpass structure is shown below



Analysis yields $W(z) = E(z) + d_i V(z)$, $R(z) = X(z) - W(z)$, $V(z) = X(z) + z^{-1}R(z)$, and
$$Y(z) = W(z) + z^{-1}R(z).$$
To determine the noise transfer function we set $X(z) = 0$ in the above equations. This leads to

$R(z) = -W(z)$, $V(z) = z^{-1}R(z) = -z^{-1}W(z)$, and hence $W(z) = E(z) - d_i z^{-1}W(z)$ or $E(z) = (1 + d_i z^{-1})W(z)$. As a result, $Y(z) = W(z) - z^{-1}W(z) = (1 - z^{-1})W(z)$. Consequently, the noise transfer function is given by

$$G(z) = \frac{Y(z)}{E(z)} = \frac{1-z^{-1}}{1+d_i z^{-1}} = \frac{z-1}{z+d_i} = 1 + \frac{-(1+d_i)}{z+d_i}.$$

Thus $\sigma_e^2 = 1 + \dfrac{(1+d_i)^2}{1-d_i^2} = \dfrac{2}{1-d_i}$.

(b) Let $G_1(z)$ be the noise transfer function for d_1, $G_2(z)$ be the noise transfer function for d_2 and $G_3(z)$ be the noise transfer function for d_3 then

$G_1(z) = \dfrac{z-1}{z+d_1} A_2(z)$, $G_2(z) = \dfrac{z-1}{z+d_2}$ and $G_3(z) = \dfrac{z-1}{z+d_3}$. From the results of part (a) it follows that the noise variances due to d_1, d_2 and d_3 are given by

$\sigma_k^2 = \dfrac{2}{1-d_k}$ for k = 1, 2, 3. Hence the total noise power at the output is given by

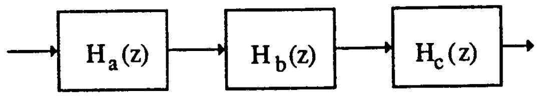
$$\sigma^2 = \frac{2}{1-d_1} + \frac{2}{1-d_2} + \frac{2}{1-d_3}.$$

9.24 Let the total noise power at the output of G(z) due to product-off be given by σ_G^2. Asuming a total of L multipliers in the realization of G(z) we get $\sigma_G^2 = \sum_{\ell=1}^{L} k_\ell \left(\frac{1}{2\pi} \int_0^{2\pi} |G_\ell(e^{j\omega})|^2 d\omega \right)$ where $G_\ell(z)$ denotes the noise transfer function due to ℓ-th noise source in G(z). Now if each delay is replaced by two delays then each of the noise transfer function becomes $G_\ell(e^{2j\omega})$. Thus the total noise power at the output due to noise sources in $G(z^2)$ is given by $\hat{\sigma}_G^2 =$ $\sum_{\ell=1}^{L} k_\ell \left(\frac{1}{2\pi} \int_0^{2\pi} |G_\ell(e^{2j\omega})|^2 d\omega \right)$. Replacing ω by $\hat{\omega}/2$ in the integral we get $\hat{\sigma}_G^2 =$ $\sum_{\ell=1}^{L} k_\ell \left(\frac{1}{2\pi} \int_0^{4\pi} |G_\ell(e^{j\hat{\omega}})|^2 \left(\frac{1}{2}\right) d\hat{\omega} \right) = \frac{1}{2} \sum_{\ell=1}^{L} k_\ell \left(\frac{1}{2\pi} \int_0^{4\pi} |G_\ell(e^{j\hat{\omega}})|^2 d\hat{\omega} \right) = \sigma_G^2$. Since $\frac{1}{2\pi} \int_0^{2\pi} |G_\ell(e^{j\omega})|^2 |A(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \int_0^{2\pi} |G_\ell(e^{j\omega})|^2 d\omega$, the total noise power at the output of the cascade is still equal to σ_0^2.

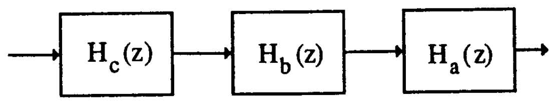
9.25 For the first factor in the numerator there are R possible choices of factors. Once this factor has been choosen, there are R – 1 choices for the next factor and continuing further we get that the total number of possible ways in which the factors in the numerator can be generated equal to $R(R-1)(R-2)\cdots 2 \times 1 = R!$. Similarly the total number of ways in which the factors in the denominator can be generated = R!. Since the numerator and denominator are generated independent of each other hence the total number of possible realizations are N = (R!)(R!) = $(R!)^2$.

9.26 **(a)** First we pair the poles closest to the unit circle with their nearest zeros resulting in the second-order section $H_a(z) = \frac{z^2+0.2z+0.9}{z^2+0.1z+0.8}$. Next, the poles that are closest to the poles of $H_a(z)$ are matched with their nearest zeros resulting in the second-order section $H_b(z) = \frac{z^2+0.3z+0.5}{z^2+0.2z+0.4}$. Finally, the remaining poles and zeros are matched yielding the second-order section $H_c(z) = \frac{z^2+0.8z+0.2}{z^2+0.6z+0.3}$.

For odering the sections to yield the smallest peak output noise due to product round-off under an $\mathcal{L}_2$-scaling rule, the sections should be placed from most peaked to least peaked as shown below.



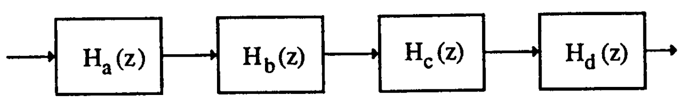
For odering the sections to yield the smallest peak output noise power due to product round-off under an $\mathcal{L}_\infty$-scaling rule, the sections should be placed from least peaked to most peaked as shown below.



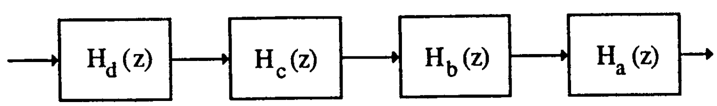
(b) The poles closest to the unit circle is given by the denominator $(z^2 - 0.2z + 0.9)$, the next closer pole is given by the factor $(z + 0.9)$, followed by the factor $(z^2 + 0.4z + 0.7)$ and finally the factor $(z + 0.2)$. Pairing these poles with their closest zeros we get the following pairings:

$H_a(z) = \frac{z^2+0.1z+0.7}{z^2-0.2z+0.9}$, $H_b(z) = \frac{z+1.1}{z+0.9}$, $H_c(z) = \frac{z^2+0.5z+0.6}{z^2+0.4z+0.7}$ and $H_d(z) = \frac{z+0.3}{z+0.2}$.

For odering the sections to yield the smallest peak output noise due to product round-off under an $\mathcal{L}_2$-scaling rule, the sections should be placed from most peaked to least peaked as shown below.



For odering the sections to yield the smallest peak output noise power due to product round-off under an $\mathcal{L}_\infty$-scaling rule, the sections should be placed from least peaked to most peaked as shown below.



9.27 SNR $= \frac{\sigma_x^2}{\sigma_0^2}$. After scaling σ_x changes to $K\sigma_x$ where K is given by Eq. (9.150).

Therefore SNR $= \frac{\sigma_x^2(1-|\alpha|)^2}{\sigma_0^2}$.

(i) For uniform density function $\sigma_x^2 = \int_{-1}^{1} \frac{1}{2}x^2dx = \frac{1}{3}$.

Thus $\text{SNR} = \dfrac{(1-|\alpha|)^2}{3\sigma_0^2}$. With b = 12 $\sigma_0^2 = \dfrac{2^{-24}}{12} = 4.967\times10^{-9}$ and $|\alpha| = 0.95$,

Hence $\text{SNR}_{dB} = 10\log_{10}\left(\dfrac{(1-|\alpha|)^2}{3\sigma_0^2}\right) = 52.24$ dB.

(ii) For Gaussian input with $\sigma_x^2 = \frac{1}{9}$. Hence $\text{SNR} = \dfrac{(1-|\alpha|)^2}{9\sigma_0^2}$. Again with b = 12 and

$|\alpha| = 0.95$, $\text{SNR}_{dB} = 10\log_{10}\left(\dfrac{(1-|\alpha|)^2}{9\sigma_0^2}\right) = 47.97$ dB.

(iii) For a sinusoidal input of known frequency, i.e. $x[n] = \sin(\omega_0 n)$.

In this case average power $= \sigma_x^2 = \frac{1}{2}$. Hence $\text{SNR} = \dfrac{(1-|\alpha|)^2\sigma_x^2}{\sigma_0^2} = \dfrac{(1-|\alpha|)^2}{2\sigma_0^2}$.

Therefore $\text{SNR}_{dB} = 10\log_{10}\left(\dfrac{(1-|\alpha|)^2}{2\sigma_0^2}\right) = 69.91$ dB.

9.28 **(a)** $H_{LP}(z) = \frac{1}{2}\{1 + A_1(z)\}$ where $A_1(z) = \dfrac{-\alpha + z^{-1}}{1-\alpha z^{-1}}$.

Hence $\left|H_{LP}(e^{j\omega})\right|^2 = \left(\dfrac{1-\alpha}{2}\right)^2 \dfrac{2(1+\cos(\omega))}{1-2\alpha\cos(\omega)+\alpha^2}$.

Proving $\left.\dfrac{\partial\left|H_{LP}(e^{j\omega})\right|}{\partial\alpha}\right|_{\omega=0} = 0$ is equivalent to proving $\left.\dfrac{\partial\left|H_{LP}(e^{j\omega})\right|^2}{\partial\alpha}\right|_{\omega=0} = 0$. Now

$$\frac{\partial\left|H_{LP}(e^{j\omega})\right|^2}{\partial\alpha} = -\frac{1-\alpha}{2}\,\frac{2(1+\cos(\omega))}{1-2\alpha\cos(\omega)+\alpha^2} - \left(\frac{1-\alpha}{2}\right)^2 \frac{2(1+\cos(\omega))(-2\cos(\omega)+\alpha)}{1-2\alpha\cos(\omega)+\alpha^2}.$$

Thus, $\left.\dfrac{\partial\left|H_{LP}(e^{j\omega})\right|^2}{\partial\alpha}\right|_{\omega=0} = \dfrac{-2}{1-\alpha} + \dfrac{2}{1-\alpha} = 0.$

(b) $H_{HP}(z) = \frac{1}{2}\{1 - A_1(z)\}$ where $A_1(z) = \dfrac{-\alpha + z^{-1}}{1-\alpha z^{-1}}$.

Hence $\left|H_{HP}(e^{j\omega})\right|^2 = \left(\dfrac{1+\alpha}{2}\right)^2 \dfrac{2(1-\cos(\omega))}{1-2\alpha\cos(\omega)+\alpha^2}$.

Now $$\frac{\partial\left|H_{HP}(e^{j\omega})\right|^2}{\partial\alpha} = -\frac{1+\alpha}{2}\,\frac{2(1-\cos(\omega))}{1-2\alpha\cos(\omega)+\alpha^2} - \left(\frac{1+\alpha}{2}\right)^2 \frac{2(1-\cos(\omega))(-2\cos(\omega)+\alpha)}{1-2\alpha\cos(\omega)+\alpha^2}.$$

Thus, $\left.\dfrac{\partial\left|H_{HP}(e^{j\omega})\right|^2}{\partial\alpha}\right|_{\omega=\pi} = \dfrac{-2}{1+\alpha} + \dfrac{2}{1+\alpha} = 0.$

9.29 **(a)** $H_{BP}(z) = \frac{1}{2}\{1 - A_2(z)\}$ where $A_2(z) = \frac{\alpha - \beta(1+\alpha)z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$, with α and β being real.

Therefore, $\left|H_{BP}(e^{j\omega})\right|^2 = \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{2(1-\cos(\omega))}{1+\beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)}\right)$. Now,

$$\frac{\partial\left|H_{BP}(e^{j\omega})\right|^2}{\partial\alpha} = \left(\frac{-2}{1-\alpha}\right)\left(\frac{1-\alpha}{2}\right)^2\left(\frac{2(1-\cos(\omega))}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right)$$

$$+\left(\frac{1-\alpha}{2}\right)^2\left(\frac{2(1-\cos(\omega))}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right)\times$$

$$\left(\frac{2\beta^2(1+\alpha)+2\alpha+2\cos(2\omega)-4\beta(1+\alpha)\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right).$$

Using the fact $\beta = \cos(\omega_0)$ we get $\left.\frac{\partial\left|H_{BP}(e^{j\omega})\right|^2}{\partial\alpha}\right|_{\omega=\omega_0} = \frac{-2}{1-\alpha} - \frac{\left(2\beta^2+2\alpha-2-2\alpha\beta^2\right)}{1+\alpha^2-2\alpha+2\alpha\beta^2-\beta^2-\alpha^2\beta^2}$

$$= \frac{-2}{1-\alpha} + \frac{2}{1-\alpha} = 0.$$

Similarly, $\frac{\partial\left|H_{BP}(e^{j\omega})\right|^2}{\partial\beta} = \left(\frac{1-\alpha}{2}\right)^2\left(\frac{2(1-\cos(\omega))}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right)\times$

$$\left(-\frac{2\beta(1+\alpha)^2-2(1+\alpha)^2\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right).$$

Again using the fact that $\beta = \cos(\omega_0)$ it can be seen that $\left.\frac{\partial\left|H_{BP}(e^{j\omega})\right|^2}{\partial\beta}\right|_{\omega=\omega_0} = 0.$

(b) For bandstop filters $H_{BS}(z) = \frac{1}{2}\{1 + A_2(z)\}$ where $A_2(z)$ is as given in (a). Thus

$\left|H_{BS}(e^{j\omega})\right|^2 = \left(\frac{1+\alpha}{2}\right)^2\left(\frac{4\beta^2+2+2\cos(2\omega)-8\beta\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right)$. Thus,

$$\frac{\partial\left|H_{BS}(e^{j\omega})\right|^2}{\partial\alpha} = \left(\frac{1+\alpha}{2}\right)\left(\frac{4\beta^2+2+2\cos(2\omega)-8\beta\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right)$$

$$-\left(\frac{1+\alpha}{2}\right)^2\left(\frac{4\beta^2+2+2\cos(2\omega)-8\beta\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right)\times\times$$

$$\left(\frac{2(1+\alpha)\beta^2+2\alpha+2\cos(2\omega)-4\beta(1+\alpha)\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right).$$

Substituting $\beta = \cos(\omega_0)$ it can be seen that

$$\left.\frac{\partial|H_{BS}(e^{j\omega})|^2}{\partial\alpha}\right|_{\omega=0}=\frac{2}{1-\alpha}-\left(\frac{1+\alpha}{2}\right)^2\left(\frac{2(1+\alpha)(\beta+1)^2}{(1+\alpha)^2(\beta+1)^2}\right)=0.$$

Similarly, $\left.\frac{\partial|H_{BS}(e^{j\omega})|^2}{\partial\alpha}\right|_{\omega=\pi}=\left(\frac{2}{1-\alpha}\right)-\left(\frac{1+\alpha}{2}\right)^2\left(\frac{2(1+\alpha)(\beta+1)^2}{(1+\alpha)^2(\beta+1)^2}\right)=0.$

Now, $\frac{\partial|H_{BS}(e^{j\omega})|^2}{\partial\beta}=\left(\frac{1+\alpha}{2}\right)^2\left(\frac{8\beta-8\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right)$

$$-\left(\frac{1+\alpha}{2}\right)^2\left(\frac{4\beta^2+2+2\cos(2\omega)-8\beta\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right)\times$$

$$\left(\frac{(1+\alpha)^2 2(\beta-\cos(\omega))}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)}\right).$$

Again substituting $\beta=\cos(\omega_0)$ it can be seen that

$\left.\frac{\partial|H_{BS}(e^{j\omega})|^2}{\partial\beta}\right|_{\omega=0}=0$ and also $\left.\frac{\partial|H_{BS}(e^{j\omega})|^2}{\partial\beta}\right|_{\omega=\pi}=0.$

9.30 For a BR transfer function G(z) realized in a parallel allpass form, its power-complementary transfer funnction H(z) is also BR atisfying the condition $|G(e^{j\omega})|^2=1-|H(e^{j\omega})|^2$. Let $\omega=\omega_o$ be a frequenncy where $|G(e^{j\omega})|$ is a maximum, i.e. $|G(e^{j\omega_o})|=1$. Then, it follows that $|H(e^{j\omega_o})|=0$. From the power-complementary condition it follows that

$2|G(e^{j\omega})|\frac{\partial|G(e^{j\omega})|}{\partial\omega}=-2|H(e^{j\omega})|\frac{\partial|H(e^{j\omega})|}{\partial\omega}$. Therefore at $\omega=\omega_o$,

$|G(e^{j\omega_o})|\left.\frac{\partial|G(e^{j\omega})|}{\partial\omega}\right|_{\omega=\omega_o}=-|H(e^{j\omega_o})|\left.\frac{\partial|H(e^{j\omega})|}{\partial\omega}\right|_{\omega=\omega_o}$, or $\left.\frac{\partial|G(e^{j\omega})|}{\partial\omega}\right|_{\omega=\omega_o}=0$ whether or not

$\left.\frac{\partial|H(e^{j\omega})|}{\partial\omega}\right|_{\omega=\omega_o}=0$. Hence, lowpassband sensitivity of G(z) does not necessarily imply low stopband sensitivity of H(z).

9.31 <u>Without error feedback</u>

The transfer function H(z) of the structure without feedback is given by

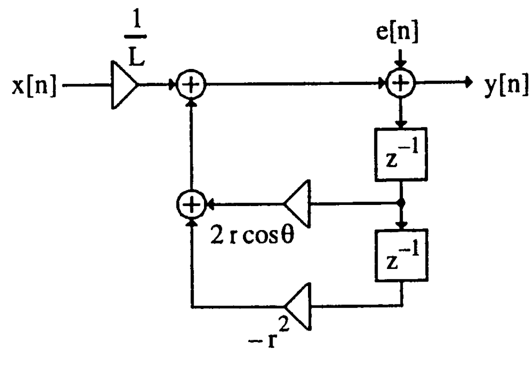
$$H(z)=\frac{1}{1+\alpha_1 z^{-1}+\alpha_1 z^{-1}}=\frac{1}{1-2r\cos\theta z^{-1}+r^2 z^{-1}},\ \text{where } r=1-\varepsilon.$$

The corresponding impulse response h[n] is given by $h[n]=\frac{r^n\sin(n+1)\theta}{\sin\theta}\cdot\mu[n]$.

To keep y[n] from overflowing we must insert a multiplier of value $\frac{1}{L}$ at the input where

$L = \sum_{n=0}^{\infty} |h[n]|$. From Eq. (9.163) we get $\dfrac{1}{(1-r)^2(1-2r\cos\theta+r^2)} \le L^2 \le \dfrac{16}{\pi^2(1-r)^2\sin^2\theta}$. (24)

The quantization noise model for H(z) is as shown below:



The output noise power is given by $\sigma_n^2 = \sigma_e^2 \sum_{n=0}^{\infty} |h[n]|^2 = \dfrac{1+r^2}{1-r^2} \cdot \dfrac{1}{r^4 - 2r^2\cos\theta + 1} \cdot \sigma_e^2$

The output signal power, assuming an input signal of variance σ_x^2 is given by

$\sigma_y^2 = \dfrac{\sigma_x^2}{L^2} \sum_{n=0}^{\infty} |h[n]|^2$. Hence, the SNR is given by $SNR = \dfrac{\sigma_y^2}{\sigma_n^2} = \dfrac{\sigma_x^2}{L^2\sigma_e^2}$. For a (b+1)-bit signed representation, $\sigma_e^2 = \dfrac{2^{-2b}}{12}$. Hence, $SNR = \dfrac{S}{N} = \dfrac{12\sigma_x^2}{L^2 2^{-2b}}$. Therefore, from the inequality of Eq. (24) we get $\dfrac{2^{-2b}}{12\sigma_x^2(1-r)^2(1-2r\cos\theta+r^2)} \le \dfrac{N}{S} \le \dfrac{16\times 2^{-2b}}{12\sigma_x^2\pi^2(1-r)^2\sin^2\theta}$.

(a) For an WSS uniformly distributed input between [– 1, 1], $\sigma_x^2 = \frac{1}{3}$. Hence,

$$\frac{2^{-2b}}{4(1-r)^2(1-2r\cos\theta+r^2)} \le \frac{N}{S} \le \frac{4}{\pi^2}\frac{2^{-2b}}{(1-r)^2\sin^2\theta}.$$

If $\varepsilon \to 0$, and $\theta \to 0$, then $(1-r)^2 \to \varepsilon^2$, $\cos 2\theta = 1 - 2\sin^2\theta \cong 1 - 2\theta^2$, and $\sin^2\theta \cong \theta^2$.

In this case we have $\dfrac{2^{-2b}}{4\varepsilon^2(\varepsilon^2+4\theta^2)} \le \dfrac{N}{S} \le \dfrac{2^{-2b}}{\pi^2\varepsilon^2\theta^2}$.

(b) For an input with a Gaussian distribution between [– 1, 1], $\sigma_x = \frac{1}{3}$.

In this case we have $3\dfrac{2^{-2b}}{4\varepsilon^2(\varepsilon^2+4\theta^2)} \le \dfrac{N}{S} \le \dfrac{3}{\pi^2}\dfrac{2^{-2b}}{\varepsilon^2\theta^2}$.

(c) For a sinusoidal input between [– 1, 1] of known frequency ω_o, $\sigma_x^2 = \frac{1}{2}$.

The output noise variance here is therefore $\sigma_n^2 = \sigma_e^2 \sum_{n=0}^{\infty} |h[n]|^2 = \dfrac{2^{-2b}}{12} \cdot \dfrac{1+r^2}{1-r^2} \cdot \dfrac{1}{r^4 - 2r^2\cos 2\theta + 1}$.

Thus, $\dfrac{N}{S}=\dfrac{1+r^2}{1-r^2}\cdot\dfrac{1}{6\left(r^4-2r^2\cos 2\theta+1\right)}=\dfrac{2^{-2b}}{24\varepsilon\left(\varepsilon^2+\theta^2\right)}\cong\dfrac{2^{-2b}}{24\varepsilon\theta^2}.$

<u>With error feedback</u>

$$G(z)=\frac{1-2z^{-1}+z^{-1}}{1-2r\cos\theta z^{-1}+r^2z^{-1}}=\frac{z^2-2z+1}{z^2-2r\cos\theta z+r^2}=1+\frac{2(r\cos\theta-1)z+1-r^2}{z^2-2r\cos\theta z+r^2}.$$

Thus, $\|G\|^2=1+\dfrac{\left(4(1-r\cos\theta)^2+(1-r^2)^2\right)(1-r^4)+8r\cos\theta(1-r^2)^2(r\cos\theta-1)}{(1-r^2)^2+2r^2(4r^2\cos^2\theta)-4(1+r^4)r^2\cos^2\theta}$

For $r=1-\varepsilon$ with $\varepsilon\to 0$, and $\theta\to 0$, we get after some manipulation $\|G\|^2=1+\dfrac{\theta^4}{4\varepsilon(\varepsilon^2+\theta^2)}$,

and $\sigma_n^2=\sigma_e^2\|G\|^2$.

Now L^2 remains the same as before since it depends only upon the denominator. Also the overall transfer function of the structure remains the same as before. The output noise power with error feedback is thus $\hat{N}=\sigma_e^2\|G\|^2$, whereas, the output noise power without error feedback is $N=\sigma_e^2\|H\|^2$. Hence, $\dfrac{\hat{N}}{N}=\dfrac{\|G\|^2}{\|H\|^2}$.

Now, $\|G\|^2=1+\dfrac{\theta^4}{4\varepsilon(\varepsilon^2+\theta^2)}$, Since $\theta>>\varepsilon$, $\|G\|^2\cong\dfrac{\theta^4}{4\varepsilon\theta^2}=\dfrac{\theta^2}{4\varepsilon}$.

Also, $\|H\|^2=\dfrac{1+r^2}{1-r^2}\cdot\dfrac{1}{r^4-2r^2\cos^2\theta+1}=\dfrac{1+(1-\varepsilon)^2}{1-(1-\varepsilon)^2}\cdot\dfrac{1}{(1-\varepsilon)^4-2(1-\varepsilon)^2(1-2\theta^2)+1}$

$$\approx\frac{1}{4\varepsilon(\theta^2+\varepsilon^2)}\approx\frac{1}{4\varepsilon\theta^2}.$$

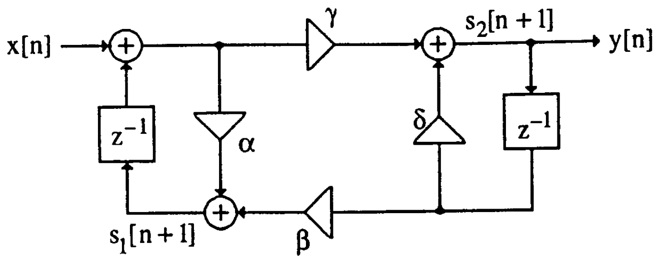
Thus, $\dfrac{\hat{N}}{N}\cong\theta^4$. As a result, with error feedback, the $\dfrac{N}{S}$ ratio gets multiplied by θ^4.

(a) input with uniform density: $\dfrac{2^{-2b}\theta^2}{16\varepsilon^2}\le\dfrac{N}{S}\le\dfrac{\theta^2}{\pi^2}\dfrac{2^{-2b}}{\varepsilon^2}$

(b) Wide-sense stationary, Gaussian density, white: $\dfrac{2^{-2b}3\theta^2}{16\varepsilon^2}\le\dfrac{N}{S}\le\dfrac{3\theta^2}{\pi^2}\dfrac{2^{-2b}}{\varepsilon^2}$

(c) sinusoid with known frequency: $\dfrac{N}{S}=\dfrac{2^{-2b}\theta^2}{24\varepsilon}$

9.32 The transfer function of the coupled-form structure shown below is given by Eq. (9.42) where $\alpha=\delta=r\cos(\theta)$ and $\beta=-\gamma=r\sin(\theta)$. Analysis yields



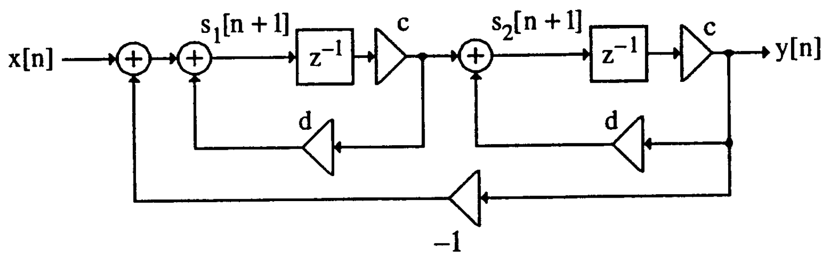
$s_1[n+1] = \alpha\big(x[n]+s_1[n]\big)+\beta s_2[n]$ and $s_2[n+1] = \gamma\big(x[n]+s_1[n]\big)+\delta s_2[n]$. Rewriting these equations in matrix form we get $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} x[n]$

Thus $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Therefore, $A^T A = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha^2+\gamma^2 & \alpha\beta+\gamma\delta \\ \alpha\beta+\gamma\delta & \beta^2+\delta^2 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}$.

Likewise, $AA^T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} \alpha^2+\beta^2 & \alpha\gamma+\beta\delta \\ \alpha\gamma+\beta\delta & \gamma^2+\delta^2 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}$.

Thus A is of normal form and hence, the structure will not support limit cycles.

9.33 The transfer function of the modified coupled-form structure shown below is given by



$s_1[n+1] = cds_1[n] - cs_2[n] + x[n]$, $s_2[n+1] = cs_1[n] + cds_2[n]$, and $y[n] = cs_2[n]$.
In matrix form these equations can be written as

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} cd & -c \\ c & cd \end{bmatrix}\begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n], \text{ and } y[n] = [0 \quad c]\begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + [0]x[n].$$

Using Eq. (4.56) the transfer function is given as $H(z) = [0 \quad c]\begin{bmatrix} z-cd & c \\ -c & z-cd \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \frac{1}{\big(z^2-2cdz+c^2(1+d^2)\big)}[0 \quad c]\begin{bmatrix} z-cd & -c \\ c & z-cd \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{c^2}{z^2-2cdz+c^2(1+d^2)}.$$

Comparing denominator of H(z) with the denominator $z^2 - 2r\cos\theta z + r^2$ of a second order transfer function we get $c = r\sin\theta$, $d = \cot\theta$. Then $A = \begin{bmatrix} cd & -c \\ c & cd \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$.

Thus $A^T A = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}$, and similarly $AA^T = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}$. Since for stability $r < 1$, A is normal form matrix, and thus the structure does not support limit cycles.

9.34 From Section 9.9.3, under zero-input conditions we have
$v[n+1] = A\,s[n]$,

$s[n+1] = \mathcal{Q}(\mathbf{v}[n+1])$.

The quadratic function $f(s[n]) = \mathbf{s}^T[n]\,\mathbf{D}\,s[n]$, where $\mathbf{D}$ is a positive-definite diagonal matrix, is related to the power stored in the delays. The changes in this quantity can provide information regarding oscillations under zero-input conditions:

$$\Delta f(s[n]) = f(s[n+1]) - f(s[n]) = -\mathbf{s}^T[n]\mathbf{D}s[n] + \mathbf{s}^T[n+1]\mathbf{D}s[n+1]$$
$$= -\mathbf{s}^T[n]\mathbf{D}s[n] + \mathbf{s}^T[n+1]\mathbf{D}s[n+1] + \mathbf{v}^T[n+1]\mathbf{D}\,\mathbf{v}[n+1] - \mathbf{v}^T[n+1]\mathbf{D}\,\mathbf{v}[n+1]$$
$$= -\mathbf{s}^T[n]\mathbf{D}s[n] + \mathbf{s}^T[n+1]\mathbf{A}^T\mathbf{D}\mathbf{A}s[n+1] - \sum_{k=1}^{N}\left(v_k^2[n+1] - \mathcal{Q}(v_k[n+1])^2\right)d_{kk}$$
$$= -\mathbf{s}^T[n]\left(\mathbf{D} - \mathbf{A}^T\mathbf{D}\mathbf{A}\right)s[n] - \sum_{k=1}^{N}\left(v_k^2[n+1] - \mathcal{Q}(v_k[n+1])^2\right)d_{kk}.$$

Now if $\mathbf{s}^T[n]\left(\mathbf{D} - \mathbf{A}^T\mathbf{D}\mathbf{A}\right)s[n] \ge 0$, and $v_k[n]$ are quantized such that $|\mathcal{Q}(v_k[n+1])| \le |v_k[n+1]|$, (i.e. using a passive quantizer), then the power stored in the delays will not increase with increasing n. So under passive quantization, limit cycles will be eliminated if $\mathbf{s}^T[n]\left(\mathbf{D} - \mathbf{A}^T\mathbf{D}\mathbf{A}\right)s[n] \ge 0$, or $\mathbf{D} - \mathbf{A}^T\mathbf{D}\mathbf{A}$ is positive-definite. For second order stable IIR filters, eigen values of $\mathbf{A} = [a_{ij}]$ are less than 1. This condition is satisfied if

$$a_{12}a_{21} \ge 0,$$

or, $\quad a_{12}a_{21} < 0$ and $|a_{11} - a_{22}| + \det(\mathbf{A}) \le 1$.

For the given structure,

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & \alpha_1 + 1 \\ \alpha_2 - 1 & \alpha_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1], \text{ and } D = 1.$$

The transfer function of the filter is given by

$$H(z) = \frac{z^2 - (\beta_1 + \beta_2)z + (1 + \beta_1 - \beta_2)}{z^2 - (\alpha_1 + \alpha_2)z + (1 + \alpha_1 - \alpha_2)}.$$

The filter is stable if $1 + \alpha_1 - \alpha_2 < 1$, and $|\alpha_1 + \alpha_2| < 1 + (1 + \alpha_1 - \alpha_2)$, or equivalently, if $\alpha_1 - \alpha_2 < 0$, and $|\alpha_1 + \alpha_2| < 2 + \alpha_1 - \alpha_2$.

Now, $a_1 a_2 = (\alpha_1 + 1)(\alpha_2 - 1) = \frac{1}{4}\left[(\alpha_1 + \alpha_2)^2 - (2 + \alpha_1 - \alpha_2)^2\right]$, and since, $(\alpha_1 + \alpha_2)^2 < (2 + \alpha_1 - \alpha_2)^2$, $a_{12}a_{21} < 0$.

Next, $|a_{11} - a_{22}| + \det(\mathbf{A}) = |\alpha_1 - \alpha_2| + \alpha_1\alpha_2 - (\alpha_1 + 1)(\alpha_2 - 1)$
$$= = -(\alpha_1 - \alpha_2) + \alpha_1\alpha_2 - (\alpha_1\alpha_2 + \alpha_2 - \alpha_1 - 1) = 1.$$

Hence, the structure does not support zero-input limit cycles.

9.35 From Section 8.3.1 we know that each computation of a DFT sample requires 2N + 4 real multiplications. Assuming that quantization noise generated from each multiplier is independent of the noise generated from other multipliers, we get $\sigma_r^2 = (2N+4)\sigma_0^2 = \dfrac{2^{-2b}(N+2)}{6}$.

9.36 SNR = $\dfrac{2^{2b}}{N^2}$. Hence, an SNR of 25 dB implies $\dfrac{2^{2b}}{N^2} = 10^{2.5}$ or $b = \frac{1}{2}\log_2\left((10)^{2.5} \times (512)^2\right) = 13.1524$. Therefore b = 14 bits should be chosen to get an SNR of 25 dB. Therefore number of bits required for each sample = 14 + 1 = 15.

9.37 Let $N = 2^{\nu}$. Consider the m^{th} stage. The output sees $4(2^{\nu-m})$ noise sources from the m^{th} stage. Each noise source has a variance reduction by a factor of $\left(\frac{1}{4}\right)^{\nu-m}$ due to multiplication by $\frac{1}{2}$ at each stage till the output. Hence the total noise variance at the output due to the noises injected in the m^{th} stage is $4\left(2^{\nu-m}\right)\left(2^{-2(\nu-m)}\right)\sigma_0^2$.

Therefore the total noise variance at the output $= \sigma^2 = \sum_{m=1}^{\nu} 4\left(2^{\nu-m}\right)\left(2^{-2(\nu-m)}\right)\sigma_0^2$

$$4\sigma_0^2 2^{-\nu}\sum_{m=1}^{\nu} 2^m = 4\frac{2^{-2b}}{12}2^{-\nu}\frac{2(2^{\nu}-1)}{2-1} = \frac{2}{3}2^{-2b}(1-2^{-\nu}) \approx \frac{2}{3}2^{-2b} \text{ for large N.}$$

9.38 $\text{SNR} = \dfrac{2^{2b}}{2N}$. Hence $b = \frac{1}{2}\log_2\left((10)^{2.5}\times 2\times 512\right) = 8.652$.
Hence we choose b + 1 = 10 bits per sample to get an SNR of 25 dB.

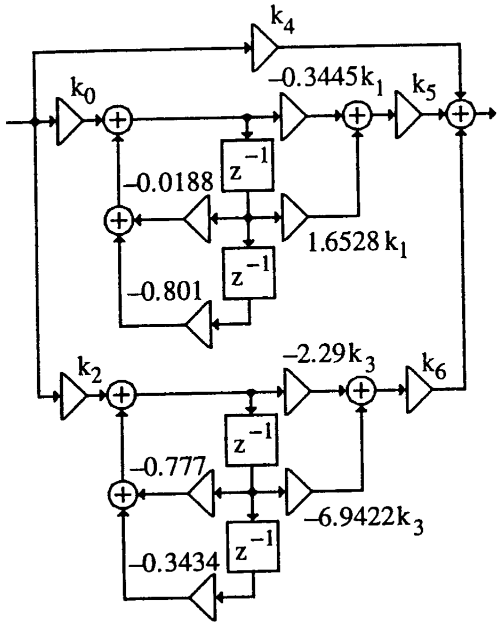
M9.1
```
wp = 0.3; ws = 0.35; Rp = 0.01; Rs = 50;
[N,wn] = ellipord(wp,ws,Rp,Rs);
[B,A] = ellip(N,Rp,Rs,wn);
B1 = 1.05*B;
A1 = [A(1) 1.05*A(2:length(A))];
[H,W] = freqz(B,A,512);
[H1,W1] = freqz(B1,A1,512);
plot(W/pi,20*log10(abs(H)),'',W1/pi,20*log10(abs(H1)),'--');
figure;
Z=[roots(B) roots(B1)];P=[roots(A) roots(A1)];
zplane(Z,P);
```

M9.2
```
N = 5; wn = 0.4; Rp = 1; Rs = 40;
[B,A] = ellip(N,Rp,Rs,wn);
z = cplxpair(roots(B)); p = cplxpair(roots(A));
disp('Factors for the numerator');
const = B(1)/A(1);
k = 1;
while k <= length(z),
      if(imag(z(k)) ~=0)
         factor = [1 -2*real(z(k)) abs(z(k))^2]
         k = k+2;
      else
         factor = [1 -z(k)]
         k = k+1;
      end
end
disp('Factors for the denominator');
k = 1;
while k <= length(p),
     if(imag(p(k)) ~=0)
         factor = [1 -2*real(p(k)) abs(p(k))^2]
         k = k+2;
       else
         factor = [1 -p(k)]
         k = k+1;
       end
end
sos = zp2sos(z,p,const);
```

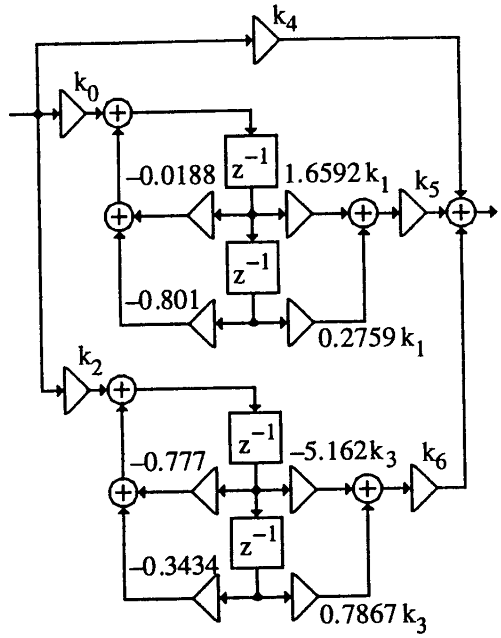
M9.3
```
[B,A] = ellip(5,1,50,0.4);
p = roots(A);
lenp = length(p);
[Y,I] = sort(angle(p));
for k = 1:lenp
    if(rem(k,2)==1)
        p1((k+1)/2) = p(I(k));
    else
            p2(k/2) = p(I(k));
    end
end
b1 = poly(p1); b2 = poly(p2);
a1 = fliplr(b1); a2 = fliplr(b2);
B1 = 0.5*(conv(b2,a1)-conv(b1,a2));
A1 = conv(b1,b2);
[H,W] = freqz(B,A,512); [H1,W] = freqz(B1,A1,512);
plot(W/pi,abs(H),'-',W/pi,abs(H1),'--');
a1 = 1.01*a1; a2 = 1.01*a2;
b1 = [b1(1) 1.01*b1(2:length(b1))];
b2 = [b2(1) 1.01*b2(2:length(b2))];
A3 = conv(b1,b2);
B3 = 0.5*conv(a1,b2)+0.5*conv(a2,b1);
B4 = 1.01*B;
A4 = [A(1) 1.01*A(2:length(B))];
[H2,W] = freqz(B3,A3,512); [H3,W] = freqz(B4,A4,512);
figure;
plot(W/pi,abs(H2),'-',W/pi,abs(H3),'--');
```

M9.4
```
num1 = input('The first factor of numerator = ');
num2 = input('The second factor of numerator =');
den1 = input('The first factor of denominator =');
den2 = input('The second factor of denominator =');
% The numerator and denominator of the scaling
% functions f1 and f2 are
f1num = 1; f1den = [den1]; f2num = num1;
f2den = conv(den1,den2); f3num = conv(num1,num2);
f3den = conv(den1,den2);
x = [1 zeros([1,511])];
% Sufficient length for impulse response
% to have decayed to nearly zero
f1 = filter(f1num,f1den,x); f2 = filter(f2num,f2den,x);
f3 = filter(f3num,f3den,x);
k1 = sqrt(sum(f1.*f1)); k2 = sqrt(sum(f2.*f2));
k3 = sqrt(sum(f3.*f3));
disp('The first scaling factor ='); disp(k1);
disp('The second scaling factor ='); disp(k2);
disp('The third scaling factor ='); disp(k3);
% The noise transfer functions
g1num = conv(num1,num2)/(k2*k3);
g1den = conv(den1,den2)/k3;
g2num = num2;      g2den = den2;
g1 = filter(g1num,g1den,x); g2 = filter(g2num,g2den,x);
var = sum(f1.*f1)*3+sum(g2.*g2)*5+3;
disp('The normalized noise variance'); disp(var);
% num1 and num2 can be interchanged to come up with the
% second realization
```

M9.5 The parallel form I structure and the parallel form II structure used for simulation are shown below:



Parallel Form I

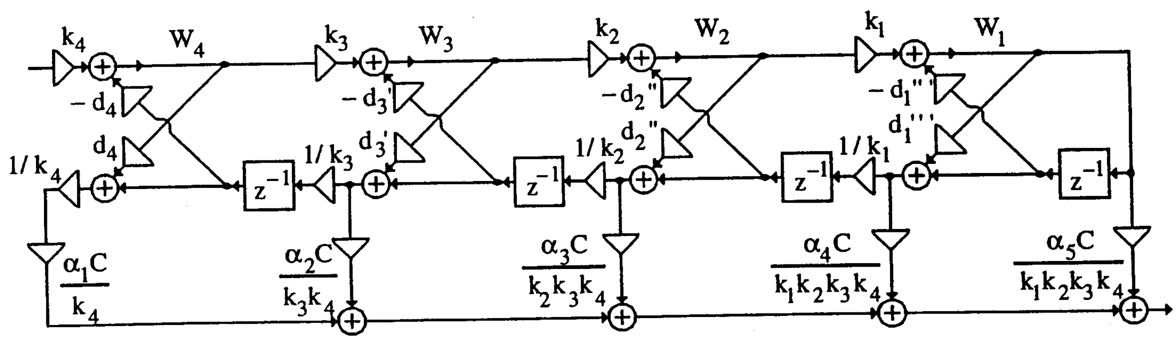


Parallel Form II

```
num1 = input('The first factor in the numerator =');
num2 = input('The second factor in the numerator =');
den1 = input('The first factor in the denominator =');
den2 = input('The second factor in the denominator =');
num = conv(num1,num2);
den = conv(den1,den2);
[r1,p1,k11] = residuez(num,den);
[r2,p2,k21] = residue(num,den);
% Simulation of structure for Eq. 9.244
R1 = [r1(1) r1(2)];P1 = [p1(1) p1(2)];
R2 =[r1(3) r1(4)];P2 = [p1(3) p1(4)];
R3 = [r2(1) r2(2)];P3 = [p2(1) p2(2)];
R4 = [r2(3) r2(4)];P4 = [p2(3) p2(4)];
[num11,den11] = residuez(R1,P1,0);
[num12,den12] = residuez(R2,P2,0);
[num21,den21] = residue(R3,P3,0);
[num22,den22] = residue(R4,P4,0);
disp('The numerators for Parallel Form I');
disp(k11); disp(num11); disp(num12);
disp('The denominators for Parallel Form I');
disp(den11); disp(den12);
disp('The numerators for Parallel Form II');
disp(k21); disp(num21); disp(num22);
disp('The denominators for Parallel Form II');
disp(den21); disp(den22);
imp = [1 zeros([1,2000])];
y0 = filter([1 0 0],den11,imp); y1 = filter(num11,den11,imp);
y2 = filter([1 0 0],den12,imp); y3 = filter(num12,den12,imp);
gamma0 = sum(y0.*conj(y0)); gamma1 = sum(y1.*conj(y1));
gamma2 = sum(y2.*conj(y2)); gamma3 = sum(y3.*conj(y3));
k0 = sqrt(1/gamma0); k1 = sqrt(gamma0/gamma1);
k2 = sqrt(1/gamma2); k3 = sqrt(gamma2/gamma3);
```

```
y = filter(num,den,imp);
gamma = sum(y.*conj(y));
k4 = sqrt(1/gamma); k5 = k4/(k0*k1); k6 = k4/(k2*k3);
disp('For parallel form I');
disp('The scaling constants are'); disp(k0);disp(k1);
disp(k2);disp(k3);disp(k4);disp(k5);disp(k6);
disp('The product roundoff noise variance');
noise = 3*(k5/k0)^2+3*(k6/k2)^2+2*k5^2+2*k6^2+3;
disp(noise);
%%%%%%%%%For Parallel From II %%%%%%%%%%%%%%%
y0 = filter([0 0 1],den21,imp);
y1 = filter(fliplr(num21),den21,imp);
y2 = filter([0 0 1],den22,imp);
y3 = filter(fliplr(num22),den22,imp);
gamma0 = sum(y0.*conj(y0)); gamma1 = sum(y1.*conj(y1));
gamma2 = sum(y2.*conj(y2)); gamma3 = sum(y3.*conj(y3));
k0 = sqrt(1/gamma0); k1 = sqrt(gamma0/gamma1);
k2 = sqrt(1/gamma2); k3 = sqrt(gamma2/gamma3);
y = filter(num,den,imp);
gamma = sum(y.*conj(y));
k4 = sqrt(1/gamma); k5 = k4/(k0*k1); k6 = k4/(k2*k3);
disp('For parallel form II');
disp('The scaling constants are');disp(k0);disp(k1);disp(k2);
disp(k3);disp(k4);disp(k5);disp(k6);
disp('The product roundoff noise variance');
noise = 3*(k5/k0)^2+3*(k6/k2)^2+2*k5^2+2*k6^2+3; disp(noise);
```

M9.6 The scaled Gray-Markel cascaded lattice structure used for simulation is shown below:




```
% Use Program_6_3.m pg. 384 to generate the lattice
% parameters and feedforward multipliers
d = [0.19149189348920    0.75953417365130    0.44348979846264
0.27506340000000];
alpha = [1.00000000000000   -3.50277000000000
4.61511974525466   -1.70693992124303 -0.90009664306164];
imp = [1 zeros([1,499])];
qold1 = 0;
for k = 1:500
     w1 = imp(k)-d(1)*qold1;
     y1(k) = w1;
     qnew1 = w1;
     qold1 = qnew1;
end
k1 = sqrt(1/(sum(y1.*conj(y1))));
imp = [1 zeros([1,499])];
qold1 = 0;qold2 = 0;
```

```
for k = 1:500
    w2 = imp(k)-d(2)*qold2*1/k1;
    w1 = k1*w2-d(1)*qold1;
    y1(k) = w1;y2(k)=w2;
    qnew1 = w1;
    qnew2 = w1*d(1)+qold1;
    qold1 = qnew1;qold2 = qnew2;
end
k2 = sqrt(1/(sum(y2.*conj(y2))));
qold1 = 0;qold2 = 0;qold3 = 0;
for k = 1:500
    w3 = imp(k)-d(3)*qold3*1/k2;
    w2 = k2*w3-d(2)*qold2*1/k1;
    w1 = k1*w2-d(1)*qold1;
    y3(k) = w3;
    qnew1 = w1;
    qnew2 = w1*d(1)+qold1;
    qnew3 = w2*d(2)+qold2*1/k1;
    qold1 = qnew1;qold2 = qnew2;qold3 = qnew3;
end
k3 = sqrt(1/sum(y3.*conj(y3)));
qold1 = 0;qold2 = 0;qold3 = 0;qold4 = 0;
for k = 1:500
    w4 = imp(k)-d(4)*qold4/k3;
    w3 = k3*w4-d(3)*qold3/k2;
    w2 = k2*w3-d(2)*qold2/k1;
    w1 = k1*w2-d(1)*qold1;
    y4(k) = w4;
    qnew1 = w1;
    qnew2 = w1*d(1)+qold1;
        qnew3 = w2*d(2)+qold2*1/k1;
        qnew4 = w3*d(3)+qold3*1/k2;
    qold1 = qnew1;qold2 = qnew2;qold3 = qnew3;qold4 = qnew4;
end
k4 = sqrt(1/sum(y4.*conj(y4)));
const = 0.135127668
% Obtained by scaling the o/p of the actual TF
disp('The scaling parameters are');
disp(k1);disp(k2);disp(k3); disp(k4);
alpha(5) = alpha(5)/(k1*k2*k3*k4);
alpha(4) = alpha(4)/(k1*k2*k3*k4);
alpha(3) = alpha(3)/(k2*k3*k4);
alpha(2) = alpha(2)/(k3*k4);
alpha(1) = alpha(1)/k4;
alpha = const*alpha;
%%%% Computation of noise variance %%%%%%
% Noise variance due to k4 and d4 = 1/k4^2= 1.08185285036642
% To compute noise variance due to k3,d3' = 1.33858225
imp = [1 zeros([1,499])];
for k = 1:500
    w4 = -d(4)*qold4/k3;
    w3 = k3*w4-d(3)*qold3/k2;
    w2 = k2*w3-d(2)*qold2/k1;
    w1 = k1*w2-d(1)*qold1;
    qnew1 = w1;
    qnew2 = w1*d(1)+qold1;
    qnew3 = w2*d(2)+qold2*1/k1;
    qnew4 = w3*d(3)+qold3*1/k2;
```

```
    y11  =  w4*d(4)+qold4/k3;
    y0(k)  =  alpha(1)*y11+alpha(2)*qnew4+alpha(3)*qnew3  +
alpha(4)*qnew2+alpha(5)*qnew1;
    qold1  =  qnew1;qold2  =  qnew2;qold3  =  qnew3;qold4  =  qnew4;
end
nv  =  sum(y0.*conj(y0));
% for k2,-d2'' nv = 3.131899935
% for k1,-d1''' nv = 1.00880596097028
% for d1''' nv = 0.95806646140013
% for d2'' nv = 2.61615077290574
% for d3' nv = 0.41493478856386
% for d4 nv = 0.01975407768839
% for 1/k1 nv = 0.75359663926391
% for 1/k2 nv = 0.58314964498424
% for 1/k3 nv = 0.09095345118133
% Total nv = 23.55888782866100
```

Chapter 10

10.1 For an input $x_1[n]$ and $x_2[n]$, the outputs of the factor-of-L up-sampler are, respectively, given by

$$x_{u1}[n]=\begin{cases}x_1[n/L], & n=0,\pm L,\pm 2L,\ldots\\ 0, & \text{otherwise,}\end{cases} \quad \text{and} \quad x_{u2}[n]=\begin{cases}x_2[n/L], & n=0,\pm L,\pm 2L,\ldots\\ 0, & \text{otherwise.}\end{cases}$$

Let $x_2[n] = x_1[n - n_o]$, where n_o is an integer. Then $x_2[n/L]=x_1[(n/L)-n_o]$. Hence,

$$x_{u2}[n]=\begin{cases}x_1[(n/L)-n_o], & n=0,\pm L,\pm 2L,\ldots\\ 0, & \text{otherwise.}\end{cases}$$

But $x_{u1}[n-n_o]=\begin{cases}x_1[(n-n_o)/L], & n-n_o=0,\pm L,\pm 2L,\ldots\\ 0, & \text{otherwise,}\end{cases}$ Since $x_{u2}[n]\neq x_{u1}[n-n_o]$, the up-sampler is a time-varying system.

10.2 Consider first the up-sampler. Let $x_1[n]$ and $x_2[n]$ be the inputs with corresponding outputs given by $y_1[n]$ and $y_2[n]$. Now, $y_1[n]=\begin{cases}x_1[n/L], & n=0,\pm L,\pm 2L,\ldots\\ 0, & \text{otherwise,}\end{cases}$ and

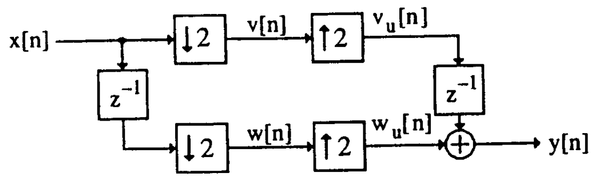
$y_2[n]=\begin{cases}x_2[n/L], & n=0,\pm L,\pm 2L,\ldots\\ 0, & \text{otherwise.}\end{cases}$ Let us now apply the input $x_3[n]=\alpha x_1[n]+\beta x_2[n]$, with the corresponding output given by $y_3[n]$, where

$$y_3[n]=\begin{cases}\alpha x_1[n/L]+\beta x_2[n/L], & n=0,\pm L,\pm 2L,\ldots\\ 0, & \text{otherwise,}\end{cases}$$

$=\begin{cases}\alpha x_1[n/L]\\ 0\end{cases}+\begin{cases}\beta x_2[n/L] & n=0,\pm L,\pm 2L,\ldots\\ 0 & \text{otherwise,}\end{cases}=\alpha y_1[n]+\beta y_2[n]$. Thus, the up-sampler is a linear system.

Now, consider the down-sampler. Let $x_1[n]$ and $x_2[n]$ be the inputs with corresponding outputs given by $y_1[n]$ and $y_2[n]$. Now, $y_1[n]=x_1[Mn]$, and $y_2[n]=x_2[Mn]$. Let us now apply the input $x_3[n]=\alpha x_1[n]+\beta x_2[n]$, with the corresponding output given by $y_3[n]$, where $y_3[n]=x_3[Mn]=\alpha x_1[Mn]+\beta x_2[Mn]$. Thus, the down-sampler is a linear system.

10.3



From the figure, $V(z)=\frac{1}{2}X(z^{1/2})+\frac{1}{2}X(-z^{1/2})$, $W(z)=\frac{z^{-1/2}}{2}X(z^{1/2})-\frac{z^{-1/2}}{2}X(-z^{1/2})$,

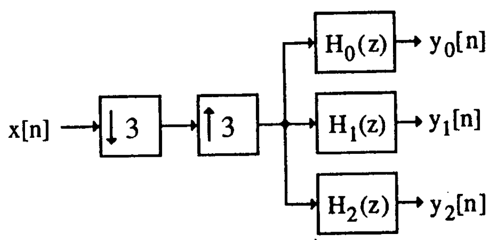
$V_u(z)=\frac{1}{2}X(z)+\frac{1}{2}X(-z)$, $W_u(z)=\frac{z^{-1}}{2}X(z)-\frac{z^{-1}}{2}X(-z)$. Hence,

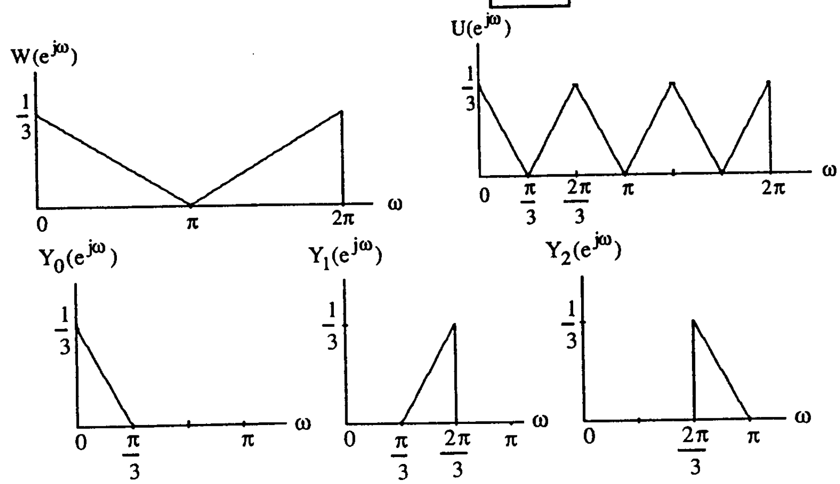
$Y(z)=z^{-1}V_u(z)+W_u(z)=z^{-1}X(z)$, or in other words, y[n] = x[n–1].

10.4 $c[n]=\frac{1}{M}\sum_{k=0}^{M-1}W_M^{kn}=\frac{1}{M}\left(\frac{1-W_M^{nM}}{1-W_M^{n}}\right)$. Hence, if $n\neq rM$, $c[n]=\frac{1}{M}\left(\frac{1-1}{1-W_M^{n}}\right)=0$. On the other hand, if n = rM, then $c[n]=\frac{1}{M}\sum_{k=0}^{M-1}W_M^{kn}=\frac{1}{M}\sum_{k=0}^{M-1}W_M^{krM}=\frac{1}{M}\sum_{k=0}^{M-1}1=1$. Thus,

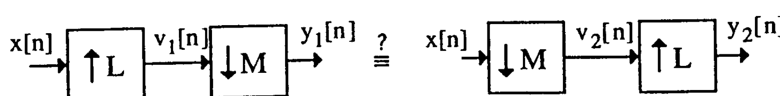
$$c[n]=\begin{cases}1, & \text{if } n=rM,\\ 0, & \text{otherwise.}\end{cases}$$

10.5





10.6

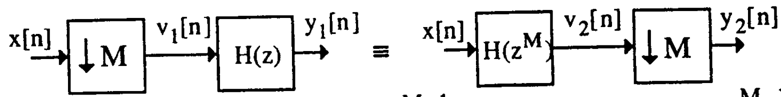


For the left-hand side figure, we have $V_1(z)=X(z^L)$, $Y_1(z)=\frac{1}{M}\sum_{k=0}^{M-1}X(z^{L/M}W_M^{kL})$.

For the right-hand side figure,we have $V_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^k)$, $Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M}W_M^k)$.

Since L and M are relatively prime, W_M^k and W_M^{kL} take the same set of values for k = 0, 1, . . ., M−1. Hence, $Y_1(z) = Y_2(z)$.

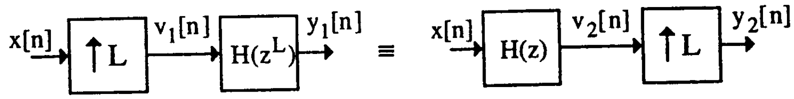
10.7



For the left-hand side figure, we have $V_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^k)$, $Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} H(z)X(z^{1/M}W_M^k)$,

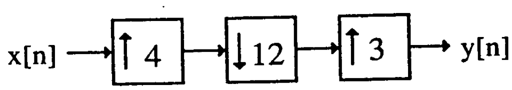
For the right-hand side figure,we have $V_2(z) = H(z^M)X(z)$, $Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} H(zW_M^{kM})X(z^{1/M}W_M^k)$

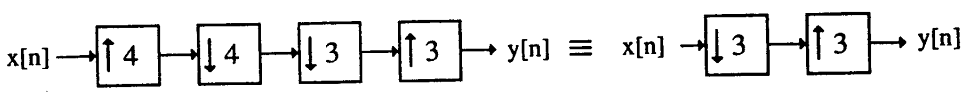
$= \frac{1}{M}\sum_{k=0}^{M-1} H(z)X(z^{1/M}W_M^k)$. Hence, $Y_1(z) = Y_2(z)$.



For the left-hand side figure, we have $V_1(z) = X(z^L)$, $Y_1(z) = H(z^L)X(z^L)$. For the right-hand side figure,we have $V_2(z) = H(z)X(z)$, $Y_2(z) = H(z^L)X(z^L)$. Hence, $Y_1(z) = Y_2(z)$.

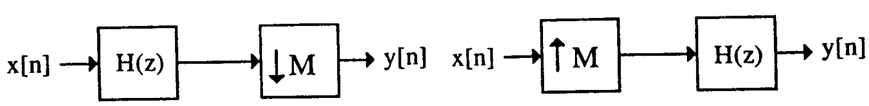
10.8



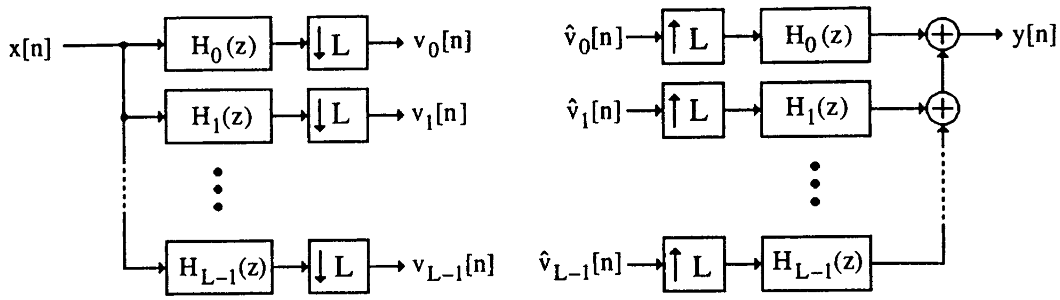


Hence, $y[n] = \begin{cases} x[n], & \text{for } n = 3r, \\ 0, & \text{otherwise.} \end{cases}$

10.9 As outlined in Section 6.3, the transpose of a digital filter structure is obtained by reversing all paths, replacing the pick-off node with an adder and vice-versa, and interchanging the input and the output nodes. Moreover, in a multirate structure, the transpose of a factor-of-M down-sampler is a factor-of-M up-sampler and vice-versa. Applying these operations to the factor-of-M decimator shown on the left-hand side, we arrive at a factor-of-M interpolator as indicated on the right-hand side in the figure below.



10.10 Applying the transpose operation to the L-channel analysis filter bank shown below on the left-hand side, we arrive at the L-channel synthesis filter bank shown below on the right-hand side.



10.11 $H(z) = G(z^{10})I(z)$. Specifications for H(z) are as follows: F_p = 180 Hz, F_s = 200 Hz, δ_p = 0.002, δ_s = 0.001. Therefore, specifications for G(z) are as follows: F_p = 1800 Hz, F_s = 2000 Hz, δ_p = 0.001, δ_s = 0.001. Here, $\Delta f = \frac{200}{12000}$. Likewise, specifications for I(z) are as follows: F_p = 180 Hz, F_s = 1000 Hz, δ_p = 0.001, δ_s = 0.001. Here, $\Delta f = \frac{820}{12000}$. Hence, from Eq. (7.13), length of G(z) is given by $N_G = \frac{-20\log_{10}\sqrt{10^{-6}} - 13}{14.6(200/12000)} = \frac{47 \times 12000}{14.6 \times 200} = 194$. Likewise, length of I(z) is given by $N_I = \frac{-20\log_{10}\sqrt{10^{-6}} - 13}{14.6(820/12000)} = \frac{(-20)(-3) - 13}{14.6(820/12000)} = 48$. Thus, $\mathcal{R}_{M,G} = 194 \times \frac{1200}{3} = 77{,}600$ muliplications/second (mps), and $\mathcal{R}_{M,I} = 48 \times \frac{12000}{10} = 57{,}600$ mps Hence, total no. of mps = 135,200. Hence the computational complexity is slightly higher in this case.

10.12 Specifications for H(z) are as follows: F_p = 350 Hz, F_s = 400 Hz, δ_p = 0.003, δ_s = 0.004. Here, $\Delta f = \frac{50}{32000}$. Hence, from Eq. (7.13), length of H(z) is given by

$$N_H = \frac{-20\log_{10}\sqrt{12 \times 10^{-6}} - 13}{14.6(50/32000)} = 1588.$$ Thus, $\mathcal{R}_{M,H} = 1588 \times \frac{32000}{40} = 1270400.$

10.13 $H(z) = G(z^{20})I(z)$.Specifications for G(z) are as follows: F_p = 7000 Hz, F_s = 8000 Hz, δ_p = 0.0015, δ_s = 0.004. Here, $\Delta f = \frac{1000}{32000}$. Likewise, specifications for I(z) are as follows: F_p = 350 Hz, F_s = 1200 Hz, δ_p = 0.0015, δ_s = 0.004. Here, $\Delta f = \frac{850}{32000}$. Hence, from Eq. (7.13), length of G(z) is given by $N_G = \frac{-20\log_{10}\sqrt{6 \times 10^{-6}} - 13}{14.6(1000/32000)} = 86$, and length of I(z) is given by $N_I = \frac{-20\log_{10}\sqrt{6 \times 10^{-6}} - 13}{14.6(850/32000)} = 102$. Thus, $\mathcal{R}_{M,G} = 86 \times 800 = 68{,}800$ and $\mathcal{R}_{M,I} = 102 \times \frac{32000}{20} = 163{,}200$. Hence, total no. of mps = 232,000.

10.14 (a) $$H_1(z) = \frac{p_0 + p_1 z^{-1}}{1 + d_1 z^{-1}} = \left(\frac{p_0 + p_1 z^{-1}}{1 + d_1 z^{-1}}\right)\left(\frac{1 - d_1 z^{-1}}{1 - d_1 z^{-1}}\right) = \left(\frac{p_0 - p_1 d_1 z^{-2}}{1 - d_1^2 z^{-2}}\right) + z^{-1}\left(\frac{p_1 - p_0 d_1}{1 - d_1^2 z^{-2}}\right).$$

(b) $$H_2(z)=\frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}}=\left(\frac{2+3.1z^{-1}+1.5z^{-2}}{(1+0.8z^{-2})+0.9z^{-1}}\right)\left(\frac{(1+0.8z^{-2})-0.9z^{-1}}{(1+0.8z^{-2})-0.9z^{-1}}\right)$$

$$=\frac{2+(3.1-1.8)z^{-1}+(1.6-0.9\times 3.1+1.5)z^{-2}+(-0.9\times 1.5+0.8\times 3.1)z^{-3}+1.5\times 0.8z^{-4}}{1+0.64z^{-4}+1.6z^{-2}-0.819z^{-2}}$$

$$=\frac{2+1.3z^{-1}+0.31z^{-2}+1.13z^{-3}+1.2z^{-4}}{1+0.79z^{-2}+0.64z^{-4}}=\frac{2+0.31z^{-2}+1.2z^{-4}}{1+0.79z^{-2}+0.64z^{-4}}+z^{-1}\frac{1.3+1.13z^{-2}}{1+0.79z^{-2}+0.64z^{-4}}.$$

(c) $$H_3(z)=\frac{2+3.1z^{-1}+1.5z^{-2}+4z^{-3}}{(1-0.5z^{-1})(1+0.9z^{-1}+0.8z^{-2})}\left(\frac{(1+0.5z^{-1})(1-0.9z^{-1}+0.8z^{-2})}{(1+0.5z^{-1})(1-0.9z^{-1}+0.8z^{-2})}\right)$$

$$=\frac{(2+3.1z^{-1}+1.5z^{-2}+4z^{-3}+0.5z^{-1}+1.55z^{-2}+0.75z^{-3}+2z^{-4})(1+0.8z^{-2}-0.9z^{-1})}{(1-0.25z^{-2})(1+0.79z^{-2}+0.64z^{-4})}$$

$$=\frac{2+2.3z^{-1}+0.96z^{-2}+5.285z^{-3}+0.165z^{-4}+2z^{-5}+1.6z^{-6}}{1+0.54z^{-2}+0.4425z^{-4}-0.16z^{-6}}$$

$$=\frac{2+0.96z^{-2}+0.165z^{-4}+1.6z^{-6}}{1+0.54z^{-2}+0.4425z^{-4}-0.16z^{-6}}+z^{-1}\frac{2.3+5.285z^{-2}+2z^{-4}}{1+0.54z^{-2}+0.4425z^{-4}-0.16z^{-6}}.$$

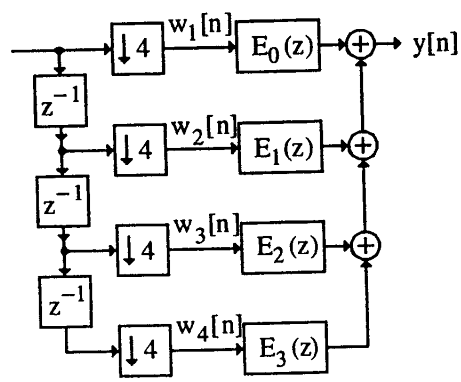
10.15 A computationally efficient realization of the factor-of-4 decimator

$$\rightarrow \boxed{H(z)} \rightarrow \boxed{\downarrow 4} \rightarrow$$

is obtained by applying a 4-branch polyphase decomposition to H(z):

$$H(z)=E_0(z^4)+z^{-1}E_1(z^4)+z^{-2}E_2(z^4)+z^{-3}E_3(z^4).$$

and then moving the down-sampler through the polyphase filters resulting in



Further reduction in computational complexity is achieved by sharing comon multipliers if H(z) is a linear-phase FIR filter. For example, for a length-16 Type I FIR transfer function

$$H(z)=h[0]+h[1]z^{-1}+h[2]z^{-2}+h[3]z^{-3}+h[4]z^{-4}+h[5]z^{-5}+h[6]z^{-6}+h[7]z^{-7}+h[7]z^{-8}$$
$$+h[6]z^{-9}+h[5]z^{-10}+h[4]z^{-11}+h[3]z^{-12}+h[2]z^{-13}+h[1]z^{-14}+h[0]z^{-15},$$

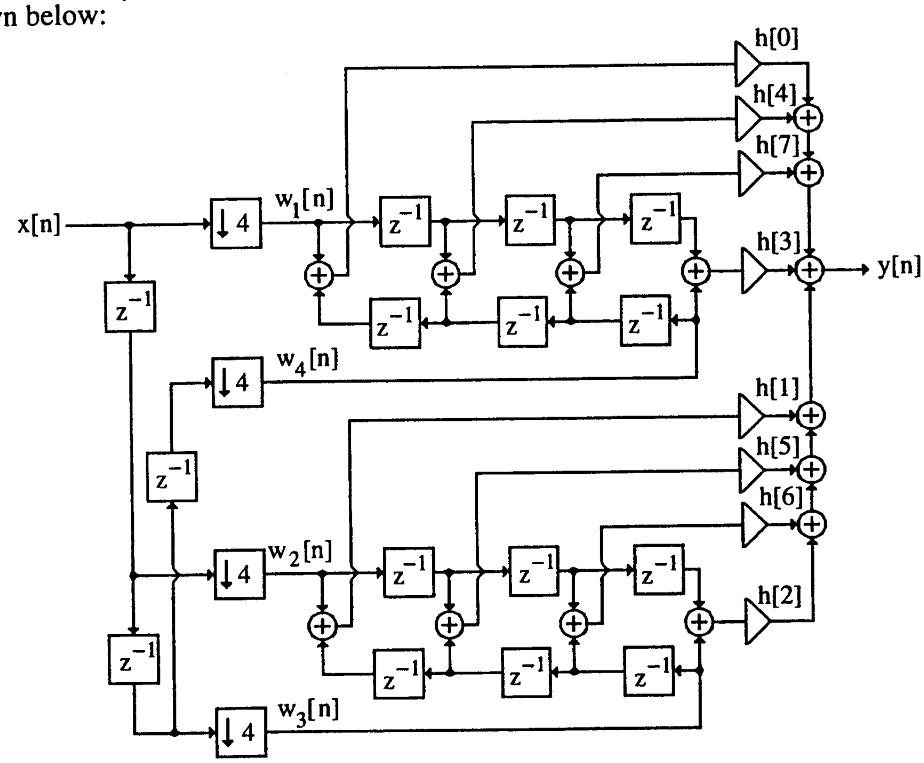
for which $E_0(z)=h[0]+h[4]z^{-1}+h[7]z^{-2}+h[3]z^{-3}$, $E_1(z)=h[1]+h[5]z^{-1}+h[6]z^{-2}+h[2]z^{-3}$,

$E_2(z) = h[2] + h[6]z^{-1} + h[5]z^{-2} + h[1]z^{-3}$, and $E_3(z) = h[3] + h[7]z^{-1} + h[4]z^{-2} + h[0]z^{-3}$.

From the above figure it follows that $Y(z) = E_0(z)W_1(z) + E_1(z)W_2(z) + E_2(z)W_3(z) + E_3(z)W_4(z)$

$$= h[0]\left(W_1(z) + z^{-3}W_4(z)\right) + h[4]\left(z^{-1}W_1(z) + z^{-2}W_4(z)\right)$$
$$+ h[7]\left(z^{-2}W_1(z) + z^{-1}W_4(z)\right) + h[3]\left(z^{-3}W_1(z) + W_4(z)\right)$$
$$+ h[1]\left(W_2(z) + z^{-3}W_3(z)\right) + h[5]\left(z^{-1}W_2(z) + z^{-2}W_3(z)\right)$$
$$+ h[6]\left(z^{-2}W_2(z) + z^{-1}W_3(z)\right) + h[2]\left(z^{-3}W_2(z) + W_3(z)\right).$$

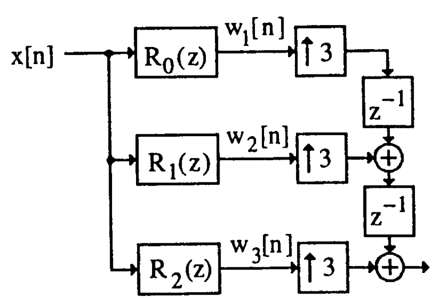
A computationally efficient factor-of-4 decimator structure based on the above equation is then as shown below:



10.16 A computationally efficient realization of the factor-of-3 interpolator is obtained by applying a 3-branch Type II polyphase decomposition to the interpolation filter $H(z)$:

$$H(z) = R_2(z^3) + z^{-1}R_1(z^3) + z^{-2}R_0(z^3),$$

and then moving the up-sampler through the polyphase filters resulting in



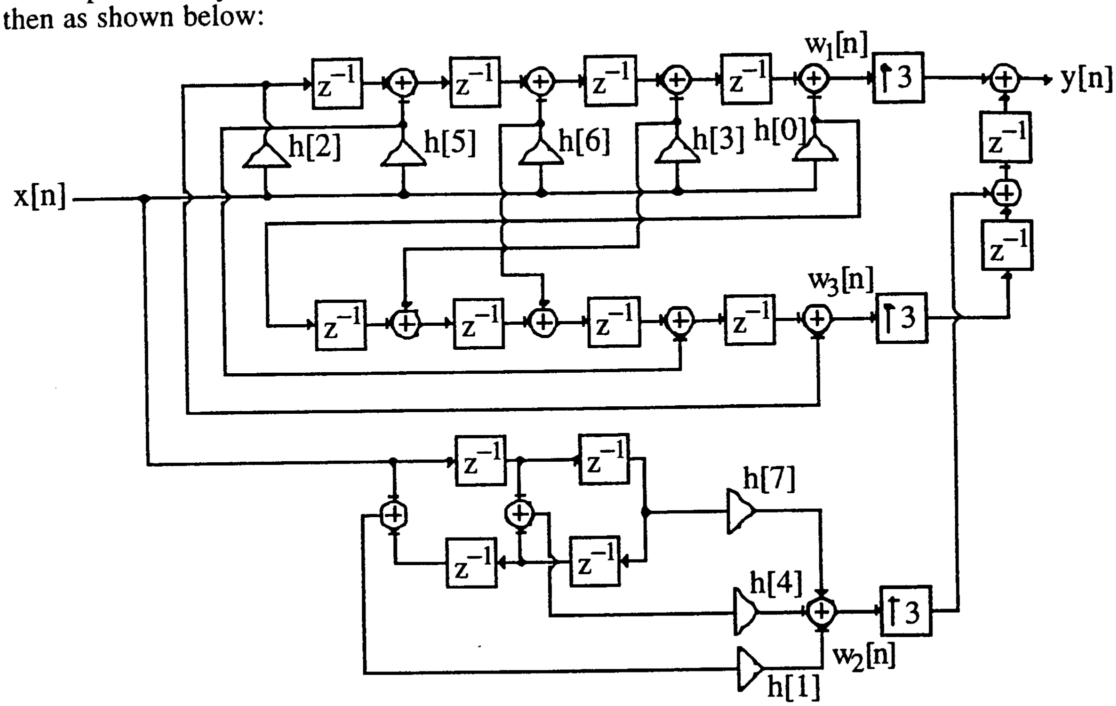
From the above figure it follows that

$W_3(z) = h[0]X(z) + h[3]z^{-1}X(z) + h[6]z^{-2}X(z) + h[5]z^{-3}X(z) + h[2]z^{-4}X(z),$

$W_1(z) = h[2]X(z) + h[5]z^{-1}X(z) + h[6]z^{-2}X(z) + h[3]z^{-3}X(z) + h[0]z^{-4}X(z),$ and

$W_2(z) = h[1]\left(X(z) + z^{-4}X(z)\right) + h[4]\left(z^{-1}X(z) + z^{-3}X(z)\right) + h[7]z^{-2}X(z).$

A computationally efficient factor-of-3 interpolator structure based on the above equations is then as shown below:



10.17 An ideal M-th band lowpass filter H(z) is characterized by a frequency response

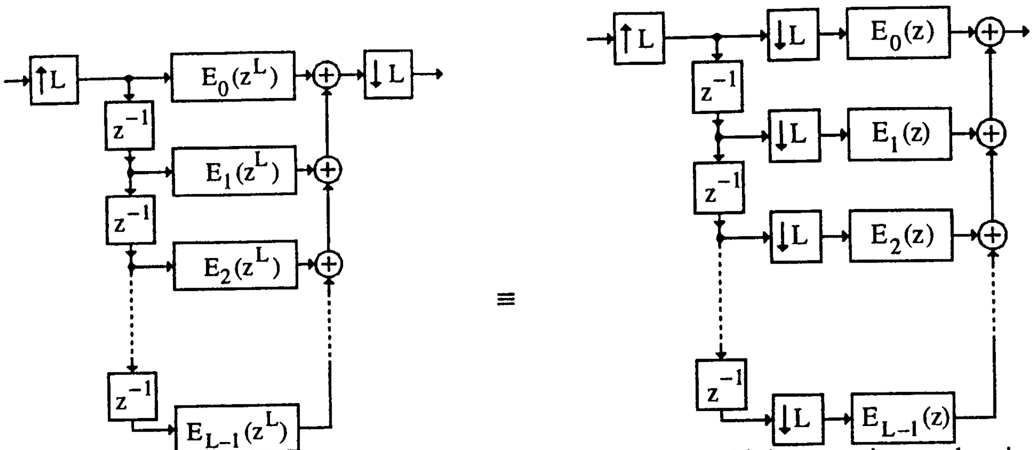
$$H(e^{j\omega}) = \begin{cases} 1, & -\frac{\pi}{M} \le \omega \le \frac{\pi}{M}, \\ 0, & \text{otherwise.} \end{cases}$$

H(z) can be expressed in an M-branch polyphase form as: $H(z) = \sum_{k=0}^{M-1} z^{-k} H_k(z^M).$

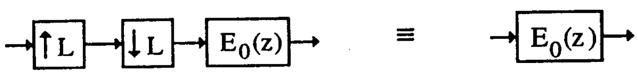
From above $\sum_{r=0}^{M-1} H(zW_M^r) = M H_0(z^M)$. Therefore, $H_0(e^{j\omega M}) = \frac{1}{M} \sum_{r=0}^{M-1} H(e^{j(\omega - 2\pi r/M)}) = \frac{1}{M}$. Or in other words, $H_0(z^M)$ is an allpass function.

Similarly for the remaining subfilters.

10.18 An equivalent realization of the structure of Figure 10.34 obtained by realizing the filter H(z) in a Type I polyphase form is as shown below on the left. By moving the down-sampler through the system and invoking the cascade equivalence #1 of Figure 10.14 we arrive at the structure shown below on the right.



which reduces to the structure shown below on the left from which we arrive at the simplified equivalent structure shown below on the right.



10.19 $H_k(z) = H_0(zW_M^k)$. Hence, $H_k(e^{j\omega}) = H_0(e^{j(\omega - \frac{2\pi k}{M})})$. Since $H_0(z)\tilde{H}_0(z)$ is an M-th band filter, hence $\sum_{k-0}^{M-1} H_0(zW_M^k)\tilde{H}_0(zW_M^k) = 1$, on the unit circle $\sum_{k=0}^{M-1}\left|H_0(e^{j(\omega-\frac{2\pi k}{M})})\right|^2 = 1$, or $\sum_{k=0}^{M-1}\left|H_k(e^{j\omega})\right|^2 = 1$. Hence, the set of M filters $\{H_k(z)\}$, k = 0, 1, . . . , M–1, satisfies the power-complementary property.

10.20 **(a)** $H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$.

Let $H_0(z^2) = \frac{1}{2}\sum_{i=0}^{\frac{N}{2}-1}(h[2i]+h[2i+1])z^{-2i}$, and $H_1(z^2) = \frac{1}{2}\sum_{i=0}^{\frac{N}{2}-1}(h[2i]-h[2i+1])z^{-2i}$. Then

$$(1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2) = \sum_{i=0}^{\frac{N}{2}-1} h[2i]z^{-2i} + \sum_{i=0}^{\frac{N}{2}-1} h[2i+1]z^{-2i+1} = H(z).$$

(b) $H(z) = (1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2)$
$= \left(H_0(z^2)+H_1(z^2)\right) + z^{-1}\left(H_0(z^2)-H_1(z^2)\right) = E_0(z^2) + z^{-1}E_1(z^2)$. Therefore, $E_0(z) = H_0(z) + H_1(z)$, and $E_1(z) = H_0(z) - H_1(z)$.

(c) Now $H(z) = \begin{bmatrix}1 & z^{-1}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}H_0(z^2)\\ H_1(z^2)\end{bmatrix} = \begin{bmatrix}1 & z^{-1}\end{bmatrix}\begin{bmatrix}H_0(z^2)+H_1(z^2)\\ H_0(z^2)-H_1(z^2)\end{bmatrix}$

$$= \left(H_0(z^2)+H_1(z^2)\right) + z^{-1}\left(H_0(z^2)-H_1(z^2)\right) = (1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2).$$

(d) If L = 2, i.e. $N = 2^2$, then we can express $H_0(z^2) = (1+z^{-2})H_{00}(z^4) + (1-z^{-2})H_{01}(z^4)$ and $H_1(z^2) = (1+z^{-2})H_{10}(z^4) + (1-z^{-2})H_{11}(z^4)$. Substituting these expressions in $H(z) = (1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2)$ we get

$$H(z) = (1+z^{-1})\left[(1+z^{-2})H_{00}(z^4) + (1-z^{-2})H_{01}(z^4)\right] + (1-z^{-1})\left[(1+z^{-2})H_{10}(z^4) + (1-z^{-2})H_{11}(z^4)\right]$$

$$= (1+z^{-1})(1+z^{-2})H_{00}(z^4) + (1+z^{-1})(1-z^{-2})H_{01}(z^4)$$
$$+(1-z^{-1})(1+z^{-2})H_{10}(z^4) + (1-z^{-1})(1-z^{-2})H_{11}(z^4)$$

$$= \begin{bmatrix}1 & z^{-1} & z^{-2} & z^{-3}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}H_{00}(z^4)\\ H_{01}(z^4)\\ H_{10}(z^4)\\ H_{11}(z^4)\end{bmatrix} = \begin{bmatrix}1 & z^{-1} & z^{-2} & z^{-3}\end{bmatrix}\mathbf{R}_4\begin{bmatrix}\hat{H}_0(z^4)\\ \hat{H}_1(z^4)\\ \hat{H}_2(z^4)\\ \hat{H}_3(z^4)\end{bmatrix}.$$

Continuing this process it is easy to establish that for $N = 2^L$, we have

$$H(z) = \begin{bmatrix}1 & z^{-1} & \cdots & z^{-(L-1)}\end{bmatrix}\mathbf{R}_L\begin{bmatrix}\hat{H}_0(z^L)\\ \hat{H}_1(z^L)\\ \vdots\\ \hat{H}_{L-1}(z^L)\end{bmatrix}.$$

10.21 Now $H(z) = \begin{bmatrix}1 & z^{-1} & z^{-2} & z^{-3}\end{bmatrix}\mathbf{R}_4\begin{bmatrix}\hat{H}_0(z^4)\\ \hat{H}_1(z^4)\\ \hat{H}_2(z^4)\\ \hat{H}_3(z^4)\end{bmatrix} = \begin{bmatrix}1 & z^{-1} & z^{-2} & z^{-3}\end{bmatrix}\begin{bmatrix}E_0(z^4)\\ E_1(z^4)\\ E_2(z^4)\\ E_3(z^4)\end{bmatrix}.$

Therefore, $\begin{bmatrix}\hat{H}_0(z)\\ \hat{H}_1(z)\\ \hat{H}_2(z)\\ \hat{H}_3(z)\end{bmatrix} = \mathbf{R}_4^{-1}\begin{bmatrix}E_0(z)\\ E_1(z)\\ E_2(z)\\ E_3(z)\end{bmatrix} = \frac{1}{4}\mathbf{R}_4\begin{bmatrix}E_0(z)\\ E_1(z)\\ E_2(z)\\ E_3(z)\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}E_0(z)\\ E_1(z)\\ E_2(z)\\ E_3(z)\end{bmatrix}.$

A length-16 Type 1 linear-phase FIR transfer function is of the form

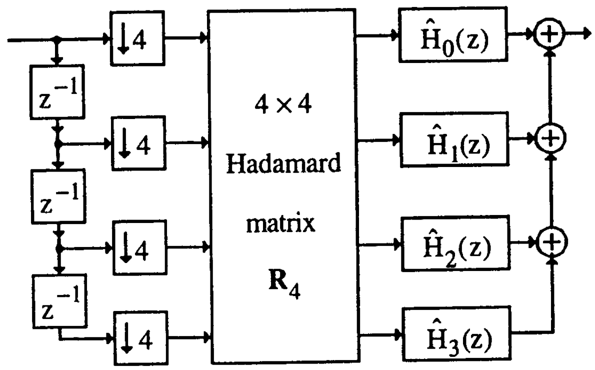
$$H(z) = h_0 + h_1z^{-1} + h_2z^{-2} + h_3z^{-3} + h_4z^{-4} + h_5z^{-5} + h_6z^{-6} + h_7z^{-7}$$
$$+h_7z^{-8} + h_6z^{-9} + h_5z^{-10} + h_4z^{-11} + h_3z^{-12} + h_2z^{-13} + h_1z^{-14} + h_0z^{-15}.$$

Hence, $E_0(z) = h_0 + h_4z^{-1} + h_7z^{-2} + h_3z^{-3}$, $E_1(z) = h_1 + h_5z^{-1} + h_6z^{-2} + h_2z^{-3}$,
$E_2(z) = h_2 + h_6z^{-1} + h_5z^{-2} + h_1z^{-3}$, $E_3(z) = h_3 + h_7z^{-1} + h_4z^{-2} + h_0z^{-3}$.

Thus, $\hat{H}_0(z) = g_0 + g_1z^{-1} + g_1z^{-2} + g_0z^{-3}$, $\hat{H}_1(z) = g_2 + g_3z^{-1} - g_3z^{-2} - g_2z^{-3}$,
$\hat{H}_2(z) = g_4 + g_5z^{-1} - g_4z^{-2} - g_5z^{-3}$, $\hat{H}_3(z) = g_6 + g_7z^{-1} + g_7z^{-2} + g_8z^{-3}$,

where $g_0 = \frac{1}{4}(h_0 + h_1 + h_2 + h_3)$, $g_1 = \frac{1}{4}(h_4 + h_5 + h_6 + h_7)$, $g_2 = \frac{1}{4}(h_0 - h_1 + h_2 - h_3)$,
$g_3 = \frac{1}{4}(h_4 - h_5 + h_6 - h_7)$, $g_4 = \frac{1}{4}(h_0 + h_1 - h_2 - h_3)$, $g_5 = \frac{1}{4}(h_4 + h_5 - h_6 - h_7)$,

$g_6 = \frac{1}{4}(h_0 - h_1 - h_2 + h_3)$, and $g_7 = \frac{1}{4}(h_4 - h_5 - h_6 + h_7)$. Observe that $\hat{H}_0(z)$ and $\hat{H}_3(z)$ are Type 1 linear-phase FIR transfer functions, where as, $\hat{H}_1(z)$ and $\hat{H}_2(z)$ are Type 2 linear-phase FIR transfer functions. A computationally efficient realization of a factor-of-4 decimator using a four-band structural subband decomposition of the decimation filter H(z) is developed below.

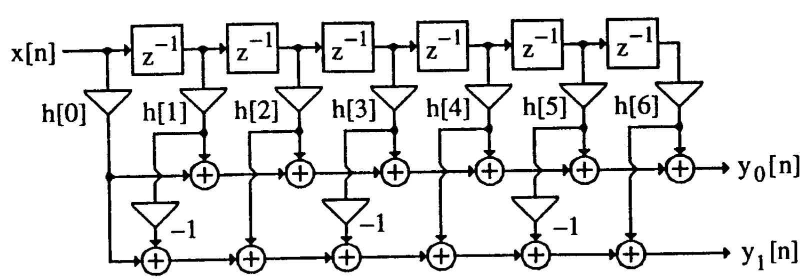


Because of the symmetric or antisymmetric impulse response, each subband filter $\hat{H}_i(z)$ can be realized using only 2 multipliers. Hence, the final realization uses only 8 multipliers. Note also that by delay-sharing, the total number of delays in implementing the four subband filters can be reduced to 3.

10.22 The uniform DFT filter bank implementation requires N multiplications for the M subfilters and an extra $\frac{M}{2}\log_2 M$ multiplications for the M-point DFT. Hence the total number of multiplications required is $N + \frac{M}{2}\log_2 M$. On the other hand if each filter in the filter bank is realized directly, the total number of multiplications required would be NM.

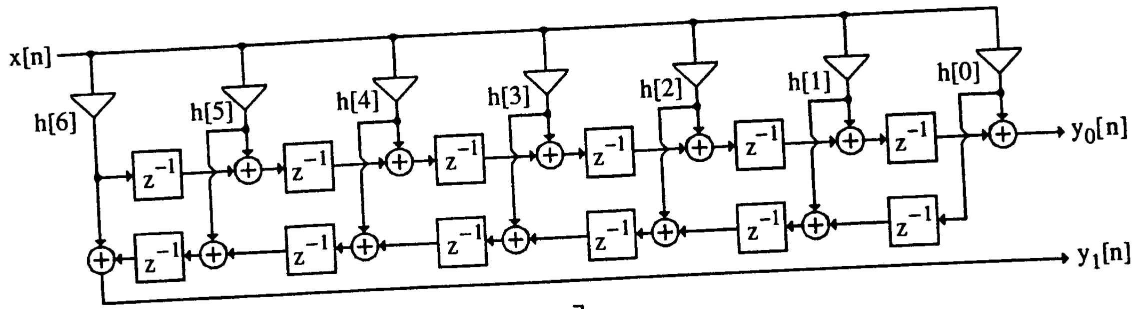
10.23 $H_0(z) = \frac{Y_0(z)}{X(z)} = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}$,

$H_1(z) = \frac{Y_1(z)}{X(z)} = h[0] - h[1]z^{-1} + h[2]z^{-2} - h[3]z^{-3} + h[4]z^{-4} - h[5]z^{-5} + h[6]z^{-6}$.



10.24 $H_0(z) = \frac{Y_0(z)}{X(z)} = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}$,

$H_1(z) = \frac{Y_1(z)}{X(z)} = h[6] + h[5]z^{-1} + h[4]z^{-2} + h[3]z^{-3} + h[2]z^{-4} + h[1]z^{-5} + h[0]z^{-6}$.



10.25 **(a)** $\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} E_0(z^4) \\ z^{-1}E_1(z^4) \\ z^{-2}E_2(z^4) \\ z^{-3}E_3(z^4) \end{bmatrix}$. Hence,

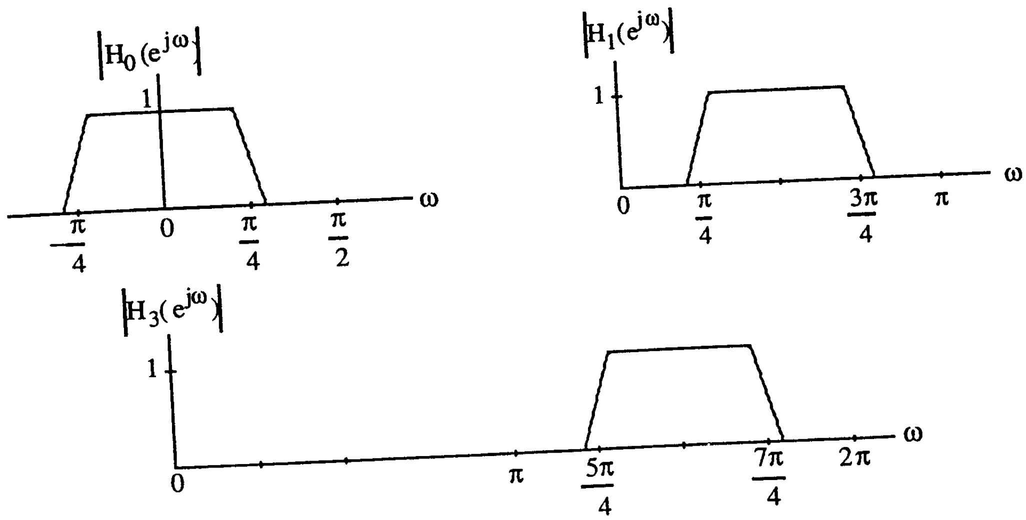
$$H_0(z) = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4) = (1 - 0.2z^{-4} + 0.9z^{-8})$$
$$+ z^{-1}(2 + 1.3z^{-4} + 3z^{-8}) + z^{-2}(4 + 2.9z^{-4} + 5.1z^{-8}) + z^{-3}(1 + 4.1z^{-4} - 1.5z^{-8})$$
$$= 1 + 2z^{-1} + 4z^{-2} + z^{-3} - 0.2z^{-4} + 1.3z^{-5} + 2.9z^{-6} + 4.1z^{-7} + 0.9z^{-8} + 3z^{-9} + 5.1z^{-10} - 1.5z^{-11},$$

$$H_1(z) = E_0(z^4) - jz^{-1}E_1(z^4) - z^{-2}E_2(z^4) + jz^{-3}E_3(z^4) = (1 - 0.2z^{-4} + 0.9z^{-8})$$
$$- jz^{-1}(2 + 1.3z^{-4} + 3z^{-8}) - z^{-2}(4 + 2.9z^{-4} + 5.1z^{-8}) + jz^{-3}(1 + 4.1z^{-4} - 1.5z^{-8})$$
$$= (1 - 4z^{-2} - 0.2z^{-4} - 2.9z^{-6} + 0.9z^{-8} - 5.1z^{-10}) - j(2z^{-1} - z^{-3} + 1.3z^{-5} - 4.1z^{-7} + 3z^{-9} + 1.5z^{-11}),$$

$$H_2(z) = E_0(z^4) - z^{-1}E_1(z^4) + z^{-2}E_2(z^4) - z^{-3}E_3(z^4) = (1 - 0.2z^{-4} + 0.9z^{-8})$$
$$- z^{-1}(2 + 1.3z^{-4} + 3z^{-8}) + z^{-2}(4 + 2.9z^{-4} + 5.1z^{-8}) - z^{-3}(1 + 4.1z^{-4} - 1.5z^{-8})$$
$$= 1 - 2z^{-1} + 4z^{-2} - z^{-3} - 0.2z^{-4} - 1.3z^{-5} + 2.9z^{-6} - 4.1z^{-7} + 0.9z^{-8} - 3z^{-9} + 5.1z^{-10} + 1.5z^{-11},$$

$$H_3(z) = E_0(z^4) + jz^{-1}E_1(z^4) - z^{-2}E_2(z^4) - jz^{-3}E_3(z^4) = (1 - 0.2z^{-4} + 0.9z^{-8})$$
$$+ jz^{-1}(2 + 1.3z^{-4} + 3z^{-8}) - z^{-2}(4 + 2.9z^{-4} + 5.1z^{-8}) - jz^{-3}(1 + 4.1z^{-4} - 1.5z^{-8})$$
$$= (1 - 4z^{-2} - 0.2z^{-4} - 2.9z^{-6} + 0.9z^{-8} - 5.1z^{-10}) + j(2z^{-1} - z^{-3} + 1.3z^{-5} - 4.1z^{-7} + 3z^{-9} + 1.5z^{-11}),$$

(b)



10.26 $Y(z) = \left(H_0(z)G_0(z) + H_1(z)G_1(z)\right)X(z)$. Now, $H_0(z) = \dfrac{1+z^{-1}}{2}$, and $H_1(z) = \dfrac{1-z^{-1}}{2}$. Choose $G_0(z) = \dfrac{1+z^{-1}}{2}$, and $G_1(z) = -\dfrac{1-z^{-1}}{2}$. Then,

$$Y(z) = \left(\frac{1}{4}(1+z^{-1})^2 - \frac{1}{4}(1-z^{-1})^2\right)X(z) = \frac{1}{4}(1+2z^{-1}+z^{-2}-1+2z^{-1}-z^{-2})X(z) = z^{-1}X(z).$$

Or in other words, $y[n] = x[n-1]$ indicating that the structure of Figure P10.5 is a perfect reconstruction filter bank.

10.27 **(a)** Since $H_0(z)$ and $H_1(z)$ are power-complementary, $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1$. Now $Y(z) = \left(H_0(z)G_0(z) + H_1(z)G_1(z)\right)X(z) = z^{-N}\left(H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1})\right)X(z)$ $= z^{-N}X(z)$. Or in other words, $y[n] = x[n-N]$ indicating that the structure of Figure P10.5 is a perfect reconstruction filter bank.

(b) If $H_0(z)$ and $H_1(z)$ are causal FIR transfer functions of order N each, $H_0(z)$ and $H_1(z)$ are polynomials in z^{-1}. As a result, $H_0(z^{-1})$ and $H_1(z^{-1})$ are polynomials in z with the highest power being z^N. Hence, $z^{-N}H_0(z^{-1})$ and $z^{-N}H_1(z^{-1})$ are polynomials in z^{-1}, making the synthesis filters $G_0(z)$ and $G_1(z)$ causal FIR transfer functions of order N each.

(c) From Figure P10.5, for perfect reconstruction we require $H_0(z)G_0(z) + H_1(z)G_1(z) = z^{-N}$. From part (a) we note that the PR condition is sdatisfied with $G_0(z) = z^{-N}H_0(z^{-1})$ and $G_1(z) = z^{-N}H_1(z^{-1})$, if $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1$, i.e. if $H_0(z)$ and $H_1(z)$ are power-complementary. The last condition is satisfied if and only if $H_0(z) = \dfrac{z^{-N/2}}{2}(z^{-n_0} + z^{n_0})$, and $H_1(z) = \dfrac{z^{-N/2}}{2}(z^{-n_0} - z^{n_0})$. As a result, $G_0(z)$ and $G_1(z)$ are also of the form $G_0(z) = \dfrac{z^{-N/2}}{2}(z^{-n_0} + z^{n_0})$, and $G_1(z) = -\dfrac{z^{-N/2}}{2}(z^{-n_0} - z^{n_0})$.

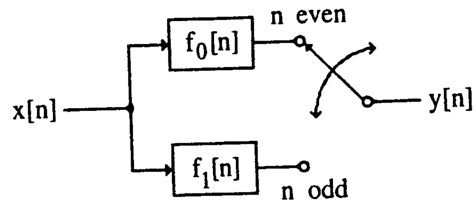
10.28 From Eq. (10.80), $Y(z) = T(z)X(z) + A(z)X(-z)$. Let $\mathcal{Z}^{-1}\{T(z)\} = t[n]$, and $\mathcal{Z}^{-1}\{A(z)\} = a[n]$. Then an inverse z-tranform of Eq. (10.80) then yields

$$y[n] = \sum_{\ell=-\infty}^{\infty} t[\ell]x[n-\ell] + \sum_{\ell=-\infty}^{\infty} a[\ell](-1)^{n-\ell}x[n-\ell] = \sum_{\ell=-\infty}^{\infty}\left(t[\ell] + (-1)^{n-\ell}a[\ell]\right)x[n-\ell].$$

Define $f_0[n] = t[n] + (-1)^{-n}a[n]$, and $f_1[n] = t[n] - (-1)^{-n}a[n]$. Then we can write

$$y[n] = \begin{cases} \sum_{\ell=-\infty}^{\infty} f_0[\ell]x[n-\ell], & \text{for n even,} \\ \sum_{\ell=-\infty}^{\infty} f_1[\ell]x[n-\ell], & \text{for n odd.} \end{cases}$$

The corresponding equivalent realization of the 2-channel QMF bank is therefore as indicated below:



As can be seen from the above representation, the QMF bank is, in general, a linear, time-varying system with a period 2. Note that if A(z) = 0, then it becomes a linear, time-invariant system.

10.29 $\hat{X}(z) = \{H_0(z)G_0(z) + H_1(z)G_1(z)\}X(z) + \{H_0(-z)G_0(z) + H_1(-z)G_1(z)\}X(-z)$

$= z^{-N}\{H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1})\}X(z) + z^{-N}\{H_0(-z)H_0(z^{-1}) + H_1(-z)H_1(z^{-1})\}X(-z).$

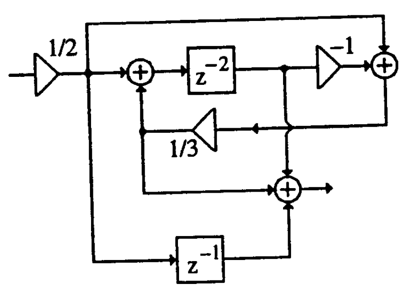
Since $H_0(z)$ and $H_1(z)$ are power-complementary, $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1$. Thus, $\hat{X}(z) = z^{-N}X(z) + z^{-N}\{H_0(-z)H_0(z^{-1}) + H_1(-z)H_1(z^{-1})\}X(-z)$. Hence, for perfect reconstruction we require $H_0(-z)H_0(z^{-1}) + H_1(-z)H_1(z^{-1}) = 0$, or $\dfrac{H_0(-z)}{H_1(-z)} = -\dfrac{H_1(z^{-1})}{H_0(z^{-1})}$.

Above implies $H_0(z) = z^{-N}H_1(-z^{-1})$, and $H_1(z) = -z^{-N}H_0(-z^{-1})$,

10.30 **(a)** $G(z) = \dfrac{1+3z^{-1}+3z^{-2}+z^{-3}}{6+2z^{-2}} = \dfrac{1}{2}\left(\dfrac{1+3z^{-2}}{3+z^{-2}} + \dfrac{3z^{-1}+z^{-3}}{3+z^{-2}}\right) = \dfrac{1}{2}\left(\dfrac{1+3z^{-2}}{3+z^{-2}} + z^{-1}\right)$

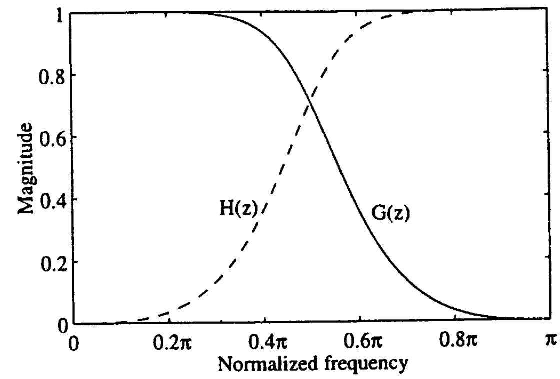
$= \dfrac{1}{2}\left(A_0(z) + A_1(z)\right)$, where $A_0(z) = \dfrac{1+3z^{-2}}{3+z^{-2}}$, and $A_1(z) = z^{-1}$.

(b)



(c) $H(z) = \dfrac{1}{2}\left(A_0(z) - A_1(z)\right) = \dfrac{1}{2}\left(\dfrac{1+3z^{-2}}{3+z^{-2}} - z^{-1}\right) = \dfrac{1}{2}\left(\dfrac{1-3z^{-1}+3z^{-2}-z^{-3}}{3+z^{-2}}\right) = \dfrac{(1-z^{-1})^3}{6+2z^{-2}}.$

(d)



10.31 Now the magnitude square function of an N-th order analog lowpass Butterworth transfer function $H_a(s)$ is given by $|H_a(j\Omega)|^2 = \dfrac{1}{1+(\Omega/\Omega_c)^{2N}}$, where Ω_c is the 3-dB cutoff frequency. For $\Omega_c = 1$, then $|H_a(j\Omega)|^2 = \dfrac{1}{1+\Omega^{2N}}$. The corresponding transfer function of an N-th order analog highpass transfer function is simply $H_a\left(\frac{1}{s}\right)$, whose magnitude square function is given by $\left|H_a\left(\frac{1}{j\Omega}\right)\right|^2 = \dfrac{\Omega^{2N}}{1+\Omega^{2N}}$. As a result, $|H_a(j\Omega)|^2 + \left|H_a\left(\frac{1}{j\Omega}\right)\right|^2 = \dfrac{1}{1+\Omega^{2N}} + \dfrac{\Omega^{2N}}{1+\Omega^{2N}} = 1.$

Now the bilinear transformation maps the analog frequency Ω to the digital frequency ω through the relation $e^{j\omega} = \dfrac{1-j\Omega}{1+j\Omega}$. As, $-e^{j\omega} = \dfrac{1-\frac{1}{j\Omega}}{1+\frac{1}{j\Omega}}$, the analog frequency $1/\Omega$ is mapped to the digital frequency $\pi + \omega$. Hence the relation $|H_a(j\Omega)|^2 + \left|H_a\left(\frac{1}{j\Omega}\right)\right|^2 = 1$, becomes

$|H_0(e^{j\omega})|^2 + |H_0(e^{j(\pi+\omega)})|^2 = |H_0(e^{j\omega})|^2 + |H_0(-e^{j\omega})|^2 = |H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$, where $H_0(e^{j\omega})$ is the frequency response of the digital lowpass filter $H_0(z)$ obtained by applying a bilinear transformation to $H_a(s)$ and $H_1(e^{j\omega})$ is the frequency response of the digital highpass filter $H_1(z)$ obtained by applying a bilinear transformation to $H_a(1/s)$. Note that a transfer function $H_0(z)$ satisfying the condition $|H_0(e^{j\omega})|^2 + |H_0(-e^{j\omega})|^2 = 1$, is called power-symmetric.

Moreover, from the relation $\Omega = \tan(\omega/2)$ it follows that the analog 3-dB cutoff frequency $\Omega_c = 1$ is mapped into the digital 3-dB cutoff frequency $\omega_c = \pi/2$. Hence, H(z) is a digital half-band filter.

Let $H_0(z) = \dfrac{P_0(z)}{D_0(z)}$, where $P_0(z)$ and $D_0(z)$ are polynomials in z^{-1}. Hence,

$H_0(e^{j\omega}) = \dfrac{P_0(e^{j\omega})}{D_0(e^{j\omega})}$. Now, $\dfrac{P_0(-e^{j\omega})P_0^*(-e^{j\omega})}{D_0(-e^{j\omega})D_0^*(-e^{j\omega})} = \dfrac{D_0(e^{j\omega})D_0^*(e^{j\omega}) - P_0(e^{j\omega})P_0^*(e^{j\omega})}{D_0(e^{j\omega})D_0^*(e^{j\omega})}$.

Note that there are no common factors between $P_0(e^{j\omega})$ and $D_0(e^{j\omega})$, and between $P_0^*(e^{j\omega})$ and $D_0(e^{j\omega})$. As a result there are no common factors between $P_0(-e^{j\omega})P_0^*(-e^{j\omega})$ and $D_0(-e^{j\omega})D_0^*(-e^{j\omega})$. This implies then $D_0(e^{j\omega})D_0^*(e^{j\omega}) = D_0(-e^{j\omega})D_0^*(-e^{j\omega})$ As a result, $D_0(e^{j\omega}) = D_0(-e^{j\omega})$, or $D_0(e^{j\omega}) = d_0(e^{j2\omega})$. Hence, $D_0(z) = d_0(z^2)$.

Since $H_0(z) = \dfrac{P_0(z)}{D_0(z)} = \dfrac{P_0(z)}{d_0(z^2)}$, it follows then $H_1(z) = \dfrac{P_0(-z)}{d_0(z^2)}$. We have shown earlier that $H_0(z)$ and $H_1(z)$ are power-complementary. Also $P_0(z)$ is a symmetric polynomial of odd order and $P_0(-z)$ is an anti-symmetric polynomial of odd order. As a result we can express $H_0(z) = \frac{1}{2}\left(\mathcal{A}_0(z) + \mathcal{A}_1(z)\right)$, and $H_1(z) = \frac{1}{2}\left(\mathcal{A}_0(z) - \mathcal{A}_1(z)\right)$. But $H_1(z) = H_0(-z)$. Hence $H_0(z) = \frac{1}{2}\left(\mathcal{A}_0(-z) - \mathcal{A}_1(-z)\right)$. It therefore follows that $\mathcal{A}_0(z) = \mathcal{A}_0(-z) = a_0(z^2)$, and $\mathcal{A}_1(z) = -\mathcal{A}_1(-z) = z^{-1}a_1(z^2)$. Thus, $H_0(z) = \frac{1}{2}\left(a_0(z^2) + z^{-1}a_1(z^2)\right)$.

10.32 $G_a(s) = \dfrac{1}{s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1}$. The corresponding digital transfer function obtained by a bilinear transformation is given by

$$H_0(z) = G_a(s)\Big|_{s=\frac{z-1}{z+1}} = \frac{(1+z^{-1})^5}{18.9443 + 12z^{-2} + 1.0557z^{-4}} = \frac{0.0528(1+z^{-1})^5}{1 + 0.6334z^{-2} + 0.0557z^{-4}}$$

$$= \frac{0.0528 + 0.2639z^{-1} + 0.5279z^{-2} + 0.5279z^{-3} + 0.2639z^{-4} + 0.0528z^{-5}}{1 + 0.6334z^{-2} + 0.0557z^{-4}}$$

$$= \frac{0.0528 + 0.5279z^{-2} + 0.2639z^{-4}}{1 + 0.6334z^{-2} + 0.0557z^{-4}} + \frac{0.2639z^{-1} + 0.5279z^{-3} + 0.0528z^{-5}}{1 + 0.6334z^{-2} + 0.0557z^{-4}}$$

$$= \frac{0.0528(1 + 9.4704z^{-2})}{1 + 0.1056z^{-2}} + z^{-1}\frac{0.2639(1 + 1.8948z^{-2})}{1 + 0.5278z^{-2}}$$

$$= \frac{1}{2}\left\{\frac{0.1056 + z^{-2}}{1 + 0.1056z^{-2}} + z^{-1}\frac{0.5278 + z^{-2}}{1 + 0.5278z^{-2}}\right\} = \frac{1}{2}\left\{\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)\right\},$$ where

$$\mathcal{A}_0(z) = \frac{0.1056 + z^{-1}}{1 + 0.1056z^{-1}}, \text{ and } \mathcal{A}_1(z) = \frac{0.5278 + z^{-1}}{1 + 0.5278z^{-1}}.$$

The corresponding power-complementary transfer function is given by

$$H_1(z) = \frac{1}{2}\left\{\mathcal{A}_0(z^2) - z^{-1}\mathcal{A}_1(z^2)\right\} = \frac{1}{2}\left\{\frac{0.1056 + z^{-2}}{1 + 0.1056z^{-2}} - z^{-1}\frac{0.5278 + z^{-2}}{1 + 0.5278z^{-2}}\right\}.$$

In the realization of a magnitude-preserving QMF bank as shown in Figure 10.47, the realization of the allpass filters $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ require 1 multiplier each, and hence the realization of the analysis (and the synthesis) filter bank requires a total of 2 multipliers.

10.33 **(a)** Total number of multipliers required is 4(2N–1). Hence, total number of multiplications per second is equal to $4(2N-1)F_T = 4(2N-1)/T$ where $F_T = 1/T$ is the sampling frequency in Hz.

(b) In Figure 10.47, $H_0(z)=\frac{1}{2}\left\{\mathcal{A}_0(z^2)+z^{-1}\mathcal{A}_1(z^2)\right\}$, and $H_1(z)=\frac{1}{2}\left\{\mathcal{A}_0(z^2)-z^{-1}\mathcal{A}_1(z^2)\right\}$.

If order of $\mathcal{A}_0(z)$ is K and the order of $\mathcal{A}_1(z)$ is L, then the order of $H_0(z)$ is 2K + 2L +1 = N. Hence K+L = (N–1)/2. The total number of multpliers needed to implement $\mathcal{A}_0(z)$ is K while the total number of multpliers needed to implement $\mathcal{A}_1(z)$ is L. Hence the total number of multipliers required to implement the structure of Figure 10.47 is 2(K+L) = N–1. However, the multipliers here are operating at half of the sampling rate of the input x[n]. As a result, the total number of multiplications per second in this case $(N-1)F_T/2 = (N-1)/2T$.

10.34 $\begin{bmatrix}H_0(z)\\H_1(z)\\H_2(z)\\H_3(z)\end{bmatrix}=\begin{bmatrix}1&3&3&1\\1&1&-1&-1\\1&-1&-1&1\\1&-3&3&-1\end{bmatrix}\begin{bmatrix}1\\z^{-1}\\z^{-2}\\z^{-3}\end{bmatrix}$. Comparing this equation with Eq. (10.135) we observe

that here $\mathbf{E}(z^4)=\begin{bmatrix}1&3&3&1\\1&1&-1&-1\\1&-1&-1&1\\1&-3&3&-1\end{bmatrix}$. To design a perfect reconstruction filter bank using the above analysis filters we thus need a synthesis filter bank for which the Type II polyphase component matrix $\mathbf{R}(z)$ satisfies the condition $\mathbf{R}(z)\mathbf{E}(z) = z^{-n_o}\mathbf{I}$. Hence, $\mathbf{R}(z) = z^{-n_o}\mathbf{E}^{-1}(z)$.

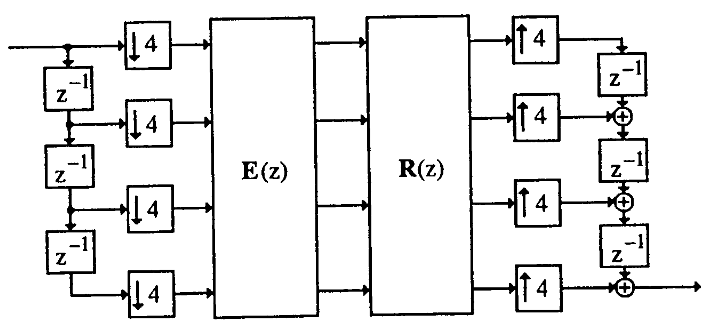
Now, $\mathbf{E}^{-1}(z)=\begin{bmatrix}0.125&0.375&0.375&0.125\\0.125&0.125&-0.125&-0.125\\0.125&-0.125&-0.125&0.125\\0.125&-0.375&0.375&-0.125\end{bmatrix}$. Then, the synthesis filters are given by

$$\begin{bmatrix}G_0(z)&G_1(z)&G_2(z)&G_3(z)\end{bmatrix}=\begin{bmatrix}z^{-3}&z^{-2}&z^{-1}&1\end{bmatrix}\begin{bmatrix}0.125&0.375&0.375&0.125\\0.125&0.125&-0.125&-0.125\\0.125&-0.125&-0.125&0.125\\0.125&-0.375&0.375&-0.125\end{bmatrix}.$$ Thus,

$G_0(z)=0.125(1+z^{-1}+z^{-2}+z^{-3})$, $\qquad G_1(z)=0.125(-3-z^{-1}+z^{-2}+3z^{-3})$,

$G_2(z)=0.125(3-z^{-1}-z^{-2}+3z^{-3})$, $\qquad G_3(z)=0.125(-1+z^{-1}-z^{-2}+z^{-3})$.

A computationally efficient realization of the 4-channel perfect reconstruction filter bank is indicated below:



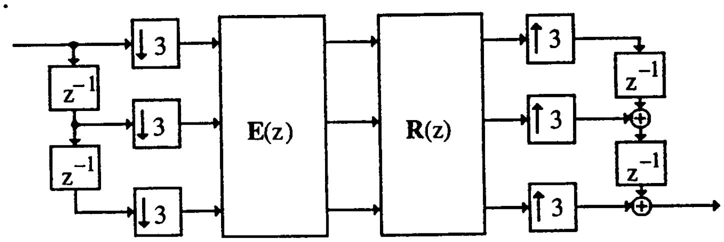
10.35 $\begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$. Rewriting this equation in the form of Eq. (10.138) we arrive at

$\begin{bmatrix} G_0(z) & G_1(z) & G_2(z) \end{bmatrix} = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$. Hence, $\mathbf{R}(z^3) = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$. For perfect reconstruction we need to find the analysis filters such that $\mathbf{E}(z) = z^{-n_o}\mathbf{R}^{-1}(z)$, where $\mathbf{E}(z)$ is the Type I polyphase component matrix. Now, $\mathbf{R}^{-1}(z) = 0.1\begin{bmatrix} 1 & 5 & -3 \\ 3 & -5 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Hence,

$\begin{bmatrix} E_0(z) \\ E_1(z) \\ E_2(z) \end{bmatrix} = 0.1\begin{bmatrix} 1 & 5 & -3 \\ 3 & -5 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$. Therefore,

$E_0(z) = 0.1(1 + 5z^{-1} - 3z^{-2}),\ E_1(z) = 0.1(3 - 5z^{-1} + z^{-2}),\ E_2(z) = 0.1(-1 + z^{-1} + z^{-2}).$

A computationally efficient realization of the 3-channel perfect reconstruction filter bank is indicated below:



10.36 The state-space representation of the k-th allpass transfer function in the analysis filter bank

$\mathcal{A}_k(z) = \dfrac{a_N + a_{N-1}z^{-1} + \cdots + a_1 z^{-(N-1)} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_{N-1}z^{-(N-1)} + a_N z^{-N}}$ is given by

$\mathbf{s}_a[n+1] = \mathbf{A}\,\mathbf{s}_a[n] + \mathbf{B}\,x[n]$, and $v[n] = \mathbf{C}\,\mathbf{s}_a[n] + D\,x[n]$, where

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{N-1} & -a_N \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$\mathbf{C} = \begin{bmatrix} (a_{N-1} - a_N a_1) & (a_{N-2} - a_N a_2) & \cdots & (a_1 - a_N a_{N-1}) & (1 - a_N^2) \end{bmatrix}$, and $D = a_N$.

(a) $$\mathbf{APA} + \mathbf{BC} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{N-1} & -a_N \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{N-1} & -a_N \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (a_{N-1} - a_N a_1) & (a_{N-2} - a_N a_2) & \cdots & (a_1 - a_N a_{N-1}) & (1 - a_N^2) \end{bmatrix},$$

$$=\begin{bmatrix} -a_N & -a_{N-1} & \cdots & -a_2 & -a_1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{N-1} & -a_N \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} (a_{N-1}-a_Na_1) & (a_{N-2}-a_Na_2) & \cdots & (a_1-a_Na_{N-1}) & (1-a_N^2) \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}=\mathbf{P}.$$

(b) $$\mathbf{APB}+\mathbf{BD}=\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{N-1} & -a_N \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}+\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}a_N$$

$$=\begin{bmatrix} -a_N & -a_{N-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}+\begin{bmatrix} a_N \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}=\mathbf{0}.$$

(c) $$\mathbf{CPA}+\mathbf{DC}=\begin{bmatrix} (a_{N-1}-a_Na_1) & (a_{N-2}-a_Na_2) & \cdots & (a_1-a_Na_{N-1}) & (1-a_N^2) \end{bmatrix}$$

$$\times\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{N-1} & -a_N \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$+a_N\begin{bmatrix} (a_{N-1}-a_Na_1) & (a_{N-2}-a_Na_2) & \cdots & (a_1-a_Na_{N-1}) & (1-a_N^2) \end{bmatrix}$$

$$=\begin{bmatrix} (a_{N-1}-a_Na_1) & (a_{N-2}-a_Na_2) & \cdots & (a_1-a_Na_{N-1}) & (1-a_N^2) \end{bmatrix}\begin{bmatrix} 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \\ -a_1 & -a_2 & \cdots & -a_{N-1} & -a_N \end{bmatrix}$$

$$+a_N\begin{bmatrix} (a_{N-1}-a_Na_1) & (a_{N-2}-a_Na_2) & \cdots & (a_1-a_Na_{N-1}) & (1-a_N^2) \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \end{bmatrix}=\mathbf{0}.$$

(d) $$\mathbf{CPB}+\mathrm{D}^2=\begin{bmatrix} (a_{N-1}-a_Na_1) & (a_{N-2}-a_Na_2) & \cdots & (a_1-a_Na_{N-1}) & (1-a_N^2) \end{bmatrix}$$

$$\times\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}+a_N^2$$

$$=\begin{bmatrix} (a_{N-1}-a_Na_1) & (a_{N-2}-a_Na_2) & \cdots & (a_1-a_Na_{N-1}) & (1-a_N^2) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}+a_N^2=1.$$

10.37 Figure 10.59(b) illustrates the scheme for realizing the unstable allpass filter in the synthesis branch. As the allpass filters in both the analysis and synthesis branches are identical, they have the same state-space representations. For simplicity we ignore the subscript k from all signal variables shown in Figure 10.59(b).

The state-space representation of the k-th allpass transfer function in the synthesis filter bank is given by

$$\mathbf{s}_s[n+1] = \mathbf{A}\mathbf{s}_s[n] + \mathbf{B}u[n], \text{ and } y[n] = \mathbf{C}\mathbf{s}_s[n] + \mathrm{D}u[n], \qquad (25)$$

whose the state-space parameter matrices are identical to those of the state-space representation of the k-th allpass transfer function in the analysis filter bank:

$$\mathbf{s}_a[n+1] = \mathbf{A}\mathbf{s}_a[n] + \mathbf{B}x[n], \text{and } v[n] = \mathbf{C}\mathbf{s}_a[n] + \mathrm{D}x[n], \qquad (26)$$

By choosing the initial values of the state-variables $\mathbf{s}_s[n]$ in the synthesis branch allpass section as a function of the final values of the state-variables $\mathbf{s}_a[n]$ in the analysis branch allpass section according to: $\mathbf{s}_s[0] = \mathbf{P}\mathbf{s}_a[M]$, where M is the length of the input x[n], we shall prove in two steps that the output y[n] is precisely the time-reversed version of x[n], i.e., $y[n] = x[M-n-1]$, for $0 \le n \le M-1$, with $u[n] = v[M-n-1]$. To this end we make use of the identities of Problem 10.36:

$$\mathbf{APA} + \mathbf{BC} = \mathbf{P}, \qquad \mathbf{APB} + \mathbf{BD} = \mathbf{0}, \qquad (27)$$

$$\mathbf{CPA} + \mathrm{D}\mathbf{C} = \mathbf{0}, \qquad \mathbf{CPB} + \mathrm{D}^2 = 1. \qquad (28)$$

<u>Step 1</u>. We first prove by induction that if $\mathbf{s}_s[0] = \mathbf{P}\mathbf{s}_a[M]$ holds then $\mathbf{s}_s[m] = \mathbf{P}\mathbf{s}_a[M-m]$, for $0 \le m \le M-1$.

Assume that $\mathbf{s}_s[\ell] = \mathbf{P}\mathbf{s}_a[M-\ell]$, for $0 \le \ell \le m$. We show that $\mathbf{s}_s[m+1] = \mathbf{P}\mathbf{s}_a[M-m-1]$. Now from Eq. (25),

$$\begin{aligned}\mathbf{s}_s[m+1] &= \mathbf{A}\mathbf{s}_s[m] + \mathbf{B}u[m] = \mathbf{AP}\,\mathbf{s}_a[M-m] + \mathbf{B}v[M-m-1] \\ &= \mathbf{AP}\left\{\mathbf{A}\mathbf{s}_a[M-m-1] + \mathbf{B}x[M-m-1]\right\} + \mathbf{B}\left\{\mathbf{C}\mathbf{s}_a[M-m-1] + \mathrm{D}x[M-m-1]\right\} \\ &= \{\mathbf{APA} + \mathbf{BC}\}\mathbf{s}_a[M-m-1] + \{\mathbf{APB} + \mathrm{D}\mathbf{C}\}x[M-m-1] = \mathbf{P}\mathbf{s}_a[M-m-1],\end{aligned}$$

using Eqs. (25), (26), and (27).

<u>Step 2</u>. We now prove $y[n] = x[M-n-1]$, for $0 \le n \le M-1$. From Eq. (25),

$$\begin{aligned}y[m] &= \mathbf{C}\mathbf{s}_s[m] + \mathrm{D}u[m] = \mathbf{C}\mathbf{s}_s[m] + \mathrm{D}v[M-m-1] = \mathbf{CP}\mathbf{s}_a[M-m] + \mathrm{D}v[M-m-1] \\ &= \mathbf{CP}\{\mathbf{A}\mathbf{s}_a[M-m-1] + \mathbf{B}x[M-m-1]\} + \mathrm{D}\{\mathbf{C}\mathbf{s}_a[M-m-1] + \mathrm{D}x[M-m-1]\} \\ &= \{\mathbf{CPA} + \mathrm{D}\mathbf{C}\}\,\mathbf{s}_a[M-m-1] + \{\mathbf{CPB} + \mathrm{D}^2\}x[M-m-1]\} = x[M-m-1]\end{aligned}$$

using Eqs. (25), (26) and (28).

10.38 $$\hat{X}(z) = \begin{bmatrix} G_0(z) & G_1(z) & \cdots & G_{L-1}(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(zW) & \cdots & H_0(zW^{L-1}) \\ H_1(z) & H_1(zW) & \cdots & H_1(zW^{L-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{L-1}(z) & H_{L-1}(zW) & \cdots & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} X(z) \\ X(zW) \\ \vdots \\ X(zW^{L-1}) \end{bmatrix}.$$

To show the system of Figure 10.50 is, in general, periodic with a period L, we need to show that if $\hat{X}_1(z)$ is the output for an input $X_1(z)$, and $\hat{X}_2(z)$ is the output for an input $X_2(z)$, then if

$X_2(z) = z^{-L}X_1(z)$, i.e. $x_2[n] = x_1[n-L]$, then the corresponding output satisfies $\hat{X}_2(z) = z^{-L}\hat{X}_1(z)$, i.e. $\hat{x}_2[n] = \hat{x}_1[n-L]$. Now,

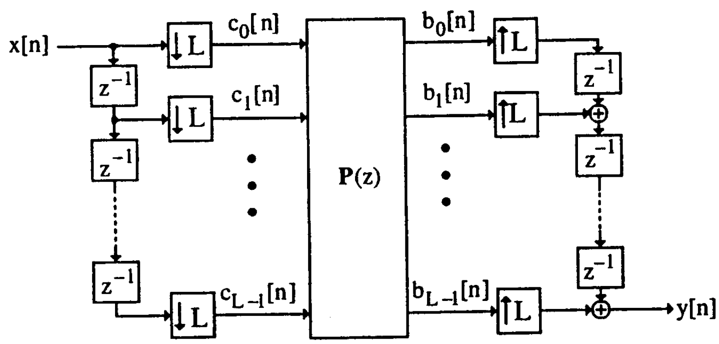
$$\hat{X}_2(z) = \begin{bmatrix} G_0(z) & G_1(z) & \cdots & G_{L-1}(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(zW) & \cdots & H_0(zW^{L-1}) \\ H_1(z) & H_1(zW) & \cdots & H_1(zW^{L-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{L-1}(z) & H_{L-1}(zW) & \cdots & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} X_2(z) \\ X_2(zW) \\ \vdots \\ X_2(zW^{L-1}) \end{bmatrix}$$

$$= \begin{bmatrix} G_0(z) & G_1(z) & \cdots & G_{L-1}(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(zW) & \cdots & H_0(zW^{L-1}) \\ H_1(z) & H_1(zW) & \cdots & H_1(zW^{L-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{L-1}(z) & H_{L-1}(zW) & \cdots & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} z^{-L}X_1(z) \\ z^{-L}W^{-L}X_1(zW) \\ \vdots \\ z^{-L}W^{-L(L-1)}X_1(zW^{L-1}) \end{bmatrix}$$

$$= z^{-L}\begin{bmatrix} G_0(z) & G_1(z) & \cdots & G_{L-1}(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(zW) & \cdots & H_0(zW^{L-1}) \\ H_1(z) & H_1(zW) & \cdots & H_1(zW^{L-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{L-1}(z) & H_{L-1}(zW) & \cdots & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_1(zW) \\ \vdots \\ X_1(zW^{L-1}) \end{bmatrix}$$

$= z^{-L}\hat{X}_1(z)$. As a result, the structure of Figure 10.50 is a time-varying system with a period L.

10.39



From the above figure it follows that we can express the z-transforms of $\{c_i[n]\}$ as

$$C_\ell(z) = \frac{1}{L}\sum_{k=0}^{L-1}(z^{1/L}W^k)^{-\ell}X(z^{1/L}W^k), \quad 0 \le \ell \le L-1, \text{ where } W = e^{-j2\pi/L}.$$

Likewise, the z-transforms of $\{b_i[n]\}$ can be expressed as

$$B_s(z) = \sum_{\ell=0}^{L-1}P_{s,\ell}(z)C_\ell(z), \quad 0 \le s \le L-1, \text{ where } P_{s,\ell}(z) \text{ denotes the } (s,\ell)\text{-th element of } \mathbf{P}(z).$$

Finally, the z-transform of the output y[n] is given by

$$Y(z) = \sum_{s=0}^{L-1}z^{-(L-1-s)}B_s(z^L) = \sum_{s=0}^{L-1}z^{-(L-1-s)}\sum_{\ell=0}^{L-1}P_{s,\ell}(z^L)C_\ell(z^L)$$

$$= \frac{1}{L}\sum_{s=0}^{L-1}z^{-(L-1-s)}\sum_{\ell=0}^{L-1}P_{s,\ell}(z^L)\sum_{k=0}^{L-1}(zW^k)^{-\ell}X(zW^k)$$

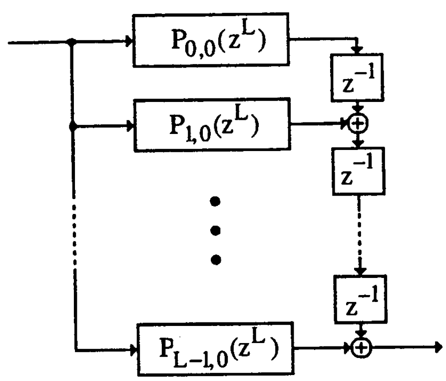
$$= \frac{1}{L}\sum_{k=0}^{L-1} X(zW^k)\sum_{\ell=0}^{L-1} W^{-k\ell}\sum_{s=0}^{L-1} z^{-\ell}z^{-(L-1-s)}P_{s,\ell}(z^L).$$

In the above expression, terms of the form $X(zW^k)$, $k \neq 0$, represent the contribution coming from aliasing. Hence, the expression for Y(z) is free from these aliasing terms for any arbitrary input x[n] if and only if

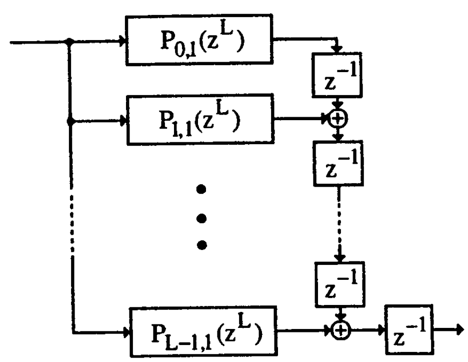
$$\sum_{\ell=0}^{L-1} W^{-k\ell}\sum_{s=0}^{L-1} z^{-\ell}z^{-(L-1-s)}P_{s,\ell}(z^L) \neq 0, \quad k \neq 0.$$

The above expression can be written in a matrix form as $\mathbf{D}^\dagger \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{L-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, where **D** is the $L \times L$ DFT matrix, T(z) is the transfer function of the alias-free system, and $V_\ell(z) = \sum_{s=0}^{L-1} z^{-\ell}z^{-(L-1-s)}P_{s,\ell}(z^L)$. Since $\mathbf{D}\mathbf{D}^\dagger = L\,\mathbf{I}$, the above matrix equation can be alternately written as $\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{L-1}(z) \end{bmatrix} = \mathbf{D}\begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. This implies that $V_\ell(z) = V(z)$, $0 \le \ell \le L-1$, as the first column of **D** has all elements equal to 1. As a result, the L-channel QMF bank is alias-free if and only if $V_\ell(z)$ is the same for all ℓ.

The two figures below show the polyphase realizations of $V_0(z)$ and $V_1(z)$.

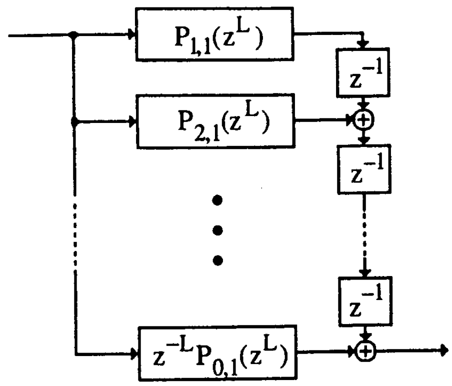


(a) $V_0(z)$



(b) $V_1(z)$

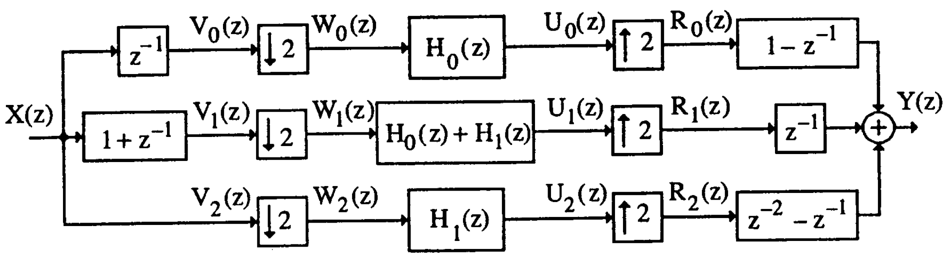
The realization of $V_1(z)$ can be redrawn as indicated below.



(c) $V_1(z)$

Because of the constraint $V_0(z) = V_1(z)$, the polyphase components in Figures (a) and (c) should be the same. From these two figures it follows that the first column of **P**(z) is an upwards-shifted version of the second column, with the topmost element appearing with a z^{-1} attached. this type of relation holds for the k-th column and the (k+1)-th column of **P**(z). As a result, **P**(z) is a pseudo-circulant matrix of the form of Eq. (10.165).

10.40 (a)



$$\mathbf{V}(z)=\begin{bmatrix}V_0(z)\\V_1(z)\\V_2(z)\end{bmatrix}=\begin{bmatrix}z^{-1}\\1+z^{-1}\\1\end{bmatrix}X(z). \quad \mathbf{W}(z)=\begin{bmatrix}W_0(z)\\W_1(z)\\W_2(z)\end{bmatrix}=\begin{bmatrix}z^{-1/2}\\1+z^{-1/2}\\1\end{bmatrix}X(z^{1/2})+\begin{bmatrix}-z^{-1/2}\\1-z^{-1/2}\\1\end{bmatrix}X(-z^{1/2}).$$

$$\mathbf{U}(z)=\begin{bmatrix}U_0(z)\\U_1(z)\\U_2(z)\end{bmatrix}=\begin{bmatrix}z^{-1/2}H_0(z)\\(1+z^{-1/2})\{H_0(z)+H_1(z)\}\\H_1(z)\end{bmatrix}X(z^{1/2})+\begin{bmatrix}-z^{-1/2}H_0(z)\\(1-z^{-1/2})\{H_0(z)+H_1(z)\}\\H_1(z)\end{bmatrix}X(-z^{1/2}).$$

$$\mathbf{R}(z)=\begin{bmatrix}R_0(z)\\R_1(z)\\R_2(z)\end{bmatrix}=\begin{bmatrix}z^{-1}H_0(z^2)\\(1+z^{-1})\{H_0(z^2)+H_1(z^2)\}\\H_1(z^2)\end{bmatrix}X(z)+\begin{bmatrix}-z^{-1}H_0(z^2)\\(1-z^{-1})\{H_0(z^2)+H_1(z^2)\}\\H_1(z^2)\end{bmatrix}X(-z).$$

$$\begin{aligned}Y(z)&=(1-z^{-1})R_0(z)+z^{-1}R_1(z)+(z^{-2}-z^{-1})R_2(z)\\&=\left[(1-z^{-1})z^{-1}H_0(z^2)+(1+z^{-1})z^{-1}\{H_0(z^2)+H_1(z^2)\}+(z^{-2}-z^{-1})H_1(z^2)\right]X(z)\\&\quad+\left[-(1-z^{-1})z^{-1}H_0(z^2)+(1-z^{-1})z^{-1}\{H_0(z^2)+H_1(z^2)\}+(z^{-2}-z^{-1})H_1(z^2)\right]X(-z)\\&=\left[2z^{-1}H_0(z^2)+2z^{-2}H_1(z^2)\right]X(z)=2z^{-1}\left[H_0(z^2)+z^{-1}H_1(z^2)\right]X(z).\end{aligned}$$ Hence,

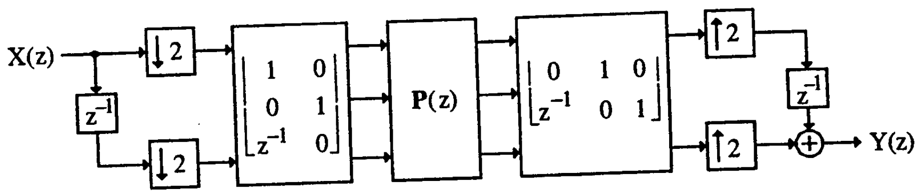
$T(z) = 2z^{-1}\left[H_0(z^2) + z^{-1}H_1(z^2)\right].$

(b) $T(z) = 2z^{-1}\left[\frac{1}{2}H(z) + \frac{1}{2}H(-z) + \frac{1}{2}H(z) - \frac{1}{2}H(-z)\right] = 2z^{-1}H(z).$

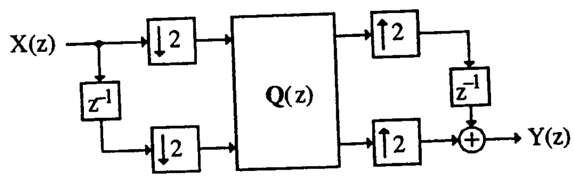
(c) Length of $H_0(z)$ = K and length of $H_1(z)$ = K.

(d) The number of multplications in the above structure is given by $3K\frac{F_T}{2}$ per second where F_T is the sampling frquency in Hz. On the other hand for a direct implementation of H(z) requires $2KF_T$ multiplications per second.

10.41 **(a)** An equivalent representation of the structure of Figure P10.7 is as indicated below:



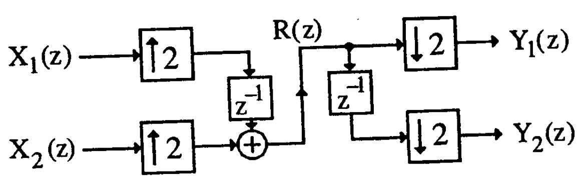
whose simplified equivalent representation is as shown below:



where $\mathbf{Q}(z) = \begin{bmatrix} 0 & 1 & 0 \\ z^{-1} & 0 & 1 \end{bmatrix} \mathbf{P}(z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ z^{-1} & 0 \end{bmatrix}$. It thus follows from Problem 10.39 that the structure of Figure P10.7 is time-invariant (i.e. alias-free) if and only if $\mathbf{Q}(z)$ is pseudo-circulant.

(b) A realization of the structure of Figure P10.7 based on critical down-sampling and critical up-sampling is shown above in Part (a).

10.42



$R(z) = z^{-1}X_1(z^2) + X_2(z^2).$

$Y_1(z) = R(z^{1/2}) + R(-z^{1/2}) = \left[z^{-1/2}X_1(z) + X_2(z)\right] + \left[-z^{-1/2}X_1(z) + X_2(z)\right] = 2X_2(z).$

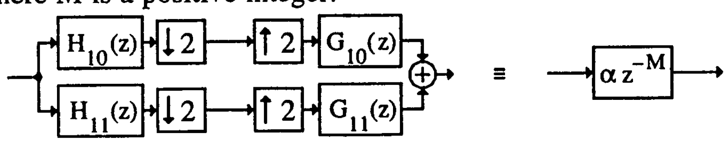
$Y_2(z) = z^{-1/2}R(z^{1/2}) - z^{-1/2}R(-z^{1/2})$

$= \left[z^{-1}X_1(z) + z^{-1/2}X_2(z)\right] + \left[z^{-1}X_1(z) - z^{-1/2}X_2(z)\right] = 2z^{-1}X_1(z).$

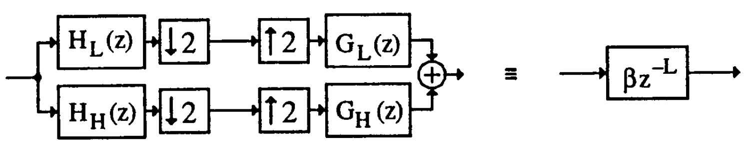
Thus, the output $y_1[n]$ is a scaled replica of the input $x_2[n]$ while the output $y_2[n]$ is a scaled replica of the delayed input $x_1[n-1]$.

10.43 Now $R(e^{j\omega}) = S(e^{j\omega}) + \delta e^{-jM\omega} = e^{-jM\omega}[\breve{S}(\omega) + \delta]$, where $\breve{S}(\omega)$ is the amplitude response of $S(z)$. However, $-\delta \le \breve{S}(\omega) \le 1 + \delta$. Hence, the amplitude response $\breve{R}(\omega)$ of $R(z)$ satisfies $0 \le \breve{R}(\omega) \le 1 + 2\delta$. Or in other words, $\breve{R}(\omega)$ is nonnegative for all ω. We can therefore factorize it as $\breve{R}(\omega) = |H_0(e^{j\omega})|^2 = H_0(e^{j\omega})H_0(e^{-j\omega})$. Thus, we can write $R(e^{j\omega}) = e^{-jM\omega}H_0(e^{j\omega})H_0(e^{-j\omega})$, and by analytic continuation write $R(z) = z^{-M}H_0(z)H_0(z^{-1}) = H_0(z)\hat{H}_0(z)$, where $\hat{H}_0(z) = z^{-M}H_0(z^{-1})$ is the mirror-image of $H_0(z)$. If $h_0[n]$ denotes the impulse response of $H_0(z)$, then the impulse response $\hat{h}_0[n]$ of $\hat{H}_0(z)$ is given by $\hat{h}_0[n] = h_0[M - n]$.

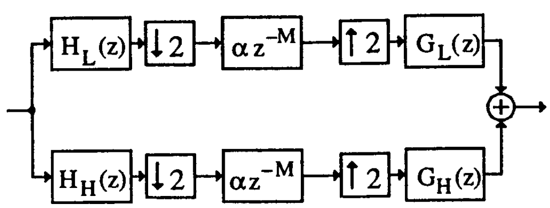
10.44 If the 2-channel QMF banks in the middle of the structure of Figure 10.61 are of perfect reconstruction type, then each of these two filter banks have a distortion transfer function of the form αz^{-M}, where M is a positive integer:



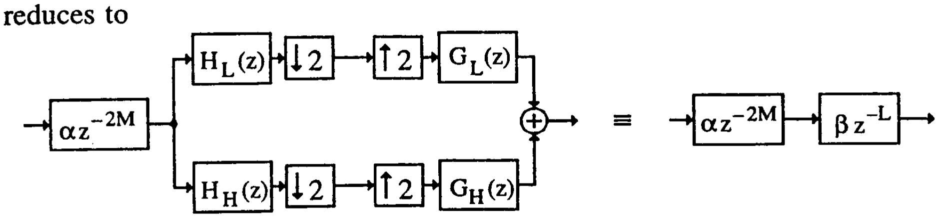
Likewise, the 2-channel analysis filter bank on the left with the 2-channel synthesis filter bank on the right form a perfect reconstruction QMF bank with a distortion tranfer function βz^{-L}, where L is a positive integer:



Hence an equivalent representation of Figure 10.61 is as indicated below

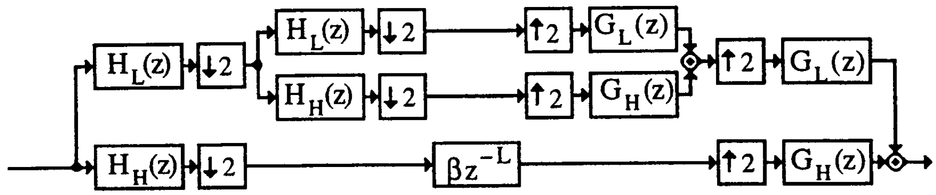


which reduces to

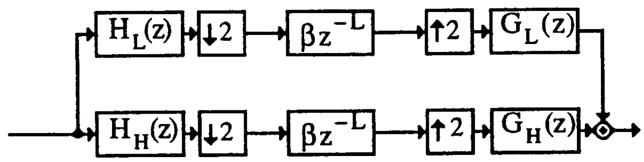


Thus, the overall structure is also of perfect reconstruction type with a distortion transfer function given by $\alpha\beta z^{-(2M+L)}$.

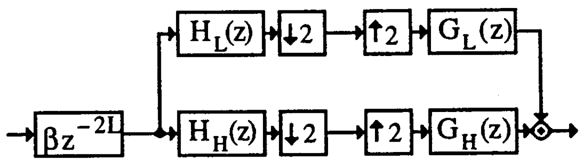
10.45 We analyze the 3-channel filter bank of Figure 10.64(b). If the 2-channel QMF bank of Figure 10.64(a) is of perfect reconstruction type with a distortion transfer function βz^{-L}, the structure of Figure 10.64(b) should be implemented as indicated below to ensure perfect reconstruction:



An equivalent representation of the above structure is as shown below:



which reduces to



verifying the perfect reconstruction property of Figure 10.64(b).

In a similar manner the perfect reconstruction property of Figure 10.64(c) can be proved.

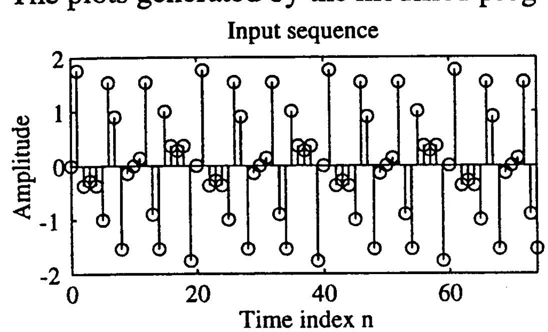
M10.1 (a) For Part (i) use the MATLAB statement

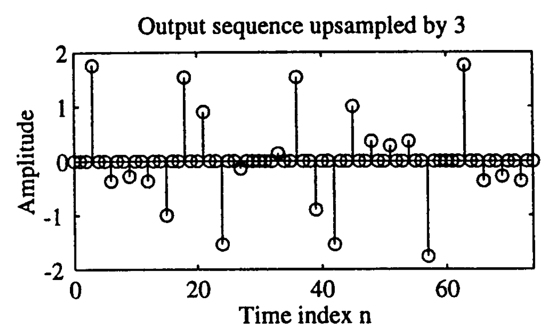
```
x = sin(2*pi*0.2*n) + sin(2*pi*0.35*n);
```

in Program 10_1 with L = 3 and N = 75, and remove the statement

```
wo = input('Input signal frequency = ');
```

The plots generated by the modified program are shown below:

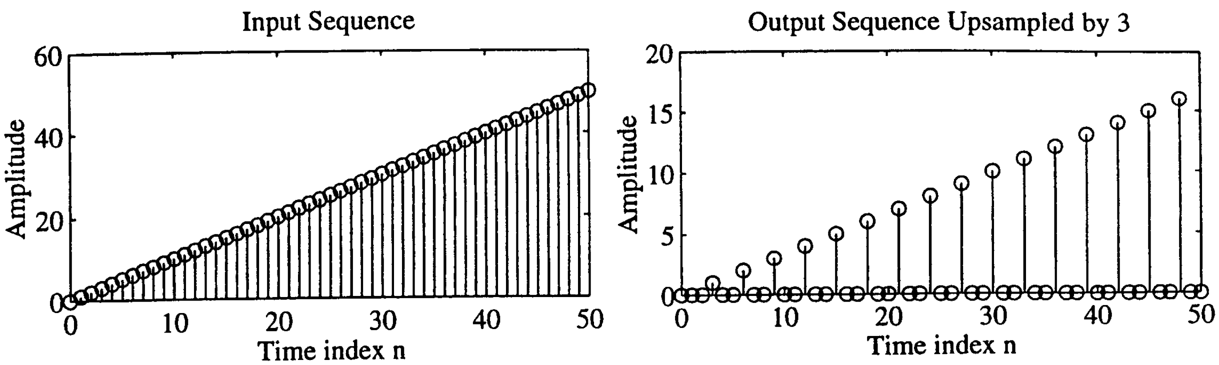




For Part (ii) use the MATLAB statement `x = n;` in Program 10_1 with L = 3 and N = 75, and remove the statement

```
wo = input('Input signal frequency = ');
```

The plots generated by the modified program are shown below:



For Part (iii) use `x = square(N, duty);` where `duty` can be chosen as `10, 20, 30, ...`

(b) Use L = 4 in the modified program as described above.

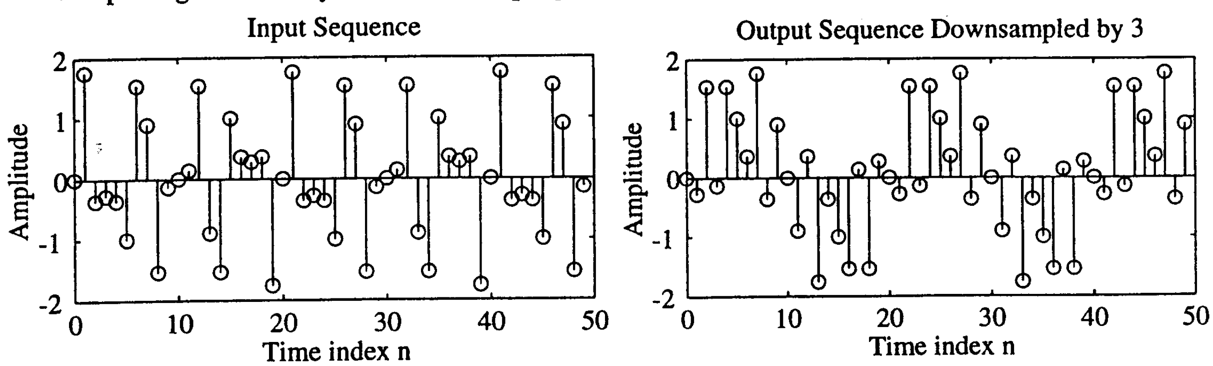
M10.2 (a) For Part (i) use the MATLAB statement

```
x = sin(2*pi*0.2*n) + sin(2*pi*0.35*n);
```

in Program 10_2 with L = 3 and N = 50, and remove the statement

```
wo = input('Input signal frequency = ');
```

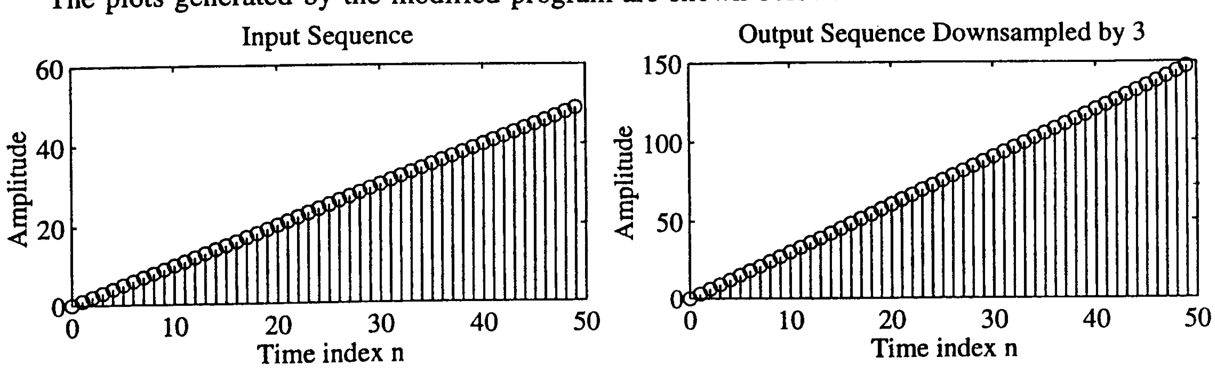
The plots generated by the modified program are shown below:



For Part (ii) use the MATLAB statement `x = m;` in Program 10_1 with L = 3 and N = 50, and remove the statement

```
wo = input('Input signal frequency = ');
```

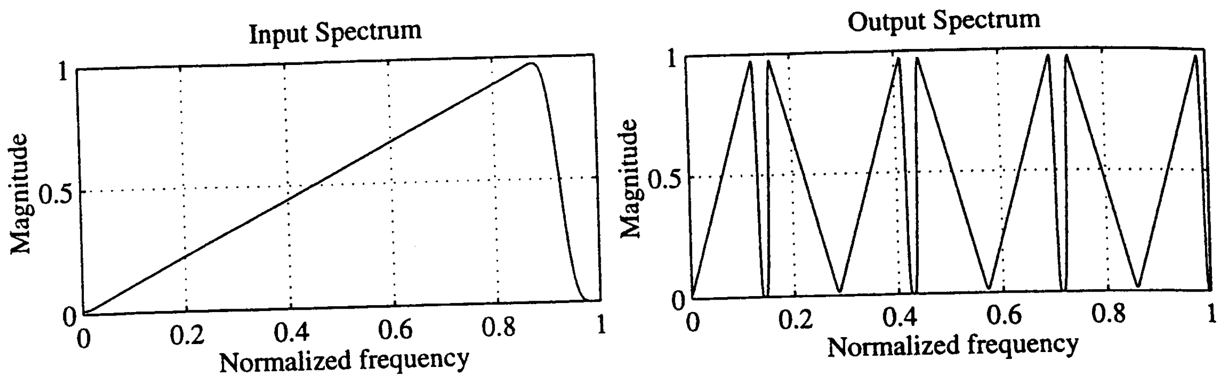
The plots generated by the modified program are shown below:



For Part (iii) use `x = square(N, duty);` where `duty` can be chosen as `10, 20, 30, ...`

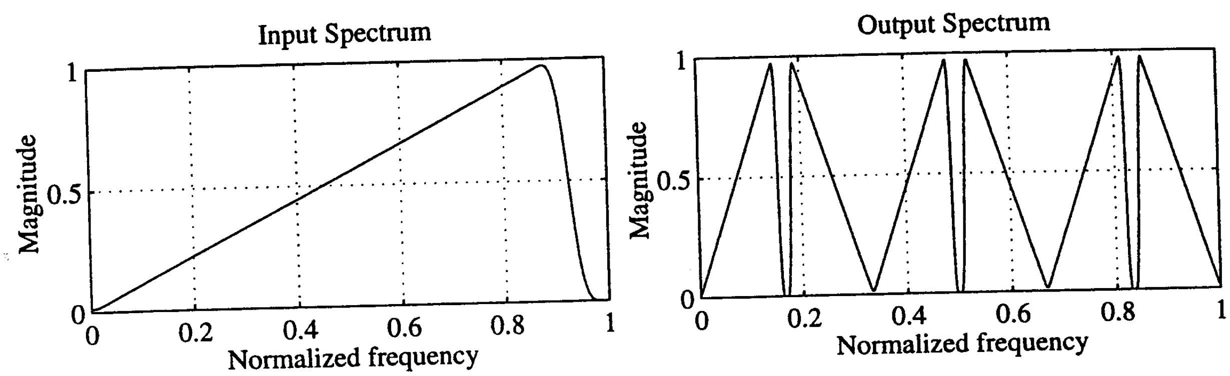
(b) Use L = 4 in the modified program as described above.

M10.3 (a) Plots generated are shown below:



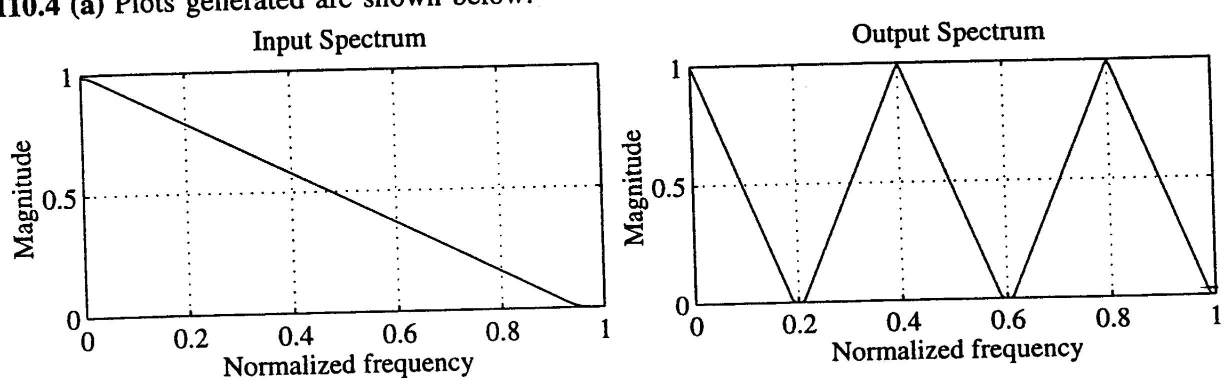
Since L = 7, the output spectrum consists of the input spectrum shrunk by a factor of 7 and L–1 = 6 aliased spectra.

(b) Plots generated are shown below:



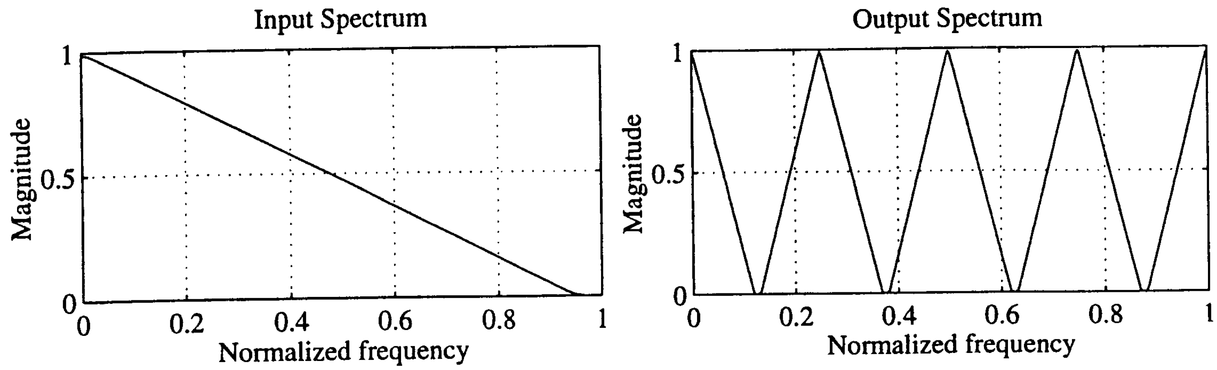
Since L = 6, the output spectrum consists of the input spectrum shrunk by a factor of 6 and L–1 = 5 aliased spectra.

M10.4 (a) Plots generated are shown below:



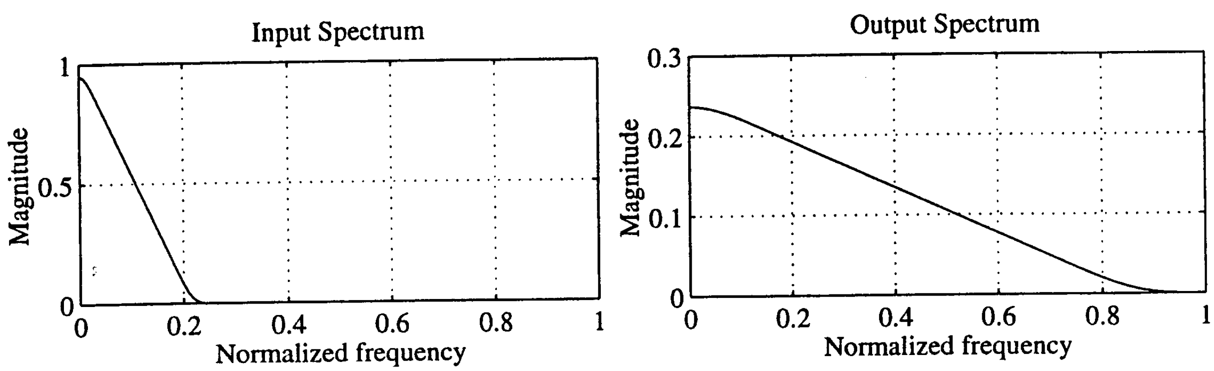
Since L = 5, the output spectrum consists of the input spectrum shrunk by a factor of 5 and L–1 = 4 aliased spectra.

(b) Plots generated are shown below:



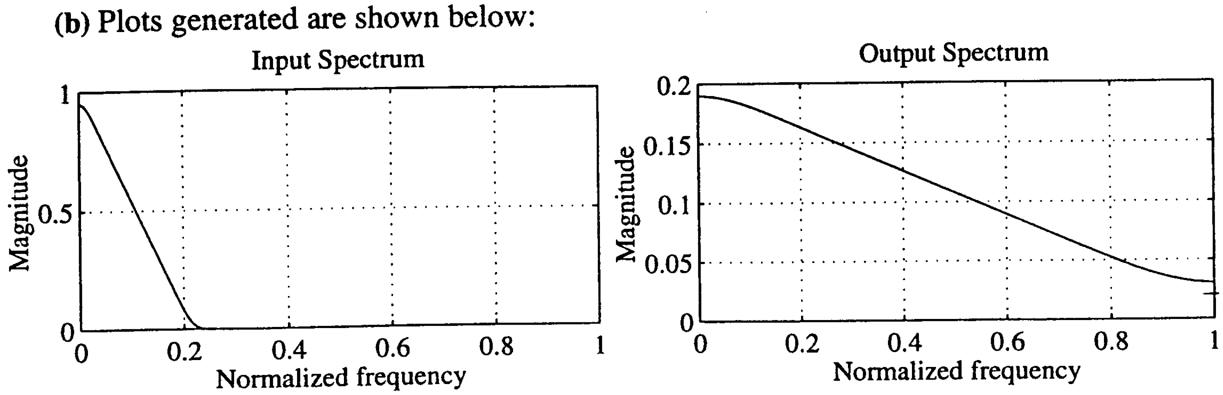
Since L = 8, the output spectrum consists of the input spectrum shrunk by a factor of 8 and L–1 = 7 aliased spectra.

M10.5 **(a)** Plots generated are shown below:



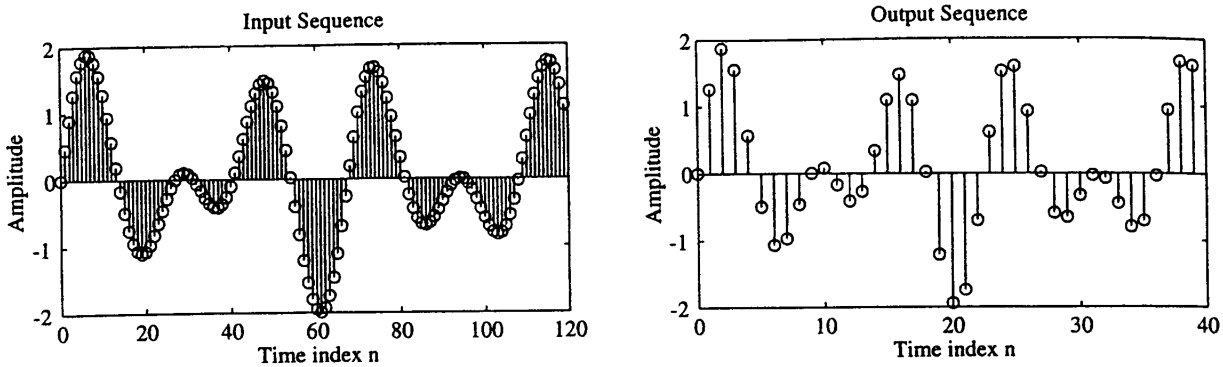
Since M = 4 and the input is bandlimited to $\pi/4$, the output spectrum is a stretched version of the input spectrum stretched by a factor of 4 and there is no aliasing. Moreover, the output spectrum is scaled by a factor 1/4 as expected.

(b) Plots generated are shown below:

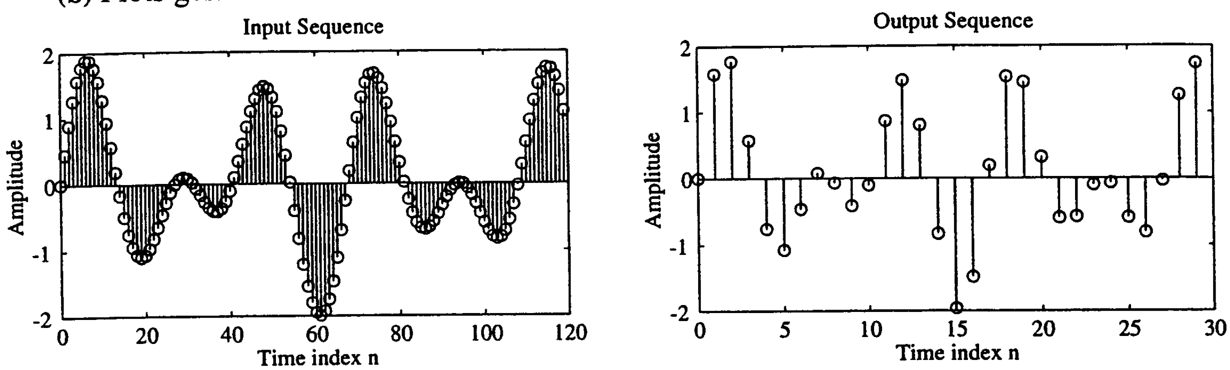


Since M = 5 and the input is bandlimited to $\pi/4$, the output spectrum is a stretched version of the input spectrum stretched by a factor of 5 and there is some visible aliasing. Moreover, the output spectrum is scaled by a factor 1/5 as expected.

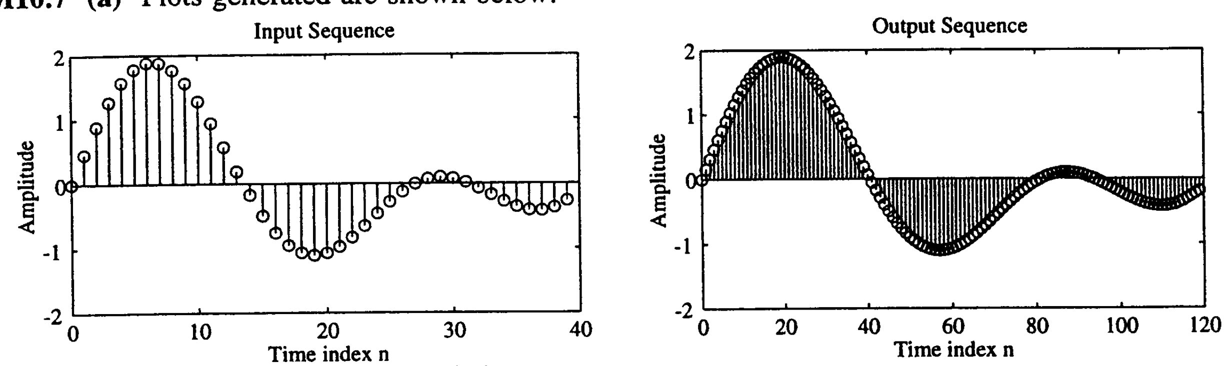
M10.6 **(a)** Plots generated are shown below:



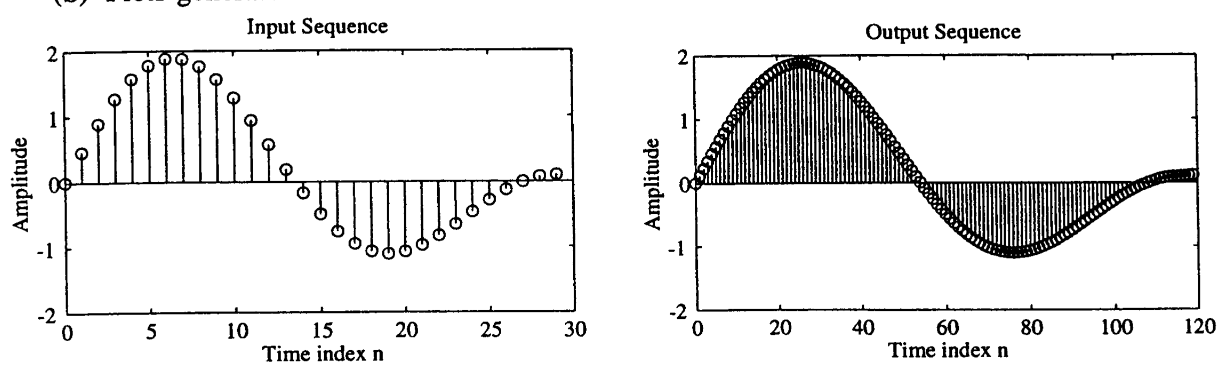
(b) Plots generated are shown below:



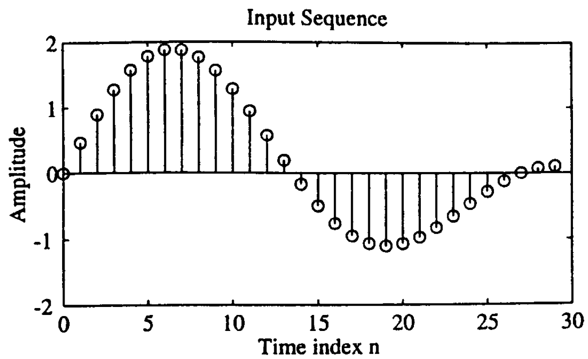
M10.7 **(a)** Plots generated are shown below:

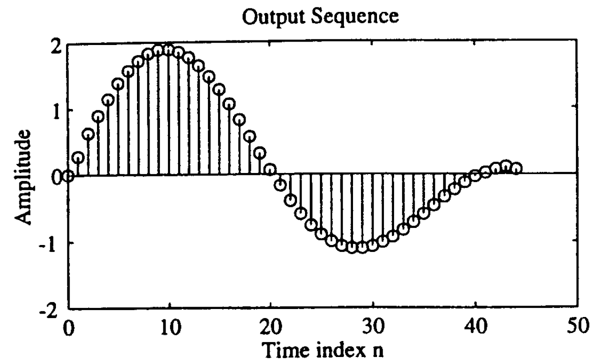


(b) Plots generated are shown below:

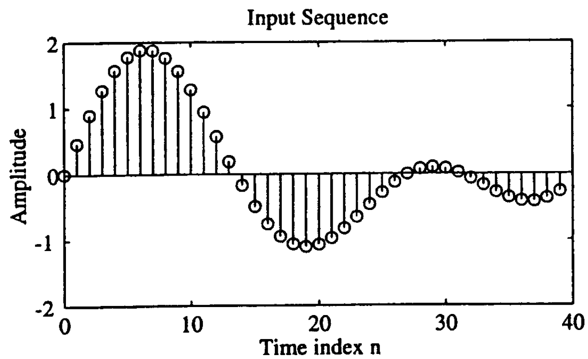


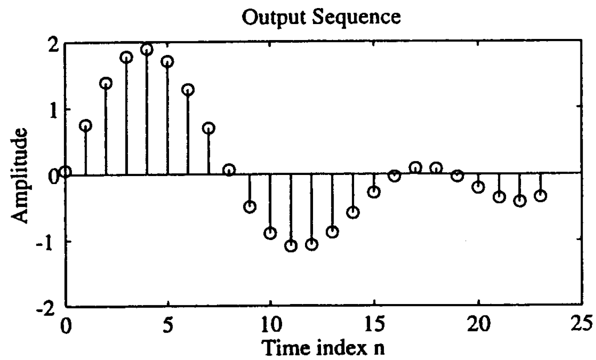
M10.8 **(a)** Plots generated are as shown below:



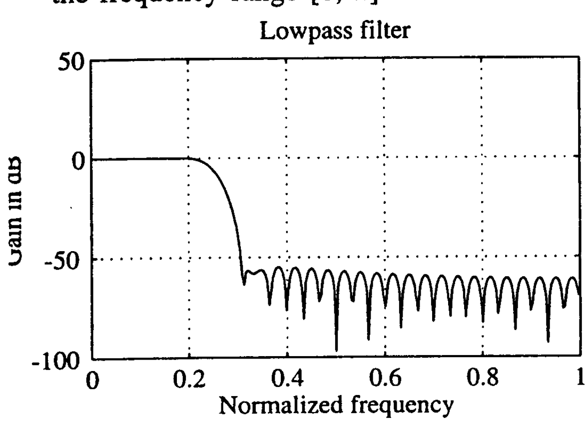


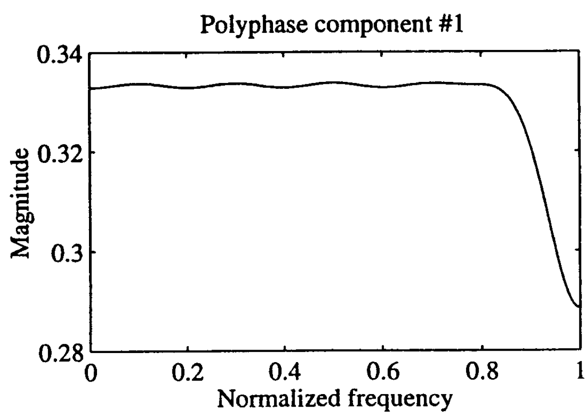
(b) Plots generated are as shown below:

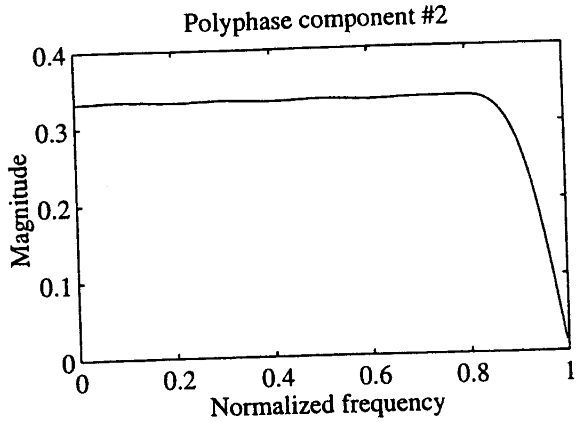


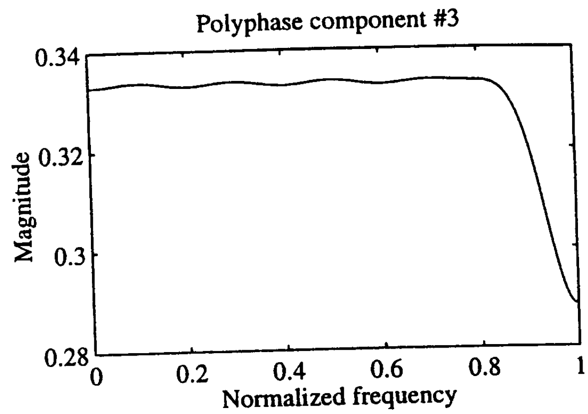


M10.9 The gain response of the length-60 FIR lowpass filter with a cutoff at $\pi/3$ and the magnitude responses of its 3 polyphase components are shown below. As can be seen from the plots, the magnitude resoponses of the polyphase components are nearly constant over the frequency range $[0, \pi]$.









M10.10 The specifications given are $\omega_s = 0.55\pi$ and $\delta_s = 0.01$. Hence, $\omega_p = \pi - 0.55\pi = 0.45\pi$, and $\delta_p = 1 - \sqrt{\delta_s^2} = 0.0202$. Therefore, $Rs = -20\log_{10}(1-\delta_p) = 0.00043432$ dB and $Rs = -20\log_{10}(\delta_s) = 40$ dB. Using the command `[b,a] = ellip(7, Rp, Rs, 0.5)` we then determine the transfer function of a 7-th order elliptic filter and plot its pole locations using `zplane(b)`. The poles of the transfer function generated are not on the imaginary axis. We next adjust the value of δ_s and found that for $\delta_s = 0.2$ the poles of the transfer function generated are on the imaginary axis. Using the pole-interlacing property we then determine the transfer functions of the two allpass filters, $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$:

$$\mathcal{A}_0(z) = \frac{0.6268 + 1.6248z^{-1} + z^{-2}}{1 + 1.6248z^{-1} + 0.6268z^{-2}}, \text{ and } \mathcal{A}_1(z) = \frac{0.9486 + z^{-1}}{1 + 0.9486z^{-1}}.$$

Hence the elliptic transfer function can be expressed as

$$G(z) = \tfrac{1}{2}\left[\mathcal{A}_1(z^2) + z^{-1}\mathcal{A}_1(z^2)\right], = \tfrac{1}{2}\left[\left(\frac{0.6268 + 1.6248z^{-1} + z^{-2}}{1 + 1.6248z^{-1} + 0.6268z^{-2}}\right) + z^{-1}\left(\frac{0.9486 + z^{-1}}{1 + 0.9486z^{-1}}\right)\right].$$

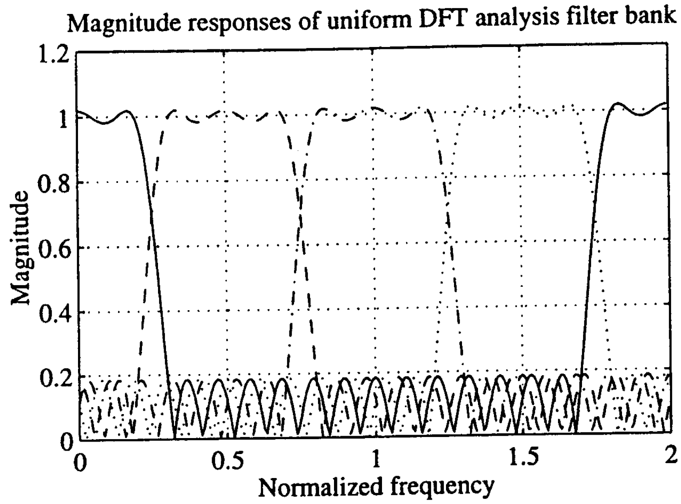
The allpass filter $\mathcal{A}_0(z)$ can be realized with only 2 multipliers and the allpass filter $\mathcal{A}_1(z)$ can be realized with only 1 multiplier. As a result G(z) requires only 3 multipliers.

M10.11 The MATLAB program used to generate the prototype lowpass filter and the analysis filters of the 4-channel uniform DFT filter bank is given below:

```
L = 21; f = [0 0.2 0.3 1]; m = [1 1 0 0]; w = [10 1];
N = 4; WN = exp(-2*pi*j/N);
plottag = ['- ';'--';'-.';': '];
h = zeros(N,L);
n = 0:L-1;
h(1,:) = remez(L-1, f, m, w);
for i = 1:N-1
  h(i+1,:) = h(1,:).*(WN.^(-i*n));
end;
clf;
for i = 1:N
  [H,w] = freqz(h(i,:), 1, 256, 'whole');
  plot(w/pi, abs(H), plottag(i,:));
  hold on;
end;
grid on;
hold off;
xlabel('Normalized frequency');ylabel('Magnitude');
```

```
title('Magnitude responses of uniform DFT analysis filter
bank');
```

The plots generated by the above program is given below:



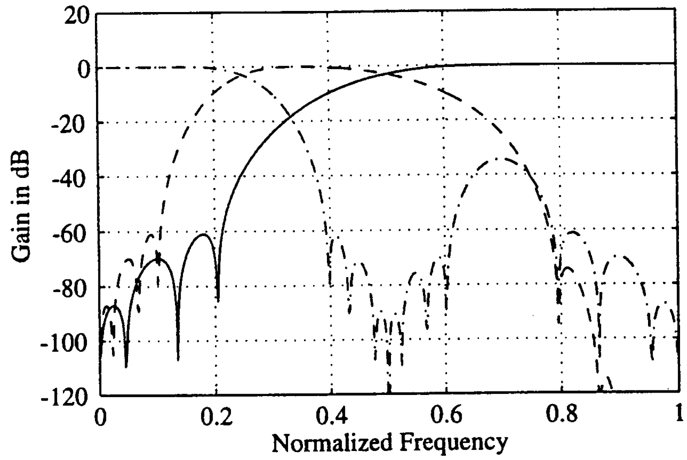
M10.12 The first 8 impulse response coefficients of Johnston's 16A lowpass filter $H_L(z)$ are given by

0.001050167, –0.005054526, –0.002589756, 0.0276414, –0.009666376, –0.09039223, 0.09779817, 0.4810284

The remaining 8 coefficients are given by flipping the coefficients left to right, From Eq. (10.157), the highpass filter in the tree-structured 3-channel filter bank is given by $H_2(z) = z^{-15}H_L(z^{-1})$. The two remaining filters are given by $H_0(z) = H_L(z)H_L(z^2)$ and $H_1(z) = H_L(z)H_H(z^2)$. The MATLAB program used to generate the gain plots of the 3 analysis filters is given by:

```
G1 = [0.10501670e-2 -0.50545260e-2 -0.25897560e-2
0.27641400e-1 -0.96663760e-2 -0.90392230e-1 0.97798170e-1
0.48102840];
G = [G1 fliplr(G1)];
n = 0:15;
H0 = (-1).^n.*G;
Hsqar = zeros(1,31); Gsqar = zeros(1,31);
Hsqar(1:2:31) = H0; Gsqar(1:2:31) = G;
H1 = conv(Hsqar,G); H2 = conv(Gsqar,G);
[h0,w0] = freqz(H0,[1]); [h1,w1] = freqz(H1,[1]); [h2,w2] =
freqz(H2,[1]);
plot(w0/pi,20*log10(abs(h0)),'b-',w1/pi,20*log10(abs(h1)),'r-
',w2/pi,20*log10(abs(h2)),'g-.');
axis([0 1 -120 20]);
grid on;
xlabel('Normalized Frequency');ylabel('Gain in dB');
```

The plots generated are given below:



Chapter 11

11.1 **(a)** resolution = $\frac{8000}{256}$ = 31.25 Hz.

(b) We need to take a $\frac{8000}{16} = 500$-point DFT.

(c) resolution = $\frac{8000}{N}$. Hence desired length N of the DFT is given by $N = \frac{8000}{128} = 62.5$. ince N must be an integer we choose N = 63 as the DFT size.

11.2 **(a)** $F_m = 4$ kHz, $R = 1024$, and $F_T = 8$ kHz.

$F_{200} = \frac{200 \times F_T}{R} = \frac{200 \times 8000}{1024} = 1562.5$ Hz, $\quad F_{350} = \frac{350 \times 8000}{1024} = 2734.4$ Hz,

$F_{824} = \frac{824 \times 8000}{1024} = 6437.5$ Hz.

(b) $F_m = 6$ kHz, $R = 1024$, and $F_T = 12$ kHz.

$F_{200} = \frac{200 \times 12000}{1024} = 2343.8$ Hz. $\quad F_{350} = \frac{350 \times 12000}{1024} = 4101.6$ Hz.

$F_{824} = \frac{824 \times 12000}{1024} = 9656.2$ Hz.

(b) $F_m = 4$ kHz, $R = 512$, and $F_T = 8$ kHz.

$F_{100} = \frac{100 \times 8000}{512} = 1562.5$ Hz. $\quad F_{200} = \frac{200 \times 8000}{512} = 3125$ Hz.

$F_{312} = \frac{312 \times 8000}{512} = 4875$ Hz.

11.3 **(a)** $F_m = 4$ kHz. Let $F_T = 8$ kHz. Then resolution = F_T/R. Hence R = $\frac{8000}{3}$ = 2667.

(b) R = 4096.

11.4 **(a)** $X_{STFT}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[n-m]\, w[m] e^{-j\omega m}$.

$$G_{STFT}(e^{j\omega},n)=\sum_{m=-\infty}^{\infty} g[n-m]w[m]e^{-j\omega m} = \sum_{m=-\infty}^{\infty}(\alpha x[n-m]+\beta y[n-m])w[m]e^{-j\omega m}$$

$$=\alpha\sum_{m=-\infty}^{\infty} x[n-m]w[m]e^{-j\omega m}+\beta\sum_{m=-\infty}^{\infty} y[n-m]w[m]e^{-j\omega m}=\alpha X_{STFT}(e^{j\omega},n)+\beta Y_{STFT}(e^{j\omega},n).$$

(b) $y[n] = x[n-n_o]$. Hence, $Y_{STFT}(e^{j\omega},n)=\sum_{m=-\infty}^{\infty} y[n-m]w[m]e^{-j\omega m}$

$$=\sum_{m=-\infty}^{\infty} x[n-n_o-m]w[m]e^{-j\omega m} = X_{STFT}(e^{j\omega},n-n_o).$$

(c) $y[n]=e^{j\omega_o}x[n]$. Hence, $Y_{STFT}(e^{j\omega},n)=\sum_{m=-\infty}^{\infty} y[n-m]w[m]e^{-j\omega m}$

$$=\sum_{m=-\infty}^{\infty} x[n-m]w[m]e^{-j(\omega-\omega_o)m} = X_{STFT}(e^{j(\omega-\omega_o)},n).$$

11.5 $X_{STFT}(e^{j\omega},n)=\sum_{m=-\infty}^{\infty} x[n-m]w[m]e^{-j\omega m}$. Replacing m in this expression with n – m we arrive at $X_{STFT}(e^{j\omega},n)=\sum_{m=-\infty}^{\infty} x[m]w[n-m]e^{-j\omega n}e^{-j\omega m}=e^{-j\omega n}\sum_{m=-\infty}^{\infty} x[m]w[n-m]\,e^{-j\omega m}$ $e^{-j\omega n}\overline{X}_{STFT}(e^{-j\omega},n)$. Hence, $\overline{X}_{STFT}(e^{j\omega},n)=e^{-j\omega n}X_{STFT}(e^{-j\omega},n)$. Thus, in computing $X_{STFT}(e^{j\omega},n)$ the input x[n] is shifted through the window w[n], whereas, in computing $\overline{X}_{STFT}(e^{j\omega},n)$ the window w[n] is shifted through the input x[n].

11.6 $\overline{X}_{STFT}(e^{j\omega},n)=\sum_{m=-\infty}^{\infty} x[m]w[n-m]e^{-j\omega m}$. Hence, by inverse DTFT we obtain

$x[m]w[n-m]=\frac{1}{2\pi}\int_0^{2\pi}\overline{X}_{STFT}(e^{j\omega},n)e^{j\omega m}\,d\omega$. Therefore,

$\sum_{n=-\infty}^{\infty} x[m]w[n-m]=\frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty}\overline{X}_{STFT}(e^{j\omega},n)e^{j\omega m}\,d\omega$, which is equivalent to

$x[m]\sum_{n=-\infty}^{\infty} w[n-m]=x[m]W[0]=\frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty}\overline{X}_{STFT}(e^{j\omega},n)e^{j\omega m}\,d\omega$, where

$W[0]=\sum_{n=-\infty}^{\infty} w[n-m]=\sum_{n=-\infty}^{\infty} w[n]$ or $x[m]=\frac{1}{2\pi W[0]}\int_0^{2\pi}\sum_{n=-\infty}^{\infty}\overline{X}_{STFT}(e^{j\omega},n)e^{j\omega m}\,d\omega.$

11.7 $X_{STFT}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[n-m]w[m]e^{-j\omega m}$. Hence,

$X_{STFT}[k, n] = \sum_{m=-\infty}^{\infty} x[n-m]w[m]e^{-j2\pi km/N} = x[n] \circledast w[n]e^{-j2\pi kn/N}$. Or in other words, $X_{STFT}[k, n]$ can be obtained by filtering x[n] by an LTI dystem with an impulse response $h_k[n] = w[n]e^{-j2\pi kn/N}$ as indicated in Figure P11.1.

11.8 $X_{STFT}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[n-m]w[m]e^{-j\omega m}$. Hence,

$\left|X_{STFT}(e^{j\omega}, n)\right|^2 = \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n-m]x[n-s]w[m]w[s]e^{-j\omega m}e^{-j\omega s}$. Thus,

$$r[k, n] = \frac{1}{2\pi}\int_0^{2\pi} \left|X_{STFT}(e^{j\omega}, n)\right|^2 e^{j\omega k}d\omega = \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n-m]x[n-s]w[m]w[s]\delta[s+k-m]$$

$$= \sum_{m=-\infty}^{\infty} x[n-m]x[n-m+k]w[m]w[m-k].$$

11.9 $\varphi_{ST}[k, n] = \sum_{m=-\infty}^{\infty} x[m]w[n-m]x[m+k]\, w[n-k-m]$.

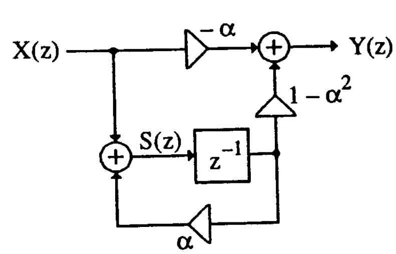
(a) $\varphi_{ST}[-k, n] = \sum_{m=-\infty}^{\infty} x[m]w[n-m]x[m-k]w[n+k-m]$.

Substitute in the above expression m – k = s, i.e. m = k +s. This yields

$$\varphi_{ST}[-k, n] = \sum_{s=-\infty}^{\infty} x[s+k]w[n-k-s]x[s]w[n-s] = \varphi_{ST}[k, n].$$

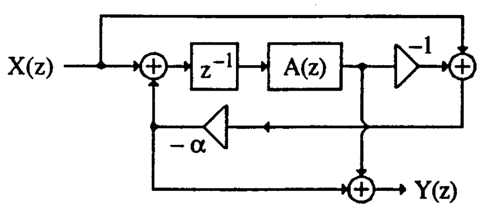
(b) Let m + k = s. Then, $\varphi_{ST}[k, n] = \sum_{s=-\infty}^{\infty} x[s-k]x[s]w[n-s+k]\, w[n-s]$. It follows from this expression that $\varphi_{ST}[k, n]$ can be computed by a convolution of $h_k[n]$ = w[n]w[n+k] with x[n]x[n – s] as indicated in Figure P11.2.

11.10



Analysis yields $S(z) = \alpha z^{-1} S(z) + X(z)$, and $Y(z) = -\alpha X(z) + (1-\alpha^2)z^{-1} S(z)$. Solving the first equation we get $S(z) = \frac{X(z)}{1-\alpha z^{-1}}$, which when substituted in the second equation yields after some algebra $\frac{Y(z)}{X(z)} = \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}$. The transfer function is thus seen to be a Type 1 allpass of the form of Eq. (6.58) and can thus be realized using any one of the single-multiplier structures of Figure 6.36.

11.11 Analysis of the structure of Figure P11.4 yields $\frac{Y(z)}{X(z)} = \frac{-\alpha + z^{-1}A(z)}{1 - \alpha z^{-1}A(z)}$, where A(z) denotes the transfer function of the "allpass reverberator". Note that this expression is similar in form to that of Eq. (6.58) with "d" replaced by "$-\alpha$" and "z^{-1}" replaced by "$z^{-1}A(z)$". Hence an efficient realization of the structure of Figure P11.4 also is obtained readily from any one of the structures of Figure 6.36. One such realization is indicated below:



11.12 $G_1(z) = \frac{K_1}{2}\{1 - A_1(z)\} + \frac{K_2}{2}\{1 + A_1(z)\} = K_2\left[\frac{1}{2}\frac{K_1}{K_2}\{1 - A_1(z)\} + \frac{1}{2}\{1 + A_1(z)\}\right]$.

Hence in this case, the ratio K_1/K_2 determines the amount of boost or cut at low frequencies, K_2 determines the amount of dc gain or attenuation at all frequencies, and α determines the 3-dB bandwidth where $\cos\omega_c = \frac{2\alpha}{1+\alpha^2}$.

```
alpha = 0.9;
K1 = [0.8];
K2 = [0.5 2];
nlp = ((1-alpha)/2)*[1 1];
dlp = [1 -alpha];
nhp = ((1+alpha)/2)*[1 -1];
dhp = [1 -alpha];
[Hlp,w] = freqz(nlp,dlp,512);
[Hhp,w] = freqz(nhp,dhp,512);
hold on
for k = 1:length(K1)
    for m = 1:length(K2)
    H = K1(k)*Hlp+K2(m)*Hhp;
     semilogx(w/pi,20*log10(abs(H)));
     xlabel('Gain, dB'); ylabel('Normalized Frequency');
    clear H;
    hold on;
    end
end
grid on
axis([.01 1 -8 8]);
```

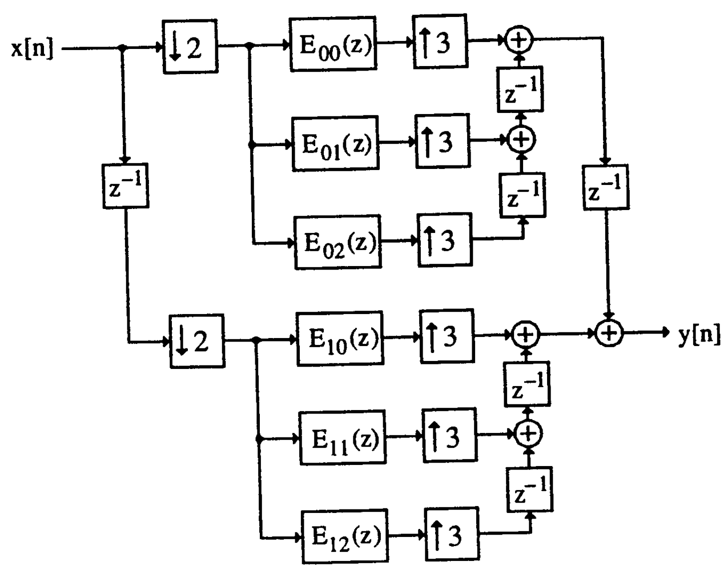
11.13 $G_2(z) = \frac{K_1}{2}\{1 - A_2(z)\} + \frac{K_2}{2}\{1 + A_2(z)\} = K_2\left[\frac{1}{2}\frac{K_1}{K_2}\{1 - A_2(z)\} + \frac{1}{2}\{1 + A_2(z)\}\right]$.

Hence in this case, the ratio K_1/K_2 determines the amount of boost or cut at low frequencies, K_2 determines the amount of dc gain or attenuation at all frequencies, α determines the

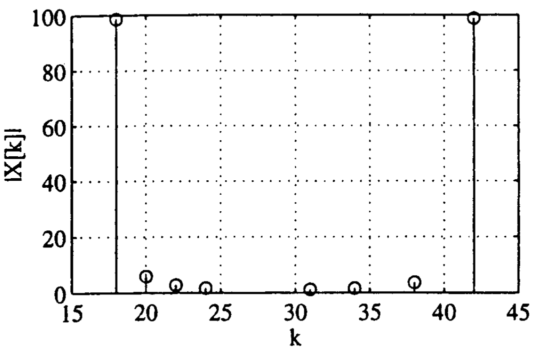
3-dB bandwidth $\Delta\omega_{3\text{-dB}} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$, and the center frequency ω_o is related to β through $\beta = \cos\omega_o$.

```
alpha=0.8;
beta=0.4;
K1=[0.9];
K2=[0.5 2];
nbp=((1-alpha)/2)*[1 0 -1];
dbp=[1 -beta*(1+alpha) alpha];
nbs=((1+alpha)/2)*[1 -2*beta 1];
dbs=dbp;
[Hlp,w]=freqz(nbp,dbp,512);
[Hhp,w]=freqz(nbs,dbs,512);
hold on
for k=1:length(K1)
     for m=1:length(K2)
     H=K1(k)*Hlp+K2(m)*Hhp;
          semilogx(w/pi,20*log10(abs(H)));
     xlabel('Gain, dB');
          ylabel('Normalized Frequency');
     clear H;
          hold on;
     end
end
grid on
axis([.01 1 -8 8]);
```

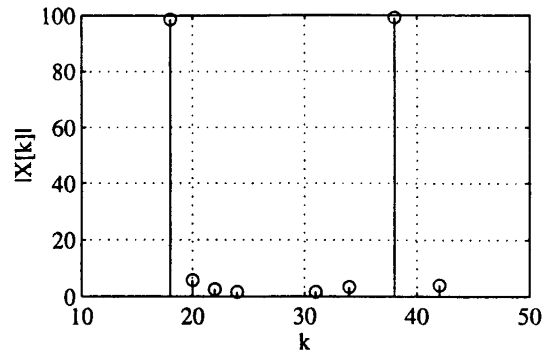
11.14 The transpose of the decimator structure of Figure 11.66 yields



M11.1 Figures below illustrate the application of Program 11_1 in detecting the touch-tone digits A and 3:



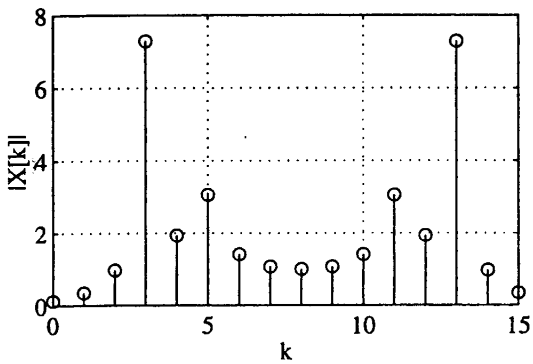
Touch-Tone Symbol = A



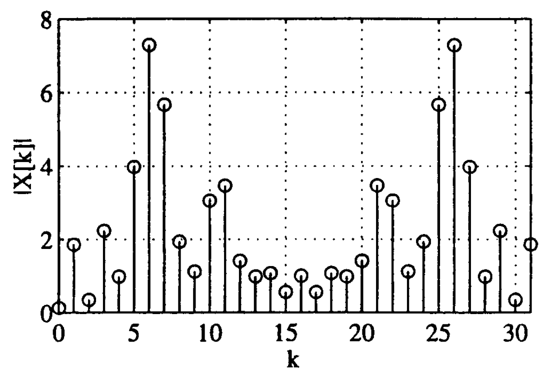
Touch-Tone Symbol = 3

M11.2 For R = 16, the two strong peaks occur at k = 3 and 5. The associated frequencies are $\omega_1 = \frac{2\pi \times 3}{16}$, or $f_1 = \frac{3}{16} = 0.1875$, and $f_2 = \frac{5}{16} = 0.3125$. For R = 32, the two strong peaks occur at k =5 and 10. The associated frequencies are $f_1 = \frac{5}{32} = 0.15625$, and $f_2 = \frac{10}{32} = 0.3125$.

R = 16

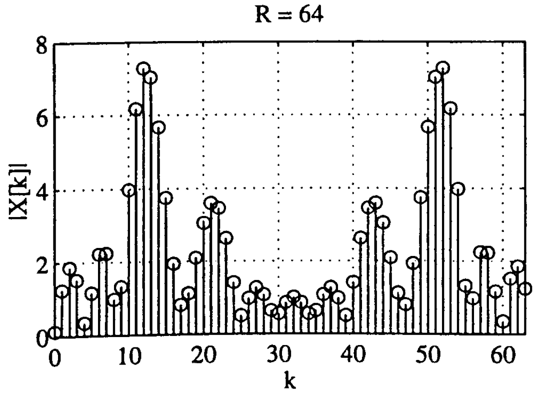


R = 32

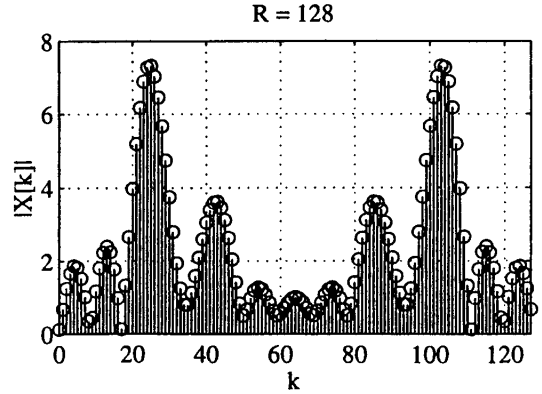


For R = 64, the two strong peaks occur at k = 11 and 22. The associated frequencies are $f_1 = 11/64 = 0.1718$, **and** $f_2 = 20/64 = 0.3125$. For R = 128, the two strong peaks occur at k = 21 and 39. The associated frequencies are $f_1 = 21/128 = 0.1641$, and $f_2 = 39/128 = 0.3047$. Moreover, the last two plots show a number of minor peaks and it is not clear by examining these plots whether or not there are other sinusoids of lesser strengths present in the sequence being analyzed.

R = 64

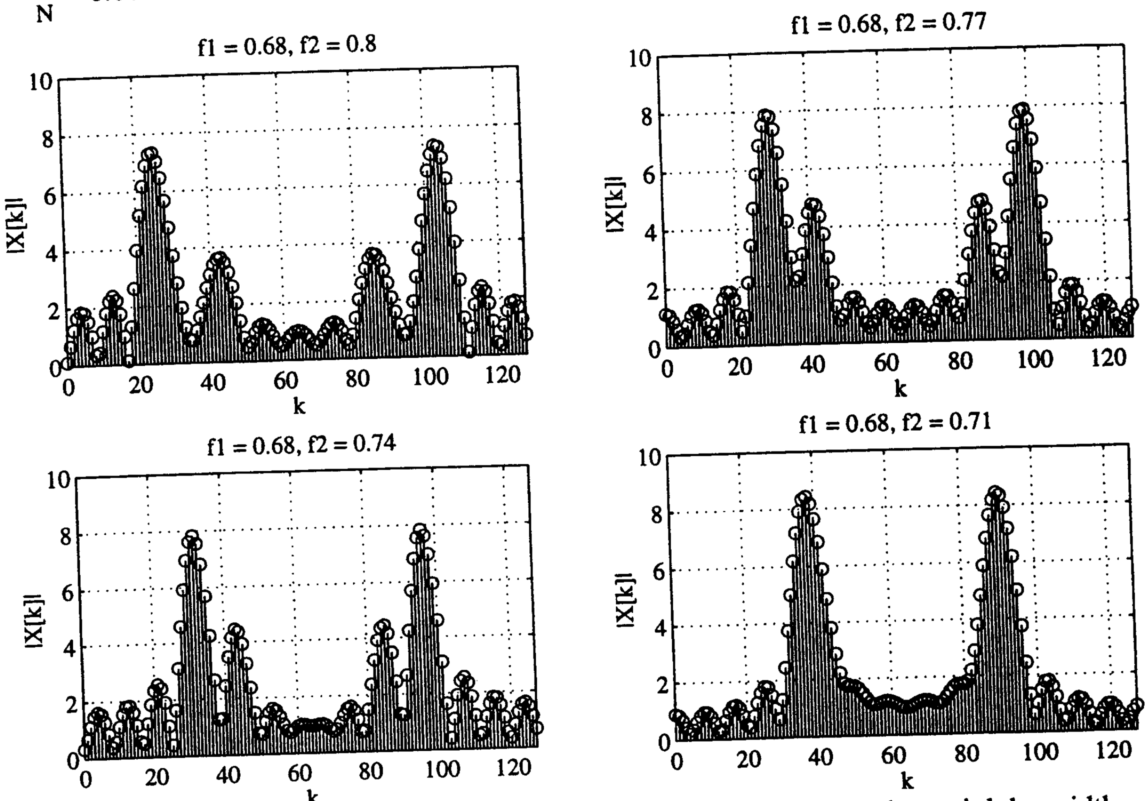


R = 128

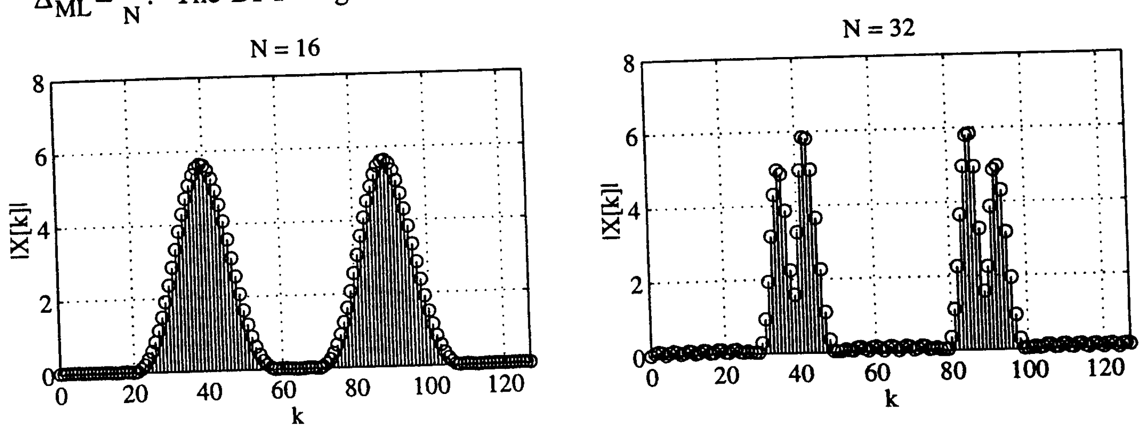


An increase in the size of the DFT increases the resolution of the spectral analysis by reducing the separation between adjacent DFT samples. Also the estimated values of the frequencies of the sinusoid get closer to the actual values of 0.167 and 0.3076 as the size of the DFT increases.

M11.3 As the separation between the two frequencies decreases, the distance between the two maximas in the DFT of the sequence decreases, and when $f_2 = 0.21$, the second sinusoid cannot be determined from the DFT plot. This is due to the use of a length-16 rectangular window to truncate the original infinite-length sequence. For a length-16 rectangular window, two adjacent sinusoids can be distinguished if their angular frequencies are apart by half the mainlobe width of $\frac{4\pi}{N}$ radians or equivalently, if their frequencies are apart by $\frac{2}{N} = 0.0625$. Note that the DFT length R = 128 is all plots.

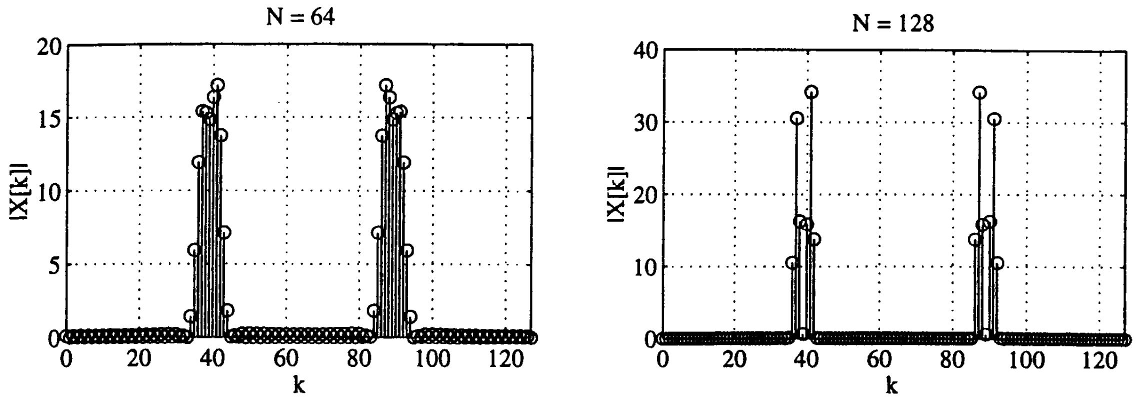


M11.4 $f_2 = 0.21$, $f_1 = 0.18$. Hence, $\Delta f = 0.03$. For a Hamming window the mainlobe width $\Delta_{ML} = \frac{8\pi}{N}$. The DFT length R = 128 in all plots.



(i) N = 16. Here it is not possible to distinguish the two sinusoids. This also can be seen from the value of $\Delta_{ML} = \frac{8}{16} = 0.5$, and hence, half of the mainlobe width is greater than Δf.

(ii) Increasing N to 32, makes the separation between the two peaks visible. However, it is difficult to identify the peaks accurately.

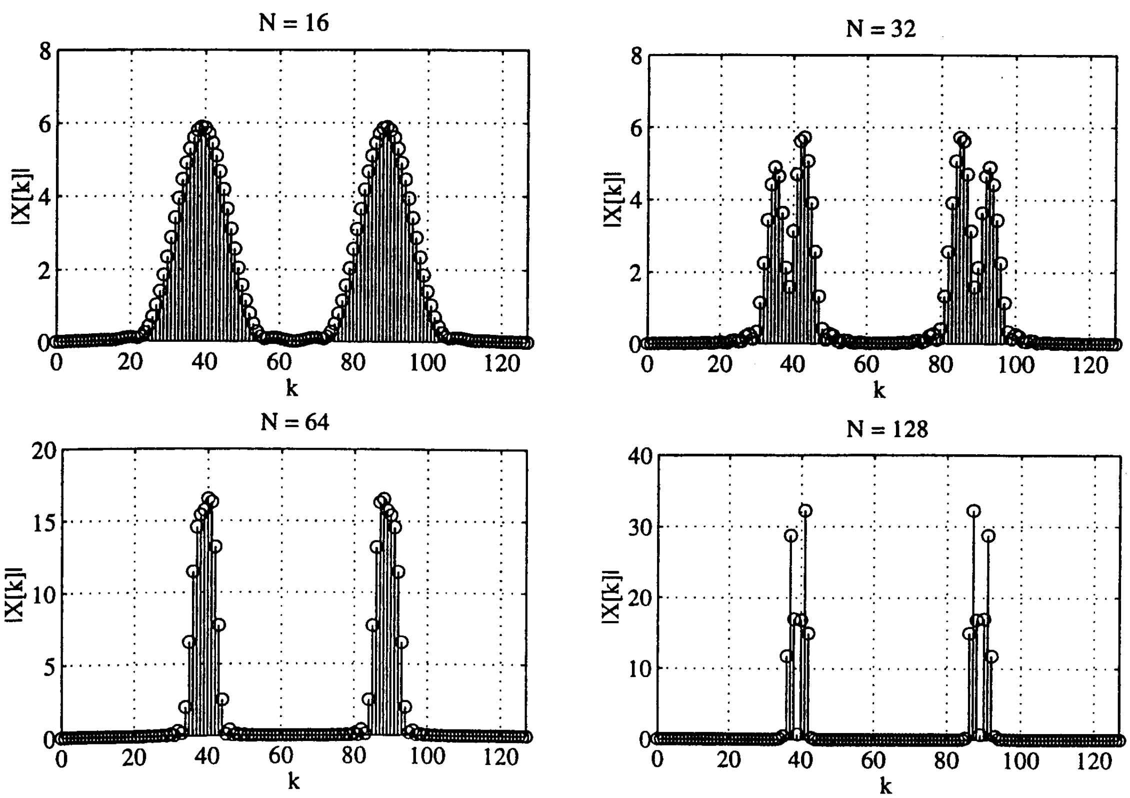


(iii) Increasing N to 64, makes the separation between the two peaks more visible. However, it is still difficult to identify the peaks accurately.

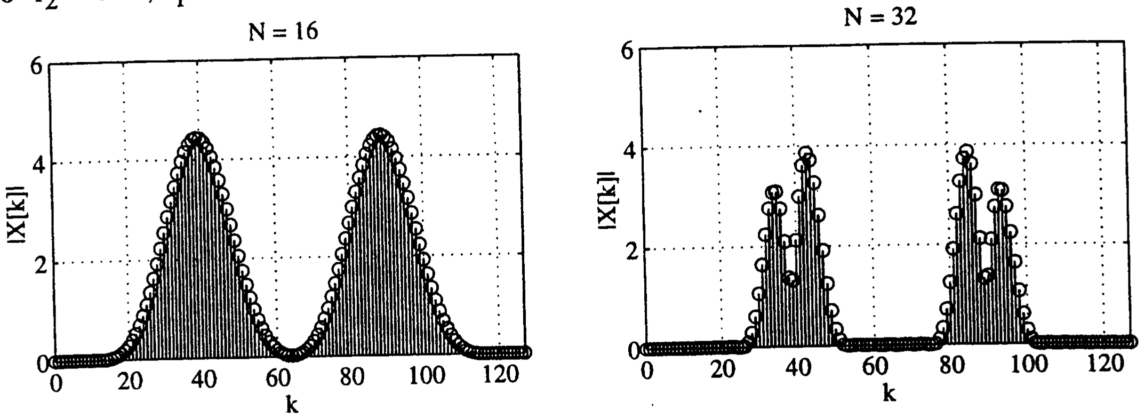
(iv) For N = 128, the separation between the two peaks clearly visible.

Note also the suppression of the minor peaks due to the use of a tapered window.

M11.5 Results are similar to that in Problem M11.4.

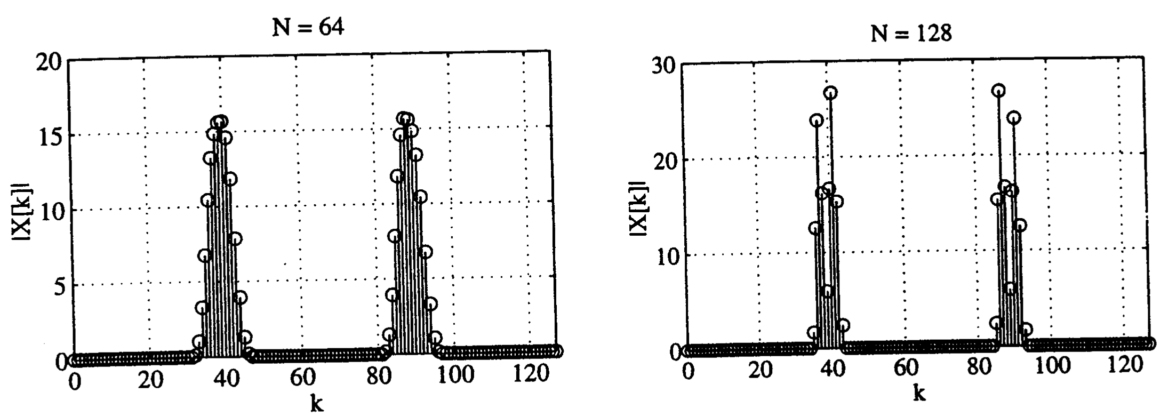


M11.6 $f_2 = 0.21$, $f_1 = 0.18$. Hence, $\Delta f = 0.03$. The DFT length $R = 128$ in all plots.



(i) For N = 16, it is difficult to identify the two sinusoids.

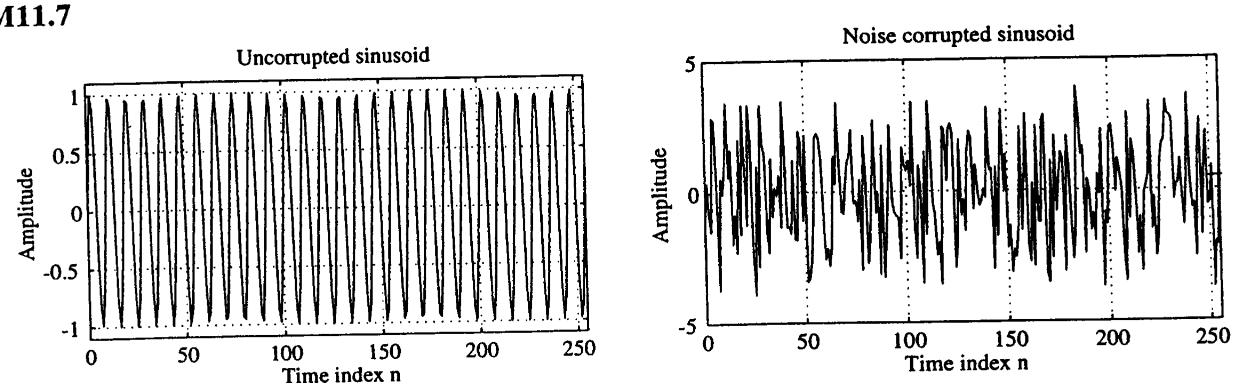
(ii) For N = 32, there are two peaks clearly visible at k = 36 and 43, respectively.

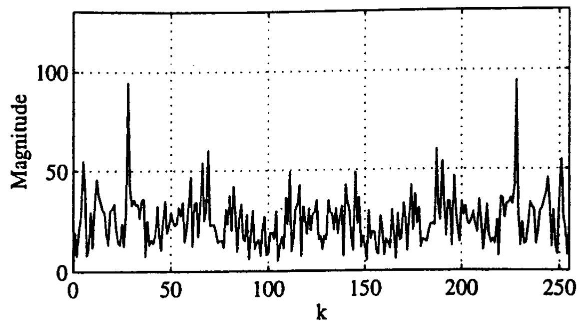


(iii) For N = 64, it is difficult to identify the two sinusoids.

(iv) For N = 128, there are two peaks clearly visible at k = 37 and 41, respectively.

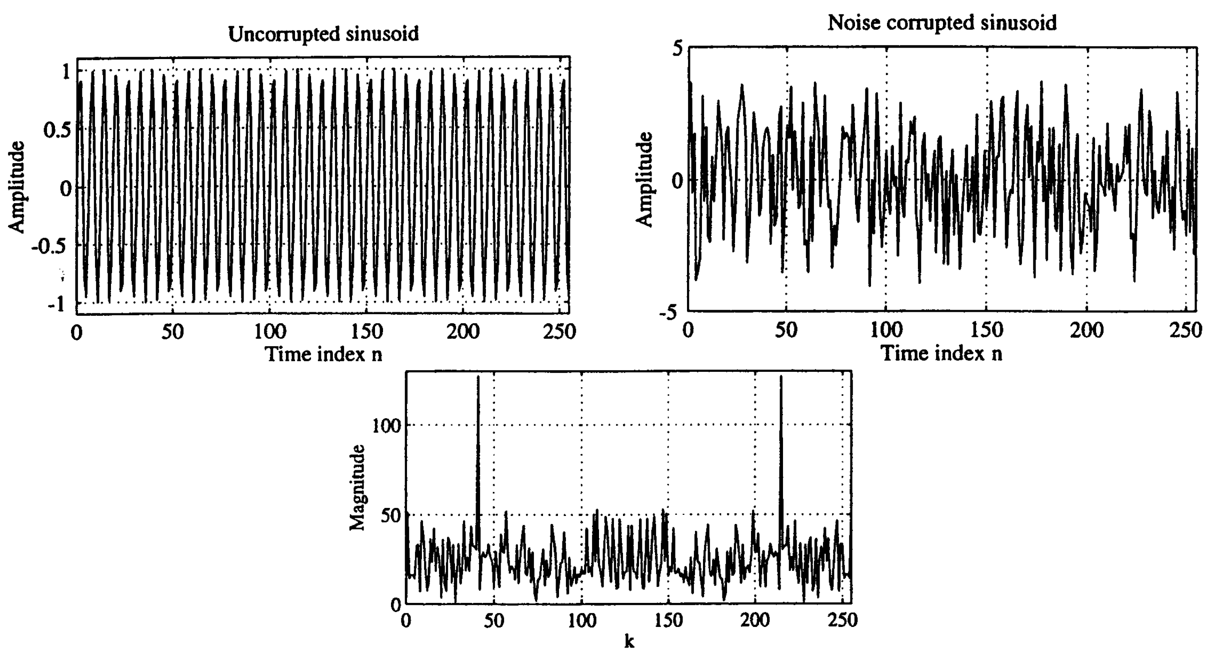
M11.7





The SNR computed by the program is –7.4147 dB. There is a peak at the frequency index 29 whose normalized frequency equivalent is equal to 29/256 = 0.1133. Hence the DFT approach has correctly identified the frequency of the sinusoid corrupted by the noise.

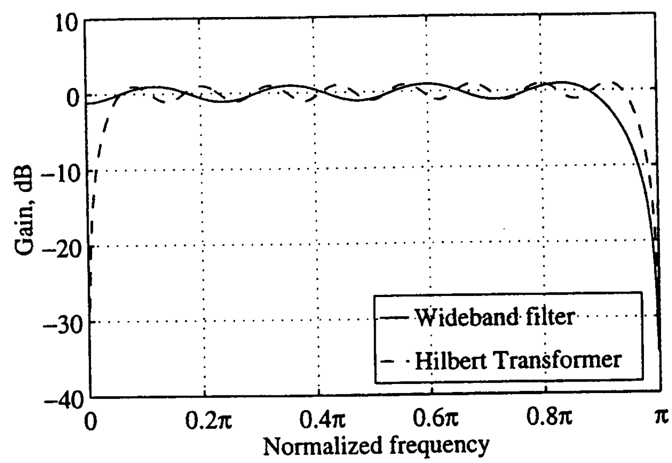
M11.8



The SNR computed by the program is –7.7938 dB. There is a peak at the frequency index 42 whose normalized frequency equivalent is equal to 42/256 = 0.1641. Hence the DFT approach has correctly identified the frequency of the sinusoid corrupted by the noise.

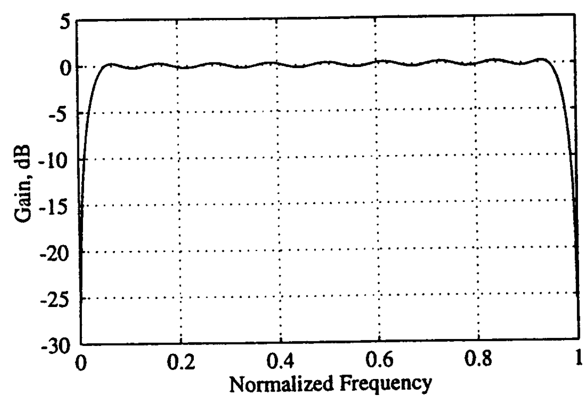
M11.9

```
N = 17;
f = [0 0.9 0.95 1];m = [1 1 0 0]; wt = [2.0 0.5];
c = remez(N,f,m,wt);
for k = 1:length(f)
     h(2*k-1) = (-1)^(k-1)*c(k);
     h(2*k) = 0.0;
end
[F,w] = freqz(c,1,512); [H,w] = freqz(h,1,512);
plot(w/pi,20*log10(abs(F)),'-',w/pi,20*log10(abs(H)),'b--');
xlabel('Normalized Frequency'); ylabel('Magnitude, dB');
grid;
legend('r-','Wideband filter','b--','Hilbert Transformer');
```



The number of multipliers required is 9.

M11.10
```
b = remez(34,[0.05 0.95],[1 1],'hilbert');
[B,w] = freqz(b,1,512);
plot(w/pi,20*log10(abs(B)));
xlabel('Normalized frequency');ylabel('Magnitude,dB');
grid on;
```

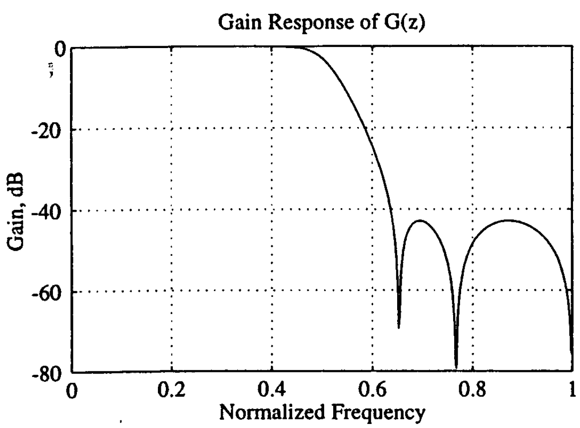


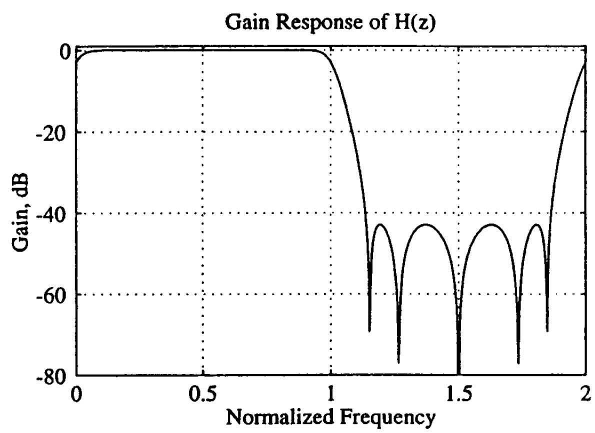
Note that in this case odd coefficients are small but not zero so number of multipliers required is 18 (due to linear phase). In M11.9 the number of multipliers required is 9.

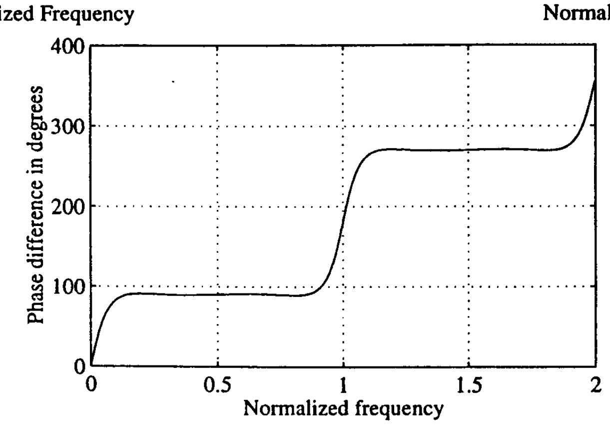
M11.11 With $\delta_s = 0.014$, the poles did not lie on the unit circle. By changing the stopband ripple value to $\delta_s = 0.0071726$, the poles are on the unit circle. This is due to the fact that with $\delta_s = 0.014$, the value of the filter order is N = 4.3. When this value is rounded up to N = 5, we need to recompute the value of δ_s of using Eqns. (5.49) and (5.50) after a bilinear transformation.

```
% Note for half band filter (1-2*delp)^2+dels^2 = 1;
dels = 0.0071726; delp = 0.5*(1-sqrt(1-dels^2));
rp = -20*log10(1-2*delp); rs = -20*log10(dels);
wp = 0.36;ws = 0.64;
ws = 1-wp;
[N,wn] = elliord(wp,ws,rp,rs);
[b,a] = ellip(N,rp,rs,wn);
zplane(b,a);
% Using the pole interlacing property we get
% A0(z) = (0.19525+z^-1)/(1+0.19525z^-1) and
% A1(z) = (0.6662 +z^-1)/(1+0.6662z^-1)
```

```
numh = [0.19525 0.6662j -1.130075 -1.130075j 0.6662
0.19525j];
denh = [1 0 -0.86145 0 0.130075];
[G,w] = freqz(b,a,512);
[H,W] = freqz(numh,denh,512,'whole');
subplot(211)
plot(w/pi,20*log10(abs(G)));
title(Gain Response of G(z)');
xlabel('Normalized Frequency'); ylabel(Gain, dB');
axis([0 1 -80 0]);grid;
subplot(212)
plot(W/pi,20*log10(abs(H)/max(abs(H))));
title(Gain Response of H(z)');
xlabel('Normalized Frequency');ylabel(Gain, dB');
axis([0 2 -80 1]);grid;
numa0 = [0.19525 0 -1]; dena0 = [1 0 -0.19525];
numa1 = [0 0.6662 0 -1]; dena1=[1 0 -0.6662];
[A0,w] = freqz(numa0,dena0,512,'whole');
[A1,w] = freqz(numa1,dena1,512,'whole');
figure;
plot(w/pi,180/pi*unwrap((angle(A0)-angle(A1))));
xlabel('Normalized frequency');
ylabel('Phase difference, degrees');
grid;
```







M11.12
```
% Choose N = 5 i.e. a fifth order lagrange polynomial
N = 5;
alpha = [0 5/4 10/4 15/4];
for k = 1:length(alpha)
```

```
        for l = 0:N
        var = 1;
        for m = 0:N
            if (m ~= l)
                var = (alpha(k)-m)/(l-m)*var;
                end
            end
        P(k,l+1) = var;
        end
end
n = 0:61;
x = sin(2*pi*0.05*n);
for k = 0:11
     ip = [x(5*k+1) x(5*k+2) x(5*k+3) x(5*k+4) x(5*k+5)
x(5*k+6)];
     for m = 1:4
          y(4*k+m) = P(m,:)*ip';
     end
end
op = y(1:31);
stem(op);
n = 0:30;
xx = sin(2*pi*0.05*5/4*n);
stem(n, xx-op);
xlabel('Time index n'); ylabel('Amplitude');title('Error');
```