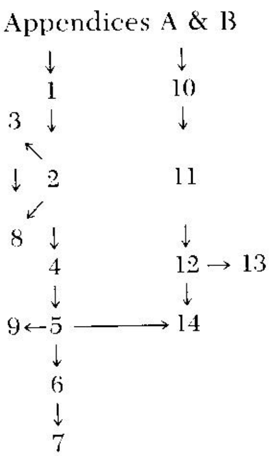
LOGICAL DEPENDENCE OF CHAPTERS



Deterministic Part = Chapters 1–9

Stochastic Part = Chapters 10–14

Introductory Part = Chapter 1 through Section 5
Chapter 2 through Section 3
Chapters 4, 5(excluding Section 3)
Chapter 7 through Section 2
Chapters 10–12

Advanced Part = Remainder of book

Parameter optimization problems

1

1.1 Problems without constraints

The simplest class of parameter optimization problems involves finding the values of m parameters $u_1, \ldots, u_m$ that minimize a performance index which is a function of these parameters,

$$L(u_1, \ldots, u_m).$$

For convenience, we shall use a more compact nomenclature; let

$$u = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{bmatrix} = \text{decision vector} \tag{1.1.1}$$

and write the performance index then as

$$L(u). \tag{1.1.2}$$

If there are no constraints on possible values of u and if the function $L(u)$ has first and second partial derivatives everywhere, *necessary conditions for a minimum* are

$$\frac{\partial L}{\partial u} = 0, \tag{1.1.3}$$

by which we mean that $\partial L/\partial u_i = 0$, $i = 1, \ldots, m$, and

$$\frac{\partial^2 L}{\partial u^2} \geq 0, \tag{1.1.4}$$

by which we mean that the $(m \times m)$-matrix whose components are $\partial^2 L/\partial u_i\, \partial u_j$ must be positive semidefinite, i.e., have eigenvalues that are zero or positive.

All points that satisfy (1.1.3) are called *stationary* points.

Sufficient conditions for a local minimum are (1.1.3) and

$$\frac{\partial^2 L}{\partial u^2} > 0 \,; \qquad (1.1.5)$$

that is, all the eigenvalues must be positive.

If (1.1.3) is satisfied but $\partial^2 L/\partial u^2 = 0$, that is, the determinant of the matrix is zero (meaning that one or more of its eigenvalues is zero), additional information is needed to establish whether or not the point is a minimum. Such a point is called a *singular* point. Note that, if L is a linear function of u, then $\partial^2 L/\partial u^2 = 0$ everywhere, and, in general, a minimum does not exist.

Examples *for* $L = L(u_1, u_2)$.

(a) ***minimum***: both eigenvalues of $\partial^2 L/\partial u_i\, \partial u_j > 0$

$$L = [u_1 u_2] \begin{bmatrix} 1, -1 \\ -1, \ \ 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

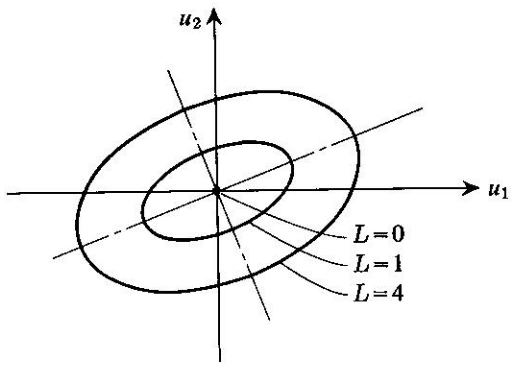


Figure 1.1.1. A minimum point.

(b) *saddlepoint*: one positive eigenvalue, one negative eigenvalue of $\partial^2 L/\partial u_i\, \partial u_j$

$$L = [u_1 u_2] \begin{bmatrix} -1,1 \\ 1,3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(c) *singular point*: one positive eigenvalue, one zero eigenvalue

$$L = (u_1 - u_2^2)(u_1 - 3u_2^2).$$

1.2 Problems with equality constraints; necessary conditions for a stationary point

A more general class of parameter optimization problems involves finding the values of m decision parameters $u_1, \ldots, u_m$ that minimize

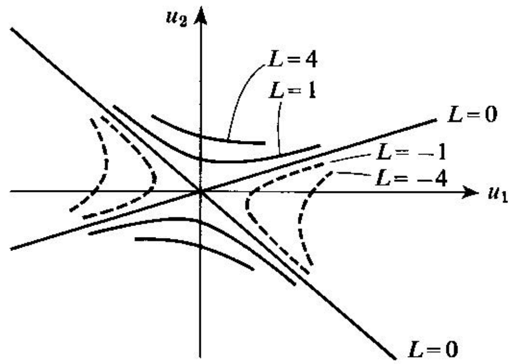


Figure 1.1.2. A saddle point.

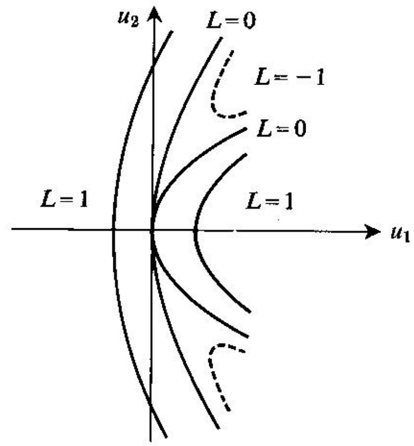


Figure 1.1.3. A singular point.

a performance index which is a scalar function of $n + m$ parameters,

$$L(x_1, \ldots x_n \,; u_1, \ldots u_m)\,,$$

where the n *state parameters* $x_1, \ldots, x_n$ are determined by the decision parameters through a set of n *constraint relations*,

$$f_1(x_1, \ldots, x_n \,; u_1, \ldots, u_m) = 0\,,$$
$$\vdots$$
$$f_n(x_1, \ldots, x_n \,; u_1, \ldots, u_m) = 0\,.$$

For convenience, we shall again use a more compact nomenclature. Let

$$u = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{bmatrix} = \text{decision vector}, \qquad x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \text{state vector},$$

$$f = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix} = \text{constraint vector}.$$

In this nomenclature, the problem may be stated as follows.

Find the decision vector u that minimizes

$$L(x,u), \tag{1.2.1}$$

where the state vector x is determined by the decision vector through the constraint relations

$$f(x,u) = 0 \qquad (n \text{ equations}). \tag{1.2.2}$$

For a given parameter optimization problem, the choice of which parameters to designate as decision parameters is not unique; it is only a matter of convenience to make a distinction between decision and state parameters. However, the choice must be such that u determines x through the constraint relations (1.2.2).

If the relations (1.2.1) and (1.2.2) are *linear* in both x and u, then, in general, a minimum does not exist. Inequality constraints on the magnitudes of x and/or u are necessary to make the problem meaningful; such problems, treated in later sections of this chapter, are called *linear programming problems* if the inequality constraints are also linear in x and u.

In the first part of this chapter we shall discuss problems that have some nonlinearity in (1.2.1) and (1.2.2). Of course, the presence of nonlinearity does not, in itself, insure that a minimum exists.

A *stationary* point is one where $dL = 0$, for arbitrary du, while holding $df = 0$ (letting dx change as it will). Now we have

$$dL = L_x\, dx + L_u\, du, \tag{1.2.3}$$

$$df = f_x\, dx + f_u\, du. \tag{1.2.4}$$

When we require that $df = 0$, then, if f_x is nonsingular (and it should be if u determines x from (1.2.2)), (1.2.4) may be solved for dx

$$dx = -f_x^{-1} f_u\, du. \tag{1.2.5}$$

Substituting (1.2.5) into (1.2.3) yields

$$dL = (L_u - L_x f_x^{-1} f_u)\, du. \tag{1.2.6}$$

Hence, if dL is to be zero for arbitrary du, *it is necessary that*

$$\boxed{L_u - L_x f_x^{-1} f_u = 0} \qquad (m \text{ equations}). \tag{1.2.7}$$

These m equations together with the n equations (1.2.2), determine the m quantities u and the n quantities x at stationary points. Note that (1.2.7) represents the partial derivative of L with respect to u, *holding f constant*, whereas L_u represents the partial derivative of L with respect to u, *holding x constant.*

Another (equivalent) approach to Equation (1.2.7) is to notice that (1.2.3) and (1.2.4), with $dL = 0$, $df = 0$, must be *consistent* linear equations in dx and du at a stationary point. If they are consistent, we should be able to find a set of n constants $\lambda_1, \ldots, \lambda_n$ such that

$$L_y + \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial_y} = 0, \tag{1.2.8}$$

where

$$y^T = (x_1, \ldots, x_n, u_1, \ldots u_m) \tag{1.2.9}$$

that is, a linear combination of rows of f_y must equal L_y. † For convenience, let

$$\lambda = \begin{bmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n \end{bmatrix} \qquad \text{and} \qquad \lambda^T = [\lambda_1, \ldots, \lambda_n].$$

Then we may write (1.2.8) and (1.2.9) as

$$L_x + \lambda^T f_x = 0 \qquad (n \text{ equations}), \tag{1.2.10}$$

$$L_u + \lambda^T f_u = 0 \qquad (m \text{ equations}). \tag{1.2.11}$$

Equation (1.2.10) may be solved for λ^T (since f_x must be nonsingular):

$$\lambda^T = -L_x (f_x)^{-1}, \tag{1.2.12}$$

which, in turn, may be substituted into (1.2.11) to yield (1.2.7).

†More generally, consistency requires that the rank of the $[(n+1) \times (n+m)]$-matrix

$$\begin{bmatrix} f_x \cdot f_u \\ L_x, L_u \end{bmatrix}$$

be less than $(n + 1)$.

The interpretation of λ may be inferred from (1.2.3) and (1.2.4) by placing $du = 0$ and eliminating dx

$$-\lambda^T = L_x(f_x)^{-1} = (\partial L/\partial f)_u ;$$

that is, the λ's are partial derivatives of L with respect to f, holding u constant, letting x change as required. This will have a special significance in optimization problems with *inequality* constraints (Section 1.7).

Still another (equivalent) approach, which we shall use many times throughout the remainder of the book, is to "adjoin" the constraints (1.2.2) to the performance index (1.2.1) by a set of n "undetermined multipliers," $\lambda_1, \ldots, \lambda_n$,* as follows:

$$H(x,u,\lambda) = L(x,u) + \sum_{i=1}^{n} \lambda_i f_i(x,u) = L(x,u) + \lambda^T f(x,u) . \quad (1.2.13)$$

Suppose we have chosen some nominal values of u and determined the corresponding values of x from (1.2.2) so that $L = H$. Differential changes in H due to differential changes in x and u are given by

$$dH = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial u} du . \quad (1.2.14)$$

Since we are interested in how H (and, hence, L) changes as the control vector u changes, it is convenient to *choose* the λ vector so that

$$\frac{\partial H}{\partial x} = \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} = 0 \Rightarrow \lambda^T = -\frac{\partial L}{\partial x}\left(\frac{\partial f}{\partial x}\right)^{-1} , \quad (1.2.15)$$

which is, of course, the same as (1.2.12).

Since x was found from (1.2.2), it follows that

$$dL \equiv dH = \frac{\partial H}{\partial u} du . \quad (1.2.16)$$

Thus, $\partial H/\partial u$ is the gradient of L with respect to u *while holding* $f(x,u) = 0$.
At a stationary point in the u-space, dL vanishes for arbitrary du; this can happen only if

$$\frac{\partial H}{\partial u} \equiv \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0 . \quad (1.2.17)$$

Thus, *necessary conditions for a stationary value of* $L(x,u)$ are

$$f(x,u) = 0 ; \quad (1.2.18)$$

*The constants $\lambda_1, \ldots, \lambda_n$ are often referred to as *Lagrange multipliers*.

$$\frac{\partial H}{\partial x} = 0 , \quad (1.2.19)$$

$$\text{where} \quad H = L(x,u) + \lambda^T f(x,u) ,$$

$$\frac{\partial H}{\partial u} = 0 , \quad (1.2.20)$$

which are $2n + m$ equations for the $2n + m$ quantities x, λ, and u.

Example 1. Find the scalar parameter u that yields a stationary value of

$$L = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right) ,$$

with the linear constraint

$$f(x,u) = x + mu - c = 0 ,$$

where x is a scalar parameter and a, b, m, and c are constants.

The curves of constant L are ellipses, with L increasing with the size of the ellipse, whereas $x + mu - c = 0$ is a fixed straight line.

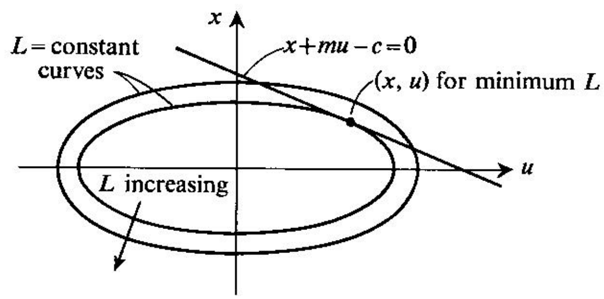


Figure 1.2.1. Example of minimization subject to a constraint.

Clearly, the minimum value of L satisfying the constraint is obtained when the ellipse is just tangent to the straight line. Analytically, the H function is

$$H = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right) + \lambda(x + mu - c) ,$$

and necessary conditions for a stationary value are

$$x + mu - c = 0 , \qquad \frac{\partial H}{\partial x} = \frac{x}{a^2} + \lambda = 0 , \qquad \frac{\partial H}{\partial u} = \frac{u}{b^2} + \lambda m = 0 .$$

These three equations for the three unknowns, x, u, λ, have a simple unique solution:

$$x = \frac{a^2c}{a^2 + m^2b^2}, \qquad u = \frac{b^2mc}{a^2 + m^2b^2}, \qquad \lambda = -\frac{c}{a^2 + m^2b^2},$$

and the minimum value of L is thus given by

$$J = L_{\min} = \frac{c^2}{2(a^2 + m^2b^2)}.$$

Note that $-\lambda = \partial J/\partial c \equiv \partial J/\partial f$.

Example 2. *Maximum steady rate of climb for aircraft.* The net force on an aircraft maintaining a steady rate of climb must be zero. If we choose force components parallel and perpendicular to the flight path (see Figure 1.2.2), this requires that

$$f_1(V,\gamma,\alpha) = T\cos(\alpha + \epsilon) - D - mg\sin\gamma = 0,$$

$$f_2(V,\gamma,\alpha) = T\sin(\alpha + \epsilon) + L - mg\cos\gamma = 0,$$

where

V = velocity,
γ = flight path angle to horizontal,
α = angle-of-attack,
m = mass of aircraft,
g = gravitational force per unit mass,
ϵ = angle between thrust axis and zero-lift axis,

and, at a given altitude,

$L = L(V,\alpha)$ = lift force,
$D = D(V,\alpha)$ = drag force,
$T = T(V)$ = thrust of engine.

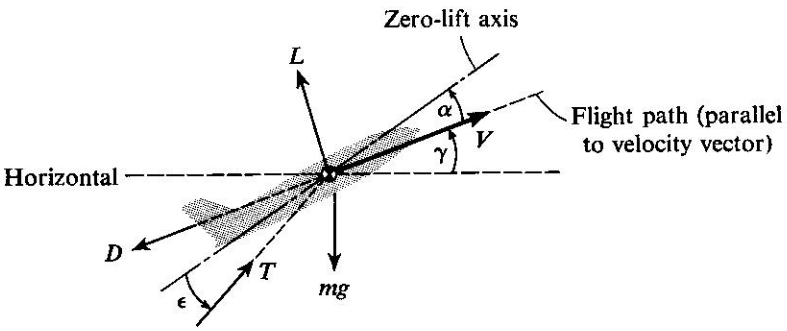


Figure 1.2.2. Force equilibrium of climbing aircraft.

The rate of climb is simply

$$V\sin\gamma.$$

We choose V and γ as state parameters and α as the control parameter since, at a given altitude, a choice of α determines V, γ from the two force equilibrium relations.

The H function is

$$H = V\sin\gamma + \lambda_1(T\cos(\alpha + \epsilon) - D - mg\sin\gamma) + \lambda_2(T\sin(\alpha + \epsilon) + L - mg\cos\gamma).$$

Hence, the necessary conditions for a stationary value of rate of climb are:

$$f_1 = T(V)\cos(\alpha + \epsilon) - D(V,\alpha) - mg\sin\gamma = 0,$$

$$f_2 = T(V)\sin(\alpha + \epsilon) - L(V,\alpha) - mg\cos\gamma = 0,$$

$$\frac{\partial H}{\partial V} = \sin\gamma + \lambda_1\left[\frac{\partial T}{\partial V}\cos(\alpha + \epsilon) - \frac{\partial D}{\partial V}\right] + \lambda_2\left[\frac{\partial T}{\partial V}\sin(\alpha + \epsilon) + \frac{\partial L}{\partial V}\right] = 0,$$

$$\frac{\partial H}{\partial \gamma} = V\cos\gamma - \lambda_1\, mg\cos\gamma + \lambda_2\, mg\sin\gamma = 0,$$

$$\frac{\partial H}{\partial \alpha} = \lambda_1\left[-T\sin(\alpha + \epsilon) - \frac{\partial D}{\partial \alpha}\right] + \lambda_2\left[T\cos(\alpha + \epsilon) + \frac{\partial L}{\partial \alpha}\right] = 0.$$

These five equations for the five unknowns, V, γ, α, λ_1, and λ_2, will, in general, have to be solved numerically for realistic lift, drag, and thrust functions.

1.3 Problems with equality constraints; sufficient conditions for a local minimum

To second order, the differential changes of $L(x,u)$ and $f(x,u)$ away from a nominal point (x,u) are

$$dL = (L_x, L_u)\begin{pmatrix} dx \\ du \end{pmatrix} + \frac{1}{2}(dx^T, du^T)\begin{pmatrix} L_{xx}, & L_{xu} \\ L_{ux}, & L_{uu} \end{pmatrix}\begin{pmatrix} dx \\ du \end{pmatrix}, \qquad (1.3.1)$$

$$df = (f_x, f_u)\begin{pmatrix} dx \\ du \end{pmatrix} + \frac{1}{2}(dx^T, du^T)\begin{pmatrix} f_{xx}, & f_{xu} \\ f_{ux}, & f_{uu} \end{pmatrix}\begin{pmatrix} dx \\ du \end{pmatrix}, \qquad (1.3.2)\dagger$$

where

$$L_{xu} = \frac{\partial}{\partial u}\left(\frac{\partial L}{\partial x}\right)^T, \qquad L_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial x}\right)^T, \qquad \text{etc.}$$

†This equation must be interpreted as applying to each component of f.

If we multiply (1.3.2) by λ^T, determined from Equation (1.2.19), where $(H_x = 0)$, and add the result to (1.3.1), we obtain

$$dL = (0, H_u)\begin{pmatrix} dx \\ du \end{pmatrix} + \frac{1}{2}(dx^T, du^T)\begin{pmatrix} H_{xx}, H_{xu} \\ H_{ux}, H_{uu} \end{pmatrix}\begin{pmatrix} dx \\ du \end{pmatrix} - \lambda^T df, \quad (1.3.3)$$

where

$$H = L(x,u) + \lambda^T f(x,u). \quad (1.3.4)$$

Let us assume that the nominal point (x,u) satisfies the constraints $f(x,u) = 0$. We wish to examine the behavior of $L(x,u)$ in an infinitesimal neighborhood of this nominal point, keeping $f(x,u) = 0$ to second order.

If we place $df = 0$ in (1.3.2), we can write for dx

$$dx = -f_x^{-1} f_u \, du + \text{second and higher-order terms in components of } du \text{ and } dx. \quad (1.3.5)$$

Now, if the nominal point is a ***stationary point***, then $H_u = 0$, and (1.3.3) becomes, with $df = 0$,

$$dL = \frac{1}{2} du^T [-f_u^T (f_x^T)^{-1}, I] \begin{bmatrix} H_{xx}, H_{xu} \\ H_{ux}, H_{uu} \end{bmatrix} \begin{bmatrix} -f_x^{-1} f_u \\ I \end{bmatrix} du \quad (1.3.6)$$

$$+ \text{ third and higher-order terms in components of } du \text{ and } dx.$$

It follows that

$$\left(\frac{\partial^2 L}{\partial u^2}\right)_{f=0} = H_{uu} - H_{ux} f_x^{-1} f_u - f_u^T (f_x^T)^{-1} H_{xu} + f_u^T (f_x^T)^{-1} H_{xx} f_x^{-1} f_u. \quad (1.3.7)$$

Thus, ***sufficient conditions for a local minimum*** are the stationarity conditions (1.2.18) through (1.2.20) and the ***positive-definiteness*** of the matrix in (1.3.7). Clearly, a ***necessary condition*** for a local minimum is that the matrix in (1.3.7) be ***positive semidefinite***. Note that (1.3.6) could have been obtained directly by considering the augmented criterion H to *second* order while considering the constraint $f = 0$ to *first* order only.

Furthermore, it is *not* necessary that H_{uu} be positive semidefinite.

Example. Find the scalar quantity u that minimizes

$$L = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right)$$

with the quadratic constraint

$$f(x,u) = c - xu = 0,$$

where x is a scalar parameter and a, b, and c are positive constants. The curves of constant L are ellipses, with L increasing with the size of the ellipse, whereas $c - xu = 0$ is a hyperbola with two branches.

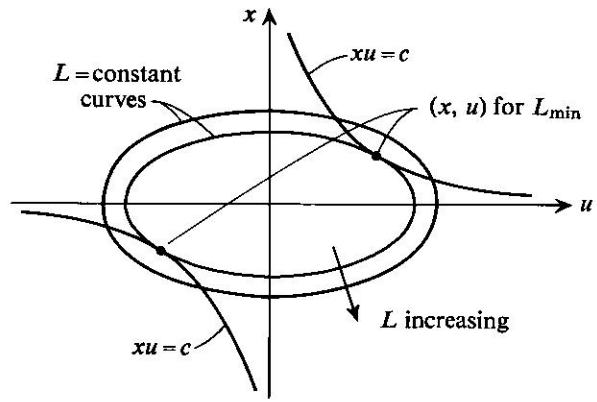


Figure 1.3.1. Example of minimization subject to a nonlinear constraint.

The minimum value of L satisfying the constraint is obtained when the ellipse is just tangent to the hyperbola. Analytically, the H function is

$$H = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right) + \lambda(c - xu),$$

and necessary conditions for a stationary value are

$$c - xu = 0, \qquad \frac{\partial H}{\partial x} = \frac{x}{a^2} - \lambda u = 0, \qquad \frac{\partial H}{\partial u} = \frac{u}{b^2} - \lambda x = 0.$$

Using these relations, we find that

$$x = \pm\sqrt{\frac{ac}{b}}, \qquad u = \pm\sqrt{\frac{bc}{a}}, \qquad \lambda = \frac{1}{ab}, \qquad J = L_{\min} = \frac{c}{ab}.$$

The sufficient condition (1.3.7) for this problem is

$$\left(-\frac{a}{b}, 1\right)\begin{bmatrix} \frac{1}{a^2}, & -\frac{1}{ab} \\ -\frac{1}{ab}, & \frac{1}{b^2} \end{bmatrix}\begin{bmatrix} -\frac{a}{b} \\ 1 \end{bmatrix} = \frac{4}{b^2} > 0,$$

which is clearly satisfied. Note that there are two points at which the same minimum value of L occurs. Note, also, that

$$\lambda = \frac{\partial J}{\partial c}.$$

Problem 1. Find the point nearest the origin on the line

$$x + 2y + 3z = 10\,, \qquad x - y + 2z = 1\,,$$

where x, y, z are rectangular coordinates; i.e., minimize

$$L = x^2 + y^2 + z^2$$

subject to the two linear constraints.

Problem 2. Find the rectangle of maximum perimeter that can be inscribed in an ellipse; i.e., maximize

$$P = 4(x + y)$$

with the constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

Problem 3. Find the rectangular parallelopiped of maximum volume that can be contained in a given ellipsoid; i.e., maximize

$$V = 8xyz$$

with the constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,.$$

Problem 4. *Quadratic performance index with linear constraints.* Show that the control vector u that minimizes the nonnegative definite quadratic form

$$L = \frac{1}{2}x^TQx + \frac{1}{2}u^TRu\,,$$

with the linear constraints

$$f(x,u) = x + Gu + c = 0\,,$$

is

$$u = -(R + G^TQG)^{-1}G^TQc\,.$$

Show, also, that the minimum value of L is

$$J = L_{\min} = \tfrac{1}{2}c^T(Q - QG(R + G^TQG)^{-1}G^TQ)\,c$$

and that

$$\begin{aligned} \lambda &= (Q - QG(R + G^TQG)^{-1}G^TQ)\,c \\ &\equiv (Q^{-1} + GR^{-1}G^T)^{-1}c \quad \text{if} \quad Q^{-1} \text{ exists;}\dagger \\ x &= -(I - G(R + G^TQG)^{-1}G^TQ)\,c\,. \end{aligned}$$

Note, also, that

$$\lambda^T = \frac{\partial J}{\partial c}\,.$$

Problem 5. *Sail setting and heading for maximum upwind velocity.* A simplified model of a sailboat moving at constant velocity is shown in Figure 1.3.2.

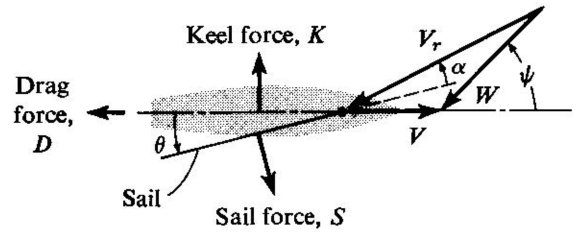


Figure 1.3.2. Force equilibrium of sailboat.

The sailboat's velocity relative to the water is V, at an angle ψ to the wind, which is blowing with velocity W relative to the water. The sail is set at an angle θ to the centerline of the boat, and the aerodynamic force S is assumed to act normal to the sail. The hydrodynamic force on the hull is resolved into components perpendicular to the centerline K and parallel to the centerline D. The magnitude of S is assumed to vary with the square of the relative wind, V_r, and the sine of the sail angle of attack, α:

$$S = C_1 V_r^2 \sin\alpha\,,$$

where C_1 is a constant and V_r and α are as defined in Figure 1.3.2. The drag is assumed to vary with the square of the boat velocity, V:

$$D = C_2 V^2\,,$$

where C_2 is a constant. For equilibrium of forces parallel to the centerline, we have

$$D = S\sin\theta\,.$$

Show that: (a) For given ψ, maximum V is obtained when $\alpha = \theta$. (b) The maximum velocity for $\psi = 180°$ (running before the wind) is $W\mu/(1 + \mu)$ and is obtained when $\theta = 90°$, where $\mu^2 = C_1/C_2$. (c) The maximum upwind velocity, $V\cos\psi$, is equal to $W\mu/4$ and is obtained when the sail setting and the heading are chosen to be

†This is known as the "matrix inversion lemma." See Section 12.2 for discussion of its importance.

$$\theta \cong [(\mu+2)^2 + 4]^{-1/2}, \qquad \psi \cong 45° .$$

Assume for this part of the problem that α and θ are small angles so that $\sin\alpha \cong \alpha$, $\sin\theta \cong \theta$, $\cos\alpha \cong 1$, $\cos\theta \cong 1$.

Problem 6. *Angle of attack and bank angle for maximum lateral range glide.* A quasisteady approximation for gliding turns of a low-speed (subsonic) glider, made with constant angle of attack and constant bank angle, gives lateral gliding range, y_f, as

$$y_f = r(1 - \cos\beta_f),$$

where

$$r = \frac{\ell}{\alpha}\frac{\cos^2\gamma}{\sin\sigma} = \text{radius of the helix},$$

$$\beta_f = \frac{z_o}{\ell}\frac{\alpha\sin\sigma}{\sin\gamma\cos\gamma} = \text{final heading angle},$$

$$\gamma = \tan^{-1}\left[\left(\alpha + \frac{\delta^2}{4\alpha}\right)\sec\sigma\right] = \text{gliding helix angle},$$

and

$\alpha = \eta\bar{\alpha}$; $\bar{\alpha}$ = angle of attack,
σ = bank angle
} (decision parameters),

z_o = initial altitude,

$\ell = \dfrac{2m\eta}{\rho S C_{L_\alpha}}$ = characteristic length ($\cong$ 10 ft for typical sailplane),

$\delta = 2(\eta C_{D_0}/C_{L_\alpha})^{1/2}$ = minimum drag to lift ratio ($\cong \frac{1}{30}$ for typical sailplane),

η = efficiency factor $(0 < \eta < 1)$.

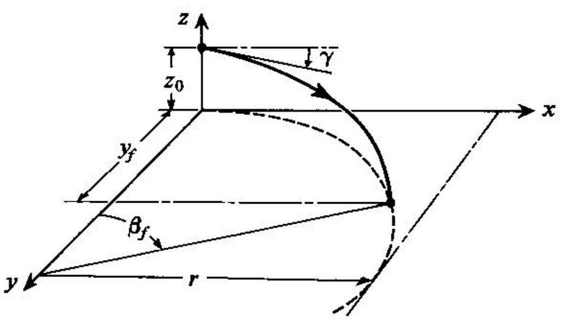


Figure 1.3.3. Geometry of flight path for lateral turn.

Show that the maximum value of y_f for a given z_o is obtained when we have

$$\tan\frac{\beta_f}{2} = \frac{\beta_f}{1 + (4\beta_f^2/\zeta^2)},$$

which may be regarded as a transcendental equation for β_f as a function of $\zeta = z_o/\ell$. The corresponding values of σ, α, and γ are obtained from

$$\tan\sigma = \frac{2\beta_f}{\zeta}, \qquad \alpha = \frac{\delta}{2\sqrt{\cos 2\sigma}}, \qquad \gamma = 2\alpha\cos\sigma .$$

Assume that α, γ, δ are $<< 1$.

[NOTE 1. Within this same approximation, the maximum value of x_f for given z_o is

$$x_f = z_o/\delta$$

and is obtained with

$$\alpha = (1/2)\delta, \qquad \sigma = 0 \Rightarrow \tan\gamma = \delta .]$$

[NOTE 2: Further definition of symbols:

m = mass of glider, V = velocity,
ρ = density of the atmosphere (approximated as constant in this problem),
C_{L_α} = lift coefficient slope,
C_{D_0} = zero-lift drag coefficient
S = reference area for coefficients,

$$\text{Lift} = C_{L_\alpha}\bar{\alpha}\frac{\rho V^2}{2}S, \qquad \text{Drag} = (C_{D_0} + \eta C_{L_\alpha}\bar{\alpha}^2)\frac{\rho V^2}{2}S .]$$

Problem 7. *Maximum steady rate of climb for an aircraft.* For the problem stated in Example 2 of Section 1.2, find the maximum steady rate of climb at sea level and at altitudes of 10,000 ft, 20,000 ft, 30,000 ft, and 40,000 ft for an airplane with weight mg = 34,000 lbs and wing area S = 530 ft². The lift, drag, and thrust characteristics are given below:

$$L = C_{L_\alpha}\alpha\frac{\rho V^2}{2}S, \qquad D = (C_{D_0} + \eta C_{L_\alpha}\alpha^2)\frac{\rho V^2}{2}S .$$

Here C_{L_α}, C_{D_0}, and η are functions of Mach number $M \equiv V/c$, as shown in Figures 1.3.4 and 1.3.5; c = speed of sound and ρ = density of the air, both of which are functions of altitude, that is, $c = c(h)$, $\rho = \rho(h)$. These functions are given in Table 1.3.1. The thrust, T, at full throttle, is a function of Mach number and altitude, as shown in Figure 1.3.5. Use $\epsilon = 3°$.

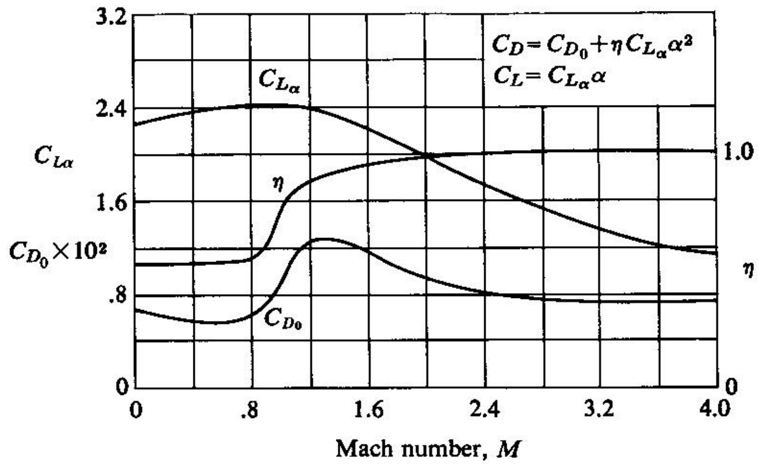


Figure 1.3.4. Drag and lift coefficients as function of Mach number.

Find, also, the altitude at which the maximum rate of climb is zero. This is called the "ceiling" of the airplane.

Table 1.3.1. Air density and speed of sound variation with altitude

Altitude ~h, ft	Speed of sound ~C, ft/sec	Air density ~ρ, slugs/ft^3
0	1,116	2,377 × 10^{-6}
5,000	1,097	2,048
10,000	1,077	1,755
15,000	1,057	1,496
20,000	1,037	1,266
25,000	1,016	1,065
30,000	994.7	889.3
36,090	968.1	706.1
40,000	968.1	585.1
45,000	968.1	460.1
50,000	968.1	361.8
55,000	968.1	284.5
60,000	968.1	223.8
70,000	968.1	138.4
80,000	968.1	85.56
82,020	968.1	77.64
90,000	984.2	51.51
100,000	1,004	31.38

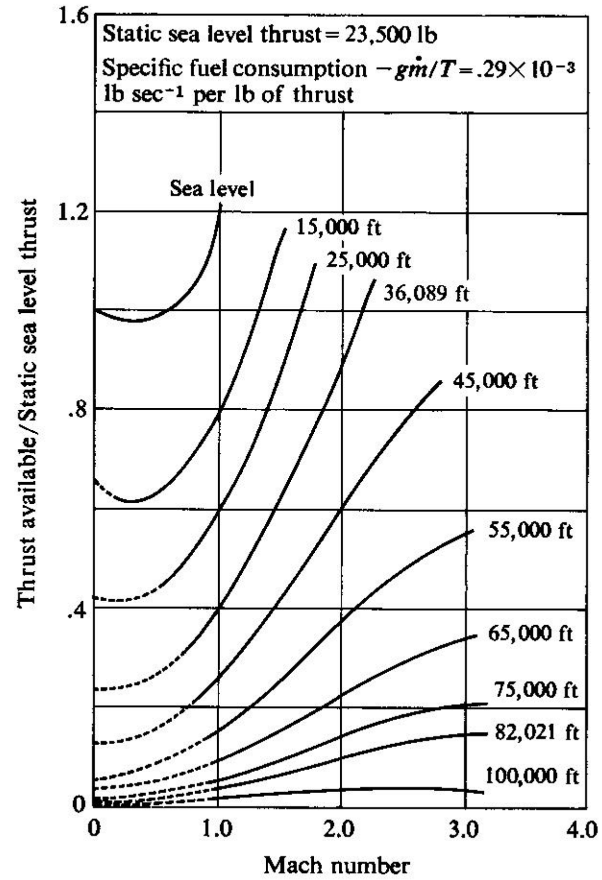


Figure 1.3.5. Thrust as function of Mach number and altitude at full throttle.

Problem 8. *Minimum fuel turn at constant altitude.* A steady turn ($\dot{V} = 0$, $\dot{r} = 0$) at constant altitude is described by

$$(C_{D_0} + \eta C_{L_\alpha}\bar{\alpha}^2)\frac{\rho V^2}{2}S = T \qquad \text{(drag = thrust)},$$

$$mg = C_{L_\alpha}\bar{\alpha}\frac{\rho V^2}{2}S\,(\cos\sigma) \qquad \text{(weight = vertical component of lift)},$$

$$mV\dot{\beta} = C_{L_\alpha}\bar{\alpha}\frac{\rho V^2}{2}S\,(\sin\sigma) \qquad \text{(turn rate} \approx \text{horizontal component of lift)},$$

where

$$\left.\begin{aligned}\bar{\alpha} &= \text{angle-of-attack,}\\ \sigma &: \text{bank angle}\end{aligned}\right\}\quad \text{decision parameters}$$

and the rest of the symbols are as defined in Problem 6.

Find $\alpha \triangleq \eta\bar{\alpha}$ and σ to minimize the fuel in making a turn from $\beta = \beta_o$ to $\beta = \beta_f$ where fuel is proportional to

$$J \triangleq \int_0^{t_f} T\,dt = \int_{\beta_0}^{\beta_f} \frac{T\,d\beta}{\dot{\beta}} = \frac{T}{\dot{\beta}}(\beta_f - \beta_o);$$

that is, minimize

$$\frac{T}{\dot{\beta}} = \frac{(C_{D_0} + \eta C_{L_\alpha}\bar{\alpha}^2)\,mV}{C_{L_\alpha}\bar{\alpha}\sin\sigma}$$

subject to $mg = C_{L_\alpha}\bar{\alpha}\left(\frac{\rho V^2}{2}\right) S\cos\sigma$.

ANSWER. $\alpha = (\sqrt{3}/2)\delta$, $\sigma = \cos^{-1}(1/\sqrt{3}) = 54.7°$, where $\delta = 2\sqrt{\eta C_{D_0}/C_{L_\alpha}}$.

Note that this implies $V = \sqrt{2g\ell/\delta}$, $\frac{L}{D} = (\sqrt{3}/2)\,(1/\delta)$ and $T = 2mg\,\delta$, where $\ell = \frac{2m\eta}{C_{L_\alpha}\rho S}$.

1.4. Neighboring optimum solutions and the interpretation of the Lagrange multipliers

Occasionally we wish to find how the optimum solution changes if some of the constants in the constraint equations are changed by small amounts.

Let us suppose that the constraints (1.2.2) are increased by infinitesimal amounts so that we have $f(x,u) = df$, where df is an infinitesimal constant vector. Then, assuming that the values of x and u for the minimal solution will be changed by infinitesimal amounts dx and du, we have, from (1.2.19), (1.2.20), (1.2.18),

$$dH_x^T = H_{xx}\,dx + H_{xu}\,du + f_x^T d\lambda = 0, \tag{1.4.1}$$

$$dH_u^T = H_{ux}\,dx + H_{uu}\,du + f_u^T d\lambda = 0, \tag{1.4.2}$$

$$df = f_x\,dx + f_u\,du, \tag{1.4.3}$$

where the partial derivatives are evaluated at the point corresponding to the original optimum solution.

The $2n + m$ relations (1.4.1), (1.4.2), and (1.4.3) determine the $2n + m$ parameters dx, du, and $d\lambda$. Since f_x must be nonsingular for du to determine dx, it follows from (1.4.3) and (1.4.1) that

$$dx = f_x^{-1}\,df - f_x^{-1}f_u\,du, \tag{1.4.4}$$

$$d\lambda = -(f_x^T)^{-1}(H_{xx}\,dx + H_{xu}\,du). \tag{1.4.5}$$

Substituting these relations into (1.4.2) and solving for du yields

$$du = -C\,df, \tag{1.4.6}$$

where

$$C = \left(\frac{\partial^2 L}{\partial u^2}\right)^{-1}_{f=0} [H_{ux} - f_u^T(f_x^T)^{-1}H_{xx}]f_x^{-1}, \tag{1.4.7}$$

and $(\partial^2 L/\partial u^2)_{f=0}$ is as defined in Equation (1.3.7). Thus, the existence of neighboring optimum solutions is guaranteed if the original stationary point was a local minimum; that is, if $(\partial^2 L/\partial u^2)_{f=0} > 0$.

By substituting (1.4.4) into (1.3.3), with $H_u = 0$, we obtain dL correct to second order. If, in addition, we substitute the expression for du from (1.4.6) into (1.3.3), we have, after some manipulation,

$$dL = -\lambda^T df + (1/2)\,df^T[f_x^{-T}H_{xx}f_x^{-1} - C^T L_{uu} C]\,df + \cdots, \tag{1.4.8}$$

where

$$L_{uu} \equiv \left(\frac{\partial^2 L}{\partial u^2}\right)_{f=0},$$

which is given by (1.3.7).

Thus, we have

$$\frac{\partial L_{\min}}{\partial f} = -\lambda^T, \tag{1.4.9}$$

$$\frac{\partial^2 L_{\min}}{\partial f^2} = f_x^{-T}H_{xx}f_x^{-1} - C^T L_{uu} C. \tag{1.4.10}$$

1.5 Numerical solution by a first-order gradient method†

Unless the relations for $L(x,u)$ and $f(x,u)$ in Section 1.2 are quite simple, numerical methods must be used to determine the values of u that minimize H. A straightforward numerical method, in common use for many years, is that of *steepest descent* for finding minima (or steepest ascent for finding maxima).

Steepest descent or gradient methods are characterized by iterative algorithms for improving estimates of the control parameters, u, so as to come closer to satisfying the stationarity conditions $\partial H/\partial u = 0$.

The steps in using the gradient method are as follows:

†Grateful acknowledgment is made to Walter F. Denham for his assistance in writing this section.

(a) Guess a set of values for u.
(b) Determine the values of x from $f(x,u) = 0$.
(c) Determine the values of λ from $\lambda^T = -(\partial L/\partial x)(\partial f/\partial x)^{-1}$.
(d) Determine the values of $\partial H/\partial u = (\partial L/\partial u) + \lambda^T(\partial f/\partial u)$, which, in general, will *not* be zero.
(e) Interpreting $\partial H/\partial u$ as a gradient vector, change the estimates of u by amounts $\Delta u = -K(\partial H/\partial u)^T$, where K is a positive scalar constant. The predicted change in the criterion, ΔJ, is thus $-K(\partial H/\partial u)(\partial H/\partial u)^T$. (The "−" is replaced by a "+" if a maximum is being sought.)
(f) Repeat steps (a) through (f), using the revised estimates of u, until $(\partial H/\partial u)(\partial H/\partial u)^T$ is very small.

There are many variations of this approach, and we will consider one of them in the next section. Graphically, the gradient method (for finding a maximum) is a hill-climbing technique in the u-space; if u is a two-component vector, we can show contours of constant J in the u-plane (see Figure 1.5.1). Starting with an initial guess of u, a sequence of changes Δu is made. At each step, Δu is in the direction of the gradient $\partial H/\partial u$ whose magnitude gives the steepest slope at that point on the hill. The choice of K, which determines the magnitude of Δu, involves judging the extent of nonlinearity so that the linearized prediction will be reasonably accurate, while at the same time trying to keep the number of steps in the sequence from becoming excessive. K should almost always be varied in the sequence.

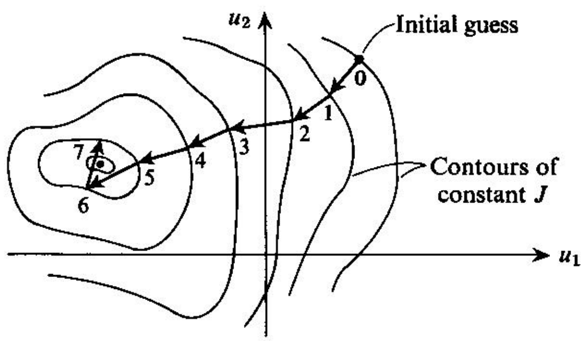


Figure 1.5.1. Typical numerical procedure for first-order gradient method.

Usually this will be done to decrease the magnitude of Δu when it is thought the minimum (or maximum) is near. As Figure 1.5.1 suggests, it is easy to overshoot the extremal point. In problems of higher dimension, the geometrical concept of hypersurfaces of constant J in the u-hyperspace provides valuable insight.

Gradient methods usually show substantial improvements in the first few iterations but have poor convergence characteristics as the optimal solution is approached. Second-order gradient methods that use the "curvature" as well as the "slope" at the nominal point are discussed in the next section; they have excellent convergence characteristics as the optimal solution is approached but may have starting difficulties associated with picking a "convex" nominal solution.†

1.6 Numerical solution by a second-order gradient method

Second-order gradient methods†† use information on the curvature as well as the slope at the nominal point in the u-space. If u is a scalar, we can sketch a simple description of the second-order gradient method as in Figure 1.6.1.

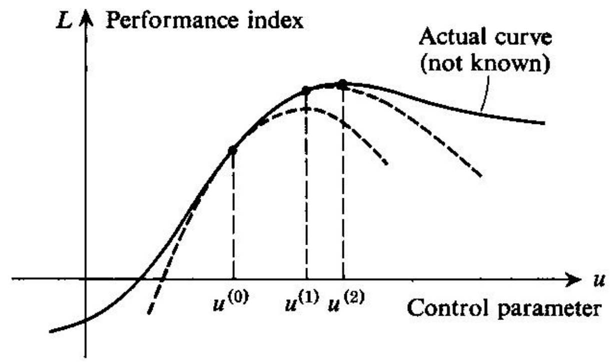


Figure 1.6.1. Typical numerical procedure for second-order ascent method.

The actual curve of performance index L vs. control parameter u could, in principle, be calculated and the maximum picked out. However, this may involve an enormous amount of computation. Instead, using the second-order gradient method, we *guess* a control parameter $u^{(o)}$ and determine $x^{(o)}$ from $f(x^{(o)},u^{(o)}) = 0$, and then $L(x^{(o)},u^{(o)})$. We then determine the first and second derivatives of L with respect to u, holding $f(x,u) = 0$ using (1.2.6) and (1.3.7) and approximate the (L vs. u)-curve by a quadratic curve (a parabola):

$$L \cong L(x^{(o)}, u^{(o)}) + \left(\frac{\partial L}{\partial u}\right)^0_{f=0} (u - u^o) + \frac{1}{2}\left(\frac{\partial^2 L}{\partial u^2}\right)^0_{f=0} (u - u^o)^2 .$$

†By this we mean that the nominal approximating quadratic surface has a minimum.
††These are often called Newton-Raphson methods.

The value of u that yields the maximum of this approximate curve is then easily determined; call it $u^{(1)}$. This value is taken as an improved guess and the process is repeated. In Figure 1.6.1 it is apparent that two steps in the iterative procedure already yield a good approximation to the maximizing value of u. In more complicated problems, more steps may be required. Also, if the initial guess $u^{(o)}$ is too far away from the maximizing value, we may find that $(\partial^2 L/\partial u^2)^o_{f=o} > 0$; i.e., the curvature has the wrong sign. In this case, the method fails completely; note, however, that the first-order gradient method may still converge.

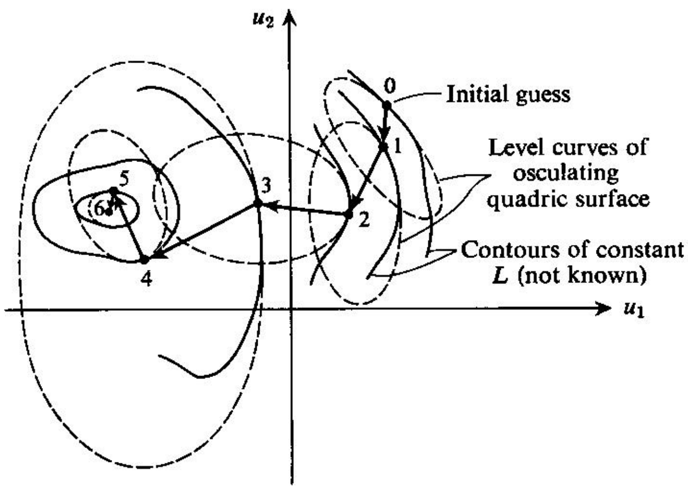


Figure 1.6.2. Two-dimensional illustration of second-order gradient method.

Figure 1.6.2 shows a two-parameter maximization problem with contours of constant performance index L, holding $f(x,u) = 0$ (unknown to the optimizer). An initial guess is made at point 0, and an *osculating quadric surface* is fitted locally to the region around 0 by determining the first and second derivatives of L, holding $f(x,u) = 0$, from (1.2.6) and (1.3.7). If this quadric surface turns out to be an elliptic paraboloid with a maximum (that is, the matrix of second derivatives is negative definite), the location of that maximum is taken as the next guess (point 1).† The procedure is repeated until we have $(\partial L/\partial u)_{f=o} = 0$, hopefully with $(\partial^2 L/\partial u^2)_{f=o} < 0$ all the way. Figure 1.6.2 shows the maximum being achieved in six steps.

†If the matrix of second derivatives is positive definite or indefinite, the procedure fails.

The constraint relations $f(x,u) = 0$ are often so complicated that numerical methods are needed just to determine x, given u. In this case a slightly more general version of the second-order gradient method may be used. Recall that necessary conditions for a stationary value of $L(x,u)$ are

$$H_x = 0\,, \tag{1.6.2}$$

$$H_u = 0\,, \tag{1.6.3}$$

$$f = 0\,, \qquad \text{where} \qquad H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u). \tag{1.6.4}$$

The steps in the generalized second-order gradient method are:

(a) Guess a set of values for x, u, and λ; call them x^o, u^o, and λ^o.
(b) Determine the values of

$$H_x(x^o,u^o,\lambda^o) = H^o_x\,, \tag{1.6.5}$$

$$H_u(x^o,u^o,\lambda^o) = H^o_u\,, \tag{1.6.6}$$

$$f(x^o,u^o) = f^o\,. \tag{1.6.7}$$

(c) Linearize the relations (1.6.2), (1.6.3), and (1.6.4) about x^o, u^o, and λ^o:

$$H^o_x + H^o_{xx}\,dx + H^o_{xu}\,du + (f^T_x)^o\,d\lambda = 0\,, \tag{1.6.8}$$

$$H^o_u + H^o_{ux}\,dx + H^o_{uu}\,du + (f^T_u)^o\,d\lambda = 0\,, \tag{1.6.9}$$

$$f^o + f^o_x\,dx + f^o_u\,du = 0\,. \tag{1.6.10}$$

(d) Solve Equations (1.6.8), (1.6.9), and (1.6.10) for dx, du, and $d\lambda$ in terms of H^o_x, H^o_u, and f^o†,
(e) Repeat (a) through (d), using, as improved guesses,

$$x^1 = x^o + dx\,, \qquad u^1 = u^o + du\,, \qquad \lambda^1 = \lambda^o + d\lambda\,.$$

The process is repeated until the necessary conditions (1.6.2), (1.6.3), and (1.6.4) are satisfied to the desired degree of accuracy.

If the method converges at all, it may converge on a minimum, a maximum, or a saddle point. To determine which of these it is, we must examine the curvature matrix given in Equation (1.3.7). If the matrix is positive definite, the point is a minimum; if the matrix is negative definite, the point is a maximum; if the matrix is not singular but is neither positive definite nor negative definite, the point is a saddle point; if the matrix is singular, we do not know the nature of the point without going to higher derivatives.

†If values of H^o_x, H^o_u, f^o are such that dx, du, $d\lambda$ obtained from step (d) are very large then ϵH^o_x, ϵH^o_u, ϵf^o may be used instead, where $0 < \epsilon < 1$.

Problem. Another variant of the second-order gradient method would be to guess only x and u, determining λ from $H_x = 0$. Work out the procedure for this variant.

1.7 Problems with inequality constraints

Parameter optimization problems involving inequality constraints require extension of the methods treated in the previous sections. An important class of problems of this type involves minimizing

$$L(y) \tag{1.7.1}$$

subject to

$$f(y) \leq 0\,, \tag{1.7.2}$$

where, in general, f and y are vectors of different dimension.†

Consider first the case in which y and f are scalars. There are two possibilities for the optimal value of y, y^o : $f(y^o) < 0$ or $f(y^o) = 0$. In the former case, the constraint is not effective and can be ignored. The situation remains the same as in Section 1.1. In the latter case, consider small perturbations about y^o; if $L(y^o)$ is a minimum, then we have

$$dL = \left.\frac{\partial L}{\partial y}\right|_{y^o} dy \geq 0 \tag{1.7.3}$$

for all *admissible* values of dy, which must satisfy

$$df = \frac{\partial f}{\partial y}\, dy \leq 0\,. \tag{1.7.4}$$

In order that Equations (1.7.3) and (1.7.4) be consistent, it is clearly necessary that either

$$\operatorname{sgn}\frac{\partial L}{\partial y} = -\operatorname{sgn}\frac{\partial f}{\partial y} \qquad \text{or} \qquad \frac{\partial L}{\partial y} = 0\,.$$

These two possibilities may be expressed in one relation as

$$\frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0\,, \qquad \lambda \geq 0\,. \tag{1.7.5}$$

†Such problems, referred to as *nonlinear programming problems*, have been treated extensively in the technical literature. We give only brief treatment in this section. We will not distinguish between state and control variables as was done in Section 1.2. Indeed, in many nonlinear programming problems, the dimension of f is greater than the dimension of y, so that it is not possible to decompose y into state and decision variables.

The two situations are shown geometrically in Figures 1.7.1(a) and (b).

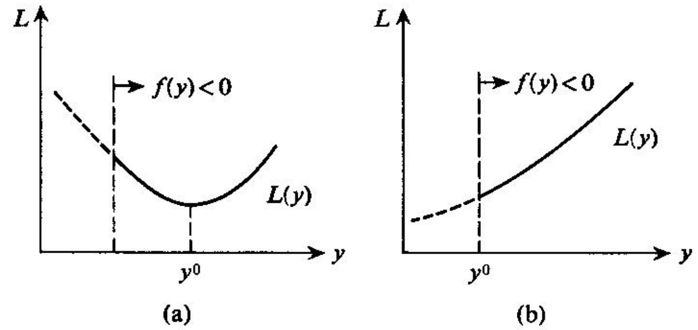


Figure 1.7.1. One-dimensional illustration of two possible types of minimum with inequality constraints.

The two cases may be treated analytically by adjoining (1.7.2) to (1.7.1):

$$H(y,\lambda) = L(y) + \lambda f(y)\,. \tag{1.7.6}$$

The necessary conditions become

$$\frac{\partial H}{\partial y} = 0 \tag{1.7.7}$$

and

$$f(y) \leq 0\,, \tag{1.7.8}$$

where

$$\lambda \begin{cases} \geq 0\,, & f(y) = 0\,, \\ = 0\,, & f(y) < 0\,. \end{cases} \tag{1.7.9}$$

When y is a vector but f is still a scalar, Equations (1.7.3), (1.7.4), and (1.7.5) remain applicable if we interpret the symbols in vector notation. We should interpret conclusion (1.7.5) to mean

$$\frac{\partial L}{\partial y} \text{ parallel to } \frac{\partial f}{\partial y} \text{ and pointing in opposite directions.} \tag{1.7.10}$$

The necessity of (1.7.10) is easily established by contradiction. Let us suppose that (1.7.10) is not true as illustrated in two dimensions in Figure 1.7.2. Then the cross-hatched region represents a region of admissible y which will yield smaller L.

This and the other situation (namely, when $f(y^o) < 0$) again can be

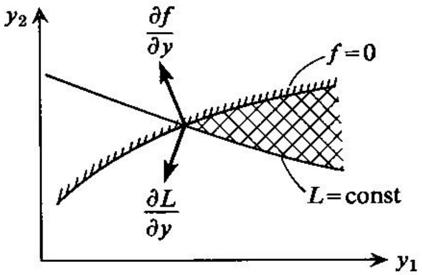


Figure 1.7.2. Two-dimensional illustration showing the necessity of Equation (1.7.10).

summarized by the necessary conditions stated by (1.7.7), (1.7.8), and (1.7.9).

In the more general case, when f itself is a vector, we can still employ (1.7.4) and (1.7.5), noting that $\partial f/\partial y$ is now a matrix. If only one component of f is effective, the problem is the same as that just treated. If two components of f are effective, the situation, in two dimensions, is as shown in Figure 1.7.3.

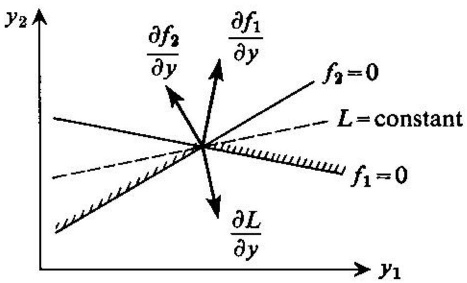


Figure 1.7.3. Two-dimensional illustration of minimization subject to two inequality constraints.

It is clear that, if y^o is to be an optimal point on $f_1 = f_2 = 0$, then $\partial L/\partial y$ must lie between the negative gradients of f_1 and f_2.† Analytically, this means that $\partial L/\partial y$ can be expressed as a *negative* linear combination of $\partial f_1/\partial y$ and $\partial f_2/\partial y$. In general, when q components are effective at a boundary optimal point, we must have

$$\frac{\partial L}{\partial y} + \lambda_1 \frac{\partial f_1}{\partial y} + \cdots + \lambda_q \frac{\partial f_q}{\partial y} = 0 \tag{1.7.11}$$

†Recall the parallelogram construction of a resultant from two components.

or

$$\frac{\partial L}{\partial y} + \lambda^T \frac{\partial f}{\partial y} = 0\,, \tag{1.7.12}$$

where

$$\lambda \begin{cases} \geq 0\,, & f(y) = 0\,, \\ = 0\,, & f(y) < 0\,. \end{cases} \tag{1.7.13†}$$

Hence, as in Section 1.2, we may define a quantity $H \equiv L + \lambda^T f$ and write (1.7.12) as $\partial H/\partial y = 0$. Equations (1.7.12) and (1.7.13) are necessary conditions for minimality. For a maximum, the sign of λ must be changed in (1.7.13). In words, *the gradient of L with respect to y at a minimum must be pointed in such a way that decrease of L can only come by violating the constraints.*

Let us suppose that y has p components and that n components of the inequality constraint are "effective," that is,

$$f_i(y) = 0\,, \qquad i = 1, \ldots, n\,. \tag{1.7.14}$$

The "ineffective" constraints, $f_i(y) < 0$, $i = n+1, \ldots,$ may be disregarded. It is clear that $p \geq n$. Next, take n of the components of y and call them x; let the remaining $p - n$ components be called u; that is,

$$y^T = (x_1, \ldots, x_n\,; u_1, \ldots, u_{p-n}) \triangleq (x^T, u^T)\,.$$

The choice must be such that

$$f_i(x,u) = 0\,, \qquad i = 1, \ldots, n \tag{1.7.15}$$

determines x when u is given. Then ***sufficient conditions*** for a local minimum of $L(y)$, with $f(y) \leq 0$, are given in Section 1.3, to which we must add the condition that $\lambda_1, \ldots, \lambda_n$ *all* be positive.‡ The latter condition is easily interpreted from Equation (1.4.8) since $-\lambda_i$ is equal to $(\partial L/\partial f_i)_u$, which must be negative (that is, $dL > 0$ for $df_i < 0$).

Example. Consider $L(y_1,y_2)$ with $f_i(y_1,y_2) \leq 0$, $i = 1,2$, and suppose the level curves are as shown in Figure 1.7.4.
It is clear that $f_2 < 0$ is "ineffective" and $f_1 = 0$. From Figure 1.7.4, we have

$$\frac{\partial L}{\partial y} + \lambda_1 \frac{\partial f_1}{\partial y} = 0\,, \qquad \lambda_1 > 0\,;$$

†Equation (1.7.13) is understood, of course, to be in the component-by-component sense.
‡For a precise statement, see G. McCormick, "Second Order Sufficient Conditions for Constrained Minimum," *SIAM Journal on Appl. Math.*, Vol. 15, No. 3, May 1967.

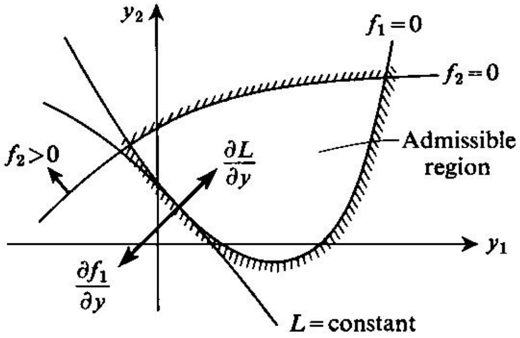


Figure 1.7.4. Example of minimization subject to inequality constraints.

that is, grad L is parallel and in the opposite direction to grad f_1. Also, the "curvature" of L along $f_1 = 0$ is such that L increases on $f_1 = 0$ away from the minimum; to show this analytically, we may let $y_1 = x$ and $y_2 = u$. Then we have

$$f_1(x,u) = 0 \Rightarrow x\,, \qquad \text{given } u,$$

and, from (1.3.7), we can compute $(\partial^2 L/\partial u^2)_{f_1=0}$ which, as we can see from Figure 1.7.4, is positive.

Equations (1.7.12) and (1.7.13) are the essence of the Kuhn-Tucker Theorem in nonlinear programming. The precise statement of the condition requires the assumption of the so-called "constraint qualification" on the set $f(y) \leq 0$ (Kuhn-Tucker, p. 483). This qualification is designed to rule out geometric situations as shown in Figure 1.7.5. At the minimum point we see that $(y_1,y_2) = (1,0)$ and $(\partial L/\partial y)$ is *not* equal to any finite linear combination $(\partial f_1/\partial y)$, $(\partial f_2/\partial y)$.

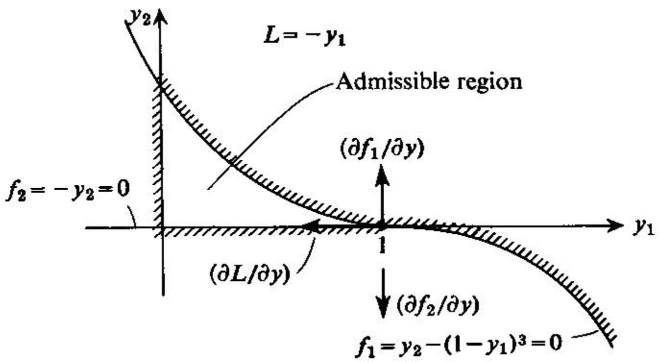


Figure 1.7.5. Example of Kuhn-Tucker constraints qualification.

Another approach to sufficiency is the ***saddle-point theorem*** of nonlinear programming. It is more elegant (but usually harder to apply) than the conditions given above since it does not require the arbitrary separation of y into x and u. Consider the function $H(y,\lambda) = L + \lambda^T f$. Suppose that it is possible to find y^o and λ^o such that they constitute a saddle point for H; that is,

$$H(y^o,\lambda) \leq H(y^o,\lambda^o) \leq H(y,\lambda^o) \qquad (1.7.16)$$

for all $\lambda \geq 0$ and $f(y) \leq 0$. Then we may conclude that y^o is a minimum point for $L(y)$ subject to $f(y) \leq 0$, regardless of the nature of L and f.

Problem 1. Prove the saddle-point theorem. [HINT: The left-hand inequality in (1.7.16) implies that $\lambda_i^o f_i(y^o) = 0$ for all i.]

Problem 2. *Aircraft cruise condition for minimum fuel consumption.* For the airplane described in Example 2, Section 1.2, and in Problem 7, Section 1.3, find the steady level-flight ($\gamma = 0$) condition for minimum fuel consumption per unit distance. Assume constant specific fuel consumption, $\sigma = .29 \times 10^{-3}$ lb sec^{-1} per lb of thrust, so that fuel consumption per unit distance is given by

$$J = \frac{\sigma T}{V},$$

where

$$T \leq T_{\max}(V,h)$$

and $T_{\max}(V,h)$ is as given graphically in Problem 7, Section 1.3.

The constraint equations are

$$L - mg + T\sin(\alpha + \epsilon) = 0\,, \qquad D - T\cos(\alpha + \epsilon) = 0\,,$$

where $L = L(V,h,\alpha)$, $D = D(V,h,\alpha)$ are as given in Problem 7, Section 1.3.

Problem 3. Write out a mathematical proof of the geometrical argument of Figure 1.7.2. In particular, show why $\lambda \geq 0$.

1.8 Linear programming problems

If both the performance index and the inequality constraints are linear functions of y, the problem is called a *linear programming problem.* Clearly, in this case, the minimum, if it exists, *must* occur on the boundary since the curvature of L is zero everywhere. Let the problem be to choose y to minimize

$$L = b^T y\,, \tag{1.8.1}$$

subject to

$$A^T y + c \leq 0\,, \tag{1.8.2}$$

where y is an n-vector and c is an m-vector, $m > n$. If A is of rank n and b^T is not collinear with any of the rows of A^T or any negative linear combinations of $n + 1$ rows of A^T, the minimum, if it exists, must occur at a *point* determined by the simultaneous satisfaction of n of the constraints $A^T y + c = 0$. This result is not surprising to anyone with geometric intuition; it is the *fundamental theorem of linear programming.*

Example 1. Minimize $L = -5y_1 - y_2$ subject to

$$f_1 = -y_1 \leq 0\,, \quad f_2 = -y_2 \leq 0\,, \quad f_3 = y_1 + y_2 - 6 \leq 0\,,$$
$$f_4 = 3y_1 + y_2 - 12 \leq 0\,, \quad f_5 = y_1 - 2y_2 - 2 \leq 0\,.$$

Figure 1.8.1 shows the admissible region, with contours of constant L.

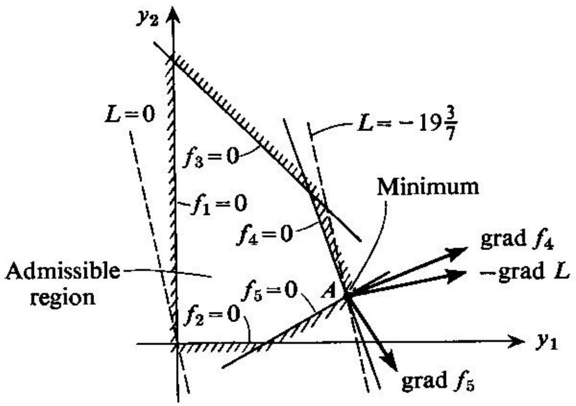


Figure 1.8.1. Solution of the linear programming problem of Example 1.

Obviously, the minimum occurs at point A, where we have $3y_1 + y_2 - 12 = 0$ and $y_1 - 2y_2 - 2 = 0 \Rightarrow y_1 = 3\frac{5}{7}$, $y_2 = \frac{6}{7} \Rightarrow L_{\min} = -19\frac{3}{7}$, and grad L can be expressed as a negative linear combination of n (but not $n - 1$) rows of A^T (namely, grad f_4 and grad f_5), as is obvious from Figure 1.8.1.

The implication of the fundamental theorem of linear programming for the *numerical solution of linear programming problems* is clear. Take n constraints at a time and treat them as equalities. Solving the equalities yields one solution (assuming admissibility), which is

either optimal or nonoptimal. In the latter case, we can discard one of the constraints and, substituting another, repeat the process with the requirement that the new solution be admissible and better. Since there are only a finite number of such possibilities, this process must eventually arrive at the optimal combination (if it exists). The method that accomplishes this is known as the ***simplex*** method. We shall say more about this in the next section.

Problem. Show that the necessary conditions for a maximum of

$$L = c^T \lambda\,,$$

subject to

$$A\lambda + b = 0\,, \qquad \lambda \geq 0\,,$$

are simply (1.8.2) of the minimization problem discussed in this section. These two problems are called ***duals*** of each other.

Example 2. ***A blending problem.*** There are many ***blending problems*** that involve finding the cheapest mixture of several materials that contains at least a certain fraction of specified ingredients. A typical problem is to find the cheapest mixture of several feeds that contains at least certain specified amounts of nutrients (proteins, fats, vitamins, etc.).† Suppose that we are considering a mixture of three feeds and we have three inequality specifications on nutrients. Table 1.8.1, below, shows the fraction of each of the three nutrients contained in each of the three feeds and the cost per unit amount of each of the three feeds.

Table 1.8.1

	Fraction of nutrient in each feed			
Feed	1	2	3	Cost
1	.06	.02	.09	15
2	.03	.04	.05	12
3	.04	.01	.03	8

Our problem is to find the cheapest mixture of the three feeds such that the fraction of nutrients one, two, three in the mixture is greater than or equal to .04, .02, and .07, respectively.

Let F_j = the fraction of feed j in the mixture, where $j = 1,2,$ or 3; these are the quantities we are trying to find (our design parameters).

†Other blending problems occur in mixing fuel oils and fertilizers.

Let N_i = fraction of nutrient i in the mixture, where i = 1,2, or 3; then we have

$$N_i = n_{i1}F_1 + n_{i2}F_2 + n_{i3}F_3 ,$$

where n_{ij} = fraction of nutrient i in feed j (the data given in Table 1.8.1). For this problem we have $N_1 \geq .04$, $N_2 \geq .02$, $N_3 \geq .07$.

Let C = cost per unit amount of the mixture and c_j = cost per unit amount of feed j (also given in Table 1.8.1). Then we have

$$C = c_1F_1 + c_2F_2 + c_3F_3 .$$

Of course, the fractions of the three feeds in the mixture must add up to one:

$$F_1 + F_2 + F_3 = 1 .$$

The problem, then, is to find the *two* quantities F_1 and F_2 (using $F_3 = 1 - F_1 - F_2$) to minimize C and satisfy the inequalities.†

$$N_i \geq \overline{N}_i , \qquad i = 1,2,3, \qquad 0 \leq F_j \leq 1 , \qquad j = 1,2$$

where $\overline{N}_i$ is the minimum allowable fraction of nutrient i in the mixture.

We draw a graph, using F_1 and F_2 as coordinates (see Figure 1.8.2). The inequalities are shown as lines with arrows perpendicular to them pointing in the "allowable" directions. In this problem, the inequalities are

(a) $N_1 = .06F_1 + .03F_2 + .04(1-F_1-F_2) \geq .04$ or $2F_1 - F_2 \geq 0$;
(b) $N_2 = .02F_1 + .04F_2 + .01(1-F_1-F_2) \geq .02$ or $F_1 + 3F_2 \geq 1$;
(c) $N_3 = .09F_1 + .05F_2 + .03(1-F_1-F_2) \geq .07$ or $3F_1 + F_2 \geq 2$;
(d) $F_3 = 1 - F_1 - F_2 \geq 0$ or $F_1 + F_2 \leq 1$;
(e) $0 \leq F_1 \leq 1$;
(f) $0 \leq F_2 \leq 1$.

Note that the inequality $F_3 = 1 - F_1 - F_2 \leq 1$ or $F_1 + F_2 \geq 0$ is redundant since we have $F_1 \geq 0$ and $F_2 \geq 0$. Which of the other inequalities are redundant? (See the graph, Figure 1.8.2.) The *feasible region* is the region in the graph where all of the inequalities are satisfied; it is clearly marked in Figure 1.8.2 and is surrounded by extra-heavy lines.

The lines of constant cost are given by setting C = different constants in

$$C = 15F_1 + 12F_2 + 8(1-F_1-F_2) \qquad \text{or} \qquad C = 8 + 7F_1 + 4F_2 .$$

†Several of these turn out to be redundant; i.e., other inequalities automatically cause them to be satisfied.

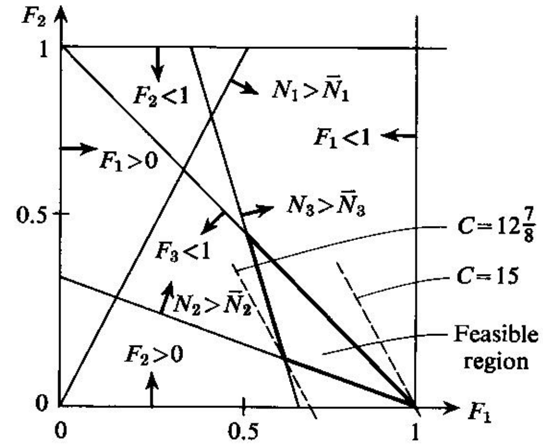


Figure 1.8.2. Solution of the linear programming problem of Example 2.

From Figure 1.8.2 it is clear that the cheapest feasible solution is one that occurs at the corner where $N_2 = \overline{N}_2$ and $N_3 = \overline{N}_3$. From above, this requires that

$$F_1 + 3F_2 = 1 , \qquad 3F_1 + F_2 = 2 .$$

These are two linear equations in two unknowns (F_1 and F_2); they are easily solved to yield

$$F_1 = \tfrac{5}{8} \qquad \text{and} \qquad F_2 = \tfrac{1}{8} .$$

The amount of feed number three is obtained by substituting the results above into $F_3 = 1 - F_1 - F_2$. This yields

$$F_3 = \tfrac{1}{4} .$$

The minimum cost is given by

$$C = 15(\tfrac{5}{8}) + 12(\tfrac{1}{8}) + 8(\tfrac{1}{4}) = 12\tfrac{7}{8} \quad \text{per unit amount of mixture.}$$

The amount of nutrient one is *above* the minimum required fraction:

$$N_1 = .06(\tfrac{5}{8}) + .03(\tfrac{1}{8}) + .04(\tfrac{1}{4}) = .05125 > .04 .$$

Notice, also, that the *most expensive feasible solution* is to use feed number one all by itself.

Example 3. *A transportation planning problem.* A grain dealer owns 50,000 bushels of wheat in Grand Forks, North Dakota, and 40,000 bushels in Chicago. He has sold 20,000 bushels to a customer in Denver, 36,000 bushels to a customer in Miami, and the remaining

34,000 bushels to a customer in New York. He wishes to determine the minimum-cost shipping schedule, given the following freight rates in cents per bushel:

Table 1.8.2

	Denver	Miami	New York
Grand Forks	42	55	60
Chicago	36	47	51

Different modes of shipment cause the rates *not* to be proportional to the distance between the cities. For convenience, we can combine our data into a table, leaving space for the answer, as follows:

Table 1.8.3

Destination / Origin	Denver	Miami	New York	
Grand Forks	42	55	60	50,000
Chicago	36	47	51	40,000
	20,000	36,000	34,000	

The figure in the upper right-hand corner of each square is the freight rate between the two cities.

Our problem is to find a *nonnegative amount* in each of the six squares so that (a) the amounts in the first row add up to 50,000 and the amounts in the second row add up to 40,000; (b) the amounts in the first, second, and third columns add up to 20,000, 36,000, and 34,000, respectively; (c) the total freight cost is a minimum; this cost is obtained by multiplying the amount in each square by the rate in the upper right-hand corner and adding these numbers together.

This problem is a little bit like a crossword puzzle, only harder, since it is not sufficient just to get the rows and columns to add up properly (a feasible solution); we must, in addition, minimize the total cost. By "cut and try" we might be able to find the solution. However, a systematic approach is apt to take less time, and, for problems with more shipping points and more destinations, a systematic approach (an algorithm) and a computer are essential.

Suppose we designate the amount from Grand Forks to Denver as "x" in thousands of bushels. Then, clearly, the amount from Chicago to Denver must be $20 - x$ (see Table 1.8.4). Similarly, let us designate the amount shipped from Grand Forks to Miami as "y"; then the amount from Chicago to Miami must be $36 - y$. Now the amount from Grand Forks to New York must be $50 - x - y$, and the amount from Chicago to New York must be $40 - (20-x) - (36-y) = x + y - 16$. (We automatically satisfy the requirement that the total shipped to New York be 34,000 since the total amount sold equals the amount owned.)

Table 1.8.4

42 x	55 y	60 $50 - x - y$	50
36 $20 - x$	47 $36 - y$	51 $x + y - 16$	40
20	36	34	

We have reduced the number of unknowns to two, x and y, which must satisfy six inequalities:

$$x \geq 0, \quad y \geq 0, \quad 50 - x - y \geq 0, \quad 20 - x \geq 0,$$
$$36 - y \geq 0, \quad x + y - 16 \geq 0.$$

We can conveniently plot all these inequalities on an (x versus y)-graph as in Figure 1.8.3. Again, as in the previous section, there is a *feasible region* where all the inequalities are satisfied.

Next, we calculate the cost in terms of x and y:

$$C = 1000/100 \times [42x + 55y + 60(50-x-y) + 36(20-x) + 47(36-y) + 51(x+y-16)],$$

or

$$C = 45{,}960 - 30x - 10y \quad \text{(in dollars)}.$$

Lines of constant cost are shown in Figure 1.8.3 as dashed lines; clearly, the feasible solution with minimum cost is at $x = 20$, $y = 30$ in Figure 1.8.3. This minimum cost solution is shown in Table 1.8.5.

Note that *no* wheat should be shipped from Chicago to Denver, even though such shipment would involve the lowest rate per bushel.

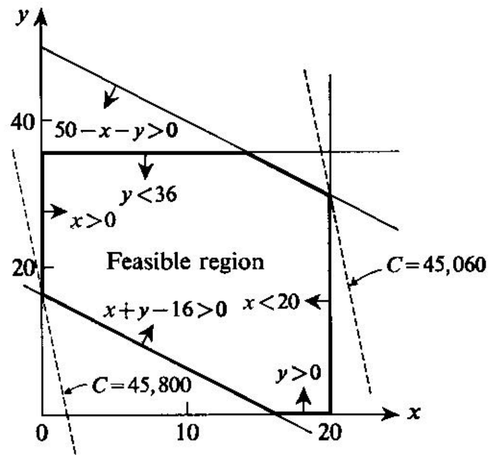


Figure 1.8.3. Solution to minimum cost shipping problem.

Table 1.8.5

42 20	55 30	60 0	50
36 0	47 6	51 34	40
20	36	34	

The difference between the best and the worst feasible solutions is only \$740 out of about \$45,000. However, this 1.6% difference could be a substantial per cent of the profit involved in the sales.

1.9 Numerical solution of problems with inequality constraints

Numerical solution of optimization problems with inequality constraints is one of the major concerns of the field of "mathematical programming." Numerous texts exist on the subject (e.g., Zoutendijk), and we shall describe only the main features of the "method of feasible directions" or "gradient projection method." This method is divided into two separate but related steps:

STEP 1. *Finding a feasible solution.* With reference to Section 1.7, locating a value of y such that $f(y) \leq 0$ is often *not* a trivial task. In problems with equality constraints, such as those treated in Sections 1.5 and 1.6, finding a feasible solution is usually straightforward since there are more variables (x and u) than there are constraint equations

($f(x,u) = 0$). In problems with inequality constraints, there are often more constraint equations (components of f) than variables (components of y). Finding a feasible solution may be approached by guessing a value for y, then considering a small perturbation, dy, which will change f according to

$$df = \frac{\partial f}{\partial y} dy\,. \tag{1.9.1}$$

If certain components of $f(y)$ are greater than zero, i.e., not feasible, we require a dy such that the corresponding components of df are less than zero. In other words, $f(y + dy)$ should be an improvement toward a feasible solution, that is,

$$F\,dy \leq 0\,, \tag{1.9.2}$$

where F contains only rows of $\partial f/\partial y$ corresponding to infeasible values of f. The problem is thus reduced to finding feasible solutions to successive linear inequalities (instead of nonlinear inequalities).

STEP 2. *Finding a feasible improvement.* If a feasible y can be found, the next step is to find a dy which is not only feasible but which also improves the performance index; that is, we must have $f(y + dy) \leq 0$ and $L(y + dy) < L(y)$. This gives rise to another set of linear inequalities like (1.9.2) above:

$$\begin{bmatrix} \dfrac{\partial L}{\partial y} \\ \hline \dfrac{\partial f}{\partial y} \end{bmatrix} dy \equiv H\,dy \leq 0\,. \tag{1.9.3}$$

Example. *Quadratic performance index with linear inequality constraints.* Minimize

$$L = \frac{(y_1 - 2)^2}{4} + (y_2 - 1)^2$$

subject to

$$3y_1 + 2y_2 - 6 \leq 0\,, \qquad y_1 > 0\,, \qquad y_2 > 0\,.$$

A sketch of the admissible region with contours of constant L is shown in Figure 1.9.1. If we guess $y_1 = y_2 = \frac{1}{2}$ to start with, we find that

$$\frac{\partial L}{\partial y_1} = -\frac{3}{4}\,, \frac{\partial L}{\partial y_2} = -1\,.$$

Since we are minimizing, the greatest improvement will be in the direction of the *negative gradient* that is shown at $A(y_1 = y_2 = \frac{1}{2})$. We

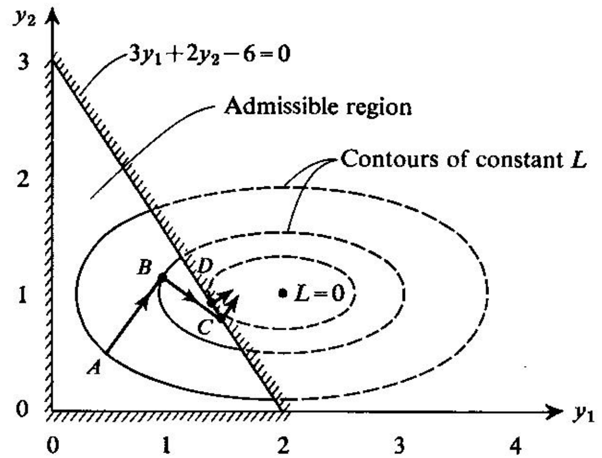


Figure 1.9.1. Solution of the quadratic programming problem of preceding example.

proceed in that direction until we reach a minimum (which may be on a boundary). In this case we reach a minimum at point B in the *interior* of the admissible region. Following the direction of the negative gradient at B, we reach point C on the constraint $3y_1 + 2y_2 - 6 = 0$. Here the negative gradient points *out of* the admissible region, so we take the component of the negative gradient *along the constraint boundary*, which is *up* in this case. Moving along the constraint, we finally arrive at the minimum at D, where the negative gradient points out of the admissible region and is perpendicular to the boundary.

As discussed in Section 1.8, *linear programming problems* have gained a great importance in recent years. Thus, it seems worthwhile to discuss briefly the special procedures applicable to them. Consider again the problem of minimizing

$$L = b^T y \tag{1.9.4}$$

subject to

$$A^T y + c \leq 0\,, \tag{1.9.5}$$

where y is an n-vector, A^T is an $(m \times n)$-matrix, $m > n$. We know that the minimum must occur at a point intersection of n hyperplanes† whose normals are in the direction of rows of A^T. We start the solution, then, by selecting n equations out of (1.9.5) and solving them

†There are abnormal situations in which the minimum lies on an "edge" rather than at a "point"; see conditions specified in Section 1.8 for a "point" solution.

(set *equal* to zero). If this point is feasible, we then examine the n "edges" leading away from this point (formed by the intersection of the n sets of $n-1$ hyperplanes we have chosen); these "edges" will have directions, away from the point, of $e^1, e^2, \ldots, e^n$, where e^i is a *unit* n-vector along the ith edge. The gradient of L is simply b^T, so we consider the projection of the edge directions on b^T; that is, we consider the scalar products $b^T e^i$, $i = 1, \ldots, n$. If all the scalar products are positive, it is not possible to move along any edge to obtain an improvement (i.e., a smaller value of L); we have the optimal solution. On the other hand, if some of the scalar products are negative, let us choose the one with the largest magnitude and move along the corresponding edge until we encounter another constraint. This new constraint and the $n-1$ old constraints that form the "edge" determine a new point at which the value of L is necessarily smaller since we moved along a *projected gradient* $b^T e^i < 0$. The process is then repeated over and over until a point is found at which we have all $b^T e^i > 0$; that is, no improvement is possible. This is the basis of the *simplex algorithm* proposed by Dantzig (1963), which uses, essentially, a method of feasible directions.

Problem. Perform one step of the above-described process for Example 2 of Section 1.8. Why are we allowed to move as far as the next constraint boundary during each step of the simplex method?

1.10 The penalty function method

Another method for the handling of equality constraints as well as inequality constraints is the so-called penalty function method. The idea is quite simple. Suppose we wish to minimize $L(y)$ subject to

$$f(y) = 0\,. \tag{1.10.1}$$

Instead of solving the desired problem directly, we consider the minimization of

$$\overline{L} = L(y) + K\|f(y)\|^2 \tag{1.10.2}$$

subject to *no* constraints, where K is very large.† If $\overline{L}$ attains a minimum at y^*, it is reasonable to expect that

$$f(y^*) \approx 0 \tag{1.10.3}$$

and

$$L(y^*) \approx L(y^o)\,, \tag{1.10.4}$$

†Other functions of $f(y)$ that are zero when $f(y) = 0$ and positive when $f(y) \neq 0$ are, of course, possible.

where y^o is the value of y that minimizes L subject to $f(y) = 0$. In fact, it is possible to show that, in some cases,

$$\lim_{K\to\infty} y^* \to y^o\,,\ \lim_{K\to\infty} L(y^*) \to L(y^o)\,. \tag{1.10.5}$$

Computationally, the penalty function method is appealing and has been used both in parameter optimization problems and in function optimization problems (see Chapters 2, 3, and 4).

Nevertheless, it is important to note that, *in practice*, the penalty function method occasionally does *not* come very close to the proper limit indicated in (1.10.5). One reason for this is as follows: The augmented performance index (1.10.2), with large K, has a long narrow "valley" containing the point y^* at the "bottom" (see Figure 1.10.1). Gradient procedures for finding this point tend to go back and forth, from one side of the narrow valley to the other side, instead of down the "long" direction of the valley. Even worse, if K is very large, the "width" of the valley becomes comparable to the numerical accuracy of the computation, and the gradient procedure breaks down completely.

Another potential source of difficulty is the creation of artificial minima that are not present in the original problem.

Example. Find y_1 and y_2 to minimize

$$L = (y_1 - 2)^2 + y_2^2$$

subject to $y_1 = 0$. Now, this is a trivial problem with the obvious answer $y_1 = y_2 = 0$. However, if we use the penalty function approach, we minimize

$$\overline{L} = (y_1 - 2)^2 + y_2^2 + K y_1^2 = y_2^2 + \left[\left(y_1 - \frac{2}{1+K}\right)^2 \Big/ \frac{1}{1+K}\right] + \frac{4K}{1+K}\,.$$

Contours of constant $\overline{L}$ are ellipses with centers at $y_1 = (2/1 + K)$, $y_2 = 0$ and semi-axes in a ratio of $\sqrt{1+K}$. Figure 1.10.1 shows contours of constant $\overline{L}$ for $K = 35$. Note the long "valley" created by the penalty function.

Inequality constraints can also be handled by the penalty function method. Suppose that, instead of (1.10.1), we have

$$f(y) \le 0\,. \tag{1.10.6}$$

Then we may consider minimizing

$$\overline{L} = L(y) + K[f(y)]^2\,\mathbf{1}[f(y)]\,, \tag{1.10.7}$$

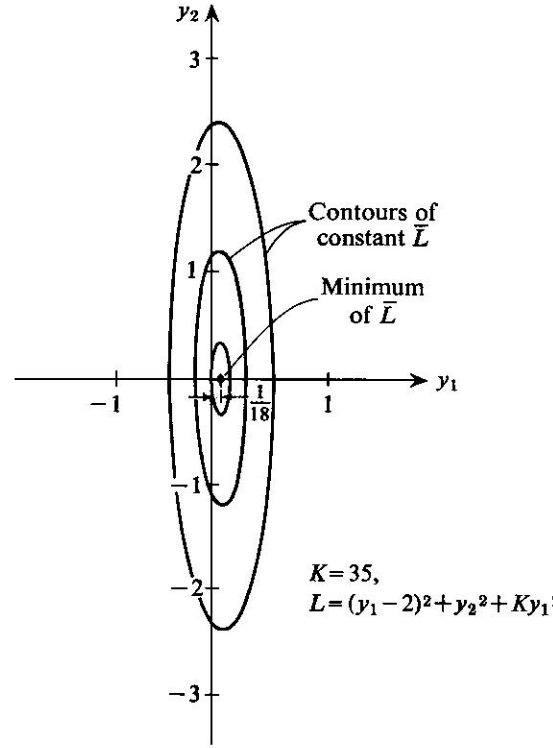


Figure 1.10.1 Cost contours created by penalty functions in the preceding example.

where $\mathbf{1}(f)$ is the unit step function defined as

$$\mathbf{1}(f) = \begin{cases} 1, & f > 0\,, \\ 0, & f < 0\,. \end{cases} \tag{1.10.8}$$

The use of penalty functions is often quite helpful during the initial stages of numerical computation on problems with complex constraints.

Optimization problems for dynamic systems

2

2.1 Single-stage systems

As an introduction to multistage systems, let us consider the simplest nontrivial multistage system, namely, the single-stage system.

A system is initially in a known state described by $x(0)$, an n-dimensional state vector. Choice of an m-dimensional control vector $u(0)$ determines a transition to a state described by $x(1)$ through the relation

$$x(1) = f^o[x(0), u(0)] , \qquad (2.1.1)$$

which is shown schematically in Figure 2.1.1.

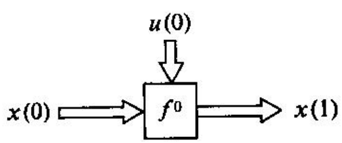


Figure 2.1.1. Flow chart for a single-stage system.

We wish to choose $u(0)$ to minimize a performance index of the form

$$J = \phi[x(1)] + L^o[x(0), u(0)] . \qquad (2.1.2)$$

This is a parameter optimization problem with equality constraints, exactly like the ones considered in Section 1.2. We shall treat it in the same way, differing only slightly in our choice of an H-function; we adjoin the constraints (2.1.1) to (2.1.2) with undetermined multipliers $\lambda(1)$:

$$\bar{J} = \phi[x(1)] + L^o[x(0), u(0)] + \lambda^T(1)\{f^o[x(0), u(0)] - x(1)\} . \qquad (2.1.3)$$

Now, let

$$H^o = L^o[x(0), u(0)] + \lambda^T(1) f^o[x(0), u(0)] , \qquad (2.1.4)$$

so that (2.1.3) may be written as

$$\bar{J} = \phi[x(1)] + H^o[x(0), u(0), \lambda(1)] - \lambda^T(1)\, x(1) . \qquad (2.1.5)$$

Next, consider infinitesimal changes in $\bar{J}$ due to infinitesimal changes in $u(0)$, $x(1)$, and $x(0)$:

$$d\bar{J} = \left[\frac{\partial \phi}{\partial x(1)} - \lambda^T(1)\right] dx(1) + \frac{\partial H^o}{\partial u(0)} du(0) + \frac{\partial H^o}{\partial x(0)} dx(0) . \qquad (2.1.6)$$

An expedient choice of $\lambda(1)$ is apparent from (2.1.6); to avoid determining $dx(1)$ in terms of $du(0)$ by differentiating (2.1.1), we *choose*

$$\lambda^T(1) = \frac{\partial \phi}{\partial x(1)} . \qquad (2.1.7)$$

As a result of this choice, (2.1.6) becomes

$$d\bar{J} = \frac{\partial H^o}{\partial u(0)} du(0) + \frac{\partial H^o}{\partial x(0)} dx(0) . \qquad (2.1.8)$$

Thus $\partial H^o/\partial u(0)$ is the gradient of J with respect to $u(0)$, holding $x(0)$ constant and satisfying (2.1.1), and $\partial H^o/\partial x(0)$ is the gradient of J with respect to $x(0)$, holding $u(0)$ constant and satisfying (2.1.1). If $x(0)$ is given, we have $dx(0) = 0$.

Clearly, a stationary value of $\bar{J}$ and, hence, J, for given values of $x(0)$, will be obtained if

$$\frac{\partial H^o}{\partial u(0)} = 0 . \qquad (2.1.9)$$

Note that (2.1.1), (2.1.7), and (2.1.9) constitute $n + n + m$ equations for determining the $n + n + m$ quantities $x(1)$, $\lambda(1)$, and $u(0)$.

2.2 Multistage systems; no terminal constraints, fixed number of stages

Optimal programming problems for multistage systems are also parameter optimization problems. Consider the multistage system described by the nonlinear difference equations:

$$x(i+1) = f^i[x(i), u(i)] ; \qquad x(0) \text{ given}, \qquad i = 0, \ldots, N-1 , \qquad (2.2.1)$$

which is nothing but a sequential set of equality constraints, where $x(i)$, a sequence of n-vectors, is determined by $u(i)$, a sequence of m-vectors. This is shown schematically in Figure 2.2.1.

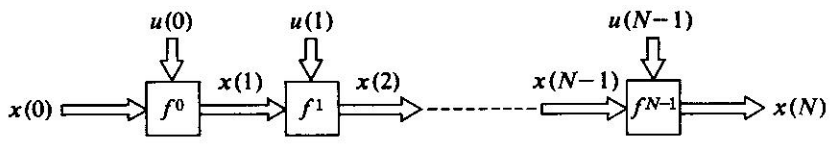


Figure 2.2.1. Flow chart for a multistage system.

Consider a performance index of the form

$$J = \phi[x(N)] + \sum_{i=0}^{N-1} L^i[x(i), u(i)]. \tag{2.2.2}$$

The problem is to find the sequence $u(i)$ that minimizes (or maximizes) J. Adjoin the system equations (2.2.1) to J with a multiplier sequence $\lambda(i)$:

$$\bar{J} = \phi[x(N)] + \sum_{i=0}^{N-1} [L^i[x(i), u(i)] + \lambda^T(i+1)\{f^i[x(i), u(i)] - x(i+1)\}]. \tag{2.2.3}$$

For convenience, define a scalar sequence H^i:

$$H^i = L^i[x(i), u(i)] + \lambda^T(i+1) f^i[x(i), u(i)], \qquad i=0, \ldots, N-1. \tag{2.2.4}$$

Also, change indices of summation on the last term in (2.2.3), obtaining

$$\bar{J} = \phi[x(N)] - \lambda^T(N)\, x(N) + \sum_{i=1}^{N-1} [H^i - \lambda^T(i)\, x(i)] + H^o. \tag{2.2.5}$$

Now consider differential changes in $\bar{J}$ due to differential changes in $u(i)$:

$$d\bar{J} = \left[\frac{\partial \phi}{\partial x(N)} - \lambda^T(N)\right] dx(N) + \sum_{i=1}^{N-1} \left\{\left[\frac{\partial H^i}{\partial x(i)} - \lambda^T(i)\right] dx(i) + \frac{\partial H^i}{\partial u(i)} du(i)\right\}$$
$$+ \frac{\partial H^o}{\partial x(0)} dx(0) + \frac{\partial H^o}{\partial u(0)} du(0). \tag{2.2.6}$$

It would be tedious to determine the differential changes $dx(i)$ produced by a given $du(i)$ sequence, so we *choose* the multiplier sequence $\lambda(i)$ so that we have

$$\lambda^T(i) - \frac{\partial H^i}{\partial x(i)} = 0 \Rightarrow \lambda^T(i) = \frac{\partial L^i}{\partial x(i)} + \lambda^T(i+1)\frac{\partial f^i}{\partial x(i)}; \qquad i = 0, \ldots, N-1, \tag{2.2.7}$$

with boundary conditions

$$\lambda^T(N) = \frac{\partial \phi}{\partial x(N)}. \tag{2.2.8}$$

Equation (2.2.6) then becomes

$$d\bar{J} = \sum_{i=0}^{N-1} \frac{\partial H^i}{\partial u(i)} du(i) + \lambda^T(0)\, dx(0). \tag{2.2.9}$$

Thus, $\partial H^i/\partial u(i)$ is the ***gradient*** of J with respect to $u(i)$ while holding $x(0)$ constant and satisfying (2.2.1), and $\lambda^T(0) = \partial H^o/\partial x(0)$ is the gradient of J with respect to $x(0)$ while holding $u(i)$ constant and satisfying (2.2.1). If $x(0)$ is given, we have $dx(0) = 0$.

For an extremum, $d\bar{J}$ must be zero for arbitrary $du(i)$; this can only happen if we have

$$\frac{\partial H^i}{\partial u(i)} = 0, \qquad i = 0, \ldots, N-1. \tag{2.2.10}$$

In summary, to find a control vector sequence $u(i)$ that produces a stationary value of the performance index J, we must solve the following difference equations:

$$x(i+1) = f^i[x(i), u(i)], \tag{2.2.11}$$

$$\lambda(i) = \left[\frac{\partial f^i}{\partial x(i)}\right]^T \lambda(i+1) + \left[\frac{\partial L^i}{\partial x(i)}\right]^T, \tag{2.2.12}$$

where $u(i)$ is determined by seeking a stationary point of H^i, which requires that we have

$$\frac{\partial H^i}{\partial u(i)} = \frac{\partial L^i}{\partial u(i)} + \lambda^T(i+1)\frac{\partial f^i}{\partial u(i)} = 0, \qquad i = 0, \ldots, N-1. \tag{2.2.13}$$

The boundary conditions for (2.2.11) and (2.2.12) are *split*; i.e., some are given for $i = 0$, and some are given for $i = N$

$$x(0) \quad \text{given} \tag{2.2.14}$$

$$\lambda(N) = \left[\frac{\partial \varphi}{\partial x(N)}\right]^T. \tag{2.2.15}$$

Such problems are called ***two-point boundary-value problems***, and they are sometimes rather difficult to solve, even with a high-speed computer. Notice that the difference equations (2.2.11) and (2.2.12) are *coupled* since $u(i)$ depends on $\lambda(i)$ through (2.2.13), and the coefficients of (2.2.12) depend, in general, on $x(i)$ and $u(i)$.

In order that J be a local minimum, not only must we have $\partial H^i/\partial u(i) = 0$, but, in addition, the second-order expression for $d\bar{J}$

with the constraint (2.2.1) must be nonnegative for all (infinitesimal) values of $du(i)$; that is, we must have $dJ \geq 0$, where (from (2.2.3))

$$dJ = \frac{1}{2} dx^T(N) \frac{\partial^2 \phi}{\partial x(N)\, \partial x(N)} dx(N) + \frac{1}{2} \sum_{i=0}^{N-1} [dx^T(i), du^T(i)] \begin{bmatrix} \dfrac{\partial^2 H^i}{\partial x(i)\, \partial x(i)}, & \dfrac{\partial^2 H^i}{\partial x(i)\, \partial u(i)} \\ \dfrac{\partial^2 H^i}{\partial u(i)\, \partial x(i)}, & \dfrac{\partial^2 H^i}{\partial u(i)\, \partial u(i)} \end{bmatrix} \begin{bmatrix} dx(i) \\ du(i) \end{bmatrix}. \tag{2.2.16}$$

The values of $dx(i)$ are determined by the sequence $du(i)$ from the differential of (2.2.1).

$$dx(i+1) = \frac{\partial f}{\partial x(i)} dx(i) + \frac{\partial f}{\partial u(i)} du(i), \qquad dx(0) = 0. \tag{2.2.17}$$

Methods for checking this criterion are given in Chapter 6.

Example. *Quadratic performance index with linear system equations. Find* the control vector sequence $u(i)$, $i = 0, \ldots, N-1$ that minimizes the quadratic form

$$J = \tfrac{1}{2} x^T(N) A(N) x(N) + \sum_{i=0}^{N-1} [\tfrac{1}{2} x^T(i) A(i) x(i) + \tfrac{1}{2} u^T(i) B(i) u(i)], \tag{2.2.18}$$

where $A(i)$ and $B(i)$ are given positive definite matrices, with the linear system equations

$$x(i+1) = \Phi(i) x(i) + \Gamma(i) u(i), \qquad x(0) \text{ given}. \tag{2.2.19}$$

SOLUTION. The H^i sequence for the problem is

$$H^i = \tfrac{1}{2} x^T(i) A(i) x(i) + \tfrac{1}{2} u^T(i) B(i) u(i) + \lambda^T(i+1) [\Phi(i) x(i) + \Gamma(i) u(i)], \tag{2.2.20}$$

where

$$\lambda^T(i) = \lambda^T(i+1) \Phi(i) + x^T(i) A(i) \quad \text{and} \quad \lambda^T(N) = x^T(N) A(N). \tag{2.2.21}$$

A stationary value of H^i with respect to $u(i)$ will occur where we have

$$\frac{\partial H^i}{\partial u(i)} = u^T(i) B(i) + \lambda^T(i+1) \Gamma(i) = 0, \tag{2.2.22}$$

$$\Rightarrow u(i) = -[B(i)]^{-1} \Gamma^T(i) \lambda(i+1). \tag{2.2.23}$$

Hence, we obtain

$$x(i+1) = \Phi(i) x(i) - \Gamma(i) [B(i)]^{-1} \Gamma^T(i) \lambda(i+1); \tag{2.2.24}$$

$$\lambda(i) = \Phi^T(i) \lambda(i+1) + A(i) x(i); \qquad i = 0, \ldots, N-1, \tag{2.2.25}$$

with boundary conditions $\lambda(N) = A(N) x(N)$ and $x(0)$ given. These are coupled sets of *linear* difference equations with two-point boundary values; the solution to this boundary-value problem yields the minimizing sequence $u(i)$ from (2.2.23).

Problem 1. Show that the two-point boundary-value problem of the Example can be solved by placing

$$\lambda(i) = S(i) x(i)$$

and determining $S(i)$ from the backward recursive relations:

$$\left.\begin{aligned} S(i) &= \Phi^T(i) M(i+1) \Phi(i) + A(i), \\ M(i+1) &= [S^{-1}(i+1) + \Gamma(i) B^{-1}(i) \Gamma^T(i)]^{-1}; \end{aligned}\right\} \quad i = N-1, \ldots, 0;$$

or

$$M(i+1) = S(i+1) - S(i+1) \Gamma(i) [B(i) + \Gamma^T(i) S(i+1) \Gamma(i)]^{-1} \Gamma^T(i) S(i+1),$$

where

$$S(N) = A(N).$$

Having determined $S(i)$, $i = N-1, \ldots, 0$, from the relations above, we obtain

$$x(i+1) = [I + \Gamma(i) B^{-1}(i) \Gamma^T(i) S(i+1)]^{-1} \Phi(i) x(i), \qquad x(0) \text{ specified.}$$

This is known as the *sweep method* for solving a linear two-point boundary-value problem. (For more on this method see Sections 6.10 and 6.11.)

Problem 2. Consider the problem of this section as a parameter optimization problem of Section 1.2 where x denotes the vector with component vectors $x(1), \ldots, x(N)$, u the vector with component vectors $u(0), \ldots, u(N-1)$, and f the vector with component vectors $x(1) - f^0, x(2) - f^1, \ldots, x(N) - f^{N-1}$. Show that the necessary conditions of Section 1.2 reduces to that of Equations (2.2.11)-(2.2.15).

Continuous systems; no terminal constraints, fixed terminal time

Optimal programming problems for continuous systems are problems in the *calculus of variations*. They may be considered as limiting cases of optimal programming problems for multistage systems in which the time increment between steps becomes small compared to times of interest. Actually, the reverse procedure is more common today; continuous systems are approximated by multistage systems for solution on digital computers. Consider the system described by the following nonlinear differential equations:

$$\dot{x} = f[x(t), u(t), t]; \quad x(t_o) \text{ given}, \quad t_o \le t \le t_f, \tag{2.3.1}$$

where $x(t)$, an n-vector function, is determined by $u(t)$, an m-vector function. Consider a performance index (scalar) of the form

$$J = \varphi[x(t_f), t_f] + \int_{t_o}^{t_f} L[x(t), u(t), t]\, dt. \tag{2.3.2}$$

The problem is to find the functions $u(t)$ that minimize (or maximize) J. Adjoin the system differential equations (2.3.1) to J with multiplier functions $\lambda(t)$:

$$\bar{J} = \varphi[x(t_f), t_f] + \int_{t_o}^{t_f} [L[x(t), u(t), t] + \lambda^T(t)\{f[x(t), u(t), t] - \dot{x}\}]\, dt. \tag{2.3.3}$$

For convenience, define a scalar function H (the *Hamiltonian*), as follows:

$$H[x(t), u(t), \lambda(t), t] = L[x(t), u(t), t] + \lambda^T(t) f[x(t), u(t), t]. \tag{2.3.4}$$

Also, integrate the last term on the right side of (2.3.3) by parts, yielding

$$\begin{aligned}\bar{J} &= \varphi[x(t_f), t_f] - \lambda^T(t_f)x(t_f) + \lambda^T(t_o)\, x(t_o) \\ &+ \int_{t_o}^{t_f} \{H[x(t), u(t), t] + \dot{\lambda}^T(t)\, x(t)\}\, dt.\end{aligned} \tag{2.3.5}$$

Now consider the *variation* in J due to variations in the control vector $u(t)$ for *fixed times t_o and t_f*,

$$\delta\bar{J} = \left[\left(\frac{\partial\varphi}{\partial x} - \lambda^T\right)\delta x\right]_{t=t_f} + [\lambda^T \delta x]_{t=t_o} + \int_{t_o}^{t_f}\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u\right] dt. \tag{2.3.6}$$

It would be tedious to determine the variations $\delta x(t)$ produced by a given $\delta u(t)$, so we *choose* the multiplier functions $\lambda(t)$ to cause the coefficients of δx in (2.3.6) to vanish:

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x}, \tag{2.3.7}$$

with boundary conditions

$$\lambda^T(t_f) = \frac{\partial\varphi}{\partial x(t_f)}. \tag{2.3.8}$$

Equation (2.3.6) then becomes

$$\delta\bar{J} = \lambda^T(t_o)\,\delta x(t_o) + \int_{t_o}^{t_f} \frac{\partial H}{\partial u}\delta u\, dt. \tag{2.3.9}$$

Thus, $\lambda^T(t_o)$ is the gradient of J with respect to variations in the initial conditions, while holding $u(t)$ constant and satisfying (2.3.1). The functions $\lambda(t)$ are also called ***influence functions*** on J of variations in $x(t)$ since t_o is arbitrary. The functions $\partial H/\partial u$ are called ***impulse response functions*** since each component of $\partial H/\partial u$ represents the variation in J due to a unit impulse (Dirac function) in the corresponding component of δu at time t, while holding $x(t_o)$ constant and satisfying (2.3.1).

For an extremum, δJ must be zero for arbitrary $\delta u(t)$; this can only happen if

$$\frac{\partial H}{\partial u} = 0, \quad t_o \le t \le t_f. \tag{2.3.10}$$

Equations (2.3.7), (2.3.8), and (2.3.10) are known as the *Euler-Lagrange equations* in the calculus of variations.

In summary, to find a control vector function $u(t)$ that produces a stationary value of the performance index J, we must solve the following differential equations:

$$\dot{x} = f(x,u,t), \tag{2.3.11}$$

$$\dot{\lambda} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda - \left(\frac{\partial L}{\partial x}\right)^T, \tag{2.3.12}$$

where $u(t)$ is determined by

$$\frac{\partial H}{\partial u} = 0 \quad \text{or} \quad \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial L}{\partial u}\right)^T = 0. \tag{2.3.13}$$

The boundary conditions for (2.3.11) and (2.3.12) are split; i.e., some are given for $t = t_o$, and some are given for $t = t_f$:

$$x(t_o) \text{ given}, \tag{2.3.14}$$

$$\lambda(t_f) = \left(\frac{\partial\varphi}{\partial x}\right)^T. \tag{2.3.15}$$

Thus, as in the multistage system optimal programming problems, we are faced with a *two-point boundary-value problem.*

A ***first integral*** of the boundary-value problem exists if L and f are ***not*** explicit functions of the time t, since we have

$$\begin{aligned}\dot{H} &= H_t + H_x\dot{x} + H_u\dot{u} + \dot{\lambda}^T f \\ &= H_t + H_u\dot{u} + (H_x + \dot{\lambda}^T)f \\ &= H_t + H_u\dot{u}.\end{aligned}$$

If L and f (hence, H) are not explicit functions of t and $u(t)$ is an optimal program (that is, $\partial H/\partial u = 0$), then we have

$$\dot{H} = 0 \quad \text{or} \quad H = \text{constant on the optimal trajectory}. \tag{2.3.16}$$

In order that J be a *local* minimum, not only must we have $\partial H/\partial u = 0$ but, in addition, the second-order expression for δJ, holding $\dot{x} - f = 0$, must be nonnegative for all values (infinitesimal) of δu; that is, we have

$$\delta J = \frac{1}{2}\left[\delta x^T \frac{\partial^2 \varphi}{\partial x^2}\delta x\right]_{t=t_f} + \frac{1}{2}\int_{t_o}^{t_f} [\delta x^T, \delta u^T] \begin{bmatrix} \dfrac{\partial^2 H}{\partial x^2}, & \dfrac{\partial^2 H}{\partial x\, \partial u} \\ \dfrac{\partial^2 H}{\partial u\, \partial x}, & \dfrac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt \geq 0, \tag{2.3.17}$$

where $\delta(\dot{x} - f) = 0$, or

$$\frac{d}{dt}(\delta x) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u, \qquad \delta x(t_o) = 0. \tag{2.3.18}$$

Equation (2.3.18) determines $\delta x(t)$ in terms of $\delta u(t)$, but in a complicated way. We will have more to say about this *second variation* in Chapter 6.

Example 1. *Hamilton's principle in mechanics.* The motion of a conservative system, from time t_o to t_f is such that the integral

$$I = \int_{t_o}^{t_f} L(u,q)\, dt \tag{2.3.19}$$

has a stationary value, where

$$\begin{aligned} L &= T(u,q) - V(q) = \text{the Lagrangian of the system}, \\ T &= \text{kinetic energy of the system}, \\ V &= \text{potential energy of the system}, \\ q &= \text{generalized coordinate vector (state of system)}, \\ u &= \dot{q} = \text{generalized velocity vector}. \end{aligned} \tag{2.3.20}$$

The Hamiltonian is then

$$H = L + \lambda^T u.\dagger \tag{2.3.21†}$$

Consequently, the Euler-Lagrange equations are

†In mechanics, H is defined as $-L + \lambda^T u$, and the vector λ is usually called p, where p is the generalized momentum vector.

$$\dot{\lambda}^T = -\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}, \tag{2.3.22}$$

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T. \tag{2.3.23}$$

Combining these last two vector equations, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \tag{2.3.24}$$

which are Lagrange's equations of motion for a conservative system.

If L is not an explicit function of time, a first integral of the motion is H = constant:

$$H = L - \frac{\partial L}{\partial u}u = T - V - \frac{\partial T}{\partial u}u = \text{const}. \tag{2.3.25}$$

Now T is a homogeneous quadratic form in u, so that we have

$$\frac{\partial T}{\partial u}u = 2T. \tag{2.3.26}$$

Hence, we have

$$-H = T + V = \text{const}; \tag{2.3.27}$$

that is, the kinetic plus potential energy is constant during the motion.

Example 2. *Variational principle for nonconservative mechanical systems.*† The motion of a nonconservative mechanical system from time t_o to t_f is such that

$$\delta\int_{t_o}^{t_f} T(u,q)\, dt + \int_{t_o}^{t_f} Q^T(q)\, \delta q\, dt = 0, \tag{2.3.28}$$

where

$$\delta\dot{q} = \delta u, \tag{2.3.29}$$

and $Q(q)$ is the generalized force vector. $Q(q)$ is defined by the fact that the *work* done on the system by these generalized forces is given by the (path-dependent) line integral

$$W = \int_{q_o}^{q_f} Q^T(q)\, dq. \tag{2.3.30}$$

The second term in (2.3.28) is the time integral of the *virtual work*; note that it is *not* $\delta\int_{t_o}^{t_f} W dt$, which prevents us from defining a Ham-

†See C. Lanczos, *The Variational Principle of Mechanics*. Toronto, Canada: University of Toronto Press, 1949, Chapter 5.

iltonian for nonconservative systems. However, we can adjoin the constraint (2.3.29) to Equation (2.3.28) with a Lagrange multiplier vector, as follows:

$$\int_{t_0}^{t_f}\left[\frac{\partial T}{\partial u}\delta u+\frac{\partial T}{\partial q}\delta q+Q^T\,\delta q+\lambda^T(\delta u-\delta\dot{q})\right]dt=0\,. \qquad (2.3.31)$$

Integrating the last term by parts, we have

$$\int_{t_0}^{t_f}\left[\left(\frac{\partial T}{\partial u}+\lambda^T\right)\delta u+\left(\frac{\partial T}{\partial q}+Q^T+\dot{\lambda}^T\right)\delta q\right]dt=0\,. \qquad (2.3.32)$$

As usual, we choose $\lambda(t)$ to make the coefficient of δq vanish:

$$\dot{\lambda}^T=-\frac{\partial T}{\partial q}-Q^T\,. \qquad (2.3.33)$$

Since u is arbitrary, the integral can vanish only if we have

$$\lambda^T=-\frac{\partial T}{\partial u}\,. \qquad (2.3.34)$$

Combining (2.3.33) and (2.3.34) to eliminate λ, we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}}\right)-\frac{\partial T}{\partial q}=Q^T\,, \qquad (2.3.35)$$

which are Lagrange's equations of motion for a nonconservative mechanical system.

Example 3. *Minimum-drag nose shape in hypersonic flow.*† The pressure drag of a body of revolution at zero angle of attack in hypersonic flow is given quite accurately by the expression

$$D=-2\pi q\int_{x=0}^{\ell}C_p(\theta)r\,dr\,, \qquad (2.3.36)$$

where

q = dynamic pressure,

x = axial distance from point of maximum radius,

$r = r(x)$ = radius of body,

$$\frac{dr}{dx}=-\tan\theta \text{ (see Figure 2.3.1)} \qquad (2.3.37)$$

$$C_p=\begin{Bmatrix}2\sin^2\theta\,;\,\theta\geq 0\\ 0\quad;\,\theta\leq 0\end{Bmatrix}=\text{pressure coefficient (Newtonian approximation),†} \qquad (2.3.38)$$

†This was the first problem ever solved in the calculus of variations; it was set up and solved in 1686 by Isaac Newton, whose model of aerodynamic forces happens to be very good at hypersonic speeds but *not* very good at subsonic speeds.

ℓ = length of body,

$r(0) = a$ = maximum radius of body,

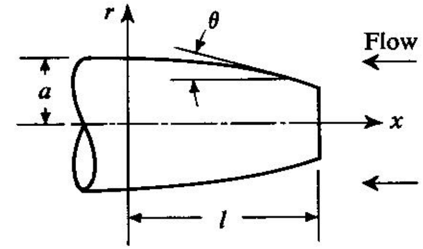


Figure 2.3.1. Nomenclature for analyzing minimum-drag nose shape.

The problem is to find $r(x)$ to minimize D for given values of q, ℓ, and a.

Let

$$-u=-\tan\theta=\frac{dr}{dx} \qquad (2.3.39)$$

be the control variable, and allow for the possibility of a blunt tip by writing (2.3.36) in the form

$$\frac{D}{4\pi q}=\frac{1}{2}[r(\ell)]^2+\int_0^{\ell}\frac{ru^3}{1+u^2}dx\,. \qquad (2.3.40)$$

The Hamiltonian of the system is, therefore,

$$H=\frac{ru^3}{1+u^2}+\lambda(-u)\,. \qquad (2.3.41)$$

The Euler-Lagrange equations are

$$\frac{d\lambda}{dx}=-\frac{\partial H}{\partial r}=-\frac{u^3}{1+u^2}\,, \qquad (2.3.42)$$

$$0=\frac{\partial H}{\partial u}=\frac{ru^2(3+u^2)}{(1+u^2)^2}-\lambda\,. \qquad (2.3.43)$$

Now the first term on the right-hand side of (2.3.40) is a function of $r(\ell)$. According to (2.3.8), the optimal value of $r(\ell)$ is such that

$$\lambda(\ell)=r(\ell)\,. \qquad (2.3.44)$$

Since $r(0)=a$ is specified, $\lambda(0)$ is not specified. Thus, the two boundary conditions for the second-order system of differential equations (2.3.37) and (2.3.42) are (2.3.44) and $r(0)=a$.

Instead of trying to solve (2.3.43) for u in terms of λ, substituting into (2.3.37) and (2.3.42) and integrating, we can take advantage of the fact that the Hamiltonian in (2.3.41) is not an explicit function of x, so that H = constant is an integral of the system. Eliminating λ between (2.3.41) and (2.3.43) gives

$$H = -\frac{2ru^3}{(1+u^2)^2} = \text{const.} \tag{2.3.45}$$

Eliminating $\lambda(\ell)$ between (2.3.43) and (2.3.44) yields

$$r(\ell)\left[1 - \frac{u^2(3+u^2)}{(1+u^2)^2}\right]_{x=\ell} = 0\,, \tag{2.3.46}$$

which is satisfied by $r(\ell) = 0$, or

$$u(\ell) = 1\,. \tag{2.3.47}$$

Using (2.3.47) in (2.3.45) at $x = \ell$, we find that

$$-H = \frac{r(\ell)}{2}\,. \tag{2.3.48}$$

Using (2.3.45) and (2.3.48), we have the radius of the body in terms of the slope u,

$$\frac{r}{r(\ell)} = \frac{(1+u^2)^2}{4u^3}\,. \tag{2.3.49}$$

From (2.3.37) and (2.3.39), we have

$$\frac{dx}{dr} = -\frac{1}{u}$$

or

$$\frac{\ell - x}{r(\ell)} = \int_1^u \frac{1}{u}\frac{d}{du}\frac{(1+u^2)^2}{4u^3}\,du\,. \tag{2.3.50}$$

Equation (2.3.50) can be integrated in terms of simple functions:

$$\frac{\ell - x}{r(\ell)} = \frac{1}{4}\left(\frac{3}{4u^4} + \frac{1}{u^2} - \frac{7}{4} - \log\frac{1}{u}\right). \tag{2.3.51}$$

Equations (2.3.49) and (2.3.51) are parametric equations for the optimum body shape. The tip radius $r(\ell)$ and the slope u_0 at $x = 0$ must be obtained by solution of the transcendental equations

$$\frac{a}{r(\ell)} = \frac{(1+u_0^2)^2}{4u_0^3}\,, \tag{2.3.52}$$

$$\frac{\ell}{r(\ell)} = \frac{1}{4}\left(\frac{3}{4u_0^4} + \frac{1}{u_0^2} - \frac{7}{4} - \log\frac{1}{u_0}\right). \tag{2.3.53}$$

Figure 2.3.2 shows some of these shapes for fixed a and several values of ℓ.

The minimum-drag coefficient is given by

$$C_D = \frac{D}{q\pi a^2} = \frac{u_0^2}{(1+u_0^2)^2}\left(3 + 10u_0^2 + 17u_0^4 + 2u_0^6 + 4u_0^4\log\frac{1}{u_0}\right). \tag{2.3.54}$$

As $a/\ell \to 0$, it is easily shown that

$$\frac{r}{a} \to \left(\frac{\ell - x}{\ell}\right)^{3/4}, \tag{2.3.55}$$

$$C_D \to \frac{27}{16}\left(\frac{a}{\ell}\right)^2, \quad \frac{a}{\ell} \to 0\,. \tag{2.3.56}$$

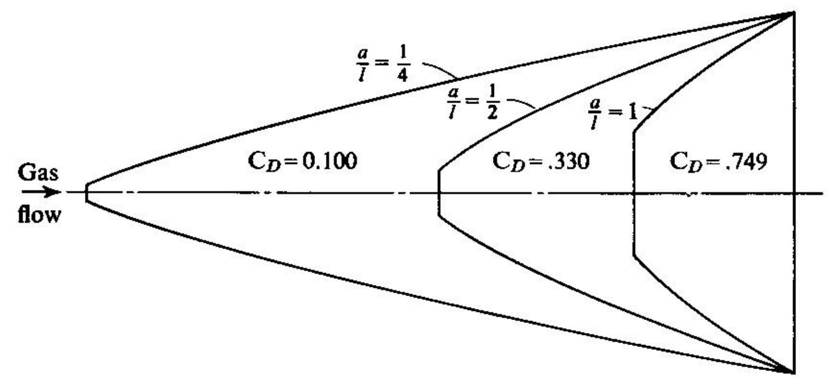


Figure 2.3.2. Minimum-drag bodies of revolution in hypersonic flow for several fineness ratios.

Continuous systems; some state variables specified at a fixed terminal time

Suppose that, in the optimization problem defined in Section 2.3, we wish to constrain *some* of the components of the state vector $x(t)$ to have prescribed values at $t = t_f$. Section 2.3 applies up to and including (2.3.7). Now, if x_i (the ith component of the vector x) is prescribed at $t = t_f$, it follows that admissible variations must produce $\delta x_i(t_f) = 0$ in (2.3.6). Thus, it is *not* necessary that $[(\partial\phi/\partial x_i) - \lambda_i^T]_{t=t_f} = 0$. Essentially, we have traded this latter boundary condition for another, namely, $x_i(t_f)$ given, so that the boundary-value problem (2.3.11)–(2.3.15) still has $2n$ boundary conditions.

Similarly, if x_k is *not* prescribed at $t = t_o$, it does *not* follow that $\delta x_k(t_o) = 0$; in fact, there will be an optimum value for $x_k(t_o)$ and it will be such that $\delta J = 0$ for arbitrary small variations of $x_k(t_o)$ around this value. For this to be the case, we choose

$$\lambda_k(t_o) = 0\,, \tag{2.4.1}$$

which simply says that the influence of small changes in $x_k(t_o)$ on J is zero. Again we have simply traded one boundary condition, $x_k(t_o)$ given, for another, (2.4.1). Boundary conditions like (2.4.1) are sometimes called "natural boundary conditions."

However, the necessary condition (2.3.13), $\partial H/\partial u = 0$, needs additional justification for the problem with terminal constraints. In Section 2.3, we derived it under the assumption that $\delta u(t)$, $t_o < t < t_f$ is arbitrary. In the present case, $\delta u(t)$ is *not* completely arbitrary; the set of admissible $\delta u(t)$ is restricted by the constraints

$$\delta x_i(t_f) = 0\,, \qquad i = 1, \ldots, q\,, \tag{2.4.2}$$

where we define "admissible" $\delta u(t)$, generally, as those $\delta u(t)$ which satisfy all constraints of the problem, for example, (2.4.2).

Now, it is still possible to determine influence functions for the performance index exactly as in Section 2.3. In this section we shall designate these influence functions with a superscript "J." However, since $x_i(t_f)$ for $i = 1, \ldots, q$ are specified, it is consistent to regard

$$\phi = \phi[x_{q+1}, \ldots, x_n]_{t=t_f}\,. \tag{2.4.3}$$

Thus (cf., Equations (2.3.7) through (2.3.9), we have (for $\delta x(t_o) = 0$)

$$\delta J = \int_{t_o}^{t_f} \left[\frac{\partial L}{\partial u} + (\lambda^{(J)})^T \frac{\partial f}{\partial u}\right] \delta u(t)\, dt\,, \tag{2.4.4}$$

where

$$\dot{\lambda}^{(J)} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda^{(J)} - \left(\frac{\partial L}{\partial x}\right)^T \tag{2.4.5}$$

$$\lambda_j^{(J)}(t_f) = \begin{cases} 0\,; & j = 1, \ldots, q \\ \left.\dfrac{\partial \phi}{\partial x_j}\right|_{t=t_f}; & j = q+1, \ldots, n\,. \end{cases} \tag{2.4.6}$$

Suppose that, instead of $J = \phi[x(t_f)] + \int_{t_o}^{t_f} L(x,u,t)\, dt$, the performance index was $J = x_i(t_f)$; i.e., the ith component of the state vector at the final time. We could then determine influence functions for $x_i(t_f)$ by specializing the relations above; we would put $\phi = x_i(t_f)$ and $L(x,u,t) = 0$. We shall designate these influence functions with a superscript "i." Analogous to Equations (2.4.4), (2.4.5), and (2.4.6), we have

$$\delta x_i(t_f) = \int_{t_o}^{t_f} (\lambda^{(i)})^T \frac{\partial f}{\partial u} \delta u(t)\, dt\,, \tag{2.4.7}$$

where

$$\dot{\lambda}^{(i)} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda^{(i)}\,, \tag{2.4.8}$$

$$\lambda_j^{(i)}(t_f) = \begin{cases} 0\,; & i \neq j\,, \\ 1\,; & i = j\,, \quad j = 1, \ldots, n \end{cases} \tag{2.4.9}$$

We could, in fact, determine q sets of such influence functions for $i = 1, \ldots, q$ (see Appendix A4).

We shall now construct a $\delta u(t)$ history that decreases J, i.e., produces $\delta J < 0$, *and* satisfies the q terminal constraints (2.4.2). Multiply each of the q equations in (2.4.7) by an undetermined constant, ν_i, and add the resulting equations to (2.4.4):

$$\delta J + \nu_i \delta x_i(t_f) = \int_{t_o}^{t_f} \left\{\frac{\partial L}{\partial u} + [\lambda^{(J)} + \nu_i \lambda^{(i)}]^T \frac{\partial f}{\partial u}\right\} \delta u\, dt\,.\dagger \tag{2.4.10}$$

Now *choose*

$$\delta u = -k\left\{\left(\frac{\partial f}{\partial u}\right)^T [\lambda^{(J)} + \nu_i \lambda^{(i)}] + \left(\frac{\partial L}{\partial u}\right)^T\right\}, \tag{2.4.11}$$

where k is a positive scalar constant, and substitute this expression into (2.4.10), as follows:

$$\delta J + \nu_i \delta x_i(t_f) = -k \int_{t_o}^{t_f} \left\|\left(\frac{\partial f}{\partial u}\right)^T (\lambda^{(J)} + \nu_i \lambda^{(i)}) + \left(\frac{\partial L}{\partial u}\right)^T\right\|^2 dt < 0\,, \tag{2.4.12}$$

which is negative unless the integrand vanishes over the whole integration interval.

Next, we determine the ν_i's so as to satisfy the terminal constraints (2.4.2). Substituting (2.4.11) into (2.4.7), we have

$$0 = \delta x_i(t_f) = -k \int_{t_o}^{t_f} [\lambda^{(i)}]^T \frac{\partial f}{\partial u}\left[\left(\frac{\partial f}{\partial u}\right)^T (\lambda^{(J)} + \nu_j \lambda^{(j)}) + \left(\frac{\partial L}{\partial u}\right)^T\right] dt$$

$$0 = \int_{t_o}^{t_f} [\lambda^{(i)}]^T \frac{\partial f}{\partial u}\left[\left(\frac{\partial f}{\partial u}\right)^T \lambda^{(J)} + \left(\frac{\partial L}{\partial u}\right)^T\right] dt + \nu_j \int_{t_o}^{t_f} [\lambda^{(i)}]^T \frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial u}\right)^T \lambda^{(j)}\, dt\,,$$

†Repeated indices indicate summation over the range of that index, for example:

$$\mu_i \delta x_i \equiv \sum_{i=1}^{q} \mu_i \delta x_i\,.$$

from which the appropriate choice of the ν_i's is

$$\nu = -Q^{-1}g\,, \tag{2.4.13}$$

where Q is a $(q \times q)$ matrix and g is a q-component vector:

$$Q_{ij} = \int_{t_0}^{t_f} (\lambda^{(i)})^T f_u f_u^T \lambda^{(j)}\, dt\,; \qquad i,j = 1, \ldots, q\,, \tag{2.4.14}$$

$$g_i = \int_{t_0}^{t_f} (\lambda^{(i)})^T \frac{\partial f}{\partial u}\left[\left(\frac{\partial f}{\partial u}\right)^T \lambda^{(J)} + \left(\frac{\partial L}{\partial u}\right)^T\right] dt\,; \qquad i = 1, \ldots, q\,. \tag{2.4.15}$$

The existence of the inverse of Q is the *controllability condition* (see Appendix B2 and Section 5.3). If Q^{-1} does not exist, it is not possible to control the system with $u(t)$ to satisfy one or more of the terminal conditions.

We have thus *constructed* a $\delta u(t)$ history that decreases the performance index *and* satisfies the terminal constraints (2.4.2); that is, $\delta u(t)$ is admissible and improving. From (2.4.12) the only case in which we cannot decrease the performance index is when

$$\frac{\partial L}{\partial u} + [\lambda^{(J)} + \nu_i \lambda^{(i)}]^T \frac{\partial f}{\partial u} = 0\,; \qquad t_0 \le t \le t_f\,. \tag{2.4.16}$$

If (2.4.16) is satisfied, we have a *stationary solution* that satisfies the terminal constraints. Now, since the influence equations (2.4.5), (2.4.6), (2.4.8), and (2.4.9) are linear, the necessary condition (2.4.16) may be written as

$$\frac{\partial H}{\partial u} = 0\,, \tag{2.4.17}$$

where

$$H = L(x,u,t) + \lambda^T(t) f(x,u,t)\,, \tag{2.4.18}$$

and

$$\dot\lambda^T = -H_x\,, \lambda_j(t_f) = \begin{cases} \nu_j\,, & j = 1, \ldots, q\,, \\ \left.\dfrac{\partial \phi}{\partial x_j}\right|_{t=t_f}\,, & j = q+1, \ldots, n\,. \end{cases} \tag{2.4.19}$$

The development in this section represents the fundamental approach to the modern calculus of variations. By construction, we arrive at the equation

$$\delta J = \int_{t_0}^{t_f} H_u(t)\delta u(t)\, dt\,, \qquad \text{where} \qquad H_u(t) = \frac{\partial H}{\partial u}\,, \tag{2.4.20}$$

and the Hamiltonian is defined in terms of the multiplier functions $\lambda(t)$ and multipliers ν. *We then show that, unless we have* $H_u(t) \equiv 0$, *it is always possible (assuming controllability; that is,* Q^{-1} *exists) to choose* ν *such that* $\delta u(t)$ *as given by (2.4.11) is* ADMISSIBLE *and* IMPROVING.†

H_u may be interpreted as a function-space gradient of the performance index with respect to the control variable $u(t)$, while holding fixed the terminal values of x_i, $i = 1, \ldots, q$ and satisfying the system of differential equations.

Example. *Maximum velocity transfer to a rectilinear path.* Consider a particle of mass m, acted upon by a thrust force of magnitude ma. We assume planar motion and use an inertial coordinate system x,y to locate the particle; the velocity components of the particle are u,v. The thrust-direction angle $\beta(t)$ is the control variable for the system (see Figure 2.4.1). The equations of motion are

$$\dot u = a\cos\beta\,,$$
$$\dot v = a\sin\beta\,,$$
$$\dot x = u\,,$$
$$\dot y = v\,,$$

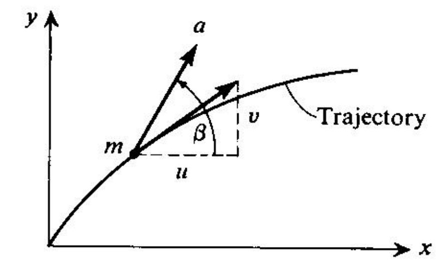


Figure 2.4.1. Nomenclature for planar motion with thrust acceleration = a.

where the thrust acceleration, a, is assumed to be a known function of time. The equations for the influence functions are particularly simple:

$$\dot\lambda_u = -\lambda_x\,, \qquad \dot\lambda_v = -\lambda_y\,, \qquad \dot\lambda_x = 0\,, \qquad \dot\lambda_y = 0\,.$$

These relations are easily integrated to yield

†For our purposes in this section, this "first-order" demonstration of the necessity of (2.4.17) is adequate. A more rigorous "second-order" demonstration is given in Section 6.3, where it is shown that the concept of "normality" is really what is needed, rather than "controllability."

$$\lambda_u = -c_1 t + c_3\,, \qquad \lambda_v = -c_2 t + c_4\,, \qquad \lambda_x = c_1\,, \qquad \lambda_y = c_2\,,$$

where $c_1,\ c_2,\ c_3,\ c_4$ are constants.

If we wish to extremalize a function of the end conditions only, then we have $L \equiv 0$, and the Hamiltonian of the system is

$$H = \lambda_u a \cos\beta + \lambda_v a \sin\beta + \lambda_x u + \lambda_y v\,,$$

which is constant for an optimal path if a is constant.

The optimality condition is

$$\frac{\partial H}{\partial \beta} = -\lambda_u \sin\beta + \lambda_v \cos\beta = 0\,.$$

Thus, the optimal control law is

$$\tan\beta = \frac{\lambda_v}{\lambda_u} = \frac{-c_2 t + c_4}{-c_1 t + c_3}\,,$$

which is often referred to as the "bilinear tangent law."

We wish to transfer the particle to a path parallel to the x-axis, a distance h away, in a given time T, arriving with the maximum value of $u(T)$. We do *not* care what the final x coordinate is (see Figure 2.4.2).

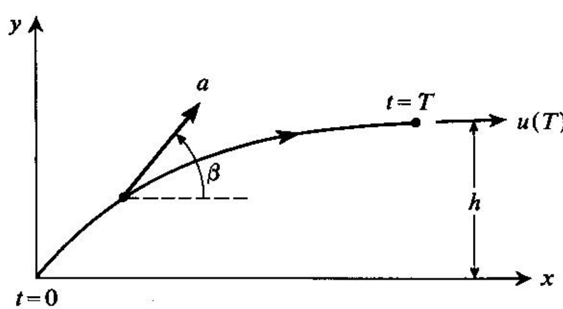


Figure 2.4.2. Nomenclature for transfer to a rectilinear path.

Thus, the boundary conditions for the problem are

$$\begin{aligned} u(0) &= 0\,, & \lambda_u(T) &= 1\,;\\ v(0) &= 0\,, & v(T) &= 0\,, \lambda_v(T) = \nu_v\,;\\ x(0) &= 0\,, & \lambda_x(T) &= 0\,;\\ y(0) &= 0\,, & y(T) &= h\,, \lambda_y(T) = \nu_y\,; \end{aligned}$$

where ν_v and ν_y are constants to be determined so that $v(T) = 0$, $y(T) = h$.

With $\lambda_x = 0$, it follows that $\lambda_u = 1$ throughout the flight, so the optimal control law becomes a "linear tangent law":

$$\tan\beta = \tan\beta_0 - ct\,, \qquad \text{where} \qquad \tan\beta_0 = \nu_v + \nu_y T\,, c = \nu_y\,.$$

For *constant-thrust acceleration*, a, the differential equations are readily integrated with the linear tangent law, using β as the independent variable instead of t, to obtain

$$u = \frac{a}{c}\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta}\,,$$

$$v = \frac{a}{c}(\sec\beta_0 - \sec\beta)\,,$$

$$x = \frac{a}{c^2}\left(\sec\beta_0 - \sec\beta - \tan\beta\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta}\right),$$

$$y = \frac{a}{2c^2}\left[(\tan\beta_0 - \tan\beta)\sec\beta_0 - (\sec\beta_0 - \sec\beta)\tan\beta - \log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta}\right].$$

The constants β_0 and c (and, hence, ν_y and ν_v) are determined by the two final boundary conditions $v = 0$, $y = h$. These relations are *implicit* and may be put into the form

$$\frac{4h}{aT^2} = \frac{1}{\sin\beta_0} - \log\frac{\sec\beta_0 + \tan\beta_0}{\sec\beta_0 - \tan\beta_0}\Big/ 2\tan^2\beta_0\,,$$

$$c = \frac{2\tan\beta_0}{T} \Rightarrow \tan\beta = \tan\beta_0\left(1 - \frac{2t}{T}\right).$$

Clearly, the one dimensionless quantity, h/aT^2, determines β_0, which, in turn, determines c. The maximum velocity $u_{\max}$ and the final value of x are then determined from

$$\frac{u_{\max}}{aT} = \frac{2x_f}{aT^2} = \log\frac{\sec\beta_0 + \tan\beta_0}{\sec\beta_0 - \tan\beta_0}\Big/ 2\tan\beta_0\,.$$

These relations are shown on Figures 2.4.3 and 2.4.4. Note, also, that

$$\nu_v = -\tan\beta_0\,, \qquad \nu_y = \frac{2\tan\beta_0}{T}\,.$$

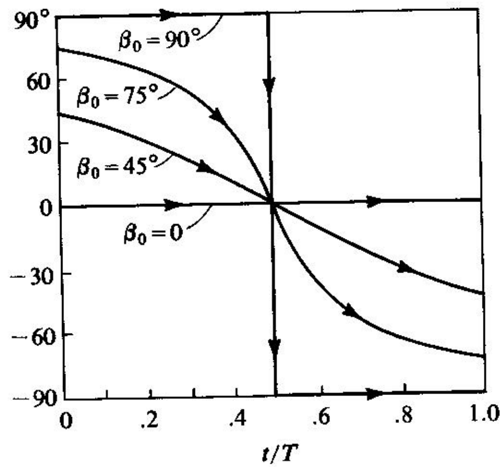


Figure 2.4.3. Thrust-angle programs for maximum velocity transfer to a rectilinear path.

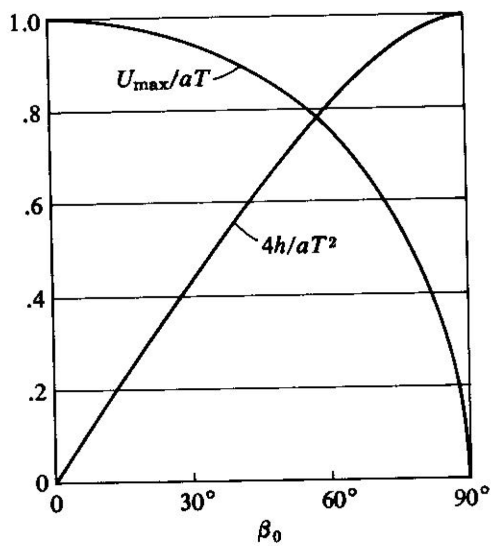


Figure 2.4.4. Maximum final velocity (U_{max}) vs. initial thrust angle (β_0), and β_0 as a function of $4h/aT^2$.

Problem 1. Consider an approximation to the optimal thrust-direction program in the example on pp. 59-62:

$$\beta = \begin{cases} \beta_1, & 0 < t < \dfrac{T}{2}, \\ -\beta_1, & \dfrac{T}{2} < t < T, \end{cases}$$

where β_1 is a constant. Note that this program gives $v(T) = 0$. Find β_1 so that $y(T) = h$ and determine $u(T)$, $x(T)$. Compare $u(T)$ with u_{max} in the example for a given h/aT^2.

ANSWER

$$\sin\beta_1 = \frac{4h}{aT^2}, \qquad u(T) = aT\cos\beta_1, \qquad x(T) = \frac{1}{2}aT^2\cos\beta_1.$$

Problem 2. *Airplane path in a wind to enclose maximum area in a given time.* An airplane has a fixed velocity V with respect to the air, and the wind velocity, u, is constant. ***Find*** the closed curve (as projected on the ground) the airplane should fly to enclose the maximum area in a given time T.

The equations of motion are

$$\dot{x} = V\cos\theta + u, \qquad \dot{y} = V\sin\theta,$$

where we have chosen the x-axis to be in the direction of the wind. If the airplane flies a closed curve, the area enclosed is given by

$$A = \oint y\,dx = \int_o^T y\dot{x}\,dt.$$

ANSWER. The closed curve is an ellipse of eccentricity $e = u/V$, minor axis parallel to the wind, and the maximum area enclosed is

$$A = \frac{V^2T^2}{4\pi}\left(1 - \frac{u^2}{V^2}\right)^{3/2}.$$

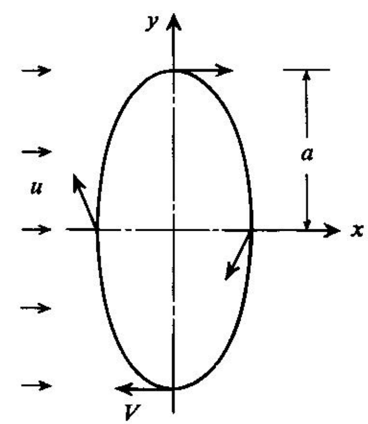


Figure 2.4.5. Airplane path in a wind to enclose maximum area in a given time.

Problem 3. ***Minimum surface of revolution connecting two coaxial circular loops.*** Given two coaxial circular loops, of radius a which are a distance 2ℓ apart, ***find*** the surface of revolution containing the two loops with minimum area. (This is the shape a soap film would take if stretched between two rings.) [HINT: Choose cylindrical coordinates r,x as shown in Figure 2.4.6. The annular element of area is

$$dA = 2\pi r\sqrt{(dr)^2 + (dx)^2}\,,$$

so the problem is to find $u(x)$ to minimize the integral.

$$A = 2\pi \int_{-\ell}^{\ell} r\sqrt{1 + u^2}\, dx\,,$$

where

$$\frac{dr}{dx} = u \qquad \text{and} \qquad r(\ell) = a\,, r(-\ell) = a\,.$$

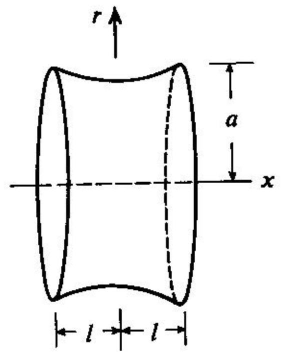


Figure 2.4.6. Minimum-area surface of revolution connecting two coaxial circular loops.

ANSWER. For $0 < \dfrac{\ell}{a} < .528$, the minimizing curve is

$$r = H\cosh\frac{x}{H}\,, \text{ where } \frac{H}{\ell} \text{ is determined by } \frac{a}{\ell}\frac{\ell}{H} = \cosh\frac{\ell}{H}\,.$$

This equation has ***two*** solutions for $0 < \ell/a < .663$ and no solution for $\ell/a > .663$. For $\ell/a > .528$, the minimizing curve is $r = 0$; i.e., two discs, each with area πa^2. The minimum area is given by

$$A_{\min} = \begin{cases} 2\pi a^2\left(\tanh\dfrac{\ell}{H} + \dfrac{\ell}{H}\operatorname{sech}^2\dfrac{\ell}{H}\right), & 0 < \dfrac{\ell}{a} < .528\,, \\ 2\pi a^2\,, & \dfrac{\ell}{a} > .528\,. \end{cases}$$

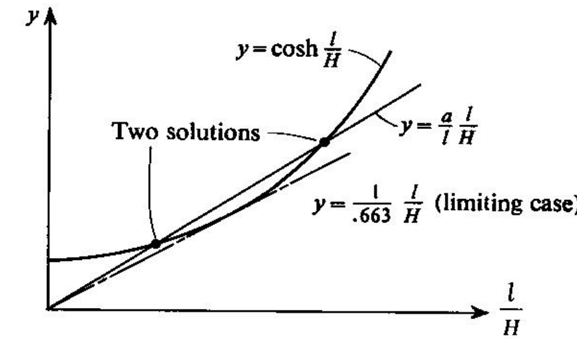


Figure 2.4.7. Solution of the minimum surface of revolution problem.

Problem 4. Find the minimum surface of revolution connecting two coaxial circular loops a distance ℓ apart, where one loop has radius a and the other loop has radius $b < a$. For each given value of b/a, show that a limiting value of $\ell/a < (\ell/a)_{\text{lim}}$ exists beyond which the minimum surface is $r = 0$; that is, two flat discs within the circular loops.

2.5 Continuous systems with functions of the state variables prescribed at a fixed terminal time

In some problems we are interested in constraining *functions* of the terminal state to have prescribed values: that is, we have

$$\psi[x(t_f),t_f] = 0 \qquad (q \text{ equations})\,, \tag{2.5.1}$$

where ψ is a q-vector ($q \le n - 1$ if $L = 0$, $q \le n$ if $L \ne 0$).

As in the previous section, we adjoin (2.5.1) to the performance index by a multiplier vector ν (a q-vector), also adjoining the system equations as in Section 2.3:

$$J = \phi[x(t_f),t_f] + \nu^T\psi[x(t_f),t_f] + \int_{t_0}^{t_f} \{L[x(t),u(t),t] + \lambda^T(f - \dot{x})\}\, dt\,. \tag{2.5.2}$$

If we define

$$\Phi = \phi + \nu^T\psi\,, \tag{2.5.3}$$

the development of Section 2.3 applies here without change. However, the final expressions for the necessary conditions for a stationary value of J satisfying (2.5.1) must be interpreted in a manner similar to that of Section 2.4; that is, we have a set of parameters, ν, which must be chosen to satisfy the q equations (2.5.1).† *In summary*, necessary conditions for J to have a stationary value are

†A controllability argument regarding the variations $\delta u(t)$ can be made, similar to the one made in Section 2.4, to justify (2.5.6).

$$\dot{x} = f(x,u,t) \qquad (n \text{ differential equations}), \tag{2.5.4}$$

$$\dot{\lambda} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda - \left(\frac{\partial L}{\partial x}\right)^T \qquad (n \text{ differential equations}), \tag{2.5.5}$$

$$\left(\frac{\partial H}{\partial u}\right)^T = \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial L}{\partial u}\right)^T = 0 \qquad (m \text{ algebraic equations}), \tag{2.5.6}$$

$$x_k(t_o) \text{ given or } \lambda_k(t_o) = 0 \quad k = 1, \ldots, n \qquad (n \text{ boundary conditions}), \tag{2.5.7}$$

$$\lambda^T(t_f) = \left(\frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right)_{t=t_f} \qquad (n \text{ boundary conditions}), \tag{2.5.8}$$

$$\psi[x(t_f),t_f] = 0 \qquad (q \text{ side conditions}). \tag{2.5.9}$$

The stationarity conditions (2.5.6) determine the m-vector $u(t)$. The $2n$ differential equations (2.5.4) and (2.5.5), with the $2n$ boundary conditions (2.5.7) and (2.5.8), form a two-point boundary-value problem with q parameters ν to be found in (2.5.8) so that the q side conditions (2.5.9) are satisfied.

Example. ***Maximum radius orbit transfer in a given time.*** Given a constant-thrust rocket engine, T = thrust, operating for a given length of time, t_f, we wish to find the thrust-direction history, $\phi(t)$, to transfer a rocket vehicle from a given initial circular orbit to the largest possible circular orbit. The nomenclature is defined in Figure 2.5.1, below.

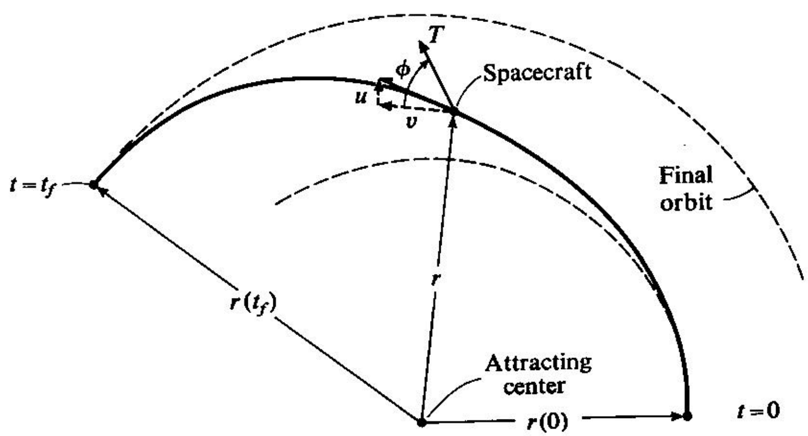


Figure 2.5.1. Maximum radius orbit transfer in a given time (or minimum time for a given final radius).

r = radial distance of spacecraft from attracting center
u = radial component of velocity
v = tangential component of velocity
m = mass of spacecraft, $\dot{m}$ = fuel consumption rate (constant)
ϕ = thrust direction angle
μ = gravitational constant of attracting center

Using this nomenclature, the problem may be stated as:
Find $\phi(t)$ to maximize $r(t_f)$ subject to

$$\dot{r} = u, \tag{2.5.10}$$

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin\phi}{m_o - |\dot{m}|t}, \tag{2.5.11}$$

$$\dot{v} = -\frac{uv}{r} + \frac{T\cos\phi}{m_o - |\dot{m}|t}, \tag{2.5.12}$$

and

$$r(0) = r_o, \tag{2.5.13}$$

$$u(0) = 0, \tag{2.5.14}$$

$$v(0) = \sqrt{\frac{\mu}{r_o}}, \tag{2.5.15}$$

$$\psi_1 = u(t_f) = 0, \tag{2.5.16}$$

$$\psi_2 = v(t_f) - \sqrt{\frac{\mu}{r(t_f)}} = 0. \tag{2.5.17}$$

The Hamiltonian is, therefore,

$$H = \lambda_r u + \lambda_u\left(\frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin\phi}{m_o - |\dot{m}|t}\right) + \lambda_v\left(-\frac{uv}{r} + \frac{T\cos\phi}{m_o - |\dot{m}|t}\right)$$

and

$$\Phi = r(t_f) + \nu_1 u(t_f) + \nu_2\left[v(t_f) - \sqrt{\frac{\mu}{r(t_f)}}\right].$$

Thus, the necessary conditions (2.5.5), (2.5.6), and (2.5.9) become

$$\dot{\lambda}_r = -\lambda_u\left(-\frac{v^2}{r^2} + \frac{2\mu}{r^3}\right) - \lambda_v\left(\frac{uv}{r^2}\right), \tag{2.5.18}$$

$$\dot{\lambda}_u = -\lambda_r + \lambda_v\frac{v}{r}, \tag{2.5.19}$$

$$\dot{\lambda}_v = -\lambda_u\frac{2v}{r} + \lambda_v\frac{u}{r}, \tag{2.5.20}$$

$$0 = (\lambda_u \cos\phi - \lambda_v \sin\phi)\frac{T}{m_o - |\dot{m}|t} \Rightarrow \tan\phi = \frac{\lambda_u}{\lambda_v}, \tag{2.5.21}$$

$$\lambda_r(t_f) = 1 + \frac{\nu_2\sqrt{\mu}}{2[r(t_f)]^{3/2}}, \tag{2.5.22}$$

$$\lambda_u(t_f) = \nu_1, \tag{2.5.23}$$

$$\lambda_v(t_f) = \nu_2. \tag{2.5.24}$$

The *six* differential equations (2.5.10), (2.5.11), (2.5.12), (2.5.18), (2.5.19), and (2.5.20) are to be solved subject to the *six* boundary conditions (2.5.13), (2.5.14), (2.5.15), (2.5.22), (2.5.23), and (2.5.24), with the choice of ν_1 and ν_2 available to satisfy the additional two boundary conditions (2.5.16) and (2.5.17). The control $\phi(t)$ is determined in terms of λ_u and λ_v from (2.5.21).

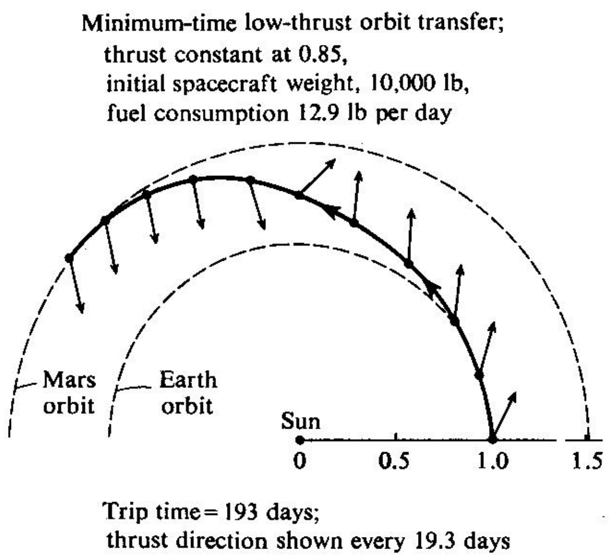


Figure 2.5.2. A particular minimum-time low-thrust orbit transfer path.

A numerical solution of this problem for

$$\frac{T/m_o}{\mu/r_o^2} = .1405, \quad |\dot{m}|\sqrt{\mu/r_o}/T = 0.533, \quad \frac{t_f}{\sqrt{r_o^3/\mu}} = 3.32$$

has been given by Kopp and McGill.† Interpreted for a 10,000-lb

†See A. V. Balakrishnan and L. W. Neustadt (eds.), *Computing Methods in Optimization Problems*. New York: Academic Press, 1964.

spacecraft moving out from the earth's orbit, the thrust would be 0.85 lb, the fuel consumption 12.9 lb/day, and the trip time 193 days. The optimal thrust direction and the resulting trajectory are shown in Figure 2.5.2. Note that the radial component of thrust is outward for the first half (roughly) of the flight, and inward for the second half.

2.6 Multistage systems; functions of the state variables specified at the terminal stage

Multistage systems, while of importance in their own right, have gained a special significance because of the use of digital computers to solve continuous problems. For numerical solution on a digital computer, the continuous optimization problems of Sections 2.3 to 2.5 must be converted to multistage optimization problems. Proper multistage formulation of such problems contributes significantly to the speed of convergence of iterative numerical solution procedures.

The following is a multistage version of the problems treated in Section 2.5. It differs from Section 2.2 only in the inclusion of terminal constraints. Find the sequence $u(0), \ldots, u(N-1)$ to minimize

$$J = \phi[x(N)] + \sum_{i=0}^{N-1} L^i[x(i),u(i)] \tag{2.6.1}$$

subject to the constraints

$$x(i+1) = f^i[x(i),u(i)], \tag{2.6.2}$$

$$\psi[x(N)] = 0, \tag{2.6.3}$$

where x is an n-vector, u is an m-vector, and ψ is a q-vector function, $q \le n$.

As in Section 2.2, we adjoin Equation (2.6.2) to J with a multiplier sequence $\lambda(i)$ and, in addition, we adjoin Equation (2.6.3) with a set of q multipliers $(\nu_1, \ldots, \nu_q) \triangleq \nu^T$:

$$\bar{J} = \phi[x(N)] + \nu^T\psi[x(N)] + \sum_{i=0}^{N-1}\{L^i[x(i),u(i)] + \lambda^T(i+1)[f^i[x(i),u(i)] - x(i+1)]\}. \tag{2.6.4}$$

For convenience, define a scalar sequence H^i and a scalar function Φ, as follows:

$$H^i = L^i[x(i),u(i)] + \lambda^T(i+1)f^i[x(i),u(i)], \tag{2.6.5}$$

$$\Phi = \phi[x(N)] + \nu^T\psi[x(N)]. \tag{2.6.6}$$

Also, change indices of summation on the last term in (2.6.4), yielding

$$\bar{J} = \Phi[x(N)] - \lambda^T(N)\,x(N) + \sum_{i=1}^{N-1} [H^i - \lambda^T(i)\,x(i)] + H^o. \qquad (2.6.7)$$

Now consider differential changes in $\bar{J}$ due to differential changes in $u(i)$:

$$d\bar{J} = \left[\frac{\partial\Phi}{\partial x(N)} - \lambda^T(N)\right] dx(N) + \sum_{i=1}^{N-1} \left\{\left[\frac{\partial H^i}{\partial x(i)} - \lambda^T(i)\right] dx(i) + \frac{\partial H^i}{\partial u(i)}\, du(i)\right\}$$
$$+ \frac{\partial H^o}{\partial x(0)}\, dx(0) + \frac{\partial H^o}{\partial u(0)}\, du(0). \qquad (2.6.8)$$

The coefficients multiplying $dx(i)(i = 0, \ldots, n)$ vanish if we *choose* the multiplier sequence $\lambda(i)$ so that we have

$$\lambda^T(i) - \frac{\partial H^i}{\partial x(i)} = 0, \qquad (2.6.9)$$

or

$$\lambda^T(i) = \frac{\partial L^i}{\partial x(i)} + \lambda^T(i+1)\frac{\partial f^i}{\partial x(i)}, \qquad i = 0, \ldots, N-1, \qquad (2.6.9a)$$

with boundary conditions

$$\lambda^T(N) = \frac{\partial\Phi}{\partial x(N)}, \qquad (2.6.10)$$

or

$$\lambda^T(N) = \frac{\partial\phi}{\partial x(N)} + \nu^T\frac{\partial\psi}{\partial x(N)}. \qquad (2.6.10a)$$

Equation (2.6.8) then becomes

$$d\bar{J} = \lambda^T(0)\, dx(0) + \sum_{i=o}^{N-1} \frac{\partial H^i}{\partial u(i)}\, du(i). \qquad (2.6.11)$$

Thus $\partial H^i/\partial u(i)$ ***is the gradient*** of $\bar{J}$ with respect to $u(i)$ while holding $x(0)$ constant and satisfying (2.6.2), and $\lambda^T(0)$ is the gradient of $\bar{J}$ with respect to $x(0)$ while holding $u(i)$ constant and satisfying (2.6.2). If $x(0)$ is given, we have $dx(0) = 0$.

For a stationary value of $\bar{J}$, $d\bar{J}$ must be zero for admissible $du(i)$. If $u(i)$ is unconstrained and H^i is differentiable with respect to $u(i)$ and the problem is "normal", this can happen only if†

$$\frac{\partial H^i}{\partial u(i)} = 0, \qquad (2.6.12)$$

†See Sections 5.3 and 6.3 for the argument concerning "normality" which relates to the existence of neighboring optimal paths.

or

$$\frac{\partial L^i}{\partial u(i)} + \lambda^T(i+1)\frac{\partial f^i}{\partial u(i)} = 0, \qquad i = 0, \ldots, N-1. \qquad (2.6.12a)$$

In summary, to find a control-vector sequence $u(i)$ that produces a stationary value of the performance index J, we must solve the "two-point boundary-value problem" defined by (2.6.2), (2.6.3), (2.6.9), (2.6.10), and (2.6.12).

These equations constitute $(2n + m)\; N + n + p$ equations for as many unknowns: $x(0), \ldots, x(N)$ where x is an n-vector; $u(0), \ldots, u(N-1)$, where u is an m-vector; $\lambda(o), \lambda(1), \ldots, \lambda(N)$, where λ is an n-vector; and ν, a p-vector.

To solve (2.6.2) and (2.6.9a) together sequentially in the forward direction, using (2.6.12a) to determine $u(i)$, it is necessary to solve (2.6.9a) for $\lambda(i+1)$ in terms of $\lambda(i)$ and $x(i)$:

$$\lambda^T(i+1) = \left[\lambda^T(i) - \frac{\partial L^i}{\partial x(i)}\right]\left[\frac{\partial f^i}{\partial x(i)}\right]^{-1}. \qquad (2.6.13)$$

The inverse of $\partial f^i/\partial x(i)$ exists since it is, essentially, the linearized transition-matrix;† however, the computation of this inverse is time-consuming.‡ The alternative of sequential backward solution offers no improvement since (2.6.2), (2.6.9a), and (2.6.12a) would have to be viewed as a set of implicit equations for $x(i)$, $\lambda(i)$, and $u(i)$, given $x(i+1)$, $\lambda(i+1)$, $u(i+1)$.

Continuous systems; some state variables specified at an unspecified terminal time (including minimum-time problems)

This is almost the same set of problems as in Section 2.4, with the important difference that the terminal time, t_f, is not specified. It is convenient to regard t_f as a *control parameter* to be chosen in addition to the *control functions*, $u(t)$, so as to minimize the performance index and satisfy the constraints. We shall show that the same necessary conditions apply as in Section 2.4, but, in addition, the following condition must be satisfied by the optimal choice of the terminal time, t_f:

$$\left(\frac{\partial\phi}{\partial t} + \lambda^T f + L\right)_{t=t_f} = 0.$$

As in Section 2.3, we adjoin the system differential equations to the performance index, as follows:

†See Appendix A3.

‡The computation of the inverse is circumverted in the algorithm given in Section 7.7.

$$J = \phi[x(t_f),t_f] + \int_{t_0}^{t_f} [L(x,u,t) + \lambda^T(t)f(x,u,t) - \lambda^T\dot{x}]\, dt\,. \qquad (2.7.1)$$

The *differential* of (2.7.1), taking into account differential changes in the terminal time, t_f, is

$$dJ = \left(\frac{\partial \phi}{\partial t}dt_f + \frac{\partial \phi}{\partial x}dx\right)_{t=t_f} + (L)_{t=t_f}dt_f + \int_{t_0}^{t_f}\left[\left(\frac{\partial L}{\partial x} + \lambda^T\frac{\partial f}{\partial x}\right)\delta x + \left(\frac{\partial L}{\partial u} + \lambda^T\frac{\partial f}{\partial u}\right)\delta u - \lambda^T\delta\dot{x}\right]dt\,. \qquad (2.7.2)$$

Integrating (2.7.2) by parts and collecting terms gives

$$dJ = \left[\left(\frac{\partial \phi}{\partial t} + L\right)dt_f + \frac{\partial \phi}{\partial x}dx\right]_{t=t_f} - [\lambda^T\delta x]_{t=t_f} + [\lambda^T\delta x]_{t=t_0} + \int_{t_0}^{t_f}\left[\left(\frac{\partial L}{\partial x} + \lambda^T\frac{\partial f}{\partial x} + \dot{\lambda}^T\right)\delta x + \left(\frac{\partial L}{\partial u} + \lambda^T\frac{\partial f}{\partial u}\right)\delta u\right]dt\,. \qquad (2.7.3)$$

Now δx, the variation in x, means "for time held fixed," so dx, the differential in x, may be written (see Figure 2.7.1)

$$dx(t_f) = \delta x(t_f) + \dot{x}(t_f)\,dt_f\,. \qquad (2.7.4)$$

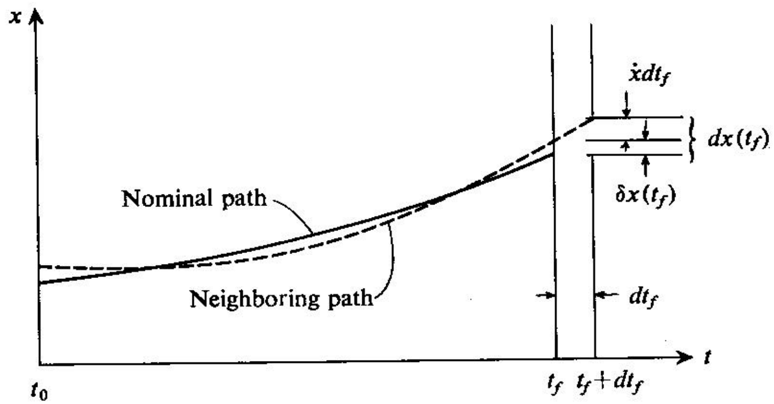


Figure 2.7.1. Relationship between $dx(t_f)$, $\delta x(t_f)$, and dt_f.

From (2.7.4), we have $\delta x(t_f) = dx(t_f) - \dot{x}(t_f)\,dt_f$; substituting this into (2.7.3) and collecting terms gives

$$dJ = \left[\left(\frac{\partial \phi}{\partial t} + L + \lambda^T\dot{x}\right)dt_f + \left(\frac{\partial \phi}{\partial x} - \lambda^T\right)dx\right]_{t=t_f} + \lambda^T(t_o)\,\delta x(t_o) + \int_{t_0}^{t_f}\left[\left(\frac{\partial L}{\partial x} + \lambda^T\frac{\partial f}{\partial x} + \dot{\lambda}^T\right)\delta x + \left(\frac{\partial L}{\partial u} + \lambda^T\frac{\partial f}{\partial u}\right)\delta u\right]dt\,. \qquad (2.7.5)$$

Now, as in Section 2.4, we consider that

$$x_i(t_f)\,; \qquad i = 1,\ldots,q \text{ are specified,} \qquad (2.7.6)$$

and, hence, it is consistent to consider ϕ to be a function only of the unspecified state variables:

$$\phi = \phi[x_j(t_f),t_f]\,; \qquad j = q+1,\ldots,n\,. \qquad (2.7.7)$$

Next, we *choose* the functions $\lambda(t) \equiv \lambda^{(j)}(t)$ to make the coefficients of $\delta x(t)$, and $dx(t_f)$ vanish in (2.7.5)

$$\dot{\lambda}^{(j)} = -\left(\frac{\partial L}{\partial x}\right)^T - \left(\frac{\partial f}{\partial x}\right)^T\lambda^{(j)}\,, \qquad (2.7.8)$$

$$\lambda_j^{(j)}(t_f) = \begin{cases} 0\,, & j = 1,\ldots,q\,, \\ \left(\dfrac{\partial \phi}{\partial x_j}\right)_{t=t_f}, & j = q+1,\ldots,n\,. \end{cases} \qquad (2.7.9)$$

This choice of $\lambda(t)$ leaves (2.7.5) in the form

$$dJ = \left(\frac{\partial \phi}{\partial t} + L + f^T\lambda^{(j)}\right)_{t=t_f} dt_f + \int_{t_0}^{t_f}\left\{\frac{\partial L}{\partial u} + [\lambda^{(j)}]^T\frac{\partial f}{\partial u}\right\}\delta u\, dt\,, \qquad (2.7.10)$$

where we have placed $\delta x(t_o) = 0$ since $x(t_o)$ is given.

Now, as in Section 2.4, let us consider the change in $x_i(t_f)$, $i = 1,\ldots,q$ for arbitrary $\delta u(t)$. Using the concept of influence (adjoint) functions (see Appendix A3) we have

$$dx_i(t_f) = [f_i]_{t=t_f}dt_f + \int_{t_0}^{t_f}[\lambda^{(i)}(t)]^T\frac{\partial f}{\partial u}\delta u\, dt\,, \qquad (2.7.11)$$

where

$$\dot{\lambda}^{(i)} = -\left(\frac{\partial f}{\partial x}\right)^T\lambda^{(i)}\,, \qquad (2.7.12)$$

$$\lambda_j^{(i)}(t_f) = \begin{cases} 1\,, & i = j\,, \\ 0\,, & i \neq j\,. \end{cases} \qquad (2.7.13)$$

Note that Equation (2.7.11) may be regarded as a special case of (2.7.10) by replacing ϕ with x_i and placing $L = 0$.

We will now construct a $\delta u(t)$ history and select a value for dt_f that produces $dJ < 0$, and satisfies $dx_i(t_f) = 0, i = 1,\ldots,q$. Multiply each of the q equations (2.7.11) by an undetermined constant, ν_i, and add the resulting equations to (2.7.10)

$$dJ + \nu_i\, dx_i(t_f) = \left\{\frac{\partial \phi}{\partial t} + L + [\lambda^{(j)}]^T f + \nu_i f_i\right\}_{t=t_f} dt_f + \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial u} + (\lambda^{(j)} + \nu_i\lambda^{(i)})^T\frac{\partial f}{\partial u}\right]\delta u\, dt\,. \qquad (2.7.14)$$

Now *choose*

$$dt_f = -k_1\left\{\frac{\partial\phi}{\partial t} + L + [\lambda^{(J)}]^T f + \nu_i f_i\right\}_{t=t_f}, \tag{2.7.15}$$

$$\delta u = -k_2\left[\left(\frac{\partial L}{\partial u}\right)^T + \left(\frac{\partial f}{\partial u}\right)^T (\lambda^{(J)} + \nu_i \lambda^{(i)})\right], \tag{2.7.16}$$

where k_1 and k_2 are positive constants; substituting (2.7.15) and (2.7.16) into (2.7.14) yields

$$dJ + \nu_i\, dx_i(t_f) = -k_1\left\|\frac{\partial\phi}{\partial t} + L + (\lambda^{(J)})^T f + \nu_i f_i\right\|^2$$
$$-k_2\int_{t_o}^{t_f}\left\|\frac{\partial L}{\partial u} + (\lambda^{(J)} + \nu_i\lambda^{(i)})^T\frac{\partial f}{\partial u}\right\|^2 dt \le 0\,, \tag{2.7.17}$$

which is negative unless the squared terms are identically zero.

Next, we determine the ν_i's so as to satisfy the terminal constraints (2.7.11) with $dx_i(t_f) = 0$, $i = 1, \ldots, q$. Substitute (2.7.15) and (2.7.16) into (2.7.11):

$$0 = -k_1\{f_i[\phi_t + L + (\lambda^{(J)})^T f + \nu_j f_j]\}_{t=t_f}$$
$$-k_2\int_{t_o}^{t_f}(\lambda^{(i)})^T f_u[L_u^T + f_u^T(\lambda^{(J)} + \nu_j\lambda^{(j)})]\,dt\,, \tag{2.7.18}$$

or

$$0 = -k_1\{f_i[\phi_t + L + (\lambda^{(J)})^T f]\}_{t=t_f} - k_2\int_{t_o}^{t_f}(\lambda^{(i)})^T f_u[L_u^T + f_u^T\lambda^{(J)}]\,dt$$
$$-\left\{k_1(f_i f_j)_{t=t_f} + k_2\int_{t_o}^{t_f}(\lambda^{(i)})^T f_u f_u^T\lambda^{(j)}\,dt\right\}\nu_j\,,$$

from which the appropriate choice of ν is

$$\nu = -\left[Q + \frac{k_1}{k_2}S\right]^{-1}\left(g + \frac{k_1}{k_2}r\right), \tag{2.7.19}$$

where

$$Q_{ij} = \int_{t_o}^{t_f}(\lambda^{(i)})^T f_u f_u^T\lambda^{(j)}\,dt\,, \qquad S_{ij} = (f_i f_j)_{t=t_f}\,,$$
$$g_i = \int_{t_o}^{t_f}(\lambda^{(i)})^T f_u[L_u^T + f_u^T\lambda^{(J)}]\,dt\,, \qquad r_i = \{f_i[\phi_t + L + (\lambda^{(J)})^T f]\}_{t=t_f}\,.$$

From (2.7.17), the only case in which we *cannot* decrease the performance index is when

$$(\phi_t + L + (\lambda^{(J)})^T f + \nu_j f_j)_{t=t_f} = 0\,, \tag{2.7.20}$$

$$L_u + (\lambda^{(J)} + \nu_i\lambda^{(i)})^T f_u = 0\,; \qquad t_o < t < t_f\,. \tag{2.7.21}$$

If (2.7.20), (2.7.21) obtain, we have a *stationary solution* that satisfies the terminal constraints.

Using (2.7.20) in (2.7.18), we see that the ν_i's for a stationary solution are independent of k_1/k_2 and are given by

$$\nu = -Q^{-1}g\,. \tag{2.7.22}$$

Thus, as in the fixed-terminal-time case, the existence of the inverse of Q is the *controllability condition.* Since the influence equations are linear, the necessary conditions (2.7.20)–(2.7.21) may be written as

$$(\phi_t + H)_{t=t_f} = 0\,, \tag{2.7.23}$$

$$\frac{\partial H}{\partial u} = 0\,; \qquad t_o < t < t_f\,, \tag{2.7.24}$$

where

$$H = L + \lambda^T f\,, \tag{2.7.25}$$

$$\dot\lambda^T = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T\frac{\partial f}{\partial x}\,, \tag{2.7.26}$$

$$\lambda_j(t_f) = \begin{cases}\nu_j\,, & j = 1, \ldots, q\,,\\ \left(\dfrac{\partial\phi}{\partial x_j}\right)_{t=t_f}, & j = q+1, \ldots, n\,.\end{cases} \tag{2.7.27}$$

We may regard the ν_i's, $i = 1, \ldots, q$, as *control parameters* that control the terminal values of x_i, $i = 1, \ldots, q$, which must have the specified values for an admissible path. Similarly, t_f is a control parameter that controls the terminal value of $\phi_t + H$, which must vanish for a stationary path.

Conceptually, the problem of unspecified terminal time can always be approached as a series of optimization problems with fixed terminal time. In other words, we consider the terminal time t_f as an additional control parameter and solve a series of identical optimization problems, as in Section 2.4, with different values of t_f. The particular value of t_f that yields the minimal value of J for the series of optimization problems must be the solution to the problem with unspecified terminal time. Thus, we can expect that all necessary conditions derived in Section 2.4 hold. There must also be one additional condition that determines the optimal value of t_f, and this is (2.7.23).

Problem 1. Consider

$$\bar{J} = \Phi(x(t_f),t_f) + \int_o^{t_f} L(x,u,t)\,dt$$

and t_f as a control parameter. What is the variation of J due to a variation of t_f when all optimality conditions in Section 2.4 are to be satisfied? From this, derive the condition

$$\frac{\partial \Phi}{\partial t_f} = -H(t_f)$$

directly. [HINT:

$$d\bar{J} = \frac{\partial \Phi}{\partial x}\frac{dx}{dt}dt_f + \frac{\partial \Phi}{\partial t_f}dt_f + L\,dt_f.]$$

MINIMUM-TIME SOLUTIONS. In many problems, the performance index of interest is the *elapsed time* to transfer the system from its initial state to a specified state. In this case, we may place

$$\phi = 0, \qquad L = 1, \tag{2.7.28}$$

which implies that

$$J = t_f - t_o. \tag{2.7.29}$$

The minimum-time control program is obtained, then, by solving the two-point boundary-value problem:

$$\dot{x} = f(x,u,t); \qquad x(t_o) \text{ given}\dagger\ (n \text{ initial conditions}), \tag{2.7.30}$$

$$\dot{\lambda} = -(f_x)^T\lambda; \qquad x_j(t_f) \text{ specified}; \qquad j = 1, \ldots, q;$$
$$\lambda_j(t_f) = 0, j = q+1, \ldots, n \ (n \text{ terminal conditions}), \tag{2.7.31}$$

$$0 = f_u^T\lambda \ (m \text{ optimality conditions}), \tag{2.7.32}$$

$$(\lambda^T f)_{t=t_f} = -1. \tag{2.7.33}$$

Note that there are $2n$ boundary conditions for the $2n$ differential equations (2.7.30)–(2.7.31), m optimality conditions (2.7.32) for the m control variables, u, and one transversality condition (2.7.33) for the terminal time, t_f. The unspecified values of $\lambda_j(t_f)$, $j = 1, \ldots, q$, which we have called ν_j above, are part of the solution.

Note, also, that at least one state variable must be specified at $t = t_o$ and at $t = t_f$ or the minimum-time problem makes no sense.

†If $x_j(t_o)$ is not specified, we have $\lambda_j(t_o) = 0$.

mple 1. *Minimum-time paths through a region of position-dependent vector velocity (Zermelo's problem).*† A ship must travel through a region of strong currents. The magnitude and direction of the currents are known as functions of position:

$$u = u(x,y) \qquad \text{and} \qquad v = v(x,y),$$

where (x,y) are rectangular coordinates and (u,v) are the velocity components of the current in the x and y directions, respectively. The magnitude of the ship's velocity relative to the water is V, a constant. The problem is to steer the ship in such a way as to minimize the time necessary to go from a point A to a point B.

The equations of motion are

$$\dot{x} = V\cos\theta + u(x,y), \tag{2.7.34}$$

$$\dot{y} = V\sin\theta + v(x,y), \tag{2.7.35}$$

where θ is the heading angle of the ship's axis relative to the (fixed) coordinate axes, and (x,y) represents the position of the ship. The Hamiltonian of the system is

$$H = \lambda_x(V\cos\theta + u) + \lambda_y(V\sin\theta + v) + 1, \tag{2.7.36}$$

So the Euler-Lagrange equations are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -\lambda_x\frac{\partial u}{\partial x} - \lambda_y\frac{\partial v}{\partial x}, \tag{2.7.37}$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x\frac{\partial u}{\partial y} - \lambda_y\frac{\partial v}{\partial y}, \tag{2.7.38}$$

$$0 = \frac{\partial H}{\partial \theta} = V(-\lambda_x\sin\theta + \lambda_y\cos\theta), \Rightarrow \tan\theta = \frac{\lambda_y}{\lambda_x}. \tag{2.7.39}$$

Since the Hamiltonian (2.7.36) is not an explicit function of time, H = constant is an integral of the system. Furthermore, since we are minimizing time, this constant must be 0. We may solve (2.7.36) and (2.7.39) for λ_x and λ_y:

$$\lambda_x = \frac{-\cos\theta}{V + u\cos\theta + v\sin\theta}, \tag{2.7.40}$$

$$\lambda_y = \frac{-\sin\theta}{V + u\cos\theta + v\sin\theta}. \tag{2.7.41}$$

†or another derivation, using vector notation, see Section 3.2, Example 2, which ats the problem in three dimensions (e.g., for an aircraft in a region of strong winds).

We may now substitute (2.7.40) and (2.7.41) into either (2.7.37) or (2.7.38) (or demand consistency between $H_\theta = 0$, $\dot{H}_\theta = 0$) to obtain

$$\dot{\theta} = \sin^2\theta \frac{\partial v}{\partial x} + \sin\theta\cos\theta\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \cos^2\theta\frac{\partial u}{\partial y}. \qquad (2.7.42)$$

This equation, solved simultaneously with (2.7.34) and (2.7.35), will give the desired minimum time paths; in order to go through a particular point B, starting at a point A, we must pick the correct value of θ_A.
Note that, if u and v are constant, (2.7.42) indicates that θ = const; that is, the minimum-time paths are straight lines.

ANALOG TO SNELL'S LAW. If we have $u = u(y)$, $v = v(y)$, then (2.7.37) becomes

$$\dot{\lambda}_x = 0 \Rightarrow \lambda_x = \text{const.} \qquad (2.7.43)$$

Equation (2.7.40) is then

$$\frac{\cos\theta}{V + u(y)\cos\theta + v(y)\sin\theta} = \text{const}, \qquad (2.7.44)$$

which is directly analogous to Snell's Law in optics since it (implicitly) gives the ship's heading, θ, in terms of the local current velocities.

Special case: linear variation of current velocity. If we have $u = -V(y/h)$, $v = 0$, and we wish to find the minimum-time path from a certain point x_o, y_o to the origin (0,0) then we may use (2.7.44) to express the optimal heading angle, θ, in terms of the final heading angle, θ_f, and the present y coordinate, as follows:

$$\frac{\cos\theta}{V - V(y/h)\cos\theta} = \frac{\cos\theta_f}{V} = \text{constant},$$
$$\cos\theta = \frac{\cos\theta_f}{1 + (y/h)\cos\theta_f}. \qquad (2.7.45)$$

It is convenient to use θ as the independent variable instead of t. From (2.7.45), we already have $y(\theta)$,

$$\frac{y}{h} = \sec\theta - \sec\theta_f. \qquad (2.7.46)$$

Equation (2.7.42) becomes

$$\frac{dt}{d\theta} = \frac{h}{V}\sec^2\theta \Rightarrow \frac{V(t_f - t)}{h} = \tan\theta - \tan\theta_f, \qquad (2.7.47)$$

where $t_f - t$ is the time to go to the origin.

Finally, (2.7.34), using (2.7.46) and (2.7.47), becomes

$$\frac{dx}{d\theta} = \frac{V\cos\theta + V(\sec\theta_f - \sec\theta)}{-(V/h)\cos^2\theta} = -h(\sec\theta + \sec\theta_f\sec^2\theta - \sec^3\theta), \qquad (2.7.48)$$

which may be integrated to give

$$\frac{x}{h} = \frac{1}{2}\Big[\sec\theta_f(\tan\theta_f - \tan\theta) - \tan\theta(\sec\theta_f - \sec\theta) + \log\frac{\tan\theta_f + \sec\theta_f}{\tan\theta + \sec\theta}\Big]. \qquad (2.7.49)$$

Now, let us suppose that we want to find the minimum-time path from $x_o/h = 3.66$, $y_o/h = -1.86$ to the origin. Equations (2.7.46) and (2.7.49) are implicit equations for θ_o and θ_f, where θ_o is the value of θ at the initial point, as follows:

$$-1.86 = \sec\theta_o - \sec\theta_f, \qquad (2.7.50)$$

$$3.66 = \tfrac{1}{2}[\sec\theta_f(\tan\theta_f - \tan\theta_o) - \tan\theta_o(\sec\theta_f - \sec\theta_o) + \sinh^{-1}(\tan\theta_f) - \sinh^{-1}(\tan\theta_o)]. \qquad (2.7.51)$$

The solution to these equations is

$$\theta_o = 105°, \qquad \theta_f = 240°.$$

From (2.7.47), the time to go from the initial point to the origin is

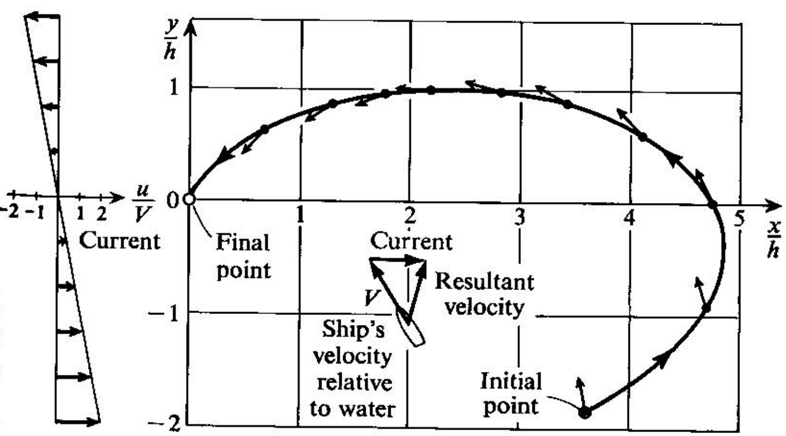


Figure 2.7.2. A minimum-time path through a region of linearly increasing current.

on a circle rolling without slipping in a horizontal direction, and that $\dot{\theta}$ = constant.

Problem 7. (Courtesy T. N. Edelbaum). Find the path of minimum time connecting two points on the surface of the earth through a tunnel in the earth. The tunnel is assumed to be evacuated, gravity is the propelling force on the particle, and friction is negligible. Note that the gravitational force per unit mass inside the earth is directed radially toward the center of the earth and increases *linearly* with radius from zero at the center.

ANSWER. The paths are hypocycloids, i.e., curves generated by a point on a small circle rolling without slipping on the *inside* of the earth's surface.

Problem 8. *Thrust-direction programming with negligible external forces.* This is one of the simplest problems, of some practical interest, in optimal programming. As such, it is useful for fixing ideas.† Consider a particle of mass m, acted upon by a thrust force of magnitude ma. Assume planar motion and use an inertial coordinate system x,y to locate the particle; the velocity components of the particle are u,v. The thrust-direction angle $\beta(t)$ is the control variable for the system (see Figure 2.4.1). The equations of motion are

$$\dot{u} = a\cos\beta\,, \qquad \dot{v} = a\sin\beta\,, \qquad \dot{x} = u\,, \qquad \dot{y} = v\,,$$

where the thrust acceleration, a, is assumed to be a known function of time. If we wish to extremalize a function of the end conditions only *or* minimize the time, *show that* the optimal control law is

$$\tan\beta = \frac{-c_2 t + c_4}{-c_1 t + c_3}\,,$$

where c_1, c_2, c_3, c_4 are constants. This is often referred to as the "bilinear tangent law."

Problem 9. *Minimum-time orbit injection* ($g = 0$). We wish to transfer the particle of Problem 8 to a path a distance h away, arriving with a velocity of U parallel to the path in the least time; we do *not* care what the final x-coordinate is (see Figure 2.4.2).
Thus, the boundary conditions for the problem are

$$u(0) = 0\,, \quad u(T) = U\,, \quad v(0) = 0\,, \quad v(T) = 0\,, \quad x(0) = 0\,,$$
$$\lambda_x(T) = 0\,, \quad y(0) = 0\,, \quad y(T) = h\,,$$
$$(\lambda_u a\cos\beta + \lambda_v a\sin\beta)_{t=T} = -1\,.$$

†See Example, Section 2.4.

Since $x(T)$ is unspecified, we have $\lambda_x = c_1 = 0$, and the optimal control law becomes a "linear tangent law":

$$\tan\beta = \tan\beta_0 - ct\,, \qquad \text{where} \qquad \tan\beta_0 = \frac{c_4}{c_3}\,, c = \frac{c_2}{c_3} \text{ (see Problem 8).}$$

For *constant-thrust acceleration*, a, and using β as the independent variable instead of t, *show that*

$$u = \frac{a}{c}\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta}\,, \qquad v = \frac{a}{c}(\sec\beta_0 - \sec\beta)\,,$$

$$x = \frac{a}{c^2}\left(\sec\beta_0 - \sec\beta - \tan\beta\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta}\right),$$

$$y = \frac{a}{2c^2}\left[(\tan\beta_0 - \tan\beta)\sec\beta_0 - (\sec\beta_0 - \sec\beta)\tan\beta - \log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta}\right],$$

$$\lambda_u = -\frac{\cos\beta_0}{a}\,, \qquad \lambda_v = -\frac{\sin\beta_0}{a}\left(1 - 2\frac{t}{T}\right), \qquad \lambda_x = 0\,,$$

$$\lambda_y = -\frac{2\sin\beta_0}{aT}\,,$$

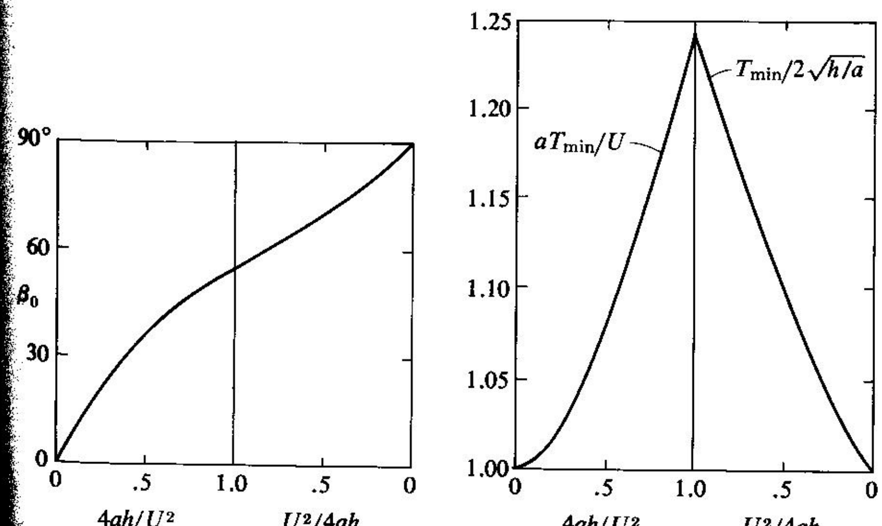


Figure 2.7.4. Initial thrust angle (β_0) and minimum time (T_{min}) as a function of $4ah/U^2$ for minimum-time transfer to a rectilinear path.

where the constants β_0 and c, as well as the final (minimum) time, T, are determined by the three final boundary conditions $v = 0$, $u = U$, $y = h$. *Show that* these relations may be put into the form

$$\frac{4ah}{U^2} = \frac{\tan\beta_0 \sec\beta_0 - \operatorname{logtan}[(\pi/4) + (1/2)\beta_0]}{\{\operatorname{logtan}[(\pi/4) + (1/2)\beta_0]\}^2},$$

$$\frac{aT}{U} = \frac{\tan\beta_0}{\operatorname{logtan}[(\pi/4) + (1/2)\beta_0]},$$

$$cT = 2\tan\beta_0 \Rightarrow \tan\beta = \left(1 - \frac{2t}{T}\right)\tan\beta_0 .$$

Clearly, the one dimensionless quantity, $4ah/U^2$ determines β_0 and, hence, also aT/U. These relationships are shown in Figure 2.7.4. Thrust-direction programs for various values of β_0 are shown in Figure 2.4.3.

Problem 10. *Minimum-time interception of a nonmaneuvering target* $(g = 0)$. Using the same equations of motion as in Problem 8, find the thrust-direction angle history, $\beta(t)$, to go to the origin, $x = y = 0$, in minimum time, starting from a given initial point, x_o, y_o, with initial velocity u_o, v_o. Assume constant-thrust acceleration a. Note that the final velocity is *not* specified, so this is an *interception* problem.

Problem 11. *Minimum-time rendezvous with a nonmaneuvering target* $(g = 0)$. This is the same as Problem 10 except that the final velocity is specified to be zero; that is, we have $u_f = v_f = 0$. This is a *rendezvous* problem. Note that the "bilinear tangent law" can be put into the form of a "linear tangent law":

$$\tan(\theta - \alpha) = \tan(\theta_f - \alpha) + c(T - t),$$

where α, θ_f, and c are parameters.

Problem 12. *Thrust-direction programming in a constant gravitational field.* If we align the y-axis in the opposite direction to the gravitational force, the only change from Problem 8 (without gravitational force) is in the vertical acceleration:

$$\dot{v} = a\sin\beta - g,$$

where g is the acceleration due to gravity. *Show that* the equations for the influence functions are unchanged, so that the "bilinear tangent law" is still the optimal law.

blem 13. *Minimum-time orbit injection* (g = *constant*). *Show that* the only change from Problem 8 (with $g = 0$) is the addition of a term $-gt$ to the vertical velocity v, and a term $-\frac{1}{2}gt^2$ to the vertical height, y. For the case of constant-thrust acceleration a, there are three quantities to be determined: the initial thrust-direction angle, β_0; the final thrust-direction angle, β_f; and the minimum time, T. *Show that* the three equations available for determining them are

$$v_f = 0 = \frac{a}{c}(\sec\beta_0 - \sec\beta_f) - gT,$$

$$y_f = h = \frac{a}{2c^2}\Big[(\tan\beta_0 - \tan\beta_f)\sec\beta_0 - (\sec\beta_0 - \sec\beta_f)\tan\beta_f - \log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta_f + \sec\beta_f}\Big] - \frac{1}{2}gT^2,$$

$$u_f = U = \frac{a}{c}\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta_f + \sec\beta_f},$$

where

$$c = \frac{\tan\beta_0 - \tan\beta_f}{T}, \qquad \tan\beta = \tan\beta_0 - ct .$$

Eliminating c and T among these equations, derive the following two equations in the two unknowns β_0 and β_f:

$$\frac{a}{g} = \frac{\tan\beta_0 - \tan\beta_f}{\sec\beta_0 - \sec\beta_f},$$

$$\frac{2ah}{U^2} = \left[\tan\beta_0\sec\beta_f - \tan\beta_f\sec\beta_0 - \log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta_f + \sec\beta_f}\right] \div \left[\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta_f + \sec\beta_f}\right]^2,$$

with

$$\frac{aT}{U} = (\tan\beta_0 - \tan\beta_f)\Big/\left[\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta_f + \sec\beta_f}\right].$$

Clearly, the quantities ah/U^2 and a/g determine β_0, β_f, aT/U, and cT.

erical Example. Figure 2.7.5 shows two example trajectories with $/g = 3$, calculated for lunar takeoffs, using the constant g approximaon. The moon's surface gravity is 5.3 ft sec^{-2}, and the radius of the oon is 938 nautical miles. Minimum-time ascent paths are shown for $h = 100$ nautical miles and $h = 50{,}000$ ft, the final velocity being circular-satellite velocity in the first case and slightly higher than circular-satellite velocity in the second case. The "characteristic velocity," ΔV_c, required is simply aT for $a =$ constant. For comparison, the impulsive injection into a 100 n mi orbit (Hohman transfer) requires a characteristic velocity of 5,780 ft sec^{-1} (5,640 at the moon's surface and 140 ft sec^{-1} at apolune). The velocity at the end of the 50,000-ft ascent is such that the spacecraft will coast to 100 n mi on the other side of the moon; there, an injection impulse of 464 ft sec^{-1} is required to put it into circular orbit at $h = 100$ n mi making the total $\Delta V_c = 6{,}584$ ft sec^{-1}. Note that the lunar surface is approximated as a parabola, which significantly extends the usefulness of the constant g approximation.

Obviously, the minimum-time "soft-landing" path with constant a (with down-range *not* specified) is the same as the minimum-time orbit-injection path run backwards.

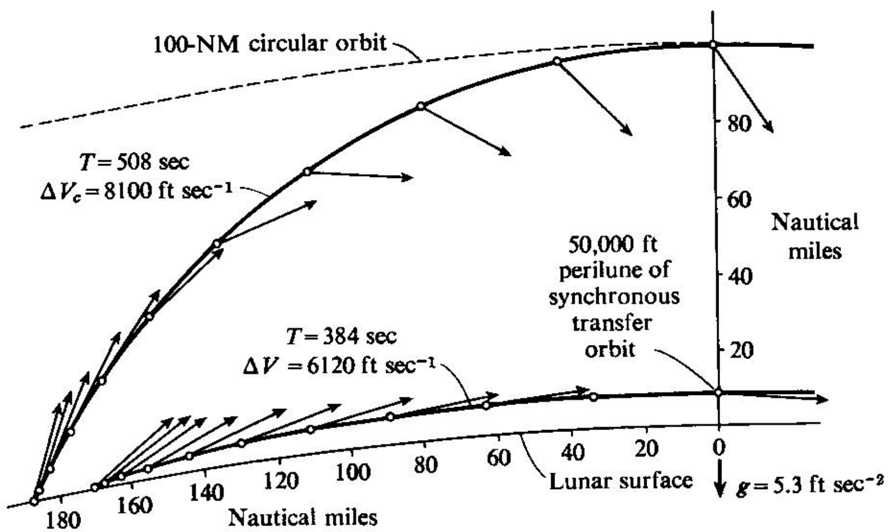


Figure 2.7.5. Minimum-time lunar take-offs (or landings) with constant-thrust acceleration = 15.9 ft sec^{-2} (thrust direction shown every tenth of total time).

Problem 14. (a) In the two-dimensional xt-plane, determine the extremal curve of stationary length which starts on the circle $x^2 + t^2 - 1 = 0$ and terminates on the line $t = T = 2$.

(b) Solve the same problem as (a) but consider that the termination is on the line $-x + t = 2\sqrt{2}$.

[NOTE: Parts (a) and (b) are *NOT* to be solved by inspection.]

Continuous systems; functions of the state variables specified at an unspecified terminal time, including minimum-time problems†

We again consider the performance index

$$J = \phi[x(t_f),t_f] + \int_{t_0}^{t_f} L[x(t),u(t),t]\,dt. \qquad (2.8.1)$$

We adjoin the constraints

$$\psi[x(t_f),t_f] = 0 \qquad (\psi \text{ a } q\text{-vector function}) \qquad (2.8.2)$$

and the system differential equations

$$\dot{x} = f[x(t),u(t),t], \qquad t_0 \text{ given} \qquad (2.8.3)$$

to the performance index with Lagrange multipliers ν and $\lambda(t)$, as follows:

$$J = [\phi + \nu^T\psi]_{t=t_f} + \int_{t_0}^{t_f} \{L(x,u,t) + \lambda^T[f(x,u,t) - \dot{x}]\}\,dt. \qquad (2.8.4)$$

The Hamiltonian is defined as

$$H = L(x,u,t) + \lambda^T(t) f(x,u,t). \qquad (2.8.5)$$

The *differential* of (2.8.4), taking into account differential changes in the terminal time, t_f, is

$$dJ = \left(\left(\frac{\partial\Phi}{\partial t} + L\right)dt + \frac{\partial\Phi}{\partial x}dx\right)_{t=t_f} + \int_{t_0}^{t_f}\left(\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u - \lambda^T\delta\dot{x}\right)dt - L|_{t=t_0}\,dt_0, \qquad (2.8.6)$$

where

$$\Phi = \phi + \nu^T\psi. \qquad (2.8.7)$$

Integrating by parts and using $\delta x = dx - \dot{x}\,dt$, we have

$$dJ = \left(\frac{\partial\Phi}{\partial t} + L + \lambda^T\dot{x}\right)_{t=t_f} dt_f + \left[\left(\frac{\partial\Phi}{\partial x} - \lambda^T\right)dx\right]_{t=t_f} + (\lambda^T\delta x)_{t=t_0} + \int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u\right]dt - L|_{t=t_0}\,dt_0. \qquad (2.8.8)$$

We now *choose* the functions $\lambda(t)$ to make the coefficients of $\delta x(t)$, $dx(t_f)$, and dt_f vanish (if t_f is not prescribed):

† J. V. Breakwell, "The Optimization of Trajectories," *SIAM Journal*, Vol. 7, 1959.

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\lambda^T \frac{\partial f}{\partial x} - \frac{\partial L}{\partial x}, \tag{2.8.9}$$

$$\lambda^T(t_f) = \left(\frac{\partial \Phi}{\partial x}\right)_{t=t_f} = \left(\frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right)_{t=t_f}, \tag{2.8.10}$$

$$\left(\frac{\partial \Phi}{\partial t} + L + \lambda^T \dot{x}\right)_{t=t_f} = \left(\frac{d\Phi}{dt} + L\right)_{t=t_f} = 0, \tag{2.8.11}$$

where

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x}\dot{x}.$$

As a result of this choice of $\lambda(t)$, (2.8.8) is simplified to

$$dJ = \int_{t_o}^{t_f} \frac{\partial H}{\partial u} \delta u \, dt + \lambda^T(t_o)\, dx\,(t_o) - H(t_o)\, dt_o. \tag{2.8.12}$$

Clearly, as before, $\lambda^T(t_o)$ is the ***influence*** vector on J of changes in initial conditions $\delta x(t_o)$, while $\partial H/\partial u$ is a set of ***impulse-response functions*** indicating how J would change as a result of unit impulses in the controls at any point in the interval $t_o \le t \le t_f$.

For a stationary value of J, clearly, we have

$$\frac{\partial H}{\partial u} = \lambda^T \frac{\partial f}{\partial u} + \frac{\partial L}{\partial u} = 0, \qquad t_o \le t \le t_f, \tag{2.8.13}\dagger$$

and if a component $x_k(t_o)$ is not specified, we have $\lambda_k(t_o) = 0$.

For ***minimum time***, $t_f - t_o$, we may let $\phi[x(t_f)t_f] = 0$ and $L = 1$, so that condition (2.8.11) becomes

$$\left(\frac{d\Phi}{dt} + 1\right)_{t=t_f} = 0. \tag{2.8.14}$$

As in Section 2.6, the q constants ν must be determined to satisfy the terminal constraints (2.8.2). The condition (2.8.14) is the extra condition needed to determine the final time t_f.

In summary, a set of necessary conditions for J to have a stationary value is

$$\dot{x} = f(x,u,t) \tag{2.8.15}$$

$$\dot{\lambda} = -\left(\frac{\partial H}{\partial x}\right)^T = -\left(\frac{\partial f}{\partial x}\right)^T \lambda - \left(\frac{\partial L}{\partial x}\right)^T \tag{2.8.16}$$

†An argument regarding admissibility, similar to the one made in Section 2.7, must be made to justify (2.8.13).

$$0 = \left(\frac{\partial H}{\partial u}\right)^T = \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial L}{\partial u}\right)^T \tag{2.8.17}$$

$$x_k(t_o) \text{ given}, \qquad \text{or} \qquad \lambda_k(t_o) = 0 \tag{2.8.18}$$

$$\lambda(t_f) = \left(\frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right)^T_{t=t_f} \tag{2.8.19}$$

$$\Omega = \left[\frac{\partial \phi}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} + \left(\frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right) f + L\right]_{t=t_f} = 0 \tag{2.8.20}$$

$$\psi[x(t_f)t_f] = 0 \tag{2.8.21}$$

The optimality condition (2.8.17) determines the m-vector $u(t)$. The solution to the $2n$ differential equations (2.8.15) and (2.8.16) and the choice of the $q + 1$ parameters ν and t_f are determined by the $2n + 1 + q$ boundary conditions (2.8.18)–(2.8.21). Needless to say, this boundary-value problem is, in general, not very easy to solve.

Notice, however, that if we were to specify ν instead of ψ, and t_f instead of Ω, (2.8.18) and (2.8.19) provide $2n$ boundary conditions for a fixed-terminal-time, two-point boundary-value problem of order $2n$. By changing values of ν and t_f, it ***may*** be possible to bring ψ and Ω to zero at $t = t_f$ (see Chapter 7, Section 3).