# MATHEMATICA BY EXAMPLE

4TH EDITION

MARTHA L. ABELL
JAMES P. BRASELTON

INCLUDES CD-ROM

Wolfram Mathematica 6

# Mathematica by Example

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## Fourth Edition

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# Preface

*Mathematica by Example* bridges the gap that exists between the very elementary handbooks available on Mathematica and those reference books written for the advanced Mathematica users. This book is an appropriate reference for all users of Mathematica and, in particular, for beginning users such as students, instructors, engineers, businesspeople, and other professionals first learning to use Mathematica. This book introduces the very basic commands and includes typical examples of applications of these commands. In addition, the text also includes commands useful in areas such as calculus, linear algebra, business mathematics, ordinary and partial differential equations, and graphics. In all cases, however, examples follow the introduction of new commands. Readers from the most elementary to advanced levels will find that the range of topics covered addresses their needs.

Taking advantage of Version 6 of Mathematica, *Mathematica by Example*, Fourth Edition, introduces the fundamental concepts of Mathematica to solve typical problems of interest to students, instructors, and scientists. The fourth edition is an extensive revision of the text. Features that make this edition easy to use as a reference and as useful as possible for the beginner include the following:

1. **Version 6 compatibility.** All examples illustrated in this book were completed using Version 6 of Mathematica. Although many computations can continue to be carried out with earlier versions of Mathematica, we have taken advantage of the new features in Version 6 as much as possible.
2. **Applications.** New applications, many of which are documented by references from a variety of fields, especially biology, physics, and engineering, are included throughout the text.
3. **Detailed table of contents.** The table of contents includes all chapter, section, and subsection headings. Along with the comprehensive index, we hope that users will be able to locate information quickly and easily.
4. **Additional examples.** We have considerably expanded the topics throughout the book. The results should be more useful to instructors, students, businesspeople, engineers, and other professionals using Mathematica on a variety of platforms. In addition, several sections have been added to make it easier for the user to locate information.

5. **Comprehensive index.** In the index, mathematical examples and applications are listed by topic or name, and commands along with frequently used options are also listed. Particular mathematical examples as well as examples illustrating how to use frequently used commands are easy to locate. In addition, commands in the index are cross-referenced with frequently used options. Functions available in the various packages are cross-referenced both by package and alphabetically.
6. **CD included.** All Mathematica code that appears in this edition is included on the CD packaged with the text.
7. **Exercises at the end of each chapter.** Each chapter of this edition concludes with a section of exercises that range from easy to difficult.

We began *Mathematica by Example* in 1990 and the first edition was published in 1991. Back then, we were on top of the world using Macintosh IIcx's with 8 megs of RAM and 40-meg hard drives. We tried to choose examples that we thought would be relevant to beginning users—typically in the context of mathematics encountered in the undergraduate curriculum. Those examples could also be carried out by Mathematica in a timely manner on a computer as powerful as a Macintosh IIcx.

Now, we are on the top of the world with iMacs with dual Intel processors complete with 2 gigs of RAM and 250-gig hard drives, which will almost certainly be obsolete by the time you read this. The examples presented in this book continue to be the ones that we think are most similar to the problems encountered by beginning users and are presented in the context of someone familiar with mathematics typically encountered by undergraduates. However, for this edition of *Mathematica by Example*, we have taken the opportunity to expand on several of our favorite examples because the machines now have the speed and power to explore them in greater detail.

Other improvements to the fourth edition include the following:

1. Throughout the text, we have attempted to eliminate redundant examples and added several interesting ones. The following changes are especially worth noting:

   **(a)** In Chapter 2, we have increased the number of parametric and polar plots in two and three dimensions. For a sample, see Examples 2.3.17, 2.3.18, 2.3.21, and 2.3.23.

   **(b)** In Chapter 3, we have improved many examples by adding additional graphics that capitalize on Mathematica's enhanced three-dimensional graphics capabilities. See especially Example 3.3.15.

(c) Chapter 4 contains several examples illustrating various techniques for quickly creating plots of bifurcation diagrams, Julia sets, and the Mandelbrot set.
(d) The graphics discussion in Chapter 5 has been increased considerably with the addition of Section 5.6, Matrices and Graphs, and the improvement of many of the examples regarding curves and surfaces in space. We have also added a brief discussion regarding the Frenet frame field and curvature and torsion of curves in space. See Examples 5.5.11 and 5.5.12.
(e) In Chapter 6, we have taken advantage of the new Manipulate function to illustrate a variety of situations and expand on many examples throughout the chapter. For example, see Example 6.2.5 for a comparison of solutions of nonlinear equations to their corresponding linear approximations.

2. We have included references that we find particularly interesting in the Bibliography, even if they are not specific Mathematica-related texts. A comprehensive list of Mathematica-related publications can be found on the Wolfram website:

   http://store.wolfram.com/catalog/books

   Also, be sure to investigate, use, and support Wolfram's MathWorld, which is simply an amazing web resource for mathematics, Mathematica, and other information.

Finally, we express our appreciation to those who assisted in this project. We express appreciation to our editor, Lauren Schultz, our production editor, Mara Vos-Sarmiento, and our project manager, Phil Bugeau, at Elsevier for providing a pleasant environment in which to work. In addition, Wolfram Research, especially Maryka Baraka, has been most helpful in providing us up-to-date information about Mathematica. Finally, we thank those close to us, especially Imogene Abell, Lori Braselton, Ada Braselton, and Mattie Braselton, for enduring with us the pressures of meeting a deadline and for graciously accepting our demanding work schedules. We certainly could not have completed this task without their care and understanding.

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*December 2007*

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# CHAPTER 1
# Getting Started

## 1.1 INTRODUCTION TO MATHEMATICA

Mathematica, first released in 1988 by Wolfram Research, Inc. http://www.wolfram.com/, is a system for doing mathematics on a computer. Mathematica combines symbolic manipulation, numerical mathematics, outstanding graphics, and a sophisticated programming language. Because of its versatility, Mathematica has established itself as the computer algebra system of choice for many computer users. Among the more than 1 million users of Mathematica, 28% are engineers, 21% are computer scientists, 20% are physical scientists, 12% are mathematical scientists, and 12% are business, social, and life scientists. Two-thirds of the users are in industry and government, and there are a small (8%) but growing number of student users. However, due to its special nature and sophistication, beginning users need to be aware of the special syntax required to make Mathematica perform in the way intended. You will find that calculations and sequences of calculations most frequently used by beginning users are discussed in detail along with many typical examples. In addition, the comprehensive index not only lists a variety of topics but also cross-references commands with frequently used options. *Mathematica by Example* serves as a valuable tool and reference to the beginning user of Mathematica as well as to the more sophisticated user, with specialized needs.

For information, including purchasing information, about Mathematica, contact:

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100 Trade Center Drive
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## A Note Regarding Different Versions of Mathematica

With the release of Version 6 of Mathematica, many new functions and features have been added to Mathematica. We encourage users of earlier versions of Mathematica to update to Version 6 as soon as possible. All examples in *Mathematica by Example*, fourth edition, were completed with Version 6. In most cases, the same results will be obtained if you are using Version 5.0 or later, although the appearance of your results will almost certainly differ from that presented here. However, particular features of Version 6 are used, and in those cases, of course, these features are not available in earlier versions. If you are using an earlier or later version of Mathematica, your results may not appear in a form identical to those found in this book: Some commands in Version 5 are not available in earlier versions of Mathematica; in later versions, some commands will certainly be changed, new commands added, and obsolete commands removed. For details regarding these changes, please refer to the **Documentation Center**. You can determine the version of Mathematica you are using during a given Mathematica session by entering either the command $Version or the command $VersionNumber. In this text, we assume that Mathematica has been correctly installed on the computer you are using. If you need to install Mathematica on your computer, please refer to the documentation that came with the Mathematica software package.

On-line help for upgrading older versions of Mathematica and installing new versions is available at the Wolfram Research, website http://www.wolfram.com/.

Details regarding what is different in Mathematica 6 from previous versions of Mathematica can be found at

http://www.wolfram.com/products/mathematica/newin6



Also, when you go to the **Documentation Center** (under **Help** in the Mathematica menu) you can choose **New in 6** to see the major differences. In addition, the upper right-hand corner of the main help page for each function will tell you if it is new in Version 6 ( ) or has been updated in Version 6 ( ).

## 1.1.1 Getting Started with Mathematica

We begin by introducing the essentials of Mathematica. The examples presented are taken from algebra, trigonometry, and calculus topics that you are familiar with to assist you in becoming acquainted with the Mathematica computer algebra system.

We assume that Mathematica has been correctly installed on the computer you are using. If you need to install Mathematica on your computer, please refer to the documentation that came with the Mathematica software package.

Start Mathematica on your computer system. Using Windows or Macintosh mouse or keyboard commands, activate the Mathematica program by selecting the Mathematica icon or an existing Mathematica document (or notebook) and then clicking or double-clicking on the icon.


If you start Mathematica by selecting the Mathematica icon, a blank untitled notebook is opened, as illustrated in the following screen shot,


along with the **Startup Palette**.


When you start typing, the thin black horizontal line near the top of the window is replaced by what you type.

Once Mathematica has been started, computations can be carried out immediately. Mathematica commands are typed and the black horizontal line is replaced by the command, which is then evaluated by pressing **Enter**. Note that pressing **Enter** or **Return** evaluates commands and pressing **Shift-Return** yields a new line. Output is displayed below input. We illustrate some of the typical steps involved in working with Mathematica in the calculations that follow. In each case, we type the command and press **Enter**. Mathematica evaluates the command, displays the result, and inserts a new horizontal line after the result. For example, typing N[, then pressing the $\pi$ key on the **Basic Math Input** palette, followed by typing, 50] and pressing the enter key

With some operating systems, **Enter** evaluates commands and **Return** yields a new line.

The **Basic MathInput** palette:


**N[π, 50]**

3.1415926535897932384626433832795028841971693993751

returns a 50-digit approximation of $\pi$. Note that both $\pi$ and Pi represent the mathematical constant $\pi$, so entering N[Pi, 50] returns the same result. For basic computations, enter them into Mathematica in the same way as you would with most scientific calculators.

The next calculation can then be typed and entered in the same manner as the first. For example, entering

```
Plot[{Sin[x], 2Cos[2x]}, {x, 0, 3π},
        PlotStyle → {GrayLevel[0], GrayLevel[0.5]}]
```

graphs the functions $y = \sin x$ and $y = 2\cos 2x$ and on the interval $[0, 3\pi]$ shown in Figure 1.1.

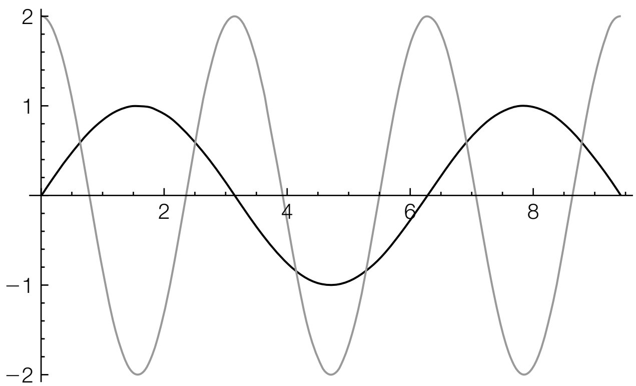



**FIGURE 1.1**

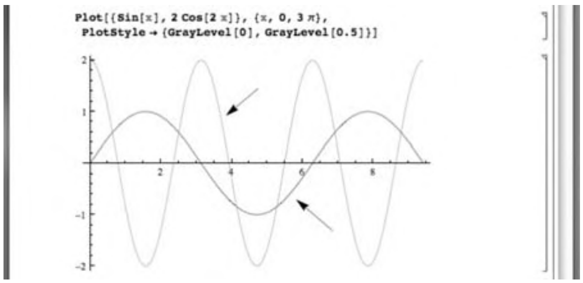
A two-dimensional plot

With Mathematica 6, you can easily add explanation to the graphic. Go to **Graphics** in the main menu, followed by **Drawings Tools**. You can use the **Drawing Tools** palette 

to quickly enhance a graphic.

In this case we select the **Arrow** button 

to add two arrows

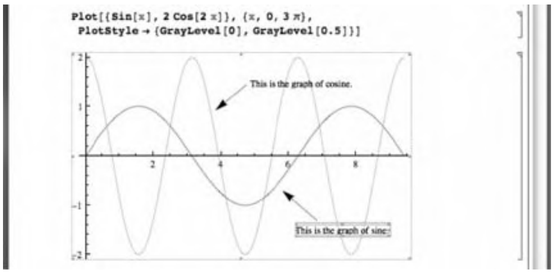


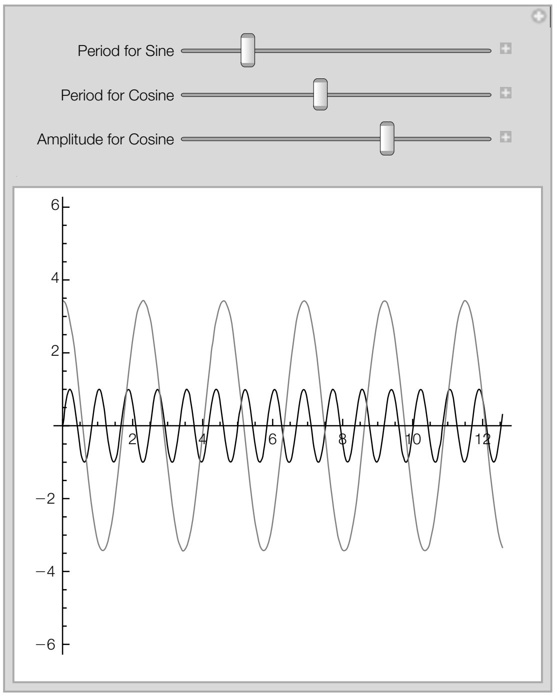

and then the **A** button 

to add some text to help identify each plot. The various elements can be modified by clicking on them and moving and/or typing as needed.

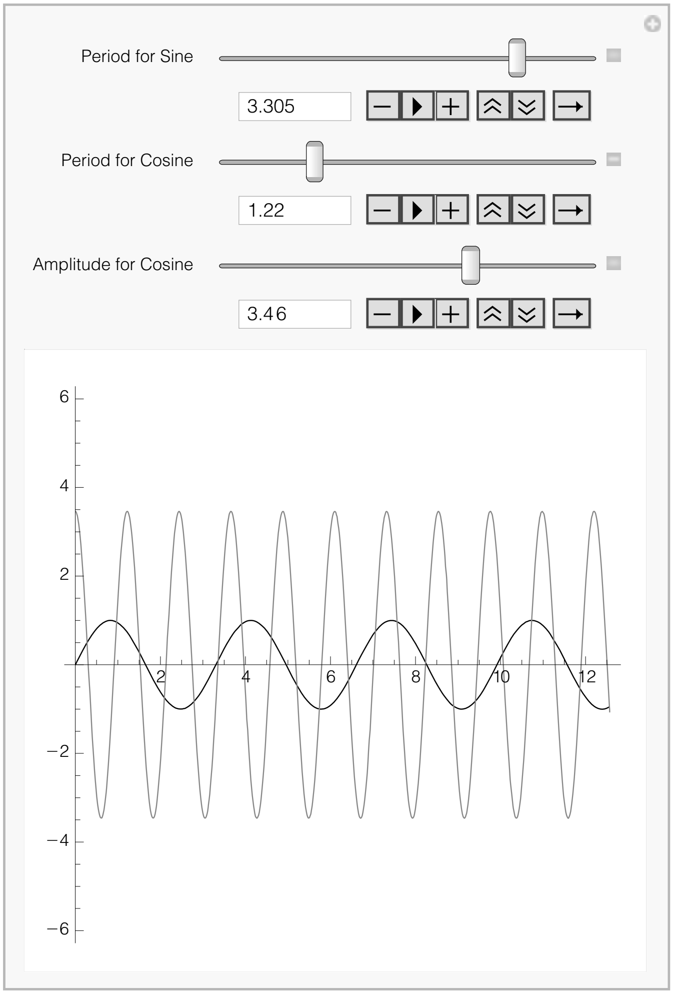
With Mathematica 6, you can use Manipulate to illustrate how changing various parameters affects a given function or functions. With the following command, we illustrate how *a* and *b* affect the period of sine and cosine and *c* affects the amplitude of cosine:

```
Manipulate[Plot[{Sin[2Pi/ax], cCos[2Pi/bx]}, {x, 0, 4π},
    PlotStyle → {GrayLevel[0], GrayLevel[.5]}, PlotRange → {− 4π/2, 4π/2},
        AspectRatio → 1], {{a, 2Pi, "Period for Sine"}, .1, 4},
            {{b, 2Pi, "Period for Cosine"}, .1, 5},
        {{c, 2Pi, "Amplitude for Cosine"}, .1, 5}]
```




Use the slide bars to adjust the values of the parameters or click on the + button to expand the options to enter values explicitly or generate an animation.




Notice that every Mathematica command begins with capital letters and the argument is enclosed by square brackets [. . .].

Use Plot3D to generate basic three-dimensional plots. Entering

**Plot3D[Sin[x + Cos[y]], {x, 0, 4π}, {y, 0, 4π}, Ticks → None, Boxed → False, Axes → None]**

graphs the function $z = \sin(x + \cos y)$ for $0 \le x \le 4\pi$ and $0 \le y \le 4\pi$ shown in Figure 1.2. To view the image from different angles, use the mouse to select the graphic and then drag to the desired angle.

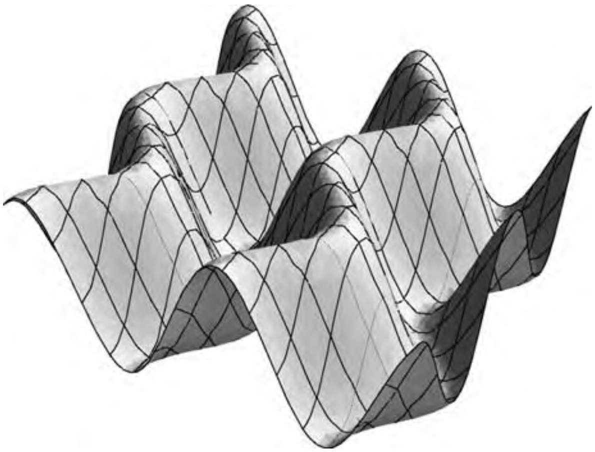
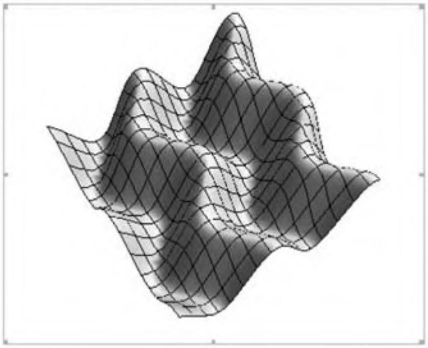


**FIGURE 1.2**

A three-dimensional plot

```
Plot3D[Sin[x + Cos[y]], {x, 0, 4 π}, {y, 0, 4 π}, Ticks → None,
 Boxed → False, Axes → None]
```



To type $x^3$ in Mathematica, press the  on the **Basic Math Input** palette, type *x* in the base position, and then click (or tab to) the exponent position and type 3. Use the **esc** key, tab button, or mouse to help you place or remove the cursor from its current location.

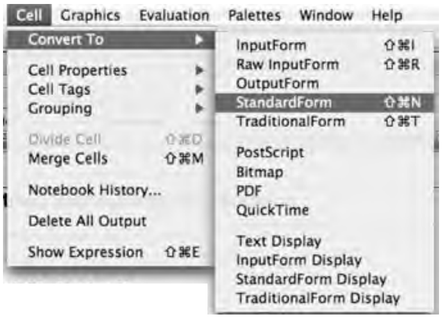
Notice that all three of the following commands

```
Solve[x^3 - 2 x + 1 == 0]
{{x → 1}, {x → 1/2 (-1 - √5)}, {x → 1/2 (-1 + √5)}}

Solve[x³ - 2 x + 1 == 0]
{{x → 1}, {x → 1/2 (-1 - √5)}, {x → 1/2 (-1 + √5)}}

Solve[x³ - 2x + 1 == 0]
{{x → 1}, {x → 1/2 (-1 - √5)}, {x → 1/2 (-1 + √5)}}
```

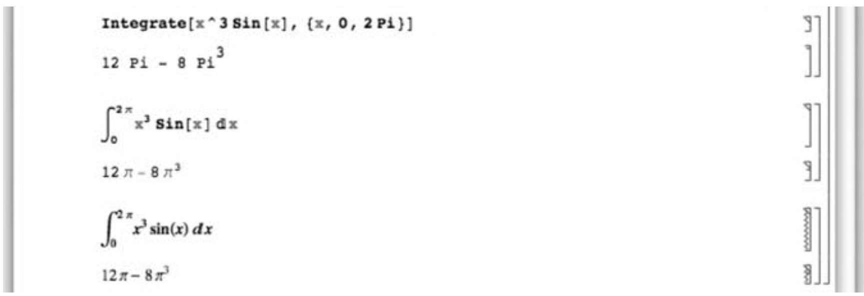
solve the equation $x^3 - 3x + 1 = 0$ for $x$.

In the first case, the input and output are in **StandardForm**; in the second case, the input and output are in **InputForm**; and in the third case, the input and output are in **TraditionalForm**. Move the cursor to the Mathematica menu,


select **Cell**, and then **ConvertTo**, as illustrated in the following screen shot:




You can change how input and output appear by using **ConvertTo** or by changing the default settings. Moreover, you can determine the form of input/output by looking at the cell bracket that contains the input/output. For example, even though all three of the following commands look different, all three evaluate $\int_0^{2\pi} x^3 \sin x\, dx$:




In the first calculation, the input is in **InputForm** and the output in **OutputForm**; in the second, the input and output are in **StandardForm**; and in the third, the input and output are in **TraditionalForm**. Throughout *Mathematica by Example*, fourth edition, we display input and output using **InputForm** (for input) or **StandardForm** (for output), unless otherwise stated.

To enter code in **StandardForm**, we often take advantage of the **Basic Math Input** palette, which is accessed by going to **Palettes** under the Mathematica menu and then selecting **BasicMathInput**. See Figure 1.3.

Use the buttons to create templates and enter special characters. Alternatively, you can access a complete list of typesetting shortcuts from Mathematica help at guide/MathematicalTypesetting in the **Documentation Center**.

Mathematica sessions are terminated by entering Quit[ ] or by selecting **Quit** from the **File** menu, or by using a keyboard shortcut, such as **command-Q**, as with other applications. They can be saved by referring to **Save** from the **File** menu.

Mathematica allows you to save notebooks (as well as combinations of cells) in a variety of formats, in addition to the standard Mathematica format.

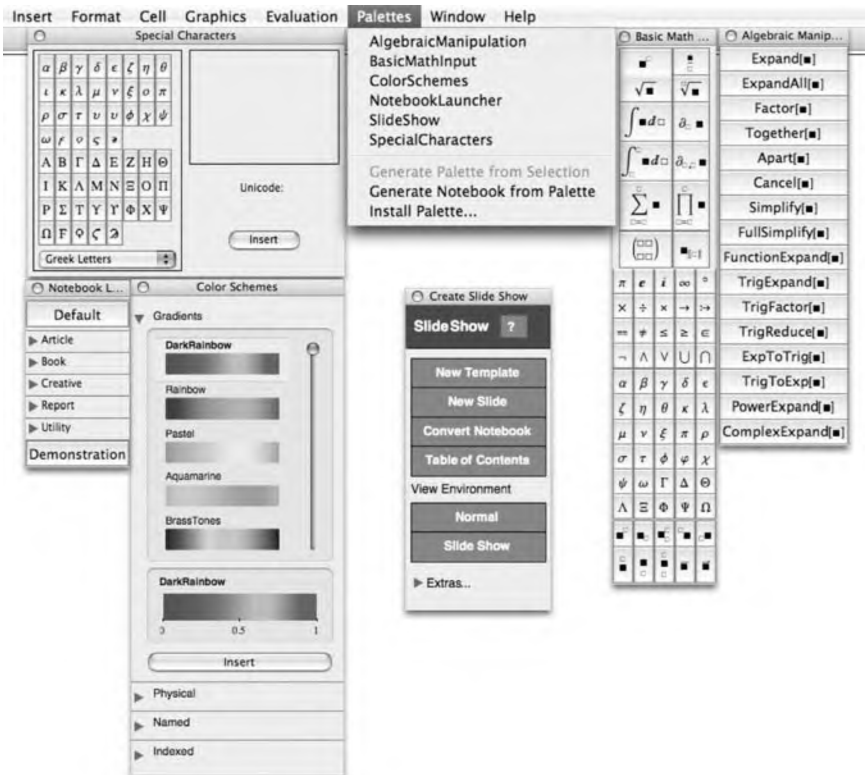
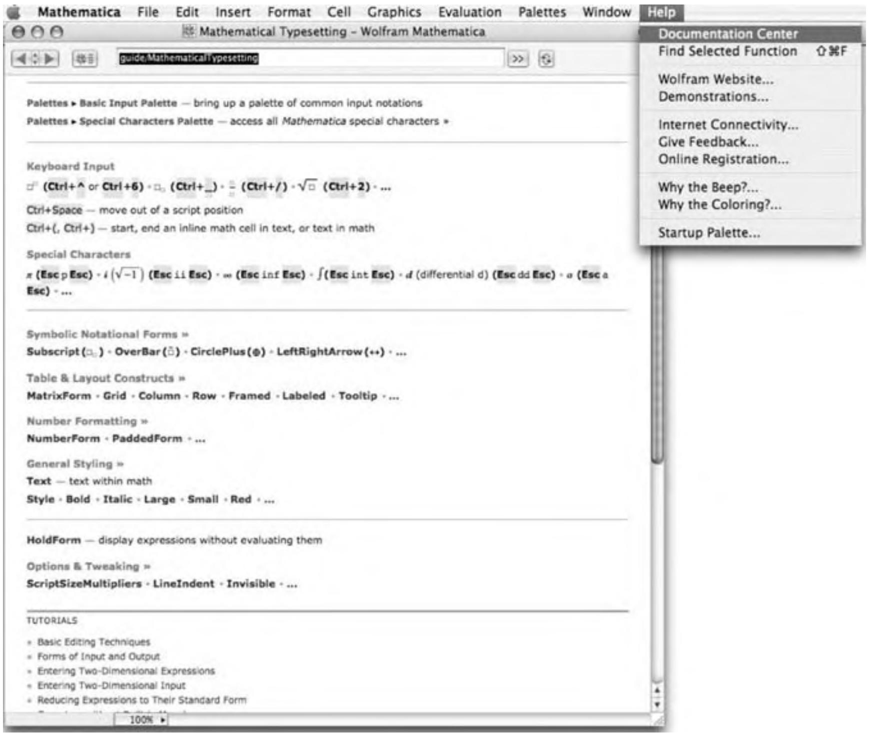
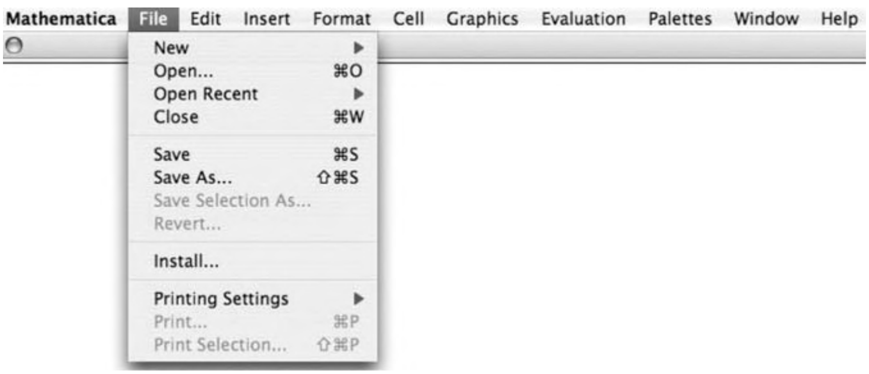



**FIGURE 1.3**

Mathematica 6 palettes




**Remark 1.1** Input and text regions in notebooks can be edited. Editing input can create a notebook in which the mathematical output does not make sense in the sequence it appears. It is also possible to simply go into a notebook and alter input without doing any recalculation. This also creates misleading notebooks. Hence, common sense and caution should be used when editing the input regions of notebooks. Recalculating all commands in the notebook will clarify any confusion.

## Preview

In order for the Mathematica user to take full advantage of this powerful software, an understanding of its syntax is imperative. The goal of *Mathematica by Example* is to introduce the reader to the Mathematica commands and sequences of commands most frequently used by beginning users. Although the rules of Mathematica syntax are far too numerous to list here, knowledge of the following five rules equips the beginner with the necessary tools to start using the Mathematica program with little trouble.

## Five Basic Rules of Mathematica Syntax

1. The arguments of *all* functions (both built-in ones and ones that you define) are given in brackets [. . .]. Parentheses (. . .) are used for grouping operations; vectors, matrices, and lists are given in braces {. . .}; and double square brackets [[. . .]] are used for indexing lists and tables.
2. Every word of a built-in Mathematica function begins with a capital letter.
3. Multiplication is represented by a ∗ or space between characters. Enter 2∗x∗y or 2x y to evaluate $2xy$ *not* 2xy.
4. Powers are denoted by a ^. Enter (8∗x^3)^(1/3) to evaluate $(8x^3)^{1/3} = 8^{1/3}(x^3)^{1/3} = 2x$ instead of 8x^1/3, which returns 8x/3.
5. Mathematica follows the order of operations *exactly*. Thus, entering (1 + x)^1/x returns $\frac{(1+x)^1}{x}$, whereas (1 + x)^(1/x) returns $(1+x)^{1/x}$. Similarly, entering x^3x returns $x^3 \cdot x = x^4$, whereas entering x^(3x) returns $x^{3x}$.

**Remark 1.2** If you get no response or an incorrect response, you may have entered or executed the command incorrectly. In some cases, the amount of memory allocated to Mathematica can cause a crash. Like people, Mathematica is not perfect and errors can occur.

# 1.2 LOADING PACKAGES

Although Mathematica contains many built-in functions, some other functions are contained in **packages** that must be loaded separately. Experienced users can create their own packages; other packages are available from user groups and **MathSource**, which electronically distributes Mathematica-related products. For information about MathSource, visit

http://library.wolfram.com/infocenter/MathSource or send the message “help” to mathsource@wri.com. If desired, you can purchase MathSource on a CD directly from Wolfram Research, or you can access MathSource from the Wolfram Research website.

With Mathematica 6, many packages included with previous versions of Mathematica have been made obsolete because their functionality has been incorporated into Mathematica, combined into a new package, or eliminated altogether. In addition to **MathSource**, you should also think about investigating Wolfram’s **MathWorld** website.

## 1.2.1 Packages Included with Older Versions of Mathematica

Packages are loaded by entering the command <<directory`packagename`, Needs[directory`packagename`], <<packagename` or Needs[packagename`], where **directory** is the location of the package **packagename**. Entering the command <<directory`Master` makes all the functions contained in all the packages in **directory** available. In this case, each package need not be loaded individually.

Over time, Wolfram Research expects that packages will become obsolete and that Mathematica will automatically load functions that are needed at startup or when called.

For most teachers and students, a function like $f(x) = (x-1)^{1/3}(x+1)^{2/3}$ is a real-valued function for all values of $x$. Nevertheless, when we ask Mathematica to plot the function with Plot,

```
Plot[(x – 1) ^ (1 / 3)(x + 1) ^ (2 / 3), {x, –2, 2}, PlotStyle → GrayLevel[0]]
```

we see in Figure 1.4 that Mathematica does not compute real values for $x$ values between $-1$ and 1 because complex roots are selected by Mathematica for the $x$ values between $-1$ and 1, which is where the values of $f(x)$ are negative.

Generally, when Mathematica computes the odd root of a negative number, it returns a complex number. (Note that % refers to the previous

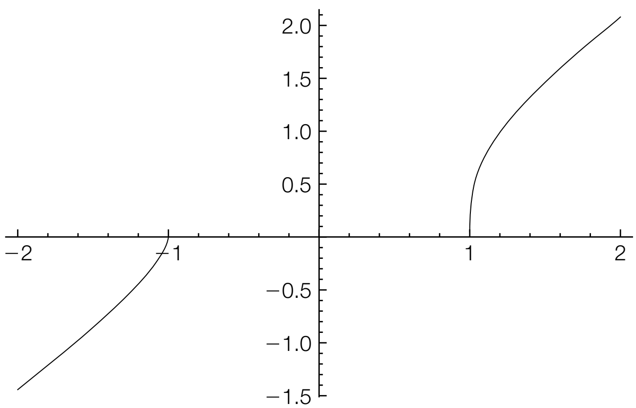



**FIGURE 1.4**

When computing odd roots of negative numbers, Mathematica returns complex values output; N[x] returns a numerical approximation of *x*, and Abs[x] returns the absolute value of the number *x*.)

**(−8) ^ (1/3)**

$2(-1)^{1/3}$

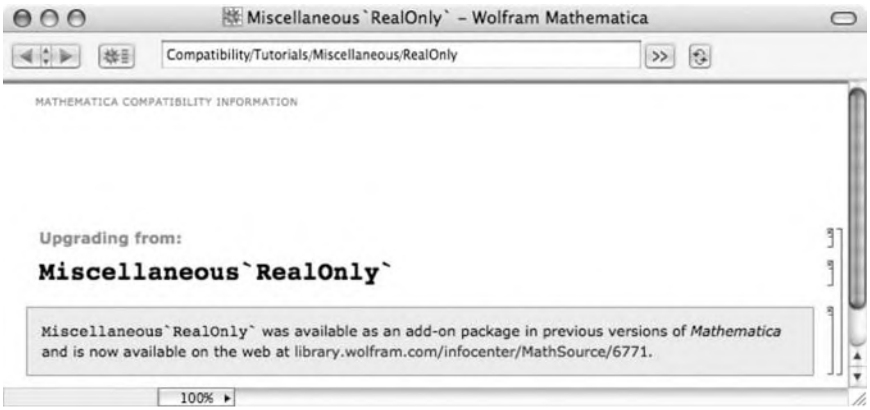
**N[%]**

1. + 1.73205i

**Abs[%]**

2.

To instruct Mathematica to select the *real* third root, we load the **RealOnly** package that is contained in the **Miscellenous** directory. Note that the **RealOnly** package has been included with many versions of Mathematica but *not* included with Mathematica 6. If you need to obtain the **RealOnly** package, you need to download it from the Wolfram website.




After loading the package, when we reenter the Plot command, Mathematica generates the expected plot, which is shown in Figure 1.5.

**<< Miscellaneous`RealOnly`**
**Plot[(x − 1) ^ (1/3)(x + 1) ^ (2/3), {x, −2, 2}, PlotStyle → GrayLevel[0]]**

## 1.2.2 Loading New Packages

One new package included with Mathematica 6 is **VectorFieldPlots**, which replaces several packages in previous versions of Mathematica.

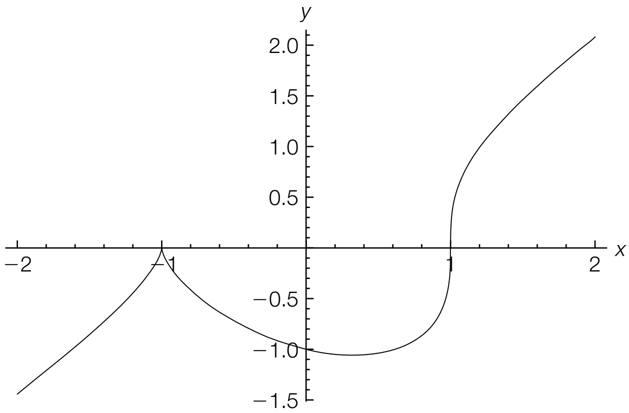



**FIGURE 1.5**

We see the real values of $f(x)$ for $-1 < x < 1$ after loading the **RealOnly** package


**Example 1.2.1** The differential equation $dy/dx = \cos(y/x)$ is a **first-order homogeneous differential equation**. Using DSolve, we see that the solution contains an integral that does not have a known closed form. The result returned by DSolve indicates

that the integral curves for the differential equation satisfy the equation contained within the brackets in the output:

```
DSolve[y'[x] == Cos[y[x]/x], y[x], x]
```

Solve::tdep: The equations appear to involve the variables to be solved for in an essentially nonalgebraic way. ⟩⟩

$$Solve\left[\int_1^{\frac{y[x]}{x}} \frac{1}{-\text{Cos}[K[1]] + K[1]}\, dK[1] == C[1] - \text{Log}[x], y[x]\right]$$

For a differential equation like this, even the function $g(x,y) = \int_1^{y/x} \frac{1}{t - \cos t}\, dt + \ln|x|$ is difficult to evaluate for particular values of $x$ and $y$, so generating a plot of the level curves of $g(x,y) = C$ (the integral curves for the differential equation) for various values of $C$ is challenging.

To see how the solutions of the differential equation behave, we plot a **direction field** or **slope field** for the equation. For this equation, the slope of a solution at $(x,y)$ satisfies $dy/dx = \cos(y/x)$. A direction field for the equation is generated by selecting a grid of $(x,y)$ points and then plotting line segments at those points with slope $dy/dx = \cos(y/x)$. With Mathematica, we can do so with the VectorFieldPlot function that is contained in the **VectorFieldPlots** package. First, we load the package with

```
<< VectorFieldPlots`;
```

Now that the package has been loaded, you can can use ? or Options to obtain information about the commands contained in the package. Finally, we generate a slope field for the equation with

```
p1 = VectorFieldPlot[{1, f[x, y]}, {x, -2Pi, 2Pi}, {y, -2Pi, 2Pi},
    PlotPoints → 25];
Show[p1, Axes → Automatic, AxesOrigin → {0, 0}]
```

Note that Mathematica returns several error messages due to the division by $0$ in the $y/x$ term that are not displayed here. The plot is displayed in Figure 1.6. From the slope field, we see that solutions of the differential equation can behave quite strangely near $x = 0$.

## 1.3 GETTING HELP FROM MATHEMATICA

Becoming competent with Mathematica can take a serious investment of time. Hopefully, messages that result from syntax errors are viewed light-heartedly. Ideally, instead of becoming frustrated, beginning Mathematica users will find it challenging and fun to locate the source of errors. Frequently, Mathematica's error messages indicate where the error(s) has occurred. In this process, it is natural that you will become more proficient

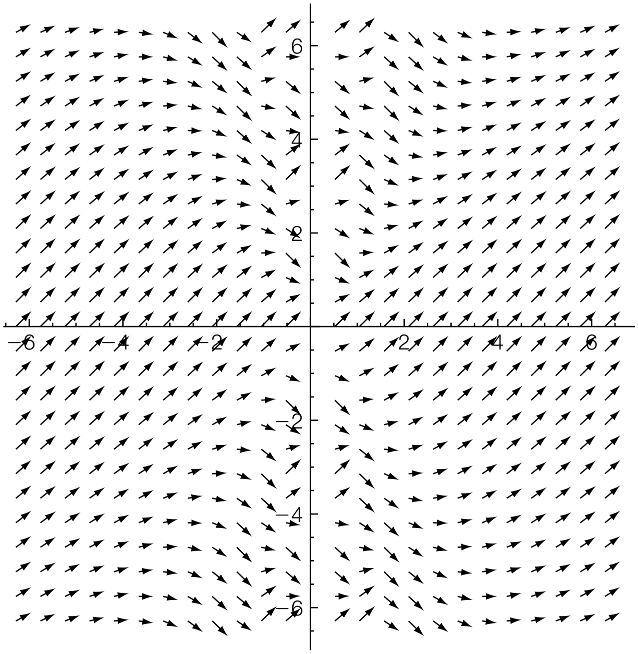


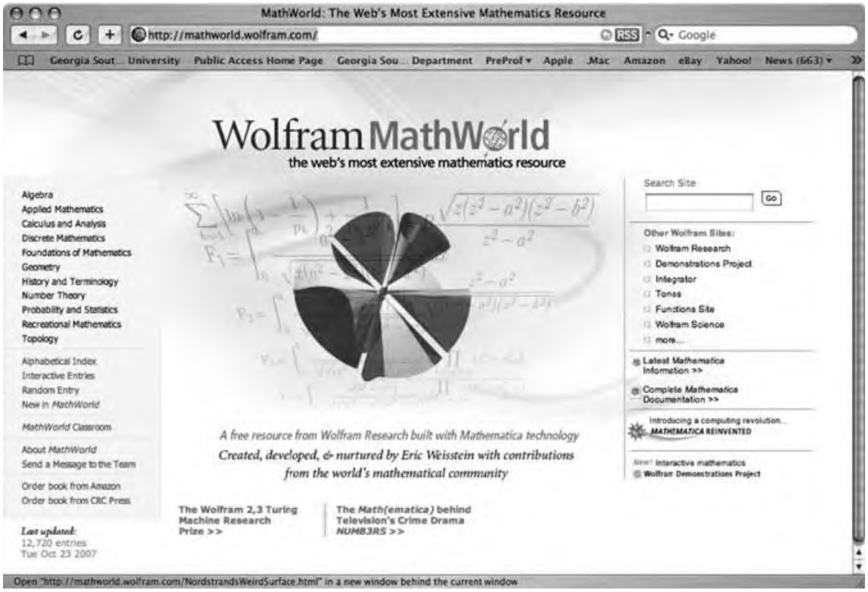

**FIGURE 1.6**

Numerically solving a differential equation such as $dy/dc = \cos(y/x)$ is difficult. To help us understand how the solutions behave, we use a slope field

with Mathematica. In addition to Mathematica's extensive help facililities, which are described next, a tremendous amount of information is available for all Mathematica users at the Wolfram Research website. Not only can you get significant Mathematica help at the Wolfram website but also you can access outstanding *mathematical* resources at Wolfram's MathWorld resource,

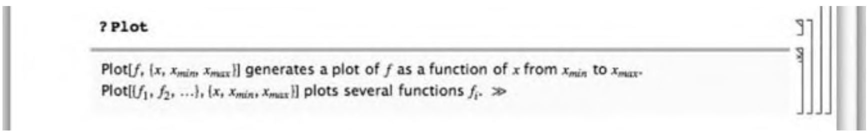
http://mathworld.wolfram.com

One way to obtain information about Mathematica commands and functions, including user-defined functions, is the command ?. ?object gives a basic description and syntax information of the Mathematica object object. ??object yields detailed information regarding syntax and options for the object object. Equivalently, Information[object] yields the information on the Mathematica object object returned by both ?object and Options[object] in addition to a list of attributes of object. Note that object may either be a user-defined object or a built-in Mathematica object.




**Example 1.3.1** Use ? and ?? to obtain information about the command Plot.

***Solution*** ?Plot uses basic information about the Plot function,

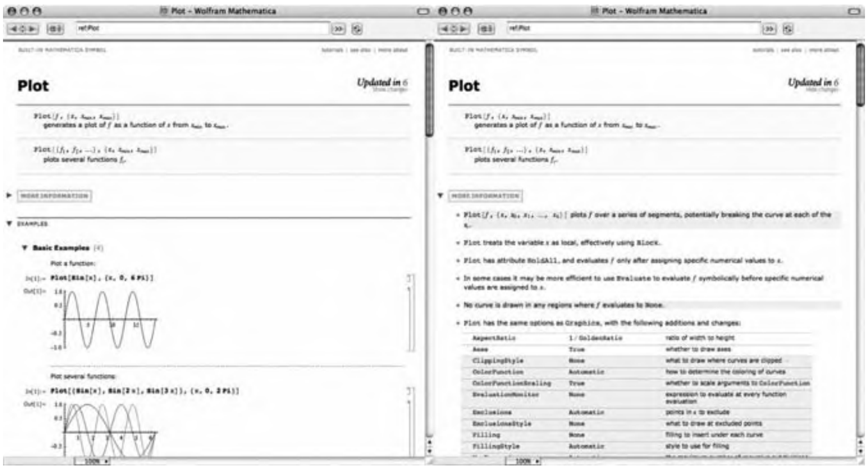



whereas ??Plot includes basic information as well as a list of options and their default values.

If you click on the >> button, Mathematica returns its extensive description of the function. Notice that the updated button in Version 6 ( ) shows that Plot has been updated. Click on **Show Changes** and then **More Information** to see the changes in Version 6.

Options[object] returns a list of the available options associated with objects along with their current settings. This is quite useful when working with a Mathematica command such as ParametricPlot, which has many options. Notice that the default value (the value automatically assumed by Mathematica) for each option is given in the output.

```
?? Plot

Plot[f, {x, xmin, xmax}] generates a plot of f as a function of x from xmin to xmax.
Plot[{f1, f2, ...}, {x, xmin, xmax}] plots several functions fi. >>

Attributes[Plot] = {HoldAll, Protected}

Options[Plot] = {AlignmentPoint → Center, AspectRatio → 1/GoldenRatio,
  Axes → True, AxesLabel → None, AxesOrigin → Automatic,
  AxesStyle → {}, Background → None, BaselinePosition → Automatic,
  BaseStyle → {}, ClippingStyle → None, ColorFunction → Automatic,
  ColorFunctionScaling → True, ColorOutput → Automatic,
  ContentSelectable → Automatic, DisplayFunction :→ $DisplayFunction,
  Epilog → {}, Evaluated → System`Private`$Evaluated,
  EvaluationMonitor → None, Exclusions → Automatic,
  ExclusionsStyle → None, Filling → None, FillingStyle → Automatic,
  FormatType :→ TraditionalForm, Frame → False, FrameLabel → None,
  FrameStyle → {}, FrameTicks → Automatic, FrameTicksStyle → {},
  GridLines → None, GridLinesStyle → {}, ImageMargins → 0.,
  ImagePadding → All, ImageSize → Automatic, LabelStyle → {},
  MaxRecursion → Automatic, Mesh → None, MeshFunctions → {#1 &},
  MeshShading → None, MeshStyle → Automatic, Method → Automatic,
  PerformanceGoal :→ $PerformanceGoal, PlotLabel → None,
  PlotPoints → Automatic, PlotRange → {Full, Automatic},
  PlotRangeClipping → True, PlotRangePadding → Automatic,
  PlotRegion → Automatic, PlotStyle → Automatic,
  PreserveImageOptions → Automatic, Prolog → {},
  RegionFunction → (True &), RotateLabel → True, Ticks → Automatic,
  TicksStyle → {}, WorkingPrecision → MachinePrecision}
```



**Example 1.3.2** Use Options to obtain a list of the options and their current settings for the command ParametricPlot.

***Solution*** The command Options [ParametricPlot] lists all the options and their current settings for the command ParametricPlot.

```
Options[ParametricPlot]
{AlignmentPoint → Center, AspectRatio → Automatic, Axes → True,
 AxesLabel → None, AxesOrigin → Automatic, AxesStyle → {},
 Background → None, BaselinePosition → Automatic, BaseStyle → {},
 BoundaryStyle → Automatic, ColorFunction → Automatic,
 ColorFunctionScaling → True, ColorOutput → Automatic,
 ContentSelectable → Automatic, DisplayFunction :→ $DisplayFunction,
 Epilog → {}, Evaluated → Automatic, EvaluationMonitor → None,
 Exclusions → Automatic, ExclusionsStyle → None,
 FormatType :→ TraditionalForm, Frame → Automatic, FrameLabel → None,
 FrameStyle → {}, FrameTicks → Automatic, FrameTicksStyle → {},
 GridLines → None, GridLinesStyle → {}, ImageMargins → 0.,
 ImagePadding → All, ImageSize → Automatic, LabelStyle → {},
 MaxRecursion → Automatic, Mesh → Automatic, MeshFunctions → Automatic,
 MeshShading → None, MeshStyle → Automatic, Method → Automatic,
 PerformanceGoal :→ $PerformanceGoal, PlotLabel → None,
 PlotPoints → Automatic, PlotRange → Automatic, PlotRangeClipping → True,
 PlotRangePadding → Automatic, PlotRegion → Automatic,
 PlotStyle → Automatic, PreserveImageOptions → Automatic, Prolog → {},
 RegionFunction → (True &), RotateLabel → True, Ticks → Automatic,
 TicksStyle → {}, WorkingPrecision → MachinePrecision}
```

The command Names["form"] lists all objects that match the pattern defined in form. For example, Names ["Plot"] returns Plot, Names["*Plot"] returns all objects that end with the string Plot, Names["Plot*"] lists all objects that begin with the string Plot, and Names["*Plot*"] lists all objects that contain the string Plot. Names["form",SpellingCorrection->True] finds those symbols that match the pattern defined in form after a spelling correction.

---

**Example 1.3.3** Create a list of all built-in functions beginning with the string Plot.

***Solution*** We use Names to find all objects that match the pattern Plot.

```
Names["Plot"]
{Plot}
```

---

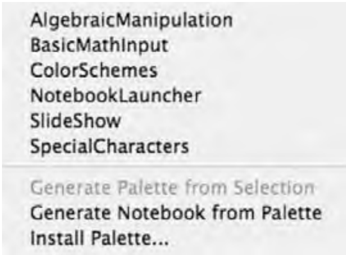
Next, we use Names to create a list of all built-in functions beginning with the string Plot.

```
Names["Plot*"]
{Plot, Plot3D, Plot3Matrix, PlotDivision,
 PlotJoined, PlotLabel, PlotMarkers, PlotPoints, PlotRange,
 PlotRangeClipping, PlotRangePadding, PlotRegion, PlotStyle}
```

---

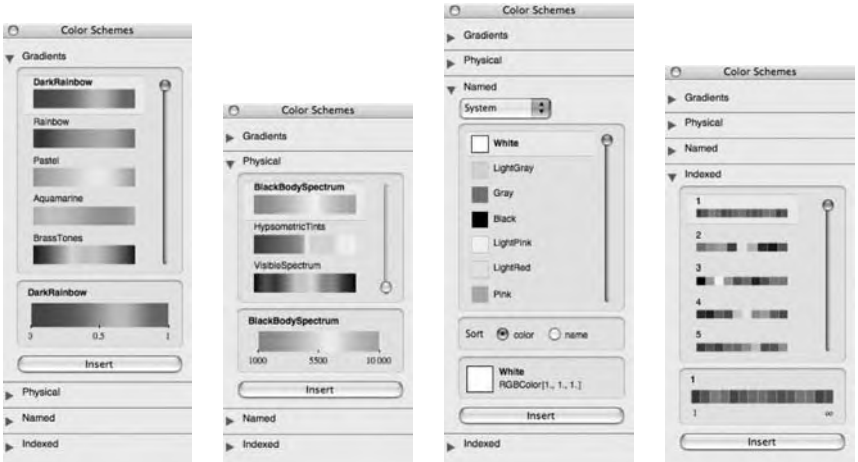
In the following, after using ? to learn about the new Mathematica 6 function ColorData we illustrate its use with a Plot command. We first go to the Mathematica menu


and select **Palettes**, followed by **ColorSchemes**.




We are given a variety of choices, which are illustrated throughout *Mathematica by Example*.

Remember that on a computer running Mathematica, these graphics will appear in color rather than in black-and-white as seen in this text.




We then use the help facilities description of the **ColorData** function to help us generate a plot of $y = \sin x$ on the interval $[0, 2\pi]$ in deep red on our computer. (Of course, the plot is dark gray in a black-and-white text such as this).

As we have illustrated, the ? function can be used in many ways. Entering ?letters∗ gives all Mathematica objects that begin with the string letters; ?∗letters∗ gives all Mathematica objects that contain the string letters; and ?∗letters gives all Mathematica commands that end in the string letters.

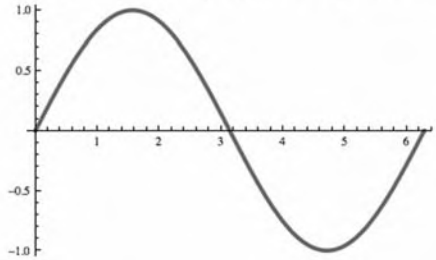
**Example 1.3.4** What are the Mathematica functions that (a) end in the string Cos, (b) contain the string Sin, and (c) begin with the string Polynomial?

***Solution*** Entering

```
?ColorData
```

ColorData["*scheme*"] gives a function that generates colors in the named color scheme when applied to parameter values.
ColorData["*scheme*", "*property*"] gives the specified property of a color scheme.
ColorData["*collection*"] gives a list of color schemes in a named collection.
ColorData[] gives a list of named collections of color schemes. »

```
Plot[Sin[x], {x, 0, 2 Pi},
 PlotStyle → {{Thickness[.01], ColorData["DarkRainbow", 1]}}]
```




```
?*Cos
```

▼ System`

| | |
|---|---|
| ArcCos | Cos |

returns all functions ending with the string **Cos**, entering

```
?*Sin*
```

▼ System`

| | |
|---|---|
| ArcSin | SingleLetterItalics |
| ArcSinh | SingleLetterStyle |
| FourierSinTransform | SingularValueDecomposition |
| IncludeSingularTerm | SingularValueList |
| InverseFourierSinTransform | SingularValues |
| Sin | Sinh |
| Sinc | SinhIntegral |
| SingleEvaluation | SinIntegral |

returns all functions containing the string **Sin**, and entering

```
?Polynomial*
```

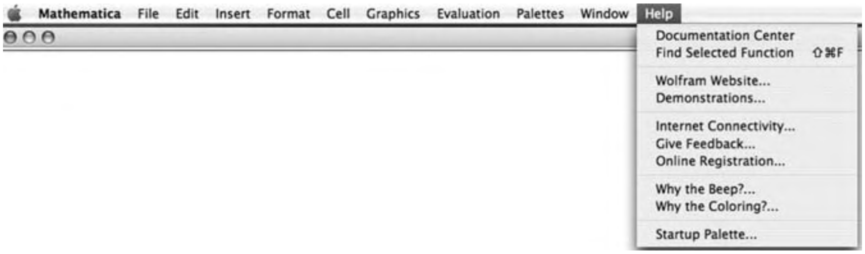
▼ System`

| | |
|---|---|
| PolynomialExtendedGCD | PolynomialQuotient |
| PolynomialForm | PolynomialQuotientRemainder |
| PolynomialGCD | PolynomialReduce |
| PolynomialLCM | PolynomialRemainder |
| PolynomialMod | Polynomials |
| PolynomialQ | |

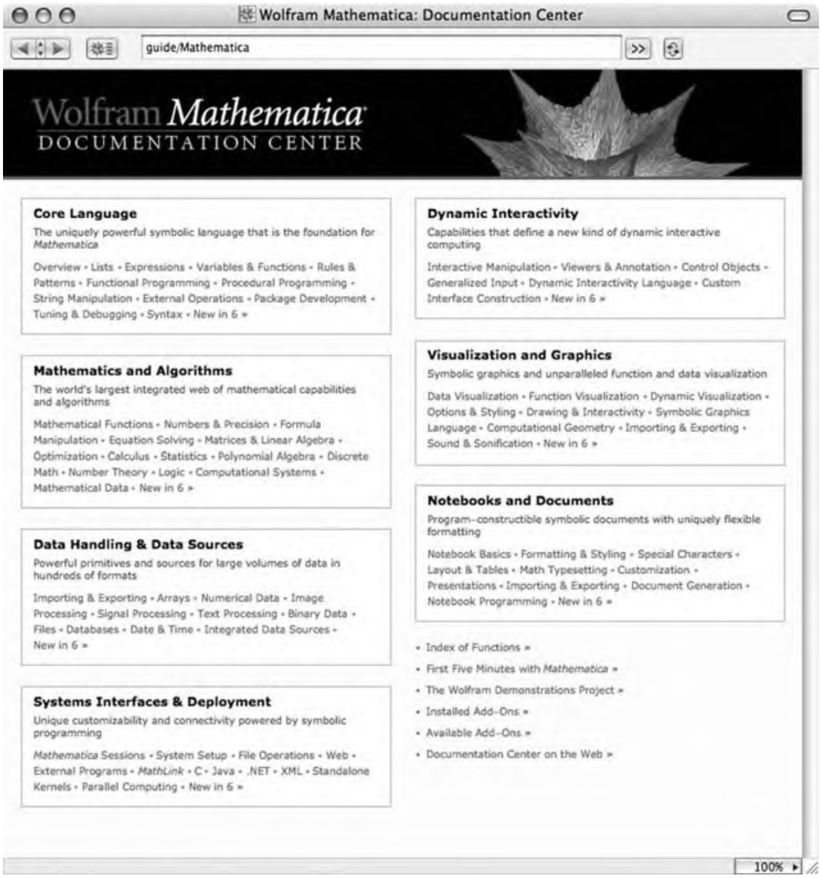
returns all functions that begin with the string **Polynomial**.

## Mathematica Help

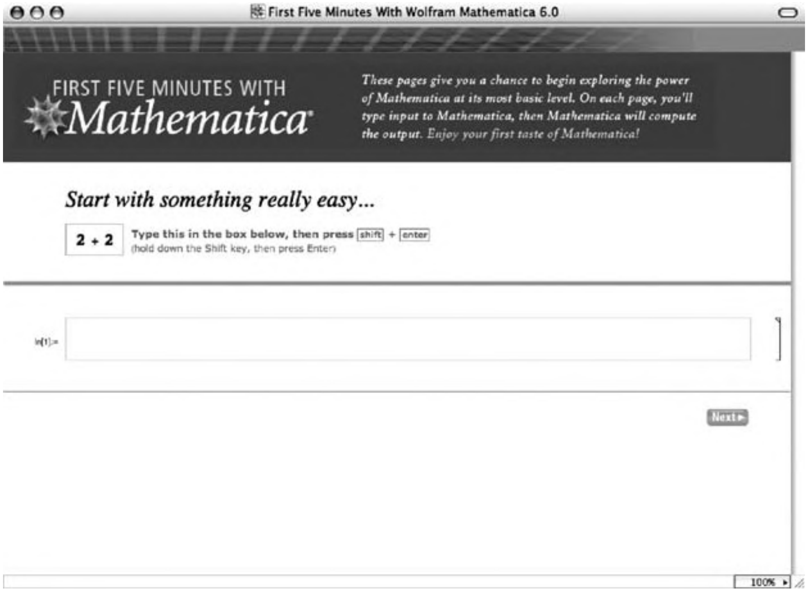
Additional help features are accessed from the Mathematica menu under **Help**. For basic information about Mathematica, go to the Mathematica menu, followed by **Help**




and select **Documentation Center**.



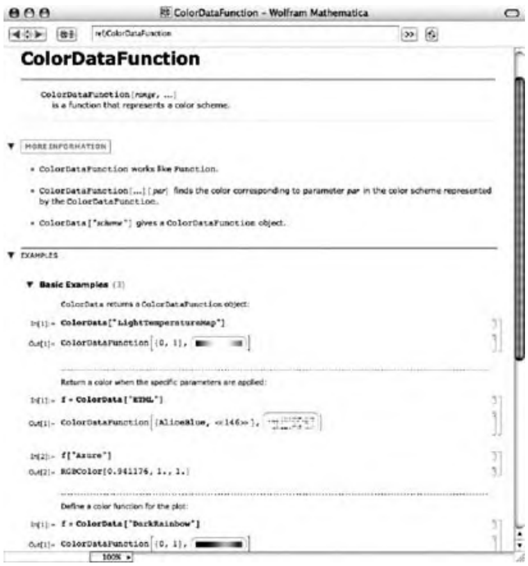
If you are a beginning Mathematica user, you may choose to select **First Five Minutes with Mathematica**.



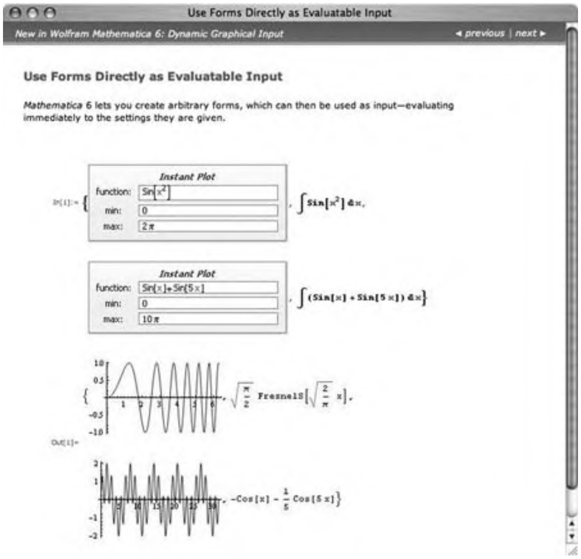

To obtain information about a particular Mathematica object or function, open the **Documentation Center**, type the name of the object, function, or topic, and press the **Go** (>>) button as we have done here with ExampleData. A typical help window contains not only a detailed description of the command and its options but also hyperlinked cross-references to related commands and can be accessed by clicking on the appropriate links.

You can also use the **Documentation Center** to search for help regarding a particular topic. In this case, we enter color schemes in the top line of the **Documentation Center** and then click on the >> button (or press **Enter**) to see all the on-line help regarding "color schemes."


Clicking on the topic will take you to the documentation for the topic. Here is what we see when we select ColorDataFunction:



As you become more proficient with Mathematica, you will want to learn to take advantage of its extensive capabilities.




Remember that Mathematica contains thousands of functions to perform many tasks. If you wish to perform a task that is not discussed here, go to the **Documentation Center** and type a few words related to what you want to do.

---

**Example 1.3.5** In this example, we investigate digit operations. *Mathematica by Example*, fourth edition, has a copyright in 2008, which has four digits.

```
IntegerDigits[2008]
{2,0,0,8}
```

As a string, the number is

```
IntegerString[2008]
2008
```

In base 2, the copyright year is

```
IntegerString[2008, 2]
11111011000
```

On the other hand, with Roman numerals the copyright year is

```
IntegerString[2008, "Roman"]
MMVIII
```

## 1.4 EXERCISES

1. Generate the same plot of $f(x) = (x-1)^{1/3}(x+1)^{2/3}$ as that shown *after* loading the **RealOnly** package *without* loading the **RealOnly** package. Hint: Abs[x] returns the absolute value of the real number $x$.
2. After using the **Documentation Center** to obtain help regarding the function Plot, describe the use of the Plot function.
3. Use Mathematica help and the **Documentation Center** to describe the ExampleData function. Use ExampleData to generate the plot of the torus shown in Figure 1.7.
4. Use ? to determine the value of the Golden Ratio, GoldenRatio.
5. Determine the proper syntax for evaluating $\lim_{x\to\pi/2} \sin x$ and evaluate the limit.
6. Load the **VectorFieldPlot** package. Use Options to learn about the options associated with VectorFieldPlot. Describe the use of three of those options. Your description should contain sufficient detail so that it is readily understandable by an intelligent classmate.
7. Find the graphing options available with Plot3D and ParametricPlot3D.
8. Determine the Mathematica objects that contain the string "gamma."
9. Do any Mathematica objects begin with the letter "z"? Do any end with "z"?
10. What Mathematica function is used to represent the inverse tangent function?
11. Create a list of *all* Mathematica objects.

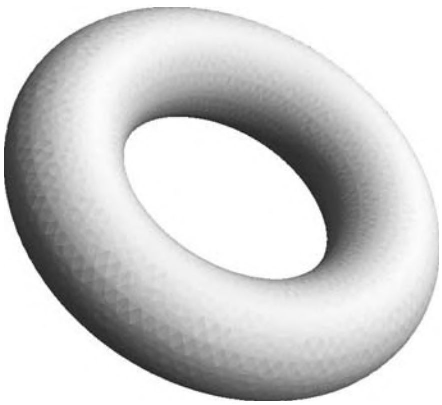


**FIGURE 1.7**

A plot of a torus generated with ExampleData

**12.** Visit MathWorld at http://mathworld.wolfram.com. Use RandomInteger to generate a random integer $n$ between 1 and 11. Visit the $n$th mathematical topic in the subject list. Then, randomly visit a subtopic followed by another subtopic. From the list of topics, choose one that sounds interesting but that you know nothing about. Follow the link and learn about the topic. Write a brief (one-page) report on your findings.

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**CHAPTER**

# 2 Basic Operations on Numbers, Expressions, and Functions

Chapter 2 introduces the essential commands of Mathematica. Basic operations on numbers, expressions, and functions are introduced and discussed.

## 2.1 NUMERICAL CALCULATIONS AND BUILT-IN FUNCTIONS

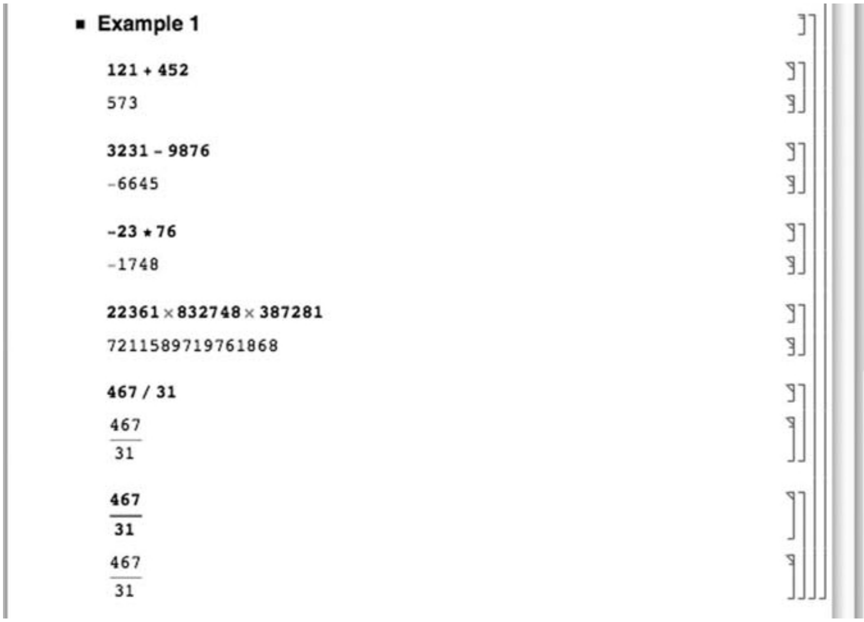
### 2.1.1 Numerical Calculations

The basic arithmetic operations (addition, subtraction, multiplication, division, and exponentiation) are performed in the natural way with Mathematica. Whenever possible, Mathematica gives an exact answer and reduces fractions.

1. "$a$ plus $b$," $a + b$, is entered as a+b;
2. "$a$ minus $b$," $a - b$, is entered as a−b;
3. "$a$ times $b$," $ab$, is entered as either a*b or a b (note the space between the symbols a and b);
4. "$a$ divided by $b$," $a/b$, is entered as a/b. Executing the command a/b results in a fraction reduced to lowest terms; and
5. "$a$ raised to the $b$th power," $a^b$, is entered as a^b.

**Example 2.1.1** Calculate (a) $121 + 542$; (b) $3231 - 9876$; (c) $(-23)(76)$; (d) $(22341)(832748)(387281)$; and (e) $\frac{467}{31}$.

***Solution*** These calculations are carried out in the following screen shot. In each case, the input is typed and then evaluated by pressing **Enter**. In the last case, the **Basic Math** template is used to enter the fraction.


The term $a^{n/m} = \sqrt[m]{a^n} = \left(\sqrt[m]{a}\right)^n$ is entered as a^(n/m). For $n/m = 1/2$, the command Sqrt[a] can be used instead. Usually, the result is returned in unevaluated form but N can be used to obtain numerical approximations to virtually any degree of accuracy. With N[expr, n], Mathematica yields a numerical approximation of expr to $n$ digits of precision, if possible. At other times, Simplify can be used to produce the expected results.

**Remark 2.1** If the expression $b$ in $a^b$ contains more than one symbol, be sure that the exponent is included in parentheses. Entering a^n/m computes $a^n/m = \frac{1}{m}a^n$, whereas entering a^(n/m) computes $a^{n/m}$.

**Example 2.1.2** Compute (a) $\sqrt{27}$ and (b) $\sqrt[3]{8^2} = 8^{2/3}$.

***Solution*** (a) Mathematica automatically simplifies $\sqrt{27} = 3\sqrt{3}$. We use N to obtain an approximation of $\sqrt{27}$. (b) Mathematica automatically simplifies $8^{2/3}$.

N[number] and number//N return numerical approximations of number.

```
Sqrt[27]
3 √3
N[Sqrt[27]]
5.19615
8^(2/3)
4
```

When computing odd roots of negative numbers, Mathematica's results are surprising to the novice. Namely, Mathematica returns a complex number. We will see that this has important consequences when graphing certain functions.

**Example 2.1.3** Calculate (a) $\frac{1}{3}\left(-\frac{27}{64}\right)^2$ and (b) $\left(-\frac{27}{64}\right)^{2/3}$.

***Solution*** (a) Because Mathematica follows the order of operations, (-27/64)^2/3 first computes $(-27/64)^2$ and then divides the result by $3$.

```
(-27/64)^2/3
243
────
4096
```

(b) On the other hand, (-27/64)^(2/3) raises $-27/64$ to the $2/3$ power. Mathematica does not automatically simplify $\left(-\frac{27}{64}\right)^{2/3}$.

```
(-27/64)^(2/3)
 9
── (-1)^(2/3)
16
```

However, when we use N, Mathematica returns the numerical version of the principal root of $\left(-\frac{27}{64}\right)^{2/3}$.

```
N[(-27/64)^(2/3)]
-0.28125 + 0.487139 i
```

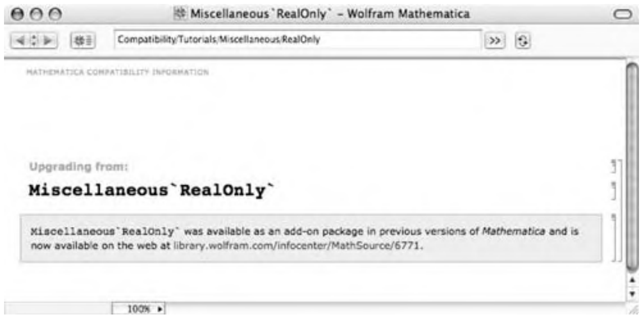
To obtain the result

$$\left(-\frac{27}{64}\right)^{2/3} = \left(\sqrt[3]{\frac{-27}{64}}\right)^2 = \left(-\frac{3}{4}\right)^2 = \frac{9}{16},$$

which would be expected by most algebra and calculus students, we first square $-27/64$ and then take the third root.

```
((-27 / 64) ^2) ^ (1 / 3)
9
──
16
```

Alternatively, download the **RealOnly** package from the Wolfram website.




Then,

```
<< Miscellaneous`RealOnly`
General::obspkg :
  Miscellaneous`RealOnly` is now obsolete. The legacy version being loaded
     may conflict with current Mathematica functionality.
     See the Compatibility Guide for updating information. >>
(-27 / 64) ^ (2 / 3)
9
──
16
```

returns the result $9/16$.

## 2.1.2 Built-in Constants

Mathematica has built-in definitions of nearly all commonly used mathematical constants and functions. To list a few, $e \approx 2.71828$ is denoted by E, $\pi \approx 3.14159$ is denoted by Pi, and $i = \sqrt{-1}$ is denoted by I. Usually, Mathematica performs complex arithmetic automatically.

Other built-in constants include $\infty$; denoted by Infinity; Euler's constant, $\gamma \approx 0.577216$, denoted by EulerGamma; Catalan's constant, approximately 0.915966, denoted by Catalan; and the golden ratio, $\frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.61803$, denoted by GoldenRatio.

**Example 2.1.4** Entering

**N[E, 50]**
2.7182818284590452353602874713526624977572470937000

returns a 50-digit approximation of $e$. Entering

**N[π, 25]**
3.141592653589793238462643

returns a 25-digit approximation of $\pi$. Entering

$$\mathbf{\frac{3+I}{4-I}}$$
$$\frac{11}{17}+\frac{7i}{17}$$

performs the division $(3+i)/(4-i)$ and writes the result in standard form.

## 2.1.3 **Built-in Functions**

Functions frequently encountered by beginning users include the exponential function, Exp[x]; the natural logarithm, Log[x]; the absolute value function, Abs[x]; the trigonometric functions Sin[x], Cos[x], Tan[x], Sec[x], Csc[x], and Cot[x]; the inverse trigonometric functions ArcSin[x], ArcCos[x], ArcTan[x], ArcSec[x], ArcCsc[x], and ArcCot[x]; the hyperbolic trigonometric functions Sinh[x], Cosh[x], and Tanh[x]; and their inverses ArcSinh[x], ArcCosh[x], and ArcTanh[x]. Generally, Mathematica tries to return an exact value unless otherwise specified with N.

Several examples of the natural logarithm and the exponential functions are given next. Mathematica often recognizes the properties associated with these functions and simplifies expressions accordingly.

**Example 2.1.5** Entering

N[number] or number//N returns approximations of number.

Exp[x] computes $e^x$. Enter E to compute $e \approx 2.718$.

**Exp[−5]//N**
0.00673795

returns an approximation of $e^{-5} = 1/e^5$. Entering

**Log[E$^3$]**
3

computes $\ln e^3 = 3$. Entering

Log[x] computes $\ln x$. $\ln x$ and $e^x$ are inverse functions ($\ln e^x = x$ and $e^{\ln x} = x$) and Mathematica uses these properties when simplifying expressions involving these functions.

**Exp[Log[π]]**

π

computes $e^{\ln \pi} = \pi$. Entering

**Abs[−5]**

5

computes $|-5| = 5$. Entering

Abs[x] returns the absolute value of $x, |x|$.

**Abs$\left[\frac{3+2I}{2-9I}\right]$**

$\sqrt{\frac{13}{85}}$

computes $|(3+2i)/(2-9i)|$. Entering

**Cos$\left[\frac{\pi}{12}\right]$**

$\frac{1+\sqrt{3}}{2\sqrt{2}}$

**N$\left[\text{Cos}\left[\frac{\pi}{12}\right]\right]$**

0.965926

computes the exact value of $\cos(\pi/12)$ and then an approximation. Although Mathematica cannot compute the exact value of tan 1000, entering

N[number] and number//N return approximations of number.

**N[Tan[1000]]**

1.47032

returns an approximation of $\tan 1000$. Similarly, entering

**N[ArcSin[1/3]]**

0.339837

returns an approximation of $\sin^{-1}(1/3)$, and entering

**ArcCos[2/3]//N**

0.841069

returns an approximation of $\cos^{-1}(2/3)$.

Mathematica is able to apply many identities that relate the trigonometric and exponential functions using the functions TrigExpand, TrigFactor, TrigReduce, TrigToExp, and ExpToTrig.

```
?TrigExpand
TrigExpand[expr] expands out trigonometric functions in expr. >>
?TrigFactor
TrigFactor[expr] factors trigonometric functions in expr. >>
?TrigReduce
TrigReduce[expr] rewrites products and powers of trigonometric functions in expr
    in terms of trigonometric functions with combined arguments. >>
?TrigToExp
TrigToExp[expr] converts trigonometric functions in expr to exponentials. >>
?ExpToTrig
ExpToTrig[expr] converts exponentials in expr to trigonometric functions. >>
```

**Example 2.1.6** Mathematica does not automatically apply the identity $\sin^2 x + \cos^2 x = 1$.

Many of the algebraic manipulation commands can be accessed from the **Algebraic-Manipulation** palette.

Algebraic Manip...
- Expand[■]
- ExpandAll[■]
- Factor[■]
- Together[■]
- Apart[■]
- Cancel[■]
- Simplify[■]
- FullSimplify[■]
- FunctionExpand[■]
- TrigExpand[■]
- TrigFactor[■]
- TrigReduce[■]
- ExpToTrig[■]
- TrigToExp[■]
- PowerExpand[■]
- ComplexExpand[■]

**Cos[x]^2 + Sin[x]^2**
Cos[x]$^2$ + Sin[x]$^2$

To apply the identity, we use Simplify. Generally, Simplify[expression] attempts to simplify expression.

**Simplify[Cos[x]^2 + Sin[x]^2]**
1

Use TrigExpand to multiply expressions or to rewrite trigonometric functions. In this case, entering

**TrigExpand[Cos[3x]]**
Cos[x]$^3$ – 3Cos[x]Sin[x]$^2$

writes $\cos 3x$ in terms of trigonometric functions with argument $x$. We use the TrigReduce function to convert products to sums.

**TrigReduce[Sin[3x]Cos[4x]]**
$\frac{1}{2}$(– Sin[x] + Sin[7x])

We use TrigExpand to write

**TrigExpand[Cos[2x]]**
Cos[x]$^2$ – Sin[x]$^2$

in terms of trigonometric functions with argument $x$. We use ExpToTrig to convert exponential expressions to trigonometric expressions.

**ExpToTrig[1/2(Exp[x] + Exp[−x])]**
Cosh[x]

Similarly, we use TrigToExp to convert trigonometric expressions to exponential expressions.

**TrigToExp[Sin[x]]**
$\frac{1}{2}ie^{-ix} - \frac{1}{2}ie^{ix}$

Usually, you can use Simplify to apply elementary identities.

**Simplify[Tan[x]^2 + 1]**
Sec[x]$^2$

## A Word of Caution

Remember that there are certain ambiguities in traditional mathematical notation. For example, the expression $\sin^2(\pi/6)$ is usually interpreted to mean "compute $\sin(\pi/6)$ and square the result." That is, $\sin^2(\pi/6) = [\sin(\pi/6)]^2$. The symbol sin is not being squared; the number $\sin(\pi/6)$ *is* squared. With Mathematica, we must be especially careful and follow the standard order of operations exactly, especially when using **InputForm**. We see that entering

```
Sin[Pi/6]^2
1
-
4
```

computes $\sin^2(\pi/6) = [\sin(\pi/6)]^2$, whereas

```
Sin^2[Pi/6]
Sin^2[π/6]
```

raises the symbol Sin to the power $2\left[\frac{\pi}{6}\right]$. Mathematica interprets

$$\text{Sin}\left[\frac{\pi}{6}\right]^2$$

$$\frac{1}{4}$$

to be the product of the symbols $\sin^2$ and $\pi/6$. However, using **TraditionalForm** we are able to evaluate $\sin^2(\pi/6) = [\sin(\pi/6)]^2$ with Mathematica using conventional mathematical notation.

$$\sin^2\left(\frac{\pi}{6}\right)$$

$$\frac{1}{4}$$

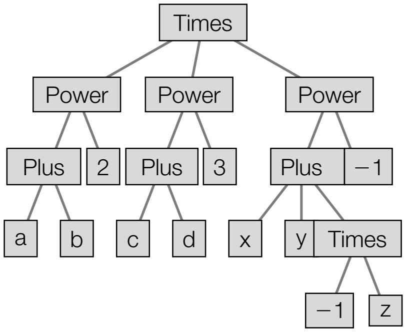
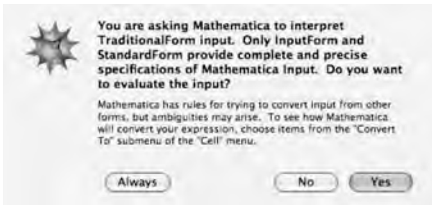



**FIGURE 2.1**

Visualizing the order in which Mathematica carries out a sequence of operations

Be aware, however, that traditional mathematical notation does contain certain ambiguities and Mathematica may not return the result you expect if you enter input using **TraditionalForm** unless you are especially careful to follow the standard order of operations, as the following warning message indicates.




**Example 2.1.7** As stated, Mathematica follows the order of operations exactly. To see how Mathematica performs a calculation, TreeForm presents the sequence graphically. For example, for the calculation $\frac{(a+b)^2(c+d)^3}{x+y-z}$, TreeForm gives us the results shown in Figure 2.1.

```
Clear[a, b, c, d, x, y, z]
TreeForm[(a + b)^2(c + d )^3/(x + y − z)]
```

## 2.2 EXPRESSIONS AND FUNCTIONS: ELEMENTARY ALGEBRA

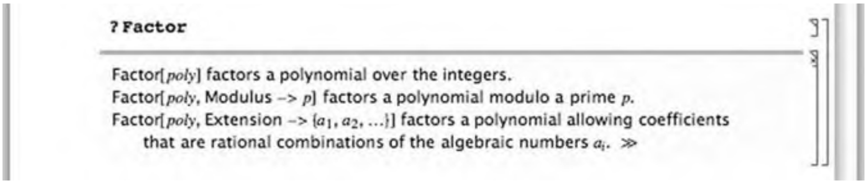
### 2.2.1 Basic Algebraic Operations on Expressions

Expressions involving unknowns are entered in the same way as numbers. Mathematica performs standard algebraic operations on mathematical expressions. For example, the commands

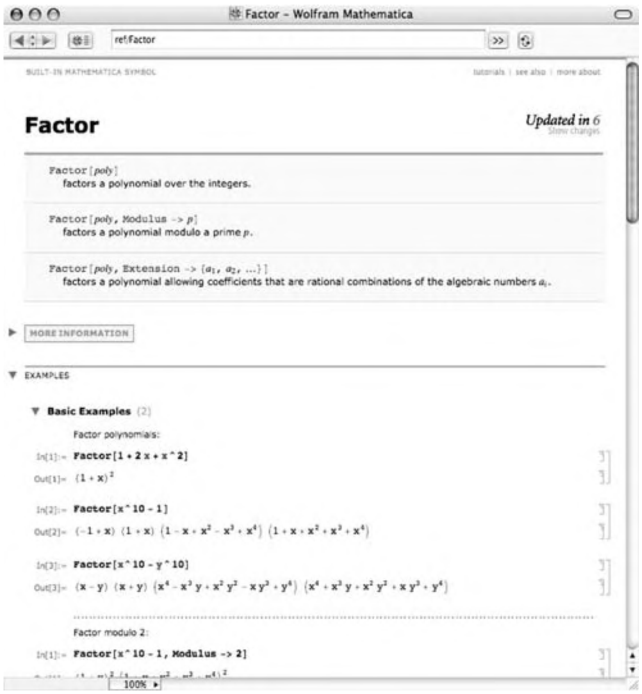
1. Factor[expression] factors expression;
2. Expand[expression] multiplies expression;

3. Together[expression] writes expression as a single fraction; and
4. Simplify[expression] performs basic algebraic manipulations on expression and returns the simplest form it finds.

For basic information about any of these commands (or any other), enter ?command as we do here for Factor,




or access the **Help Browser** as we do here for Factor.




When entering expressions, be sure to include a space or * between variables to denote multiplication.

---

**Example 2.2.1** (a) Factor the polynomial $12x^2 + 27xy - 84y^2$. (b) Expand the expression $(x+y)^2(3x-y)^3$. (c) Write the sum $\frac{2}{x^2} - \frac{x^2}{2}$ as a single fraction.

***Solution*** The result obtained with Factor indicates that $12x^2 + 27xy - 84y^2 = 3(4x - 7y)(x + 4y)$. When typing the command, be sure to include a space, or *, between the x and y terms to denote multiplication. xy represents an expression, whereas x y or x*y denotes x multiplied by y.

**Factor[12x² + 27xy – 84y²]**

3(4x – 7y)(x + 4y)

We use Expand to compute the product $(x + y)^2(3x - y)^3$ and Together to express $\frac{2}{x^2} - \frac{x^2}{2}$ as a single fraction.

**Expand[(x + y)²(3x – y)³]**

$27x^5 + 27x^4y - 18x^3y^2 - 10x^2y^3 + 7xy^4 - y^5$

**Together[2/x² – x²/2]**

$\frac{4 - x^4}{2x^2}$

---

Factor[x^2 - 3] returns $x^2 - 3$.

To factor an expression such as $x^2 - 3 = x^2 - (\sqrt{3})^2 = (x - \sqrt{3})(x + \sqrt{3})$, use Factor with the Extension option.

**Factor[x^2 – 3, Extension → {Sqrt[3]}]**

$-(\sqrt{3} - x)(\sqrt{3} + x)$

Similarly, use Factor with the Extension option to factor expressions such as $x^2 + 1 = x^2 - i^2 = (x + i)(x - i)$.

**Factor[x^2 + 1]**

$1 + x^2$

**Factor[x^2 + 1, Extension → {I}]**

$(-i + x)(i + x)$

Mathematica does not automatically simplify $\sqrt{x^2}$ to the expression $x$

**Sqrt[x^2]**

$\sqrt{x^2}$

because without restrictions on $x$, $\sqrt{x^2} = |x|$. The command PowerExpand [expression] simplifies expression assuming that all variables are positive. Alternatively, you can use Assumptions to tell Mathematica to assume that $x > 0$.

**PowerExpand[Sqrt[x^2]]**

x

**Simplify[Sqrt[x^2], Assumptions → x>0]**

x

Thus, entering

```
Simplify[Sqrt[a^2b^4]]
```

$\sqrt{a^2b^4}$

returns $\sqrt{a^2b^4}$, but entering

```
PowerExpand[Sqrt[a^2b^4]]
```

$ab^2$

```
Simplify[Sqrt[a^2b^4], Assumptions →
    {a > 0, b > 0}]
```

$ab^2$

returns $ab^2$.

In general, a space is not needed between a number and a symbol to denote multiplication when a symbol follows a number. That is, 3dog means "3 times variable dog," dog3 is a variable with name dog3. Mathematica interprets 3 dog, dog*3, and dog 3 as "3 times variable dog." However, when multiplying two variables, either include a space or * between the variables.

1. cat dog means "variable cat times variable dog."
2. cat*dog means "variable cat times variable dog."
3. But, catdog is interpreted as a variable catdog.

The command Apart[expression] computes the partial fraction decomposition of expression; Cancel[expression] factors the numerator and denominator of expression and then reduces expression to lowest terms.

---

**Example 2.2.2** (a) Determine the partial fraction decomposition of $\frac{1}{(x-3)(x-1)}$. (b) Simplify $\frac{x^2-1}{x^2-2x+1}$.

***Solution*** Apart is used to see that $\frac{1}{(x-3)(x-1)} = \frac{1}{2(x-3)} - \frac{1}{2(x-1)}$. Then, Cancel is used to find that $\frac{x^2-1}{x^2-2x+1} = \frac{(x-1)(x+1)}{(x-1)^2} = \frac{x+1}{x-1}$. In this calculation, we have assumed that $x \neq 1$, an assumption made by Cancel but not by Simplify.

**Apart**$\left[\frac{1}{(x-3)(x-1)}\right]$

$\frac{1}{2(-3+x)} - \frac{1}{2(-1+x)}$

**Cancel**$\left[\frac{x^2-1}{x^2-2x+1}\right]$

$\frac{1+x}{-1+x}$

---

In addition, Mathematica has several built-in functions for manipulating parts of fractions.

1. Numerator[fraction] yields the numerator of fraction.
2. ExpandNumerator[fraction] expands the numerator of fraction.
3. Denominator[fraction] yields the denominator of fraction.
4. ExpandDenominator[fraction] expands the denominator of fraction.

---

**Example 2.2.3** Given $\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}$, (a) factor both the numerator and denominator; (b) reduce $\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}$ to lowest terms; and (c) find the partial fraction decomposition of $\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}$.

***Solution*** The numerator of $\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}$ is extracted with Numerator. We then use Factor together with %, which is used to refer to the most recent output, to factor the result of executing the Numerator command.

**Numerator**$\left[\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}\right]$

$-2-x+2x^2+x^3$

**Factor[%]**

$(-1+x)(1+x)(2+x)$

Similarly, we use Denominator to extract the denominator of the fraction. Again, Factor together with % is used to factor the previous result, which corresponds to the denominator of the fraction.

**Denominator**$\left[\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}\right]$

$-4-4x+x^2+x^3$

**Factor[%]**

$(-2+x)(1+x)(2+x)$

Cancel is used to reduce the fraction to lowest terms.

**Cancel**$\left[\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}\right]$

$\frac{-1+x}{-2+x}$

Finally, Apart is used to find its partial fraction decomposition.

**Apart**$\left[\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}\right]$

$1+\frac{1}{-2+x}$

---

You can also take advantage of the **AlgebraicManipulation** palette, which is accessed by going to **Palettes** under the Mathematica menu, followed by **AlgebraicManipulation**, to evaluate expressions.

---

**Example 2.2.4** Simplify $\dfrac{2(x-3)^2(x+1)}{3(x+1)^{4/3}}+2(x-3)(x+1)^{2/3}$.

***Solution*** First, we type the expression.

$$\frac{2\ (x-3)^2\ (x+1)}{3\ (x+1)^{4/3}}+2\ (x-3)\ \ (x+1)^{2/3}$$

Then, select the expression.

$$\frac{2\ (x-3)^2\ (x+1)}{3\ (x+1)^{4/3}}+2\ (x-3)\ \ (x+1)^{2/3}$$

Move the cursor to the palette and click on Simplify. Mathematica simplifies the expression.

$$\frac{8\ (-3+x)\ x}{3\ (1+x)^{1/3}}$$

---

## 2.2.2 Naming and Evaluating Expressions

In Mathematica, objects can be named. Naming objects is convenient: We can avoid typing the same mathematical expression repeatedly (as we did in Example 2.2.3) and named expressions can be referenced throughout a notebook or Mathematica session. Every Mathematica object can be named—expressions, functions, graphics, and so on can be named with Mathematica. Objects are named by using a single equals sign (=).

Because every built-in Mathematica function begins with a capital letter, we adopt the convention that *every* mathematical object we name in this text will begin with a *lowercase* letter. Consequently, we will be certain to avoid any possible ambiguity with any built-in Mathematica objects.

With Mathematica 6, the default option is to display *known* objects in black and unknown objects in blue. Thus, in the following screen shot,

```
Fraction x y
Apart
apart
2 Pi pi π
7 Plot Expand Cancel
E e E
```

Fraction, x, y, apart, pi, e, and E are in blue; Apart, 2, Pi, $\pi$, 7, Plot, Expand, Cancel, and E are in black.

To automatically update named variables, Dynamic[x] returns the current value of $x$.

Thus, Dynamic[x] returns dog.

```
x = dog
dog

Dynamic[x]
dog
```

However, when we enter $x = 7$ afterwards, Dynamic[x] is automatically updated to the new value of $x$.

```
x = dog
dog

Dynamic[x]
7

x = 7
7
```

Expressions are easily evaluated using ReplaceAll, which is abbreviated with /. and obtained by typing a backslash (/) followed by a period (.), together with Rule, which is abbreviated with -> and obtained by typing a forward slash (/) followed by a greater than sign (>). For example, entering the command

**x^2/ . x –> 3**

returns the value of the expression $x^2$ if $x = 3$. Note, however, that this does not assign the symbol x the value 3: entering x=3 assigns $x$ the value 3.

---

**Example 2.2.5** Evaluate $\dfrac{x^3 + 2x^2 - x - 2}{x^3 + x^2 - 4x - 4}$ if $x = 4$, $x = -3$, and $x = 2$.

***Solution*** To avoid retyping $\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}$, we define **fraction** to be $\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}$.

Of course, you can simply copy and paste this expression if you neither want to name it nor retype it.

**fraction =** $\mathbf{\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}}$

$\frac{-2-x+2x^2+x^3}{-4-4x+x^2+x^3}$

/. is used to evaluate **fraction** if $x = 4$ and then if $x = -3$.

If you include a semicolon (;) at the end of the command, the resulting output is suppressed.

**fraction/.x−> 4**

$\frac{3}{2}$

**fraction/.x−>−3**

$\frac{4}{5}$

When we try to replace each $x$ in **fraction** by $2$, we see that the result is undefined: division by $0$ is always undefined.

```
fraction /. x → -2

Power::infy : Infinite expression 1/0 encountered. ≫

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

Indeterminate
```

However, when we use **Cancel** to first simplify and then use **ReplaceAll** to evaluate,

```
fraction2 = Cancel[fraction]

-1 + x
------
-2 + x

fraction2 /. x → -2

3
-
4
```

we see that the result is $3/4$. The result indicates that $\lim_{x\to-2}\frac{x^3+2x^2-x-2}{x^3+x^2-4x-4}=\frac{3}{4}$. We confirm this result with **Limit**.

```
Limit[fraction, x → -2]

3
-
4
```

Generally, **Limit[f[x], x->a]** attempts to compute $\lim_{x\to a} f(x)$. The **Limit** function is discussed in more detail in the next chapter.

### 2.2.3 **Defining and Evaluating Functions**

It is important to remember that functions, expressions, and graphics can be named anything that is not the name of a built-in Mathematica function or command. As previously indicated, every built-in Mathematica object begins with a capital letter so every user-defined function, expression, or other object in this text will be assigned a name using lowercase letters exclusively. This way, the possibility of conflicting with a built-in Mathematica command or function is completely eliminated. Because definitions of functions and names of objects are frequently modified, we introduce the command Clear. Clear[expression] clears all definitions of expression, if any. You can see if a particular symbol has a definition by entering ?symbol.

In Mathematica, an elementary function of a single variable, $y = f(x) =$ *expression in x*, is typically defined using the form

**f[x_] = expression in x** or **f[x_] := expression in x.**

Notice that when you first define a function, you must always enclose the argument in square brackets ([...]) and place an underline (or blank) "_" after the argument on the left-hand side of the equals sign in the definition of the function.

---

**Example 2.2.6** Entering

**f[x_] = x/(x^2 + 1)**

$\frac{x}{1+x^2}$

defines and computes $f(x) = x/\left(x^2+1\right)$. Entering

**f[3]**

$\frac{3}{10}$

computes $f(3) = 3/\left(3^2+1\right) = 3/10$. Entering

**f[a]**

$\frac{a}{1+a^2}$

computes $f(a) = a/\left(a^2+1\right)$. Entering

**f[3 + h]**

$\frac{3+h}{1+(3+h)^2}$

computes $f(3+h) = (3+h)/\left((3+h)^2+1\right)$. Entering

**n1 = Simplify[(f[3 + h] − f[3])/h]**

$-\frac{8+3h}{10\left(10+6h+h^2\right)}$

computes and simplifies $\dfrac{f(3+h)-f(3)}{h}$ and names the result n1. Entering

**n1/.h→0**

$-\frac{2}{25}$

evaluates n1 if $h = 0$. Entering

**n2 = Together[(f[a + h] – f[a])/h]**

$\dfrac{1-a^2-ah}{(1+a^2)(1+a^2+2ah+h^2)}$

computes and simplifies $\dfrac{f(a+h)-f(a)}{h}$ and names the result n2. Entering

**n2/.h→0**

$\dfrac{1-a^2}{(1+a^2)^2}$

evaluates n2 if $h = 0$.

Often, you will need to evaluate a function for the values in a **list**,

$$\text{list} = \{a_1, a_2, a_3, \ldots, a_n\}.$$

Once $f(x)$ has been defined, Map [f,list] returns the list

$$\{f(a_1), f(a_2), f(a_3), \ldots, f(a_n)\}.$$

Also,

The Table function will be discussed in more detail as needed.

1. Table [f[n], {n, n1, n2}] returns the list

$$\{f(n_1), f(n_1+1), f(n_1+2), \ldots, f(n_2)\}.$$

2. Table [{n, f[n]},{n, n1, n2}] returns the list of ordered pairs

$$\{(n_1, f(n_1)), (n_1+1, f(n_1+1)), (n_1+2, f(n_1+2)), \ldots, (n_2, f(n_2))\}.$$

---

**Example 2.2.7** Entering

**Clear[h]**
**h[t_] = (1 + t)^(1/t);**
**h[1]**
2

defines $h(t) = (1+t)^{1/t}$ and then computes $h(1) = 2$. Because division by 0 is always undefined, $h(0)$ is undefined.

```
h[0]
Power::infy : Infinite expression 1/0 encountered. >>
∞::indet : Indeterminate expression 1^ComplexInfinity encountered. >>
Indeterminate
```

RandomReal[{a, b}] returns a random real number between $a$ and $b$; RandomReal[{a, b}, n] returns $n$ random real numbers between $a$ and $b$.

However, $h(t)$ is defined for all $t > 0$. In the following, we use RandomReal together with Table to generate six random numbers "close" to 0 and name the resulting list t1. Because we are using RandomReal, your results will almost certainly differ from those here.

```
t1 = Table[RandomReal[{0, 10^-n}], {n, 0, 5}]
{0.457711, 0.0446146, 0.00848021,
0.000465453, 0.0000566835, 1.6690247776250502*^-6}
```

We then use Map to compute $h(t)$ for each of the values in the list t1.

```
Map[h, t1]
{2.27817, 2.66002, 2.70684, 2.71765, 2.7182, 2.71828}
```

From the result, we might correctly deduce that $\lim_{t\to 0^+}(1+t)^{1/t} = e$.

In each of these cases, do not forget to include the blank (or underline) (_) on the left-hand side of the equals sign in the definition of each function. Remember to always include arguments of functions in square brackets.

---

**Example 2.2.8** Entering

Including a semicolon at the end of a command suppresses the resulting output.

```
Clear[f]
f[0] = 1;
f[1] = 1;
f[n_] := f[n-1] + f[n-2]
```

defines the recursively defined function defined by $f(0) = 1$, $f(1) = 1$, and $f(n) = f(n-1) + f(n-2)$. For example, $f(2) = f(1) + f(0) = 1 + 1 = 2$; $f(3) = f(2) + f(1) = 2 + 1 = 3$. We use Table to create a list of ordered pairs $(n, f(n))$ for $n = 0, 1, \ldots, 10$.

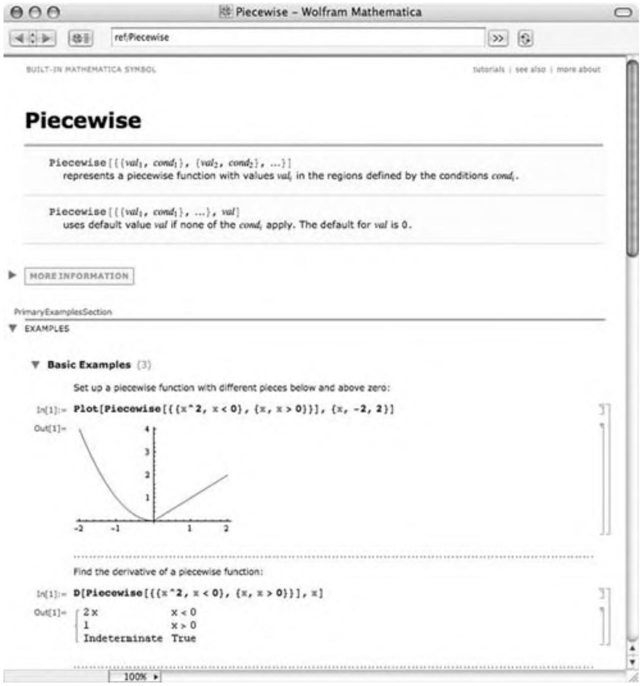
The $f_n$ we have defined here returns the **Fibonacci number** $F_n$. Fibonacci[n] also returns the $n$th Fibonacci number.

```
Table[{n, f[n]}, {n, 0, 10}]
{{0, 1}, {1, 1}, {2, 2}, {3, 3}, {4, 5},
   {5, 8}, {6, 13}, {7, 21}, {8, 34}, {9, 55}, {10, 89}}
```

In the preceding examples, the functions were defined using each of the forms f[x_]:=... and f[x_]=.... As a practical matter, when defining "routine" functions with domains consisting of sets of real numbers and ranges consisting of sets of real numbers, either form can be used. Defining a function using the form f[x_]=... instructs Mathematica to define f and then compute and return f[x] **(immediate assignment)**; defining a function using the form f[x_]:=... instructs Mathematica to define f. In this case, f[x] is not computed and, thus, Mathematica returns no output **(delayed assignment)**. The form f[x_]:=... should be used when Mathematica cannot evaluate f[x] unless x is a particular value, as with recursively defined functions or piecewise-defined functions, which we will discuss soon.

Generally, if attempting to define a function using the form f[x_] = . . . produces one or more error messages, use the form f[x_] := . . . instead.

To define piecewise-defined functions, we usually use Condition (/;) as illustrated in the following example. In simple situations, we take advantage of Piecewise.




**Example 2.2.9** Entering

**Clear[f]**
**f[t_] := Sin[1/t]/; t > 0**

defines $f(t) = \sin(1/t)$ for $t > 0$. Entering

**f[1/(10Pi)]**
0

is evaluated because $1/(10\pi) > 0$. However, both of the following commands are returned unevaluated. In the first case, $-1$ is not greater than $0(f(t)$ is not defined for $t \leq 0)$. In the second case, Mathematica does not know the value of $a$ so it cannot determine if it is or is not greater than $0$.

**f[−1]**
f[−1]

```
f[a]
f[a]
```

Entering

```
f[t_] := -t/; t ≤ 0
```

defines $f(t) = -t$ for $t \leq 0$. Now, the domain of $f(t)$ is all real numbers. That is, we have defined the piecewise-defined function

$$f(t) = \begin{cases} \sin(1/t), & t > 0 \\ -t, & t \leq 0 \end{cases}.$$

We can now evaluate $f(t)$ for any real number $t$.

```
f[2/(5Pi)]
1

f[0]
0

f[-10]
10
```

However, $f(a)$ still returns unevaluated because Mathematica does not know if $a \leq 0$ or if $a > 0$.

```
f[a]
f[a]
```

Recursively defined functions are handled in the same way. The following example shows how to define a periodic function.

---

**Example 2.2.10** Entering

```
Clear[g]
g[x_] := x/; 0 ≤ x < 1
g[x_] := 1/;1 ≤ x < 2
g[x_] := 3-x/;2 ≤ x < 3
g[x_] := g[x-3]/;x ≥ 3
```

defines the recursively defined function $g(x)$. For $0 \leq x < 3$, $g(x)$ is defined by

$$g(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 3 - x, & 2 \leq x < 3. \end{cases}$$

For $x \geq 3$, $g(x) = g(x-3)$. Entering

```
g[7]
1
```

computes $g(7) = g(4) = g(1) = 1$. We use Table to create a list of ordered pairs $(x, g(x))$ for 25 equally spaced values of $x$ between 0 and 6.

```
Table[{x, g[x]}, {x, 0, 6, 6/24}]
```

$$\{\{0,0\},\{\tfrac{1}{4},\tfrac{1}{4}\},\{\tfrac{1}{2},\tfrac{1}{2}\},\{\tfrac{3}{4},\tfrac{3}{4}\},\{1,1\},\{\tfrac{5}{4},1\},\{\tfrac{3}{2},1\},\{\tfrac{7}{4},1\},\{2,1\},\{\tfrac{9}{4},\tfrac{3}{4}\},\{\tfrac{5}{2},\tfrac{1}{2}\},\{\tfrac{11}{4},\tfrac{1}{4}\},\{3,0\},\{\tfrac{13}{4},\tfrac{1}{4}\},\{\tfrac{7}{2},\tfrac{1}{2}\},\{\tfrac{15}{4},\tfrac{3}{4}\},\{4,1\},\{\tfrac{17}{4},1\},\{\tfrac{9}{2},1\},\{\tfrac{19}{4},1\},\{5,1\},\{\tfrac{21}{4},\tfrac{3}{4}\},\{\tfrac{11}{2},\tfrac{1}{2}\},\{\tfrac{23}{4},\tfrac{1}{4}\},\{6,0\}\}$$

We will discuss additional ways to define, manipulate, and evaluate functions as needed. However, Mathematica's extensive programming language allows a great deal of flexibility in defining functions, many of which are beyond the scope of this text. These powerful techniques are discussed in detail in texts such as Gaylord, Kamin, and Wellin's *Introduction to Programming with Mathematica* [9], Gray's *Mastering Mathematica: Programming Methods and Applications* [12], and Maeder's *The Mathematica Programmer II* and *Programming in Mathematica* [15, 16].

## 2.3 GRAPHING FUNCTIONS, EXPRESSIONS, AND EQUATIONS

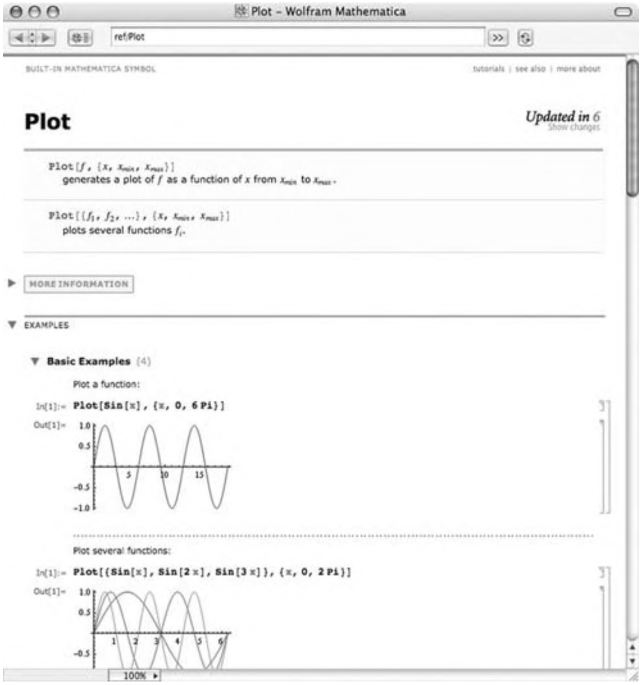
One of the best features of Mathematica is its graphics capabilities. In this section, we discuss methods of graphing functions, expressions, and equations and several of the options available to help graph functions.

### 2.3.1 Functions of a Single Variable

The commands

**Plot[f[x], {x, a, b}] and Plot [f[x], {x, a, x1, x2, . . ., xn, b}]**

graph the function $y = f(x)$ on the intervals $[a, b]$ and $[a, x_1) \cup (x_1, x_2) \cup \ldots \cup (x_n, b]$, respectively. Mathematica returns information about the basic syntax of the Plot command with ?Plot or use the **Documentation Center** to obtain detailed information regarding Plot.




Remember that every Mathematica object can be assigned a name, including graphics. Show[p1,p2, ..., pn] displays the graphics p1, p2, ..., pn together.

**Example 2.3.1** Graph $y = \sin x$ for $-\pi \le x \le 2\pi$.

***Solution*** Entering

**p1 = Plot[Sin[x], {x, −Pi, 2Pi}]**

graphs $y = \sin x$ for $-\pi \le x \le 2\pi$ and names the result p1. The plot is shown in Figure 2.2(a). With

**p1b = Plot[Cos[x], {x, −2Pi, 2Pi},**
**ColorFunction → "ValentineTones",**
**PlotStyle → Thickness[.025]]**

we create a slightly thicker plot of $y = \cos x$ and shade the plot using the ValentineTones color gradient. See Figure 2.2(b).

Show[p1,p2, ..., pn] shows the graphics p1, ..., pn. You can also use Show to rerender graphics. Using Show with the Epilog option together with Inset, we place a small version of the cosine plot in the sine plot. See Figure 2.2(c).

**p1c = Show[p1,**
**Epilog → Inset[p1b, {Pi/2, −1/2}, Automatic, 5]]**

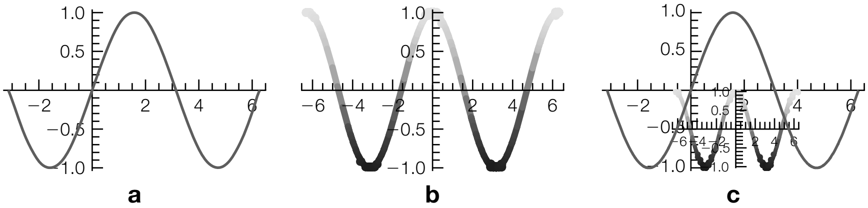


FIGURE 2.2

(a) $y = \sin x$ for $-\pi \le x \le 2\pi$. (b) A "reddish" plot of $y = \cos x$ for $-2\pi \le x \le 2\pi$. (c) Combining two graphics with Epilog and Inset

Multiple graphics can be shown in rows, columns, or grids using GraphicsRow, GraphicsColumn, or GraphicsGrid, respectively. Thus,

```
Show[GraphicsRow[{p1, p1b, p1c}]]
```

generates Figure 2.2.

---

**Be careful when graphing functions with discontinuities. Often, Mathematica will catch discontinuities. In other cases, it does not and you might need to use the Exclusions option to generate a more accurate plot.**

**Example 2.3.2** Graph $s(t)$ for $0 \le t \le 5$, where $s(t) = 1$ for $0 \le t < 1$ and $s(t) = 1 + s(t-1)$ for $t \ge 1$.

***Solution*** After defining $s(t)$,

```
s[t_] := 1/;0 ≤ t < 1
s[t_] := 1 + s[t − 1]/;t ≥ 1
```

we use Plot to graph $s(t)$ for $0 \le t \le 5$ in Figure 2.3(a).

```
p1 = Plot[s[t], {t, 0, 5}, AspectRatio → Automatic]
```

Of course, Figure 2.3(a) is not completely precise: Vertical lines are never the graphs of functions. In this case, discontinuities occur at $t = 1, 2, 3, 4,$ and $5$. If we were to redraw the figure by hand, we would erase the vertical line segments and then for emphasis place open dots at $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, and $(5,5)$ and then closed dots at $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, and $(5,6)$. In cases like this in which Plot does not automatically detect discontinuities, you can specify them with Exclusions. See Figure 2.3(b).

```
p2 = Plot[s[t], {t, 0, 5}, Exclusions → {1, 2, 3, 4}]
Show[GraphicsRow[{p1, p2}]]
```

To fine-tune graphics, use the **Drawing Tools** and **Graphics Inspector** palettes, which are accessed under **Graphics** in the menu. In this case, we add the closed dots at the left endpoints.

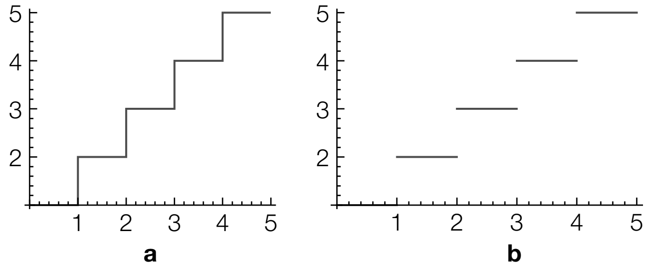
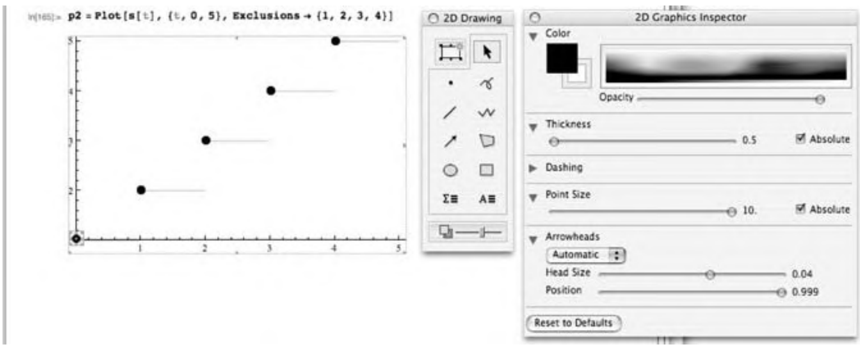
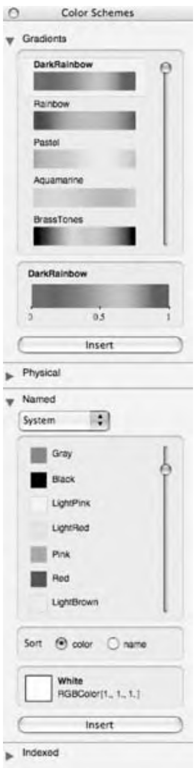



**FIGURE 2.3**

(a) $s(t) = 1 + s(t-1),\ 0 \le t \le 5$. (b) Catching the discontinuities




Entering Options[Plot] lists all Plot options and their default values. The most frequently used options include PlotStyle, DisplayFunction, AspectRatio, PlotRange, PlotLabel, and AxesLabel.

1. PlotStyle controls the color and thickness of a plot. PlotStyle-> GrayLevel[w], where $0 \le w \le 1$, instructs Mathematica to generate the plot in GrayLevel[w]. GrayLevel[0] corresponds to black and GrayLevel[1] corresponds to white. Color plots can be generated using RGBColor. RGBColor[1,0,0] corresponds to red, RGBColor[0,1,0] corresponds to green, and RGBColor[0,0,1] corresponds to blue. You can also use any of the named colors listed on the **Color Schemes** palette.

   PlotStyle->Dashing[a1,a2, . . . , an] indicates that successive segments be dashed with repeating lengths of $a_1,\ a_2,\ \ldots,\ a_n$. The thickness of the plot is controlled with PlotStyle->Thickness[w], where $w$ is the fraction of the total width of the graphic. For a single plot, the PlotStyle options are combined with PlotStyle->{{option1, option2, . . . , optionn}}.

2. A plot is not displayed when the option DisplayFunction->Identity is included or when a semicolon (;) is included at the end of the command. Including the option DisplayFunction->$DisplayFunction in Show or Plot commands instructs Mathematica to display graphics.
3. The ratio of height to width of a plot is controlled by AspectRatio. The default is 1/GoldenRatio. Generally, a plot is drawn to scale when the option AspectRatio->Automatic is included in the Plot or Show command.
4. PlotRange controls the horizontal and vertical axes. PlotRange->{c,d} specifies that the vertical axis displayed corresponds to the interval $c \leq y \leq d$, whereas PlotRange->{{a,b},{c,d}} specifes that the horizontal axis displayed corresponds to the interval $a \leq x \leq b$ and that the vertical axis displayed corresponds to the interval $c \leq y \leq d$.
5. PlotLabel->"titleofplot" labels the plot titleofplot.
6. AxesLabel->{"xaxislabel","yaxislabel"} labels the $x$-axis with xaxislabel and the $y$-axis with yaxislabel.

---

**Example 2.3.3** Graph $y = \sin x$, $y = \cos x$, and $y = \tan x$ together with their inverse functions.

***Solution*** In p2 and p3, we use Plot to graph $y = \sin^{-1} x$ and $y = x$, respectively. Neither plot is displayed because we include a semicolon at the end of the Plot commands. p1, p2, and p3 are displayed together with Show in Figure 2.4. The plot is shown to scale; the graph of $y = \sin x$ is in black, $y = \sin^{-1} x$ is in gray, and $y = x$ is dashed.

Be sure you have completed the previous example immediately before entering the following commands.

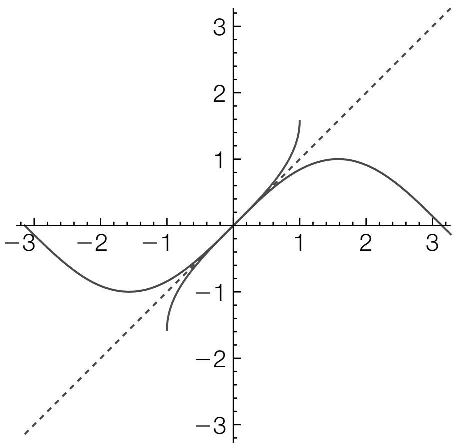



**FIGURE 2.4**

$y = \sin x$, $y = \sin^{-1} x$, and $y = x$

```
p2 = Plot[ArcSin[x], {x, -1, 1}, PlotStyle → GrayLevel[.3]];
p3 = Plot[x, {x, -Pi, 2Pi}, PlotStyle → Dashing[{ . 01}]];
p4 = Show[p1, p2, p3, PlotRange → {{-Pi, Pi}, {-Pi, Pi}}, AspectRatio →
        Automatic]
```

The command Plot[{f1[x],f2[x],. . .,fn[x]},{x,a,b}] plots $f_1(x), f_2(x), \ldots, f_n(x)$ together for $a \le x \le b$. Simple **PlotStyle** options are incorporated with PlotStyle->{**option1**, **option2**, . . ., **optionn**}, where **optioni** corresponds to the plot of $f_i(x)$. Multiple options are incorporated using PlotStyle->{{**options1**}, {**options2**}, . . ., {**optionsn**}}, where **optionsi** are the options corresponding to the plot of $f_i(x)$.

In the following, we use **Plot** to graph $y = \cos x$, $y = \cos^{-1} x$, and $y = x$ together. The plot in Figure 2.5 is shown to scale; the graph of $y = \cos x$ is in black, $y = \cos^{-1} x$ is in gray, and $y = x$ is dashed.

```
r4 = Plot[{Cos[x], ArcCos[x], x}, {x, -Pi, Pi},
   PlotStyle → {GrayLevel[0],
      GrayLevel[.3], Dashing[{ . 01}]},
         PlotRange → {-Pi, Pi}, AspectRatio → Automatic]
```

We use the same idea to graph $y = \tan x$, $y = \tan^{-1} x$, and $y = x$ in Figure 2.6

```
q4 = Plot[{Tan[x], ArcTan[x], x}, {x, -Pi, Pi},
    PlotStyle → {GrayLevel[0],
      GrayLevel[.3], Dashing[{ . 01}]},
         PlotRange → {-Pi, Pi}, AspectRatio → Automatic]
```

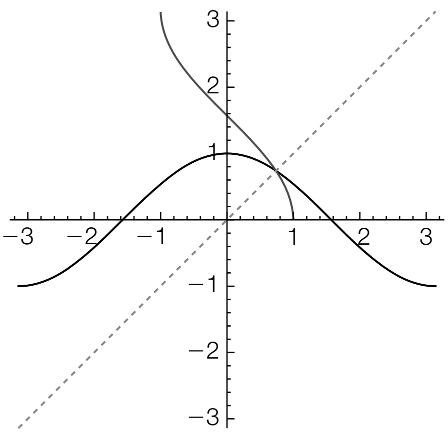



**FIGURE 2.5**

$y = \cos x$, $y = \cos^{-1} x$, and $y = x$

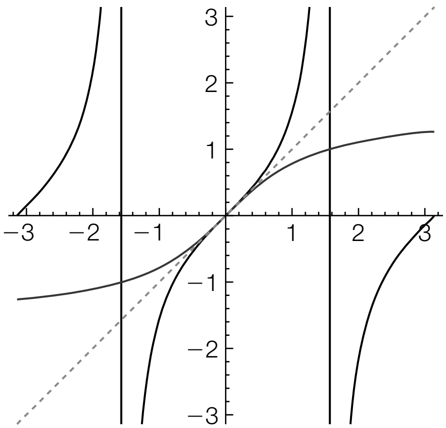



**FIGURE 2.6**

$y = \tan x$, $y = \tan^{-1} x$, and $y = x$

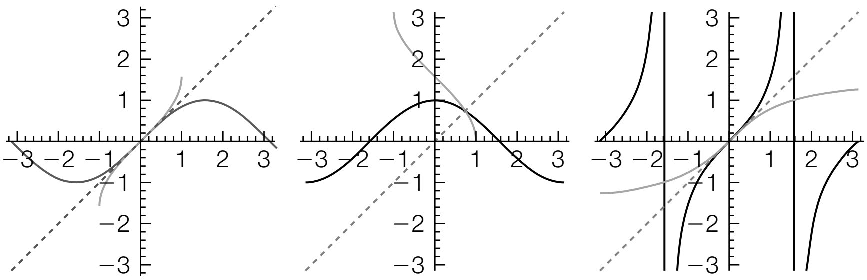



**FIGURE 2.7**

The elementary trigonometric functions and their inverses

Use **Show** together with **GraphicsRow** to display graphics in rectangular arrays. Entering

**Show[GraphicsRow[{p4, r4, q4}]]**

shows the three plots **p4**, **r4**, and **q4** in a row as shown in Figure 2.7.

**The previous example illustrates the graphical relationship between a function and its inverse.**

**Example 2.3.4** **(Inverse Functions).** $f(x)$ and $g(x)$ are **inverse functions** if

$$f(g(x)) = g(f(x)) = x.$$

If $f(x)$ and $g(x)$ are inverse functions, their graphs are symmetric about the line $y = x$. The command

**Composition[f1,f2,f3,...,fn,x]**

computes the composition

$$\left(f_1 \circ f_2 \circ \cdots \circ f_n\right)(x) = f_1\left(f_2\left(\cdots\left(f_n(x)\right)\right)\right).$$

For two functions $f(x)$ and $g(x)$, it is usually easiest to compute the composition $f(g(x))$ with f[g[x]] or f[x]//g.

Show that

$$f(x) = \frac{-1-2x}{-4+x} \quad \text{and} \quad g(x) = \frac{4x-1}{x+2}$$

are inverse functions.

***Solution*** After defining $f(x)$ and $g(x)$,

*$f(x)$ and $g(x)$ are not returned because a semicolon is included at the end of each command.*

```
f[x_] = (-1-2x)/(-4+x);
g[x_] = (4x-1)/(x+2);
```

we compute and simplify the compositions $f(g(x))$ and $g(f(x))$. Because both results are $x$, $f(x)$ and $g(x)$ are inverse functions.

```
f[g[x]]
```

$$\frac{-1-\frac{2(-1+4x)}{2+x}}{-4+\frac{-1+4x}{2+x}}$$

```
Simplify[f[g[x]]]
x
Simplify[g[f[x]]]
x
```

To see that the graphs of $f(x)$ and $g(x)$ are symmetric about the line $y = x$, we use Plot to graph $f(x)$, $g(x)$, and $y = x$ together in Figure 2.8. Because Tooltip is being applied to the set of functions being plotted, you can identify each curve by sliding the cursor over the curve: when the cursor is placed over a curve, Mathematica displays its definition.

```
Plot[Tooltip[{f[x], g[x], f[g[x]]}], {x, -10, 10},
  PlotStyle → {GrayLevel[0], GrayLevel[.3],
    Dashing[{.01}]}, PlotRange → {-10, 10},
      AspectRatio → Automatic]
```

In the plot, observe that the graphs of $f(x)$ and $g(x)$ are symmetric about the line $y = x$. The plot also illustrates that the domain and range of a function and its inverse are interchanged: $f(x)$ has domain $(-\infty, 4) \cup (4, \infty)$ and range $(-\infty, -2) \cup (-2, \infty)$; $g(x)$ has domain $(-\infty, -2) \cup (-2, \infty)$ and range $(-\infty, 4) \cup (4, \infty)$.

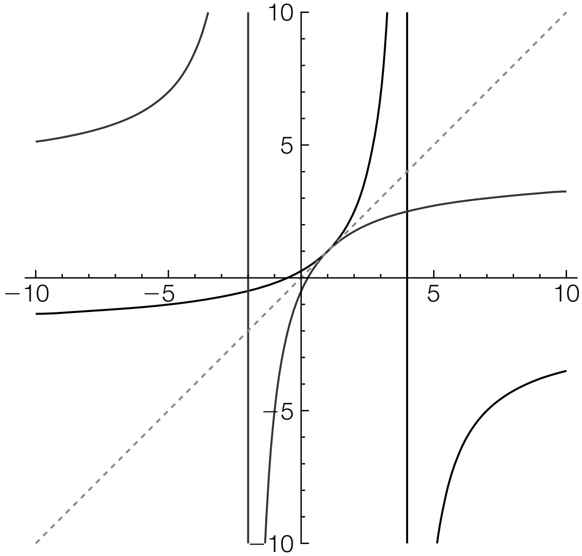



**FIGURE 2.8**

$f(x)$ in black, $g(x)$ in gray, and $y = x$ dashed

For repeated compositions of a function with itself, Nest[f,x,n] computes the composition

$$\underbrace{\left(f \circ f \circ f \circ \cdots \circ f\right)}_{n \text{ times}}(x) = \underbrace{\left(f\left(f\left(f \cdots\right)\right)\right)}_{n \text{ times}}(x) = f^n(x).$$

---

**Example 2.3.5** Graph $f(x)$, $f^{10}(x)$, $f^{20}(x)$, $f^{30}(x)$, $f^{40}(x)$, and $f^{50}(x)$ if $f(x) = \sin x$ for $0 \le x \le 2\pi$.

***Solution*** After defining $f(x) = \sin x$,

```
f[x_] = Sin[x]
Sin[x]
```

we graph $f(x)$ in p1 with Plot

```
p1 = Plot[f[x], {x, 0, 2Pi}];
```

and then illustrate the use of Nest by computing $f^5(x)$.

```
Nest[f, x, 5]
Sin[Sin[Sin[Sin[Sin[x]]]]]
```

Next, we use Table together with Nest to create the list of functions

$$\left\{f^{10}(x), f^{20}(x), f^{30}(x), f^{40}(x), f^{50}(x)\right\}.$$

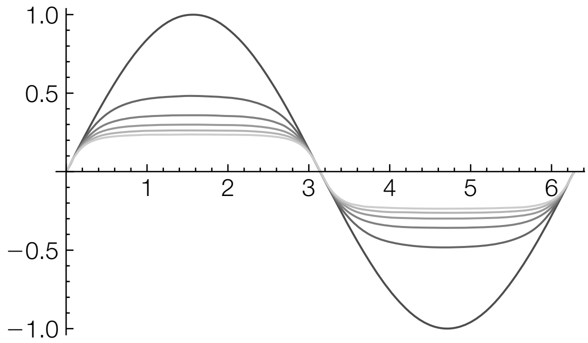



**FIGURE 2.9**

$f(x)$ in black; the graphs of $f^{10}(x)$, $f^{20}(x)$, $f^{30}(x)$, $f^{40}(x)$, and $f^{50}(x)$ are successively lighter—the graph of $f^{50}(x)$ is the lightest

Because the resulting output is rather long, we include a semicolon at the end of the Table command to suppress the resulting output.

Table[f[i],{i,a,b,istep}] computes $f(i)$ for $i$ values from $a$ to $b$ using increments of *istep*.

**toplot = Table[Nest[f, x, n], {n, 10, 50, 10}];**

We then graph the functions in toplot on the interval $[0, 2\pi]$ with Plot, applying the Tooltip function to the list being plotted so they can easily be identified. Last, we use Show to display p1 and p2 together in Figure 2.9.

**p2 = Plot[Tooltip[toplot], {x, 0, 2Pi}];**
**Show[p1, p2]**

In the plot, we see that repeatedly composing sine with itself has a flattening effect on $y = \sin x$.

---

The command

**ListPlot[{{x1, y1}, {x2, y2}, . . ., {xn, yn}}]**

plots the list of points $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$. The size of the points in the resulting plot is controlled with the option PlotStyle->PointSize[w], where $w$ is the fraction of the total width of the graphic. For two-dimensional graphics, the default value is 0.008.

---

**Remark 2.2** The command

**ListPlot[y1, y2, ..., yn]**

plots the list of points $\{(1, y_1), (2, y_2), \ldots, (n, y_n)\}$.

**Example 2.3.6** Graph $y = \dfrac{\sqrt{9 - x^2}}{x^2 - 4}$.

***Solution*** We use Plot to generate the basic graph of $y$ shown in Figure 2.10(a).

**p1 = Plot[Sqrt[9 – x^2]/(x^2 – 4), {x, –5, 5}]**

Observe that the domain of $y$ is $[-3, -2) \cup (-2, 2) \cup (2, 3]$. A better graph of $y$ is obtained by plotting $y$ for $-3 \le x \le 3$ and shown in Figure 2.10(b). We then use the PlotRange option to specify that the displayed horizontal axis corresponds to $-7 \le x \le 7$ and that the displayed vertical axis corresponds to $-7 \le y \le 7$. The graph is drawn to scale because we include the option AspectRatio->Automatic. In this case, Mathematica does not generate any error messages. Mathematica uses a point-plotting scheme to generate graphs. Coincidentally, Mathematica does not sample $x = \pm 2$ and thus does not generate any error messages.

**p2 = Plot[Sqrt[9 – x^2]/(x^2 – 4), {x, –5, 5}, PlotRange → {{–7, 7}, {–7, 7}},
AspectRatio → Automatic]**

To see the endpoints in the plot, we use ListPlot to plot the points $(-3, 0)$ and $(3, 0)$. The points are slightly enlarged in Figure 2.10(c) because we increase their size using PointSize.

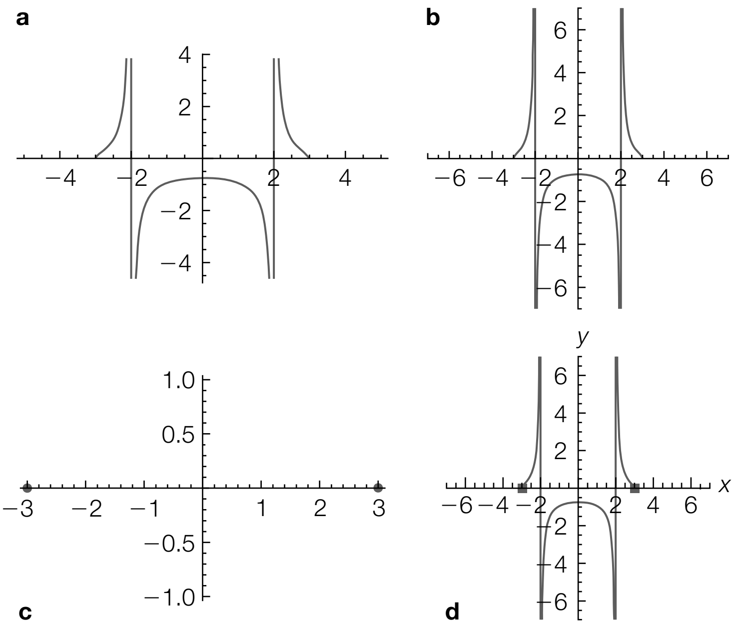



**FIGURE 2.10**

The four plots p1, p2, p3, and p4 combined into a single graphic

**p3 = ListPlot[{{−3, 0}, {3, 0}}, PlotStyle → PointSize[.02]]**

Finally, we use Show to display p2 and p3 together in Figure 2.10(d), where we have labeled the axes using the AxesLabel option.

**p4 = Show[p2, p3, AxesLabel → {"x", "y"}]**

The sequence of plots shown in Figure 2.10, which combines p1, p2, p3, and p4 into a single graphic, is generated using Show together with GraphicsGrid.

**Show[GraphicsGrid[{{p1, p2}, {p3, p4}}]]**

---

When graphing functions involving odd roots, Mathematica's results may be surprising to the beginner. The key is to load the **RealOnly** package located at the Wolfram website first *or* remember that Mathematica follows the order of operations *exactly* and understand that without restrictions on $x$, $\sqrt{x^2} = |x|$.

---

**Example 2.3.7** Graph $y = x^{1/3}(x-2)^{2/3}(x+1)^{4/3}$.

***Solution*** Entering

**p1 = Plot[x^(1/3)(x−2)^(2/3)(x + 1)^(4/3),
{x, −3, 5}, PlotRange → {−4, 4},
AspectRatio → Automatic]**

not does not produce the graph we expect (see Figure 2.11(a)) because many of us consider $y = x^{1/3}(x-2)^{2/3}(x+1)^{4/3}$ to be a real-valued function with domain $(-\infty, \infty)$. Generally, Mathematica does return a real number when computing the odd root of a negative number. For example, $x^3 = -1$ has three solutions:

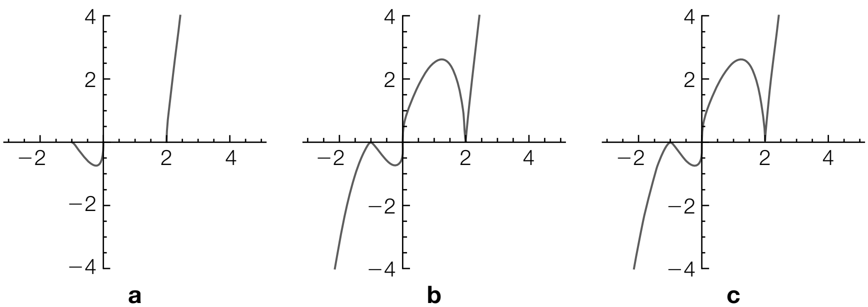



**FIGURE 2.11**

Three plots of $y = x^{1/3}(x-2)^{2/3}(x+1)^{4/3}$

Solve is discussed in more detail in the next section.

```
s1 = Solve[x^3 + 1==0]
```

$\{\{x \to -1\}, \{x \to (-1)^{1/3}\}, \{x \to -(-1)^{2/3}\}\}$

N[number] returns an approximation of number.

```
N[s1]
```

$\{\{x \to -1.\}, \{x \to 0.5 + 0.866025i\}, \{x \to 0.5 - 0.866025i\}\}$

When computing an odd root of a negative number, Mathematica has many choices (as illustrated above) and chooses a root with positive imaginary part—the result is not a real number.

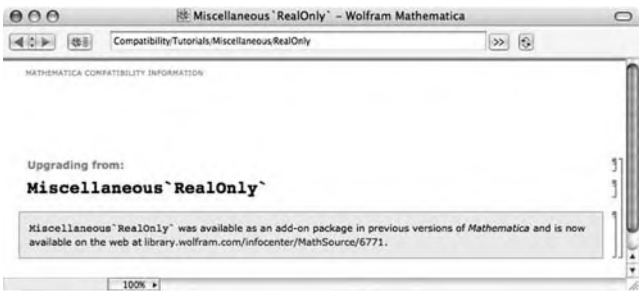
```
(-1)^(1/3)//N
```

$0.5 + 0.866025i$

To obtain real values when computing odd roots of negative numbers, first let $sign(x) = \begin{cases} x/|x|, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$. Sign[x] returns $sign(x)$. Then, for the reduced fraction $n/m$ with $m$ odd, $x^{n/m} = \begin{cases} sign(x)|x|^{n/m}, & \text{if } n \text{ is odd} \\ |x|^{n/m}, & \text{if } n \text{ is even} \end{cases}$. See Figure 2.11(b).

```
p2 = Plot[Sign[x]Abs[x]^(1/3)Abs[x - 2]^(2/3)Abs[x + 1]^(4/3),
        {x, -3, 5}, PlotRange -> {-4, 4}, AspectRatio -> Automatic]
```

Alternatively, load the **RealOnly** package that is located in the **Miscellaneous** folder or directory if you have an older version of Mathematica or that can be downloaded from the Wolfram website if you only have version 6 or later.




After the **RealOnly** package has been loaded, reentering the **Plot** command produces the expected graph. See Figure 2.11c.

```
<< Miscellaneous`RealOnly`
p3 = Plot[x^(1/3)(x - 2)^(2/3)(x + 1)^(4/3), {x, -3, 5},
     PlotRange -> {-4, 4}, AspectRatio -> Automatic]
Show[GraphicsRow[{p1, p2, p3}]]
```

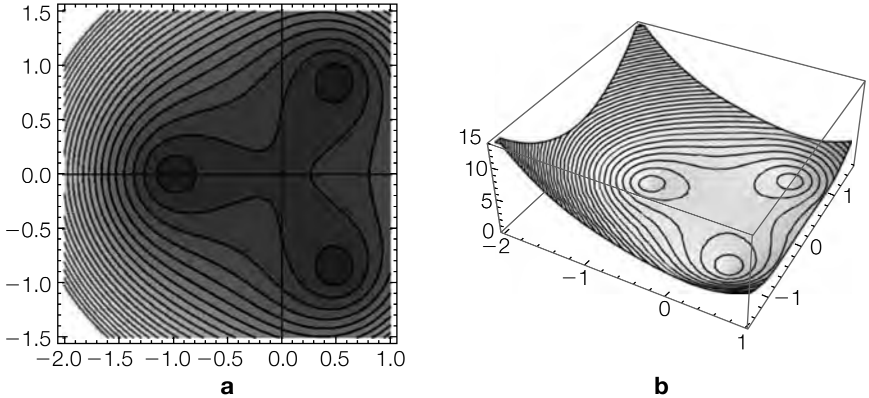



**FIGURE 2.12**

(a) Contour plot of $f(x,y)$, (b) 3D plot of $f(x,y)$

A comprehensive discussion of Mathematica's extensive graphics capabilities cannot be reasonably covered in a single text, so our approach is to address issues that might be of interest or present a different point of view to the novice. In the previous example, we saw that $x^3 + 1 = 0$ has three solutions, two of which are complex. To visualize this graphically, observe that the zeros of $z^3 + 1 = 0$ are the level curves of $f(x,y) = \left|(x + iy)^3 + 1\right|$ ($x$, $y$ real) corresponding to 0. In a plot of $f(x,y)$, the solutions are the zeros. Soon, we will discuss ContourPlot and Plot3D. For now, we remark that

```
cp1 = ContourPlot[Abs[(x + Iy)^3 + 1], {x, −2, 1}, {y, −3/2, 3/2},
    Contours → 30, Axes → True]
p13d = Plot3D[Abs[(x + Iy)^3 + 1], {x, −2, 1}, {y, −3/2, 3/2},
    Axes → True, PlotRange → {0, 15}, MeshFunctions −>{#3&}, Mesh → 35]
Show[GraphicsRow[{cp1, p13d}]]
```

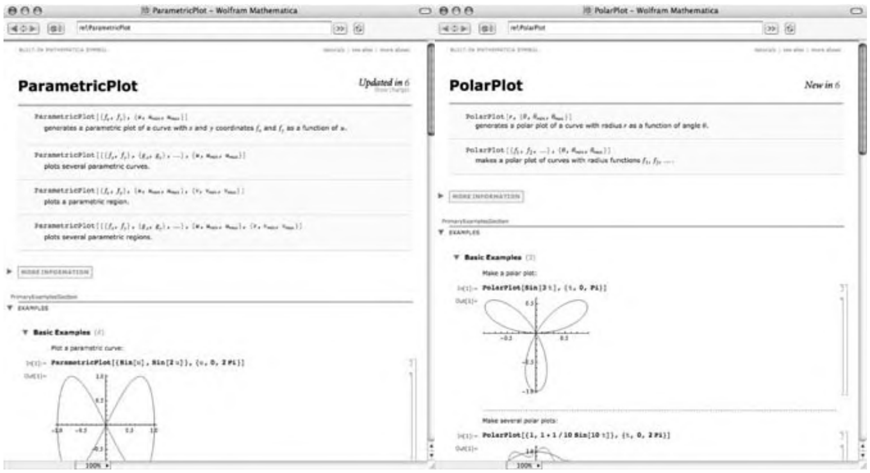
generates several level curves of $f(x,y)$ (Figure 2.12(a)) and a three-dimensional (3D) plot of $f(x,y)$ (Figure 2.12(b)) that help us see the zeros of the original equation. In the 3D plot, note how we use the MeshFunctions option to generate contours.

## 2.3.2 Parametric and Polar Plots in Two Dimensions

ParametricPlot has the same options as Plot.

To graph the parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, use

```
ParametricPlot[{x[t], y[t]}, {t, a, b}]
```




and to graph the polar function $r = r(\theta)$, $\alpha \le \theta \le \beta$, use

**PolarPlot[r[theta], {theta, alpha, beta}].**

---

**Example 2.3.8** **(The Unit Circle).** The **unit circle** is the set of points $(x, y)$ exactly 1 unit from the origin, $(0, 0)$, and, in rectangular coordinates, has equation $x^2 + y^2 = 1$. The unit circle is the classic example of a relation that is neither a function of $x$ nor a function of $y$. The top half of the unit circle is given by $y = \sqrt{1 - x^2}$ and the bottom half is given by $y = -\sqrt{1 - x^2}$.

```
p1 = Plot[{Sqrt[1 - x^2], -Sqrt[1 - x^2]}, {x, -1, 1},
    PlotRange → {{-3/2, 3/2}, {-3/2, 3/2}},
        AspectRatio → Automatic];
```

Each point $(x, y)$ on the unit circle is a function of the angle, $t$, that subtends the $x$-axis, which leads to a parametric representation of the unit circle, $\begin{cases} x = \cos t, \\ y = \sin t, \end{cases}$ $0 \le t \le 2\pi$, which we graph with ParametricPlot.

```
p2 = ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2Pi},
    PlotRange → {{-3/2, 3/2}, {-3/2, 3/2}},
        AspectRatio → Automatic];
```

Using the change of variables $x = r\cos t$ and $y = r\sin t$ to convert from rectangular to polar coordinates, a polar equation for the unit circle is $r = 1$. We use PolarPlot to graph $r = 1$.

```
p3 = PolarPlot[1, {t, 0, 2Pi},
    PlotRange → {{-3/2, 3/2}, {-3/2, 3/2}},
        AspectRatio → Automatic];
```

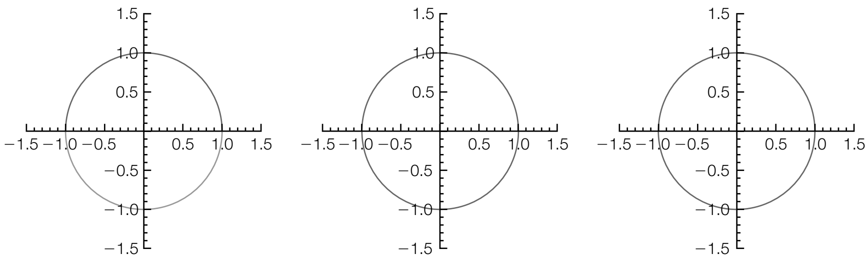



**FIGURE 2.13**

The unit circle generated with Plot, ParametricPlot, and PolarPlot

We display **p1**, **p2**, and **p3** side-by-side using **Show** together with **GraphicsRow** in Figure 2.13. Of course, they all look the same.

```
Show[GraphicsRow[{p1, p2, p3}]]
```

**Example 2.3.9** Graph the parametric equations

$$\begin{cases} x = t + \sin 2t, \\ y = t + \sin 3t, \end{cases} \qquad -2\pi \le t \le 2\pi.$$

***Solution*** After defining $x$ and $y$, we use ParametricPlot to graph the parametric equations in Figure 2.14(a).

```
x[t_] = t + Sin[2t];
y[t_] = t + Sin[3t];
p1 = ParametricPlot[{x[t], y[t]}, {t, −2Pi, 2Pi},
     AspectRatio → Automatic]
```

In Figure 2.14(b), we illustrate how to use the PlotStyle option to increase the thickness of the plot. Color is introduced using ColorFunction together with ColorData. We choose to use the SolarColors gradient to produce our plot.

```
x[t_] = t + Sin[2t];
y[t_] = t + Sin[3t];
p2 = ParametricPlot[{x[t], y[t]}, {t, −2Pi, 2Pi}, PlotStyle → Thickness[.02],
     AspectRatio → Automatic, ColorFunction → (ColorData
        ["SolarColors"][#3]&)]
```

ParametricPlot can also be used to parametrically plot a region. In Figure 2.14(c), we plot $(r^2x(t), r^2y(t))$ for $-2\pi \le t \le 2\pi$ and $0 \le r \le 2$.

```
x[t_] = t + Sin[2t];
y[t_] = t + Sin[3t];
p3 = ParametricPlot[r^2{x[t], y[t]}, {t, −2Pi, 2Pi}, {r, 0, 2},
```

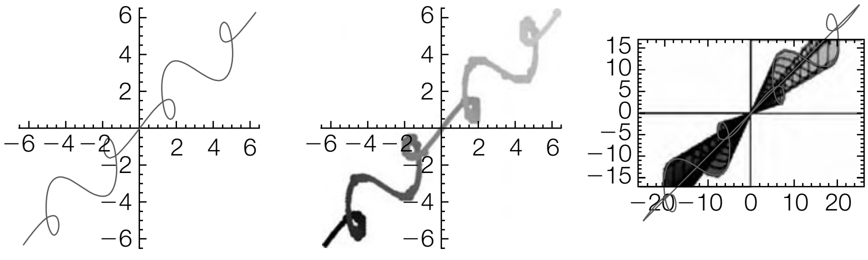



**FIGURE 2.14**

(a) $(x(t), y(t))$, $-2\pi \le t \le 2\pi$, (b) Adding some color and increasing the thickness, (c) Adding a second parameter

```
    AspectRatio → Automatic, ColorFunction → (ColorData
        ["SolarColors"][#3]&)]
Show[GraphicsRow[{p1, p2, p3}]]
```

In the following example, the equations involve integrals.

**Remark 2.3** Topics from calculus are discussed in Chapter 3. For now, we state that Integrate[f[x],{x,a,b}] attempts to evaluate $\int_a^b f(x)\,dx$.

**Example 2.3.10** **(Cornu Spiral).** The **Cornu spiral** (or **clothoid**) (see [11] and [20]) has parametric equations

$$x = \int_0^t \sin\left(\frac{1}{2}u^2\right)\,du \quad \text{and} \quad y = \int_0^t \cos\left(\frac{1}{2}u^2\right)\,du.$$

Graph the Cornu spiral.

***Solution*** We begin by defining $x$ and $y$. Notice that Mathematica can evaluate these integrals, even though the results are in terms of the **FresnelS** and **FresnelC** functions, which are defined in terms of integrals:

$$\text{FresnelS[t]} = \int_0^t \sin\left(\frac{\pi}{2}u^2\right)\,du \quad \text{and} \quad \text{FresnelC[t]} = \int_0^t \cos\left(\frac{\pi}{2}u^2\right)\,du.$$

**x[t_] = Integrate[Sin[u^2/2], {u, 0, t}]**

$\sqrt{\pi}\text{FresnelS}\left[\frac{t}{\sqrt{\pi}}\right]$

**y[t_] = Integrate[Cos[u^2/2], {u, 0, t}]**

$\sqrt{\pi}\text{FresnelC}\left[\frac{t}{\sqrt{\pi}}\right]$

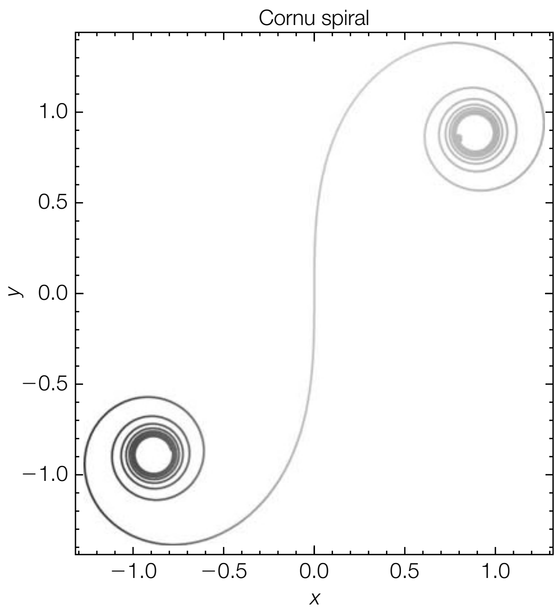



**FIGURE 2.15**

The Cornu spiral

We use ParametricPlot to graph the Cornu spiral in Figure 2.15. The option AspectRatio->Automatic instructs Mathematica to generate the plot to scale; PlotLabel->"Cornu spiral" labels the plot.

```
ParametricPlot[{x[t], y[t]}, {t, −10, 10}, AspectRatio → Automatic,
  PlotStyle → Thickness[.01], PlotLabel −>"Cornuspiral",
    Frame → True, FrameLabel → {x, y},
      ColorFunction → (ColorData["SouthwestColors"][#1]&)]
```

Observe that the graph of the polar equation $r = f(\theta)$, $\alpha \le \theta \le \beta$ is the same as the graph of the parametric equations

$$x = f(\theta)\cos\theta \qquad \text{and } y = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta$$

so both ParametricPlot and PolarPlot can be used to graph polar equations.

**Example 2.3.11** Graph (a) $r = \sin(8\theta/7)$, $0 \le \theta \le 14\pi$; (b) $r = \theta\cos\theta$, $-19\pi/2 \le \theta \le 19\pi/2$; (c) ("The Butterfly") $r = e^{\cos\theta} - 2\cos 4\theta + \sin^5(\theta/12)$, $0 \le \theta \le 24\pi$; and (d) ("The Lituus") $r^2 = 1/\theta$, $0.1 \le \theta \le 10\pi$.

***Solution*** For (a) and (b) we use ParametricPlot. First define $r$ and then use ParametricPlot to generate the graph of the polar curve. No graphics are displayed because we place a semicolon at the end of each command.

```
Clear[r]
r[θ_] = Sin[8θ/7];
pp1 = ParametricPlot[{r[θ]Cos[θ], r[θ]Sin[θ]},
    {θ, 0, 14Pi}, AspectRatio → Automatic];
```

For (b), we use the option PlotRange->{{-30,30},{-30,30}} to indicate that the range displayed on both vertical and horizontal axes corresponds to the interval $[-30, 30]$. To help (a) ensure that the resulting graphic appears "smooth," we increase the number of points that Mathematica samples when generating the graph by including the option PlotPoints->200.

```
Clear[r]
r[θ_] = θCos[θ];
pp2 = ParametricPlot[{r[θ]Cos[θ], r[θ]Sin[θ]},
  {θ, -19π/2, 19π/2}, PlotRange → {{-30, 30}, {-30, 30}},
    AspectRatio → Automatic, PlotPoints → 200];
```

For (c) and (d), we use PolarPlot. Using standard mathematical notation, we know that $\sin^5(\theta/12) = (\sin(\theta/12))^5$. However, when defining $r$ with Mathematica, be sure you use the form `Sin(θ/12)^5`, not `Sin^5[θ/12]`, which Mathematica will not interpret in the way intended.

```
Clear[r]
r[θ_] = Exp[Cos[θ]] - 2Cos[4θ] + Sin[θ/12]^5;
pp3 = PolarPlot[r[θ], {θ, 0, 24π}, PlotPoints → 200,
  PlotRange → {{-4, 5}, {-4.5, 4.5}},
    AspectRatio → Automatic];
Clear[r]
pp4 = PolarPlot[{Sqrt[1/θ], -Sqrt[1/θ]}, {θ, .1, 10π},
    AspectRatio → Automatic, PlotRange → All];
```

Finally, we use Show together with GraphicsGrid to display all four graphs as a graphics array in Figure 2.16. pp1 and pp2 are shown in the first row and pp3 and pp4 are shown in the second.

```
Show[GraphicsGrid[{{pp1, pp2}, {pp3, pp4}}]]
```

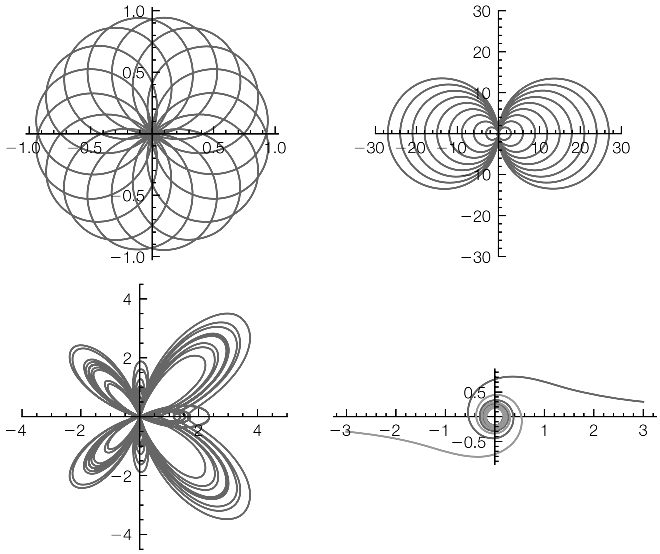



**FIGURE 2.16**

Graphs of four polar equations

## 2.3.3 Three-Dimensional and Contour Plots: Graphing Equations

An elementary function of two variables, $z = f(x, y) =$ *expression in x and y*, is typically defined using the form

**f[x_, y_] = expression in x and y.**

For delayed evaluation, use f[x_,y_]:=... rather than f[x_,y_]=... (immediate evaluation). Once a function has been defined, a basic graph is generated with Plot3D:

**Plot3D[f[x, y], {x, a, b}, {y, c, d}]**

graphs $f(x, y)$ for $a \leq x \leq b$ and $c \leq y \leq d$.

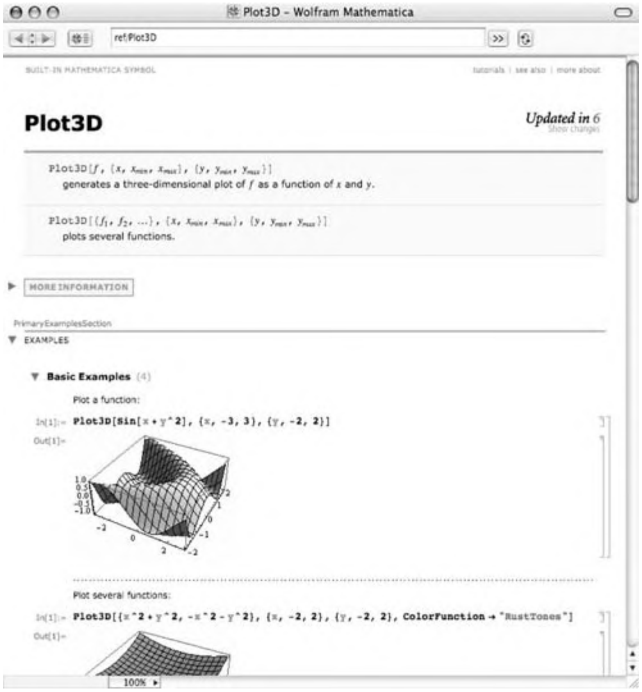
For details regarding Plot3D and its options, enter ?Plot3D or ??Plot3D or access the **Documentation Center** to obtain information about the Plot3D command, as we do here.

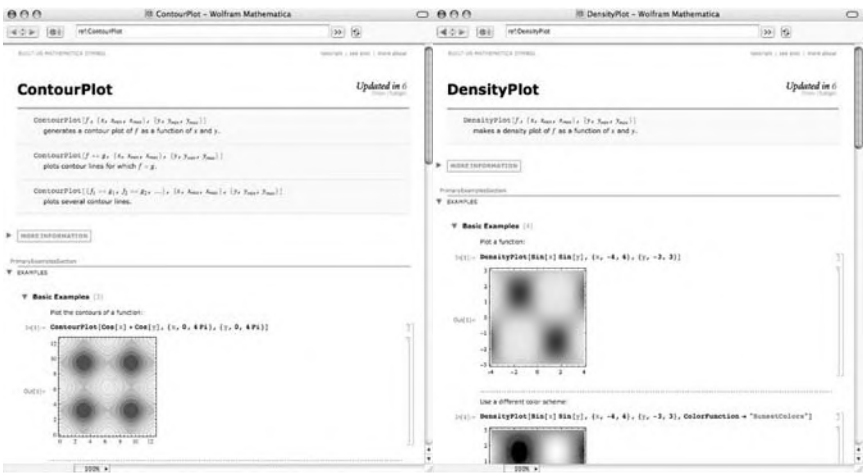
Graphs of several level curves of $z = f(x, y)$ are generated with

**ContourPlot[f[x, y], {x, a, b}, {y, c, d}].**

A density plot of $z = f(x, y)$ is generated with

**DensityPlot[f[x, y], {x, a, b}, {y, c, d}].**




For details regarding ContourPlot (DensityPlot) and its options, enter ?ContourPlot (?DensityPlot) or ??ContourPlot (??DensityPlot) or access the **Documentation Center**.

**Example 2.3.12** Let $f(x,y) = \dfrac{x^2 y}{x^4 + 4y^2}$. (a) Calculate $f(1,-1)$. (b) Graph $f(x,y)$ and several contour plots of $f(x,y)$ on a region containing $(0,0)$.

***Solution*** After defining $f(x,y)$, we evaluate $f(1,-1) = -1/5$.

```
Clear[f]
f[x_, y_] = x^2y/(x^4 + 4y^2)
```

$\frac{x^2 y}{x^4 + 4y^2}$

```
f[1, 1]
```

$-\frac{1}{5}$

Next, we use Plot3D to graph $f(x,y)$ for $-1/2 \le x \le 1/2$ and $-1/2 \le y \le 1/2$ in Figure 2.17. We illustrate the use of the Axes, Boxed, PlotPoints, MeshFunctions, PlotStyle, and ColorFunction options.

```
p1 = Plot3D[f[x, y], {x, -1/2, 1/2}, {y, -1/2, 1/2},
      Axes → Automatic, Boxed → False, PlotPoints → 50]
```

Use MeshFunctions to modify the standard rectangular grid. In Figure 2.17(b), we use the level curves of the function for the grid.

To adjust the viewing angle of three-dimensional graphics, select the graphic and drag to the desired viewing angle.

```
p2 = Plot3D[f[x, y], {x, -1/2, 1/2}, {y, -1/2, 1/2},
    Axes → Automatic, Boxed → False, MeshFunctions->{#3&},
      PlotPoints → 50]
```

We use the GrayTones color gradient to shade the graph (Figure 2.17(c))

```
p3 = Plot3D[f[x, y], {x, -1/2, 1/2}, {y, -1/2, 1/2},
    Axes → Automatic, Boxed → False, MeshFunctions->{#3&},
      PlotPoints → 50, ColorFunction->(ColorData["GrayTones"][#3]&)]
```

Use Opacity to make a "clear" plot (Figure 2.17(d)). We use Show together with GraphicsGrid to display all four plots together in Figure 2.17.

```
p4 = Plot3D[f[x, y], {x, -1/2, 1/2}, {y, -1/2, 1/2},
    Axes → Automatic, Boxed → False, MeshFunctions->{#3&},
      PlotPoints → 50, ColorFunction->(ColorData["GrayTones"][#3]&),
    PlotStyle → Opacity[.3]]
Show[GraphicsGrid[{{p1, p2}, {p3, p4}}]]
```

Four contour plots are generated with ContourPlot. The second through fourth illustrate the use of the PlotPoints, Frame, ContourShading, Axes, AxesOrigin, ColorFunction, and Contours options (see Figure 2.18).

```
cp1 = ContourPlot[f[x, y], {x, -1/2, 1/2}, {y, -1/2, 1/2},
       PlotPoints → 50]
cp2 = ContourPlot[f[x, y], {x, -1/2, 1/2}, {y, -1/2, 1/2},
```

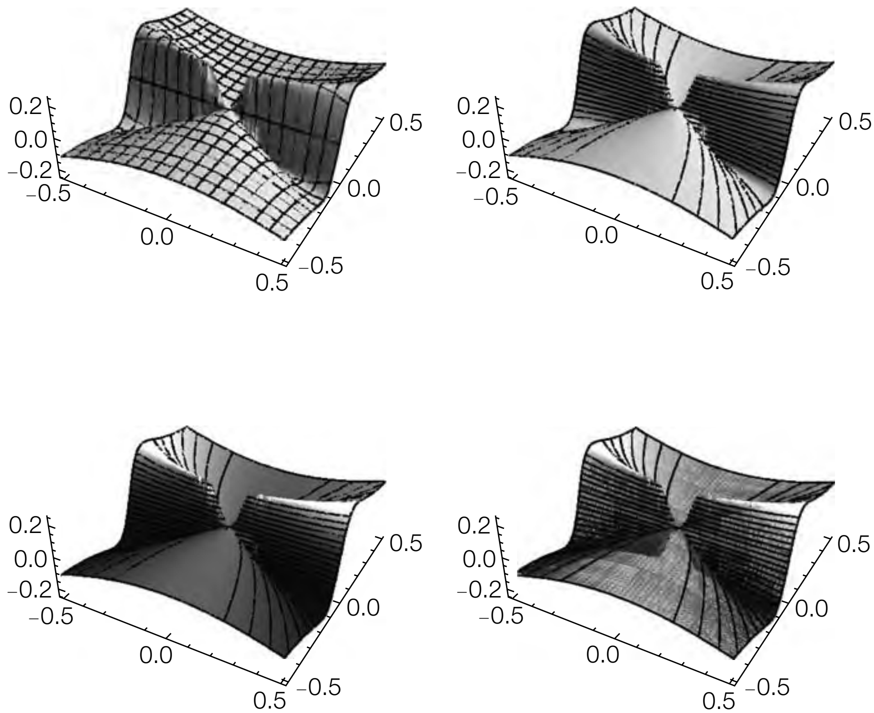
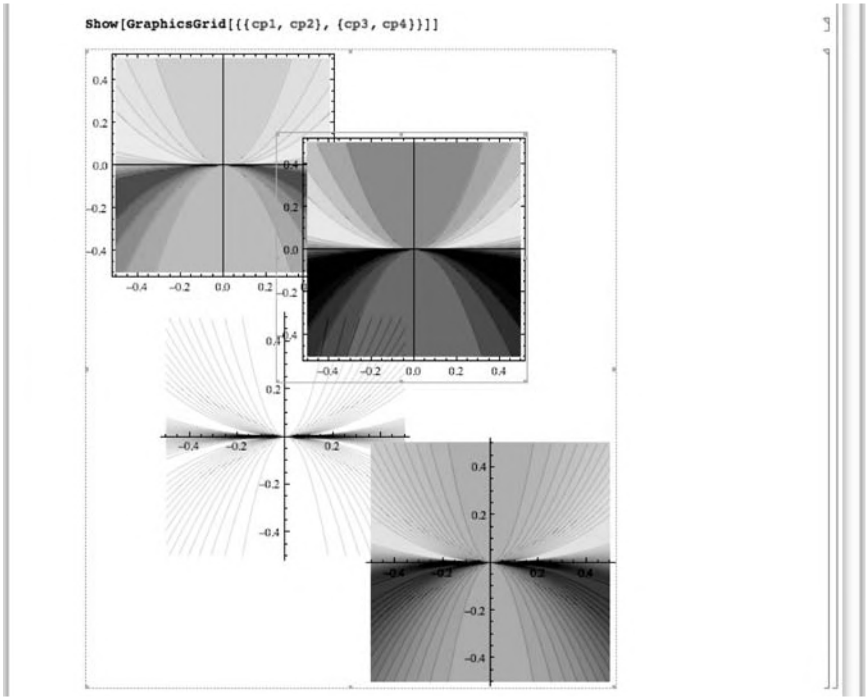



**FIGURE 2.17**

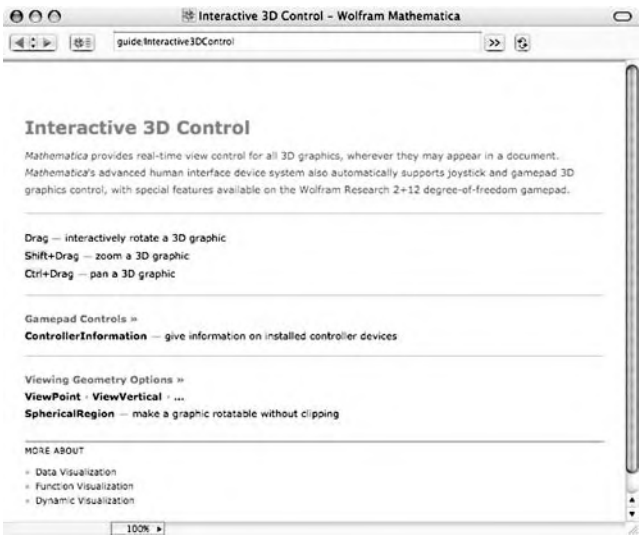
Three-dimensional plot of $f(x, y)$: Upper left is the basic plot generated with Plot3D; in upper right, we use contour lines to determine the mesh; in lower left, we use the GrayTones color gradient to shade the plot; in lower right, we create a transparent plot with Opacity

```
    Axes → Automatic, PlotPoints → 50,
        ColorFunction –>ColorData["GrayTones"]]

cp3 = ContourPlot[f[x, y], {x, – 1/2, 1/2}, {y, – 1/2, 1/2},
    Axes → Automatic, PlotPoints → 50, Contours → 30,
        ContourShading → False, Frame → False,
            Axes → Automatic, AxesOrigin → {0, 0}]

cp4 = ContourPlot[f[x, y], {x, – 1/2, 1/2}, {y, – 1/2, 1/2},
    Axes → Automatic, PlotPoints → 50, Contours → 30,
        Frame → False, ColorFunction → "CandyColors",
            Axes → Automatic, AxesOrigin → {0, 0}]

Show[GraphicsGrid[{{cp1, cp2}, {cp3, cp4}}]]
```

Figure 2.18 shows the graphics array generated with the previous commands. With Mathematica 6, if you want to adjust your array, drag and move the objects within the graphic.




With Mathematica 6, you can adjust the viewing angle of a 3D graphic by selecting the graphic and dragging it to the desired position.
Manually, use the ViewPoint option.



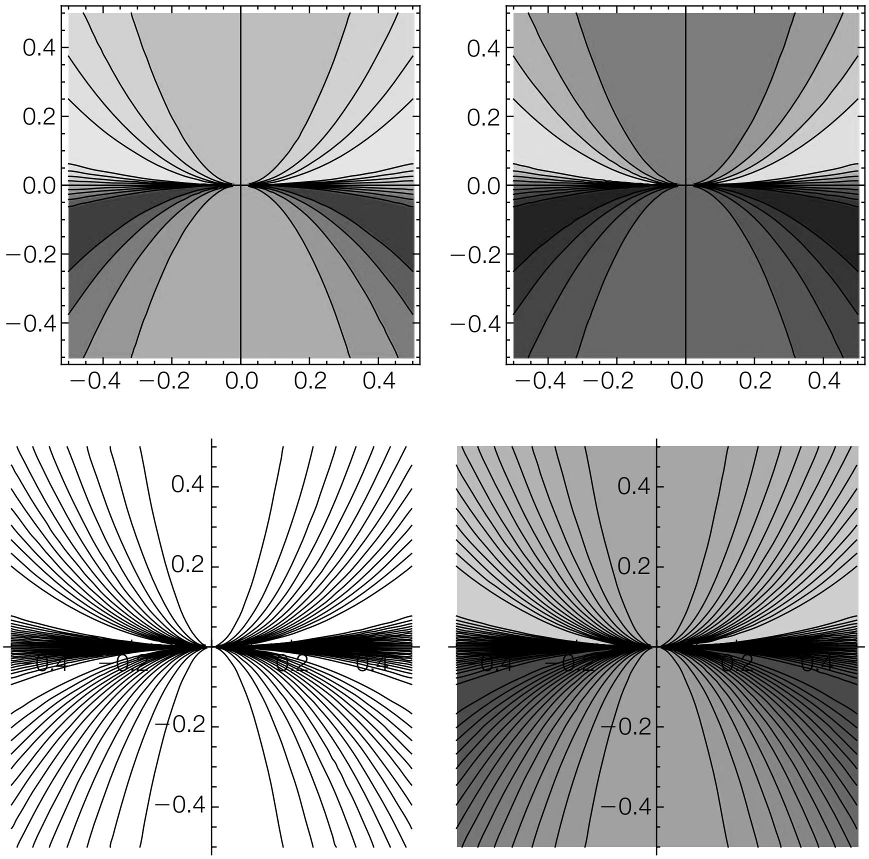
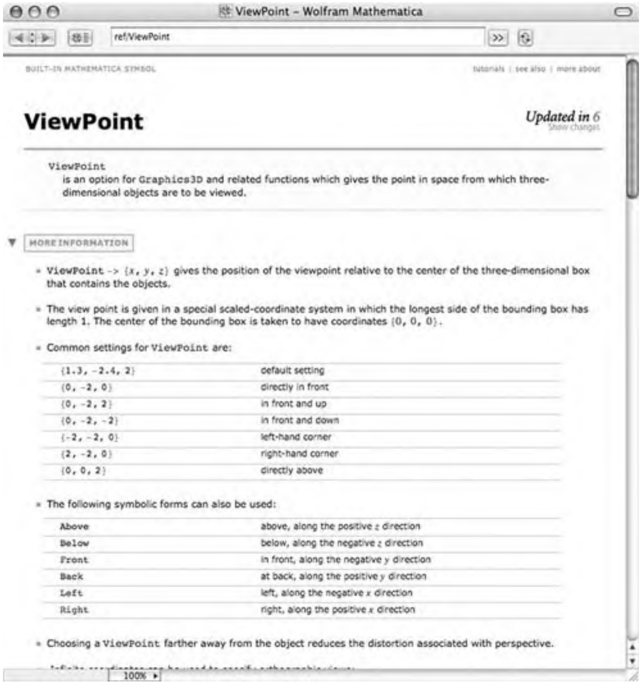



**FIGURE 2.18**

Four contour plots of $f(x,y)$: upper left, the basic plot generated by ContourPlot; upper right, introduce a coloring function; lower left, eliminate the coloring function and increase the number of contours; lower right, add color and increase the number of contours

Figure 2.19 shows four different views of the graph of $g(x,y) = x\sin y + y\sin x$ for $0 \le x \le 5\pi$ and $0 \le y \le 5\pi$. The options AxesLabel, BoxRatios, ViewPoint, PlotPoints, Shading, and Mesh are also illustrated.

```
Clear[g]
g[x_, y_] = x Sin[y] + y Sin[x];
p1 = Plot3D[g[x, y], {x, 0, 5Pi}, {y, 0, 5Pi},
      PlotPoints → 60, AxesLabel → {"x", "y", "z"}];
p2 = Plot3D[g[x, y], {x, 0, 5Pi}, {y, 0, 5Pi},
   PlotPoints → 60, ViewPoint->{-2.846, -1.813, 0.245},
      Boxed → False, BoxRatios → {1, 1, 1},
      AxesLabel → {"x", "y", "z"}];
```




```
p3 = Plot3D[g[x, y], {x, 0, 5π}, {y, 0, 5π},
   PlotPoints → 60, ViewPoint → {1.488, −1.515, 2.634},
      AxesLabel → {"x", "y", "z"}, ColorFunction → (White&)];
p4 = Plot3D[g[x, y], {x, 0, 5Pi}, {y, 0, 5Pi},
   PlotPoints → 60, AxesLabel → {"x", "y", "z"},
      Mesh → False, BoxRatios → {2, 2, 3},
      ViewPoint−>{−1.736, 1.773, −2.301}];
Show[GraphicsGrid[{{p1, p2}, {p3, p4}}]]
```

ContourPlot is especially useful when graphing equations. The graph of the equation $f(x, y) = C$, where $C$ is a constant, is the same as the contour plot of $z = f(x, y)$ corresponding to $C$. That is, the graph of $f(x, y) = C$ is the same as the level curve of $z = f(x, y)$ corresponding to $z = C$.

**Example 2.3.13** Graph the unit circle, $x^2 + y^2 = 1$.

***Solution*** We first graph $z = x^2 + y^2$ for $-4 \le x \le 4$ and $-4 \le y \le 4$ with Plot3D in Figure 2.20(a).

```
p1 = Plot[{Sqrt[1 − x^2], −Sqrt[1 − x^2]}, {x, −1, 1},
    PlotRange → {{−3/2, 3/2}, {−3/2, 3/2}},
       AspectRatio → Automatic];
```

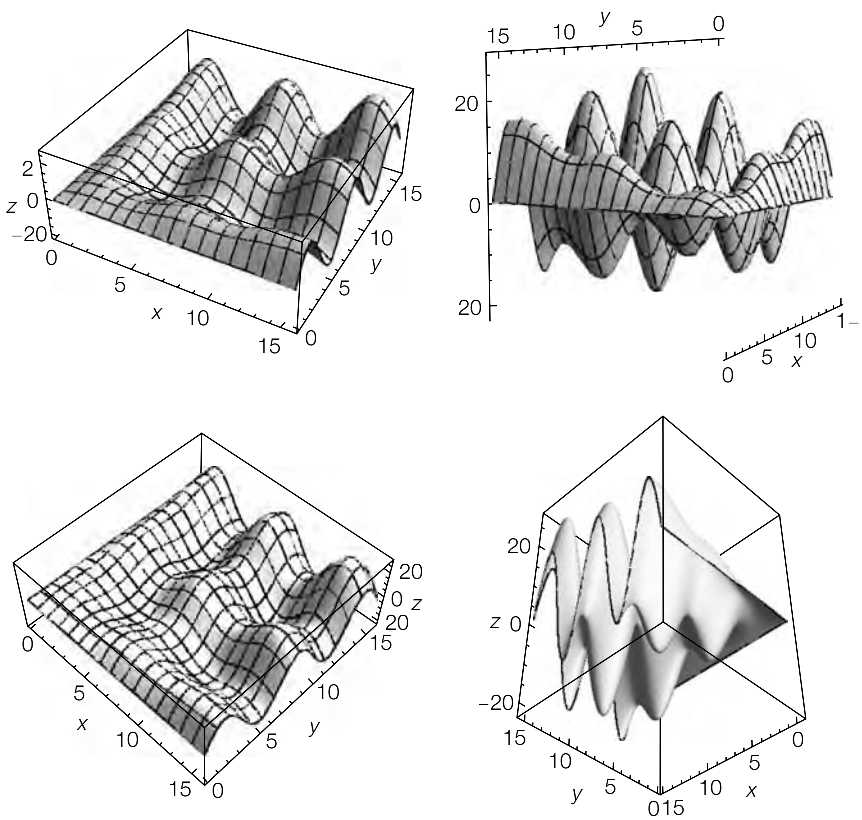



**FIGURE 2.19**

Four different plots of $g(x, y) = x \sin y + y \sin x$ for $0 \le x \le 5\pi$

The graph of $x^2 + y^2 = 1$ is the graph of $z = x^2 + y^2$ corresponding to $z = 1$ as well as the graph of $(\cos t, \sin t)$ for $0 \le t \le 2\pi$. We use ParametricPlot to graph these parametric equations in Figure 2.20.

```
p2 = ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2Pi},
   PlotRange → {{−3/2, 3/2}, {−3/2, 3/2}},
      AspectRatio → Automatic];
```

For the unit circle, it is probably easiest to convert to polar coordinates and use PolarPlot.

When converting from rectangular to polar coordinates, use $x = r\cos\theta$ and $y = r\sin\theta$.

```
p3 = PolarPlot[1, {t, 0, 2Pi},
   PlotRange → {{−3/2, 3/2}, {−3/2, 3/2}},
      AspectRatio → Automatic];
Show[GraphicsRow[{p1, p2, p3}]]
```

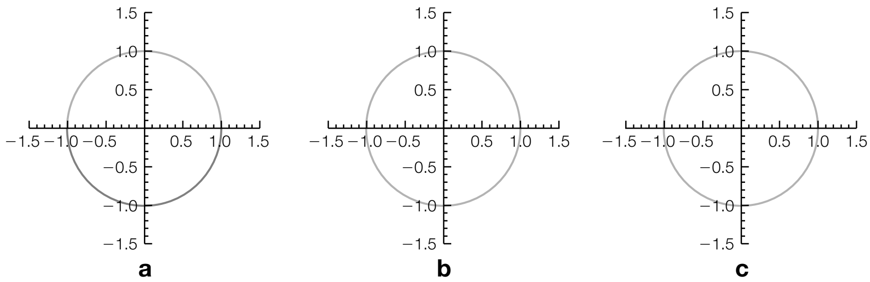


**FIGURE 2.20**

Three different ways of generating plots of the unit circle—all plots are the same in the end

Use ContourPlot to graph equations of the form $f(x, y) = g(x, y)$ with ContourPlot[f[x,y]==g[x,y],{x,a,b},{y,c,d}].

**Example 2.3.14** Graph the equation $y^2 - 2x^4 + 2x^6 - x^8 = 0$ for $-1.5 \le x \le 1.5$.

***Solution*** We define lhseq to be the left-hand side of the equation $y^2 - 2x^4 + 2x^6 - x^8 = 0$ and then use ContourPlot to graph eq for $-1.5 \le x \le 1.5$ in Figure 2.21.

```
Clear[x, y]
lhseq = y^2 - x^4 + 2x^6 - x^8;
cp1 = ContourPlot[lhseq==0, {x, -2, 2}, {y, -2, 2},
      AspectRatio → Automatic]
cp2 = ContourPlot[lhseq==0, {x, -2, 2}, {y, -2, 2},
   AspectRatio → Automatic, Frame → False,
      Axes → Automatic, AxesLabel → {x, y}]
Show[GraphicsRow[{cp1, cp2}]]
```

Equations can be plotted together, as with the commands Plot and Plot3D, with

**ContourPlot[{eq1, eq2, ..., eqn}, {x, a, b}, {y, c, d}].**

**Example 2.3.15** Graph the equations $x^2 + y^2 = 1$ and $4x^2 - y^2 = 1$ for $-1.5 \le x \le 1.5$.

***Solution*** We use ContourPlot to graph the equations together on the same axes in Figure 2.22. The graph of $x^2 + y^2 = 1$ is the unit circle, whereas the graph of $4x^2 - y^2 = 1$ is a hyperbola.

```
cp1 = ContourPlot[{x^2 + y^2==1, 4x^2 - y^2==1},
    {x, -3/2, 3/2}, {y, -3/2, 3/2}, Frame → False,
      Axes → Automatic]
```

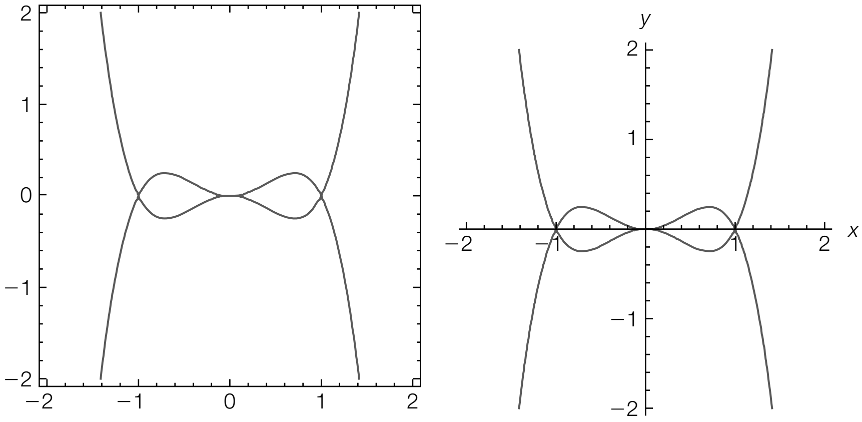



**FIGURE 2.21**

Two plots of $y^2 - 2x^4 + 2x^6 - x^8 = 0$

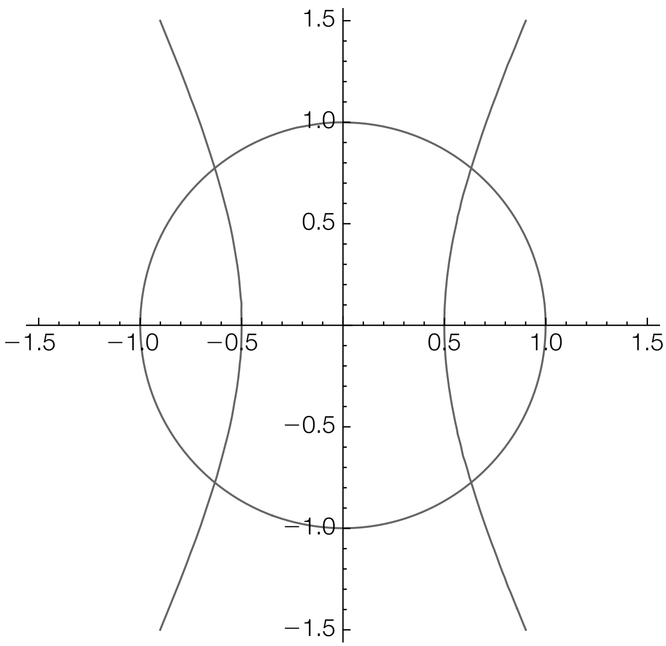



**FIGURE 2.22**

Plots of $x^2 + y^2 = 1$ and $4x^2 - y^2 = 1$

**Example 2.3.16** **(Conic Sections).** A **conic section** is a graph of the equation

Also see Example 2.3.19.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Except when the conic is degenerate, the conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is a (an)

1. **Ellipse** or **circle** if $B^2 - 4AC < 0$;
2. **Parabola** if $B^2 - 4AC = 0$; or
3. **Hyperbola** if $B^2 - 4AC > 0$.

Graph the conic section $ax^2 + bxy + cy^2 = 1$ for $-4 \le x \le 4$ and for $a$, $b$, and $c$ equal to all possible combinations of $-1$, $1$, and $2$.

***Solution*** We begin by defining conic to be the equation $ax^2 + bxy + cy^2 = 1$ and then use Permutations to produce all possible orderings of the list of numbers $\{-1, 1, 2\}$, naming the resulting output vals.

Permutations[list] returns a list of all possible orderings of the list list.

```
Clear[a, b, c, x, y, p]
conic = ax^2 + bxy + cy^2==1;
    vals = Permutations[{-1, 1, 2}]
{{-1, 1, 2}, {-1, 2, 1}, {1, -1, 2},
        {1, 2, -1}, {2, -1, 1}, {2, 1, -1}}
```

Next we define the function p. Given a1, b1, and c1, p defines toplot to be the equation obtained by replacing *a, b*, and *c* in conic by a1, b1, and c1, respectively. Then, toplot is graphed for $-4 \le x \le 4$. p returns a graphics object.

```
Clear[p]
p[{a1_, b1_, c1_}] := Module[{toplot},
    toplot = Evaluate[conic/ . {a → a1, b → b1, c → c1}];
        ContourPlot[Evaluate[toplot],
        {x, -5, 5}, {y, -5, 5}, Frame → False,
            Axes → Automatic, Ticks → None]
    ]
```

We then use Map to compute p for each ordered triple in vals. The resulting output, named graphs, is a set of six graphics objects.

```
graphs = Map[p, vals];
```

Partition is then used to partition graphs into three element subsets. The resulting array of graphics objects named toshow is displayed with Show and GraphicsGrid in Figure 2.23.

```
Show[GraphicsGrid[Partition[graphs, 3]]]
```

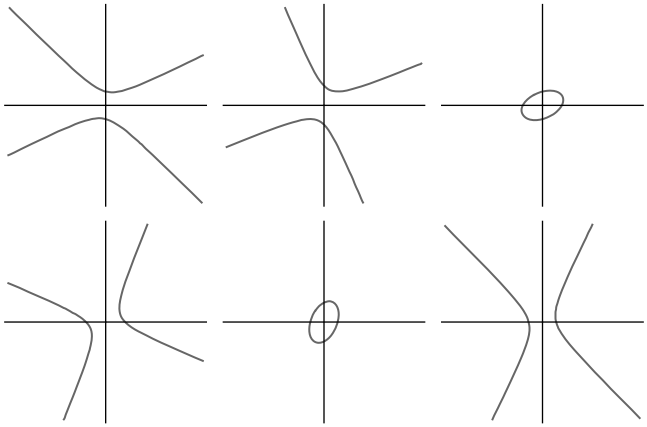


**FIGURE 2.23**
Plots of six conic sections

## 2.3.4 Parametric Curves and Surfaces in Space

The command

**ParametricPlot3D[{x[t], y[t], z[t]}, {t, a, b}]**

generates the three-dimensional curve $\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t), \end{cases} \quad a \le t \le b,$ and the command

**ParametricPlot3D[{x[u, v], y[u, v], z[u, v]}, {u, a, b}, {v, c, d}]**

plots the surface $\begin{cases} x = x(u, v), \\ y = y(u, v), \\ z = z(u, v), \end{cases} \quad a \le u \le b,\ c \le v \le d.$

Entering Information[ParametricPlot3D] or ??ParametricPlot3D returns a description of the ParametricPlot3D command along with a list of options and their current settings.

**Example 2.3.17** **(Umbilic Torus NC)**. A parametrization of **umbilic torus NC** is given by $\mathbf{r}(s, t) = x(s, t)\mathbf{i} + y(s, t)\mathbf{j} + z(s, t)\mathbf{k},\ -\pi \le s \le \pi,\ -\pi \le t \le \pi,$ where

$$x = \left[7 + \cos\left(\frac{1}{3}s - 2t\right) + 2\cos\left(\frac{1}{3}s + t\right)\right]\sin s$$
$$y = \left[7 + \cos\left(\frac{1}{3}s - 2t\right) + 2\cos\left(\frac{1}{3}s + t\right)\right]\cos s$$

and

$$z = \sin\left(\frac{1}{3}s - 2t\right) + 2\sin\left(\frac{1}{3}s + t\right).$$

Graph the torus.

***Solution*** We define $x$, $y$, and $z$.

```
c = 3;
a = 1;
x[s_, t_] = (7 + Cos[s/3 - 2t] + 2Cos[s/3 + t])Sin[s];
y[s_, t_] = (7 + Cos[s/3 - 2t] + 2Cos[s/3 + t])Cos[s];
z[s_, t_] = Sin[s/3 - 2t] + 2Sin[s/3 + t];
r[s_, t_] = {x[s, t], y[s, t], z[s, t]};
```

The torus is then graphed with ParametricPlot3D, DensityPlot, and ContourPlot in Figure 2.24. In the plots, we illustrate the Mesh, MeshFunctions, PlotPoints, and PlotRange options. All four plots are shown together with Show and GraphicsGrid. Notice that DensityPlot and ContourPlot yield very similar results: A basic density plot is like a basic contour plot without the contour lines.

```
threedp1uta = ParametricPlot3D[r[s, t], {s, -Pi, Pi},
    {t, -Pi, Pi}, PlotPoints->{30, 30},
        AspectRatio->1, AxesLabel->{"x", "y", "z"},
            PlotRange->{{-12, 12}, {-12, 12}, {-3, 3}},
            BoxRatios->{4, 4, 1}, Mesh → False, PlotStyle → Opacity[.9]]
threedp1utb = ParametricPlot3D[r[s, t], {s, -Pi, Pi},
    {t, -Pi, Pi}, PlotPoints->{50, 50},
        AspectRatio->1, AxesLabel->{"x", "y", "z"},
            PlotRange->{{-12, 12}, {-12, 12}, {-3, 3}},
            BoxRatios->{4, 4, 1},
            MeshFunctions->{#3&}, Mesh → 10]
threedp1utc = DensityPlot[r[s, t], {s, -Pi, Pi},
    {t, -Pi, Pi}, PlotPoints->{100, 100},
        AspectRatio->1,
            MeshFunctions->{#3&}, Mesh → 10]
threedp1utd = ContourPlot[r[s, t], {s, -Pi, Pi},
    {t, -Pi, Pi}, PlotPoints->{100, 100},
        AspectRatio->1,
            MeshFunctions->{#3&}, Mesh → 10]
Show[GraphicsGrid[{{threedp1uta, threedp1utb},
        {threedp1utc, threedp1utd}}]]
```

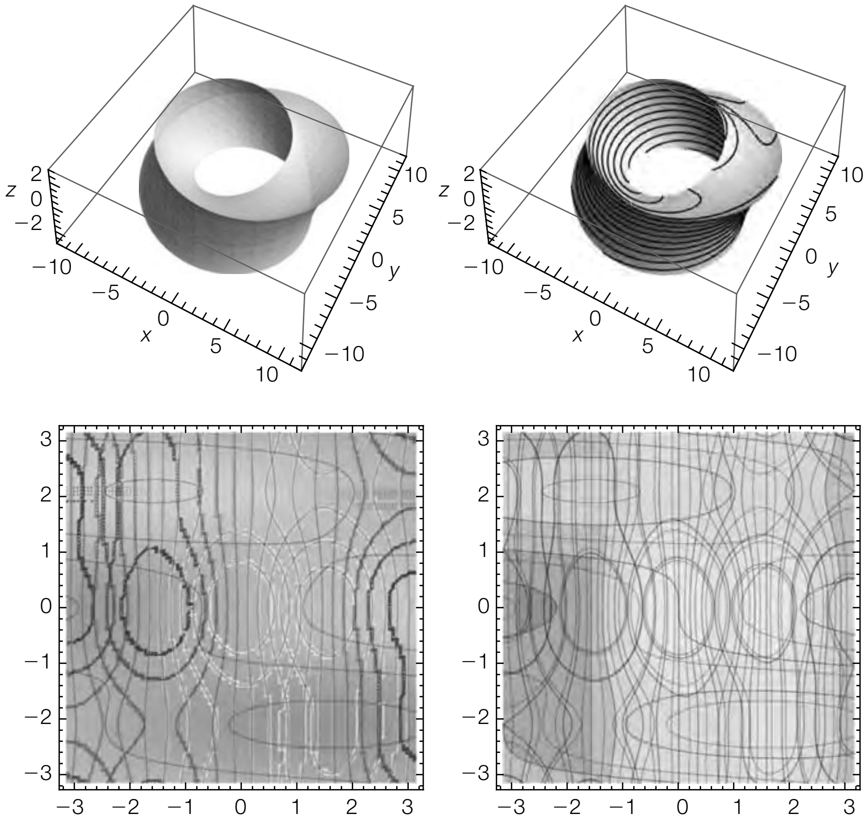



**FIGURE 2.24**

On the top row, two plots of umbilic torus; on the bottom, comparing a density plot (left) to a contour plot (right)

**Example 2.3.18 (Gray's Torus Example).** A parametrization of an **elliptical torus** is given by

This example is explored in detail in Sections 8.2 and 11.4 of Gray's *Modern Differential Geometry of Curves and Surfaces* [11], an indispensible reference for those who use Mathematica's graphics extensively.

$$x = (a + b\cos v)\cos u, \quad y = (a + b\cos v)\sin u, \quad z = c\sin v$$

For positive integers $p$ and $q$, the curve with parametrization

$$x = \left(a + b\cos qt\right)\cos pt, \quad y = \left(a + b\cos qt\right)\sin pt, \quad z = c\sin qt$$

winds around the elliptical torus and is called a **torus knot**.

Plot the torus if $a = 8$, $b = 3$, and $c = 5$ and then graph the torus knots for $p = 2$ and $q = 5$, $p = 1$ and $q = 10$, and $p = 2$ and $q = 3$.

***Solution*** We begin by defining torus and torusknot.

```
torus[a_, b_, c_][p_, q_][u_, v_] :=
    {(a + b Cos[u])Cos[v], (a + b Cos[u])Sin[v], c Sin[u]}
torusknot[a_, b_, c_][p_, q_][t_] :=
    {(a + b Cos[qt])Cos[pt], (a + b Cos[qt])Sin[pt], cSin[qt]}
```

Next, we use ParametricPlot3D to generate all four graphs

```
pp1 = ParametricPlot3D[Evaluate[torus[8, 3, 5][2, 5][u, v]],
    {u, 0, 2Pi}, {v, 0, 2Pi}, PlotPoints → 60];
pp2 = ParametricPlot3D[Evaluate[torusknot[8, 3, 5][2, 5][t]],
    {t, 0, 3Pi}, PlotPoints → 200];
pp3 = ParametricPlot3D[Evaluate[torusknot[8, 3, 5][1, 10][t]],
    {t, 0, 3Pi}, PlotPoints → 200];
pp4 = ParametricPlot3D[Evaluate[torusknot[8, 3, 5][2, 3][t]],
    {t, 0, 3Pi}, PlotPoints → 200];
```

and show the result as a graphics array with Show and GraphicsGrid in Figure 2.25.

```
Show[GraphicsGrid[{{pp1, pp2}, {pp3, pp4}}]]
```

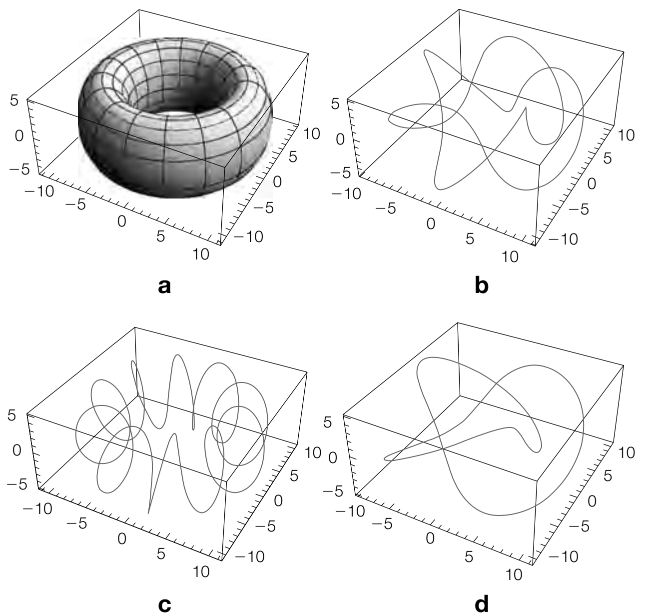



**FIGURE 2.25**

(a) An elliptical torus. (b) This knot is also known as the trefoil knot. (c) The curve generated by torusknot[8,3,5][2,3][1,10] is not a knot. (d) The torus knot with $p = 2$ and $q = 3$

If we take advantage of a few options, such as eliminating the mesh (Mesh->False) and increasing the opacity (PlotStyle->Opacity[.4]), we can produce a graphic of the knot on the torus. After using the PlotStyle option together with Opacity, we produce a nearly transparent torus. Then, each knot is plotted. To ensure smooth plots, we increase the number of points plotted with PlotPoints and also increase the thickness of the curve with Thickness.

```
pp1 = ParametricPlot3D[Evaluate[torus[8, 3, 5][2, 5][u, v]],
    {u, 0, 2Pi}, {v, 0, 2Pi}, PlotPoints → 60,
        Mesh → False, PlotStyle → Opacity[.4],
            ColorFunction → "AlpineColors"];
pp2 = ParametricPlot3D[Evaluate[torusknot[8, 3, 5][2, 5][t]],
        {t, 0, 3Pi}, PlotPoints → 200, PlotStyle → {{Thickness[.01]}}];
pp3 = ParametricPlot3D[Evaluate[torusknot[8, 3, 5][1, 10][t]],
        {t, 0, 3Pi}, PlotPoints → 200, PlotStyle → {{Thickness[.01]}}];
pp4 = ParametricPlot3D[Evaluate[torusknot[8, 3, 5][2, 3][t]],
        {t, 0, 3Pi}, PlotPoints → 200, PlotStyle → {{Thickness[.01]}}];
```

We use Show twice together with GraphicsRow to first display the torus with each knot and then display all three graphics side-by-side in Figure 2.26.

```
Show[GraphicsRow[{Show[{pp1, pp2}], Show[{pp1, pp3}], Show[{pp1, pp4}]}]]
```

---

**Example 2.3.19** **(Quadric Surfaces)**. The **quadric surfaces** are the three-dimensional objects corresponding to the conic sections in two dimensions. A **quadric surface** is a graph of

Also see Example 2.3.16.

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where $A$–$J$ are constants.

The intersection of a plane and a quadric surface is a conic section.

Several of the basic quadric surfaces, in standard form, and a parametrization of the surface are listed in the following table.

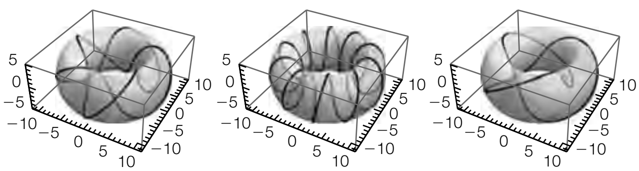



**FIGURE 2.26**

The knots in Figure 2.25 on the torus

| Name | Parametric Equations |
|---|---|
| *Ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ | $\begin{cases} x = a\cos t\cos r, \\ y = b\cos t\sin r, \\ z = c\sin t, \end{cases}$ $-\pi/2 \le t \le \pi/2,\ -\pi \le r \le \pi$ |
| *Hyperboloid of One Sheet* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ | $\begin{cases} x = a\sec t\cos r, \\ y = b\sec t\sin r, \\ z = c\tan t, \end{cases}$ $-\pi/2 < t < \pi/2,\ -\pi \le r \le \pi$ |
| *Hyperboloid of Two Sheets* $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ | $\begin{cases} x = a\sec t, \\ y = b\tan t\cos r, \\ z = c\tan t\sin r, \end{cases}$ $-\pi/2 < t < \pi/2$ or $\pi/2 < t < 3\pi/2,\ -\pi \le r \le \pi$ |

Graph the ellipsoid with equation $\frac{1}{16}x^2+\frac{1}{4}y^2+z^2=1$, the hyperboloid of one sheet with equation $\frac{1}{16}x^2+\frac{1}{4}y^2-z^2=1$, and the hyperboloid of two sheets with equation $\frac{1}{16}x^2-\frac{1}{4}y^2-z^2=1$.

***Solution*** A parametrization of the ellipsoid with equation $\frac{1}{16}x^2+\frac{1}{4}y^2+z^2=1$ is given by

$$x = 4\cos t\cos r, \quad y = 2\cos t\sin r, \quad z = \sin t, \quad -\pi/2 \le t \le \pi/2,\ -\pi \le r \le \pi,$$

which is graphed with ParametricPlot3D.

```
Clear[x, y, z]
x[t_, r_] = 4Cos[t]Cos[r];
y[t_, r_] = 2Cos[t]Sin[r];
z[t_, r_] = Sin[t];
pp1 = ParametricPlot3D[{x[t, r], y[t, r], z[t, r]}, {t, -Pi/2, Pi/2},
      {r, -Pi, Pi}, PlotPoints -> 30];
```

A parametrization of the hyperboloid of one sheet with equation $\frac{1}{16}x^2+\frac{1}{4}y^2-z^2=1$ is given by

$$x = 4\sec t\cos r, \quad y = 2\sec t\sin r, \quad z = \tan t, \quad -\pi/2 < t < \pi/2,\ -\pi \le r \le \pi.$$

Because $\sec t$ and $\tan t$ are undefined if $t = \pm\pi/2$, we use ParametricPlot3D to graph these parametric equations on a subinterval of $[-\pi/2, \pi/2]$, $[-\pi/3, \pi/3]$.

```
Clear[x, y, z]
x[t_, r_] = 4Sec[t]Cos[r];
```

```
y[t_, r_] = 2Sec[t]Sin[r];
z[t_, r_] = Tan[t];
pp2 = ParametricPlot3D[{x[t, r], y[t, r], z[t, r]}, {t, -Pi/3, Pi/3},
        {r, -Pi, Pi}, PlotPoints → 30];
```

pp1 and pp2 are shown together in Figure 2.27 using Show and GraphicsRow.

```
Show[GraphicsRow[{pp1,pp2}]]
```

For (c), we take advantage of the ContourPlot3D function:

```
ContourPlot3D[f[x, y, z], {x, a, b}, {y, c, d}, {z, u, v}]
```

graphs several level surfaces of $w = f(x, y, z)$.

We use ContourPlot3D to graph the equation $\frac{1}{16}x^2 - \frac{1}{4}y^2 - z^2 - 1 = 0$ in Figure 2.28(a), illustrating the use of the PlotPoints, Axes, AxesLabel, and BoxRatios options. In Figure 2.28(b), several level surfaces are drawn that illustrate the use of the Opacity function with the ContourStyle and Mesh options.

```
cp3d1 = ContourPlot3D[x^2/16 - y^2/4 - z^2 - 1==0,
   {x, -10, 10}, {y, -8, 8}, {z, -2, 2},
      PlotPoints → {8, 8, 8}, Axes → Automatic,
         AxesLabel → {"x", "y", "z"}, BoxRatios → {2, 1, 1}]
```

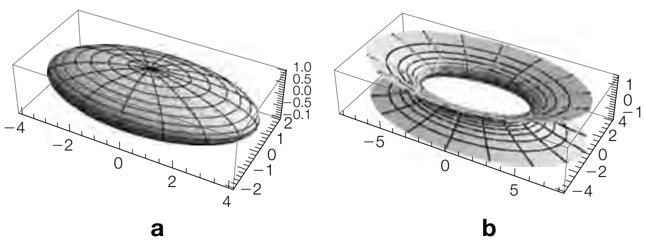


**FIGURE 2.27**

(a) Plot of $\frac{1}{16}x^2 + \frac{1}{4}y^2 + z^2 = 1$. (b) Plot of $\frac{1}{16}x^2 + \frac{1}{4}y^2 - z^2 = 1$

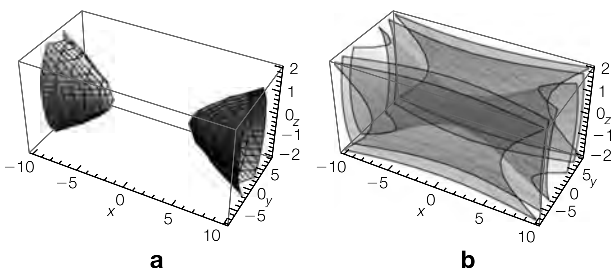


**FIGURE 2.28**

(a) Plot of $\frac{1}{16}x^2 - \frac{1}{4}y^2 - z^2 = 1$ generated with ContourPlot3D. (b) Several level surfaces of $f(x, y, z) = \frac{1}{16}x^2 - \frac{1}{4}y^2 - z^2$

```
cp3d2 = ContourPlot3D[x^2/16 – y^2/4 – z^2 – 1,
    {x, – 10, 10}, {y, – 8, 8},
        {z, – 2, 2}, PlotPoints → {8, 8, 8}, Axes → Automatic,
            AxesLabel → {"x", "y", "z"},
            BoxRatios → {2, 1, 1}, Mesh → False,
    ContourStyle → Opacity[.5]]
Show[GraphicsRow[{cp3d1, cp3d2}]]
```

ContourPlot3D is especially useful in plotting equations involving three variables $x$, $y$, and $z$ for which it is difficult to solve for one variable as a function of the other two.

**Example 2.3.20** **(Cross-Cap)**. The **Cross-Cap** has equation

$$4x^2\left(x^2+y^2+z^2+z\right)+y^2\left(y^2+z^2-1\right)=0.$$

We ContourPlot3D to generate the plot of the cross-cap shown in Figure 2.29.

```
ContourPlot3D[4x^2(x^2 + y^2 + z^2 + z) +
    y^2(y^2 + z^2 – 1)==0, {x, – 1, 1}, {y, – 1, 1},
        {z, – 1, 1}, Mesh → False, Boxed → False,
        Axes → None, ColorFunction → (ColorData["BrightBands"][#3]&),
            ContourStyle → Opacity[.8]]
```

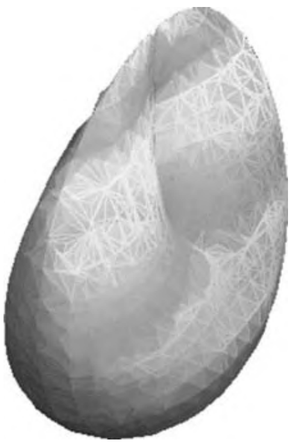


**FIGURE 2.29**

The Cross-Cap

**Example 2.3.21** A homotopy from the **Roman surface** to the **Boy surface** is given by

If $f$ and $g$ are functions from $X$ to $Y$, a **homotopy** from $f$ to $g$ is a continuous function $H$ from $X \times [0,1]$ to $Y$ satisfying $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

$$x(u,v) = \frac{\sqrt{2}\cos(2u)\,\cos^2 v + \cos u\,\sin(2v)}{2 - \alpha\sqrt{2}\sin(3u)\,\sin(2v)},$$

$$y(u,v) = \frac{\sqrt{2}\sin(2u)\,\cos^2 v + \sin u\,\sin(2v)}{2 - \alpha\sqrt{2}\sin(3u)\,\sin(2v)}, \text{ and}$$

$$z(u,v) = \frac{3\cos^2 v}{2 - \alpha\sqrt{2}\sin(3u)\,\sin(2v)}.$$

Tables and lists are discussed in more detail in Chapters 4 and 5.

Here, $\alpha = 0$ gives the Roman surface and $\alpha = 1$ gives the Boy surface.

To see the homotopy we first define $x$, $y$, and $z$.

```
x[α_][u_, v_] = (Sqrt[2]Cos[2u]Cos[v]^2 + Cos[u]Sin[2v])/
        (2 - α Sqrt[2]Sin[3u]Sin[2v]);
y[α_][u_, v_] = (Sqrt[2]Sin[2u]Cos[v]^2 + Sin[u]Sin[2v])/
        (2 - α Sqrt[2]Sin[3u]Sin[2v]);
z[α_][u_, v_] = 3Cos[v]^2/
        (2 - α Sqrt[2]Sin[3u]Sin[2v]);
```

We then use Table together with ParametricPlot3D to parametrically plot $x$, $y$, and $z$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ for nine equally spaced values of $\alpha$ between 0 and 1. Note that if the semicolon is omitted at the end of the command, the nine plots are displayed.

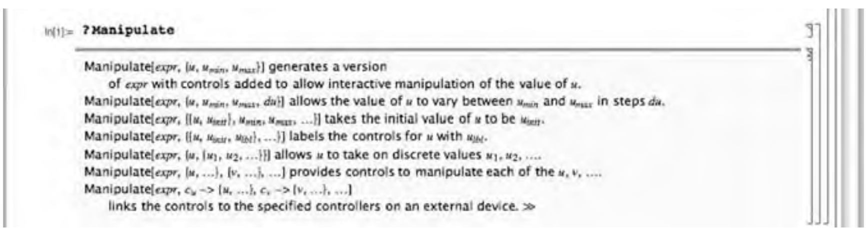
```
smalltable = Table[ParametricPlot3D[
    {x[α][u, v], y[α][u, v], z[α][u, v]},
        {u, 0, 2Pi}, {v, 0, 2Pi}, Boxed → False, Axes → None,
            PlotRange → {{-2, 5/2}, {-2, 2}, {0, 7/2}}],
        {α, 0, 1, 1/8}];
```

We then use Partition to partition smalltable into three element subsets. The resulting $3 \times 3$ array of graphics is shown as a grid with Show together with GraphicsGrid in Figure 2.30.

To adjust the viewing angles of three-dimensional plots, select the graphic and drag to the desired viewing angle.

```
Show[GraphicsGrid[Partition[smalltable, 3]]]
```

Another way of seeing the transformation is to use Manipulate. Manipulate is *very* powerful.



In[1]:= ?Manipulate

Manipulate[*expr*, {*u*, $u_{min}$, $u_{max}$}] generates a version of *expr* with controls added to allow interactive manipulation of the value of *u*.
Manipulate[*expr*, {*u*, $u_{min}$, $u_{max}$, *du*}] allows the value of *u* to vary between $u_{min}$ and $u_{max}$ in steps *du*.
Manipulate[*expr*, {{*u*, $u_{init}$}, $u_{min}$, $u_{max}$, ...}] takes the initial value of *u* to be $u_{init}$.
Manipulate[*expr*, {{*u*, $u_{init}$, $u_{lbl}$}, ...}] labels the controls for *u* with $u_{lbl}$.
Manipulate[*expr*, {*u*, {$u_1$, $u_2$, ...}}] allows *u* to take on discrete values $u_1$, $u_2$, ....
Manipulate[*expr*, {*u*, ...}, {*v*, ...}, ...] provides controls to manipulate each of the *u*, *v*, ....
Manipulate[*expr*, $c_u$ -> {*u*, ...}, $c_v$ -> {*v*, ...}, ...] links the controls to the specified controllers on an external device. >>

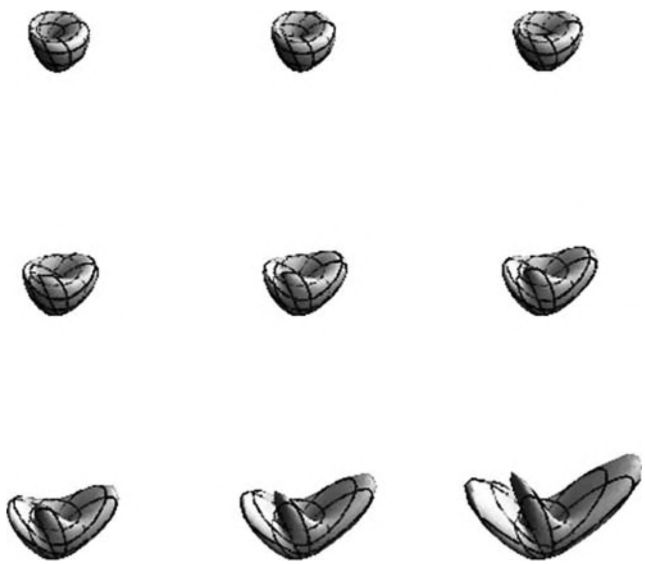


**FIGURE 2.30**

Seeing the Roman surface continuously transform to the Boy surface

In its most basic form, Manipulate[f[x],{x,a,b}] creates an interactive display of $f(x)$ for $x$ values from $a$ to $b$. Because the previous commands depended only on $\alpha$, we combine the commands into a single Manipulate object that depends on $\alpha$.

```
Manipulate[
Clear[x, y, z];
x[α_][u_, v_] = (Sqrt[2]Cos[2u]Cos[v]^2 + Cos[u]Sin[2v]) /
        (2 − αSqrt[2]Sin[3u]Sin[2v]);
y[α_][u_, v_] = (Sqrt[2]Sin[2u]Cos[v]^2 + Sin[u]Sin[2v]) /
        (2 − αSqrt[2]Sin[3u]Sin[2v]);
z[α_][u_, v_] = 3Cos[v]^2 /
        (2 − αSqrt[2]Sin[3u]Sin[2v]);
ParametricPlot3D[
    {x[α][u, v], y[α][u, v], z[α][u, v]},
        {u, 0, 2Pi}, {v, 0, 2Pi}, PlotPoints → 50,
            Boxed → False, Axes → None,
        PlotRange → {{−2.5, 2.5}, {−2.5, 2.5}, {0, 3.5}}],
    {α, 0, 1}]
```

Several images from the result are shown in Figure 2.31.

Manipulation of graphics is discussed in more detail in Chapter 5. Here, we simply illustrate a few quick ways to manipulate a basic jpeg that illustrates a few of the features of Mathematica 6.

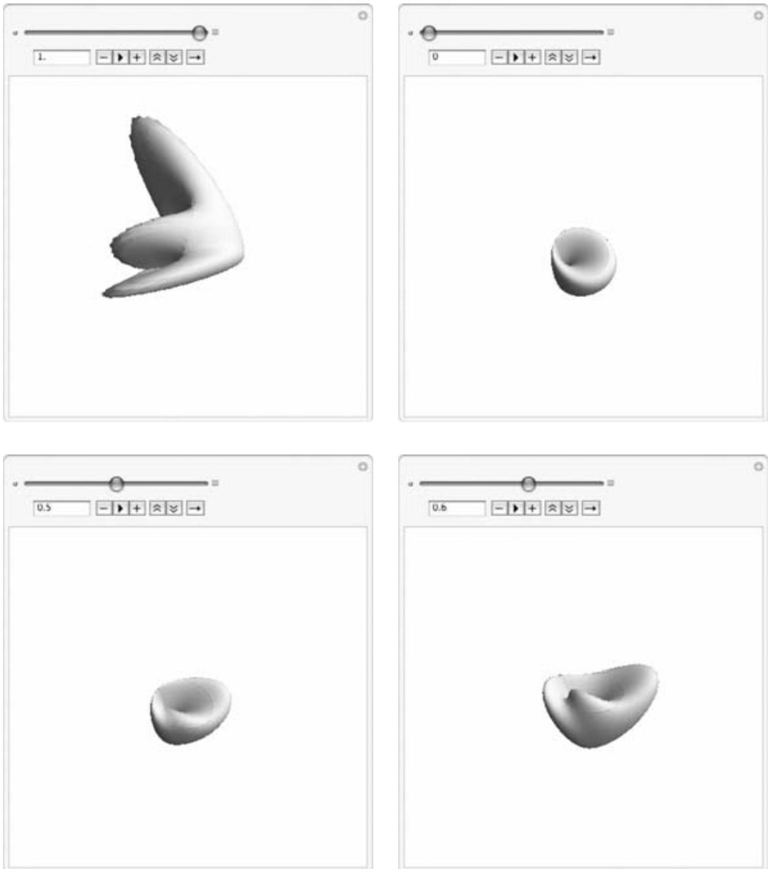


**FIGURE 2.31**

With Manipulate we can create an animation of the transformation of the Roman surface to the Boy surface or inspect the plot for various values of $\alpha$

**Example 2.3.22** We use Import to import a few graphics into Mathematica. The four graphs are displayed in a row using Show and GraphicsRow in Figure 2.32.

```
p1 = Import["house1.jpg"];
p2 = Import["house2.jpg"];
p3 = Import["county1.jpg"];
p4 = Import["county2.jpg"];
Show[GraphicsRow[{p1, p2, p3, p4}]]
```

The underlying structure of a jpeg is contained in the first element of the first part of the graphic. Part and manipulation of matrices is discussed in more detail in

**FIGURE 2.32**

Importing elementary graphics into Mathematica

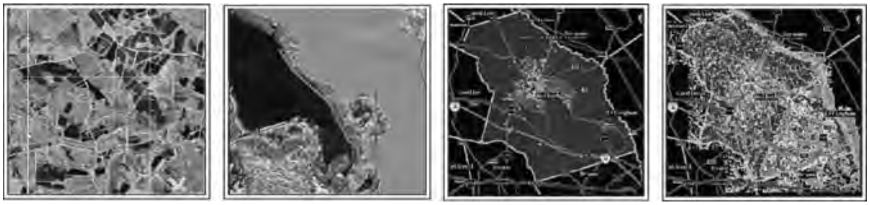


**FIGURE 2.33**

Visualizing data with ArrayPlot may not produce results identical to the original without additional adjustments

Chapter 5. Regardless, you should be able to import your jpeg into Mathematica and adjust the following code to achieve your desired results.

With just the basic data, ArrayPlot does not reproduce the imported graphics as we see in Figure 2.33.

```
Show[GraphicsRow[
        Map[ArrayPlot[Reverse[#[[1, 1]]]]&, {p1, p2, p3, p4}]]]
```

However, with some manipulation, you can reveal interesting detail. First, we use ReliefPlot to help us see the terrain of the image.

```
q1 = Flatten[p1[[1, 1]], 1];
q1b = Table[q1[[i, 1]], {i, 1, Length[q1]}];
q1c = Partition[q1b, Length[p1[[1, 1, 1]]]];

r1 = ReliefPlot[q1c, ColorFunction → "GreenBrownTerrain"]
```

A different view is obtained by choosing a different ColorFunction.

```
q2 = Flatten[p2[[1, 1]], 1];
q2b = Table[q2[[i, 2]], {i, 1, Length[q2]}];
q2c = Partition[q2b, Length[p2[[1, 1, 1]]]];

r2 = ReliefPlot[q2c, ColorFunction → "GrayTones"]
```

ReliefPlot and ArrayPlot return similar graphics. Here are two images of Southeast Georgia generated with ArrayPlot.

```
q3 = Flatten[p3[[1, 1]], 1];
q3b = Table[q3[[i, 1]], {i, 1, Length[q3]}];
q3c = Partition[q3b, Length[p3[[1, 1, 1]]]];
```

```
r3 = ArrayPlot[Reverse[q3c], ColorFunction → "FallColors"]
```

```
q4 = Flatten[p4[[1, 1]], 1];
q4b = Table[q4[[i, 1]], {i, 1, Length[q4]}];
q4c = Partition[q4b, Length[p4[[1, 1, 1]]]];
```

```
r4 = ArrayPlot[Reverse[q4c], ColorFunction → "StarryNightColors"]
```

All four images are displayed together using Show with GraphicsGrid in Figure 2.34.

```
Show[GraphicsGrid[{{r1, r2}, {r3, r4}}]]
```

## 2.3.5 Miscellaneous Comments

Be sure to take advantage of **MathWorld** for a huge number of resources related to graphics and Mathematica.

Clearly, Mathematica's graphics capabilities are extensive and *volumes* could be written about them. You can see many commands that we have

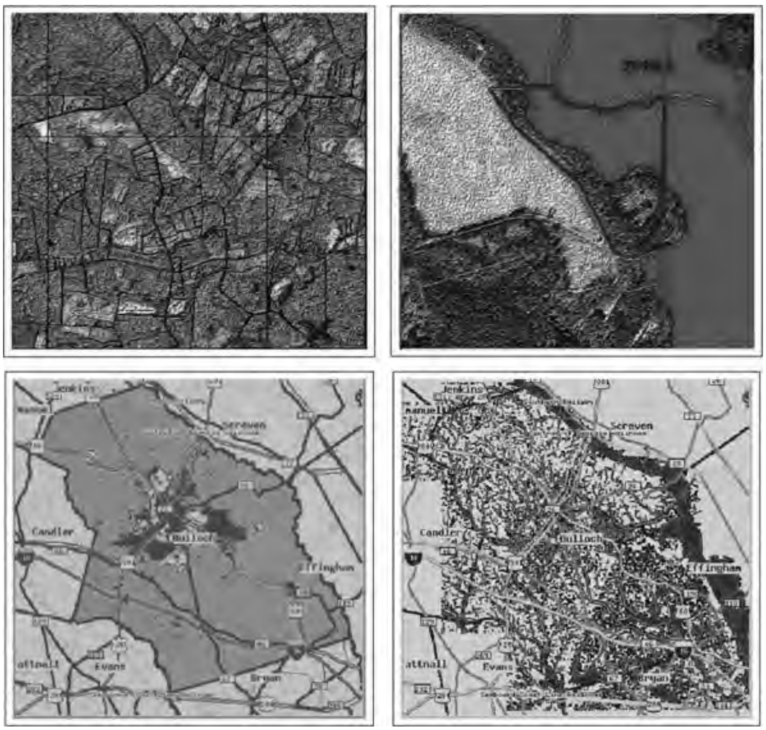



**FIGURE 2.34**

Using ReliefPlot and ArrayPlot to adjust elementary graphics not discussed here by using ? to see those commands that contain the string Plot.

```
In[10]:= ?*Plot*
```

▼ System`

| | | |
|---|---|---|
| ArrayPlot | ListPlot3D | PlotLabel |
| ContourPlot | ListPointPlot3D | PlotMarkers |
| ContourPlot3D | ListPolarPlot | PlotPoints |
| DateListPlot | ListSurfacePlot3D | PlotRange |
| DensityPlot | LogLinearPlot | PlotRangeClipping |
| GraphPlot | LogLogPlot | PlotRangePadding |
| GraphPlot3D | LogPlot | PlotRegion |
| LayeredGraphPlot | MatrixPlot | PlotStyle |
| ListContourPlot | MaxPlotPoints | PolarPlot |
| ListContourPlot3D | ParametricPlot | RegionPlot |
| ListDensityPlot | ParametricPlot3D | RegionPlot3D |
| ListLinePlot | Plot | ReliefPlot |
| ListLogLinearPlot | Plot3D | RevolutionPlot3D |
| ListLogLogPlot | Plot3Matrix | SphericalPlot3D |
| ListLogPlot | PlotDivision | TreePlot |
| ListPlot | PlotJoined | |

You can obtain detailed information regarding any of these commands from the **Documentation Center** by clicking on the command's name.

For now, we briefly mention a few of the ones not discussed previously. To plot lists of numbers or lists of ordered pairs, use ListPlot, which is discussed in Chapter 4. For matrices and other arrays, use MatrixPlot or ArrayPlot.

---

**Example 2.3.23** **(Cellular Automaton)**. Very loosely speaking, a **cellular automaton** is a discrete function that assigns values to subsequent rows based on the values of the cells in the previous row(s). For a concise discussion of cellular automaton, refer to Weisstein,[1] CellularAutomatan is a powerful command that allows you to investigate (quite complicated) cellular automaton. In its simplest form, CellularAutomaton[rule, initialvalues, n] returns the first $n$ generations of the cellular automaton following the specified rule and having the indicated initial values.

The simplest cellular automaton are called **elementary cellular automaton**.[2] Based on basic counting principals, there are 256 elementary cellular automatons. They are cataloged by number. With

---

[1] Weisstein, Eric W., "Cellular Automaton." From *MathWorld—A Wolfram Web Resource,* http://mathworld.wolfram.com/CellularAutomaton.html.

[2] Weisstein, Eric W., "Elementary Cellular Automaton." From *MathWorld—A Wolfram Web Resource,* http://mathworld.wolfram.com/ElementaryCellularAutomaton.html.

```
CellularAutomaton[146, {{1}, 0}, 5]
{{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0},
  {0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0},
    {0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0}, {1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1}}
```

we calculate the first five generations of the elementary cellular automaton with a 1 at position 0 on generation 0 using Rule 146. To calculate the first 100 generations, we use CellularAutomaton[146, 1,0, 100]+. The resulting array is rather large, so we use ArrayPlot to visualize it in Figure 2.35(a). Using our color scheme, the cells with value 1 are shaded red and those with 0 are light green.

```
a1 = ArrayPlot[CellularAutomaton[146, {{1}, 0}, 100],
  ColorFunction → "NeonColors", AspectRatio → 1]
```

In this case, the grid is initially spaced so that positions 1, 11, 21, 31, and 41 have the value 1. The first three generations using Rule 146 are calculated.

```
CellularAutomaton[146,
  {SparseArray[{1 → 1, 11 → 1, 21 → 1, 31 → 1, 41 → 1}], 0}, 3]
{{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
    {0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0,
      0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0},
    {0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0,
  1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0},
    {1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1,
      0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1}}
```

How the situation evolves over 100 generations is more easily seen using ArrayPlot. See Figure 2.35(b).

```
a2 = ArrayPlot[CellularAutomaton[146,
  {SparseArray[{1 → 1, 11 → 1, 21 → 1, 31 → 1, 41 → 1}], 0}, 100],
    ColorFunction → "NeonColors", AspectRatio → 1]
Show[GraphicsRow[{a1, a2}]]
```

Of the 256 elementary cellular automaton, many are equivalent. To see that some of them are equivalant, we create a plot of the 256 elementary cellular automaton for 50 generations as done with Rule 146. All 256 plots are shown on the left in Figure 2.36(a). With Union, we remove and sort the ones that are identically equal. Those are shown on the right in Figure 2.36(b).

```
t1 = Table[ArrayPlot[CellularAutomaton[i, {{1}, 0}, 50]],
    {i, 0, 255}];
t2 = Partition[t1, 16];
p1 = Show[GraphicsGrid[t2]];
t3 = Union[t1];
t4 = Partition[t3, 12];
```

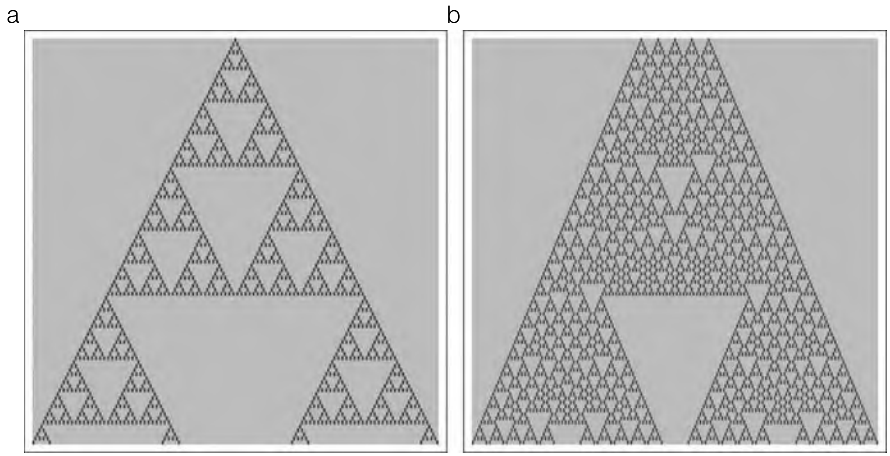



**FIGURE 2.35**

The evolution of two cellular automaton evolving according to Rule 146

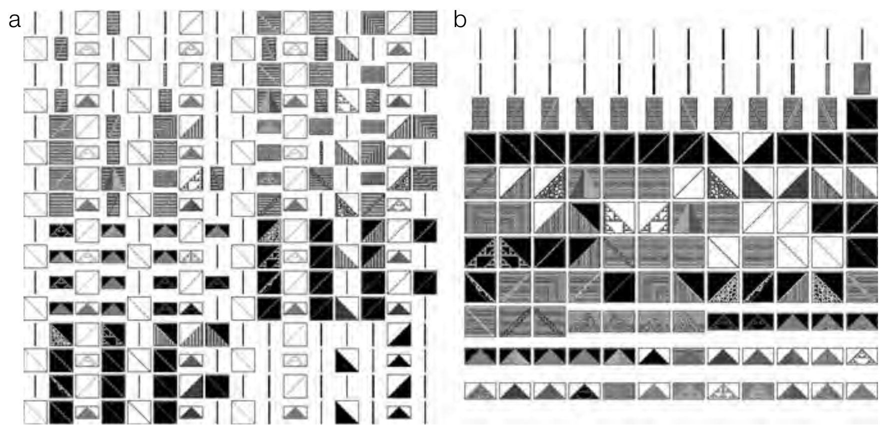



**FIGURE 2.36**

(a) The first 50 generations for the 256 elementary cellular automaton. (b) Removal of the identical ones

```
p2 = Show[GraphicsGrid[t4]];
Show[GraphicsRow[{p1, p2}]]
```

To see the plots together with the rule number, use Table. Each order pair returned consists of the rule number and the 50 generation plot. To display the ordered pairs in an organized fashion, we use Grid. Of course, the result is quite large, so just a portion of the actual grid is displayed in Figure 2.37.

```
t5 = Table[{i, ArrayPlot[CellularAutomaton[i, {{1}, 0}, 50]]},
      {i, 0, 255}];
```

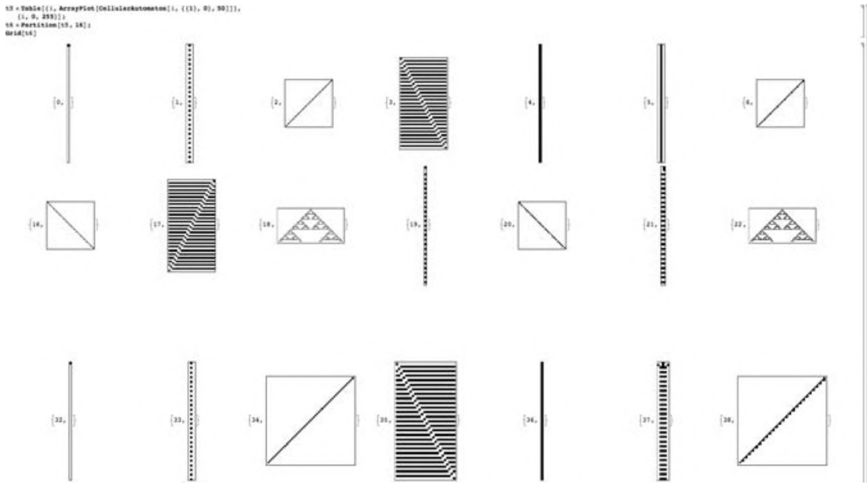


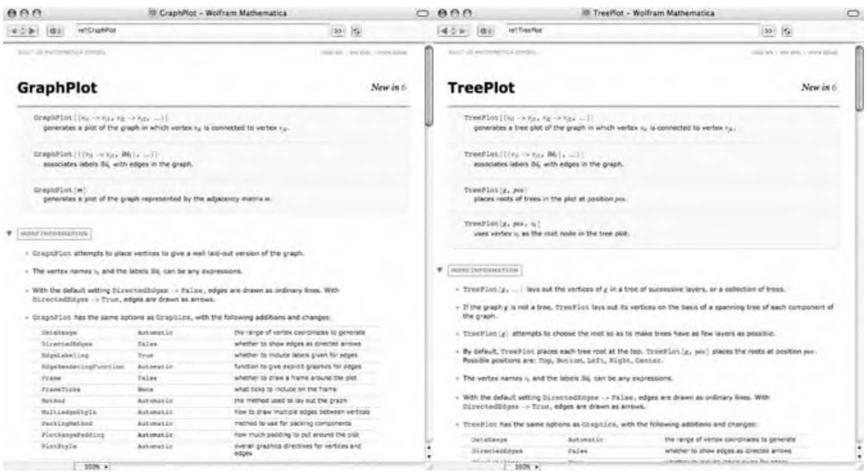
**FIGURE 2.37**

Seeing the automaton together with its rule number

**t6 = Partition[t5, 16];**
**Grid[t6]**

Note that MatrixPlot and ArrayPlot are discussed in more detail in Chapter 5.

For graphs of the form points or nodes connected by edges (graph theory), you can use GraphPlot to help investigate some problems. For trees, use TreePlot.



**Example 2.3.24** **Graceful graphs** do not have multiple edges or loops.

We generate $O$ with **GraphPlot** and display the result in Figure 2.38(a).

```
gp1 = GraphPlot[{{0->12, "12"}, {12->1, "11"}, {1->0, "1"}, {0->9, "9"},
    {1->9, "8"}, {1->6, "5"}, {12->6, "6"}, {2 → 12, "10"},
        {0->2, "2"}, 6->9, 9 → 10, 10 → 6},
            VertexLabeling → True, AspectRatio → 1]
```

$C_4$ is shown in Figure 2.38(b).

```
gp2 = GraphPlot[{0 → 4, 4 → 2, 2 → 3, 3 → 0}, DirectedEdges → True,
        VertexLabeling → True, AspectRatio → 1]
```

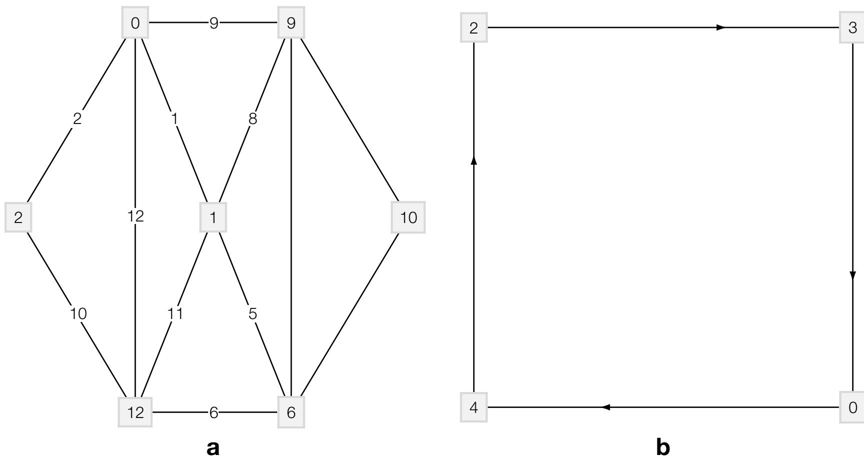



**FIGURE 2.38**

(a) $O$. (b) $C_4$

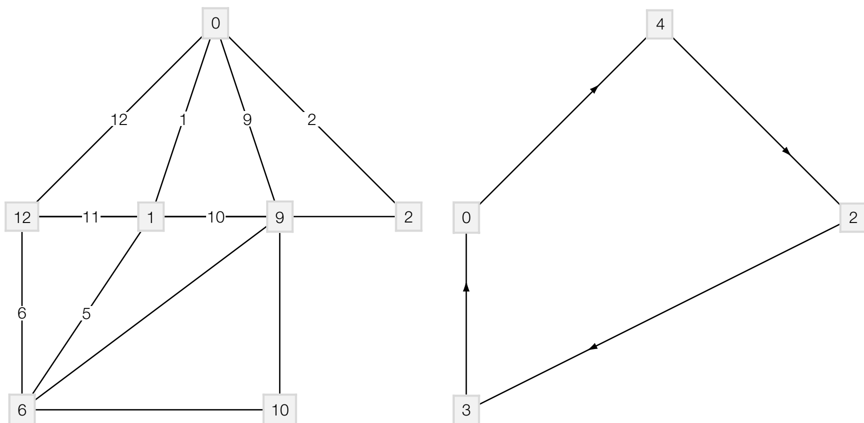



**FIGURE 2.39**

Using **TreePlot** instead of **GraphPlot**

Replacing **GraphPlot** with **TreePlot** gives us Figure 2.39.

```
tp1 = TreePlot[{{0->12, "12"}, {12->1, "11"}, {1->0, "1"}, {0->9, "9"},
    {1->9, "8"}, {1->6, "5"}, {12->6, "6"}, {2 → 12, "10"},
       {0->2, "2"}, 6->9, 9 → 10, 10 → 6},
          VertexLabeling → True, AspectRatio → 1]
tp2 = TreePlot[{0 → 4, 4 → 2, 2 → 3, 3 → 0}, DirectedEdges → True,
       VertexLabeling → True, AspectRatio → 1]
```

# 2.4 SOLVING EQUATIONS

## 2.4.1 Exact Solutions of Equations

Mathematica can find exact solutions to many equations and systems of equations, including exact solutions to polynomial equations of degree four or less. Because a single equals sign "=" is used to name objects and assign values in Mathematica, equations in Mathematica are of the form

**left – hand side==right – hand side.**

The "double-equals" sign "==" between the left-hand side and right-hand side specifies that the object is an equation. For example, to represent the equation $3x + 7 = 4$ in Mathematica, type 3x+7==4. The command Solve[lhs==rhs,x] solves the equation lhs = rhs for $x$. If the only unknown in the equation lhs = rhs is $x$ and Mathematica does not need to use inverse functions to solve for $x$, the command Solve[lhs==rhs] solves the equation lhs = rhs for $x$. Hence, to solve the equation $3x + 7 = 4$, both the commands Solve[3x+7==4] and Solve[3x+7==4, x] return the same result.

**Example 2.4.1** Solve the equations $3x + 7 = 4$, $(x^2 - 1)/(x - 1) = 0$ and $x^3 + x^2 + x + 1 = 0$.

***Solution*** In each case, we use **Solve** to solve the indicated equation. Be sure to include the double equals sign "==" between the left- and right-hand sides of each equation. Thus, the result of entering

```
Solve[3x + 7==4]
{{x → -1}}
```

means that the solution of $3x + 7 = 4$ is $x = 1$, and the result of entering

```
Solve[(x^2-1)/(x-1)==0]
{{x → -1}}
```

means that the solution of $\frac{x^2 - 1}{x - 1} = 0$ is $x = -1$. On the other hand, the equation $x^3 + x^2 + x + 1 = 0$ has two imaginary roots. We see that entering

```
Solve[x^3 + x^2 + x + 1==0]
{{x → -1}, {x → -i}, {x → i}}
```

yields all three solutions. Thus, the solutions of $x^3 + x^2 + x + 1 = 0$ are $x = -1$ and $x = \pm i$. Remember that the Mathematica symbol I represents the complex number $i = \sqrt{-1}$. In general, Mathematica can find the exact solutions of any polynomial equation of degree four or less.

Lists and tables are discussed in more detail in Chapters 4 and 5.

**Observe that the results of a Solve command are a list.**

**Mathematica can also solve equations involving more than one variable for one variable (literal equations) in terms of other unknowns.**

**Example 2.4.2** (a) Solve the equation $v = \pi r^2/h$ for $h$. (b) Solve the equation $a^2 + b^2 = c^2$ for $a$.

***Solution*** These equations involve more than one unknown, so we must specify the variable for which we are solving in the **Solve** commands. Thus, entering

```
Solve[v==Pi r^2/h, h]
{{h → πr²/v}}
```

solves the equation $v = \pi r^2/h$ for $h$. (Be sure to include a space or * between $\pi$ and $r$.) Similarly, entering

```
Solve[a^2 + b^2==c^2, a]
{{a → -√(-b² + c²)}, {a → √(-b² + c²)}}
```

solves the equation $a^2 + b^2 = c^2$ for $a$.

**If Mathematica needs to use inverse functions to solve an equation, you must be sure to specify the variable(s) for which you want Mathematica to solve.**

---

**Example 2.4.3** Find a solution of $\sin^2 x - 2\sin x - 3 = 0$.

***Solution*** When the command Solve[Sin[x]^2-2Sin[x]-3==0] is entered, Mathematica solves the equation for Sin[x]. However, when the command

**Solve[Sin[x] ^ 2 – 2Sin[x] – 3==0, x]**

is entered, Mathematica attempts to solve the equation for $x$. In this case, Mathematica succeeds in finding one solution.

**Solve[Sin[x]$^2$ – 2Sin[x] – 3==0]**
{{Sin[x] → –1}, {Sin[x] → 3}}

In fact, this equation has infinitely many solutions of the form $x = \frac{1}{2}(4k-1)\pi$, $k = 0, \pm1, \pm2, \ldots$; $\sin x = 3$ has no solutions.

```
Solve[Sin[x]^2 - 2 Sin[x] - 3 == 0, x]

Solve::ifun :
 Inverse functions are being used by Solve, so some solutions may
   not be found; use Reduce for complete solution information. >>

{{x → -π/2}, {x → ArcSin[3]}}
```

---

**The example indicates that it is especially important to be careful when dealing with equations involving trigonometric functions.**

---

**Example 2.4.4** Let $f(\theta) = \sin 2\theta + 2\cos\theta$, $0 \le \theta \le 2\pi$. (a) Solve $f'(\theta) = 0$. (b) Graph $f(\theta)$ and $f'(\theta)$.

***Solution*** After defining $f(\theta)$, we use D to compute $f'(\theta)$ and then use Solve to solve $f'(\theta) = 0$.

D[f[x],x] computes $f'(x)$; D[f[x],{x,n}] computes $f^{(n)}(x)$. Topics from calculus are discussed in more detail in Chapter 3.

**f[θ_] = Sin[2θ] + 2Cos[θ];**
**df = f′[θ]**
2Cos[2$\theta$] – 2Sin[$\theta$]

```
Solve[df == 0, θ]

Solve::ifun :
 Inverse functions are being used by Solve, so some solutions may not be
   found; use Reduce for complete solution information. >>

{{θ → -π/2}, {θ → π/6}, {θ → 5π/6}}
```

Notice that $-\pi/2$ is not between $0$ and $2\pi$. Moreover, $\pi/6$ and $5\pi/6$ are *not* the only solutions of $f'(\theta) = 0$ between $0$ and $2\pi$. Proceeding by hand, we use the identity $\cos 2\theta = 1 - 2\sin^2\theta$ and factor

$$\begin{aligned} 2\cos 2\theta - 2\sin\theta &= 0 \\ 1 - 2\sin^2\theta - \sin\theta &= 0 \\ 2\sin^2\theta + \sin\theta - 1 &= 0 \\ (2\sin\theta - 1)(\sin\theta + 1) &= 0 \end{aligned}$$

expression /. x->y replaces all occurrences of $x$ in *expression* by $y$.

so $\sin\theta = 1/2$ or $\sin\theta = -1$. Because we are assuming that $0 \le \theta \le 2\pi$, we obtain the solutions $\theta = \pi/6$, $5\pi/6$, or $3\pi/2$. We perform the same steps with Mathematica.

**s1 = TrigExpand[df]**

2Cos[θ]^2 − 2Sin[θ] − 2Sin[θ]^2

**s2 = s1/.Cos[θ]^2 → 1 − Sin[θ]^2**

−2Sin[θ] − 2Sin[θ]^2 + 2 (1 − Sin[θ]^2)

**Factor[s2]**

−2(1 + Sin[θ])(−1 + 2Sin[θ])

Finally, we graph $f(\theta)$ and $f'(\theta)$ with Plot in Figure 2.40. Note that the plot is drawn to scale because we include the option AspectRatio->Automatic.

**p1 = Plot[{f[θ], df}, {θ, 0, 2π}, AspectRatio → Automatic]**

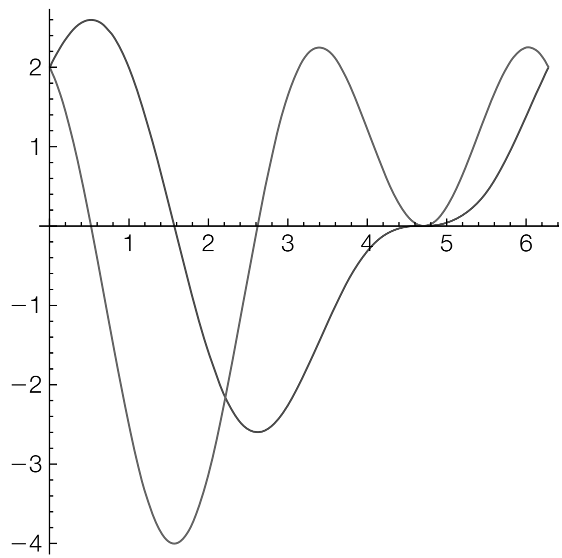



**FIGURE 2.40**

Graphs of $f(\theta)$ and $f'(\theta)$

We can also use Solve to find the solutions, if any, of various types of systems of equations. Entering

**Solve[{lhs1==rhs1, lhs2==rhs2}, {x, y}]**

solves a system of two equations for $x$ and $y$, whereas entering

**Solve[{lhs1==rhs1, lhs2==rhs2}]**

attempts to solve the system of equations for all unknowns. In general, Solve can find the solutions to a system of linear equations. In fact, if the systems to be solved are inconsistent or dependent, Mathematica's output indicates so.

---

**Example 2.4.5** Solve each system:

(a) $\begin{cases} 3x - y = 4 \\ x + y = 2 \end{cases}$; (b) $\begin{cases} 2x - 3y + 4z = 2 \\ 3x - 2y + z = 0 \\ x + y - z = 1 \end{cases}$; (c) $\begin{cases} 2x - 2y - 2z = -2 \\ -x + y + 3z = 0 \\ -3x + 3y - 2z = 1 \end{cases}$; and

(d) $\begin{cases} -2x + 2y - 2z = -2 \\ 3x - 2y + 2z = 2 \\ x + 3y - 3z = -3 \end{cases}$.

***Solution*** In each case, we use Solve to solve the given system. For (a), the result of entering

```
Solve[{3x - y==4, x + y==2}, {x, y}]
{{x → 3/2, y → 1/2}}
```

means that the solution of $\begin{cases} 3x - y = 4 \\ x + y = 2 \end{cases}$ is $(x, y) = (3/2, 1/2)$, which is the point of intersection of the lines with equations $3x - y = 4$ and $x + y = 2$. See Figure 2.41(a).

```
cp1 = ContourPlot[{3x - y==4, x + y==2},
   {x, -1, 2}, {y, -1, 2}, Frame → False,
      Axes → Automatic, AxesOrigin → {0, 0},
         AxesLabel → {x, y}] .1 in
```

(b) We can verify that the results returned by Mathematica are correct. First, we name the system of equations sys and then use Solve to solve the system of equations naming the result sols.

```
sys = {2x - 3y + 4z==2, 3x - 2y + z==0,
      x + y - z==1};
   sols = Solve[sys, {x, y, z}]
{{x → 7/10, y → 9/5, z → 3/2}}
```

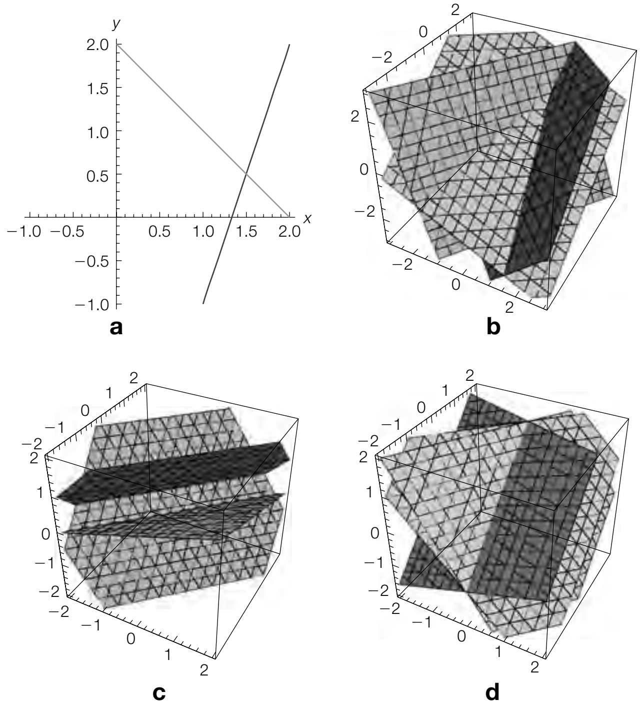


**FIGURE 2.41**

(a) Two intersecting lines. (b) Three planes that intersect in a single point. (c) These three planes have no point in common. (d) The intersection of these three planes is a line

We verify the result by substituting the values obtained with Solve back into sys with ReplaceAll (/.).

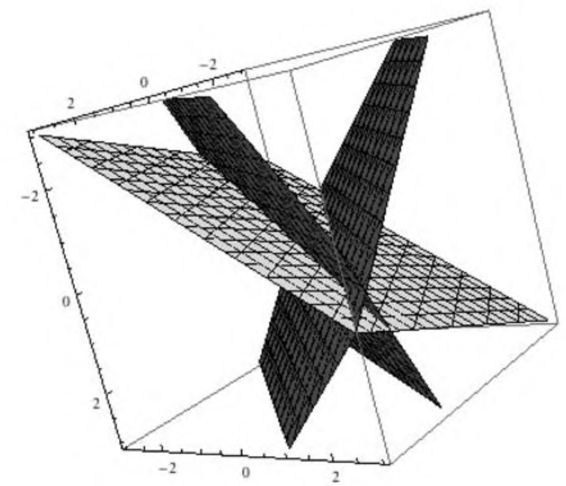
```
sys/.sols
{{True, True, True}}
```

means that the solution of $\begin{cases} 2x - 3y + 4z = 2 \\ 3x - 2y + z = 0 \\ x + y - z = 1 \end{cases}$ is $(x, y, z) = (7/10, 9/5, 3/2)$, which is the point of intersection of the planes with equations $2x - 3y + 4z = 2$, $3x - 2y + z = 0$, $x + y - z = 1$. See Figure 2.41(b).

```
cp2a = ContourPlot3D[{2x - 3y + 4z==2, 3x - 2y + z==0,
    x + y - z==1}, {x, -3, 3},
      {y, -3, 3}, {z, -3, 3}]
```

To better see the intersection point, click within the graphic and then drag to an appropriate viewing angle.

```
cp2a = ContourPlot3D[{2 x - 3 y + 4 z == 2, 3 x - 2 y + z == 0,
   x + y - z == 1}, {x, -3, 3},
  {y, -3, 3}, {z, -3, 3}]
```




(c) When we use **Solve** to solve this system, Mathematica returns {}, which indicates that the system has no solution; the system is inconsistent.

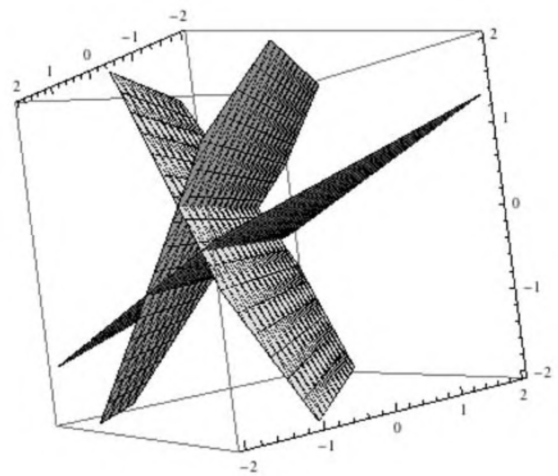
```
Solve[{2x - 2y - 2z == -2, -x + y + 3z == 0,
 -3x + 3y - 2z == 1}]
{}
```

To see that the planes with equations $2x - 2y - 2z = -2$, $-x + y + 3z = 0$ and $-3x + 3y - 2z = 1$ have no points in common, graph them within Figure 2.41(c).

```
cp3a = ContourPlot3D[{2x - 2y - 2z == -2,
    -x + y + 3z == 0, -3x + 3y - 2z == 1}, {x, -2, 2},
      {y, -2, 2}, {z, -2, 2}]
```

To better see that the planes do not intersect, we click and drag the graphic to an appropriate viewing angle.

```
cp3a = ContourPlot3D[{2 x - 2 y - 2 z == -2,
   -x + y + 3 z == 0, -3 x + 3 y - 2 z == 1}, {x, -2, 2},
  {y, -2, 2}, {z, -2, 2}]
```




(d) On the other hand, when we use Solve to solve this system, Mathematica's result indicates that the system has infinitely many solutions. That is, all ordered triples of the form $\{(0, z-1, z)|z \text{ real}\}$ are solutions of the system.

```
Solve[{-2 x + 2 y - 2 z == -2, 3 x - 2 y + 2 z == 2,
  x + 3 y - 3 z == -3}]
Solve::svars :
 Equations may not give solutions for all "solve" variables. >>
{{x → 0, y → -1 + z}}
```

We see that the intersection of the three planes is a line with ContourPlot3D. See Figure 2.41(d).

```
cp3a = ContourPlot3D[{2x − 2y − 2z== − 2,
    3x − 2y + 2z==2, x + 3y − 3z== − 3}, {x, − 2, 2},
      {y, − 2, 2}, {z, − 2, 2}]
Show[GraphicsGrid[{{cp1, cp2a}, {cp3a, cp4a}}]]
```

---

We can often use Solve to find solutions of a nonlinear system of equations as well.

**Example 2.4.6** Solve the systems

(a) $\begin{cases} 4x^2 + y^2 = 4 \\ x^2 + 4y^2 = 4 \end{cases}$ and (b) $\begin{cases} \frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1 \\ y = mx \end{cases}$ ($a$, $b$ greater than zero) for $x$ and $y$.

***Solution*** The graphs of the equations are both ellipses. We use ContourPlot to graph each equation, naming the results cp1 and cp2, respectively, and then use Show to display both graphs together in Figure 2.42(a). The solutions of the system correspond to the intersection points of the two graphs. Alternatively, the solutions of the system correspond to the intersection points of the level curves of $f(x, y) = 4x^2 + y^2 - 4$ and $g(x, y) = x^2 + 4y^2 - 4$ corresponding to $0$. See Figure 2.42(b).

```
cp1 = ContourPlot [4x^2 + y^2 - 4, {x, -3, 3}, {y, -3, 3}, Contours → {0},
        ContourShading → False, PlotPoints → 60 ];
cp2 = ContourPlot [x^2 + 4y^2 - 4, {x, -3, 3}, {y, -3, 3}, Contours → {0},
        ContourShading → False, PlotPoints → 60 ];
cp3 = Show[cp1, cp2, Frame → False, Axes → Automatic, AxesOrigin → {0, 0}]
cp4 = Plot3D [{4x^2 + y^2 - 4, x^2 + 4y^2 - 4}, {x, -10, 10},
    {y, -10, 10}, BoxRatios → {1, 1, 1}, MeshFunctions->{#3&},
        ColorFunction → (ColorData["Rainbow"][#3]&),
            PlotStyle → {Opacity[.4], Opacity[.8]} ]
Show[GraphicsRow[{cp3, cp4}]]
```

Finally, we use Solve to find the solutions of the system.

```
Solve [{4x^2 + y^2==4, x^2 + 4y^2==4}]
```

$\left\{\left\{x \to -\frac{2}{\sqrt{5}}, y \to -\frac{2}{\sqrt{5}}\right\}, \left\{x \to -\frac{2}{\sqrt{5}}, y \to \frac{2}{\sqrt{5}}\right\}, \left\{x \to \frac{2}{\sqrt{5}}, y \to -\frac{2}{\sqrt{5}}\right\}, \left\{x \to \frac{2}{\sqrt{5}}, y \to \frac{2}{\sqrt{5}}\right\}\right\}$

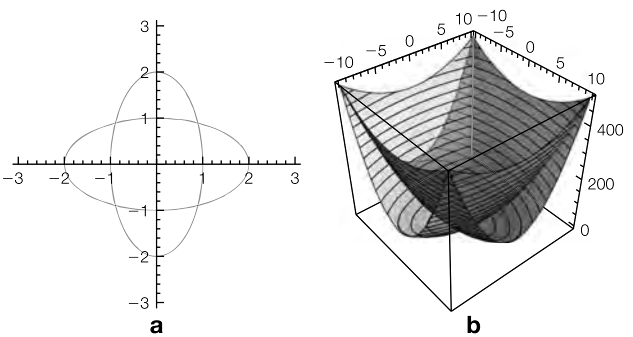



**FIGURE 2.42**

(a) Graphs of $4x^2 + y^2 = 4$ and $x^2 + 4y^2 = 4$. (b) Three-dimensional plots of $f(x, y)$ and $g(x, y)$ together with their level curves shown as contours

For (b), we also use **Solve** to find the solutions of the system. However, because the unknowns in the equations are $a$, $b$, $m$, $x$, and $y$, we must specify that we want to solve for $x$ and $y$ in the **Solve** command.

**Solve** $\left[\{\frac{x^2}{a^2}+\frac{y^2}{b^2}==1, y==mx\}, \{x, y\}\right]$

$\left\{\left\{y\to -\frac{abm}{\sqrt{b^2+a^2m^2}}, x\to -\frac{ab}{\sqrt{b^2+a^2m^2}}\right\}, \left\{y\to \frac{abm}{\sqrt{b^2+a^2m^2}}, x\to \frac{ab}{\sqrt{b^2+a^2m^2}}\right\}\right\}$

---

**Although Mathematica can find the exact solution to every polynomial equation of degree four or less, exact solutions to some equations may not be meaningful. In those cases, Mathematica can provide approximations of the exact solutions using either the** N[expression] **or the** expression // N **commands.**

---

**Example 2.4.7** Approximate the solutions to the equations (a) $x^4 - 2x^2 = 1 - x$; and (b) $1 - x^2 = x^3$.

***Solution*** Each of these is a polynomial equation with degree less than five so **Solve** will find the exact solutions of each equation. However, the solutions are quite complicated, so we use **N** to obtain approximate solutions of each equation. For (a), entering

**N** $\left[\textbf{Solve}\left[x^4-2x^2==1-x\right]\right]$

{{x→0.182777 − 0.633397i}, {x→0.182777 + 0.633397i},
{x→ − 1.71064}, {x→1.34509}}

{{x→0.182777 − 0.633397i},
{x→0.182777 + 0.633397i}, {x→ − 1.71064}, {x→1.34509}}

first finds the exact solutions of the equation $x^4 - 2x^2 = 1 - x$ and then computes approximations of those solutions. The resulting output is the list of approximate solutions. For (b), entering

**Solve** $\left[1-x^2==x^3, x\right]$ **//N**

{{x→0.754878}, {x→ − 0.877439 + 0.744862i},
{x→ − 0.877439 − 0.744862i}}

{{x→0.754878}, {x→ − 0.877439 + 0.744862i},
{x→ − 0.877439 − 0.744862i}}

first finds the exact solutions of the equation $1 - x^2 = x^3$ and then computes approximations of those solutions. The resulting output is the list of approximate solutions.

---

### 2.4.2 Approximate Solutions of Equations

When solving an equation is either impractical or impossible, Mathematica provides several functions, including FindRoot, NRoots, and NSolve, to approximate solutions of equations. NRoots and NSolve numerically approximate the roots of any polynomial equation. The command NRoots[poly1{==}poly2, x] approximates the solutions of the polynomial equation poly1{==}poly2, where both poly1 and poly2 are polynomials in $x$. The syntax for NSolve is the same as the syntax of NRoots.

FindRoot attempts to approximate a root to an equation provided that a "reasonable" guess of the root is given. FindRoot works on functions other than polynomials. The command

**FindRoot[lhs==rhs, {x, firstguess}]**

searches for a numerical solution to the equation lhs==rhs, starting with $x$ =firstguess. To locate more than one root, FindRoot must be used several times. One way of obtaining firstguess (for real-valued solutions) is to graph both lhs and rhs with Plot, find the point(s) of intersection, and estimate the $x$-coordinates of the point(s) of intersection. Generally, NRoots is easier to use than FindRoot when trying to approximate the roots of a polynomial.

---

**Example 2.4.8** Approximate the solutions of $x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7 = 0$.

***Solution*** Because $x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7 = 0$ is a polynomial equation, we may use NRoots to approximate the solutions of the equation. Thus, entering

```
NRoots[x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7==0, x]
x==-2.74463||x==-0.880858||x==0.41452-1.19996i||x==0.41452+
1.19996i||x==1.79645
    x==-2.74463||x==-0.880858||x==0.41452-1.19996i||
        x==0.41452 + 1.19996i||x==1.79645
```

approximates the solutions of $x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7 = 0$. The symbol || appearing in the result represents "or."

We obtain equivalent results with NSolve.

```
NSolve[x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7==0, x]
{{x→-2.74463}, {x→-0.880858}, {x→0.41452-1.19996i},
{x→0.41452 + 1.19996i}, {x→1.79645}}
    {{x→-2.74463}, {x→-0.880858}, {x→0.41452-1.19996i},
        {x→0.41452 + 1.19996i}, {x→1.79645}}
```

FindRoot may also be used to approximate each root of the equation. However, to use FindRoot, we must supply an initial approximation of the solution that we wish to approximate. The real solutions of $x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7 = 0$ correspond

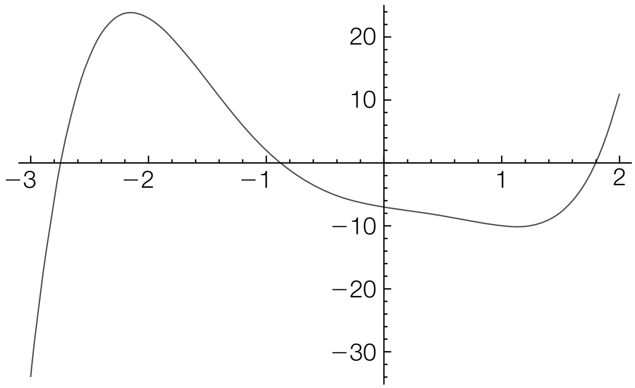



**FIGURE 2.43**

Graph of $f(x) = x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7$

to the values of $x$ where the graph of $f(x) = x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7$ intersects the $x$-axis. We use Plot to graph $f(x)$ in Figure 2.43.

```
Plot[x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7, {x, -3, 2}]
```

We see that the graph intersects the $x$-axis near $x \approx -2.5$, $-1$, and $1.5$. We use these values as initial approximations of each solution. Thus, entering

```
FindRoot[x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7==0, {x, -2.5}]
{x→-2.74463}
```

approximates the solution near $-2.5$, entering

```
FindRoot[x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7==0, {x, -1}]
{x→-0.880858}
```

approximates the solution near $-1$, and entering

```
FindRoot[x^5 + x^4 - 4x^3 + 2x^2 - 3x - 7==0, {x, 2}]
{x→1.79645}
```

approximates the solution near $1.5$. Note that FindRoot may be used to approximate complex solutions as well. To obtain initial guesses, observe that the solutions of $f(z) = 0$, $z = x + iy$, $x$, $y$ real, are the level curves of $w = |f(z)|$ that are points. In Figure 2.44, we use ContourPlot to graph various level curves of $w = |f(x + iy)|$, $-2 \le x \le 2$, $-2 \le y \le 2$. In the plot, observe that the two complex solutions occur at $x \pm iy \approx 0.5 \pm 1.2i$.

```
f[z_] = z^5 + z^4 - 4z^3 + 2z^2 - 3z - 7;
ContourPlot[Abs[f[x + Iy]], {x, -2, 2}, {y, -2, 2},
    ContourShading → False, Contours → 60,
```

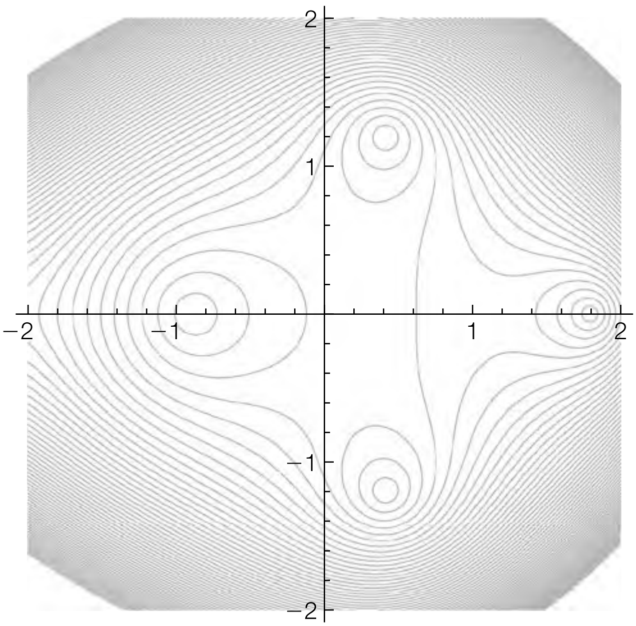



**FIGURE 2.44**

Level curves of $w = |f(x + iy)|$, $-2 \le x \le 2$, $-2 \le y \le 2$

**PlotPoints → 200, Frame → False, Axes → Automatic,**
**AxesOrigin → {0, 0}]**

Thus, entering

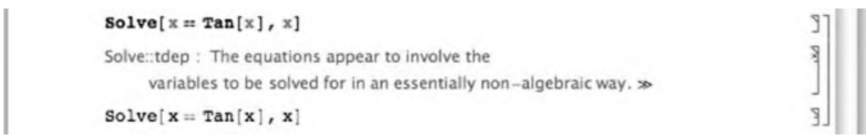
**FindRoot[x^5 + x^4 − 4x^3 + 2x^2 − 3x − 7==0, {x, 0.5 + I}]**
{x→0.41452 + 1.19996i}

approximates the solution near $x + iy \approx 0.5 + 1.2i$. For polynomials with real coefficients, complex solutions occur in conjugate pairs so the other complex solution is approximately $0.41452 - 1.19996i$.

---

**Example 2.4.9** Find the first three nonnegative solutions of $x = \tan x$.

***Solution*** We attempt to solve $x = \tan x$ with Solve.



```
Solve[x == Tan[x], x]
Solve::tdep : The equations appear to involve the
    variables to be solved for in an essentially non-algebraic way. >>
Solve[x == Tan[x], x]
```

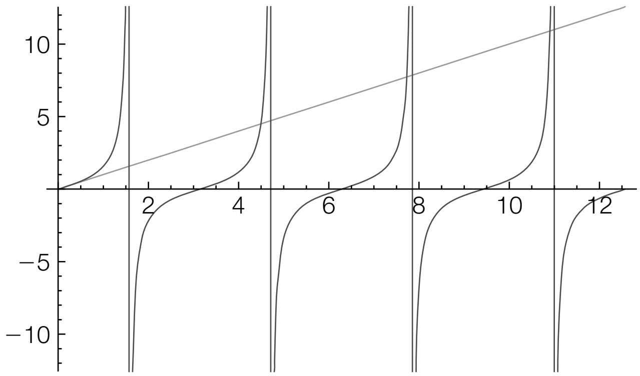



**FIGURE 2.45**

$y = x$ and $y = \tan x$

We next graph $y = x$ and $y = \tan x$ together in Figure 2.45.

```
Plot[Tooltip[{x, Tan[x]}], {x, 0, 4Pi},
        PlotRange → {−4Pi, 4Pi}]
```

Remember that vertical lines are never the graphs of functions. In this case, they represent the vertical asymptotes at odd multiples of $\pi/2$.

In the graph, we see that $x = 0$ is a solution. This is confirmed with FindRoot.

```
FindRoot[x==Tan[x], {x, 0}]
{x→0.}
```

The second solution is near 4, whereas the third solution is near 7. Using FindRoot together with these initial approximations locates the second two solutions.

```
FindRoot[x==Tan[x], {x, 4}]
{x→4.49341}
FindRoot[x==Tan[x], {x, 7}]
{x→7.72525}
```

---

FindRoot can also be used to approximate solutions to systems of equations. (Although NRoots can solve a polynomial equation, NRoots cannot be used to solve a system of polynomial equations.) When approximations of solutions of systems of equations are desired, use either Solve and N together, when possible, or FindRoot.

---

**Example 2.4.10** Approximate the solutions to the system of equations $\begin{cases} x^2 + 4xy + y^2 = 4 \\ 5x^2 - 4xy + 2y^2 = 8 \end{cases}$.

***Solution*** We begin by using ContourPlot to graph each equation in Figure 2.46. From the resulting graph, we see that $x^2 + 4xy + y^2 = 4$ is a hyperbola, $5x^2 - 4xy + 2y^2 = 8$ is an ellipse, and there are four solutions to the system of equations.

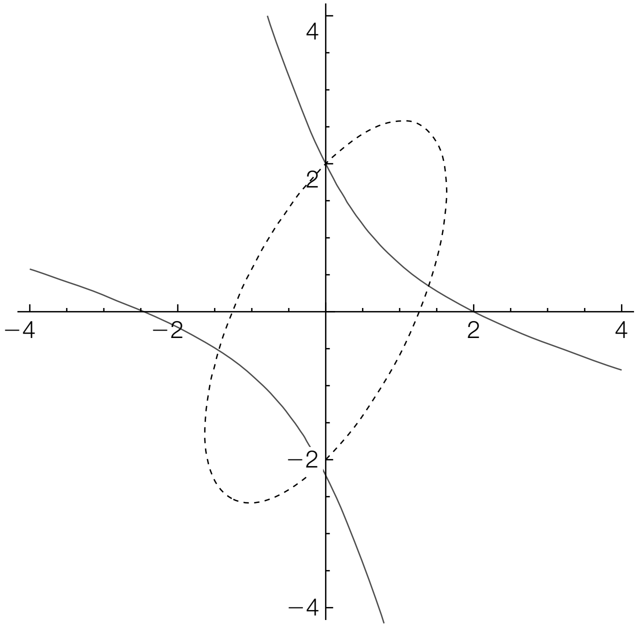



**FIGURE 2.46**

Graphs of $x^2 + 4xy + y^2 = 4$ and $5x^2 - 4xy + 2y^2 = 8$

```
cp1 = ContourPlot[x^2 + 4xy + y^2 − 4==0, {x, −4, 4}, {y, −4, 4},
        PlotPoints → 60, ContourShading → False];
cp2 = ContourPlot[5x^2 − 4xy + 2y^2 − 8==0, {x, −4, 4},
    {y, −4, 4}, PlotPoints → 60, ContourStyle −>Dashing[{0.01}],
        ContourShading → False];
Show[cp1, cp2, Frame → False, Axes → Automatic,
        AxesOrigin → {0, 0}]
```

From the graph we see that possible solutions are $(0, 2)$ and $(0, -2)$. In fact, substituting $x = 0$ and $y = -2$ and $x = 0$ and $y = 2$ into each equation verifies that these points are both exact solutions of the equation. The remaining two solutions are approximated with FindRoot.

```
FindRoot [{x^2 + 4xy + y^2==4, 5x^2 − 4xy + 2y^2==8} ,
        {x, 1}, {y, 0.25}]
{x→1.39262, y→0.348155}
FindRoot [{x^2 + 4xy + y^2==4, 5x^2 − 4xy + 2y^2==8},
        {x, − 1}, {y, −0.25}]
{x→−1.39262, y→−0.348155}
```

## 2.5 EXERCISES

1. Evaluate the following:

   (a) $432 + 701$
   (b) $251 \times 8197$
   (c) $\sqrt{116281}$
   (d) $\sqrt[3]{157464}$
   (e) $679/42$
   (f) $\sin(\pi/12)$
   (g) $\cos(11\pi/12)$
   (h) $\left|\dfrac{2+i}{5-3i}\right|$

2. Solve $x^3 - 8x^2 + 19x - 12 = \frac{1}{2}x^2 - x - \frac{1}{8}$. Confirm your result graphically.
3. Solve $-3x^2 + 12x - 5 = 2x^2 - 4x - 3$. Confirm your result graphically.
4. Find a 10-digit approximation of Euler's constant, denoted by EulerGamma.
5. Use TrigExpand to write $\sin 5x$ in terms of $\sin x$ and $\cos x$.
6. Use ExpToTrig to rewrite $e^x - e^{-x}$.
7. Use TrigToExp to rewrite $\cos it$.
8. Factor $15x^5 + 73x^4 - 621x^3 - 297x^2 + 2486x + 504$ to find the zeros of this polynomial. Compare these results with those obtained using Solve, FindRoot, or NRoots.
9. Use PowerExpand to simplify $\sqrt{a^4b^6c^{-8}}$. Compare this result with that obtained using Simplify and explain the difference.
10. Solve $\exp(-(x/4)^2)\cos(x/\pi) = \sin(x^{3/2}) + \frac{5}{4}$. Confirm your result graphically.
11. Graph the cross-cap, $4x^2\left(x^2 + y^2 + z^2 + z\right) + y^2\left(y^2 + z^2 - 1\right)$, by graphing it as functions of (a) $y$ and $z$, (b) $x$ and $y$, and (c) $x$ and $z$.
12. Determine the partial fraction decomposition of the following.

    (a) $\dfrac{6x - 18}{x^2 - 2x - 8}$
    (b) $4\dfrac{2x^2 + x + 28}{x^3 - 4x^2 + 16x - 64}$
    (c) $\dfrac{17x^2 + 2x}{x^4 + 5x^2 + 4}$

13. Let $f(x) = \begin{cases} -x, & x \le -1 \\ \sin \pi x, & -1 < x \le 1. \\ \frac{1}{2}x^2, & x > 1 \end{cases}$ Plot $f(x)$ and $f'(x)$ on the interval $[-5, 5]$.

**14.** Plot $\cos 2x$ and $2\sin x$ on $-\pi \le x \le \pi$. Use FindRoot to determine where the two curves intersect on $[-\pi, \pi]$.

**15.** Consider Example 2.3.5 using $f(x) = \cos x$ with $n = 2, 4, \ldots, 12$. Describe the graph of $f^n(x)$ as $n$ increases.

**16.** Graph the **tooth surface**, which is defined by the equation $x^4 + y^4 + z^4 - (x^2 + y^2 + z^2) = 0$.

**17.** Find a 20-digit approximation of Catalan's constant, denoted by Catalan.

**18.** Use ParametricPlot to graph (a) $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$ and (b) $x = t\cos t/2$, $y = t\sin t/2$, $0 \le t \le 12\pi$.

**19.** Use PolarPlot to graph (a) $r = 2\cos\theta$, $0 \le \theta \le \pi$ and (b) $r = \cos 2\theta$, $0 \le \theta \le 2\pi$.

**20.** Use PolarPlot to investigate the differences in the graph of $r = \cos n\theta$ and $r = \sin n\theta$ for $n$ an odd or even integer. *Question:* What happens when $n$ is a noninteger rational number?

**21.** Graph the level curves of the following: (a) $f(x,y) = x^2 - y^2$, (b) $f(x,y) = \sin(xy)$, and (c) $f(x,y) = x\cos y$.

**22.** Graph the function $f(x,y) = \sin\left(x^2 + y^2\right)$. Use the Interactive 3D control to rotate the graph in order to investigate the level curves of the function. Compare your findings to those obtained with ContourPlot.

**23.** A parametrization $(X, Y, Z)$ of Boy's surface is given by

$$X = g_1/g \qquad Y = g_2/g \qquad Z = g_3/g,$$

where $g_1 = -\frac{3}{2}\operatorname{Im}\left(\frac{z(1 - z^4)}{z^6 + \sqrt{5}z^3 - 1}\right)$, $g_2 = -\frac{3}{2}\operatorname{Re}\left(\frac{z(1 + z^4)}{z^6 + \sqrt{5}z^3 - 1}\right)$, $g_3 = \operatorname{Im}\left(\frac{1 + z^6}{z^6 + \sqrt{5}z^3 - 1}\right) - \frac{1}{2}$, and $g = {g_1}^2 + {g_2}^2 + {g_3}^2$. The complex number $z = a + bi$ satisfies $|z| \le 1$. Plot Boy's surface. (See Figure 2.47.)

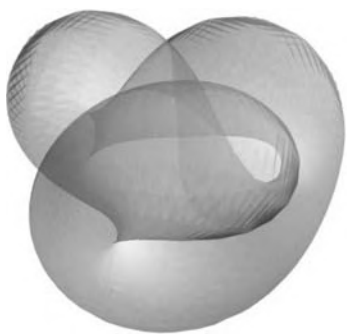


**FIGURE 2.47**

Boy's surface

# CHAPTER 3
# Calculus

Chapter 3 introduces Mathematica's built-in calculus functions. The examples used to illustrate the various functions are similar to examples typically seen in a traditional calculus sequence. If you have trouble typing commands correctly, use the buttons on the **BasicMathInput** palette to help you create templates in standard mathematical notation that you can evaluate.

## 3.1 LIMITS AND CONTINUITY

One of the first topics discussed in calculus is that of limits. Mathematica can be used to investigate limits graphically and numerically. In addition, the Mathematica command Limit[f[x], x->a] attempts to compute the limit of $y = f(x)$ as $x$ approaches $a$, $\lim_{x\to a} f(x)$, where $a$ can be a finite number, $\infty$ (Infinity), or $-\infty$ (–Infinity). The arrow "->" is obtained by typing a minus sign "–" followed by a greater than sign ">".

**Remark 3.1** To define a function of a single variable, $f(x) = \textit{expression in } x$, enter f[x_] = expression in x. To generate a basic plot of $y = f(x)$ for $a \le x \le b$, enter Plot[f[x], {x, a, b}].

Clear[f] clears all prior definitions of $f$, if any. Clearing function definitions before defining new ones helps eliminate any possible confusion and/or ambiguities.

### 3.1.1 Using Graphs and Tables to Predict Limits

**Example 3.1.1** Use a graph and table of values to investigate $\lim_{x\to 0} \dfrac{\sin 3x}{x}$.

***Solution*** We clear all prior definitions of $f$, define $f(x) = (\sin 3x)/x$, and then graph $y = f(x)$ on the interval $[-\pi, \pi]$ with Plot.

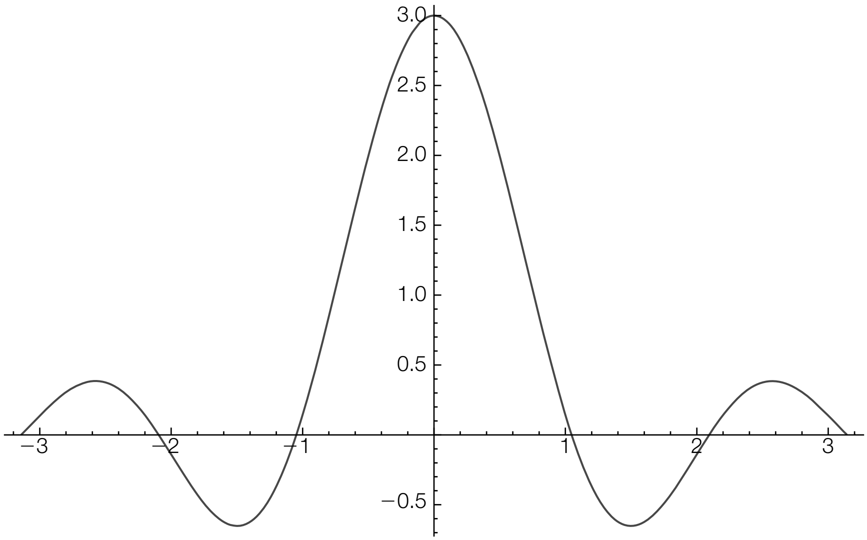




**FIGURE 3.1**

Graph of $f(x) = (\sin 3x)/x$ on the interval $[-\pi, \pi]$

```
Clear[f]
f[x_] = Sin[3x]/x;
Plot[f[x], {x, −π, π}]
```

From the graph shown in Figure 3.1, we might, correctly, conclude that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$. Further evidence that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$ can be obtained by computing the values of $f(x)$ for values of $x$ "near" $x = 0$. In the following, we use RandomReal to define xvals to be a table of six "random" real numbers. The first number in xvals is between $-1$ and $1$, the second between $-1/10$ and $1/10$, and so on.

RandomReal[{a,b}] returns a "random" real number between $a$ and $b$. Because we are generating "random" numbers, your results will differ from those obtained here.

```
xvals = Table [RandomReal [{−10^−n, 10^−n}], {n, 0, 5}]
0.424046, −0.0850248, . . .
```

We then use Map to compute the value of $f(x)$ for each $x$ in xvals. We use Table to display the results in tabular form. Generally, list[[i]] returns the $i$th element of list while Table[f[i],{i,start,finish,stepsize}] computes each value of $f(i)$ from start to finish in increments of stepsize. TableForm attempts to display a table form in a standard format such as the row-and-column format that follows.

Map[f,{x1,x2,…,xn}] returns the list $\{f(x_1), f(x_2), \ldots, f(x_n)\}$.

```
fvals = Map[f, xvals]
{2.25384, 2.96757, 2.99995, 3., 3., 3.

pairs = Table[{xvals[[i]], fvals[[i]]}, {i, 1, 6}];
TableForm[pairs]
```

| | |
|---|---|
| 0.424046 | 2.25384 |
| −0.0850248 | 2.96757 |
| 0.00334803 | 2.99995 |
| 0.0000981987 | 3. |
| 0.0000376656 | 3. |
| −2.914605226592692`^ − 6 | 3. |

From these values, we might again correctly deduce that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$. Of course, these results do not prove that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$, but they are helpful in convincing us that $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$.

---

**For piecewise-defined functions, you can either use Mathematica's *conditional command* (/;) to define the piecewise-defined function or use** Piecewise.

---

**Example 3.1.2** If $h(x) = \begin{cases} x^2 + x, & \text{if } x \le 0 \\ 1 - x, & \text{if } 0 < x < 3 \\ 2x^2 - 15x + 25, & \text{if } 3 \le x \le 5 \\ 15 - 2x, & \text{if } x > 5 \end{cases}$, compute the following limits:

(a) $\lim_{x\to 0} h(x)$, (b) $\lim_{x\to 3} h(x)$, (c) $\lim_{x\to 5} h(x)$.

***Solution*** We use Mathematica's conditional command, /;, to define $h$. We must use delayed evaluation (:=) because $h(x)$ cannot be computed unless Mathematica is given a particular value of $x$. The first line of the following defines $h(x)$ to be $x^2 + x$ for $x \le 0$, the second line defines $h(x)$ to be $1 - x$ for $0 < x < 3$, and so on. In the Plot command, {x, -2, 0, 3, 5, 6} instructs Mathematica to graph the function on $[-2, 0]$, then $[0, 3]$, then $[3, 5]$, and finally $[5, 6]$. Notice that Mathematica accidentally connects $(0, 0)$ to $(0, 1)$ and then $(5, 0)$ to $(5, 5)$. (See Figure 3.2(a)). The delayed evaluation is also incompatible with Mathematica's Limit function.

The plots p1 and p2 are not displayed because a semicolon is included at the end of each Plot command.

```
Clear[h]
h[x_] := x^2 + x/;x ≤ 0
h[x_] := 1 − x/;0 < x < 3
h[x_] := 2x^2 − 15x + 25/;3 ≤ x ≤ 5
h[x_] := 15 − 2x/;x > 5
p1 = Plot[h[x], {x, −2, 0, 3, 5, 6}];
```

To avoid these problems, we redefine $h$ using Mathematica's Piecewise function as follows:

```
Clear[h]
h[x_] := Piecewise[{{x^2 + x, x < 0}, {1 − x, 0 < x < 3}, {2x^2 − 15x + 25,
        3<=x ≤ 5}, {15 − 2x, x > 5}}];
```

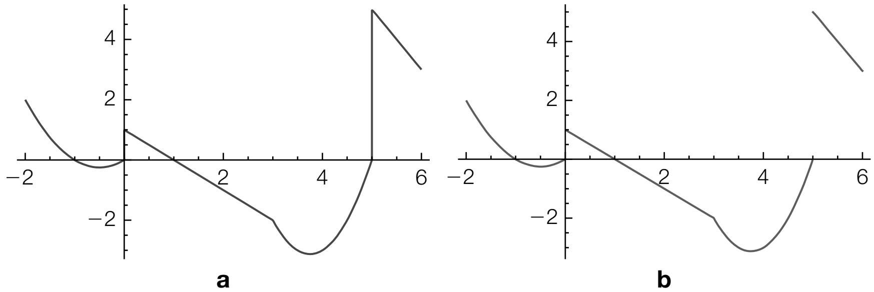



**FIGURE 3.2**

(a) Plot does not catch the breaks in the piecewise defined function. (b) If you use Piecewise, Plot can catch jumps.

```
p2 = Plot[h[x], {x, −2, 0, 3, 5, 6}, PlotRange → All];
Show[GraphicsRow[{p1, p2}]]
```

Notice that when we execute the Plot command, Mathematica "catches" the breaks between $(0, 0)$ and $(0, 1)$ and then $(5, 0)$ and $(5, 5)$ shown in Figure 3.2(b).

From Figure 3.2, we see that $\lim_{x\to 0} h(x)$ does not exist, $\lim_{x\to 3} h(x) = -2$, and $\lim_{x\to 5} h(x)$ does not exist.

---

When limits exist, you can often use Limit[f[x], x->a] (where $a$ may be ±Infinity) to compute $\lim_{x\to a} f(x)$. Thus, for the previous example we see that

```
Limit[h[x], x → 3]
−2
```

is correct. On the other hand,

```
Limit[h[x], x → 5]
5
```

is incorrect. We check by computing the right-hand limit, $\lim_{x\to 5^+} h(x)$, using the Direction->−1 option in the Limit command and then the left limit, $\lim_{x\to 5^-} h(x)$, using the Direction->1 in the Limit command.

```
Limit[h[x], x → 5, Direction → 1]
0
```

```
Limit[h[x], x → 5, Direction → −1]
5
```

We follow the same procedure for $x = 0$

```
Limit[h[x], x → 0]
1
```

```
Limit[h[x], x → 0, Direction → 1]
0

Limit[h[x], x → 0, Direction → -1]
1
```

### 3.1.2 Computing Limits

Some limits involving rational functions can be computed by factoring the numerator and denominator.

**Example 3.1.3** Compute $\lim_{x\to-9/2}\dfrac{2x^2+25x+72}{72-47x-14x^2}$.

***Solution*** We define frac1 to be the rational expression $\dfrac{2x^2+25x+72}{72-47x-14x^2}$. We then attempt to compute the value of frac1 if $x=-9/2$ by using ReplaceAll (/.) to evaluate frac1 if $x=-9/2$ but see that it is undefined.

```
frac1 /. x → -9/2

Power::infy : Infinite expression 1/0 encountered. >>

∞::indet :
 Indeterminate expression 0 ComplexInfinity encountered. >>

Indeterminate
```

Factoring the numerator and denominator with Factor, Numerator, and Denominator, we see that

$$\lim_{x\to-9/2}\frac{2x^2+25x+72}{72-47x-14x^2}=\lim_{x\to-9/2}\frac{(x+8)(2x+9)}{(8-7x)(2x+9)}=\lim_{x\to-9/2}\frac{x+8}{8-7x}.$$

The fraction $(x+8)/(8-7x)$ is named frac2 and the limit is evaluated by computing the value of frac2 if $x=-9/2$,

```
Factor[Numerator[frac1]]
Factor[Denominator[frac1]]
(8 + x)(9 + 2x)
-(9 + 2x)(-8 + 7x)

frac2 = Cancel[frac1]
(-8 - x)/(-8 + 7x)

frac2/.x → -9/2
7/79
```

or by using the Limit function on the original fraction.

```
Limit[frac1, x → -9/2]
7/79
```

We conclude that

$$\lim_{x\to -9/2} \frac{2x^2+25x+72}{72-47x-14x^2} = \frac{7}{79}.$$

---

As stated previously, Limit[f[x], x->a] attempts to compute $\lim_{x\to a} f(x)$, Limit[f[x], x->a,Direction->1] attempts to compute $\lim_{x\to a^-} f(x)$, and Limit[f[x], x->a, Direction->−1] attempts to compute $\lim_{x\to a^+} f(x)$. Generally, $a$ can be a number, ±Infinity ($\pm\infty$), or another symbol.

Thus, entering

**Limit$\left[\frac{3x^2-7x-20}{21x^2+14x-35}, x \to -\frac{5}{3}\right]$**

$\frac{17}{56}$

computes $\lim_{x\to -9/2} \frac{2x^2+25x+72}{72-47x-14x^2} = 7/79$.

---

**Example 3.1.4** Calculate each limit: (a) $\lim_{x\to -5/3} \dfrac{3x^2-7x-20}{21x^2+14x-35}$; (b) $\lim_{x\to 0} \dfrac{\sin x}{x}$; (c) $\lim_{x\to\infty} \left(1+\dfrac{z}{x}\right)^x$; (d) $\lim_{x\to 0} \dfrac{e^{3x}-1}{x}$; (e) $\lim_{x\to\infty} e^{-2x}\sqrt{x}$; and (f) $\lim_{x\to 1^+} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$.

***Solution*** In each case, we use Limit to evaluate the indicated limit. Entering

**Limit$\left[\frac{3x^2-7x-20}{21x^2+14x-35}, x \to -\frac{5}{3}\right]$**

$\frac{17}{56}$

computes $\lim_{x\to -5/3} \dfrac{3x^2-7x-20}{21x^2+14x-35} = \frac{17}{56}$, and entering

**Limit$\left[\frac{\text{Sin[x]}}{x}, x \to 0\right]$**

1

computes $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. Mathematica represents $\infty$ by Infinity. Thus, entering

**Limit[(1 + z/x)^x, x → Infinity]**

$e^z$

computes $\lim_{x\to\infty} \left(1+\dfrac{z}{x}\right)^x = e^z$. Entering

**Limit[(Exp[3x] − 1)/x, x → 0]**

3

computes $\lim_{x\to 0} \dfrac{e^{3x}-1}{x} = 3$. Entering

**Limit[Exp[−2x]Sqrt[x], x → Infinity]**

0

computes $\lim_{x\to\infty} e^{-2x}\sqrt{x} = 0$, and entering

Because $\ln x$ is undefined for $x \le 0$, a right-hand limit is mathematically necessary, even though Mathematica's Limit function computes the limit correctly without the distinction.

**Limit[1/Log[x] – 1/(x – 1), x → 1, Direction → – 1]**

$\frac{1}{2}$

computes $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\frac{1}{2}$.

### 3.1.3 One-Sided Limits

As illustrated previously, Mathematica can compute certain one-sided limits. The command Limit[f[x], x->a, Direction->1] attempts to compute $\lim_{x\to a^-} f(x)$, where as Limit[f[x], x->a, Direction->–1] attempts to compute $\lim_{x\to a^+} f(x)$.

**Example 3.1.5** Compute (a) $\lim_{x\to 0^+} |x|/x$; (b) $\lim_{x\to 0^-} |x|/x$; (c) $\lim_{x\to 0^+} e^{-1/x}$; and (d) $\lim_{x\to 0^-} e^{-1/x}$.

***Solution*** Even though $\lim_{x\to 0} |x|/x$ does not exist, $\lim_{x\to 0^+} |x|/x = 1$ and $\lim_{x\to 0^-} |x|/x = -1$, as we see using **Limit** together with the **Direction->1** and **Direction->–1** options, respectively.

**Limit$\left[\frac{\textbf{Abs[x]}}{\textbf{x}}\right.$, x → 0, Direction → 1$\left.\right]$**

**Limit$\left[\frac{\textbf{Abs[x]}}{\textbf{x}}\right.$, x → 0, Direction → – 1$\left.\right]$**

–1

1

The **Direction->–1** and **Direction->1** options are used to calculate the correct values for (c) and (d), respectively. For (c), we have

**Limit$\left[\frac{1}{x}\right.$, x → 0, Direction → – 1$\left.\right]$**

∞

Technically, $\lim_{x\to 0} e^{-1/x}$ does not exist (see Figure 3.3(a)), so the following is incorrect.

**Limit[Exp[– 1/x], x → 0]**

0

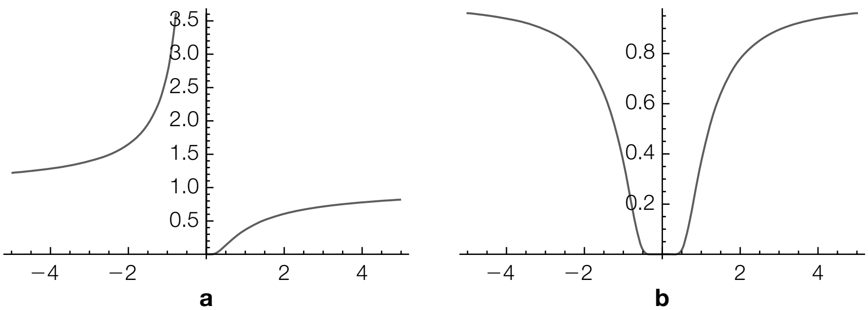



**FIGURE 3.3**

(a) Graph of $y = e^{-1/x}$. (b) Graph of $y = e^{-1/x^2}$

However, using Limit together with the Direction option gives the correct left and right limits.

```
Limit[Exp[-1/x], x → 0, Direction → 1]
∞

Limit[Exp[-1/x], x → 0, Direction → -1]
0
```

We confirm these results by graphing $y = e^{-1/x}$ with Plot in Figure 3.3(a). In (b), we also show the graph of $y = e^{-1/x^2}$ in Figure 3.3(b), which is discussed in the exercises.

```
p1 = Plot[Exp[-1/x], {x, -5, 5}];
p2 = Plot[Exp[-1/x^2], {x, -5, 5}];
Show[GraphicsRow[{p1, p2}]]
```

---

The Limit command together and its options (Direction->1 and Direction->−1) are "fragile" and should be used with caution because the results are unpredictable. It is wise to check or confirm results using a different technique for nearly all problems encountered.

### 3.1.4 Continuity

**Definition 1.** *The function $y = f(x)$ is **continuous** at $x = a$ if*

1. $\lim_{x\to a} f(x)$ *exists;*
2. $f(a)$ *exists; and*
3. $\lim_{x\to a} f(x) = f(a)$.

Be careful with regard to this. For example, since $\lim_{x\to 0^-}\sqrt{x}$ does not exist, many would say that $f(x) = \sqrt{x}$ is *right continuous* at $x = 0$.

Note that the third item in the definition means that both (1) and (2) are satisfied. However, if either (1) or (2) is not satisfied, the function is not continuous at the number in question. The function $y = f(x)$ is **continuous** on the open interval $I$ if $f(x)$ is continuous at each number $a$ contained in the interval $I$. Loosely speaking, the "standard" set of functions (polynomials, rational, trigonometric, etc.) are continuous on their domains.

---

**Example 3.1.6** For what value(s) of $x$, if any, are each of the following functions continuous? (a) $f(x) = x^3 - 8x$; (b) $f(x) = \sin 2x$; (c) $f(x) = (x-1)/(x+1)$; and (d) $f(x) = \sqrt{(x-1)/(x+1)}$.

***Solution*** (a) Polynomial functions are continuous for all real numbers. In interval notation, $f(x)$ is continuous on $(-\infty, \infty)$. (b) Because the sine function is continuous for all real numbers, $f(x) = \sin 2x$ is continuous for all real numbers. In interval notation, $f(x)$ is continuous on $(-\infty, \infty)$. (c) The rational function $f(x) = (x-1)/(x+1)$ is continuous for all $x \neq -1$. In interval notation, $f(x)$ is continuous on $(-\infty, -1) \cup (-1, \infty)$. (d) $f(x) = \sqrt{(x-1)/(x+1)}$ is continuous if the radicand is nonnegative. In

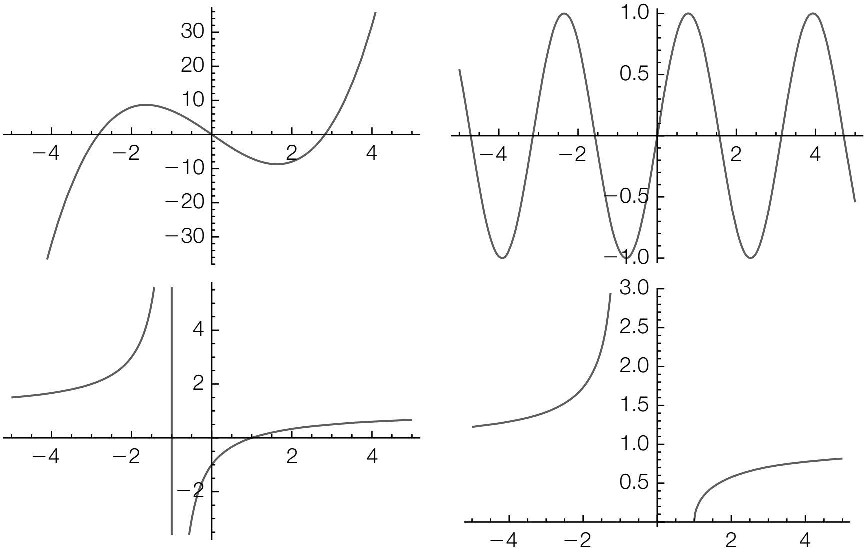



**FIGURE 3.4**

Polynomials, trigonometric, rational, and root functions are usually continuous on their domains

interval notation, $f(x)$ is strictly continuous on $(-\infty, -1) \cup (1, \infty)$ but some might say that $f(x)$ is continuous on $(-\infty, -1) \cup [1, \infty)$, where it is understood that $f(x)$ is *right continuous* at $x = 1$. We see this by graphing each function with the following commands. See Figure 3.4. Note that in p3, the vertical line is *not* a part of the graph of the function—it is a vertical asymptote. If you were to redraw the figure by hand, the vertical line would *not* be a part of the graph.

```
p1 = Plot[x^3 - 8x, {x, -5, 5}];
p2 = Plot[Sin[2x], {x, -5, 5}];
p3 = Plot[((x - 1)/(x + 1)), {x, -5, 5}];
p4 = Plot[Sqrt[(x - 1)/(x + 1)], {x, -5, 5}];
Show[GraphicsGrid[{{p1, p2}, {p3, p4}}]]
```

Computers are finite state machines, so handling "interesting" functions can be problematic, especially when one must distinguish between rational and irrational numbers. We assume that if $x = p/q$ is a *rational* number ($p$ and $q$ integers), $p/q$ is a reduced fraction. One way of tackling these sorts of problems is to view rational numbers as ordered pairs, $\{a, b\}$. If $a$ and $b$ are integers, Mathematica automatically reduces $a/b$ so Denominator[a/b] or a/b//Denominator returns the denominator of the reduced fraction; Numerator[a/b] or a/b//Numerator returns the numerator of the reduced fraction. If you want to see the points $(x, f(x))$ for which $x$ is rational, we use ListPlot.

**Example 3.1.7** Let $f(x) = \begin{cases} 1/q, & \text{if } x = p/q \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$.

Create a representative graph of $f(x)$.

***Solution*** You cannot see points: The measure of the rational numbers is $0$, and the measure of the irrational numbers is the **continuum**, $C$. A true graph of $f(x)$ would look like the graph of $y = 0$. In the context of the example, we want to see how the graph of $f(x)$ looks for rational values of $x$. We use a few points to illustrate the technique by using Table and Flatten to generate a set of ordered pairs.

Flatten[list,n] flattens list to level n.

In Mathematica, an ordered pair $(a, b)$ is represented by $\{a, b\}$.

```
t1 = Flatten[Table[{n, m}, {n, 1, 5}, {m, 1, 5}], 1]
{{1, 1}, {1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 1}, {2, 2},
  {2, 3}, {2, 4}, {2, 5}, {3, 1}, {3, 2}, {3, 3}, {3, 4}, {3, 5}, {4, 1},
    {4, 2}, {4, 3}, {4, 4}, {4, 5}, {5, 1},
      {5, 2}, {5, 3}, {5, 4}, {5, 5}}
```

Next, we defined a function $f$. Assuming that $a$ and $b$ are integers, given an ordered pair $\{a, b\}$, $f(\{a, b\})$ returns the point $\{a/b, 1/($Reduced denominator of $a/b)\}$

```
f[{a_, b_}] := {a/b, 1/(a/b//Denominator)}
```

We use Map to compute the value of $f$ for each ordered pair in t1. The resulting list is named t2.

**t2 = Map[f, t1]**

$\{\{1,1\},\{\frac{1}{2},\frac{1}{2}\},\{\frac{1}{3},\frac{1}{3}\},\{\frac{1}{4},\frac{1}{4}\},\{\frac{1}{5},\frac{1}{5}\},\{2,1\},\{1,1\},$
$\{\frac{2}{3},\frac{1}{3}\},\{\frac{1}{2},\frac{1}{2}\},\{\frac{2}{5},\frac{1}{5}\},\{3,1\},\{\frac{3}{2},\frac{1}{2}\},\{1,1\},$
$\{\frac{3}{4},\frac{1}{4}\},\{\frac{3}{5},\frac{1}{5}\},\{4,1\},\{2,1\},\{\frac{4}{3},\frac{1}{3}\},\{1,1\},$
$\{\frac{4}{5},\frac{1}{5}\},\{5,1\},\{\frac{5}{2},\frac{1}{2}\},\{\frac{5}{3},\frac{1}{3}\},\{\frac{5}{4},\frac{1}{4}\},\{1,1\}\}$

Notice that t2 contains duplicate entries. We can remove them using Flatten, but doing so does not affect the plot shown in Figure 3.5(a).

```
p1 = ListPlot[t2, PlotRange → {{0, 3}, {0, 1}}, AspectRatio → 1];
```

To generate a "prettier" plot, we repeat the procedure using more points. After entering each command, the results are not displayed because we include a semicolon (;) at the end of each. See Figure 3.5(b).

```
t3 = Flatten[Table[{n, m}, {n, 1, 300}, {m, 1, 200}], 1];
t4 = Map[f, t3];
p2 = ListPlot[t4, PlotRange → {{0, 3}, {0, 1}}, AspectRatio → 1];
```

This function is interesting because it is continuous at the irrationals and discontinuous at the rationals.

We can consider other functions in similar contexts. In the following, the $y$-coordinate is the numerator rather than the denominator. See Figure 3.5(c).

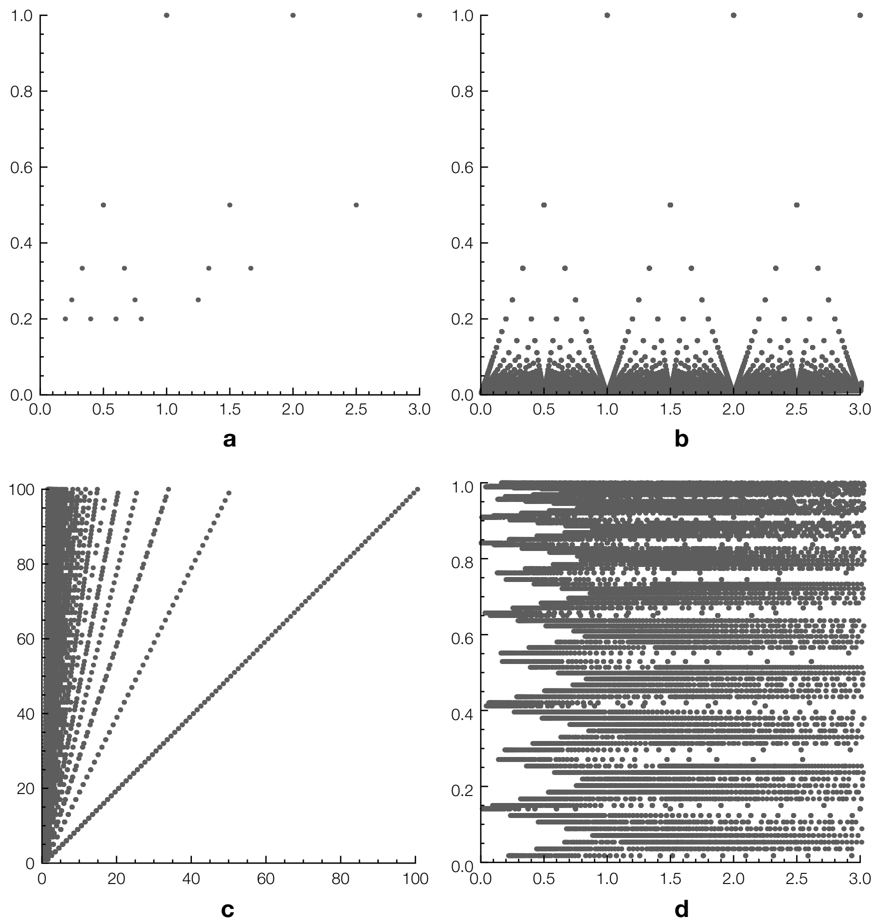



**FIGURE 3.5**

(a) After step 1. (b) After step 2. (c) Examining the numerator rather than the denominator. (d) The sine of the numerator

```
Clear[f]
f[{a_, b_}] := {a/b, a/b//Numerator};
t3 = Flatten[Table[{n, m}, {n, 1, 100}, {m, 1, 100}], 1];
t4 = Map[f, t3];
p3 = ListPlot[t4, PlotRange → {{0, 100}, {0, 100}}, AspectRatio → 1];
```

With Mathematica, we can modify commands to investigate how changing parameters affect a given situation. In the following, we compute the sine of $p$ if $x = p/q$. See Figure 3.5(d).

```
Clear[f]
f[{a_, b_}] := {a/b, Sin[(a/b//Numerator)]};
```

```
t5 = Flatten[Table[{n, m}, {n, 1, 300}, {m, 1, 200}], 1];
t6 = Map[f, t5];
p4 = ListPlot[t6, PlotRange → {{0, 3}, {0, 1}}, AspectRatio → 1];
Show[GraphicsGrid[{{p1, p2}, {p3, p4}}]]
```

# 3.2 DIFFERENTIAL CALCULUS

## 3.2.1 Definition of the Derivative

**Definition 2.** *The* **derivative** *of $y = f(x)$ is*

$$y' = f'(x) = \frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \tag{3.1}$$

*provided the limit exists.*

Assuming that $(a, f(a))$ and $(a+h, f(a+h))$ exist, the line with equation $y = \frac{f(a+h)-f(a)}{h}(x-a) + f(a)$ is the secant containing the two points. Assuming the derivative exists, as $h$ approaches 0, the secants approach the tangent. Hence, if the limit exists, the derivative gives us the slope of a function at that particular value of $x$.
The Limit command can be used along with Simplify to compute the derivative of a function using the definition of the derivative.

**Example 3.2.1** Use the definition of the derivative to compute the derivative of (a) $f(x) = x + 1/x$ and (b) $g(x) = 1/\sqrt{x}$.

***Solution*** For (a), we first define $f$, compute the difference quotient, $\left(f(x+h) - f(x)\right)$, simplify the difference quotient with Simplify, and use Limit to calculate the derivative.

```
f[x_] = x + 1/x;
step1 = (f[x + h] – f[x])/h
step2 = Simplify[step1]
Limit[step2, h → 0]
```

$\frac{h - \frac{1}{x} + \frac{1}{h+x}}{h}$

$\frac{-1 + hx + x^2}{x(h+x)}$

$1 - \frac{1}{x^2}$

For (b), we use the same approach as in (a) but use Together rather than Simplify to reduce the complex fraction.

```
step1 = (g[x + h] – g[x])/h
step2 = Together[step1]
```

```
Limit[step2, h → 0]
```

$$\frac{-\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{h+x}}}{h}$$

$$\frac{\sqrt{x}-\sqrt{h+x}}{h\sqrt{x}\sqrt{h+x}}$$

$$-\frac{1}{2x^{3/2}}$$

If the derivative of $y=f(x)$ exists at $x=a$, a geometric interpretation of $f'(a)$ is that $f'(a)$ is the slope of the line tangent to the graph of $y=f(x)$ at the point $(a, f(a))$.

To motivate the definition of the derivative, many calculus texts choose a value of $x$, $x=a$, and then draw the graph of the secant line passing through the points $(a, f(a))$ and $(a+h, f(a+h))$ for "small" values of $h$ to show that as $h$ approaches 0, the secant line approaches the tangent line. An equation of the secant line passing through the points $(a, f(a))$ and $(a+h, f(a+h))$ is given by

$$y-f(a)=\frac{f(a+h)-f(a)}{(a+h)-a}(x-a) \quad \text{or} \quad y=\frac{f(a+h)-f(a)}{h}(x-a)+f(a).$$

**Example 3.2.2** If $f(x)=x^2-4x$, graph $f(x)$ together with the secant line containing $(1, f(1))$ and $(1+h, f(1+h))$ for various values of $h$.

***Solution*** We begin by considering a particular $h$ value. We choose $h=0.4$. We then define $f(x)=x^2-4x$. In p1, we graph $f(x)$ in black on the interval $[-1, 5]$, in p2 we place a blue point at $(1, f(1))$ and a green point at $(1.4, f(1.4)$, in p3 we graph the tangent to $y=f(x)$ at $(1, f(1))$ in red, in p4 we graph the secant containing $(1, f(1))$ and $(1.4, f(1.4)$ in purple, and finally we show all four graphics together with Show in Figure 3.6.

Remember that when a semicolon is placed at the end of a command, the resulting output is not displayed. The names of the colors that Mathematica knows are listed in the **ColorSchemes** palette followed by "Known" and then "System."

```
f[x_] = x^2 - 4x;
p1 = Plot[f[x], {x, -1, 5}, PlotStyle → Black];
p2 = Graphics[{PointSize[.03], Blue, Point[{1, f[1]}],
     Green, Point[{1 + .4, f[1 + .4]}]}];
p3 = Plot[f'[1](x - 1) + f[1], {x, -1, 5}, PlotStyle → Red];
p4 = Plot[(f[1 + .4] - f[1])/.4(x - 1) + f[1], {x, -1, 5},
     PlotStyle → Purple];
Show[p1, p2, p3, p4, PlotRange → {{-1, 5}, {-6, 6}},
     AspectRatio → 1]]
```

We now generalize the previous set of commands for arbitrary $h \neq 0$ values. $g(h)$ shows plots of $y=x^2-4x$, the tangent at $(1, f(1))$, and the secant containing $(1, f(1))$ and $(1+h, f(1+h))$.

```
Clear[f, g];
f[x_] = x^2 - 4x;
g[h_] := Module[{p1, p2, p3, p4},
p1 = Plot[f[x], {x, -1, 5}, PlotStyle → Black];
```

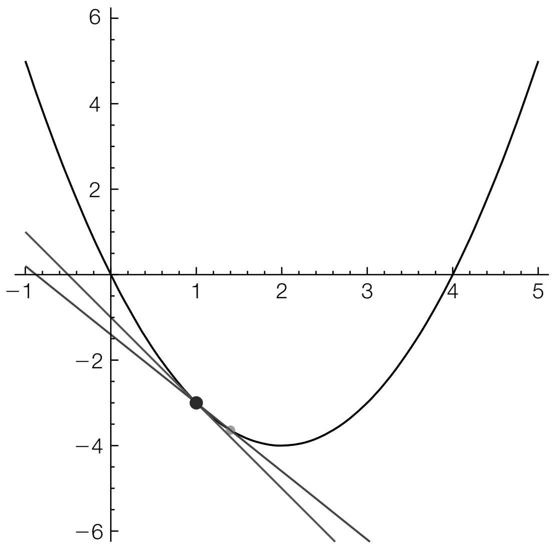



**FIGURE 3.6**

Plots of $y = x^2 - 4x$, the tangent at $(1, f(1))$, and the secant containing $(1, f(1))$ and $(1 + h, f(1 + h))$ if $h = 0.4$

```
p2 = Graphics[{PointSize[.03], Blue, Point[{1, f[1]}],
      Green, Point[{1 + h, f[1 + h]}]}];
p3 = Plot[f′[1](x − 1) + f[1], {x, −1, 5}, PlotStyle → Red];
p4 = Plot[( f[1 + h] − f[1])/h(x − 1) + f[1], {x, −1, 5},
      PlotStyle → Purple];
Show[p1, p2, p3, p4, PlotRange → {{−1, 5}, {−6, 6}}, AspectRatio → 1]]
```

Table[f[x],{x,start,stop,stepsize}] creates a table of $f(x)$ values beginning with *start* and ending with *stop* using increments of *stepsize*. Given a table, Partition[table,n] partitions the table into *n* element subgroups. Thus, if a table, t1, has nine elements, Partition[t1, 3] creates a $3 \times 3$ grid; three sets of three elements each.

Using Table followed by GraphicsGrid, we can create a table of graphics for various values of *h* like that shown in Figure 3.7. With Table, the dimensions of the grid displayed on your computer are based on the size of the active Mathematica window. To control the dimensions of the grid, we use GraphicsGrid together with Partition and Show.

```
t1 = Table[g[k], {k, 1, .0001, −(1 −.0001)/8}]
Show[GraphicsGrid[Partition[t1, 3]]]
```

Do works in the same way as Table. Rather than creating a table (or list), Do performs the action repeatedly. Thus, you can use Do to create an animation of the secants approaching the tangent.

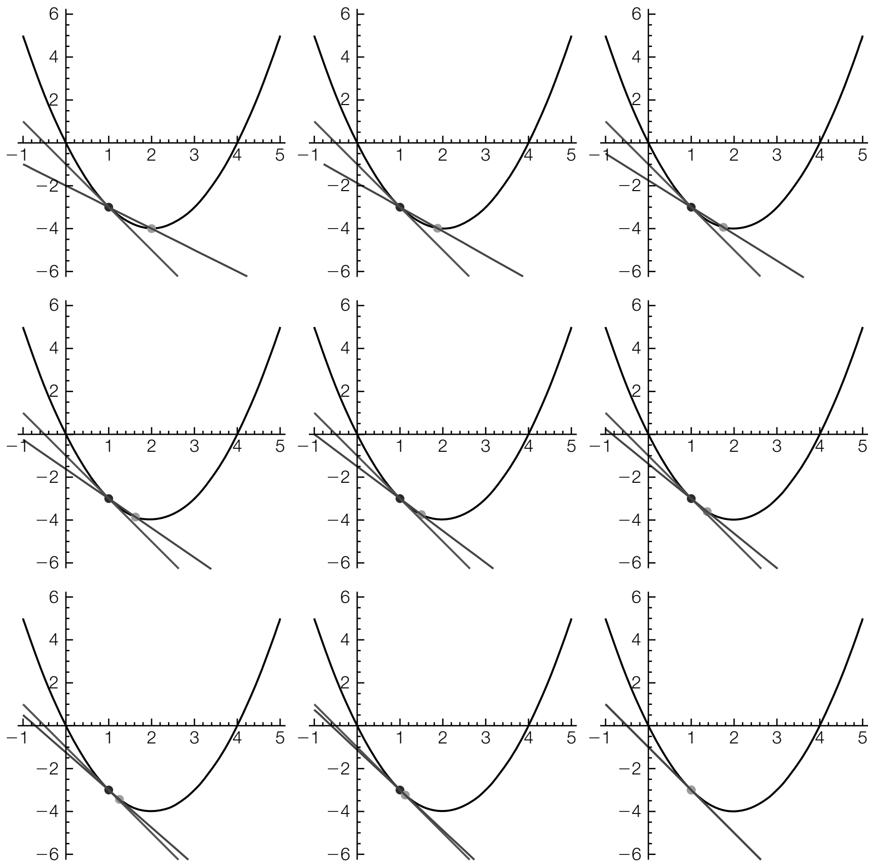
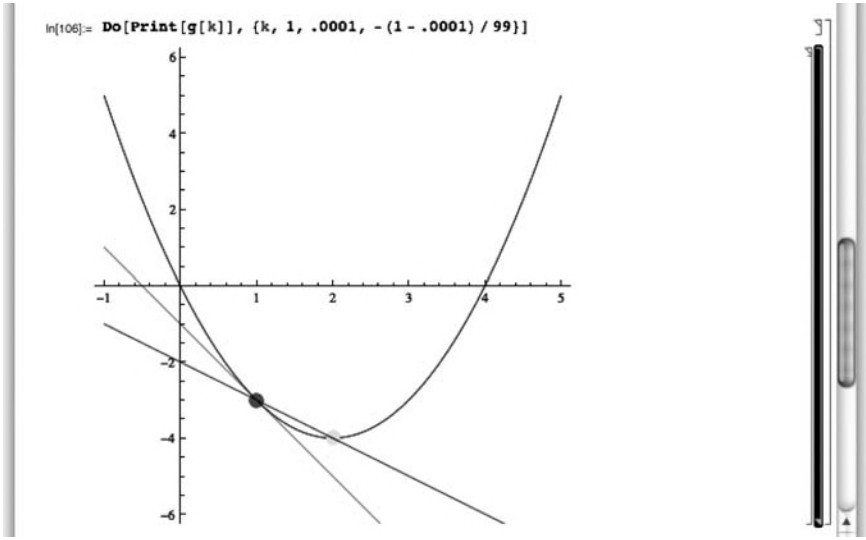


**FIGURE 3.7**

Plots of $y = x^2 - 4x$, the tangent at $(1, f(1))$, and the secant containing $(1, f(1))$ and $(1 + h, f(1 + h))$ for various values of $h$

Thus, entering computes $g(k)$ and displays the result for $100$ equally spaced values of $k$ starting with $k = 1$ and ending with $k = 0.0001$. To animate the results, select the graphics as indicated. Go to the Mathematica menu, select **Graphics** followed by **Rendering** and then **Animate Selected Graphics**




Keyboard shortcuts are usually operating system dependent.

or use a keyboard shortcut to animate the graphics. After animating the selection, you can control the animation (speed, direction, and so on) with the buttons  displayed in the lower left-hand corner of the Mathematica notebook.

With Mathematica 6, you can use Manipulate to help generate animations and images that you can adjust based on changing parameter values.

To illustrate how to do so, we begin by redefining $f$ and then defining $m(a, h)$. Given $a$ and $h$ values, $m(a, h)$ plots $f(x)$ for $-10 \le x \le 10$ (p1), plots a blue point $(a, f(a))$ and a green point at $(a+h, f(a+h))$ (p2), plots $f'(a)(x-a)+f(a)$ (the tangent to the graph of $f(x)$ at $(a, f(a))$) for $-1 \le x \le 5$ in red (p3), the secant containing $(a, f(a))$ and $(a+h, f(a+h))$ for $-10 \le x \le 10$ in purple (p4), and finally displays all four graphics together with Show. Using PlotRange, we indicate that the horizontal axis displays $x$ values between $-10$ and $10$, at the vertical axis displays $y$ values between $-10$ and $10$; AspectRatio->1 means that the ratio of the lengths of the $x$ to $y$ axes is $1$. Thus, the plot scaling is correct. Note that when we use Module to define $m$, p1, p2, p3, and p4 are *local* to the function $m$. This means that if you have such objects defined elsewhere in your Mathematica notebook, those objects are not affected when you compute $m$.

```
Clear[m, f ];
f[x_] = x^2 - 4x;
m[a_, h_] := Module[{p1, p2, p3, p4},
      p1 = Plot[f[x], {x, -10, 10}, PlotStyle → Black];
      p2 = Graphics[{PointSize[.03], Blue, Point[{a, f[a]}],
          Green, Point[{a + h, f[a + h]}]}];
      p3 = Plot[f ′[a](x - a) + f[a], {x, -1, 5}, PlotStyle → Red];
```

```
p4 = Plot[(f[a + h] − f[a])/h(x − a) + f[a], {x, −10, 10},
    PlotStyle → Purple];
Show[p1, p2, p3, p4, PlotRange → {{−10, 10}, {−10, 10}},
    AspectRatio → 1]]
```

Now we use Manipulate to create a "mini" program. The sliders (centered at $a = 0$ and $h = 0.5$ with a range from $-10$ to $10$ and $-1$ to $1$, respectively) allow you to see how changing $a$ and $h$ affects the plot. See Figure 3.8.

```
Manipulate[m[a, h], {{a, 0}, −10, 10}, {{h, .5}, −1, 1}]
```

Figure 3.8 illustrates the special case in which $f(x) = x^2 - 4x$. To illustrate the same concept using a "standard" set of functions (polynomials, rational, root, and trig), we first define the functions

```
quad[x_] = (x + 2)^2 − 2;
cubic[x_] = −1/10x(x^2 − 25);
rational[x_] = 50/((x + 5)(x − 5));
root[x_] = 3Sqrt[x + 5];
sin[x_] = 5Sin[x];
```

and then we adjust m by defining a few of these "standard" and then defining the function mmore. mmore performs the same actions as $m$ but does so for the

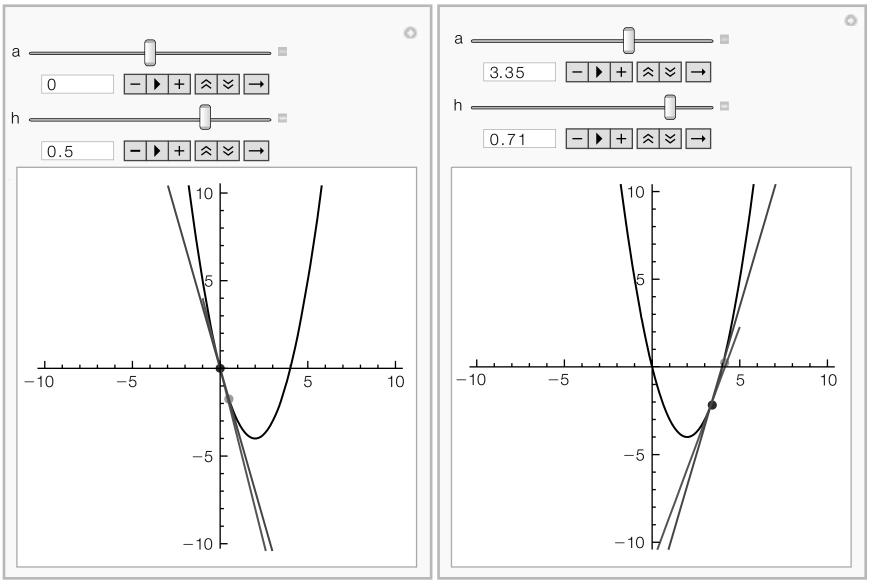



**FIGURE 3.8**

With Manipulate, we can perform animations and see how a function changes depending on parameter values function selected. We then use **Manipulate** to create an object that shows the secant (in purple), the tangent (in red) for the selected function, $a$ value, and $b$ value. See Figure 3.9.

```
Clear[mmore];
mmore[f_, a_, h_] := Module[{p1, p2, p3, p4},
    p1 = Plot[f[x], {x, -10, 10}, PlotStyle → Black];
    p2 = Graphics[{PointSize[.03], Blue, Point[{a, f[a]}],
        Green, Point[{a + h, f[a + h]}]}];
    p3 = Plot[f′[a](x - a) + f[a], {x, -10, 10}, PlotStyle → Red];
    p4 = Plot[( f[a + h] - f[a])/h(x - a) + f[a], {x, -10, 10},
        PlotStyle → Purple];
Show[p1, p2, p3, p4, PlotRange → {{-10, 10}, {-10, 10}},
    AspectRatio → 1]]

Manipulate[mmore[f, a, h], {{f, quad},
  {quad, cubic, rational, root, sin }},
    {{a, 0}, -10, 10}, {{h, 1}, -2, 2}]
```

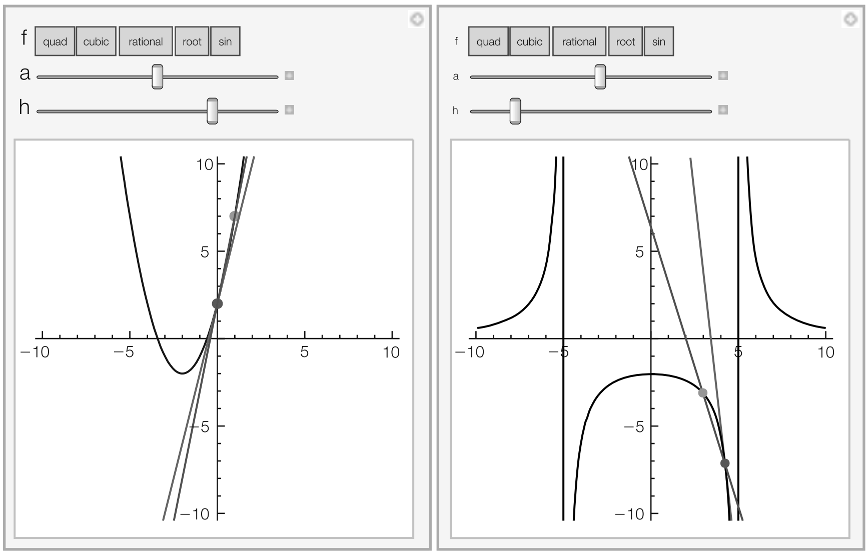



**FIGURE 3.9**

With this **Manipulate** object, we see how various functions, $a$ values, and $b$ values affect the secant to $y = f(x)$ passing through $(a, f(a))$ and $(a + b, f(a + b))$ and the tangent to $y = f(x)$ at $(a, f(a))$

### 3.2.2 **Calculating Derivatives**

The functions D and $'$ are used to differentiate functions. Assuming that $y = f(x)$ is differentiable,

1. D[f[x],x] computes and returns $f'(x) = df/dx$,
2. f'[x] computes and returns $f'(x) = df/dx$,
3. f''[x] computes and returns $f^{(2)}(x) = d^2f(x)/dx^2$, and
4. D[f[x],x,n] computes and returns $f^{(n)}(x) = d^nf(x)/dx^n$.
5. You can use the button located on the **BasicMathInput** palette to create templates to compute derivatives.

Figure 3.10 illustrates various ways of computing derivatives using the $'$ symbol, D, and the $\partial$ symbol.

Mathematica knows the numerous differentiation rules, including the product, quotient, and chain rules. Thus, entering

**Clear[f, g]**
**D[f[x]g[x], x]**
g[x]f′[x] + f[x]g′[x]

shows us that $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)$; entering

Throughout the text, input is in bold and output is not; output follows input.

**Together[D[f[x]/g[x], x]]**

$$\frac{g[x]f'[x] - f[x]g'[x]}{g[x]^2}$$

**D[f[g[x]], x]**

shows us that $\frac{d}{dx}(f(x)/g(x)) = (f'(x)g(x) - f(x)g'(x))/(g(x))^2$; and entering

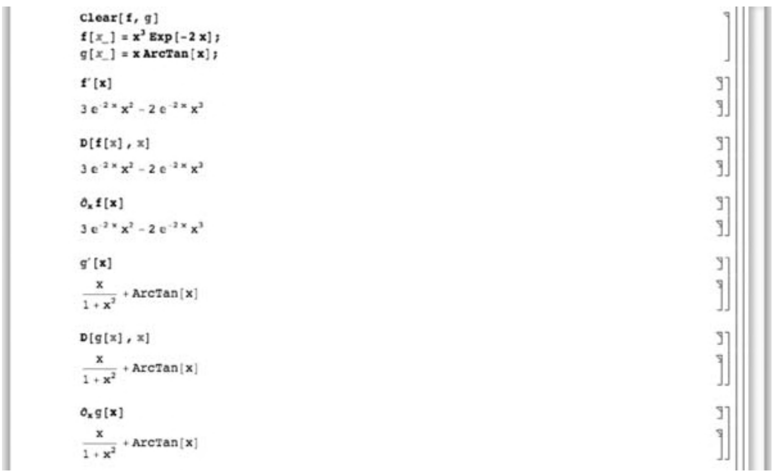



**FIGURE 3.10**

You can use $'$, D, and $\partial$ to compute derivatives of functions

```
D[f[g[x]], x]
f′[g[x]]g′[x]
```

shows us that $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$.

---

**Example 3.2.3** Compute the first and second derivatives of (a) $y = x^4 + \frac{4}{3}x^3 - 3x^2$, (b) $f(x) = 4x^5 - \frac{5}{2}x^4 - 10x^3$, (c) $y = \sqrt{e^{2x} + e^{-2x}}$, and (d) $y = (1 + 1/x)^x$.

***Solution*** For (a), we use D.

```
D[x^4 + 4/3x^3 - 3x^2, {x, 2}]
-6 + 8x + 12x^2
```

For (b), we first define $f$ and then use ′ together with Factor to calculate and factor $f'(x)$ and $f''(x)$.

```
f[x_] = 4x^5 - 5/2x^4 - 10x^3;
Factor[f′[x]]
10x^2(1 + x)(-3 + 2x)
Factor[f″[x]]
10x(-6 - 3x + 8x^2)
```

For (c), we use Simplify together with D to calculate and simplify $y'$ and $y''$.

```
D[Sqrt[Exp[2x] + Exp[-2x]], {x, 2}]//Simplify
```

$$\frac{\sqrt{e^{-2x} + e^{2x}}\left(1 + 6e^{4x} + e^{8x}\right)}{\left(1 + e^{4x}\right)^2}$$

By hand, (d) would require logarithmic differentiation. The second derivative would be particularly difficult to compute by hand. Mathematica quickly computes and simplifies each derivative.

```
Simplify[D[(1 + 1/x)^x, x]]
```

$$\frac{\left(1 + \frac{1}{x}\right)^x\left(-1 + (1 + x)\mathrm{Log}\left[1 + \frac{1}{x}\right]\right)}{1 + x}$$

```
Simplify[D[(1 + 1/x)^x, {x, 2}]]
```

$$\frac{\left(1+\frac{1}{x}\right)^x\left(-1 + x - 2x(1 + x)\mathrm{Log}\left[1 + \frac{1}{x}\right] + x(1 + x)^2\mathrm{Log}\left[1 + \frac{1}{x}\right]^2\right)}{x(1 + x)^2}$$

---

Map and operations on lists are discussed in more detail in Chapter 4.

The command Map[f,list] applies the function f to each element of the list list. Thus, if you are computing the derivatives of a large number of functions, you can use Map together with D.

---

**Remark 3.2** A built-in Mathematica function is threadable if f[list] returns the same result as Map[f,list]. Many familiar functions such as D and Integrate are threadable.

**Example 3.2.4** Compute the first and second derivatives of $\sin x$, $\cos x$, $\tan x$, $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$.

***Solution*** Notice that lists are contained in braces. Thus, entering

```
Map[D[#, x]&, {Sin[x], Cos[x], Tan[x],
    ArcSin[x], ArcCos[x], ArcTan[x]}]
```

$$\left\{\text{Cos}[x], -\text{Sin}[x], \text{Sec}[x]^2, \frac{1}{\sqrt{1-x^2}}, -\frac{1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}\right\}$$

computes the first derivative of the three trigonometric functions and their inverses. In this case, we have applied a *pure function* to the list of trigonometric functions and their inverses. Given an argument #, D[#,x]& computes the derivative of # with respect to $x$. The & symbol is used to mark the end of a *pure* function. Similarly, entering

```
Map[D[#, {x, 2}]&, {Sin[x], Cos[x], Tan[x],
    ArcSin[x], ArcCos[x], ArcTan[x]}]
```

$$\left\{-\text{Sin}[x], -\text{Cos}[x], 2\text{Sec}[x]^2\text{Tan}[x], \frac{x}{(1-x^2)^{3/2}}, -\frac{x}{(1-x^2)^{3/2}}, -\frac{2x}{(1+x^2)^2}\right\}$$

computes the second derivative of the three trigonometric functions and their inverses. Because D is threadable, the same results are obtained with the following commands:

```
D[{Sin[x], Cos[x], Tan[x],
  ArcSin[x], ArcCos[x], ArcTan[x]}, x]
```

$$\left\{\text{Cos}[x], -\text{Sin}[x], \text{Sec}[x]^2, \frac{1}{\sqrt{1-x^2}}, -\frac{1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}\right\}$$

```
D[{Sin[x], Cos[x], Tan[x],
  ArcSin[x], ArcCos[x], ArcTan[x]}, {x, 2}]
```

$$\left\{-\text{Sin}[x], -\text{Cos}[x], 2\text{Sec}[x]^2\text{Tan}[x], \frac{x}{(1-x^2)^{3/2}}, -\frac{x}{(1-x^2)^{3/2}}, -\frac{2x}{(1+x^2)^2}\right\}$$

With DynamicModule, we create a simple dynamic that lets you compute the first and second derivatives of basic functions and plot them on a standard viewing window, $[- -5, 5] \times [-5, 5]$. The layout of Figure 3.11 is primarily determined by Panel, Column, and Grid.

```
Panel[DynamicModule[{f = x^2},
  Column[{InputField[Dynamic[f]], Grid[{{"FirstDerivative",
  Panel[Dynamic[D[f, x]//Simplify]]},
    {"SecondDerivative", Panel[Dynamic[D[f, {x, 2}]//Simplify]]}}],
  Dynamic[Plot[Evaluate[Tooltip[{f, D[f, x], D[f, {x, 2}]}]],
{x, -5, 5}, PlotRange → {-5, 5},
  AspectRatio → Automatic]]}]], ImageSize → {300, 300}]
```

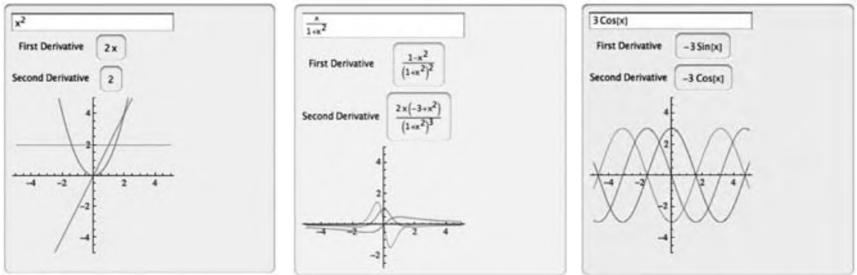



**FIGURE 3.11**

Seeing the relationship between the first and second derivative of a function and the original function

## 3.2.3 Implicit Differentiation

**If an equation contains two variables, $x$ and $y$, implicit differentiation can be carried out by explicitly declaring $y$ to be a function of $x$, $y = y(x)$, and using D or by using the Dt command.**

**Example 3.2.5** Find $y' = dy/dx$ if (a) $\cos\left(e^{xy}\right) = x$ and (b) $\ln\left(x/y\right) + 5xy = 3y$.

***Solution*** For (a) we illustrate the use of D. Notice that we are careful to specifically indicate that $y = y(x)$. First we differentiate with respect to $x$.

```
Clear[x, y]
s1 = D[Cos[Exp[xy[x]]] - x, x]
```

$$-1 - e^{xy[x]}\text{Sin}\left[e^{xy[x]}\right]\left(y[x] + xy'[x]\right)$$

and then we solve the resulting equation for $y' = dy/dx$ with Solve.

```
Solve[s1==0, y'[x]]
```

$$\left\{\left\{y'[x] \to -\frac{e^{-xy[x]}\text{Csc}\left[e^{xy[x]}\right]\left(1 + e^{xy[x]}\text{Sin}\left[e^{xy[x]}\right]y[x]\right)}{x}\right\}\right\}$$

For (b), we use Dt. When using Dt, we interpret Dt[x]= 1 and Dt[y]= $y' = dy/dx$. Thus, entering

```
s2 = Dt[Log[x/y] + 5xy - 3y]
```

$$5y\text{Dt}[x] - 3\text{Dt}[y] + 5x\text{Dt}[y] + \frac{y\left(\frac{\text{Dt}[x]}{y} - \frac{x\text{Dt}[y]}{y^2}\right)}{x}$$

```
s3 = s2/.{Dt[x] → 1, Dt[y] → dydx}
```

$$-3\text{dydx} + 5\text{dydxx} + 5y + \frac{\left(-\frac{\text{dydxx}}{y^2} + \frac{1}{y}\right)y}{x}$$

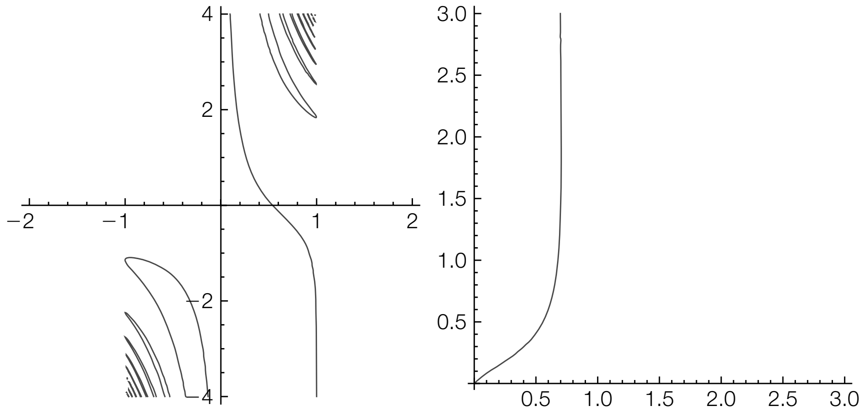



**FIGURE 3.12**

On the left, $\cos\left(e^{xy}\right) = x$ for $-2 \le x \le 2$ and $-4 \le y \le 4$; on the right, $\ln\left(x/y\right) + 5xy = 3y$ for $0.01 \le x \le 3$ and $0.01 \le y \le 3$.

and solving for dydx with Solve

**Solve[s3==0, dydx]**

$\left\{\left\{\text{dydx} \to -\frac{\text{y}(1+5\text{xy})}{\text{x}(-1-3\text{y}+5\text{xy})}\right\}\right\}$

shows us that if $\ln\left(x/y\right) + 5xy = 3y$, $y' = \frac{dy}{dx} = -\frac{(1+5xy)y}{(5xy-3y-1)x}$.

To graph each equation, we use ContourPlot. Generally, given an equation of the form $f(x,y) = g(x,y)$, the command

**ContourPlot[f[x, y]==g[x, y], {x, a, b}, {y, c, d}]**

attempts to plot the graph of $f(x,y) = g(x,y)$ on the rectangle $[a,b] \times [c,d]$. Using Show together with GraphicsRow, we show the two graphs side-by-side in Figure 3.12.

```
cp1 = ContourPlot[Cos[Exp[xy]]==x, {x, −2, 2}, {y, −4, 4}, PlotPoints → 120,
      Frame → False, Axes → Automatic, AxesOrigin → {0, 0}];
cp2 = ContourPlot[Log[x/y] + 5xy==3y, {x, .01, 3}, {y, .01, 3},
      PlotPoints → 120,
Frame → False, Axes → Automatic, AxesOrigin → {0, 0}];
Show[GraphicsRow[{cp1, cp2}]]
```

## 3.2.4 Tangent Lines

If $f'(a)$ exists, a typical interpretation of $f'(a)$ is that $f'(a)$ is the slope of the line tangent to the graph of $y = f(x)$ at the point $(a, f(a))$. In this case, an equation of the tangent is given by

$$y - f(a) = f'(a)(x-a) \quad \text{or} \quad y = f'(a)(x-a) + f(a).$$

**Example 3.2.6** Find an equation of the line tangent to the graph of $f(x) = \sin x^{1/3} + \cos^{1/3} x$ at the point with $x$-coordinate $x = 5\pi/3$.

***Solution*** Because we will be graphing a function involving odd roots of negative numbers, we begin by loading the **RealOnly** package contained in the **Miscellaneous** folder (or directory). We then define $f(x)$ and compute $f'(x)$.

If this package was not included with your version of Mathematica, you may want to download it from the Wolfram website.

```
<< Miscellaneous`RealOnly`
f[x_] = Sin[x^(1/3)] + Cos[x]^(1/3);
f'[x]
```

$$\frac{\text{Cos}\left[x^{1/3}\right]}{3x^{2/3}} - \frac{\text{Sin}[x]}{3\text{Cos}[x]^{2/3}}$$

Then, the slope of the line tangent to the graph of $f(x)$ at the point with $x$-coordinate $x = 5\pi/3$ is

```
f'[5Pi/3]
```

$$\frac{1}{2^{1/3}\sqrt{3}} + \frac{\text{Cos}\left[\left(\frac{5\pi}{3}\right)^{1/3}\right]}{3^{1/3}(5\pi)^{2/3}}$$

```
f'[5Pi/3]//N
```

0.440013

while the $y$-coordinate of the point is

```
f[5Pi/3]
```

$$\frac{1}{2^{1/3}} + \text{Sin}\left[\left(\frac{5\pi}{3}\right)^{1/3}\right]$$

```
f[5Pi/3]//N
```

1.78001

Thus, an equation of the line tangent to the graph of $f(x)$ at the point with $x$-coordinate $x = 5\pi/3$ is

$$y - \left(\frac{1}{\sqrt[3]{2}} + \sin\sqrt[3]{5\pi/3}\right) = \left(\frac{\cos\sqrt[3]{5\pi/3}}{\sqrt[3]{3}\sqrt[3]{25\pi^2}} + \frac{1}{\sqrt[3]{2}\sqrt{3}}\right)\left(x - \frac{5\pi}{3}\right),$$

as shown in Figure 3.13.

```
p1 = Plot[f[x], {x, 0, 4Pi}, PlotStyle → Black];
p2 = ListPlot[{{5Pi/3, f[5Pi/3]}//N}, PlotStyle → PointSize[.03]];
```

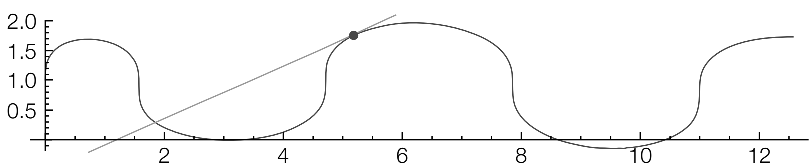



**FIGURE 3.13**

$f(x) = \sin x^{1/3} + \cos^{1/3} x$ together with its tangent at the point $(5\pi/3, f(5\pi/3))$

```
p3 = Plot[f'[5Pi/3](x-5Pi/3) + f[5Pi/3], {x, 0, 4Pi},
PlotStyle->GrayLevel[.6]];

Show[p1, p2, p3, AspectRatio->Automatic,
    DisplayFunction → $DisplayFunction]
```

### *Tangent Lines of Implicit Functions*

**Example 3.2.7** Find equations of the tangent line and normal line to the graph of $x^2y - y^3 = 8$ at the point $(-3, 1)$. Find and simplify $y'' = d^2y/dx^2$.

***Solution*** We evaluate $y' = dy/dx$ if $x = -3$ and $y = 1$ to determine the slope of the tangent line at the point $(-3, 1)$. Note that we cannot (easily) solve $x^2y - y^3 = 8$ for $y$, so we use implicit differentiation to find $y' = dy/dx$:

By the product and chain rules, $\frac{d}{dx}(x^2y) = \frac{d}{dx}(x^2)y + x^2\frac{d}{dx}(y) = 2x \cdot y + x^2 \cdot \frac{dy}{dx} = 2xy + x^2y'$.

$$\frac{d}{dx}\left(x^2y - y^3\right) = \frac{d}{dx}(8)$$
$$2xy + x^2y' - 3y^2y' = 0$$
$$y' = \frac{-2xy}{x^2 - 3y^2}.$$

```
eq = x^2y-y^3==8
x²y-y³==8
s1 = Dt[eq]
2xyDt[x] + x²Dt[y] - 3y²Dt[y]==0
s2 = s1/.Dt[x] → 1
2xy + x²Dt[y] - 3y²Dt[y]==0
s3 = Solve[s2, Dt[y]]
{{Dt[y] → -2xy/(x²-3y²)}}
```

Notice that s3 is a **list.** The formula for $y' = dy/dx$ is the second part of the first part of the first part of s3 and extracted from s3 with

**Lists** are discussed in more detail in Chapter 4.

```
s3[[1,1,2]]
-2xy/(x²-3y²)
```

We then use ReplaceAll (/.) to find that the slope of the tangent at $(-3, 1)$ is

```
s3[[1, 1, 2]]/.{x → -3, y → 1}
1
```

The slope of the normal is $-1/1 = -1$. Equations of the tangent and normal are given by

$$y - 1 = 1(x + 3) \qquad \text{and} \qquad y - 1 = -1(x + 3),$$

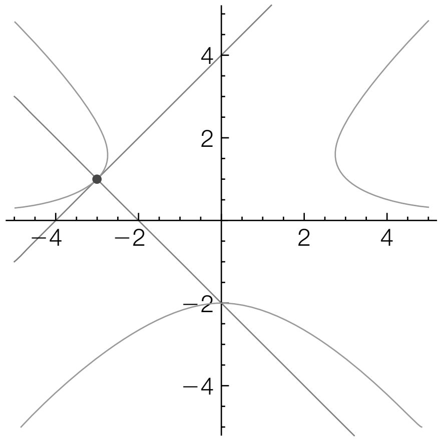



**FIGURE 3.14**

Graphs of $x^2y - y^3 = 8$ (in black) and the tangent and normal at $(-3, 1)$ (in gray)

respectively. See Figure 3.14.

```
cp1 = ContourPlot[x^2y - y^3 - 8, {x, -5, 5}, {y, -5, 5}, Contours → {0},
        ContourShading → False, PlotPoints → 200];
p1 = ListPlot[{{-3, 1}}, PlotStyle → PointSize[.03]];
p2 = Plot[{(x + 3) + 1, -(x + 3) + 1}, {x, -5, 5}, PlotStyle → Gray];
Show[cp1, p1, p2, Frame → False, Axes → Automatic,
      AxesOrigin → {0, 0}, AspectRatio → Automatic,
DisplayFunction → $DisplayFunction]
```

To find $y'' = d^2y/dx^2$, we proceed as follows:

```
s4 = Dt[s3[[1, 1, 2]]]//Simplify
```

$$-\frac{2(x^2+3y^2)(-y\,Dt[x]+x\,Dt[y])}{(x^2-3y^2)^2}$$

```
s5 = s4/.Dt[x] → 1/.s3[[1]]//Simplify
```

$$\frac{6y(x^2-y^2)(x^2+3y^2)}{(x^2-3y^2)^3}$$

The result means that

$$y'' = \frac{d^2y}{dx^2} = \frac{6\left(x^2y - y^3\right)\left(x^2+3y^2\right)}{\left(x^2-3y^2\right)^3}.$$

Because $x^2y - y^3 = 8$, the second derivative is further simplified to

$$y'' = \frac{d^2y}{dx^2} = \frac{48\left(x^2+3y^2\right)}{\left(x^2-3y^2\right)^3}.$$

### *Parametric Equations and Polar Coordinates*

For the parametric equations $\{x = f(t), y = g(t)\}$, $t \in I$,

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

and

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d/dt(dy/dx)}{dx/dt}.$$

If $\{x = f(t), y = g(t)\}$ has a tangent line at the point $(f(a), g(a))$, parametric equations of the tangent are given by

$$x = f(a) + tf'(a) \qquad \text{and} \qquad y = g(a) + tg'(a). \tag{3.2}$$

If $f'(a)$, $g'(a) \neq 0$, we can eliminate the parameter from (3.2)

$$\frac{x - f(a)}{f'(a)} = \frac{y - g(a)}{g'(a)}$$

$$y - g(a) = \frac{g'(a)}{f'(a)}(x - f(a))$$

and obtain an equation of the tangent line in point-slope form.

```
l = Solve[x[a] + tx′[a]==cx, t]
r = Solve[y[a] + ty′[a]==cy, t]
```

$\left\{\left\{t \to \frac{cx - x[a]}{x'[a]}\right\}\right\}$

$\left\{\left\{t \to \frac{cy - y[a]}{y'[a]}\right\}\right\}$

---

**Example 3.2.8 (The Cycloid).** The **cycloid** has parametric equations

$$x = t - \sin t \qquad \text{and} \qquad y = 1 - \cos t.$$

Graph the cycloid together with the line tangent to the graph of the cycloid at the point $(x(a), y(a))$ for various values of $a$ between $-2\pi$ and $4\pi$.

***Solution*** After defining $x$ and $y$, we use $'$ to compute $dy/dt$ and $dx/dt$. We then compute $dy/dx = (dy/dt)/(dx/dt)$ and $d^2y/dx^2$.

```
x[t_] = t - Sin[t];
y[t_] = 1 - Cos[t];
dx = x′[t]
dy = y′[t]
dydx = dy/dx
```

1 − Cos[t]

Sin[t]

$\frac{\text{Sin}[t]}{1 - \text{Cos}[t]}$

```
dypdt = Simplify[D[dydx, t]]
```

$\frac{1}{-1+\mathrm{Cos}[t]}$

```
secondderiv = Simplify[dypdt/dx]
```

$-\frac{1}{(-1+\mathrm{Cos}[t])^2}$

We then use ParametricPlot to graph the cycloid for $-2\pi \le t \le 4\pi$, naming the resulting graph p1.

```
p1 = ParametricPlot[{x[t], y[t]}, {t, -2Pi, 4Pi},
   PlotStyle → {{Black, Thickness[.01]}}];
```

Next, we use Table to define toplot to be 40 tangent lines (3.2) using equally spaced values of $a$ between $-2\pi$ and $4\pi$. We then graph each line toplot and name the resulting graph p2. Finally, we show p1 and p2 together with the Show function. The resulting plot is shown to scale because the lengths of the $x$- and $y$-axes are equal and we include the option AspectRatio->1. In the graphs, notice that on intervals for which $dy/dx$ is defined, $dy/dx$ is a decreasing function and, consequently, $d^2y/dx^2 < 0$. (See Figure 3.15.)

```
toplot = Table[{x[a] + tx′[a], y[a] + ty′[a]}, {a, -2Pi, 4Pi, 6Pi/39}];
p2 = ParametricPlot[Evaluate[toplot], {t, -2, 2}, PlotStyle → Gray];
Show[p1, p2, AspectRatio → 1, PlotRange → {-3Pi, 3Pi}]
```

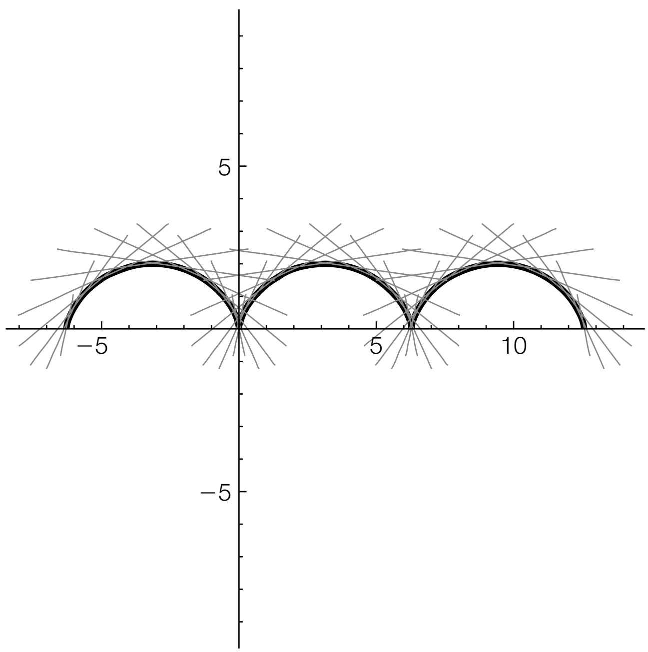



**FIGURE 3.15**

The cycloid with various tangents

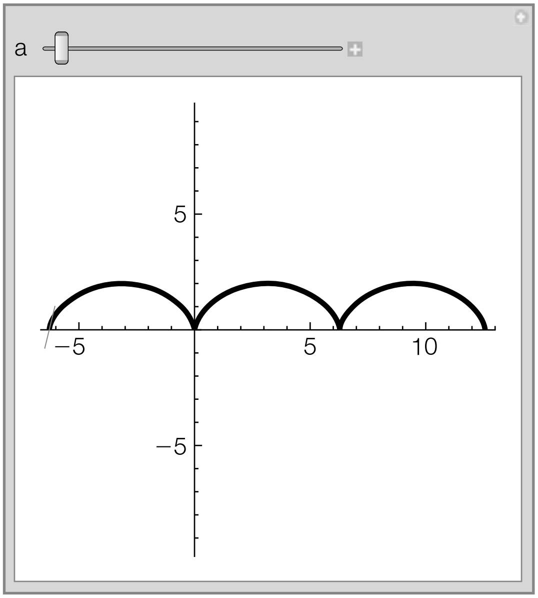



**FIGURE 3.16**

Using Manipulate you can animate the tangents

With Manipulate, you can animate the the tangents. (See Figure 3.16.)

```
Manipulate[x[t_] = t − Sin[t]; y[t_] = 1 − Cos[t];
    y[t_] = Module[{p1, p2}, p1 = ParametricPlot[{x[t], y[t]}, {t, −2Pi, 4Pi},
    PlotStyle → {{Black, Thickness[.01]}}];
p2 = ParametricPlot[{x[a] + tx′[a], y[a] + ty′[a]}, {t, −2, 2},
    PlotStyle → Gray];
Show[p1, p2, AspectRatio → 1, PlotRange → {{−2Pi, 4Pi}, {−3Pi, 3Pi}}]],
    {{a, 1}, −2Pi, 4Pi}]
```

**Example 3.2.9 (Orthogonal Curves).** Two lines $L_1$ and $L_2$ with slopes $m_1$ and $m_2$, respectively, are **orthogonal** if their slopes are negative reciprocals: $m_1 = -1/m_2$.

Extended to curves, we say that the curves $C_1$ and $C_2$ are **orthogonal** at a point of intersection if their respective tangent lines to the curves at that point are orthogonal.

Show that the family of curves with equation $x^2 + 2xy - y^2 = C$ is orthogonal to the family of curves with equation $y^2 + 2xy - x^2 = C$.

***Solution*** We begin by defining eq1 and eq2 to be equations $x^2 + 2xy - y^2 = C$ and $y^2 + 2xy - x^2 = C$, respectively. Then, use Dt to differentiate and Solve to find $y' = dy/dx$.

```
eq1 = x^2 + 2xy − y^2==c;
eq2 = y^2 + 2xy − x^2==c;
```

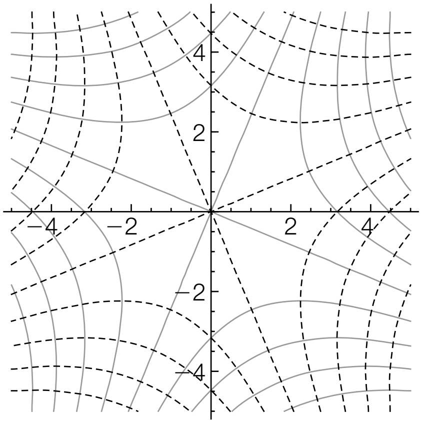



**FIGURE 3.17**

$x^2 + 2xy - y^2 = C$ and $y^2 + 2xy - x^2 = C$ for various values of $C$

```
Simplify[Solve[Dt[eq1, x], Dt[y, x]]/.Dt[c, x] → 0]
{{Dt[y, x] → -(x+y)/(x-y)}}
Simplify[Solve[Dt[eq2, x], Dt[y, x]]/.Dt[c, x] → 0]
{{Dt[y, x] → (x-y)/(x+y)}}
```

Because the derivatives are negative reciprocals, we conclude that the curves are orthogonal. We confirm this graphically by graphing several members of each family with ContourPlot and showing the results together. (See Figure 3.17.)

```
cp1 = ContourPlot[x^2 + 2xy - y^2, {x, -5, 5}, {y, -5, 5},
    ContourShading → False];
cp2 = ContourPlot [y^2 + 2xy - x^2, {x, -5, 5}, {y, -5, 5},
    ContourShading → False,
    ContourStyle  → Dashing[{0.01}]] ;
Show[cp1, cp2, Frame → False, Axes → Automatic,
    AxesOrigin → {0, 0}]
```

**Theorem 1. (The Mean-Value Theorem for Derivatives)** *If $y = f(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is at least one value of c between a and b for which*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or, equivalently,} \quad f(b) - f(a) = f'(c)(b - a). \tag{3.3}$$

**Example 3.2.10** Find all number(s) $c$ that satisfy the conclusion of the mean-value theorem for $f(x) = x^2 - 3x$ on the interval $[0, 7/2]$.

***Solution*** By the power rule, $f'(x) = 2x - 3$. The slope of the secant containing $(0, f(0))$ and $(7/2, f(7/2))$ is

$$\frac{f(7/2) - f(0)}{7/2 - 0} = \frac{1}{2}.$$

Solving $2x - 3 = 1/2$ for $x$ gives us $x = 7/4$.

```
f[x_] = x^2 - 3x
-3x + x^2
Solve[f'[x]==0, x]
{{x → 3/2}}
Solve[f'[x]==(f[7/2] - f[0])/(7/2 - 0)]
{{x → 7/4}}
```

$x = 7/4$ satisfies the conclusion of the mean-value theorem for $f(x) = x^2 - 3x$ on the interval $[0, 7/2]$, as shown in Figure 3.18.

```
p1 = Plot[f[x], {x, -1, 4}];
p2 = Plot[f[x], {x, 0, 7/2}, PlotStyle → Thickness[.02]];
p3 = ListPlot[{{0, f[0]}, {7/4, f[7/4]}, {7/2, f[7/2]}},
    PlotStyle → PointSize[.05]];
p4 = Plot[{f'[7/4](x - 7/4) + f[7/4], (f[7/2] - f[0])/(7/2 - 0)x + f[0]},
    {x, -2, 4}, PlotStyle → {Dashing[{.01}], Dashing[{.02}]}];
Show[p1, p2, p3, p4, DisplayFunction → $DisplayFunction,
    AspectRatio → Automatic, PlotRange → {-3, 3}]
```

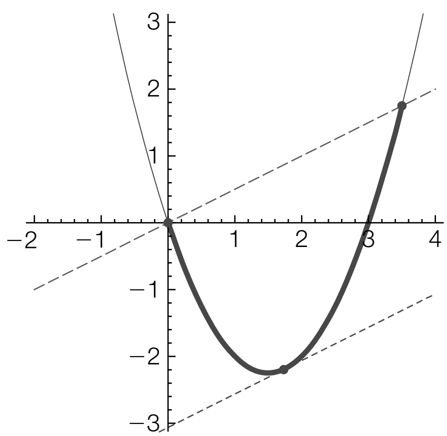



**FIGURE 3.18**

Graphs of $f(x) = x^2 - 3x$, the secant containing $(0, f(0))$ and $(7/2, f(7/2))$, and the tangent at $(7/4, f(7/4))$

### 3.2.5 The First Derivative Test and Second Derivative Test

Examples 3.2.11 and 3.2.12 illustrate the following properties of the first and second derivative.

**Theorem 2.** *Let $y = f(x)$ be continuous on $[a, b]$ and differentiable on $(a, b)$.*

1. *If $f'(x) = 0$ for all $x$ in $(a, b)$, then $f(x)$ is constant on $[a, b]$.*
2. *If $f'(x) > 0$ for all $x$ in $(a, b)$, then $f(x)$ is increasing on $[a, b]$.*
3. *If $f'(x) < 0$ for all $x$ in $(a, b)$, then $f(x)$ is decreasing on $[a, b]$.*

For the second derivative, we have the following theorem.

**Theorem 3.** *Let $y = f(x)$ have a second derivative on $(a, b)$.*

1. *If $f''(x) > 0$ for all $x$ in $(a, b)$, then the graph of $f(x)$ is concave up on $(a, b)$.*
2. *If $f''(x) < 0$ for all $x$ in $(a, b)$, then the graph of $f(x)$ is concave down on $(a, b)$.*

The **critical points** correspond to those points on the graph of $y = f(x)$ where the tangent line is horizontal or vertical; the number $x = a$ is a **critical number** if $f'(a) = 0$ or $f'(x)$ does not exist if $x = a$. The **inflection points** correspond to those points on the graph of $y = f(x)$ where the graph of $y = f(x)$ is neither concave up nor concave down. Theorems 2 and 3 help establish the first derivative test and second derivative test.

**Theorem 4. (First Derivative Test)** *Let $x = a$ be a critical number of a function $y = f(x)$ continuous on an open interval $I$ containing $x = a$. If $f(x)$ is differentiable on $I$, except possibly at $x = a$, $f(a)$ can be classified as follows.*

1. *If $f'(x)$ changes from positive to negative at $x = a$, then $f(a)$ is a **relative maximum**.*
2. *If $f'(x)$ changes from negative to positive at $x = a$, then $f(a)$ is a **relative minimum**.*

**Theorem 5. (Second Derivative Test)** *Let $x = a$ be a critical number of a function $y = f(x)$ and suppose that $f''(x)$ exists on an open interval containing $x = a$.*

1. *If $f''(a) < 0$, then $f(a)$ is a relative maximum.*
2. *If $f''(a) > 0$, then $f(a)$ is a relative minimum.*

**Example 3.2.11** Graph $f(x) = 3x^5 - 5x^3$.

***Solution*** We begin by defining $f(x)$ and then computing and factoring $f'(x)$ and $f''(x)$.

```
f[x_] = 3x^5 - 5x^3;
d1 = Factor[f'[x]]
d2 = Factor[f''[x]]
```

$15(-1+x)x^2(1+x)$

$30x\left(-1+2x^2\right)$

By inspection, we see that the critical numbers are $x = 0,\ 1,$ and $-1$ while $f''(x) = 0$ if $x = 0,\ 1/\sqrt{2},$ or $-1/\sqrt{2}$. Of course, these values can also be found with **Solve** as done next in **cns** and **ins**, respectively.

```
cns = Solve[d1==0]
ins = Solve[d2==0]
```

$\{\{x \to -1\}, \{x \to 0\}, \{x \to 0\}, \{x \to 1\}\}$

$\left\{\{x \to 0\}, \left\{x \to -\frac{1}{\sqrt{2}}\right\}, \left\{x \to \frac{1}{\sqrt{2}}\right\}\right\}$

We find the critical and inflection points by using /. (**Replace All**) to compute $f(x)$ for each value of $x$ in **cns** and **ins**, respectively. The result means that the critical points are $(0, 0)$, $(1, -2)$ and $(-1, 2)$; the inflection points are $(0, 0)$, $(1/\sqrt{2}, -7\sqrt{2}/8)$, and $(-1/\sqrt{2}, 7\sqrt{2}/8)$. We also see that $f''(0) = 0$, so Theorem 5 cannot be used to classify $f(0)$. On the other hand, $f''(1) = 30 > 0$, and $f''(-1) = -30 < 0$, so by Theorem 5, $f(1) = -2$ is a relative minimum and $f(-1) = 2$ is a relative maximum.

```
cps = {x, f[x]}/.cns
```

$\{\{-1, 2\}, \{0, 0\}, \{0, 0\}, \{1, -2\}\}$

```
f''[x]/.cns
```

$\{-30, 0, 0, 30\}$

```
ips={x,f[x]}/.ins
```

$\left\{\{0, 0\}, \left\{-\frac{1}{\sqrt{2}}, \frac{7}{4\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{7}{4\sqrt{2}}\right\}\right\}$

We can graphically determine the intervals of increase and decrease by noting that if $f'(x) > 0$ $(f'(x) < 0)$, $a|f'(x)|/f'(x) = a$ $(a|f'(x)|/f'(x) = -a)$. Similarly, the intervals for which the graph is concave up and concave down can be determined by noting that if $f''(x) > 0$ $(f''(x) < 0)$, $a|f''(x)|/f''(x) = a$ $(a|f''(x)|/f''(x) = -a)$. We use **Plot** to graph $|f'(x)|/f'(x)$ and $2|f''(x)|/f''(x)$ (different values are used so we can differentiate between the two plots) in Figure 3.19.

```
Plot[{Abs[d1]/d1, 2Abs[d2]/d2}, {x, -2, 2}, PlotRange -> {-3, 3}]
```

From the graph, we see that $f'(x) > 0$ for $x$ in $(-\infty, -1) \cup (1, \infty)$, $f'(x) < 0$ for $x$ in $(-1, 1)$, $f''(x) > 0$ for $x$ in $(-1/\sqrt{2}, 0) \cup (1/\sqrt{2}, \infty)$, and $f''(x) < 0$ for $x$ in $(-\infty, -1/\sqrt{2}) \cup (0, 1/\sqrt{2})$. Thus, the graph of $f(x)$ is

- increasing and concave down for $x$ in $(-\infty, -1)$,
- decreasing and concave down for $x$ in $(-1, -1/\sqrt{2})$,

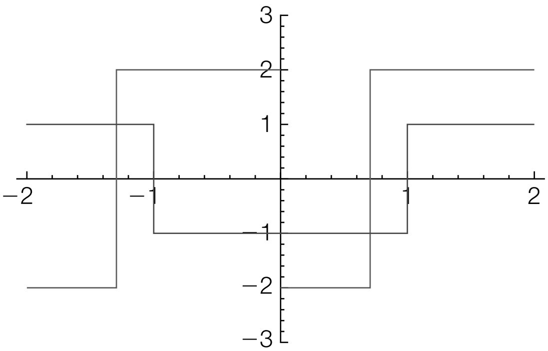



**FIGURE 3.19**

Graphs of $|f'(x)|/f'(x)$ and $2|f''(x)|/f''(x)$

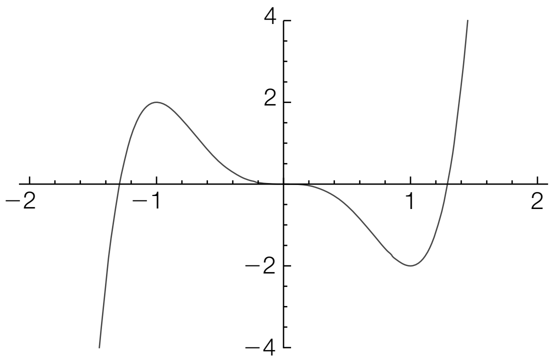



**FIGURE 3.20**

$f(x)$ for $-2 \le x \le 2$ and $-4 \le y \le 4$

- decreasing and concave up for $x$ in $(-1/\sqrt{2}, 0)$,
- decreasing and concave down for $x$ in $(0, 1\sqrt{2})$,
- decreasing and concave up for $x$ in $(1/\sqrt{2}, 1)$, and
- increasing and concave up for $x$ in $(1, \infty)$.

We also see that $f(0) = 0$ is neither a relative minimum nor maximum. To see all points of interest, our domain must contain $-1$ and $1$ while our range must contain $-2$ and $2$. We choose to graph $f(x)$ for $-2 \le x \le 2$; we choose the range displayed to be $-4 \le y \le 4$. (See Figure 3.20.)

**Plot[f[x], {x, −2, 2}, PlotRange → {−4, 4}]**

---

Remember to be especially careful when working with functions that involve odd roots.

**Example 3.2.12** Graph $f(x) = (x-2)^{2/3}(x+1)^{1/3}$.

***Solution*** We begin by defining $f(x)$ and then computing and simplifying $f'(x)$ and $f''(x)$ with $'$ and Simplify.

```
Clear[f]
f[x_] = (x-2)^(2/3)(x + 1)^(1/3);
d1 = Simplify[f'[x]]
d2 = Simplify[f''[x]]
```

$$\frac{x}{(-2+x)^{1/3}(1+x)^{2/3}}$$

$$-\frac{2}{(-2+x)^{4/3}(1+x)^{5/3}}$$

By inspection, we see that the critical numbers are $x = 0$, $2$, and $-1$. We cannot use Theorem 5 to classify $f(2)$ and $f(-1)$ because $f''(x)$ is undefined if $x = 2$ or $-1$. On the other hand, $f''(0) < 0$, so $f(0) = 2^{2/3}$ is a relative maximum. By hand, we make a sign chart to see that the graph of $f(x)$ is

- increasing and concave up on $(-\infty, -1)$,
- increasing and concave down on $(-1, 0)$,
- decreasing and concave down on $(0, 2)$, and
- increasing and concave down on $(2, \infty)$.

Hence, $f(-1) = 0$ is neither a relative minimum nor maximum, whereas $f(2) = 0$ is a relative minimum by Theorem 4. To graph $f(x)$, we load the **RealOnly** package and then use Plot to graph $f(x)$ for $-2 \le x \le 3$ in Figure 3.21.

```
<< Miscellaneous`RealOnly`
f[0]
Plot[f[x], {x, -2, 3}]
```

$2^{2/3}$

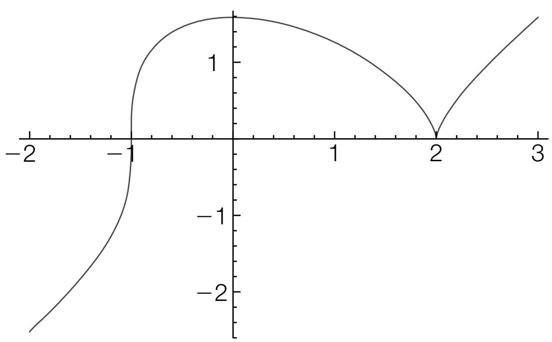



**FIGURE 3.21**

$f(x)$ for $-2 \le x \le 3$

The previous examples illustrate that if $x = a$ is a critical number of $f(x)$ and $f'(x)$ makes a ***simple change in sign*** from positive to negative at $x = a$, then $(a, f(a))$ is a relative maximum. If $f'(x)$ makes a simple change in sign from negative to positive at $x = a$, then $(a, f(a))$ is a relative minimum. Mathematica is especially useful in investigating interesting functions for which this may not be the case.

**Example 3.2.13** Consider

$$f(x) = \begin{cases} x^2 \sin^2\left(\dfrac{1}{x}\right), x \neq 0 \\ 0, x = 0. \end{cases}$$

$x = 0$ is a critical number because $f'(x)$ does not exist if $x = 0$. The point $(0, 0)$ is both a relative and absolute minimum, even though $f'(x)$ does not make a simple change in sign at $x = 0$, as illustrated in Figure 3.22.

```
f[x_] = (xSin[1/x])^2;
f'[x]//Factor
-2Sin[1/x] (Cos[1/x] - xSin[1/x])
p1 = Plot[f[x], {x, -0.1, 0.1}];
p2 = Plot[f'[x], {x, -0.1, 0.1}];
Show[GraphicsRow[{p1, p2}]]
```

Notice that the derivative "oscillates" infinitely many times near $x = 0$, so the first derivative test cannot be used to classify $(0, 0)$.

The functions Maximize and Minimize can be used to assist with finding extreme values. For a function of a single variable Maximize[f[x], x] (Minimize[f[x], x]) attempts to find the maximum (minimum) values of $f(x)$;

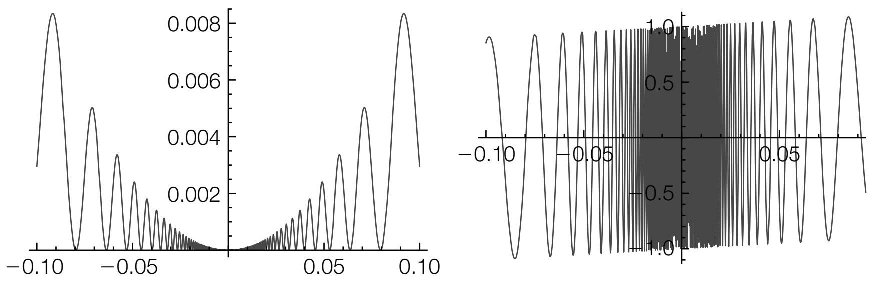



**FIGURE 3.22**

$f(x) = \left[x \sin\left(\frac{1}{x}\right)\right]^2$ and $f'(x)$ for $-0.1 \leq x \leq 0.1$

Maximize[f[x],a<=x<=b,x] (Minimize[f[x],a<=x<=b,x]) attempts to find the maximum (minimum) values of $f(x)$ on $[a, b]$.

---

**Example 3.2.14** Consider $f(x) = \frac{1}{10}\left(-12x + 3x^2 + 2x^3\right)$. After defining $f(x)$, we plot $f(x)$ and $f'(x)$ together in Figure 3.23.

```
f[x_] = 1/10 (-12x + 3x^2 + 2x^3) ;
  Plot[Tooltip[{f[x], f'[x]}], {x, -4, 4}, PlotRange → {-4, 4},
  AspectRatio → Automatic]
```

With Maximize, we see that $f(x)$ does not have a maximum on its domain. However, when we restrict the interval to $-3 \le x \le 2$, Maximize finds the relative maximum at $x = -2$.

```
Maximize[f[x], x]
Maximize::natt : The maximum is not attained at any point satisfying the given constraints.
{∞, {x → ∞}}

Maximize[{f[x], -3 ≤ x ≤ 2}, x]
{20, {x → -2}}
```

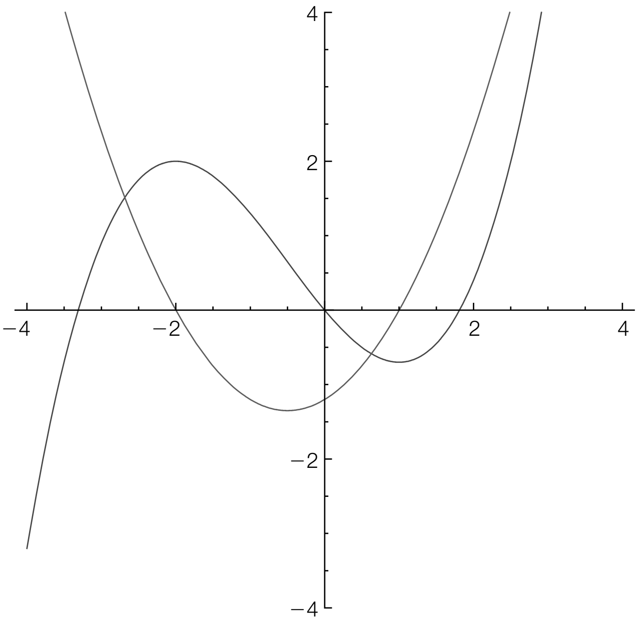



**FIGURE 3.23**

$f(x)$ has one relative maximum and one relative minimum but no absolute extreme values

Similarly, with **Minimize** we see that the $f(x)$ does not have a minimum value on its domain but find the relative minimum when we restrict the interval to $-3 \le x \le 2$.

```
Minimize[f[x], x]
Minimize::natt : The minimum is not attained at any point satisfying the given constraints.
{-∞, {x → -∞}}

Minimize[{f[x], -3 ≤ x ≤ 2}, x]
{-7, {x → 1}}
```

However, with **Solve**, we easily find the two zeros of $f'(x)$ that we see in Figure 3.23.

**Solve[f'[x]==0, x]**
{{x → −2}, {x → 1}}

When using Maximize or Minimize you should verify your results using another method.

---

**Example 3.2.15** The function $f(x) = x/(x^2 + 1)$ is continuous on $(-\infty, \infty)$ and $\lim_{x\to\pm\infty} f(x) = 0$. Thus, $f(x)$ has an absolute minimum and maximum value on its domain. In this case,

```
Maximize[x / (x^2 + 1), x]
{1/2, {x → 1}}

Minimize[x / (x^2 + 1), x]
{-1/2, {x → -1}}
```

gives us the absolute maximum and minimum values of $f(x)$ and the $x$-values where they occur. On the other hand, $f(x) = x^4 - x^2$ is continuous on $(-\infty, \infty)$ and $\lim_{x\to\pm\infty} f(x) = \infty$. Thus, $f(x)$ has an absolute minimum on its domain. Because the derivative of a fourth-degree polynomial is a third-degree polynomial, we know that $f'(x)$ has three zeros, two of which probably correspond to relative minimums. Because the graph of $f(x)$ is symmetric with respect to the $y$-axis, we further suspect that the absolute minimum is obtained twice—at each relative minimum. **Maximize** and **Minimize** give us the following results.

A polynomial of degree $n$ has $n$ zeros (counting multiplicity).

```
Maximize[x^4 - x^2, x]
Maximize::natt : The maximum is not attained at any point satisfying the given constraints.
{∞, {x → -∞}}

Minimize[x^4 - x^2, x]
{-1/4, {x → -1/√2}}
```

Note that the result returned by **Maximize** is correct. Similarly, the result returned by **Minimize** is correct, but a complete answer would indicate that the absolute minimum value occurs at both $x = -1/\sqrt{2}$ and $x = 1/\sqrt{2}$.

---

**Example 3.2.16** The function $f(x) = (x+1)^2/(x-2)$ has a vertical asymptote at $x = 2$. From the derivative,

```
f[x_] = (x + 1)^2/(x-2);
d1 = Simplify[f'[x]]
cns = Solve[f'[x]==0]
```

$\frac{-5-4x+x^2}{(-2+x)^2}$

$\{\{x \to -1\}, \{x \to 5\}\}$

```
f[x]/.cns
```

$\{0,12\}$

we find two critical numbers, one of which is a relative maximum and one is a relative minimum. See Figure 3.24.

```
Plot[Tooltip[{f[x], f'[x]}], {x, -6, 10}]
```

On the other hand, **Maximize** and **Minimize** return confusing results because the function is undefined if $x = 2$. The function has relative extreme values but not absolute extreme values.

```
Maximize[f[x], x]
Maximize::natt : The maximum is not attained at any point satisfying the given constraints.
{∞, {x → 2}}

Minimize[f[x], x]
Minimize::natt : The minimum is not attained at any point satisfying the given constraints.
{-∞, {x → 2}}
```

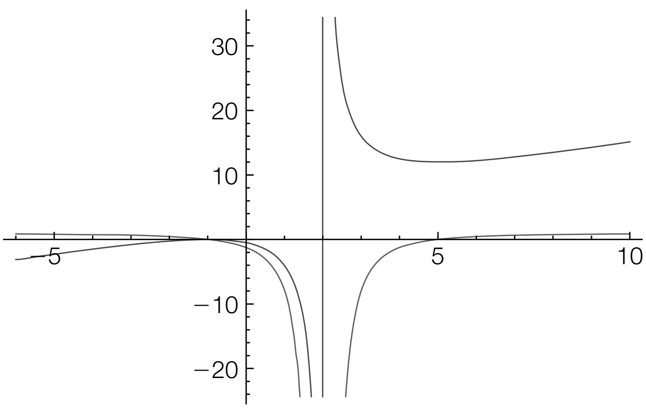



**FIGURE 3.24**

A function for which a relative minimum has a function value greater than the function value of a relative maximum

For periodic functions, such as sine and cosine, Maximize and Minimize generally do not indicate *all* extreme values.

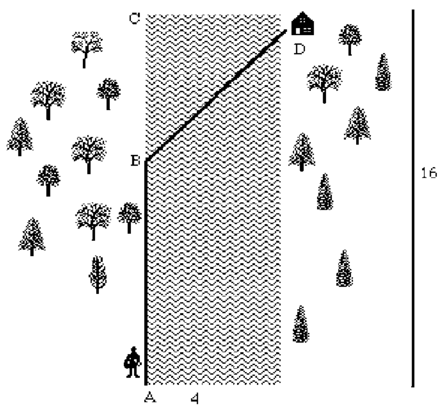
**Maximize[Sin[x], x]**
$\{1, \{x \to \frac{\pi}{2}\}\}$

**Maximize[Cos[x], x]**
$\{1, \{x \to 0\}\}$

### 3.2.6 Applied Max/Min Problems

Mathematica can be used to assist in solving maximization/minimization problems encountered in a differential calculus course.

**Example 3.2.17** A woman is located on one side of a body of water 4 miles wide. Her position is directly across from a point on the other side of the body of water 16 miles from her house, as shown in the following figure.




If she can move across land at a rate of 10 miles per hour and move over water at a rate of 6 miles per hour, find the least amount of time for her to reach her house.

***Solution*** From the figure, we see that the woman will travel from $A$ to $B$ by land and then from $B$ to $D$ by water. We wish to find the least time for her to complete the trip.

Let $x$ denote the distance $BC$, where $0 \le x \le 16$. Then, the distance $AB$ is given by $16 - x$ and, by the Pythagorean theorem, the distance $BD$ is given by $\sqrt{x^2 + 4^2}$. Because rate × time = distance, time = distance / rate. Thus, the time to travel from $A$ to $B$ is $\frac{1}{10}(16 - x)$, the time to travel from $B$ to $D$ is $\frac{1}{6}\sqrt{x^2 + 16}$, and the total time to complete the trip, as a function of $x$, is

$$time(x) = \frac{1}{10}(16 - x) + \frac{1}{6}\sqrt{x^2 + 16}, \quad 0 \le x \le 16.$$

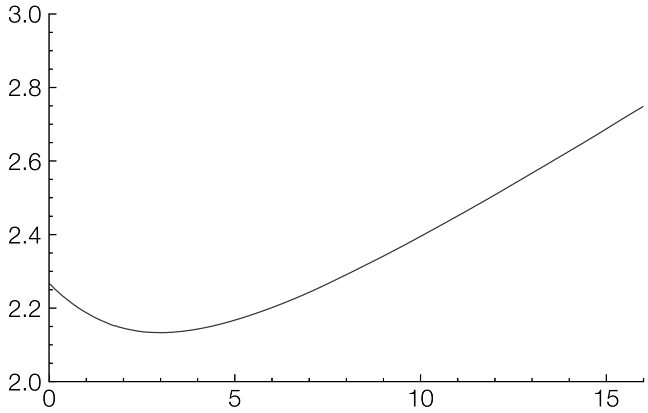



**FIGURE 3.25**

Plot of $time(x) = \frac{1}{10}(16 - x) + \frac{1}{6}\sqrt{x^2 + 16}, \quad 0 \leq x \leq 16$

We must minimize the function *time*. First, we define time and then verify that time has a minimum by graphing time on the interval $[0, 16]$ in Figure 3.25.

**Clear[time]**

**time[x_] = $\frac{16-x}{10} + \frac{1}{6}\sqrt{x^2 + 16}$;**

**Plot[time[x], {x, 0, 16}, PlotRange → {{0, 16}, {2, 3}}]**

Next, we compute the derivative of time and find the values of $x$ for which the derivative is $0$ with Solve. The resulting output is named critnums using ReplaceAll (\.).

**Together [time′[x]]**

$\frac{5x - 3\sqrt{16+x^2}}{30\sqrt{16+x^2}}$

**critnums = Solve [time′[x]==0]**

{{x → 3}}

At this point, we can calculate the minimum time by calculating time[3].

**time[3]**

$\frac{32}{15}$

Alternatively, we demonstrate how to find the value of time[x] for the value(s) listed in critnums.

**time[x]/.x → 3**

$\frac{32}{15}$

Regardless, we see that the minimum time to complete the trip is $32/15$ hours.

One of the more interesting applied max/min problems is the *beam problem*. We present two solutions.

**Example 3.2.18** **(The Beam Problem).** Find the exact length of the longest beam that can be carried around a corner from a hallway 2 feet wide to a hallway that is 3 feet wide. (See Figure 3.26.)

***Solution*** We assume that the beam has negligible thickness. Our first approach is algebraic. Using Figure 3.26, which is generated with

Graphics primitives such as Point, Line, and Text are discussed in more detail in Chapter 7.

```
f[x_] = x + 2;
p1 = Plot[f[x], {x, 0, 4}, PlotStyle->Thickness[.01], PlotRange->{0, 6}];

p2 =
Graphics[Line[{{1, 0}, {1, f[1]}, {4, f[1]}, {4, f[4]}, {4, f[4]},
   {0, f[4]},
      {0, 0}, {1, 0}}]];
p3 = Graphics[{Text["2", {.5, .2}], {Text["3", {3.8, 4.5}]}}];
p4 = Graphics[{Dashing[{0.01, 0.01}], Line[{{0, f[0]}, {1, f[0]}}]}];
p5 = Graphics[{Text["θ", {.5, 2.25}], Text["θ", {1.5, 3.25}]}];
p6 = Graphics[{Text["x", {.9, 2.35}], Text["y", {2.5, 3.25}]}];
Show[p1, p2, p3, p4, p5, p6, Axes->None]
```

and the Pythagorean theorem, the total length of the beam is

$$L = \sqrt{2^2 + x^2} + \sqrt{y^2 + 3^2}.$$

By similar triangles,

$$\frac{y}{3} = \frac{2}{x} \quad \text{so} \quad y = \frac{6}{x}$$

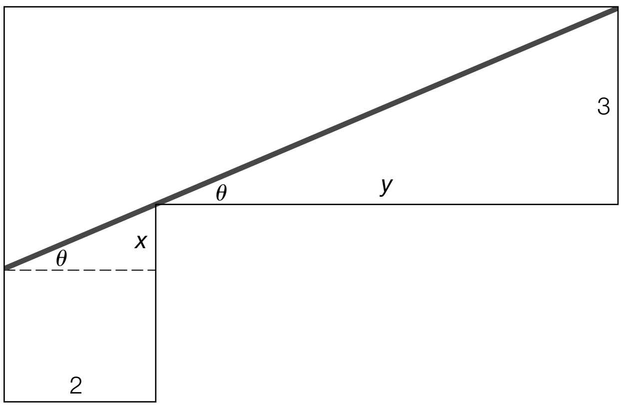



**FIGURE 3.26**

The length of the beam is found using similar triangles and the length of the beam, $L$, becomes

$$L(x) = \sqrt{4 + x^2} + \sqrt{9 + \frac{36}{x^2}}, \quad 0 < x < \infty.$$

Observe that the length of the longest beam is obtained by *minimizing* $L$. (Why?)

We ignore negative and imaginary values because length must be a nonnegative real number.

**Clear[l];**
**l[x_] = Sqrt[2^2 + x^2] + Sqrt[y^2 + 3^2]/.y->6/x**

$\sqrt{9 + \frac{36}{x^2}} + \sqrt{4 + x^2}$

We use two different methods to solve $L'(x) = 0$. Differentiating

**l'[x]**

$-\frac{36}{\sqrt{9+\frac{36}{x^2}}x^3} + \frac{x}{\sqrt{4+x^2}}$

**Solve $\left[-12\sqrt{4+x^2} + x^4\sqrt{\frac{4+x^2}{x^2}}\text{==0, x}\right]$**

$\{\{x \to -2i\}, \{x \to 2i\}, \{x \to -2^{2/3}3^{1/3}\}, \{x \to 2^{2/3}3^{1/3}\}\}$

**p1 = x^8(9 + 36/x^2) − 1296(4 + x^2)//Expand//Factor**

$9\left(4 + x^2\right)\left(-12 + x^3\right)\left(12 + x^3\right)$

and solving $L'(x) = 0$ gives us

**Solve[p1==0, x]**

$\{\{x \to -2i\}, \{x \to 2i\}, \{x \to -(-3)^{1/3}2^{2/3}\}, \{x \to (-3)^{1/3}2^{2/3}\},$
$\{x \to -(-2)^{2/3}3^{1/3}\},$

**N $[2^{2/3}3^{1/3}]$**

2.28943

**l $[2^{2/3}3^{1/3}]$**

$\sqrt{9 + 32^{2/3}3^{1/3}} + \sqrt{4 + 22^{1/3}3^{2/3}}$

**N[%]**

7.02348

It follows that the length of the beam is $L\left(2^{2/3}3^{1/3}\right) = \sqrt{9 + 3 \cdot 2^{2/3} \cdot 3^{1/3}} + \sqrt{4 + 2 \cdot 2^{1/3} \cdot 3^{2/3}} = \sqrt{13 + 9 \cdot 2^{2/3} \cdot 3^{1/3} + 6 \cdot 2^{1/3} \cdot 3^{2/3}} \approx 7.02$. (See Figure 3.27).

**Plot[l[x], {x, 0, 20}, PlotRange->{0, 20}, AspectRatio->Automatic,**
**AxesLabel->{"x", "y"}]**

Our second approach uses right triangle trigonometry. In terms of $\theta$, the length of the beam is given by

$$L(\theta) = 2\csc\theta + 3\sec\theta, \quad 0 < \theta < \pi/2.$$

Differentiating gives us

$$L'(\theta) = -2\csc\theta\cot\theta + 3\sec\theta\tan\theta.$$

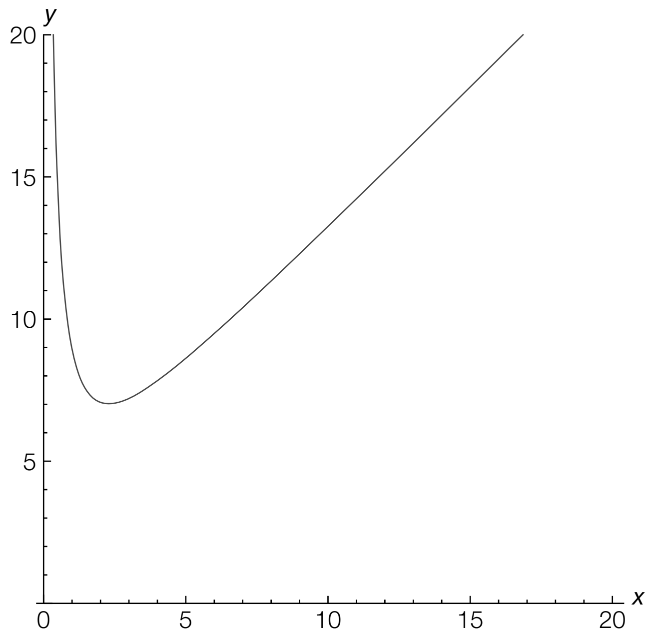



**FIGURE 3.27**

Graph of $L(x)$

To avoid typing the $\theta$ symbol, we define $L$ as a function of $t$.

```
Clear[l]
l[t_] = 2Csc[t] + 3Sec[t]
2Csc[t] + 3Sec[t]
```

We now solve $L'(\theta) = 0$. First, multiply through by $\sin\theta$ and then by $\tan\theta$.

$$\begin{aligned} 3\sec\theta\tan\theta &= 2\csc\theta\cot\theta \\ \tan^2\theta &= \frac{2}{3}\cot\theta \\ \tan^3\theta &= \frac{2}{3} \\ \tan\theta &= \sqrt[3]{\frac{2}{3}}. \end{aligned}$$

In this case, observe that we cannot compute $\theta$ exactly. However, we do not need to do so. Let $0 < \theta < \pi/2$ be the unique solution of $\tan\theta = \sqrt[3]{2/3}$. See Figure 3.28. Using the identity $\tan^2\theta + 1 = \sec^2\theta$, we find that $\sec\theta = \sqrt{1 + \sqrt[3]{4/9}}$. Similarly, because $\cot\theta = \sqrt[3]{3/2}$ and $\cot^2\theta + 1 = \csc^2\theta$, $\csc\theta = \sqrt[3]{3/2}\sqrt{1 + \sqrt[3]{4/9}}$. Hence, the length of the beam is

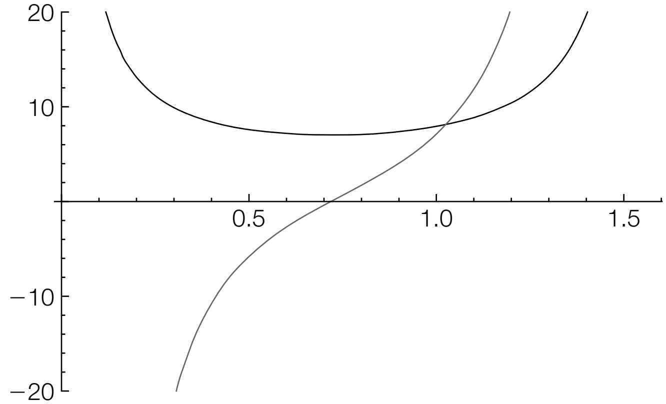



**FIGURE 3.28**

Graph of $L(\theta)$ and $L'(\theta)$

$$L(\theta) = 2\sqrt[3]{\frac{3}{2}}\sqrt{1+\sqrt[3]{\frac{4}{9}}} + 3\sqrt{1+\sqrt[3]{\frac{4}{9}}} \approx 7.02.$$

When you use Tooltip, scrolling the cursor over the plot will identify the plot for you.

```
Plot[Tooltip[{l[t], l'[t]}], {t, 0, Pi/2}, PlotRange ->{-20, 20},
    PlotStyle ->{Black, Gray}]
```

In the next two examples, the constants do not have specific numerical values.

**Example 3.2.19** Find the volume of the right circular cone of maximum volume that can be inscribed in a sphere of radius $R$.

***Solution*** Try to avoid three-dimensional figures unless they are absolutely necessary. For this problem, a cross section of the situation is sufficient. See Figure 3.29, which is created with

```
p1 = ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2Pi}];
p2 =
Graphics[
  {Line[{{0, 1}, {Cos[4Pi/3], Sin[4Pi/3]}, {Cos[5Pi/3], Sin[5Pi/3]},
    {0, 1}}], PointSize[.02], Point[{0, 0}],
      Line[{{Cos[4Pi/3], Sin[4Pi/3]}, {0, 0}, {0, 1}}],
    Line[{{0, 0}, {0, Sin[4Pi/3]}}]}];
p3 = Graphics[{Text["R", {-.256, -.28}], Text["R", {-.04, .5}],
    Text["y", {-.04, -.5}], Text["x", {-.2, -.8}]}];
Show[p1, p2, p3, AspectRatio ->Automatic, Ticks ->None, Axes ->None]
```

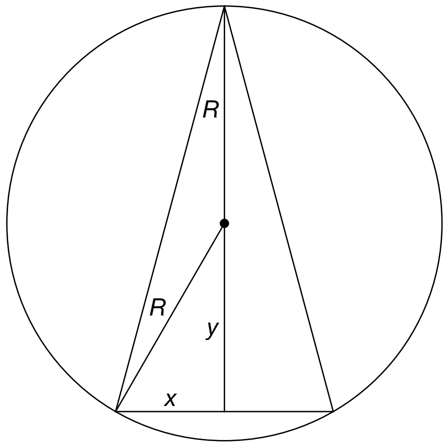



**FIGURE 3.29**

Cross section of a right circular cone inscribed in a sphere

The volume, $V$, of a right circular cone with radius $r$ and height $h$ is $V = \frac{1}{3}\pi r^2 h$. Using the notation in Figure 3.29, the volume is given by

$$V = \frac{1}{3}\pi x^2 (R + y). \tag{3.4}$$

However, by the Pythagorean theorem, $x^2 + y^2 = R^2$ so $x^2 = R^2 - y^2$ and equation 3.4 becomes

$$V = \frac{1}{3}\pi \left(R^2 - y^2\right)(R + y) = \frac{1}{3}\pi \left(R^3 + R^2 y - R y^2 - y^3\right), \tag{3.5}$$

```
s1 = Expand[(r^2-y^2)(r + y)]
r^3 + r^2y - ry^2 - y^3
```

Remember that $R$ is a constant.

where $0 \le y \le R$. $V(y)$ is continuous on $[0, R]$, so it will have minimum and maximum values on this interval. Moreover, the minimum and maximum values occur either at the endpoints of the interval or at the critical numbers on the interior of the interval. Differentiating equation (3.5) with respect to $y$ gives us

$$\frac{dV}{dy} = \frac{1}{3}\pi \left(R^2 - 2Ry - 3y^2\right) = \frac{1}{3}\pi (R - 3y)(R + y)$$

```
s2 = D[s1, y]
r^2 - 2ry - 3y^2
```

and we see that $dV/dy = 0$ if $y = \frac{1}{3}R$ or $y = -R$.

```
Factor[s2]
(r - 3y)(r + y)
```

```
Solve[s2==0, y]
{{y → -r}, {y → r/3}}
```

We ignore $y = -R$ because $-R$ is not in the interval $[0, R]$. Note that $V(0) = V(R) = 0$. The maximum volume of the cone is

$$V\left(\frac{1}{3}R\right) = \frac{1}{3}\pi \cdot \frac{32}{27}R^3 = \frac{32}{81}\pi R^2 \approx 1.24R^3.$$

**s3=s1/.y−>r/3//Together**

$\frac{32r^3}{27}$

**s3*1/3Pi**

$\frac{32\pi r^3}{81}$

**N[%]**

$1.24112r^3$

---

**Example 3.2.20** **(The Stayed-Wire Problem).** Two poles $D$ feet apart with heights $L_1$ feet and $L_2$ feet are to be stayed by a wire as shown in Figure 3.30. Find the minimum amount of wire required to stay the poles, as illustrated in Figure 3.30, which is generated with

```
p1 = Graphics[Line[{{0, 0}, {0, 4}, {3.5, 0}, {9, 5.5}, {9, 0}, {0, 0}}]];
p2 = Graphics[{Text["L1", { . 2, 2}], Text["L2", {8.8, 2.75}],
   Text["x", {1.75, .2}], Text["x", {1.75, .2}],
```

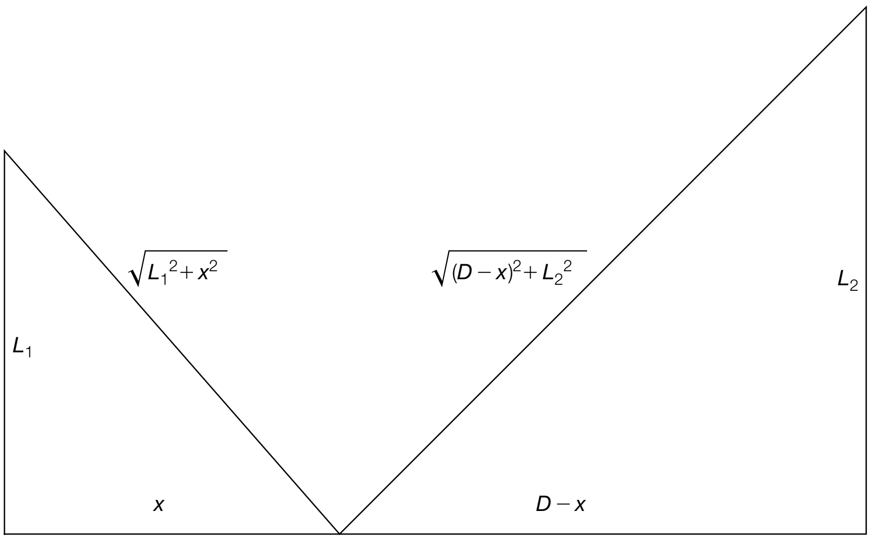



**FIGURE 3.30**

When the wire is stayed to minimize the length, the result is two similar triangles

```
Text["√(L1^2 + x^2)", {1.75, 2.75}], Text["√((D - x)^2 + L2^2)", {5.5, 2.75}],
    Text["D - x", {6.5, .2}]}];
Show[p1, p2]
```

***Solution*** Using the notation in Figure 3.30, the length of the wire, $L$, is

$$L(x) = \sqrt{{L_1}^2 + x^2} + \sqrt{{L_2}^2 + (D - x)^2}, \qquad 0 \le x \le D. \tag{3.6}$$

In the special case that $L_1 = L_2$, the length of the wire to stay the beams is minimized when the wire is placed halfway between the two beams, at a distance $D/2$ from each beam. Thus, we assume that the lengths of the beams are different; we assume that $L_1 < L_2$, as illustrated in Figure 3.30. We compute $L'(x)$ and then solve $L'(x) = 0$.

PowerExpand[expr] expands out all products and powers assuming the variables are real and positive. That is, with PowerExpand we obtain that $\sqrt{x^2} = x$ rather than $\sqrt{x^2} = |x|$.

```
Clear[l]
l[x_] = Sqrt[x^2 + l1^2] + Sqrt[(d - x)^2 + l2^2]
```

$\sqrt{l2^2 + (d-x)^2} + \sqrt{l1^2 + x^2}$

```
l'[x]//Together
```

$\frac{-d\sqrt{l1^2+x^2} + x\sqrt{l1^2+x^2} + x\sqrt{d^2+l2^2-2dx+x^2}}{\sqrt{l1^2+x^2}\sqrt{d^2+l2^2-2dx+x^2}}$

```
l[0]//PowerExpand
```

$l1 + \sqrt{d^2 + l2^2}$

```
l[d]//PowerExpand
```

$\sqrt{d^2 + l1^2} + l2$

```
Solve[l'[x]==0, x]
```

$\{\{x \to \frac{dl1}{l1 - l2}\}, \{x \to \frac{dl1}{l1 + l2}\}\}$

The result indicates that $x = L_1 D/\left(L_1 + L_2\right)$ minimizes $L(x)$. (Note that we ignore the other value because $L_1 - L_2 < 0$.) Moreover, the triangles formed by minimizing $L$ are similar triangles.

```
l1/(dl1/(l1 + l2)) //Simplify
```

$\frac{l1 + l2}{d}$

```
l2/(d - dl1/(l1 + l2)) //Simplify
```

$\frac{l1 + l2}{d}$

## 3.2.7 Antidifferentiation

### *Antiderivatives*

$f(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$. The symbol

$$\int f(x)\,dx$$

means "find all antiderivatives of $f(x)$." Because all antiderivatives of a given function differ by a constant, we usually find an antiderivative, $f(x)$, of $f(x)$ and then write

$$\int f(x)\,dx = F(x) + C,$$

where $C$ represents an arbitrary constant. The command

**Integrate[f[x],x]**

attempts to find an antiderivative, $F(x)$, of $f(x)$. Instead of using Integrate, you might prefer to use the button on the **BasicMathInput** palette to help you evaluate antiderivatives. Mathematica does not include the "$+C$" that we include when writing $\int f(x)\,dx = f(x) + C$. In the same way as D can differentiate many functions, Integrate can antidifferentiate many functions. However, antidifferentiation is a fundamentally difficult procedure so it is not difficult to find functions $f(x)$ for which the command Integrate[f[x],x] returns unevaluated.

---

**Example 3.2.21** Evaluate each of the following antiderivatives: (a) $\int \frac{1}{x^2} e^{1/x}\,dx$, (b) $\int x^2 \cos x\,dx$, (c) $\int x^2 \sqrt{1+x^2}\,dx$, (d) $\int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1}\,dx$, and (e) $\int \frac{\sin x}{x}\,dx$.

***Solution*** Entering

**Integrate[1/x^2 Exp[1/x], x]**

$-e^{\frac{1}{x}}$

shows us that $\int \frac{1}{x^2} e^{1/x}\,dx = -e^{1/x} + C$. To use the button, first click on the button, fill in the blanks, and press Enter.

```
∫□d□

∫1/x^2 Exp[1/x] d□

∫1/x^2 Exp[1/x] dx

∫1/x^2 Exp[1/x] dx

-e^(1/x)
```

Notice that Mathematica does not automatically include the arbitrary constant, $C$. When computing several antiderivatives, you can use Map to apply Integrate to a list of antiderivatives. However, because Integrate is threadable,

**Map[Integrate[#, x]&, list]**

returns the same result as Integrate[list, x], which we illustrate to compute (b), (c), and (d).

**Integrate[{x^2Cos[x], x^2Sqrt[1 + x^2], (x^2 − x + 2)/(x^3 − x^2 + x − 1)}, x]**

$\left\{2x\text{Cos}[x] + (-2 + x^2)\,\text{Sin}[x], \frac{1}{8}\left(\sqrt{1+x^2}\,(x + 2x^3) - \text{ArcSinh}[x]\right),\right.$
$\left. -\text{ArcTan}[x] + \text{Log}[-1 + x]\right\}$

For (e), we see that there is not a "closed form" antiderivative of $\int \frac{\sin x}{x}dx$ and the result is given in terms of a definite integral, the **sine integral function**:

$$Si(x) = \int_0^x \frac{\sin t}{t}dt.$$

**Integrate[Sin[x]/x,x]**
SinIntegral[x]

---

## u-*Substitutions*

Usually, the first antidifferentiation technique discussed is the method of ***u*-substitution**. Suppose that $f(x)$ is an antiderivative of $f(x)$. Given

$$\int f\left(g(x)\right) g'(x)\,dx,$$

we let $u = g(x)$ so that $du = g'(x)\,dx$. Then,

$$\int f\left(g(x)\right) g'(x)\,dx = \int f(u)\,du = F(u) + C = F\left(g(x)\right) + C,$$

where $F(x)$ is an antiderivative of $f(x)$. After mastering $u$-substitutions, the **integration by parts formula**,

$$\int u\,dv = uv - \int v\,du, \tag{3.7}$$

is introduced.

---

**Example 3.2.22** Evaluate $\int 2^x\sqrt{4^x - 1}\,dx$.

***Solution*** We use **Integrate** to evaluate the antiderivative. Notice that the result is *very* complicated.

**Integrate[2^x Sqrt[4^x − 1], x]**

$(2^x$
$\left(2^{1+2x}\text{Log}[2] - \text{Log}[4] - \sqrt{1-4^x}\text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{\text{Log}[2]}{\text{Log}[4]}, \frac{\text{Log}[8]}{\text{Log}[4]}, 4^x\right]\right.$
$\left.\text{Log}[4]\right)\right) \Big/ \left(\sqrt{-1+4^x}\text{Log}[2]\text{Log}[16]\right)$

Proceeding by hand, we let $u = 2^x$. Then, $du = 2^x \ln 2\,dx$ or, equivalently, $\frac{1}{\ln 2}du = 2^x\,dx$

```
D[2^x, x]
```

$2^x \text{Log}[2]$

so $\int 2^x \sqrt{4^x - 1}\, dx = \frac{1}{\ln 2} \int \sqrt{u^2 - 1}\, du$. We now use Integrate to evaluate $\int \sqrt{u^2 - 1}\, du$

```
s1 = Integrate[Sqrt[u^2 - 1], u]
```

$\frac{1}{2} u \sqrt{-1 + u^2} - \frac{1}{2} \text{Log}\left[u + \sqrt{-1 + u^2}\right]$

and then /. (ReplaceAll)/ to replace $u$ with $2^x$.

```
s1/.u → 2^x
```

$2^{-1+x} \sqrt{-1 + 2^{2x}} - \frac{1}{2} \text{Log}\left[2^x + \sqrt{-1 + 2^{2x}}\right]$

Clearly, proceeding by hand results in a significantly simpler antiderivative than using Integrate directly.

---

As we did with derivatives, with DynamicModule, we create a simple dynamic that lets you compute the derivative and antiderivative of basic functions and plot them on a standard viewing window, $[- -5, 5] \times [-5, 5]$. The layout of Figure 3.31 is primarily determined by Panel, Column, and Grid.

```
Panel[DynamicModule[{f = x^2},
   Column[{InputField[Dynamic[f]], Grid[{{"FirstDerivative",
      Panel[Dynamic[D[f, x]//Simplify]]},
         {"Antiderivative",
      Panel[Dynamic[Integrate[f, x]//Simplify]]}}],
Dynamic[Plot[Evaluate[Tooltip[{f, D[f, x],
      Integrate[f, x]}]], {x, -5, 5}, PlotRange → {-5, 5},
        AspectRatio → Automatic]]}]], ImageSize → {300, 300}]
```

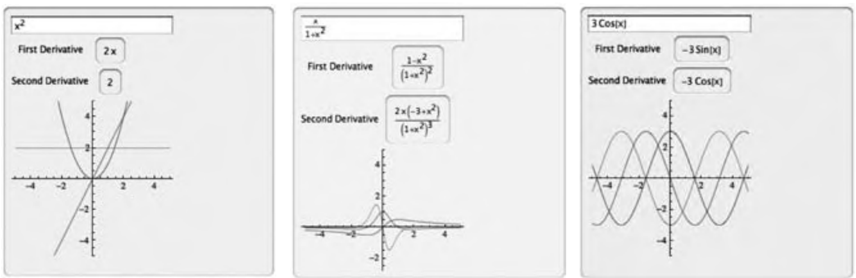



**FIGURE 3.31**

Seeing the relationship between the derivative and antiderivative of a function and the original function

# 3.3 INTEGRAL CALCULUS

## 3.3.1 Area

In integral calculus courses, the definite integral is frequently motivated by investigating the area under the graph of a positive continuous function on a closed interval. Let $y = f(x)$ be a nonnegative continuous function on an interval $[a, b]$ and let $n$ be a positive integer. If we divide $[a, b]$ into $n$ subintervals of equal length and let $\left[x_{k-1}, x_k\right]$ denote the $k$th subinterval, the length of each subinterval is $(b-a)/n$ and $x_k = a + k\frac{b-a}{n}$. The area bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the $y$-axis can be approximated with the sum

$$\sum_{k=1}^{n} f\left(x_k^{*}\right)\frac{b-a}{n}, \tag{3.8}$$

where $x_k^{*} \in [x_{k-1}, x_k]$. Typically, we take $x_k^{*} = x_{k-1} = a + (k-1)\frac{b-a}{n}$ (the left endpoint of the $k$th subinterval), $x_k^{*} = x_{k-1} = a + k\frac{b-a}{n}$ (the right endpoint of the $k$th subinterval), or $x_k^{*} = \frac{1}{2}\left(x_{k-1} + x_k\right) = a + \frac{1}{2}(2k-1)\frac{b-a}{n}$ (the midpoint of the $k$th subinterval). For these choices of $x_k^{*}$, (3.8) becomes

$$\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + (k-1)\frac{b-a}{n}\right) \tag{3.9}$$

$$\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right), \text{and} \tag{3.10}$$

$$\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{1}{2}(2k-1)\frac{b-a}{n}\right), \tag{3.11}$$

respectively. If $y = f(x)$ is increasing on $[a, b]$, (3.9) is an under approximation and (3.10) is an upper approximation: (3.9) corresponds to an approximation of the area using $n$ inscribed rectangles; (3.10) corresponds to an approximation of the area using $n$ circumscribed rectangles. If $y = f(x)$ is decreasing on $[a, b]$, (3.10) is an under approximation and (3.9) is an upper approximation: (3.10) corresponds to an approximation of the area using $n$ inscribed rectangles; (3.9) corresponds to an approximation of the area using $n$ circumscribed rectangles.

In the following example, we define the functions leftsum[f[x],a,b,n], middlesum[f[x],a,b,n], and rightsum[f[x],a,b,n] to compute (3.9), (3.11), and (3.10), respectively, and leftbox[f[x],a,b,n], middlebox[f[x],a,b,n], and rightbox[f[x],a,b,n] to generate the corresponding graphs. After you have defined these functions, you can use them with functions $y = f(x)$ that you define.

**Remark 3.3** To define a function of a single variable, $f(x) = expression\ in\ x$, enter f[x_]=expression in x. To generate a basic plot of $y = f(x)$ for $a \le x \le b$, enter Plot[f[x],{x,a,b}].

**Example 3.3.1** Let $f(x) = 9 - 4x^2$. Approximate the area bounded by the graph of $y = f(x)$, $x = 0$, $x = 3/2$, and the $y$-axis using (a) 100 inscribed and (b) 100 circumscribed rectangles. (c) What is the exact value of the area?

***Solution*** We begin by defining and graphing $y = f(x)$ in Figure 3.32.

```
f[x_] = 9 - 4x^2;
Plot[f[x], {x, 0, 3/2}]
```

The first derivative, $f'(x) = -8x$, is negative on the interval, so $f(x)$ is decreasing on $[0, 3/2]$. Thus, an approximation of the area using 100 inscribed rectangles is given by (3.10) whereas an approximation of the area using 100 circumscribed rectangles is given by (3.9). After definining **leftsum**, **rightsum**, and **middlesum**, these values are computed using **leftsum** and **rightsum**. The use of **middlesum** is illustrated as well. Approximations of the sums are obtained with N.

N[number] returns a numerical approximation of number.

```
leftsum[f_, a_, b_, n_] := Module[{ },
    (b - a)/n Sum[f/.x->a + (k - 1)(b - a)/n, {k, 1, n}]];
rightsum[f_, a_, b_, n_] := Module[{ },
    (b - a)/n Sum[f/.x->a + k(b - a)/n, {k, 1, n}]];
middlesum[f_, a_, b_, n_] := Module[{ },
    (b - a)/n Sum[f/.x->a + 1/2(2k - 1)(b - a)/n, {k, 1, n}]];
```

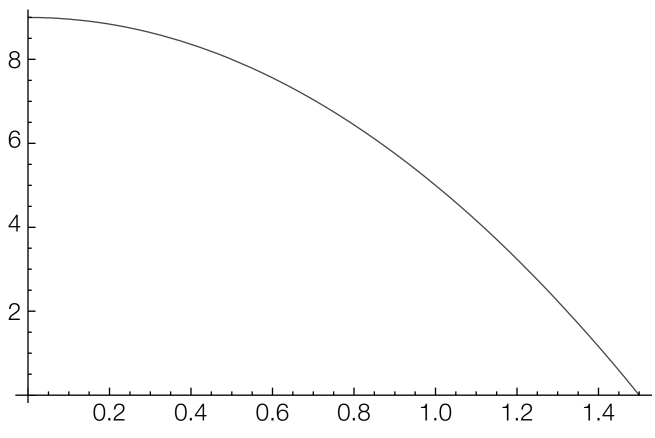



**FIGURE 3.32**

$f(x)$ for $0 \le x \le 3/2$

```
l100 = leftsum[f[x], 0, 3/2, 100]
N[%]
r100 = rightsum[f[x], 0, 3/2, 100]
N[%]
m100 = middlesum[f[x], 0, 3/2, 100]
N[%]
```

$\frac{362691}{40000}$

9.06728

$\frac{357291}{40000}$

8.93228

$\frac{720009}{80000}$

9.00011

Observe that these three values appear to be close to 9. In fact, 9 is the exact value of the area of the region bounded by $y = f(x)$, $x = 0$, $x = 3/2$, and the $y$-axis. To help us see why this is true, we define leftbox, middlebox, and rightbox and then use these functions to visualize the situation using $n = 4$, 16, and 32 rectangles in Figure 3.33.

It is not important that you understand the syntax of these three functions at this time. Once you have entered the code, you can use them to visualize the process for your own functions, $y = f(x)$.

```
leftbox[f_, a_, b_, n_, opts___] := Module[{z, p1, recs, ls},
   z[k_] = a + (b − a)k/n;
   p1 = Plot[f, {x, a, b}, PlotRange → All,
       PlotStyle ->{{Thickness[.01], GrayLevel[.3]}},
       DisplayFunction ->Identity];
   recs = Table[Rectangle[{z[k − 1], 0}, {z[k], f/.x ->z[k − 1]}], {k, 1, n}];
   ls =
      Table[Line[{{z[k − 1], 0}, {z[k − 1], f/.x -> z[k − 1]},
      {z[k], f/.x -> z[k − 1]},
      {z[k], 0}}], {k, 1, n}];
   Show[Graphics[{GrayLevel[.8], recs}], Graphics[ls], p1, opts,
       Axes ->Automatic, AspectRatio → 1,
           DisplayFunction − > $DisplayFunction]]
```

```
rightbox[f_, a_, b_, n_, opts___] := Module[{z, p1, recs, ls},
   z[k_] = a + (b − a)k/n;
   p1 = Plot[f, {x, a, b}, PlotRange → All,
       PlotStyle -> {{Thickness[.01], GrayLevel[.3]}},
       DisplayFunction -> Identity];
   recs = Table[Rectangle[{z[k − 1], 0}, {z[k], f/.x -> z[k]}],
       {k, 1, n}];
   ls =
      Table[Line[{{z[k − 1], 0}, {z[k − 1], f/.x -> z[k]}, {z[k], f/.x -> z[k]},
      {z[k], 0}}], {k, 1, n}];
   Show[Graphics[{GrayLevel[.8], recs}], Graphics[ls], p1, opts,
       Axes -> Automatic, AspectRatio → 1,
           DisplayFunction -> $DisplayFunction]]
```

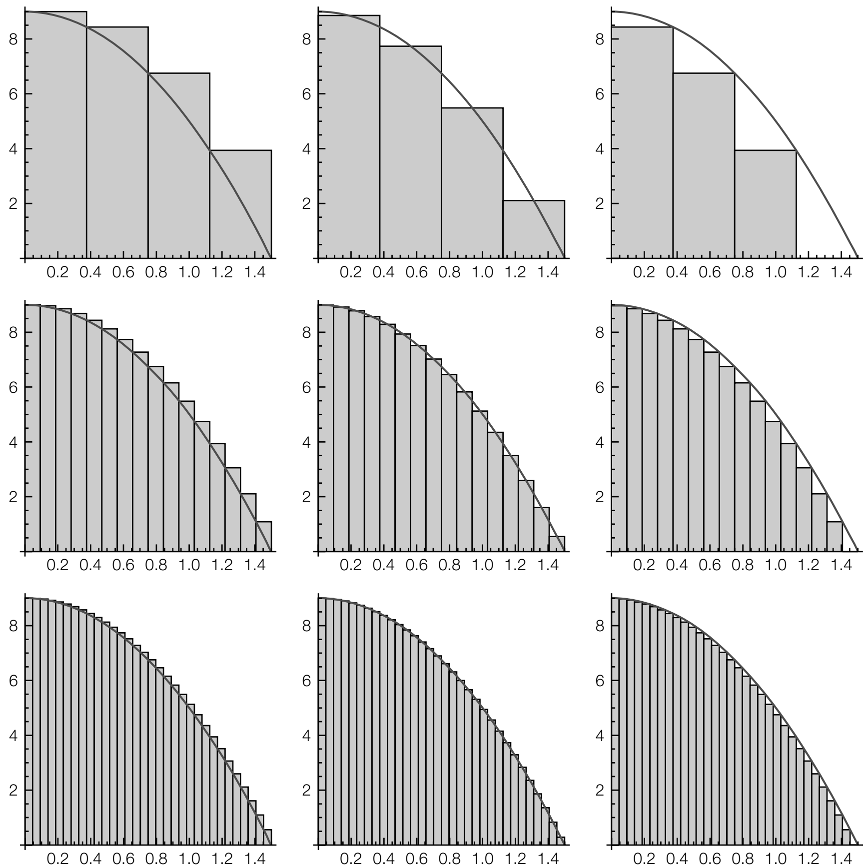



**FIGURE 3.33**

$f(x)$ with 4, 16, and 32 rectangles

```
middlebox[f_, a_, b_, n_, opts___] := Module[{z, p1, recs, ls},
  z[k_] = a + (b - a)k/n;
  p1 = Plot[f, {x, a, b}, PlotRange → All,
      PlotStyle -> {{Thickness[.01], GrayLevel[.3]}},
      DisplayFunction -> Identity];
  recs = Table[Rectangle[{z[k - 1], 0}, {z[k], f/.x -> 1/2(z[k - 1] + z[k])}],
      {k, 1, n}];
  ls = Table[Line[{{z[k - 1], 0}, {z[k - 1], f/.x -> 1/2(z[k - 1] + z[k])},
     {z[k], f/.x -> 1/2(z[k - 1] + z[k])}, {z[k], 0}}], {k, 1, n}];
  Show[Graphics[{GrayLevel[.8], recs}], Graphics[ls], p1, opts,
      Axes -> Automatic, AspectRatio → 1,
      DisplayFunction -> $DisplayFunction]]
```

```
somegraphs = {{leftbox[f[x], 0, 3/2, 4, DisplayFunction → Identity],
   middlebox[f[x], 0, 3/2, 4, DisplayFunction → Identity],
   rightbox[f[x], 0, 3/2, 4, DisplayFunction → Identity]},
   {leftbox[f[x], 0, 3/2, 16, DisplayFunction → Identity],
   middlebox[f[x], 0, 3/2, 16, DisplayFunction → Identity],
   rightbox[f[x], 0, 3/2, 16, DisplayFunction → Identity]},
   {leftbox[f[x], 0, 3/2, 32, DisplayFunction → Identity],
   middlebox[f[x], 0, 3/2, 32, DisplayFunction → Identity],
   rightbox[f[x], 0, 3/2, 32, DisplayFunction → Identity]}};
Show[GraphicsGrid[somegraphs]]
```

Notice that as $n$ increases, the under approximations increase while the upper approximations decrease.

These graphs help convince us that the limit of the sum as $n \to \infty$ of the areas of the inscribed and circumscribed rectangles is the same. We compute the exact value of Eq. (3.9) with leftsum, evaluate and simplify the sum with Simplify, and compute the limit as $n \to \infty$ with Limit. We see that the limit is $9$.

```
ls = leftsum[f[x], 0, 3/2, n]
ls2 = Simplify[ls]
Limit[ls2, n -> Infinity]
```

$\frac{9(1+n)(-1+4n)}{4n^2}$

$\frac{9(1+n)(-1+4n)}{4n^2}$

9

Similar calculations are carried out for (3.10) and again we see that the limit is $9$. We conclude that the exact value of the area is $9$.

```
rs = rightsum[f[x], 0, 3/2, n]
rs2 = Simplify[rs]
Limit[rs2, n -> Infinity]
```

$\frac{9(-1+n)(1+4n)}{4n^2}$

$\frac{9(-1+n)(1+4n)}{4n^2}$

9

For illustrative purposes, we confirm this result with middlesum.

```
ms = middlesum[f[x], 0, 3/2, n]
ms2 = Simplify[ms]
Limit[ms2, n -> Infinity]
```

$\frac{9(1+8n^2)}{8n^2}$

$9 + \frac{9}{8n^2}$

9

As illustrated previously, with Manipulate, you can experiment with different functions and different $n$ values. First, we define a set of "typical" functions.

```
quad[x_] = 100 - x^2;
cubic[x_] = 4/9 x^3 - 49/9x^2 + 100;
rational[x_] = 100/(x^2 + 1);
root[x_] = Sqrt[10 - x];
sin[x_] = 75Sin[Pi x/5];
```

Next, we use **Manipulate** to create an object that allows us to experiment with how "typical" functions react to changes in $n$ using left, middle, and right-hand endpoint approximations for computations of Riemann sums. In the resulting **Manipulate** object, $n = 4$ rectangles is the default; you can choose $n$-values from $0$ to $100$. The value of the corresponding Riemann sum is shown below the graphic. See Figure 3.34.

How does the Manipulate object change if you remove Transpose from the command?

```
Manipulate[Show[GraphicsGrid[{{leftbox[f[x], 0, 10, n],
Graphics[{Inset[leftsum[f[x], 0, 10, n]//N, {0, 0}]}]},
  {middlebox[f[x], 0, 10, n],
  Graphics[{Inset[middlesum[f[x], 0, 10, n]//N, {0, 0}]}]},
```

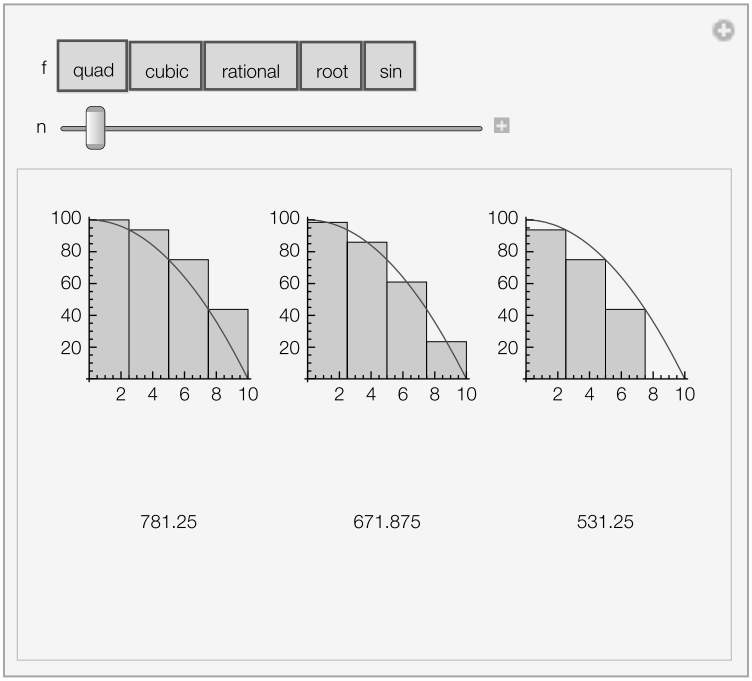



**FIGURE 3.34**

With **Manipulate**, we can investigate Riemann sum approximations and their graphical representations for various functions

```
    {rightbox[f[x], 0, 10, n],
Graphics[{Inset[rightsum[f[x], 0, 10, n] / /N, {0, 0}]}]}}//
Transpose]], {{f, quad}, {quad, cubic, rational, root, sin }},
    {{n, 4}, 0, 100, 1}]
```

---

### 3.3.2 The Definite Integral

In integral calculus courses, we formally learn that the **definite integral** of the function $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f\left(x_k{}^*\right) \Delta x_k, \tag{3.12}$$

provided that the limit exists. In equation (3.12), $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ is a partition of $[a, b]$, $\|P\|$ is the **norm** of $P$,

$$\|P\| = \max\{x_k - x_{k-1} | k = 1, 2, \ldots, n\},$$

$\Delta x_k = x_k - x_{k-1}$, and $x_k{}^* \in \left[x_{k-1}, x_k\right]$.

*The Fundamental Theorem of Calculus* provides the fundamental relationship between differentiation and integration.

**Theorem 6.** ***(The Fundamental Theorem of Calculus).*** *Suppose that $y = f(x)$ is continuous on $[a, b]$.*

1. *If $f(x) = \int_a^x f(t)\,dt$, then $F$ is an antiderivative of $f$: $F'(x) = f(x)$.*
2. *If $G$ is any antiderivative of $f$, then $\int_a^b f(x)\,dx = G(b) - G(a)$.*

Mathematica's Integrate command can compute many definite integrals. The command

```
Integrate[f[x],{x,a,b}]
```

attempts to compute $\int_a^b f(x)\,dx$. Because integration is a fundamentally difficult procedure, it is easy to create integrals for which the exact value cannot be found explicitly. In those cases, use N to obtain an approximation of its value or obtain a numerical approximation of the integral directly with

```
NIntegrate[f[x],{x,a,b}]
```

In the same way as you use the button to compute antiderivatives, you can use the button to compute definite integrals. If the result returned is unevaluated, use N to obtain a numerical approximation of the value of the integral or use NIntegrate.

**Example 3.3.2** Evaluate (a) $\int_1^4 \left(x^2+1\right)/\sqrt{x}\,dx$; (b) $\int_0^{\sqrt{\pi/2}} x\cos x^2\,dx$; (c) $\int_0^{\pi} e^{2x}\sin^2 2x\,dx$; (d) $\int_0^1 \frac{2}{\sqrt{\pi}}e^{-x^2}\,dx$; and (e) $\int_{-1}^0 \sqrt[3]{u}\,du$.

***Solution*** We evaluate (a)–(c) directly with Integrate.

```
Integrate[(x^2 + 1)/Sqrt[x], {x, 1, 4}]
```

$\frac{72}{5}$

```
Integrate[x Cos[x^2], {x, 0, Sqrt[Pi/2]}]
```

$\frac{1}{2}$

```
Integrate[Exp[2x]Sin[2x]^2,{x,0,Pi}]
```

$\frac{1}{5}\left(-1+e^{2\pi}\right)$

For (d), the result returned is in terms of the **error function**, Erf[x], which is defined by the integral

$$\mathrm{Erf}[x] = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

```
Integrate[2/Sqrt[Pi]Exp[-x^2], {x, 0, 1}]
```

Erf[1]

We use N to obtain an approximation of the value of the definite integral.

```
Integrate[2/Sqrt[Pi]Exp[-x^2], {x, 0, 1}]//N
```

0.842701

(e) Recall that Mathematica does not return a real number when we compute odd roots of negative numbers, so the following result would be surprising to many students in an introductory calculus course because it is complex.

See Chapter 2, Example 2.1.3.

```
Integrate[u^(1/3), {u, -1, 0}]
```

$\frac{3}{4}(-1)^{1/3}$

Therefore, we load the **RealOnly** package contained in the **Miscellaneous** directory so that Mathematica returns the real-valued third root of $u$.

```
<< Miscellaneous`RealOnly`
Integrate[u^(1/3), {u, -1, 0}]
```

$-\frac{3}{4}$

Improper integrals are computed using Integrate in the same way as other definite integrals.

**Example 3.3.3** Evaluate (a) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$; (b) $\int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}} e^{-x^2}\,dx$; (c) $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$; (d) $\int_0^{\infty} \frac{1}{x^2+x^4}\,dx$; (e) $\int_2^4 \frac{1}{\sqrt[3]{(x-3)^2}}\,dx$; and (f) $\int_{-\infty}^{\infty} \frac{1}{x^2+x-6}\,dx$.

***Solution*** (a) This is an improper integral because the integrand is discontinuous on the interval $[0, 1]$ but we see that the improper integral converges to $-4$.

**Integrate[Log[x]/Sqrt[x], {x, 0, 1}]**
−4

(b) This is an improper integral because the interval of integration is infinite but we see that the improper integral converges to $2$.

**Integrate[2/Sqrt[Pi]Exp[−x^2], {x, −Infinity, Infinity}]**
2

(c) This is an improper integral because the integrand is discontinuous on the interval of integration and because the interval of integration is infinite but we see that the improper integral converges to $\pi/2$.

**Integrate[1/(x , Sqrt[x^2 − 1]), {x, 1, Infinity}]**
$\frac{\pi}{2}$

(d) As with (c), this is an improper integral because the integrand is discontinuous on the interval of integration and because the interval of integration is infinite but we see that the improper integral diverges to $\infty$.

```
Integrate[1 / (x^2 + x^4), {x, 0, Infinity}]

Integrate::idiv : Integral of 1/(x^2 + x^4) does not converge on {0, ∞}. >>

Integrate::idiv : Integral of 1/(x^2 + x^4) does not converge on {0, ∞}. >>

Integrate::idiv : Integral of 1/(x^2 + x^4) does not converge on {0, ∞}. >>

General::stop :
 Further output of Integrate::idiv will be suppressed during this calculation. >>
```
$\int_0^{\infty} \frac{1}{x^2+x^4}\,dx$

(e) Recall that Mathematica does not return a real number when we compute odd roots of negative numbers, so the following result would be surprising to many students in an introductory calculus course because it contains imaginary numbers.

You do not need to reload the **RealOnly** package if you have already loaded it during your *current* Mathematica session.

**Integrate[1/(x−3)^(2/3), {x, 2, 4}]**
$3-3(-1)^{1/3}$

Therefore, we load the **RealOnly** package contained in the **Miscellaneous** directory so that Mathematica returns the real-valued third root of $x-3$.

```
<< Miscellaneous`RealOnly`
Integrate[1/(x-3)^(2/3), {x, 2, 4}]
6
```

(f) In this case, Mathematica warns us that the improper integral diverges.

```
Integrate[1 / (x^2 + x^4), {x, 0, Infinity}]

Integrate::idiv : Integral of 1/(x^2 + x^4) does not converge on {0, ∞}. >>

Integrate::idiv : Integral of 1/(x^2 + x^4) does not converge on {0, ∞}. >>

Integrate::idiv : Integral of 1/(x^2 + x^4) does not converge on {0, ∞}. >>

General::stop :
 Further output of Integrate::idiv will be suppressed during this calculation. >>

∫_0^∞ 1/(x^2 + x^4) dx
```

To help us understand why the improper integral diverges, we note that $\dfrac{1}{x^2+x-6}=\dfrac{1}{5}\left(\dfrac{1}{x-2}-\dfrac{1}{x+3}\right)$ and

$$\int \frac{1}{x^2+x-6}\,dx=\int \frac{1}{5}\left(\frac{1}{x-2}-\frac{1}{x+3}\right)dx=\frac{1}{5}\ln\left(\frac{x-2}{x+3}\right)+C.$$

```
Integrate[1/(x^2 + x-6), x]
1/5 Log[-2 + x] - 1/5 Log[3 + x]
```

Hence, the integral is improper because the interval of integration is infinite and because the integrand is discontinuous on the interval of integration so

```
s1 = Integrate[1 / (x^2 + x - 6), {x, -Infinity, Infinity}]

Integrate::idiv : Integral of 1/(-6 + x + x^2) does not converge on {-∞, ∞}. >>

∫_-∞^∞ 1/(-6 + x + x^2) dx

Integrate[1 / (x^2 + x - 6), x]

1/5 Log[-2 + x] - 1/5 Log[3 + x]

Integrate[1 / (x^2 + x - 6), {x, -Infinity, -4}]

Log[6]/5

Integrate[1 / (x^2 + x - 6), {x, -4, -3}]

Integrate::idiv : Integral of 1/(-6 + x + x^2) does not converge on {-4, -3}. >>

∫_-4^-3 1/(-6 + x + x^2) dx
```

$$\int_{-\infty}^{\infty} \frac{1}{x^2+x-6}\,dx = \int_{-\infty}^{-4} \frac{1}{x^2+x-6}\,dx + \int_{-4}^{-3} \frac{1}{x^2+x-6}\,dx + \int_{-3}^{0} \frac{1}{x^2+x-6}\,dx + \int_{0}^{2} \frac{1}{x^2+x-6}\,dx + \int_{2}^{3} \frac{1}{x^2+x-6}\,dx + \int_{3}^{\infty} \frac{1}{x^2+x-6}\,dx. \tag{3.13}$$

Evaluating each of these integrals,

```
Integrate[1 / (x^2 + x - 6), {x, -3, 0}]
Integrate::idiv : Integral of 1/(-6 + x + x^2) does not converge on {-3, 0}. >>
∫_{-3}^{0} 1/(-6 + x + x^2) dx

Integrate[1 / (x^2 + x - 6), {x, 0, 2}]
Integrate::idiv : Integral of 1/(-6 + x + x^2) does not converge on {0, 2}. >>
∫_{0}^{2} 1/(-6 + x + x^2) dx

Integrate[1 / (x^2 + x - 6), {x, 2, 3}]
Integrate::idiv : Integral of 1/(-6 + x + x^2) does not converge on {2, 3}. >>
∫_{2}^{3} 1/(-6 + x + x^2) dx

Integrate[1 / (x^2 + x - 6), {x, 3, Infinity}]
Log[6]/5
```

we conclude that the improper integral diverges because at least one of the improper integrals in (3.13) diverges.

---

**In many cases, Mathematica can help illustrate the steps carried out when computing integrals using standard methods of integration such as *u*-substitutions and integration by parts.**

---

**Example 3.3.4** Evaluate (a) $\int_e^{e^3} \frac{1}{x\sqrt{\ln x}}\,dx$ and (b) $\int_0^{\pi/4} x \sin 2x\,dx$.

***Solution*** (a) We let $u = \ln x$. Then, $du = 1/x\,dx$ so $\int_e^{e^3} \frac{1}{x\sqrt{\ln x}}\,dx = \int_1^3 \frac{1}{\sqrt{u}}du = \int_1^3 u^{-1/2}du$, which we evaluate with **Integrate**.

```
Integrate[1/Sqrt[u], {u, 1, 3}]
2(-1 + √3)
```

The new lower limit of integration is 1 because if $x = e$, $u = \ln e = 1$. The new upper limit of integration is 3 because if $x = e^3$, $u = \ln e^3 = 3$.

To evaluate (b), we let $u = x \Rightarrow du = dx$ and $dv = \sin 2x\,dx \Rightarrow v = -\frac{1}{2}\cos 2x$.

```
u = x;
dv = Sin[2x];
du = D[x, x]
v = Integrate[Sin[2x], x]
1
-1/2 Cos[2x]
```

The results mean that

$$\int_0^{\pi/4} x \sin 2x\,dx = -\frac{1}{2}x\cos 2x\Big]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \cos 2x\,dx$$
$$= 0 + \frac{1}{2}\int_0^{\pi/4} \cos 2x\,dx.$$

The resulting indefinite integral is evaluated with Integrate.

```
Integrate[x Sin[2x], x]
-1/2 xCos[2x] + 1/4 Sin[2x]
```

Alternatively, we can illustrate the integration by parts calculation.

```
u = x;
dv = Sin[2x];
du = D[x, x]
v = Integrate[Sin[2x], x]
1
-1/2 Cos[2x]
```

```
uv - Integrate[v du, x]
-1/2 xCos[2x] + 1/4 Sin[2x]
```

We use Integrate to evaluate the definite integral.

```
Integrate[x Sin[2x], {x, 0, Pi/4}]
1/4
```

## 3.3.3 Approximating Definite Integrals

Because integration is a fundamentally difficult procedure, Mathematica is unable to compute a "closed form" of the value of many definite integrals. In these cases, numerical integration can be used to obtain an approximation of the definite integral using N together with Integrate or NIntegrate:

NIntegrate[f[x],{x,a,b}]

attempts to approximate $\int_a^b f(x)\,dx$.

**Example 3.3.5** Evaluate $\int_0^{\sqrt[3]{\pi}} e^{-x^2} \cos x^3 \, dx$.

***Solution*** In this case, Mathematica is unable to evaluate the integral with Integrate.

We use the button to complete the Integrate command.

**i1 = $\int_0^{\pi^{1/3}}$Exp[−x^2]Cos[x^3]dx**

$\int_0^{\pi^{1/3}} e^{-x^2} \text{Cos}\left[x^3\right] \, \text{dx}$

An approximation is obtained with N.

**N[i1]**

0.701566

Instead of using Integrate followed by N, you can use NIntegrate to numerically evaluate the integral.

**NIntegrate[Exp[−x^2]Cos[x^3], {x, 0, π^(1/3)}]**

0.701566

returns the same result as that obtained using Integrate followed by N.

In some cases, you may wish to investigate particular numerical methods that can be used to approximate integrals. To implement numerical methods such as Simpson's rule or the trapezoidal rule, redefine the function leftsum (middlesum or rightsum) discussed previously to perform the calculation for the desired method.

### 3.3.4 Area

Suppose that $y = f(x)$ and $y = g(x)$ are continuous on $[a, b]$ and that $f(x) \geq g(x)$ for $a \leq x \leq b$. The **area** of the region bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is

$$A = \int_a^b \left[ f(x) - g(x) \right] dx. \tag{3.14}$$

**Example 3.3.6** Find the area between the graphs of $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$.

***Solution*** We graph $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$ in Figure 3.35 with Plot. The graph of $y = \cos x$ is dashed. Observe that including the option Filling->{1->{2}} fills the region *between* the two plots.

We display a portion of Mathematica's error message because it indicates that Mathematica cannot find all solutions of the equation. In this case, $\sin x = \cos x$ has infinitely many solutions.

**Plot[{Sin[x], Cos[x]}, {x,0,2π}, PlotStyle → {Black, Dashing[{0.01}]},
Filling → {1 → {2}}, AspectRatio → Automatic]**

To find the upper and lower limits of integration, we must solve the equation $\sin x = \cos x$ for $x$.

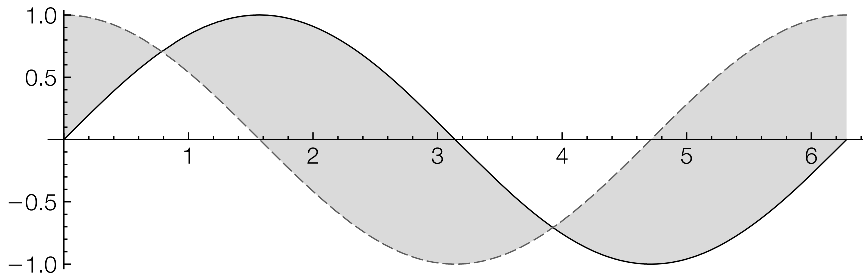



**FIGURE 3.35**

$y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$.

```
Solve[Sin[x] == Cos[x], x]
Solve::ifun :
 Inverse functions are being used by Solve, so some solutions may not
   be found; use Reduce for complete solution information. >>
{{x → -3π/4}, {x → π/4}}
```

For us the solutions of interest are valid for $0 \le x \le 2\pi$, which are $x = \pi/4$ and $x = 5\pi/4$. We check that these are valid solutions of $\sin x = \cos x$ with ==; in each case the returned result is True.

**Sin [π/4]==Cos [π/4]**
**Sin [5π/4]==Cos [5π/4]**
True
True

Hence, the area of the region between the graphs is given by

$$A = \int_0^{\pi/4} [\cos x - \sin x]\, dx + \int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\, dx + \int_{5\pi/4}^{2\pi} [\cos x - \sin x]\, dx. \quad (3.15)$$

Notice that if we take advantage of symmetry we can simplify (3.15) to

$$A = 2\int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\, dx. \quad (3.16)$$

We evaluate (3.16) with Integrate to see that the area of the region between the two graphs is $4\sqrt{2}$.

**$\int_0^{\frac{\pi}{4}}$(Cos[x] – Sin[x]) dx + $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$(Sin[x] – Cos[x]) dx + $\int_{\frac{5\pi}{4}}^{2\pi}$(Cos[x] – Sin[x]) dx**

$4\sqrt{2}$

In cases in which we cannot calculate the points of intersection of two graphs exactly, we can frequently use FindRoot to approximate the points of intersection.

**Example 3.3.7** Let

$$p(x) = \frac{3}{10}x^5 - 3x^4 + 11x^3 - 18x^2 + 12x + 1$$

and

$$q(x) = -4x^3 + 28x^2 - 56x + 32.$$

Approximate the area of the region bounded by the graphs of $y = p(x)$ and $y = q(x)$.

***Solution*** After defining $p$ and $q$, we graph them on the interval $[-1, 5]$ in Figure 3.36 to obtain an initial guess of the intersection points of the two graphs.

When you use Tooltip, you can slide your cursor over a plot and the function being graphed is displayed.

```
Clear[p,q]
p[x_] = 3x^5/10 - 3x^4 + 11x^3 - 18x^2 + 12x + 1;
q[x_] = -4x^3 + 28x^2 - 56x + 32;
Plot[ Tooltip[{p[x], q[x]}], {x, -1, 5}, PlotStyle -> {Black, Gray}]
```

The $x$-coordinates of the three intersection points are the solutions of the equation $p(x) = q(x)$. Although Mathematica can solve this equation exactly, approximate solutions are more useful for the problem and obtained with NSolve.


```
Clear[p,q]
p[x_] = 3x^5/10 - 3x^4 + 11x^3 - 18x^2 + 12x + 1;
q[x_] = -4x^3 + 28x^2 - 56x + 32;
Plot[ Tooltip[{p[x], q[x]}], {x, -1, 5}, PlotStyle -> {Black, Gray}]
```

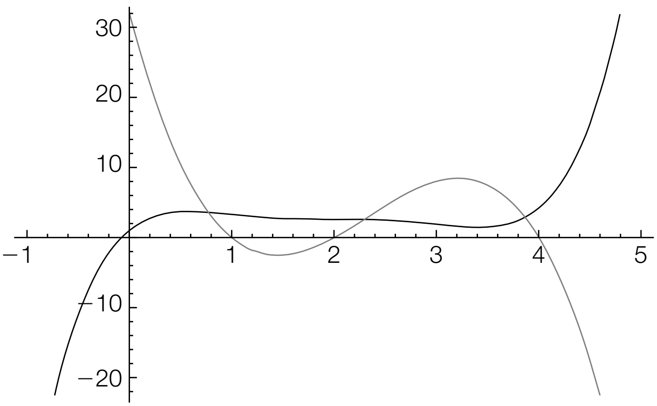




**FIGURE 3.36**

$p$ and $q$ on the interval $[-1, 5]$.

```
intpts = NRoots[p[x]==q[x], x]
x==0.772058||x==1.5355 − 3.57094i||x==1.5355 + 3.57094i||
   x==2.29182||x==3.86513
```

The numbers are extracted from the list with Part ([[...]]). For example, $0.772058$ is the second part of the first part of intpts. Counting from left to right, $2.29182$ is the second part of the fourth part of intpts.

```
x1 = intpts[[1, 2]]
x2 = intpts[[4, 2]]
x3 = intpts[[5, 2]]
0.772058
2.29182
3.86513
```

Using the roots to the equation $p(x) = q(x)$ and the graph we see that $p(x) \geq q(x)$ for $0.772 \leq x \leq 2.292$ and $q(x) \geq p(x)$ for $2.292 \leq x \leq 3.865$. Hence, an approximation of the area bounded by $p$ and $q$ is given by the sum

$$\int_{0.772}^{2.292} \left[p(x) - q(x)\right]\, dx + \int_{2.292}^{3.865} \left[q(x) - p(x)\right]\, dx.$$

These two integrals are computed with Integrate and NIntegrate. As expected, the two values are the same.

```
∫_{x1}^{x2}(p[x] − q[x]) dx + ∫_{x2}^{x3}(q[x] − p[x]) dx
12.1951
NIntegrate[p[x] − q[x], {x, x1, x2}] + NIntegrate[q[x] − p[x], {x, x2, x3}]
12.1951
```

We conclude that the area is approximately $12.195$.

Graphically, $y$ is a function of $x$, $y = y(x)$, if the graph of $y = y(x)$ passes the vertical line test.

### *Parametric Equations*

If the curve, $C$, defined parametrically by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$ is a nonnegative continuous function of $x$ and $x(a) < x(b)$, the area under the graph of $C$ and above the $x$-axis is

$$\int_{x(a)}^{x(b)} y\, dx = \int_a^b y(t)x'(t)dt.$$

**Example 3.3.8** **(The Astroid).** Find the area enclosed by the **astroid** $x = \sin^3 t$, $y = \cos^3 t$, $0 \leq t \leq 2\pi$.

***Solution*** We begin by defining $x$ and $y$ and then graphing the astroid with ParametricPlot in Figure 3.37.

```
x[t_] = Sin[t]^3;
y[t_] = Cos[t]^3;
ParametricPlot[{x[t], y[t]}, {t, 0, 2Pi}, AspectRatio ->Automatic]
```

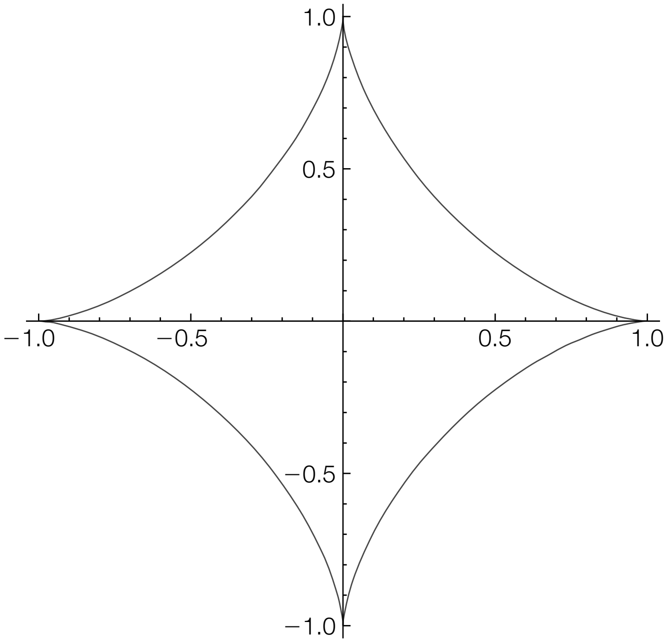



**FIGURE 3.37**

The astroid $x = \sin^3 t,\ y = \cos^3 t,\ 0 \le t \le 2\pi$.

Observe that $x(0) = 0$ and $x(\pi/2) = 1$, and the graph of the astroid in the first quadrant is given by $x = \sin^3 t,\ y = \cos^3 t,\ 0 \le t \le \pi/2$. Hence, the area of the astroid in the first quadrant is given by

$$\int_0^{\pi/2} y(t)x'(t)\,dt = 3\int_0^{\pi/2} \sin^2 t\cos^4 t\,dt$$

and the total area is given by

$$A = 4\int_0^{\pi/2} y(t)x'(t)\,dt = 12\int_0^{\pi/2} \sin^2 t\cos^4 t\,dt = \frac{3}{8}\pi \approx 1.178,$$

which is computed with Integrate and then approximated with N.

**area = 4Integrate[y[t]x′[t], {t, 0, Pi/2}]**

$\frac{3\pi}{8}$

**N[area]**

1.1781

### *Polar Coordinates*

For problems involving "circular symmetry," it is often easier to work in polar coordinates. The relationship between $(x, y)$ in rectangular coordinates and $(r, \theta)$ in polar coordinates is given by

$$x = r\cos\theta \qquad y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

If $r = f(\theta)$ is continuous and nonnegative for $\alpha \le \theta \le \beta$, then the **area** $A$ of the region enclosed by the graphs of $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta \left[f(\theta)\right]^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta.$$

---

**Example 3.3.9** **(Lemniscate of Bernoulli).** The **lemniscate of Bernoulli** is given by

$$\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right),$$

where $a$ is a constant. (a) Graph the lemniscate of Bernoulli if $a = 2$. (b) Find the area of the region bounded by the lemniscate of Bernoulli.

***Solution*** This problem is much easier solved in polar coordinates, so we first convert the equation from rectangular to polar coordinates with ReplaceAll (/.) and then solve for $r$ with Solve.

```
lofb = (x^2 + y^2)^2==a^2(x^2 - y^2);
topolar = lofb/.{x->rCos[t], y->rSin[t]}
```

$(r^2\text{Cos}[t]^2 + r^2\text{Sin}[t]^2)^2 == a^2(r^2\text{Cos}[t]^2 - r^2\text{Sin}[t]^2)$

```
Solve[topolar, r]//Simplify
```

$\left\{\{r \to 0\}, \{r \to 0\}, \left\{r \to -\sqrt{a^2\text{Cos}[2t]}\right\}, \left\{r \to \sqrt{a^2\text{Cos}[2t]}\right\}\right\}$

These results indicate that an equation of the lemniscate in polar coordinates is $r^2 = a^2\cos 2\theta$. The graph of the lemniscate is then generated in Figure 3.38 (top) using PolarPlot. The portion of the lemniscate in quadrant one is obtained by graphing $r = 2\cos 2\theta$, $0 \le \theta \le \pi/4$.

```
p1 = PolarPlot[{-2Sqrt[Cos[2t]], 2Sqrt[Cos[2t]]}, {t, 0, 2Pi}];
p2 = PolarPlot[2Sqrt[Cos[2t]], {t, 0, Pi/4}];
Show[GraphicsColumn[{p1, p2}]]
```

Then, taking advantage of symmetry, the area of the lemniscate is given by

$$A = 2\cdot\frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\, d\theta = 2\int_0^{\pi/4} r^2\, d\theta = 2\int_0^{\pi/4} a^2\cos 2\theta\, d\theta = a^2,$$

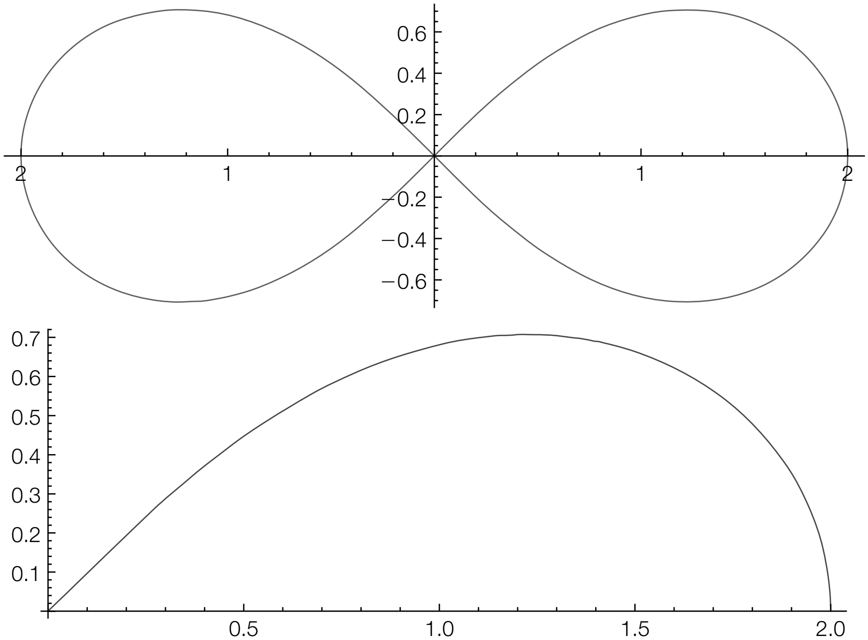



**FIGURE 3.38**

(Top) The lemniscate. (Bottom) The portion of the lemniscate in quadrant 1

which we calculate with Integrate.

```
Integrate[2a^2 Cos[2t], {t, 0, Pi/4}]
a^2
```

## 3.3.5 Arc Length

Let $y = f(x)$ be a function for which $f'(x)$ is continuous on an interval $[a, b]$. Then the **arc length** of the graph of $y = f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx. \tag{3.17}$$

The resulting definite integrals used for determining arc length are usually difficult to compute because they involve a radical. In these situations, Mathematica is helpful with approximating solutions to these types of problems.

**Example 3.3.10** Find the length of the graph of $y = \frac{x^4}{8} + \frac{1}{4x^2}$ from (a) $x = 1$ to $x = 2$ and from (b) $x = -2$ to $x = -1$.

***Solution*** With no restrictions on the value of $x$, $\sqrt{x^2} = |x|$. Generally, Mathematica does not automatically algebraically simplify $\sqrt{(dy/dx)^2 + 1}$ because Mathematica does not know if $x$ is positive or negative.

```
y[x_] = x^4/8 + 1/(4x^2);
i1 = Factor[y'[x]^2 + 1]
```

$$\frac{(1+x^2)^2(1-x^2+x^4)^2}{4x^6}$$

```
i2 = PowerExpand[Sqrt[i1]]
```

$$\frac{(1+x^2)(1-x^2+x^4)}{2x^3}$$

PowerExpand[expr] simplifies radicals in the expression expr assuming that all variables are positive.

In fact, for (b), $x$ is negative so $\frac{1}{2}\sqrt{\frac{(x^6+1)^2}{x^6}} = -\frac{1}{2}\frac{x^6+1}{x^3}$. Mathematica simplifies $\frac{1}{2}\sqrt{\frac{(x^6+1)^2}{x^6}} = \frac{1}{2}\frac{x^6+1}{x^3}$ and correctly evaluates the arc length integral (3.17) for (a).

```
Integrate[Sqrt[y'[x]^2 + 1], {x, 1, 2}]
```

$$\frac{33}{16}$$

For (b), we compute the arc length integral (3.17).

```
Integrate[Sqrt[y'[x]^2 + 1], {x, -2, -1}]
```

$$\frac{33}{16}$$

As we expect, both values are the same.

$C$ is **smooth** if both $x'(t)$ and $y'(t)$ are continuous on $(a, b)$ and not simultaneously zero for $t \in (a, b)$.

### *Parametric Equations*

If the smooth curve, $C$, defined parametrically by $x = x(t)$, $y = y(t)$, $t \in [a, b]$ is traversed exactly once as $t$ increases from $t = a$ to $t = b$, the arc length of $C$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{3.18}$$

**Example 3.3.11** Find the length of the graph of $x = \sqrt{2}t^2$, $y = 2t - \frac{1}{2}t^3$, $-2 \le t \le 2$.

***Solution*** For illustrative purposes, we graph $x = \sqrt{2}t^2$, $y = 2t - \frac{1}{2}t^3$ for $-3 \le t \le 3$ and $-2 \le t \le 2$ (thickened) in Figure 3.39.

```
x[t_] = t^2 Sqrt[2]; y[t_] = 2t - 1/2t^3;
p1 = ParametricPlot[{x[t], y[t]}, {t, -3, 3}];
```

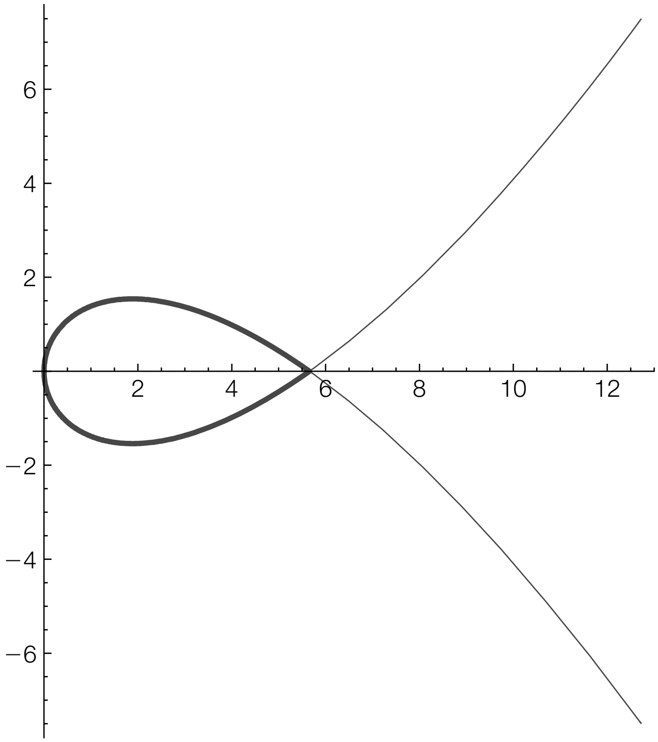



**FIGURE 3.39**

$x = \sqrt{2}t^2$, $y = 2t - \frac{1}{2}t^3$

```
p2 = ParametricPlot[{x[t], y[t]}, {t, -2, 2}, PlotStyle->Thickness[.01]];
Show[p1, p2, PlotRange->All]
```

Mathematica is able to compute the exact value of the arc length (3.18), although the result is quite complicated. For length considerations, the result of entering the i1 command is not displayed here.

**Factor[x′[t]^2 + y′[t]^2]**

$\frac{1}{4}\left(4-4t+3t^2\right)\left(4+4t+3t^2\right)$

**i1 = Integrate[2 Sqrt[x′[t]^2 + y′[t]^2], {t, 0, 2}]**

A more meaningful approximation is obtained with N or using NIntegrate.

**N[i1]**

13.7099 + 0.i

**NIntegrate[2Sqrt[x′[t]^2 + y′[t]^2], {t, 0, 2}]**

13.7099

We conclude that the arc length is approximately 13.71.

### *Polar Coordinates*

**If the smooth polar curve $C$ given by $r = f(\theta)$, $\alpha \le \theta \le \beta$ is traversed exactly once as $\theta$ increases from $\alpha$ to $\beta$, the arc length of $C$ is given by**

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \tag{3.19}$$

**Example 3.3.12** Find the length of the graph of $r = \theta$, $0 \le \theta \le 10\pi$.

***Solution*** We begin by defining $r$ and then graphing $r$ with PolarPlot in Figure 3.40.

```
r[t_] = t;
PolarPlot[r[t], {t, 0, 10Pi}, AspectRatio ->Automatic]
```

Using (3.19), the length of the graph of $r$ is given by $\int_0^{10\pi} \sqrt{1+\theta^2}\, d\theta$. The exact value is computed with Integrate

```
ev = Integrate[Sqrt[r'[t]^2 + r[t]^2], {t, 0, 10Pi}]
```

$5\pi\sqrt{1+100\pi^2} + \frac{1}{2}\text{ArcSinh}[10\pi]$

and then approximated with N.

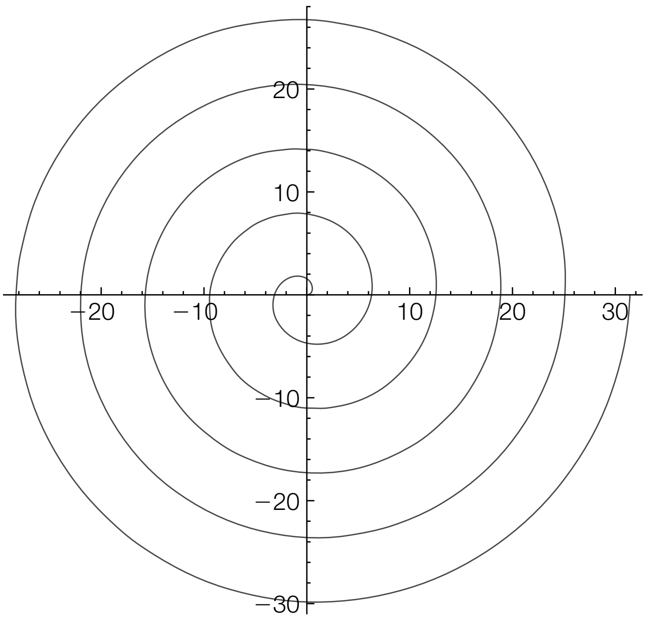



**FIGURE 3.40**

$r = \theta$ for $0 \le \theta \le 10\pi$

```
N[ev]
495.801
```

We conclude that the length of the graph is approximately 495.8.

### 3.3.6 Solids of Revolution

#### *Volume*

Let $y = f(x)$ be a nonnegative continuous function on $[a, b]$. The **volume** of the solid of revolution obtained by revolving the region bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the $x$-axis about the $x$-axis is given by

$$V = \pi \int_a^b [f(x)]^2 \, dx. \tag{3.20}$$

If $0 \le a < b$, the **volume** of the solid of revolution obtained by revolving the region bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the $x$-axis about the $y$-axis is given by

$$V = 2\pi \int_a^b x f(x) \, dx. \tag{3.21}$$

**Example 3.3.13** Let $g(x) = x \sin^2 x$. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = g(x)$, $x = 0$, $x = \pi$, and the $x$-axis about (a) the $x$-axis and (b) the $y$-axis.

***Solution*** After defining $g$, we graph $g$ on the interval $[0, \pi]$ in Figure 3.41(a).

With Mathematica 6, for three-dimensional graphics, you can adjust the viewpoint by clicking on the three-dimensional graphics object and dragging to the desired viewing angle.

```
g[x_] = xSin[x]^2;
p1 = Plot[g[x], {x, 0, Pi}, AspectRatio ->Automatic];
```

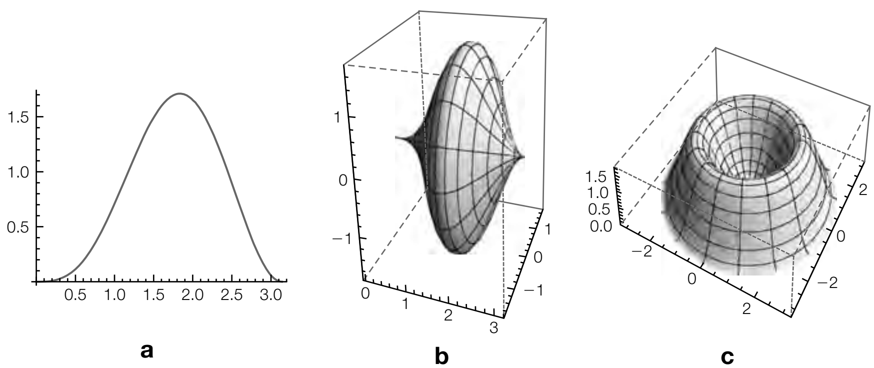



**FIGURE 3.41**

(a) $g(x)$ for $0 \le x \le \pi$. (b) $g(x)$ revolved about the $x$-axis. (c) $g(x)$ revolved about the $y$-axis

The volume of the solid obtained by revolving the region about the $x$-axis is given by (3.20), whereas the volume of the solid obtained by revolving the region about the $y$-axis is given by (3.21). These integrals are computed with Integrate and named xvol and yvol, respectively. We use N to approximate each volume.

```
xvol = Integrate[Pi g[x]^2, {x, 0, Pi}]
N[xvol]
```

$\frac{1}{64}\pi^2\left(-15+8\pi^2\right)$
9.86295

```
yvol = Integrate[2 Pi x g[x], {x, 0, Pi}]
N[yvol]
```

$\frac{1}{6}\pi^2\left(-3+2\pi^2\right)$
27.5349

We can use ParametricPlot3D to visualize the resulting solids by parametrically graphing the equations given by

$$\begin{cases} x = r\cos t \\ y = r\sin t \\ z = g(r) \end{cases}$$

for $r$ between $0$ and $\pi$ and $t$ between $-\pi$ and $\pi$ to visualize the graph of the solid obtained by revolving the region about the $y$-axis and by parametrically graphing the equations given by

$$\begin{cases} x = r \\ y = g(r)\cos t \\ z = g(r)\sin t \end{cases}$$

for $r$ between $0$ and $\pi$ and $t$ between $-\pi$ and $\pi$ to visualize the graph of the solid obtained by revolving the region about the $x$-axis. (See Figures 3.41(b) and 3.41(c).) In this case, we identify the $z$-axis as the $y$-axis. Notice that we are simply using polar coordinates for the $x$- and $y$-coordinates, and the height above the $x,y$-plane is given by $z = g(r)$ because $r$ is replacing $x$ in the new coordinate system.

```
p2 = ParametricPlot3D[{r, g[r]Cos[t], g[r]Sin[t]}, {r, 0, Pi},
    {t, 0, 2Pi}, PlotPoints -> {30, 30}];
p3 = ParametricPlot3D[{r Cos[t], r Sin[t], g[r]}, {r, 0, Pi}, {t, 0, 2Pi},
    PlotPoints -> {30, 30}];
```

p1, p2, and p3 are shown together side-by-side in Figure 3.41 using Show together with GraphicsRow.

```
Show[GraphicsRow[{p1, p2, p3}]]
```

---

We now demonstrate a volume problem that requires the method of disks.

**Example 3.3.14** Let $f(x) = e^{-(x-3)^2 \cos[4(x-3)]}$. Approximate the volume of the solid obtained by revolving the region bounded by the graphs of $y = f(x)$, $x = 1$, $x = 5$, and the $x$-axis about the $x$-axis.

***Solution*** Proceeding as in the previous example, we first define and graph $f$ on the interval $[1, 5]$ in Figure 3.42(a).

```
f[x_] = Exp[-(x-3)^2Cos[4(x-3)]];
p1 = Plot[f[x],{x,1,5},AspectRatio->Automatic];
```

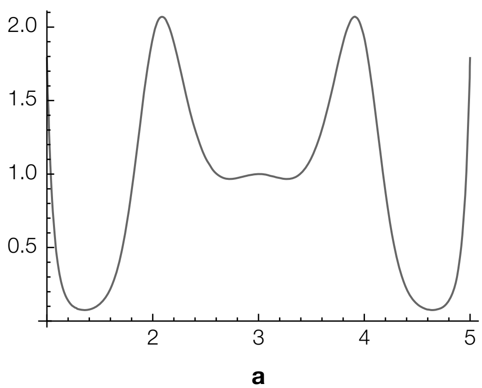
In this case, an approximation is desired so we use NIntegrate to approximate the integral $V = \int_1^5 \pi \left[f(x)\right]^2 \, dx$.

```
NIntegrate[Pi f[x]^2,{x,1,5}]
16.0762
```

In the same manner as before, ParametricPlot3D can be used to visualize the resulting solid by graphing the set of equations given parametrically by

$$\begin{cases} x = r \\ y = f(r)\cos t \\ z = f(r)\sin t \end{cases}$$

for $r$ between 1 and 5 and $t$ between 0 and $2\pi$. In this case, polar coordinates are used in the $y$, $z$-plane, with the distance from the $x$-axis given by $f(x)$. Because $r$ replaces $x$ in the new coordinate system, $f(x)$ becomes $f(r)$ in these equations. See Figure 3.42(b).



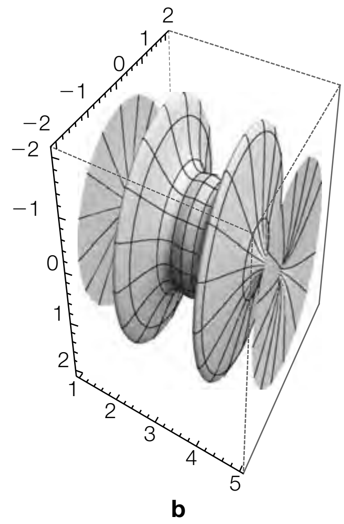




**FIGURE 3.42**

(a) $f(x)$ for $1 \le x \le 5$. (b) $f(x)$ revolved about the $x$-axis

```
p2 = ParametricPlot3D[{r, f[r]Cos[t], f[r]Sin[t]}, {r, 1, 5},
     {t, 0, 2Pi}, PlotPoints ->{45, 35}];

Show[GraphicsRow[{p1, p2}]]
```

When revolving a curve about the $y$-axis, you can use RevolutionPlot3D rather than the parametrization given previously.

**Example 3.3.15** Let $f(x) = \exp\left(-2(x-2)^2\right) + \exp\left(-(x-4)^2\right)$ for $0 \le x \le 6$. Find the minimum and maximum values of $f(x)$ on $[0, 6]$.

***Solution*** (a) Although Maximize and Minimize cannot find the exact maximum and minimum values, using N or NMaximize and NMinimize gives accurate approximations.

NMaximize and NMinimize work in the same way as Maximize and Minimize but return approximations rather than exact results.

```
f[x_] = Exp[-2(x-2)^2] + Exp[-(x-4)^2];
Maximize[f[x], x]
```
Maximize $\left[e^{-(-4+x)^2} + e^{-2(-2+x)^2}, x\right]$

```
Maximize[f[x], x]//N
```
{1.01903, {x → 2.01962}}

```
NMaximize[{f[x], 0 ≤ x ≤ 6}, x]
```
{1.00034, {x → 3.99864}}

```
Minimize[{f[x], 0 ≤ x ≤ 6}, x]
```
Minimize $\left[\left\{e^{-(-4+x)^2} + e^{-2(-2+x)^2}, 0 \le x \le 6\right\}, x\right]$

```
Minimize[{f[x], 0 ≤ x ≤ 6}, x]//N
```
{0.495486, {x → 2.92167}}

```
NMinimize[{f[x], 0 ≤ x ≤ 6}, x]//N
```
{0.495486, {x → 2.92167}}

We double check these results by graphing $f(x)$ and $f'(x)$ in Figure 3.43 and then using FindRoot to approximate the critical numbers.

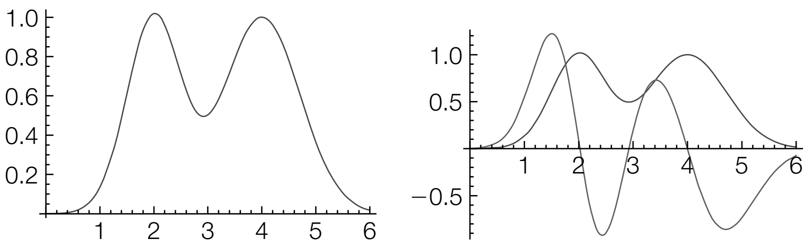



**FIGURE 3.43**

We use the graph of $f'(x)$ to help us estimate the initial values to approximate the critical numbers with FindRoot

```
pf1 = Plot[f[x], {x, 0, 6}]
pf2 = Plot[Tooltip[{f[x], f′[x]}], {x, 0, 6}]
Show[GraphicsRow[{pf1, pf2}]]
Map[FindRoot[f′[x]==0, {x, #1}]&, {2, 3, 4}]
```

{{x → 2.01962}, {x → 2.92167}, {x → 3.99864}}

(b) Mathematica finds the exact volume of the solids although the results are expressed in terms of the **Error function**, Erf.

```
Integrate[Pi x f[x], {x, 0, 6}]
```

$$\frac{1}{4e^{32}}\pi\left(-1+2e^{16}+e^{24}-2e^{28}+2e^{32}\sqrt{\pi}\,(4\mathrm{Erf}[2]+4\mathrm{Erf}[4]+\sqrt{2}(\mathrm{Erf}[2\sqrt{2}]+\mathrm{Erf}[4\sqrt{2}]))\right)$$

```
NIntegrate[Pi x f[x], {x, 0, 6}]
```

30.0673

```
Integrate[Pi f[x]^2, {x, 0, 6}]
```

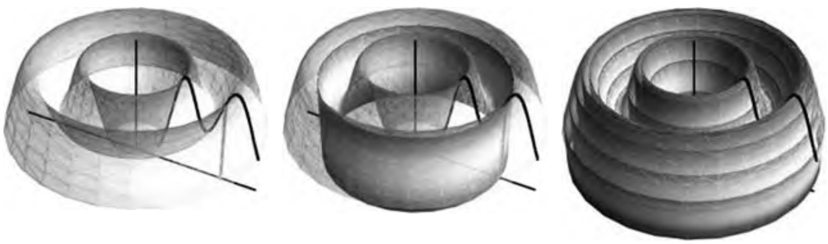
$$\frac{1}{12e^{8/3}}\pi^{3/2}\left(3e^{8/3}(\mathrm{Erf}[4]+\mathrm{Erf}[8]+\sqrt{2}(\mathrm{Erf}[2\sqrt{2}]+\mathrm{Erf}[4\sqrt{2}]))+4\sqrt{3}\left(\mathrm{Erf}\left[\tfrac{8}{\sqrt{3}}\right]+\mathrm{Erf}\left[\tfrac{10}{\sqrt{3}}\right]\right)\right)$$

```
NIntegrate[Pi f[x]^2, {x, 0, 6}]
```

7.1682

To visualize the solid revolved about the $y$-axis, we use RevolutionPlot3D in p1. We generate the curve in p2, a set of axes, and a representative “slice” of the curve. See Figure 3.44(a). Next, we show the solid together with a representative shell. See Figure 3.44(b).

```
p1 = RevolutionPlot3D[f[x], {x, 1, 5},
   BoxRatios → {2, 2, 1}, PlotRange → {{-5, 5}, {-5, 5}, {0, 5/4}},
     Mesh → None, PlotStyle → Opacity[.4],
        ColorFunction → “LightTemperatureMap”];
p2 = ParametricPlot3D[{x, 0, Exp[-2(x-2)^2] + Exp[-(x-4)^2]},
    {x, 1, 5}, {t, 0, 2Pi},
```



a b c

**FIGURE 3.44**

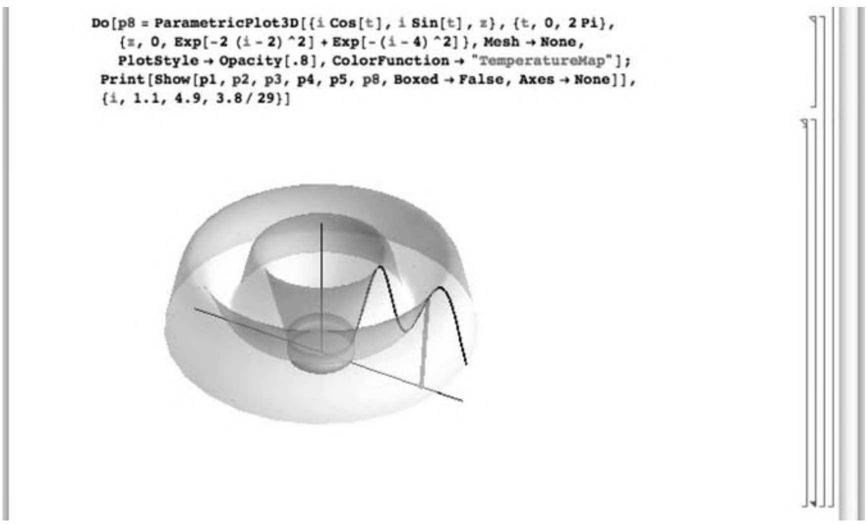
(a) The solid. (b) The solid with a “typical” shell. (c) Several shells

```
        PlotStyle → Thickness[.05], BoxRatios → {2, 2, 1},
          Axes → Automatic, Boxed → False];
p3 = ParametricPlot3D[{x, 0, 0}, {x, −5, 5}, {t, 0, 2Pi},
    PlotStyle → {Gray, Thickness[.075]}, BoxRatios → {2, 2, 1},
      Axes → Automatic, Boxed → False];
p4 = ParametricPlot3D[{0, 0, x}, {x, 0, 5/4}, {t, 0, 2Pi},
    PlotStyle → {Gray, Thickness[.1]}, BoxRatios → {2, 2, 1},
      Axes → Automatic, Boxed → False];
p5 = Graphics3D[{Gray, Thickness[.01], Line[{{3.6, 0, 0},
    {3.6, 0, Exp[−2(3.6−2)^2] + Exp[−(3.6−4)^2]}}]}];
        p6 = ParametricPlot3D[{3.6Cos[t], 3.6Sin[t], z}, {t, 0, 2Pi},
          {z, 0, Exp[−2(3.6−2)^2] + Exp[−(3.6−4)^2]}, Mesh → None,
        PlotStyle → Opacity[.8], ColorFunction → "TemperatureMap"];
g1 = Show[p1, p2, p3, p4, p5, Boxed → False, Axes → None]
g2 = Show[p1, p2, p3, p4, p5, p6, Boxed → False, Axes → None]
```

Finally, we show the solid together with several shells in Figure 3.44(c).

```
p7 = Table[ParametricPlot3D[{sp[[i]]Cos[t], sp[[i]]Sin[t], z},
     {t, 0, 2Pi},
      {z, 0, Exp[−2(sp[[i]] − 2)^2] + Exp[−(sp[[i]] − 4)^2]},
        Mesh → None,
      PlotStyle → Opacity[.8],
      ColorFunction → "TemperatureMap"], {i, 1, Length[sp]}];
g3 = Show[p1, p2, p3, p4, p5, p7, Boxed → False, Axes → None]
Show[GraphicsRow[{g1, g2, g3}]]
```

With a **Do** loop you can generate an animation of the process.



For revolving $f(x)$ about the $x$-axis, we proceed in much the same way. First, we plot $f(x)$ with a set of axes in space.

```
f[x_] = Exp[-2(x-2)^2] + Exp[-(x-4)^2];
p1 = ParametricPlot3D[{x, 0, f[x]}, {x, 0, 6}, PlotStyle → {Thick, Black},
      PlotRange → {{0, 6}, {-3/2, 3/2}, {-3/2, 3/2}}, BoxRatios → {1, 1, 1}];
p1b = ParametricPlot3D[{x, 0, -f[x]}, {x, 0, 6}, PlotStyle → {Thick, Black},
      PlotRange → {{0, 6}, {-3/2, 3/2}, {-3/2, 3/2}}, BoxRatios → {1, 1, 1}];
p2 = ParametricPlot3D[{x, 0, 0}, {x, 0, 6}, {t, 0, 2Pi},
    PlotStyle → {Gray, Thickness[.075]},
      PlotRange → {{0, 6}, {-3/2, 3/2}, {-3/2, 3/2}}, BoxRatios → {1, 1, 1}];
p3 = ParametricPlot3D[{0, 0, x}, {x, -3/2, 3/2}, {t, 0, 2Pi},
    PlotStyle → {Gray, Thickness[.1]},
      PlotRange → {{0, 6}, {-3/2, 3/2}, {-3/2, 3/2}}, BoxRatios → {1, 1, 1}];
Show[p1, p1b, p2, p3]
```

Next, we generate a basic plot of the solid in p4 and then a set of disks inside the solid in t3d.

```
p4 = ParametricPlot3D[{r, f[r]Cos[t], f[r]Sin[t]},
    {r, 0, 6}, {t, 0, 2Pi}, PlotRange → {{0, 6}, {-3/2, 3/2}, {-3/2, 3/2}},
      BoxRatios → {1, 1, 1}]
t3d = Table[ParametricPlot3D[{x, rf[x]Cos[t], rf[x]Sin[t]}, {r, 0, 1},
    {t, 0, 2Pi}, PlotRange → {{0, 6}, {-3/2, 3/2}, {-3/2, 3/2}},
      BoxRatios → {1, 1, 1}, ColorFunction → "TemperatureMap", Mesh → 5],
        {x, 0, 6, 6/14}];
```

Two variations of the solid are plotted in p5 and p6. In each case, we use MeshFunctions to have the contour lines (mesh) correspond to $f(x)$ values rather than the rectangular default mesh. In p6 the solid is made transparent with the Opacity option.

```
p5 = ParametricPlot3D[{r, f[r]Cos[t], f[r]Sin[t]},
    {r, 0, 6}, {t, 0, 2Pi}, PlotRange → {{0, 6}, {-3/2, 3/2}, {-3/2, 3/2}},
      BoxRatios → {1, 1, 1}, MeshFunctions ->{#1&}, Mesh → 60]
p6 = ParametricPlot3D[{r, f[r]Cos[t], f[r]Sin[t]},
    {r, 0, 6}, {t, 0, 2Pi}, PlotRange → {{0, 6}, {-3/2, 3/2},
      {-3/2, 3/2}},
    BoxRatios → {1, 1, 1}, MeshFunctions ->{#1&}, Mesh → 25,
      PlotStyle → Opacity[.2],
        MeshStyle → {Gray, Thick}];
Show[p1, p1b, p2, p3, p6, t3d]
```

Several combinations of the images are shown in Figures 3.45 and 3.46.

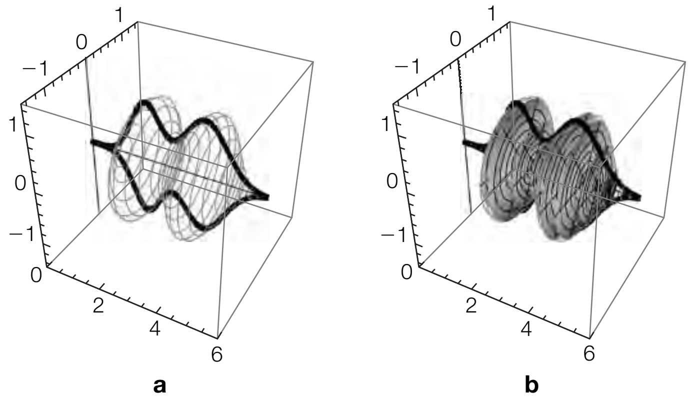



**FIGURE 3.45**

(a) Seeing $f(x)$ on the solid. (b) Disks in the solid

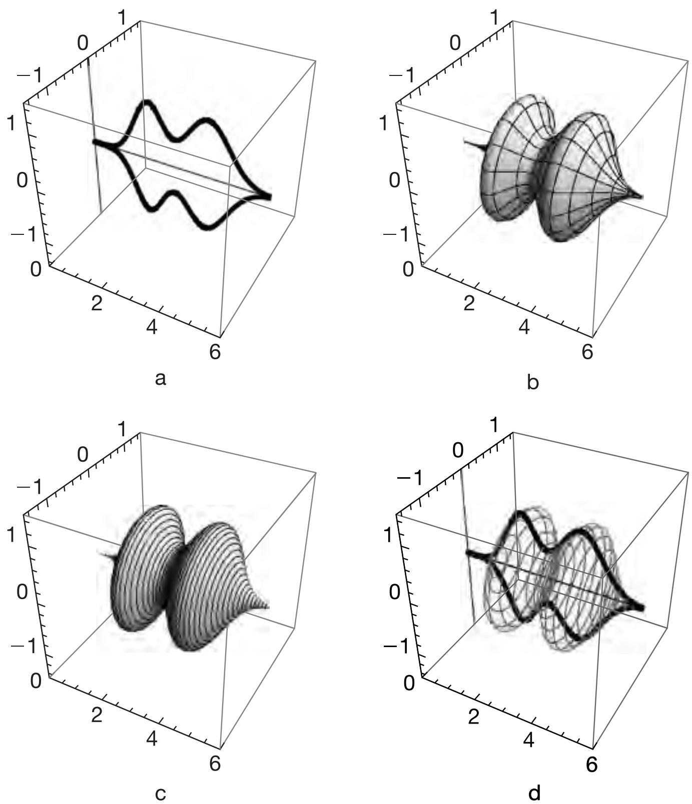



**FIGURE 3.46**

(a) $f(x)$ in space. (b) The basic solid. (c) Contours based on $f(x)$ values. (d) Seeing $f(x)$ on the solid

```
Show[GraphicsRow[{Show[p1, p1b, p2, p3, p6],
   Show[p1, p1b, p2, p3, p6, t3d]}]]
Show[GraphicsGrid[{{Show[p1, p1b, p2, p3], p4},
   {p5, Show[p1, p1b, p2, p3, p6]}}]]
```

---

To help identify regions, RegionPlot[constraints,{x,a,b},{y,a,b}] attempts to shade the region in the rectangle $[a, b] \times [c, d]$ that satisfies the constraints in constraints.

---

**Example 3.3.16** Let $g(x) = \sqrt{x}$, $h(x) = x^2$, and $R$ be the region bounded by the graphs of $g(x)$ and $h(x)$. Find the volume of the solid obtained by revolving $R$ about (a) the $x$-axis and (b) the $y$-axis.

***Solution*** We illustrate the use of RegionPlot to help us see $R$. See Figure 3.47.

```
g[x_] = Sqrt[x];
h[x_] = x^2;
p1a = Plot[{Sqrt[x], x^2}, {x, 0, 2},
      PlotRange → {{0, 2}, {0, 2}}, AspectRatio → Automatic]
p1b = RegionPlot[x^2 ≤ y ≤ Sqrt[x], {x, 0, 2}, {y, 0, 2}]
Show[{p1a, p1b}]
Show[GraphicsRow[{p1a, p1b, Show[{p1a, p1b}]}]]
```

We plot the solids with ParametricPlot3D and contour lines along the function values using the MeshFunctions option in Figure 3.48.

```
p4 = ParametricPlot3D[{{r, g[r]Cos[t], g[r]Sin[t]},
   {r, h[r]Cos[t], h[r]Sin[t]}},
     {r, 0, 1}, {t, 0, 2Pi}, PlotRange → {{0, 3/2}, {−5/4, 5/4}, {−5/4, 5/4}},
      BoxRatios → {1, 1, 1}, MeshFunctions −>{#1&}]
```

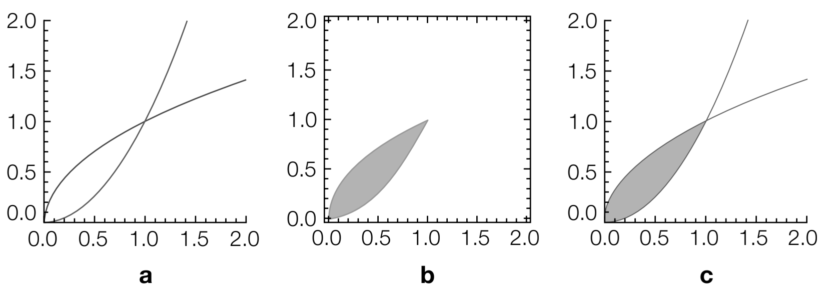



**FIGURE 3.47**

(a) Graphs of $f(x)$ and $g(x)$. (b) The region in $[0, 2] \times [0, 2]$ for which $x^2 \le y \le \sqrt{x}$. (c) The two plots displayed together

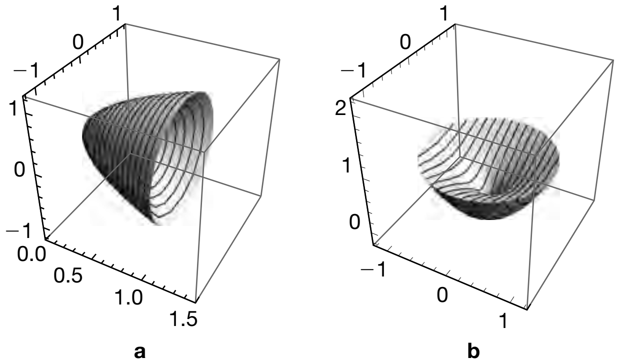



**FIGURE 3.48**

(a) The solid formed by revolving $R$ about the $x$-axis. (b) The solid formed by revolving $R$ about the $y$-axis

```
p5 = ParametricPlot3D[{{rCos[t], rSin[t], g[r]},
    {rCos[t], rSin[t], h[r]}},
        {r, 0, 1}, {t, 0, 2Pi}, PlotRange → {{−5/4, 5/4}, {−5/4, 5/4}, {−1/4, 9/4}},
         BoxRatios → {1, 1, 1}, MeshFunctions −>{#1&}]
Show[GraphicsRow[{p4, p5}]]
```

The volume of each solid is then found with Integrate and approximated with N.

```
Integrate[Pi(g[x]^2 − h[x]^2), {x, 0, 1}]
```

$\frac{3\pi}{10}$

```
N[%]
```

0.942478

```
Integrate[Pix(g[x] − h[x]), {x, 0, 1}]
```

$\frac{3\pi}{20}$

```
N[%]
```

0.471239

## *Surface Area*

Let $y = f(x)$ be a nonnegative function for which $f'(x)$ is continuous on an interval $[a, b]$. Then the **surface area** of the solid of revolution obtained by revolving the region bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the $x$-axis about the $x$-axis is given by

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + \left[f'(x)\right]^2}\, dx. \tag{3.22}$$

**Example 3.3.17** **(Gabriel's Horn). Gabriel's horn** is the solid of revolution obtained by revolving the area of the region bounded by $y = 1/x$ and the $x$-axis for $x \geq 1$ about the $x$-axis. Show that the surface area of Gabriel's horn is infinite but that its volume is finite.

***Solution*** After defining $f(x) = 1/x$, we use ParametricPlot3D to visualize a portion of Gabriel's horn in Figure 3.49.

```
f[x_] = 1/x;
ParametricPlot3D[{r, f[r]Cos[t], f[r]Sin[t]}, {r, 1, 10}, {t, 0, 2Pi},
    PlotPoints->{40, 40}, ViewPoint->{-1.509, -2.739, 1.294}]
```

Using Eq. (3.22), the surface area of Gabriel's horn is given by the improper integral

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi \lim_{L\to\infty} \int_1^L \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx.$$

```
step1 = Integrate[2Pi f[x]Sqrt[1 + f'[x]^2], {x, 1, capl}]
```

$2\pi\text{If}\left[\frac{\text{Im[capl]}^2}{(-1+\text{Re[capl]})^2} \leq 1\&\&(\text{Re[capl]} \geq 0||\text{Im[capl]} \neq 0),\right.$

$\frac{1}{2}\left(\sqrt{2}-\text{ArcSinh[1]}\right) + \frac{\sqrt{1+\frac{1}{\text{capl}^4}}\left(-\sqrt{1+\text{capl}^4}+\text{capl}^2\text{ArcSinh}\left[\text{capl}^2\right]\right)}{2\sqrt{1+\text{capl}^4}},$

$\text{Integrate}\left[\frac{\sqrt{1+\frac{1}{x^4}}}{x}, \{x, 1, \text{capl}\},\right.$

$\left.\left.\text{Assumptions} \to !\left(\frac{\text{Im[capl]}^2}{(-1+\text{Re[capl]})^2} \leq 1\&\&(\text{Re[capl]} \geq 0||\text{Im[capl]} \neq 0)\right)\right]\right]$

```
Limit[step1, capl->Infinity]
```

$\infty$

On the other hand, using Eq. (3.20) the volume of Gabriel's horn is given by the improper integral

$$V = 2\pi \int_1^\infty \frac{1}{x^2}\,dx = \pi \lim_{L\to\infty} \int_1^L \frac{1}{x^2}\,dx,$$

which converges to $\pi$.

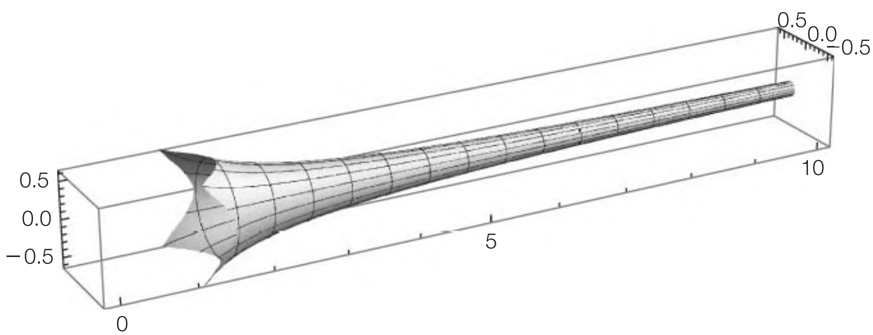



**FIGURE 3.49**

A portion of Gabriel's horn

```
step1 = Integrate[Pi f[x]^2, {x, 1, capl}]
πIf[Re[capl] ≥ 0||Im[capl] ≠ 0, 1 − 1/capl,
 Integrate[1/x^2, {x, 1, capl}, Assumptions →!(Re[capl] ≥ 0||Im[capl] ≠ 0)]]

Limit[step1, capl −>Infinity]
π

Integrate[Pi f[x]^2, {x, 1, Infinity}]
π
```

## 3.4 SERIES

### 3.4.1 Introduction to Sequences and Series

Sequences and series are usually discussed in the third quarter or second semester of introductory calculus courses. Most students find that it is one of the most difficult topics covered in calculus. A **sequence** is a function with domain consisting of the positive integers. The **terms** of the sequence $\{a_n\}$ are $a_1, a_2, a_3, \ldots$. The $n$th term is $a_n$; the $(n+1)$st term is $a_{n+1}$. If $\lim_{n\to\infty} a_n = L$, we say that $\{a_n\}$ **converges** to $L$. If $\{a_n\}$ does not converge, $\{a_n\}$ **diverges**. We can sometimes prove that a sequence converges by applying the following theorem.

**Theorem 7.** *Every bounded monotonic sequence converges.*

A sequence $\{a_n\}$ is monotonic if $\{a_n\}$ is increasing ($a_{n+1} \geq a_n$ for all $n$) or decreasing ($a_{n+1} \leq a_n$ for all $n$).

In particular, Theorem 7 gives us the following special cases.

1. If $\{a_n\}$ has positive terms and is eventually decreasing, $\{a_n\}$ converges.
2. If $\{a_n\}$ has negative terms and is eventually increasing $\{a_n\}$ converges.

After you have defined a sequence, use Table to compute the first few terms of the sequence.

1. Table[a[n],{n,1,m}] returns the list $\{a_1, a_2, a_3, \ldots, a_m\}$.
2. Table[a[n],{n,k,m}] returns $\{a_k, a_{k+1}, a_{k+2}, \ldots, a_m\}$.

**Example 3.4.1** If $a_n = \dfrac{50^n}{n!}$, show that $\lim_{n\to\infty} a_n = 0$.

**Remark 3.4** An extensive database of integer sequences can be found at the On Line Encyclopedia of Integer Sequences,

**http://www.research.att.com/~njas/sequences/Seis.html**

***Solution*** We remark that the symbol $n!$ in the denominator of $a_n$ represents the **factorial sequence**:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1.$$

We begin by defining $a_n$ and then computing the first few terms of the sequence with Table.

```
a[n_] := 50^n/n!;
afewterms = Table[a[n], {n, 1, 10}]
```

$$\left\{50, 1250, \frac{62500}{3}, \frac{781250}{3}, \frac{7812500}{3}, \frac{195312500}{9}, \frac{9765625000}{63}, \frac{61035156250}{63}, \frac{3051757812500}{567}, \frac{15258789062500}{567}\right\}$$

```
N[afewterms]
```

$\{50., 1250., 20833.3, 260417., 2.60417 \times 10^6, 2.17014 \times 10^7, 1.5501 \times 10^8, 9.68812 \times 10^8, 5.38229 \times 10^9, 2.69114 \times 10^{10}\}$

The first few terms increase in magnitude. In fact, this is further confirmed by graphing the first few terms of the sequence with ListPlot in Figure 3.50(a). Based on the graph and the values of the first few terms we might incorrectly conclude that the sequence diverges.

```
p1 = ListPlot[afewterms];
```

However, notice that $a_{n+1} = \dfrac{50}{n+1} a_n \Rightarrow \dfrac{a_{n+1}}{a_n} = \dfrac{50}{n+1}$. Because $50/(n+1) < 1$ for $n > 49$, we conclude that the sequence is decreasing for $n > 49$. Because it has positive terms, it is bounded below by $0$ so the sequence converges by Theorem 7. Let $L = \lim_{n\to\infty} a_n$. Then,

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{50}{n+1} a_n \\ L &= \lim_{n\to\infty} \frac{50}{n+1} \cdot L \\ L &= 0. \end{aligned}$$

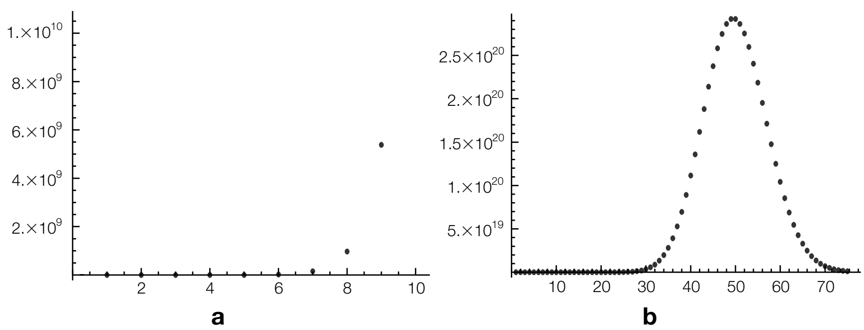



**FIGURE 3.50**

(a) The first few terms of $a_n$ (b) The first 75 terms of $a_n$

When we graph a larger number of terms, it is clear that the limit is 0. (See Figure 3.50(b).) It is a good exercise to show that for any real value of $x$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

```
p2 = ListPlot[Evaluate[Table[a[k], {k, 1, 75}]]];
Show[GraphicsRow[{p1, p2}]]
```

An **infinite series** is a series of the form

$$\sum_{k=1}^{\infty} a_k, \tag{3.23}$$

where $\{a_n\}$ is a sequence. The $n$th **partial sum** of (3.23) is

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n. \tag{3.24}$$

Notice that the partial sums of the series (3.23) form a sequence $\{s_n\}$. Hence, we say that the infinite series (3.23) **converges** to $L$ if the sequence of partial sums $\{s_n\}$ converges to $L$ and write

$$\sum_{k=1}^{\infty} a_k = L.$$

The infinite series (3.23) **diverges** if the sequence of partial sums diverges. Given the infinite series (3.23),

Sum[a[k], {k, 1, n}]

calculates the $n$th partial sum (3.24). In *some* cases, if the infinite series (3.23) converges,

Sum[a[k], {k, 1, Infinity}]

can compute the value of the infinite sum. In addition to using Sum to compute finitie and infinite sums, you can use the button on the **BasicMathInput** palette to calculate sums. You should think of the Sum function as a "fragile" command and be certain to carefully examine its results.

---

**Example 3.4.2** Determine whether each series converges or diverges. If the series converges, find its sum. (a) $\sum_{k=1}^{\infty}(-1)^{k+1}$; (b) $\sum_{k=2}^{\infty}\frac{2}{k^2-1}$ (c) $\sum_{k=0}^{\infty} ar^k$.

***Solution*** For (a), we compute the $n$th partial sum (3.24) in sn with Sum.

```
sn = Sum[(-1)^(k + 1), {k, 1, n}]
```

$\frac{1}{2}\left(1-(-1)^n\right)$

Notice that the odd partial sums are 1: $s_{2n+1} = \frac{1}{2}\left((-1)^{2n+1+1}+1\right) = \frac{1}{2}(1+1) = 1$, whereas the even partial sums are 0: $s_{2n} = \frac{1}{2}\left((-1)^{2n+1}+1\right) = \frac{1}{2}(-1+1) = 0$. We confirm that the limit of the partial sums does not exist with Limit. Mathematica's result indicates that it cannot determine the limit. The series diverges.

**Limit[sn, n → Infinity]**

$\frac{1}{2}\left(1 - e^{2i\text{Interval}[\{0,\pi\}]}\right)$

Similarly, when we attempt to compute the infinite sum with Sum, Mathematica is able to determine that the partial sums diverge, which means that the infinite series diverges.

**Sum[(−1)^(k + 1), {k, 1, Infinity}]**

Sum::div : Sum does not converge.⟩⟩

$\sum_{k=1}^{\infty}(-1)^{1+k}$

For (b), we have a *telescoping series*. Using partial fractions,

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{2}{k^2-1} &= \sum_{k=2}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)\\ &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n}\right)\\ &\quad+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\cdots,\end{aligned}$$

we see that the $n$th partial sum is given by

$$s_n = \frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}$$

and $s_n \to 3/2$ as $n \to \infty$ so the series converges to $3/2$:

$$\sum_{k=2}^{\infty}\frac{2}{k^2-1} = \frac{3}{2}.$$

Apart computes the partial fraction decomposition of a rational expression.

We perform the same steps with Mathematica using Sum, Apart, and Limit.

**sn = Sum[1/(k − 1) − 1/(k + 1), {k, 2, n}]**

$\frac{(-1+n)(2+3n)}{2n(1+n)}$

**Apart[sn]**

$\frac{3}{2}-\frac{1}{n}-\frac{1}{1+n}$

**Limit[sn, n → Infinity]**

$\frac{3}{2}$

(c) A series of the form $\sum_{k=0}^{\infty} ar^k$ is called a **geometric series**. We compute the $n$th partial sum of the geometric series with Sum.

**sn = Sum[ar^k, {k, 0, n}]**

$\frac{a\left(-1+r^{1+n}\right)}{-1+r}$

When using **Limit** to determine the limit of $s_n$ as $n \to \infty$, we see that Mathematica returns the limit unevaluated because Mathematica does not know the value of $r$.

**Limit[sn, n → Infinity]**

$\text{Limit}\left[\frac{a(-1+r^{1+n})}{-1+r}, n \to \infty\right]$

In fact, the geometric series diverges if $|r| \geq 1$ and converges if $|r| < 1$. Observe that if we simply compute the sum with **Sum**, Mathematica returns $a/(1-r)$, which is correct if $|r| < 1$ but incorrect if $|r| \geq 1$.

**Sum[ar^k, {k, 0, Infinity}]**

$\frac{a}{1-r}$

However, the result of entering

**Sum[(−5/3)^k, {k, 0, Infinity}]**

Sum::div : Sumdoesnotconverge.⟩⟩
$\sum_{k=0}^{\infty}\left(-\frac{5}{3}\right)^k$

is correct because the series $\sum_{k=0}^{\infty}\left(-\frac{5}{3}\right)^k$ is geometric with $|r| = 5/3 \geq 1$ and, consequently, diverges. Similarly,

**Sum[9(1/10)^k, {k, 1, Infinity}]**

1

is correct because $\sum_{k=1}^{\infty} 9\left(\frac{1}{10}\right)^k$ is geometric with $a = 9/10$ and $r = 1/10$ so the series converges to

$$\frac{a}{1-r} = \frac{9/10}{1-1/10} = 1.$$

---

## 3.4.2 Convergence Tests

Frequently used convergence tests are stated in the following theorems.

**Theorem 8 (The Divergence Test).** *Let $\sum_{k=1}^{\infty} a_k$ be an infinite series. If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.*

**Theorem 9 (The Integral Test).** *Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms. If $f(x)$ is a decreasing continuous function for which $f(k) = a_k$ for all k, then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.*

**Theorem 10 (The Ratio Test).** *Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms and let $\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.*

1. *If $\rho < 1$, $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $\rho > 1$, $\sum_{k=1}^{\infty} a_k$ diverges.*
3. *If $\rho = 1$, the ratio test is inconclusive.*

**Theorem 11 (The Root Test).** *Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms and let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.*

1. *If $\rho < 1$, $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $\rho > 1$, $\sum_{k=1}^{\infty} a_k$ diverges.*
3. *If $\rho = 1$, the root test is inconclusive.*

**Theorem 12 (The Limit Comparison Test).** *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be infinite series with positive terms and let $L = \lim_{k\to\infty} \frac{a_k}{b_k}$. If $0 < L < \infty$, then either both series converge or both series diverge.*

---

**Example 3.4.3** Determine whether each series converges or diverges. (a) $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$; (b) $\sum_{k=1}^{\infty} \frac{1}{k^p}$; (c) $\sum_{k=1}^{\infty} \frac{k}{3^k}$; (d) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$; (e) $\sum_{k=1}^{\infty} \left(\frac{k}{4k+1}\right)^k$; and (f) $\sum_{k=1}^{\infty} \frac{2\sqrt{k}+1}{(\sqrt{k}+1)(2k+1)}$.

***Solution*** (a) Using **Limit**, we see that the limit of the terms is $e \neq 0$ so the series diverges by the the divergence test, Theorem 8.

```
Limit[(1 + 1/k)^k, k → Infinity]
e
```

It is a very good exercise to show that the limit of the terms of the series is $e$ by hand. Let $L = \lim_{k\to\infty} (1+1/k)^k$. Take the logarithm of each side of this equation and apply L'Hôpital's rule:

$$\ln L = \lim_{k\to\infty} \ln\left(1+\frac{1}{k}\right)^k$$

$$\ln L = \lim_{k\to\infty} k \ln\left(1+\frac{1}{k}\right)$$

$$\ln L = \lim_{k\to\infty} \frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}}$$

$$\ln L = \lim_{k\to\infty} \frac{\frac{1}{1+\frac{1}{k}} \cdot -\frac{1}{k^2}}{-\frac{1}{k^2}}$$

$$\ln L = 1.$$

Exponentiating yields $L = e^{\ln L} = e^1 = e$. (b) A series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ $(p > 0)$ is called a ***p*-series**. Let $f(x) = x^{-p}$. Then, $f(x)$ is continuous and decreasing for $x \geq 1$, $f,(k) = k^{-p}$ and

$$\int_1^{\infty} x^{-p}dx = \begin{cases} \infty, & \text{if } p \le 1 \\ 1/(p-1), & \text{if } p > 1 \end{cases}$$

so the $p$-series converges if $p > 1$ and diverges if $p \le 1$. If $p = 1$, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the **harmonic series**.

**s1 = Integrate[x^(−p), {x, 1, Infinity}]**

If $\left[\text{Re}[p] > 1, \frac{1}{-1+p}, \text{Integrate }\left[x^{-p}, \{x, 1, \infty\}, \text{Assumptions } \rightarrow \text{Re}[p] \le 1\right]\right]$

(c) Let $f(x) = x \cdot 3^{-x}$. Then, $f(k) = k \cdot 3^{-k}$ and $f(x)$ is decreasing for $x > 1/\ln 3$.

**f[x_] = x3^(−x);**

**Factor[f′[x]]**

$-3^{-x}(-1 + x\text{Log}[3])$

**Solve[−1 + xLog[3]==0, x]**

$\left\{\left\{x \rightarrow \frac{1}{\text{Log}[3]}\right\}\right\}$

Using Integrate, we see that the improper integral $\int_1^{\infty} f(x)\, dx$ converges.

**ival = Integrate[f[x], {x, 1, Infinity}]**

**N[ival]**

$\frac{1+\text{Log}[3]}{3\text{Log}[3]^2}$

0.579592

Thus, by the integral test, Theorem 9, we conclude that the series converges. Note that when applying the integral test, if the improper integral converges, its value is *not* the value of the sum of the series. In this case, we see that Mathematica is able to evaluate the sum with Sum and the series converges to $3/4$.

**Sum[k3^(−k), {k, 1, Infinity}]**

$\frac{3}{4}$

(d) If $a_k$ contains factorial functions, the ratio test is a good first test to try. After defining $a_k$ we compute

$$\begin{aligned} \lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{\dfrac{[(k+1)!]^2}{[2(k+1)]}}{\dfrac{(k!)^2}{(2k)!}} \\ &= \lim_{k\to\infty} \frac{(k+1)! \cdot (k+1)!}{k! \cdot k!} \frac{(2k)!}{(2k+2)!} \\ &= \lim_{k\to\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = \lim_{k\to\infty} \frac{(k+1)}{2(2k+1)} = \frac{1}{4}. \end{aligned}$$

Because $1/4 < 1$, the series converges by the ratio test. We confirm these results with Mathematica.

**Remark 3.5** Use FullSimplify instead of Simplify to simplify expressions involving factorials.

```
a[k_] = (k!)^2/(2k)!;
s1 = FullSimplify[a[k + 1]/a[k]]
```
$\frac{1+k}{2+4k}$

```
Limit[s1, k → Infinity]
```
$\frac{1}{4}$

We illustrate that we can evaluate the sum using Sum and approximate it with N as follows.

```
ev = Sum[a[k], {k, 1, Infinity}]
```
$\frac{1}{27}\left(9+2\sqrt{3}\pi\right)$

```
N[ev]
```
0.7364

(e) Because

$$\lim_{k\to\infty}\sqrt[k]{\left(\frac{k}{4k+1}\right)^k}=\lim_{k\to\infty}\frac{k}{4k+1}=\frac{1}{4}<1,$$

the series converges by the root test.

```
a[k_] = (k/(4k + 1))^k;
Limit[a[k]^(1/k), k → Infinity]
```
$\frac{1}{4}$

As with (d), we can approximate the sum with N and Sum.

```
ev = Sum[a[k], {k, 1, Infinity}]
```
$\sum_{k=1}^{\infty}\left(\frac{k}{1+4k}\right)^k$

```
N[ev]
```
0.265757

(f) We use the limit comparison test and compare the series to $\sum_{k=1}^{\infty}\frac{\sqrt{k}}{k\sqrt{k}}=\sum_{k=1}^{\infty}\frac{1}{k}$, which diverges because it is a $p$-series with $p=1$. Because

$$0<\lim_{k\to\infty}\frac{\frac{2\sqrt{k}+1}{(\sqrt{k}+1)(2k+1)}}{\frac{1}{k}}=1<\infty$$

and the harmonic series diverges, the series diverges by the limit comparison test.

```
a[k_] = (2Sqrt[k] + 1)/((Sqrt[k] + 1)(2k + 1));
b[k_] = 1/k;
Limit[a[k]/b[k], k → Infinity]
```
1

### 3.4.3 **Alternating Series**

An **alternating series** is a series of the form

$$\sum_{k=1}^{\infty}(-1)^k a_k \qquad \text{or} \qquad \sum_{k=1}^{\infty}(-1)^{k+1} a_k \tag{3.25}$$

where $\{a_k\}$ is a sequence with positive terms.

**Theorem 13 (Alternating Series Test).** *If* $\{a_k\}$ *is decreasing and* $\lim_{k\to\infty} a_k = 0$*, the alternating series* (3.25) *converges.*

The alternating series (3.25) **converges absolutely** if $\sum_{k=1}^{\infty} a_k$ converges.

**Theorem 14.** *If the alternating series* (3.25) *converges absolutely, it converges.*

If the alternating series (3.25) converges but does not converge absolutely, we say that it **conditionally converges**.

---

**Example 3.4.4** Determine whether each series converges or diverges. If the series converges, determine whether the convergence is conditional or absolute. (a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$; (b) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(k+1)!}{4^k(k!)^2}$; and (c) $\sum_{k=1}^{\infty}(-1)^{k+1}\left(1+\frac{1}{k}\right)^k$.

***Solution*** (a) Because $\{1/k\}$ is decreasing and $1/k \to 0$ as $k \to \infty$, the series converges. The series does not converge absolutely because the harmonic series diverges. Hence, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, which is called the **alternating harmonic series**, converges conditionally. We see that this series converges to ln 2 with Sum.

```
a[k_] = (-1)^(k + 1)/k;
Sum[a[k], {k, 1, Infinity}]
Log[2]
```

(b) We test for absolute convergence first using the ratio test. Because

$$\lim_{k\to\infty} \frac{\dfrac{((k+1)+1)!}{4^{k+1}[(k+1)!]^2}}{\dfrac{(k+1)!}{4^k(k!)^2}} = \lim_{k\to\infty} \frac{k+2}{4(k+1)^2} = 0 < 1,$$

```
a[k_] = (k + 1)!/(4^k(k!)^2);
s1 = FullSimplify[a[k + 1]/a[k]]
Limit[s1, k → Infinity]
(2+k)/(4(1+k)^2)
0
```

the series converges absolutely by the ratio test. Absolute convergence implies convergence so the series converges. (c) Because $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e$,

$\lim_{k\to\infty}(-1)^{k+1}\left(1+\frac{1}{k}\right)^k$ does not exist, so the series diverges by the divergence test. We confirm that the limit of the terms is not zero with **Limit**.

**Sum[(−1)^(k + 1)a[k], {k, 1, Infinity}]**

$\frac{-3+4e^{1/4}}{4e^{1/4}}$

**a[k_] = (−1)^(k + 1)(1 + 1/k)^k;**

**Sum[a[k], {k, 1, Infinity}]**

Sum::div : Sum does not converge.⟩⟩

$\sum_{k=1}^{\infty}(-1)^{1+k}\left(1+\frac{1}{k}\right)^k$

**Limit[a[k], k → Infinity]**

$-e^{2i\text{Interval}[\{0,\pi\}]}$

---

## 3.4.4 Power Series

Let $x_0$ be a number. A **power series** in $x - x_0$ is a series of the form

$$\sum_{k=0}^{\infty} a_k \left(x - x_0\right)^k . \tag{3.26}$$

A fundamental problem is determining the values of $x$, if any, for which the power series converges, the **interval of convergence**.

**Theorem 15.** *For the power series* (3.26), *exactly one of the following is true.*

1. *The power series converges absolutely for all values of $x$. The interval of convergence is $(-\infty, \infty)$.*
2. *There is a positive number $r$ so that the series converges absolutely if $x_0 - r < x < x_0 + r$. The series may or may not converge at $x = x_0 - r$ and $x = x_0 + r$. The interval of convergence will be one of $\left(x_0 - r, x_0 + r\right)$, $\left[x_0 - r, x_0 + r\right)$, $\left(x_0 - r, x_0 + r\right]$, or $\left[x_0 - r, x_0 + r\right]$.*
3. *The series converges only if $x = x_0$. The interval of convergence is $\{x_0\}$.*

---

**Example 3.4.5** Determine the interval of convergence for each of the following power series. (a) $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$; (b) $\sum_{k=0}^{\infty} \frac{k!}{1000^k}(x-1)^k$; and (c) $\sum_{k=1}^{\infty} \frac{2^k}{\sqrt{k}}(x-4)^k$.

***Solution*** (a) We test for absolute convergence first using the ratio test. Because

$$\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}}{(2(k+1)+1)!} x^{2(k+1)+1}}{\frac{(-1)^k}{(2k+1)!} x^{2k+1}} \right| = \lim_{k\to\infty} \frac{1}{2(k+1)(2k+3)} x^2 = 0 < 1$$

```
a[x_, k_] = (-1)^k/(2k + 1)!x^(2k + 1);
s1 = FullSimplify[a[x, k + 1]/a[x, k]]
Limit[s1, k → Infinity]
```

$-\frac{x^2}{6+10k+4k^2}$

0

for all values of $x$, we conclude that the series converges absolutely for all values of $x$; the interval of convergence is $(-\infty, \infty)$. In fact, we will see later that this series converges to $\sin x$:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots,$$

which means that the partial sums of the series converge to $\sin x$. Graphically, we can visualize this by graphing partial sums of the series together with the graph of $y = \sin x$. Note that the partial sums of a series are a recursively defined function: $s_n = s_{n-1} + a_n$, $s_0 = a_0$. We use this observation to define p to be the $n$th partial sum of the series. We use the form p[x_,n_]:=p[x,n]=... so that Mathematica remembers the partial sums computed. That is, once p[x,3] is computed, Mathematica need not recompute p[x,3] when computing p[x,4].

In Figure 3.51 (top) we graph $p_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ together with $y = \sin x$ for $n = 1$, 5, and 10. In the graphs, notice that as $n$ increases, the graphs of $p_n(x)$ more closely resemble the graph of $y = \sin x$.

When you use Tooltip, placing the cursor over the plot shows you the function being plotted.

```
Clear[p]
p[x_, 0] = a[x, 0];
p[x_, n_] := p[x, n] = p[x, n - 1] + a[x, n]
p[x, 2]
```

$x - \frac{x^3}{6} + \frac{x^5}{120}$

```
p1 = Plot[Tooltip[{Sin[x], p[x, 1], p[x, 5], p[x, 10]}],
{x, -2Pi, 2Pi}, PlotRange → {-Pi, Pi}, AspectRatio → Automatic];
```

We use Manipulate to investigate how $n$ affects the situation with

```
p2 = Manipulate[Plot[Tooltip[{Sin[x], p[x, n]}],
   {x, -4Pi, 4Pi}, PlotRange → {-Pi, Pi}, AspectRatio → Automatic],
     {{n, 5}, 1, 25, 1}];
Show[GraphicsColumn[{p1, p2}]]
```

(b) As in (a), we test for absolute convergence first using the ratio test:

$$\lim_{k\to\infty} \left| \frac{\frac{(k+1)k!}{1000^{k+1}}(x-1)^{k+1}}{\frac{k!}{1000^k}(x-1)^k} \right| = \frac{1}{1000}(k+1)|x-1| = \begin{cases} 0, & \text{if } x = 1 \\ \infty, & \text{if } x \neq 1. \end{cases}$$

```
a[x_, k_] = k!/1000^k(x - 1)^k;
s1 = FullSimplify[a[x, k + 1]/a[x, k]]
Limit[s1, k → Infinity]
```

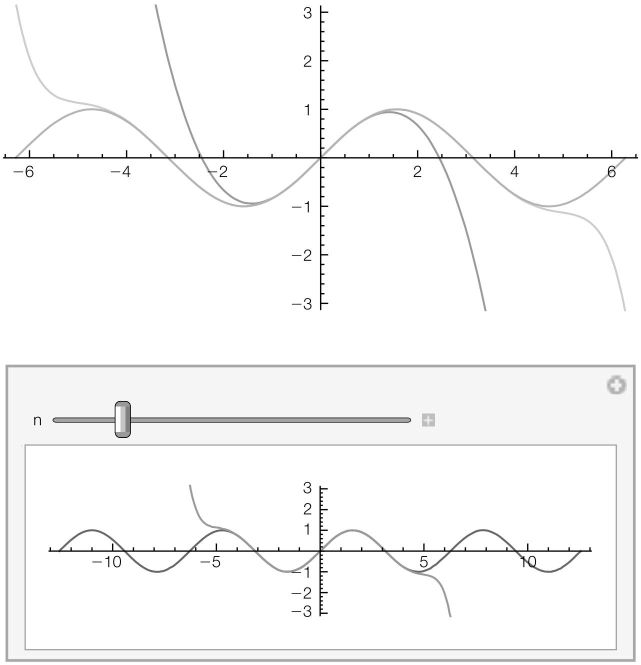



**FIGURE 3.51**

(Top) $y = \sin x$ together with the graphs of $p_1(x)$, $p_5(x)$, and $p_{10}(x)$. (Bottom) Using Manipulate to investigate the situation

$$\frac{(1+k)(-1+x)}{1000}$$
$$(-1+x)\infty$$

Be careful of your interpretation of the result of the **Limit** command because Mathematica does not consider the case $x = 1$ separately: If $x = 1$, the limit is $0$. Because $0 < 1$, the series converges by the ratio test.

The series converges only if $x = 1$; the interval of convergence is $\{1\}$. You should observe that if you graph several partial sums for "small" values of $n$, you might incorrectly conclude that the series converges.

(c) Use the ratio test to check absolute convergence first:

$$\lim_{k\to\infty}\left|\frac{\dfrac{2^{k+1}}{\sqrt{k+1}}(x-4)^{k+1}}{\dfrac{2^k}{\sqrt{k}}(x-4)^k}\right| = \lim_{k\to\infty} 2\sqrt{\frac{k}{k+1}}|x-4| = 2|x-4|.$$

By the ratio test, the series converges absolutely if $2|x-4| < 1$. We solve this inequality for $x$ with **Reduce** to see that $2|x-4| < 1$ if $7/2 < x < 9/2$.

```
Clear[a, s1, k]
a[x_, k_] = 2^k/Sqrt[k](x-4)^k;
```

```
s1 = Simplify[Abs[a[x, k + 1]/a[x, k]]]
Limit[s1, k → Infinity]
```

$2\text{Abs}\left[\sqrt{\frac{k}{1+k}}(-4+x)\right]$

$2\text{Abs}[-4+x]$

```
Reduce[2Abs[x-4] < 1, x]
```

$\frac{7}{2} < \text{Re}[x] < \frac{9}{2} \&\& -\frac{1}{2}\sqrt{-63+32\text{Re}[x]-4\text{Re}[x]^2} < \text{Im}[x] <$
$\frac{1}{2}\sqrt{-63+32\text{Re}[x]-4\text{Re}[x]^2}$

From the output, we see that for real values of $x$, the inequality is satisfied for $7/2 < x < 9/2$. We check $x = 7/2$ and $x = 9/2$ separately. If $x = 7/2$, the series becomes $\sum_{k=1}^{\infty}(-1)^k \frac{1}{\sqrt{k}}$, which converges conditionally.

```
Simplify[a[x, k]/.x → 7/2]
```

$\frac{(-1)^k}{\sqrt{k}}$

On the other hand, if $x = 9/2$,

```
Simplify[a[x, k]/.x → 9/2]
```

$\frac{1}{\sqrt{k}}$

the series is $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which diverges. We conclude that the interval of convergence is $[7/2, 9/2)$.

## 3.4.5 Taylor and Maclaurin Series

Let $y = f(x)$ be a function with derivatives of all orders at $x = x_0$. The **Taylor series** for $f(x)$ about $x = x_0$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{3.27}$$

The **Maclaurin series** for $f(x)$ is the Taylor series for $f(x)$ about $x = 0$. If $y = f(x)$ has derivatives up to at least order $n$ at $x = x_0$, the $n$th degree **Taylor polynomial** for $f(x)$ about $x = x_0$ is

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{3.28}$$

The $n$th degree **Maclaurin polynomial** for $f(x)$ is the $n$th degree Taylor polynomial for $f(x)$ about $x = 0$. Generally, finding Taylor and Maclaurin series using the definition is a tedious task at best.

**Example 3.4.6** Find the first few terms of (a) the Maclaurin series and (b) the Taylor series about $x = \pi/4$ for $f(x) = \tan x$.

***Solution*** (a) After defining $f(x) = \tan x$, we use **Table** together with **/.** and **D** to compute $f^{(k)}(0)/k!$ for $k = 0, 1, \ldots, 8$.

Use **Short** to obtain an abbreviated result. Many terms will be missing, but with **Short**, you will see the beginning and end of your result.

```
f[x_] = Tan[x];
t1 = Table[{k, D[f[x], {x, k}], D[f[x], {x, k}]/.x → 0}, {k, 0, 8}];
Short[t1]
```

{{0, Tan[x], 0}, ⟨⟨7⟩⟩, {8, 7936Sec[x]$^8$Tan[x] + ⟨⟨1⟩⟩ + ⟨⟨1⟩⟩
+128⟨⟨1⟩⟩$^2$Tan[x]$^7$, 0}}

To see these results in tabular form, enter

```
t1//TableForm
```

For length considerations, the resulting output is not shown here. Another way of approaching the problem is to use **Manipulate**. See Figures 3.52 and 3.53.

```
Manipulate[{k, D[f[x], {x, k}]//FullSimplify, D[f[x], {x, k}]/.x → 0,
        D[f[x], {x, k}]/.x → 0//N}, {{k, 5}, 0, 25, 1}]
```

Using the values in the table or from the **Manipulate** object, we apply the definition to see that the Maclaurin series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots$$

For (b), we repeat (a) using $x = \pi/4$ instead of $x = 0$

```
Manipulate[{k, D[f[x], {x, k}]//FullSimplify, D[f[x], {x, k}]/.x → Pi/4,
    D[f[x], {x, k}]/.x → Pi/4//N}, {{k, 1}, 0, 25, 1}]
```

and then apply the definition to see that the Taylor series about $x = \pi/4$ is

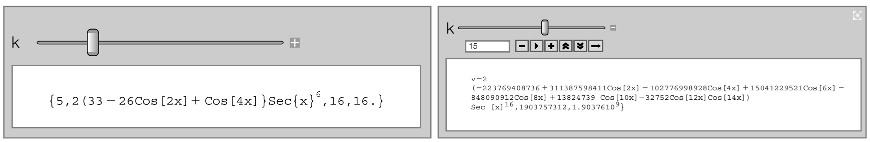



**FIGURE 3.52**

With **Manipulate**, we can adjust the function and function values

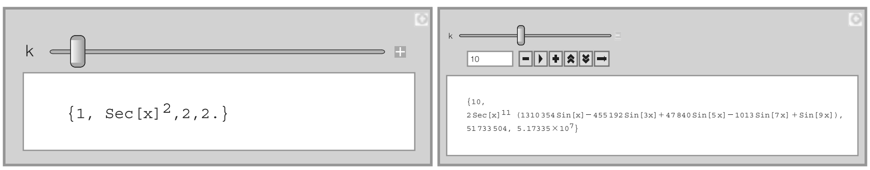



**FIGURE 3.53**

We use **Manipulate** to investigate series for the tangent function

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = 1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x-\frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x-\frac{\pi}{4}\right)^4 + \frac{64}{15}\left(x-\frac{\pi}{4}\right)^5 + \frac{244}{45}\left(x-\frac{\pi}{4}\right)^6 + \ldots$$

From the series, we can see various Taylor and Maclaurin polynomials. For example, the third Maclaurin polynomial is

$$p_3(x) = x + \frac{1}{3}x^3$$

and the fourth degree Taylor polynomial about $x = \pi/4$ is

$$p_4(x) = 1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x-\frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x-\frac{\pi}{4}\right)^4.$$

---

**The command Series[f[x],{x,x0,n}] computes (3.27) to (at least) order $n-1$. Because of the $O$-term in the result that represents the terms that are omitted from the power series for $f(x)$ expanded about the point $x = x_0$, the result of entering a Series command is not a function that can be evaluated if $x$ is a particular number. We remove the remainder ($O$-) term of the power series Series[f[x],{x,x0,n}] with the command Normal and can then evaluate the resulting polynomial for particular values of $x$.**

---

**Example 3.4.7** Find the first few terms of the Taylor series for $f(x)$ about $x = x_0$. (a) $f(x) = \cos x$, $x = 0$; (b) $f(x) = 1/x^2$, $x = 1$.

***Solution*** Entering

**Series[Cos[x], {x, 0, 4}]**

$1 - \frac{x^2}{2} + \frac{x^4}{24} + O[x]^5$

computes the Maclaurin series to order 4. Entering

**Series[Cos[x], {x, 0, 14}]**

$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} - \frac{x^{14}}{87178291200} + O[x]^{15}$

computes the Maclaurin series to order 14. In this case, the Maclaurin series for $\cos x$ converges to $\cos x$ for all real $x$. To graphically see this, we define the function p. Given $n$, p[n] returns the Maclaurin polynomial of degree $n$ for $\cos x$.

**p[n_] := Series[Cos[x], {x, 0, n}]//Normal**
**p[3]**

$1 - \frac{x^2}{2}$

We then graph $\cos x$ together with the Maclaurin polynomial of degree $n = 2$, 4, 8, and 16 on the interval $[-3\pi/2, 3\pi/2]$ in Figure 3.54. Notice that as $n$ increases, the graph of the Maclaurin polynomial more closely resembles the graph of $\cos x$.

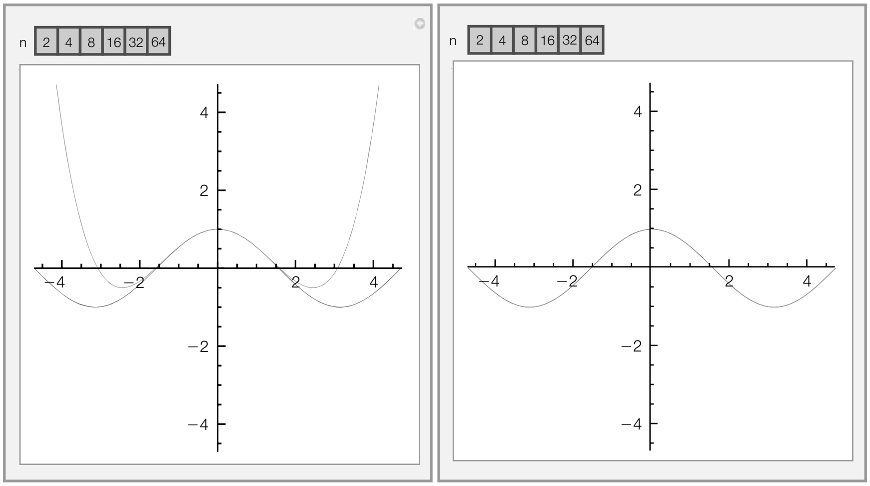



**FIGURE 3.54**

Using **Manipulate** to investigate graphs of $y = \cos x$ together with plots of several of its Maclaurin polynomials

We would see the same pattern if we increased the length of the interval and the value of $n$.

```
Manipulate[Plot[Evaluate[Tooltip[{Cos[x], p[n]}]],
        {x, −3Pi/2, 3Pi/2}, PlotRange → {{−3Pi/2, 3Pi/2}, {−3Pi/2, 3Pi/2}},
            AspectRatio → Automatic],
                {{n, 4}, {2, 4, 8, 16, 32, 64}, ControlType → Setter}]
```

(b) After defining $f(x) = 1/x^2$, we compute the first 10 terms of the Taylor series for $f(x)$ about $x = 1$ with **Series**.

```
f[x_] = 1/x^2;
p10 = Series[f[x], {x, 1, 10}]
```

$1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4-6(x-1)^5+7(x-1)^6-8(x-1)^7$
$\quad +9(x-1)^8-10(x-1)^9+11(x-1)^{10}+O[x-1]^{11}$

$1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4-6(x-1)^5+7(x-1)^6-8(x-1)^7$
$\quad +9(x-1)^8-10(x-1)^9+11(x-1)^{10}+O[x-1]^{11}$

In this case, the pattern for the series is relatively easy to see: The Taylor series for $f(x)$ about $x = 1$ is

$$\sum_{k=0}^{\infty}(-1)^k(k+1)(x-1)^k.$$

This series converges absolutely if

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}(k+2)(x-1)^{k+1}}{(-1)^k(k+1)(x-1)^k}\right| = |x-1| < 1$$

or $0 < x < 2$. The series diverges if $x = 0$ and $x = 2$. In this case, the series converges to $f(x)$ on the interval $(0, 2)$.

```
a[x_, k_] = (-1)^k(k + 1)(x - 1)^k;
s1 = FullSimplify[Abs[a[x, k + 1]/a[x, k]]]
Abs[((2+k)(-1+x))/(1+k)]

s2 = Limit[s1, k → Infinity]
Abs[-1 + x]

Reduce[s2 < 1, x]
0 < Re[x] < 2&&-√(2Re[x]-Re[x]^2) < Im[x] < √(2Re[x]-Re[x]^2)
```

To see this, we use **Manipulate** graph $f(x)$ together with the Taylor polynomial for $f(x)$ about $x = 1$ of degree $n$ for large $n$. Regardless of the size of $n$, the graphs of $f(x)$ and the Taylor polynomial closely resemble each other on the interval $(0, 2)$ — but not at the endpoints or outside the interval. (See Figure 3.55.)

```
p[n_] := Series[f[x], {x, 1, n}]//Normal

Manipulate[Plot[Evaluate[Tooltip[{f[x], p[n]}]],
    {x, 0, 3}, PlotRange → {{0, 3}, {-3/2, 3/2}}, AspectRatio → Automatic],
    {{n, 10}, {5, 10, 15, 25, 30, 35, 40, 45, 50, 55, 60}, ControlType → Setter}]
```

---

## 3.4.6 Taylor's Theorem

Taylor's theorem states the relationship between $f(x)$ and the Taylor series for $f(x)$ about $x = x_0$.

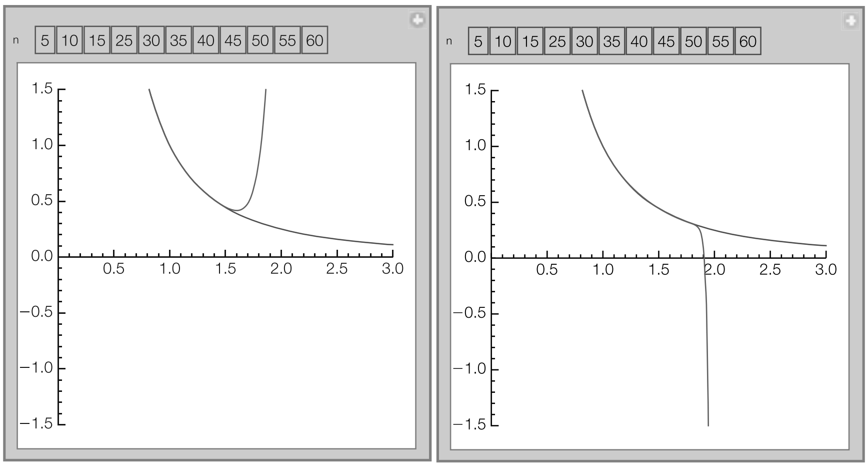



**FIGURE 3.55**

Graphs of $f(x)$ together with the various Taylor polynomials about $x = 1$

**Theorem 16 (Taylor's Theorem).** *Let $y = f(x)$ have (at least) $n+1$ derivatives on an interval $I$ containing $x = x_0$. Then, for every number $x \in I$, there is a number $z$ between $x$ and $x_0$ so that*

$$f(x) = p_n(x) + R_n(x),$$

*where $p_n(x)$ is given by equation (3.28) and*

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}\left(x - x_0\right)^{n+1}. \tag{3.29}$$

---

**Example 3.4.8** Use Taylor's theorem to show that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

***Solution*** Let $f(x) = \sin x$. Then, for each value of $x$, there is a number $z$ between $0$ and $x$ so that $\sin x = p_n(x) + R_n(x)$, where $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ and $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$. Regardless of the value of $n$, $f^{(n+1)}(z)$ is one of $\sin z$, $-\sin z$, $\cos z$, or $-\cos z$, which are all bounded by $1$. Then,

$$\left|\sin x - p_n(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}\right|$$

$$\left|\sin x - p_n(x)\right| \le \frac{1}{(n+1)!} |x|^{n+1}$$

and $x^n/n! \to 0$ as $n \to \infty$ for all real values of $x$.

You should remember that the number $z$ in $R_n(x)$ is guaranteed to exist by Taylor's theorem. However, from a practical standpoint, you would rarely (if ever) need to compute the $z$ value for a particular $x$ value.

For illustrative purposes, we show the difficulties. Suppose we wish to approximate $\sin \pi/180$ using the Maclaurin polynomial of degree $4$, $p_4(x) = x - \frac{1}{6}x^3$, for $\sin x$. The fourth remainder is $R_4(x) = \frac{1}{120} \cos z\, x^5$.

The Maclaurin polynomial of degree 4 for $\sin x$ is $\sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^4 = 0 + x + 0 \cdot x^2 + \frac{-1}{3!} x^3 + 0 \cdot x^4$.

```
Clear[f]
f[x_] = Sin[x];
r5 = D[f[z], {z, 5}]/5!x^5
```

$\frac{1}{120}x^5\text{Cos}[z]$

If $x = \pi/180$, there is a number $z$ between $0$ and $\pi/180$ so that

$$\left|R_4\left(\frac{\pi}{180}\right)\right| = \frac{1}{120} \cos z \left(\frac{\pi}{180}\right)^5$$
$$\le \frac{1}{120}\left(\frac{\pi}{180}\right)^5 \approx 0.135 \times 10^{-10},$$

which shows us that the maximum the error can be is $\frac{1}{120}\left(\frac{\pi}{180}\right)^5 \approx 0.135 \times 10^{-10}$.

**maxerror = N[1/120*(Pi/180)^5]**

1.349601623163255`*^-11

Abstractly, the exact error can be computed. By Taylor's theorem, $z$ satisfies

$$f\left(\frac{\pi}{180}\right) = p_4\left(\frac{\pi}{180}\right) + R_4\left(\frac{\pi}{180}\right)$$

$$\sin\frac{\pi}{180} = \frac{1}{180}\pi - \frac{1}{34992000}\pi^3 + \frac{1}{22674816000000}\pi^5\cos z$$

$$0 = \frac{1}{180}\pi - \frac{1}{34992000}\pi^3 + \frac{1}{22674816000000}\pi^5\cos z - \sin\frac{\pi}{180}.$$

We graph the right-hand side of this equation with Plot in Figure 3.56. The exact value of $z$ is the $z$-coordinate of the point where the graph intersects the $z$-axis.

**p4 = Series[f[x], {x, 0, 4}]//Normal**

$x - \frac{x^3}{6}$

**exval = Sin[Pi/180]**
**p4b = p4/.x → Pi/180**
**r5b = r5/.x → Pi/180**

$\text{Sin}\left[\frac{\pi}{180}\right]$

$\frac{\pi}{180} - \frac{\pi^3}{34992000}$

$\frac{\pi^5 \text{Cos}[z]}{22674816000000}$

**toplot = r5b + p4b − exval;**
**Plot[toplot, {z, 0, Pi/180}]**

We can use FindRoot to approximate $z$, if we increase the number of digits carried in floating point calculations with WorkingPrecision.

**exz = FindRoot[toplot==0, {z, 0, .004}, WorkingPrecision → 32]**

{z → 0.0038086149165541606417429516417308}

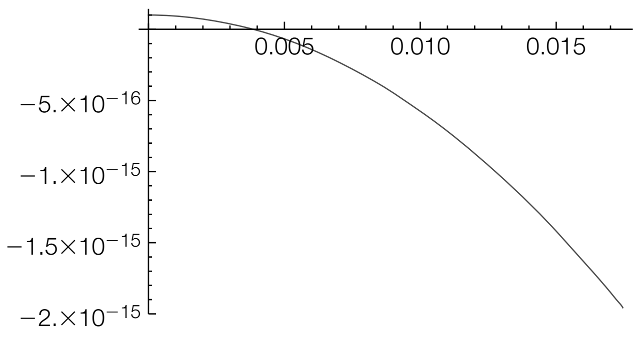



**FIGURE 3.56**

Finding $z$

Alternatively, we can compute the exact value of $z$ with Solve

Because Mathematica uses inverse functions in this calculation, it issues several warning messages that we have omitted for length considerations.

**cz = Solve[toplot==0, z]**

$$\left\{\left\{z \to \text{ArcCos}\left[\frac{648000\left(-194400\pi+\pi^3+34992000\text{Sin}\left[\frac{\pi}{180}\right]\right)}{\pi^5}\right]\right\},\right.$$
$$\left.\left\{z \to \text{ArcCos}\left[\frac{648000\left(-194400\pi+\pi^3+34992000\text{Sin}\left[\frac{\pi}{180}\right]\right)}{\pi^5}\right]\right\}\right\}$$

and then approximate the result with N.

**N[cz]**

$\{\{z \to -0.00384232\}, \{z \to 0.00384232\}\}$

---

## 3.4.7 Other Series

In calculus, we learn that the power series $f(x) = \sum_{k=0}^{\infty} a_k \left(x - x_0\right)^k$ is differentiable and integrable on its interval of convergence. However, for series, that are not power series, this result is not generally true. For example, in more advanced courses, we learn that the function

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sin\left(3^k x\right)$$

is continuous for all values of $x$ but nowhere differentiable. We can use Mathematica to help us see why this function is not differentiable. Let

$$f_n(x) = \sum_{k=0}^{n} \frac{1}{2^k} \sin\left(3^k x\right).$$

Notice that $f_n(x)$ is defined recursively by $f_0(x) = \sin x$ and $f_n(x) = f_{n-1}(x) + \frac{1}{2^n} \sin\left(3^n x\right)$. We use Mathematica to recursively define $f_n(x)$.

```
Clear[f]
f[0] = Sin[x];
f[k_] := f[k] = f[k - 1] + Sin[3^k x]/2^k
```

We define $f_n(x)$ using the form

```
f[n_] := f[n] = ...
```

so that Mathematica "remembers" the values it computes. Thus, to compute f[5], Mathematica uses the previously computed values, namely f[4], to compute f[5]. Note that we can produce the same results by defining $f_n(x)$ with the command

f[n_] := . . .

However, the disadvantage of defining $f_n(x)$ in this manner is that Mathematica does not "remember" the previously computed values and thus takes longer to compute $f_n(x)$ for larger values of $n$.

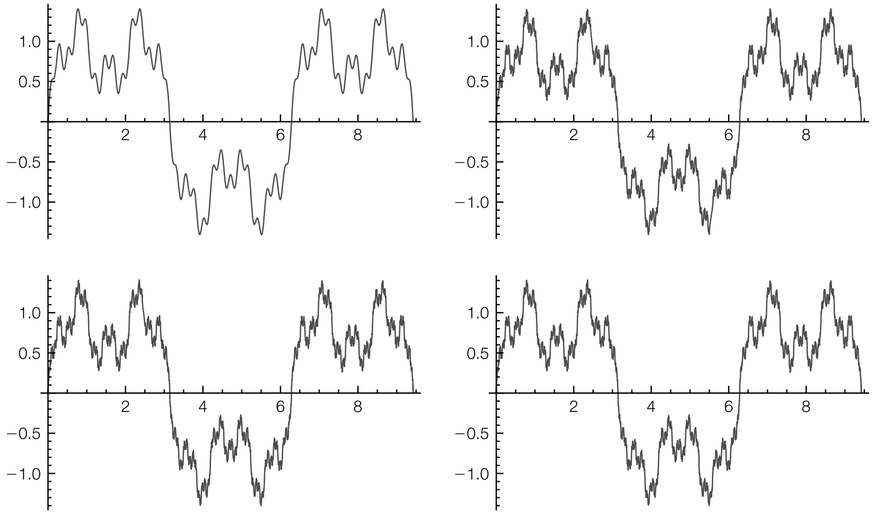



**FIGURE 3.57**

Approximating a function that is continuous everywhere but nowhere differentiable

Next, we use Table to generate $f_3(x)$, $f_6(x)$, $f_9(x)$, and $f_{12}(x)$.

```
tograph = Table[f[n], {n, 3, 12, 3}];
```

We now graph each of these functions and show the results as a graphics array with GraphicsGrid in Figure 3.57. (Note that you do not need to include the option DisplayFunction->Identity to suppress the resulting output unless you forget to include the semicolon at the end of the command.)

```
graphs = Table[Plot[Evaluate[tograph[[i]]], {x, 0, 3π},
        DisplayFunction → Identity], {i, 1, 4}];
toshow = Partition[graphs, 2];
Show[GraphicsGrid[toshow]]
```

From these graphs, we see that for large values of $n$, the graph of $f_n(x)$, although actually smooth, appears "jagged" and thus we might suspect that $f(x) = \lim_{n\to\infty} f_n(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sin\left(3^k x\right)$ is indeed continuous everywhere but nowhere differentiable.

## 3.5 MULTIVARIABLE CALCULUS

Mathematica is useful in investigating functions involving more than one variable. In particular, the graphical analysis of functions that depend on two (or more) variables is enhanced with the help of Mathematica's graphics capabilities.

### 3.5.1 Limits of Functions of Two Variables

**Mathematica's graphics and numerical capabilities are helpful in investigating limits of functions of two variables.**

---

**Example 3.5.1** Show that the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

***Solution*** We begin by defining $f(x,y) = (x^2-y^2)/(x^2+y^2)$. Next, we use Plot3D to graph $z = f(x,y)$ for $-1/2 \le x \le 1/2$ and $-1/2 \le y \le 1/2$. ContourPlot is used to graph several level curves on the same rectangle. (See Figure 3.58.) (To define a function of two variables, $f(x,y) = \textit{expression in } x \textit{ and } y$, enter f[x_,y_] = expression in x and y. Plot3D[f[x,y], {a,x,b},{y,c,d}] generates a basic graph of $z = f(x,y)$ for $a \le x \le b$ and $c \le y \le d$)

```
Clear[f]
f[x_, y_] = (x^2−y^2)/(x^2 + y^2);
p1 = Plot3D[f[x, y], {x, −2, 2}, {y, −2, 2}, PlotPoints → 40];
c1 = ContourPlot[f[x, y], {x, −2, 2}, {y, −2, 2}, ContourShading → False,
  Axes → Automatic, AxesOrigin → {0, 0}, PlotPoints → 60, Contours → 20];
Show[GraphicsRow[{p1, c1}]]
```

When you slide the cursor over the contours in the contour plot, the contour values are displayed.

From the graph of the level curves, we suspect that the limit does not exist because we see that near $(0,0)$, $z = f(x,y)$ attains many different values. We obtain further evidence that the limit does not exist by computing the value of $z = f(x,y)$ for various points chosen randomly near $(0,0)$. We use Table and RandomReal to generate 10 ordered pairs $(x,y)$ for $x$ and $y$ "close to" 0. Because RandomReal is included in the calculation, your results will almost certainly be different from those here.

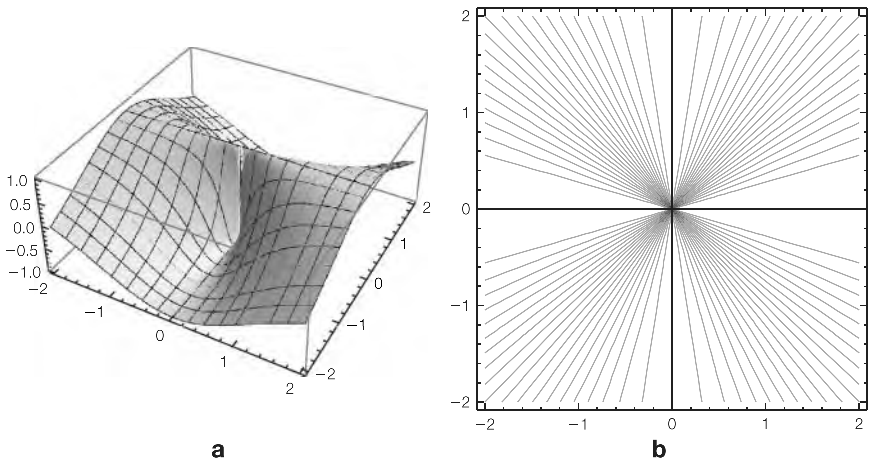



**FIGURE 3.58**

(a) Three-dimensional and (b) contour plots of $f(x,y)$

```
pts = Table[RandomReal[{-10^-i, 10^-i}], {i, 1, 10}, {2}]
{{0.08560753251471961, -0.06804168963904592},
{0.004371826092799417, 0.005941850437676466},
{-0.0009189852964073855, 0.0009026088173797l},
{-0.00006532557511289984, 0.00006629693177194204},
{-9.419651789936537*^-6, -1.2215350865693182*^-6},
{-5.743976531479158*^-7, 5.766874942905546*^-7},
{-7.626065924242957*^-8, 5.4979380353427926*^-8},
{-8.100264683497016*^-10, -3.253471996451157*^-9},
{5.270631463415014*^-12, 7.862252513620563*^-10},
{-3.633733884546907*^-11, -3.351445796649152*^-11}}
```

Next, we define a function $g$ that given an ordered pair $(x, y)$ ({x,y} in Mathematica), $g((x, y))$ returns the ordered triple $(x, y, f(x, y))$ (x,y,f[x,y] in Mathematica).

```
g[{x_, y_}] = {x, y, f[x, y]}
```

$\left\{x, y, \frac{x^2-y^2}{x^2+y^2}\right\}$

We then use Map to apply $g$ to the list pts.

```
Map[g, pts]//TableForm
```

| | | |
|---|---|---|
| 0.0856075 | −0.0680417 | 0.225699 |
| 0.00437183 | 0.00594185 | −0.297559 |
| −0.000918985 | 0.000902609 | 0.0179789 |
| −0.0000653256 | 0.0000662969 | −0.0147589 |
| −9.419651789936537′*^−6 | −1.2215350865693182′*^−6 | 0.966923 |
| −5.743976531479158′*^−7 | 5.766874942905546′*^−7 | −0.00397856 |
| −7.626065924242957′*^−8 | 5.4979380353427926′*^−8 | 0.316002 |
| −8.100264683497016′*^−10 | −3.253471996451157′*^−9 | −0.883261 |
| 5.270631463415014′*^−12 | 7.862252513620563′*^−10 | −0.99991 |
| −3.633733884546907′*^−11 | −3.351445796649152′*^−11 | 0.0806931 |

From the third column, we see that $z = f(x, y)$ does not appear to approach any particular value for points chosen randomly near $(0, 0)$. In fact, along the line $y = mx$ we see that $f(x, y) = f(x, mx) = \left(1 - m^2\right)/\left(1 + m^2\right)$. Hence, as $(x, y) \to (0, 0)$ along $y = mx$, $f(x, y) = f(x, mx) \to \frac{1-m^2}{1+m^2}$. Thus, $f(x, y)$ does not have a limit as $(x, y) \to (0, 0)$.

We choose lines of the form $y = mx$ because near $(0, 0)$ the level curves of $z = f(x, y)$ look like lines of the form $y = mx$.

```
v1 = Simplify[f[x, mx]]
```

$\frac{1-m^2}{1+m^2}$

```
v1/.m → 0
v1/.m → 1
v1/.m → 1/2
```

1

0

$\frac{3}{5}$

In some cases, you can establish that a limit does not exist by converting to polar coordinates. For example, in polar coordinates, $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ becomes $f(r\cos\theta,\ r\sin\theta) = 2\cos^2\theta - 1$

**Simplify[f[rCos[t], rSin[t]]]**
Cos[2t]

and

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0} 2\cos^2\theta - 1 = 2\cos^2\theta - 1 = \cos 2\theta$$

depends on $\theta$.

## 3.5.2 Partial and Directional Derivatives

Partial derivatives of functions of two or more variables are computed with Mathematica using D. For $z = f(x, y)$,

1. D[f[x,y],x] computes $\frac{\partial f}{\partial x} = f_x(x, y)$,
2. D[f[x,y],y] computes $\frac{\partial f}{\partial y} = f_y(x, y)$,
3. D[f[x,y],{x,n}] computes $\frac{\partial^n f}{\partial x^n}$,
4. D[f[x,y],y,x] computes $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}(x, y)$, and
5. D[f[x,y],{x,n},{y,m}] computes $\frac{\partial^{n+m} f}{\partial^n x \partial^m y}$.
6. You can use the button located on the **BasicMathInput** palette to create templates to compute partial derivatives.

The calculations are carried out similarly for functions of more than two variables.

---

**Example 3.5.2** Calculate $f_x(x, y)$, $f_y(x, y)$, $f_{xy}(x, y)$, $f_{yx}(x, y)$, $f_{xx}(x, y)$, and $f_{yy}(x, y)$ if $f(x, y) = \sin\sqrt{x^2 + y^2 + 1}$.

***Solution*** After defining $f(x, y) = \sin\sqrt{x^2 + y^2 + 1}$,

**f[x_, y_] = Sin[Sqrt[x^2 + y^2 + 1]];**

we illustrate the use of D to compute the partial derivatives. Entering

**D[f[x, y], x]**

$$\frac{\text{xCos}\left[\sqrt{1+x^2+y^2}\right]}{\sqrt{1+x^2+y^2}}$$

computes $f_x(x, y)$. Entering

**D[f[x, y], y]**

$$\frac{y\text{Cos}\left[\sqrt{1+x^2+y^2}\right]}{\sqrt{1+x^2+y^2}}$$

computes $f_y(x, y)$. Entering

**D[f[x, y], x, y]//Together**

$$\frac{-xy\text{Cos}\left[\sqrt{1+x^2+y^2}\right]-xy\sqrt{1+x^2+y^2}\text{Sin}\left[\sqrt{1+x^2+y^2}\right]}{(1+x^2+y^2)^{3/2}}$$

computes $f_{yx}(x, y)$. Entering

**D[f[x, y], y, x]//Together**

$$\frac{-xy\text{Cos}\left[\sqrt{1+x^2+y^2}\right]-xy\sqrt{1+x^2+y^2}\text{Sin}\left[\sqrt{1+x^2+y^2}\right]}{(1+x^2+y^2)^{3/2}}$$

computes $f_{xy}(x, y)$. Remember that under appropriate assumptions, $f_{xy}(x, y) = f_{yx}(x, y)$. Entering

**D[f[x, y], {x, 2}]//Together**

$$\frac{\text{Cos}\left[\sqrt{1+x^2+y^2}\right]+y^2\text{Cos}\left[\sqrt{1+x^2+y^2}\right]-x^2\sqrt{1+x^2+y^2}\text{Sin}\left[\sqrt{1+x^2+y^2}\right]}{(1+x^2+y^2)^{3/2}}$$

computes $f_{xx}(x, y)$. Entering

**D[f[x, y], {y, 2}]//Together**

$$\frac{\text{Cos}\left[\sqrt{1+x^2+y^2}\right]+x^2\text{Cos}\left[\sqrt{1+x^2+y^2}\right]-y^2\sqrt{1+x^2+y^2}\text{Sin}\left[\sqrt{1+x^2+y^2}\right]}{(1+x^2+y^2)^{3/2}}$$

computes $f_{yy}(x, y)$.

---

The **directional derivative** of $z = f(x, y)$ in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta,$$

provided that $f_x(x, y)$ and $f_y(x, y)$ both exist.

The vectors **i** and **j** are defined by $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.

Calculus of vector-valued functions is discussed in more detail in Chapter 5.

If $f_x(x, y)$ and $f_y(x, y)$ both exist, the **gradient** of $f(x, y)$ is the vector-valued function

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \left\langle f_x(x, y), f_y(x, y)\right\rangle.$$

Notice that if $\mathbf{u} = \langle\cos\theta, \sin\theta\rangle$,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \langle\cos\theta, \sin\theta\rangle.$$

---

**Example 3.5.3** Let $f(x, y) = 6x^2y - 3x^4 - 2y^3$. (a) Find $D_{\mathbf{u}} f(x, y)$ in the direction of $\mathbf{v} = \langle 3, 4\rangle$. (b) Compute $D_{\langle 3/5, 4/5\rangle} f\left(\frac{1}{3}\sqrt{9 + 3\sqrt{3}}, 1\right)$. (c) Find an equation of the line tangent to the graph of $6x^2y - 3x^4 - 2y^3 = 0$ at the point $\left(\frac{1}{3}\sqrt{9 + 3\sqrt{3}}, 1\right)$.

***Solution*** After defining $f(x, y) = 6x^2y - 3x^4 - 2y^3$, we graph $z = f(x, y)$ with Plot3D in Figure 3.59, illustrating the PlotPoints, PlotRange, and ViewPoint options.

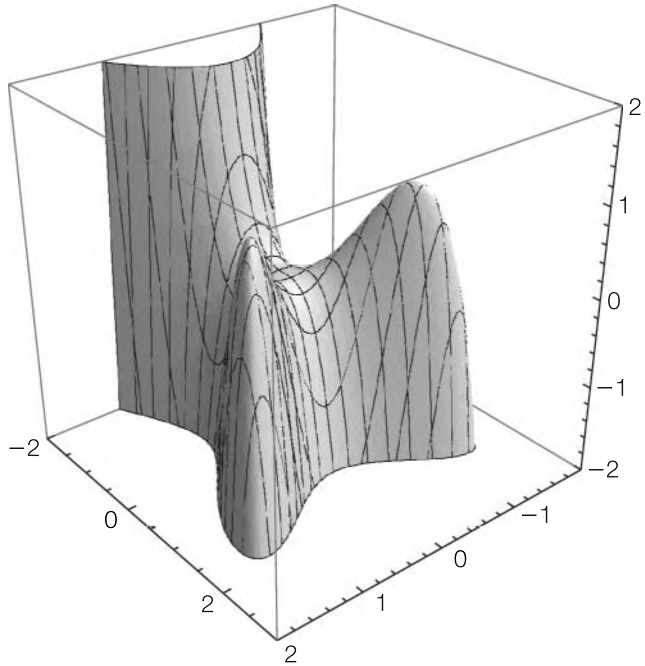



**FIGURE 3.59**

$f(x, y) = 6x^2y - 3x^4 - 2y^3$ for $-2 \le x \le 2$ and $-2 \le y \le 3$

```
f[x_, y_] = 6x^2y - 3x^4 - 2y^3;
  Plot3D[f[x, y], {x, -2, 2}, {y, -2, 3}, PlotPoints → 50,
    PlotRange → {{-2, 2}, {-2, 3}, {-2, 2}},
      BoxRatios → {1, 1, 1}, ViewPoint → {1.887, 2.309, 1.6},
    ClippingStyle → None]
```

(a) A unit vector, **u**, in the same direction as **v** is

$$\mathbf{u} = \left\langle \frac{3}{\sqrt{3^2+4^2}}, \frac{4}{\sqrt{3^2+4^2}} \right\rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

```
v = {3, 4};
u = v/Sqrt[v . v]
```

$\left\{\frac{3}{5}, \frac{4}{5}\right\}$

Then, $D_{\mathbf{u}} f(x, y) = \left\langle f_x(x, y), f_y(x, y) \right\rangle \cdot \mathbf{u}$, calculated in du.

```
gradf = {D[f[x, y], x], D[f[x, y], y]}
```

$\left\{-12x^3 + 12xy, 6x^2 - 6y^2\right\}$

```
du = Simplify[gradf . u]
```

$-\frac{12}{5}\left(-2x^2 + 3x^3 - 3xy + 2y^2\right)$

(b) $D_{\langle 3/5,4/5\rangle} f\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)$ is calculated by evaluating **du** if $x=\frac{1}{3}\sqrt{9+3\sqrt{3}}$ and $y=1$.

```
du1 = du/.{x → 1/3Sqrt[9 + 3Sqrt[3]], y → 1}//Simplify
```

$\frac{4}{5}\left(2\sqrt{3}-3\sqrt{3+\sqrt{3}}\right)$

(c) The gradient is evaluated if $x=\frac{1}{3}\sqrt{9+3\sqrt{3}}$ and $y=1$.

```
nvec = gradf/.{x → 1/3Sqrt[9 + 3Sqrt[3]], y → 1}//Simplify
```

$\left\{-4\sqrt{3+\sqrt{3}},2\sqrt{3}\right\}$

Generally, $\nabla f(x,y)$ is perpendicular to the level curves of $z=f(x,y)$, so

$$\text{nvec}=\nabla f\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)=\left\langle f_x\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right),f_y\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)\right\rangle$$

An equation of the line $L$ containing $(x_0,y_0)$ and perpendicular to $\mathbf{n}=\langle a,b\rangle$ is $a\left(x-x_0\right)+b\left(y-y_0\right)=0$.

is perpendicular to $f(x,y)=0$ at the point $\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)$. Thus, an equation of the line tangent to the graph of $f(x,y)=0$ at the point $\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)$ is

$$f_x\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)\left(x-\frac{1}{3}\sqrt{9+3\sqrt{3}}\right)+f_y\left(\frac{1}{3}\sqrt{9+3\sqrt{3}},1\right)(y-1)=0,$$

which we solve for $y$ with Solve. We confirm this result by graphing $f(x,y)=0$ using ContourPlot in conf and then graphing the tangent line in tanplot. tanplot and conf are shown together with Show in Figure 3.60.

```
conf = ContourPlot[f[x, y]==0, {x, −2, 2}, {y, −2, 2}, PlotPoints → 60,
    ContourShading → False, Frame → False, Axes → Automatic,
        AxesOrigin → {0, 0}];
tanline = Solve[nvec[[1]](x − 1/3Sqrt[9 + 3Sqrt[3]]) + nvec[[2]](y − 1)==0,
    y]
```

$\left\{\left\{y\to-\frac{2+\sqrt{3}-2\sqrt{3+\sqrt{3}}x}{\sqrt{3}}\right\}\right\}$

```
Evaluate[y[x]/.tanline[[1]]]
```

$\left(-\frac{2+\sqrt{3}-2\sqrt{3+\sqrt{3}}x}{\sqrt{3}}\right)[x]$

```
tanplot = Plot[Evaluate[y/.tanline], {x, −2, 2}];
Show[conf, tanplot, PlotRange → {{−2, 2}, {−2, 3}}, AspectRatio →
Automatic]
```

More generally, we use ContourPlot together with the PlotGradientField function, which is contained in the **VectorFieldPlots** package, to illustrate that the gradient vectors are perpendicular to the level curves of $z=f(x,y)$ in Figure 3.61.

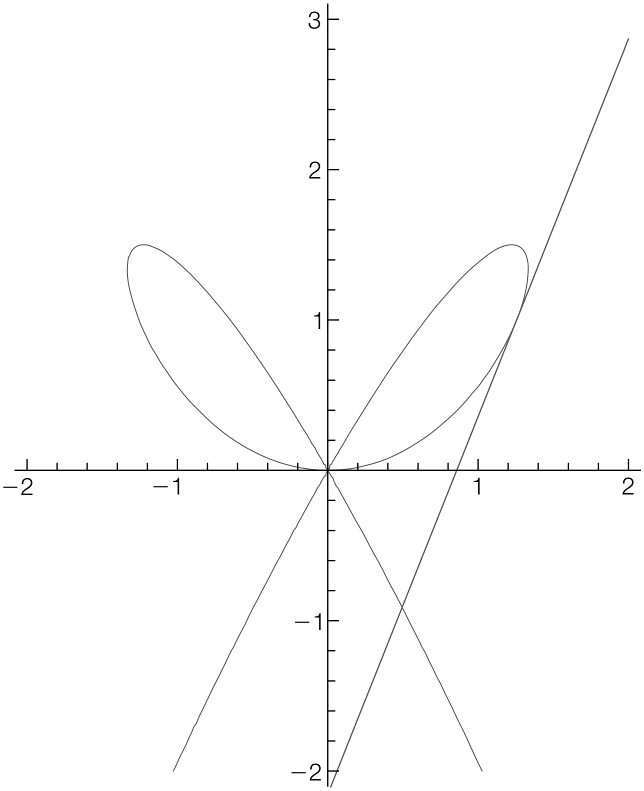



**FIGURE 3.60**

Level curve of $f(x, y)$ together with a tangent line

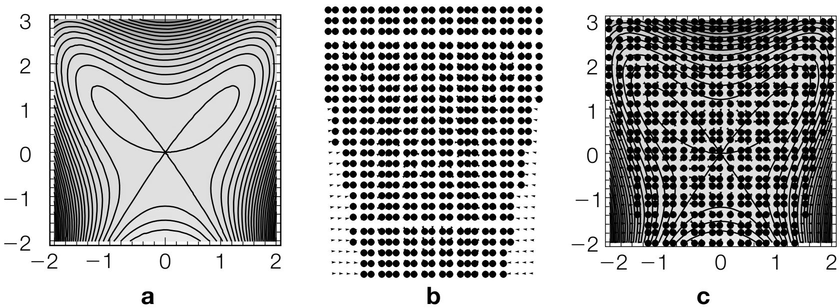



**FIGURE 3.61**

(a) Level curves of $z = f(x, y)$. (b) Gradient field of $z = f(x, y)$. (c) The gradient together with several level curves

```
<< VectorFieldPlots`;

p1 = ContourPlot[f[x, y], {x, -2, 2}, {y, -2, 3},
  Contours → 25, ContourStyle → Black]

p2 = PlotGradientField[f[x, y], {x, -2, 2}, {y, -2, 3}, PlotPoints → 25]

Show[p1, p2]

Show[GraphicsRow[{p1, p2, Show[{p1, p2}]}]]
```

---

**Example 3.5.4** Let

$$f(x,y) = (y-1)^2 e^{-(x+1)^2-y^2} - \frac{10}{3}\left(-x^5 + \frac{1}{5}y - y^3\right) e^{-x^2-y^2} - \frac{1}{9}e^{-x^2-(y+1)^2}.$$

Calculate $\nabla f(x,y)$ and then graph $\nabla f(x,y)$ together with several level curves of $f(x,y)$.

***Solution*** We begin by defining and graphing $z = f(x,y)$ with Plot3D in Figure 3.62(a).

```
Clear[f]
f[x_, y_] = (y-1)^2Exp[-(x + 1)^2-y^2]-
    10/3(-x^5 + 1/5y-y^3)Exp[-x^2-y^2]-
        1/9Exp[-x^2-(y + 1)^2];
p1 = Plot3D[f[x, y], {x, -3, 3}, {y, -3, 3},
    ViewPoint → {-1.99, 2.033, 1.833},
        PlotRange → All];
conf = ContourPlot[f[x, y], {x, -3, 3}, {y, -3, 3},
    PlotPoints → 60, Contours → 30, ContourShading → False,
        Frame → False, Axes → Automatic,
        AxesOrigin → {0, 0}];
```

In the three-dimensional plot, notice that $z$ appears to have six relative extrema: three relative maxima and three relative minima. We also graph several level curves of $f(x,y)$ with ContourPlot and name the resulting graphic conf.

Next we calculate $f_x(x,y)$ and $f_y(x,y)$ using Simplify and D. The gradient is the vector-valued function $\langle f_x(x,y), f_y(x,y)\rangle$.

```
gradf = {D[f[x, y], x], D[f[x, y], y]}//Simplify
```

$$\left\{-\frac{2}{9}e^{-2x-x^2-(1+y)^2}\left(-e^{2x}x + 9e^{2y}(1+x)(-1+y)^2 + 3e^{1+2x+2y}x\left(-25x^3 + 10x^5 - 2y + 10y^3\right)\right),\right.$$
$$\left.-\frac{2}{9}e^{-2x-x^2-(1+y)^2}\left(-e^{2x}(1+y) + 9e^{2y}\left(1-2y^2+y^3\right) + e^{1+2x+2y}\left(3 + 30x^5y - 51y^2 + 30y^4\right)\right)\right\}$$

To graph the gradient, we use PlotGradientField, which is contained in the VectorFieldPlots package. We use PlotGradientField to graph the gradient, naming the resulting graphic gradfplot. gradfplot and conf are displayed together using Show in Figure 3.62(b).

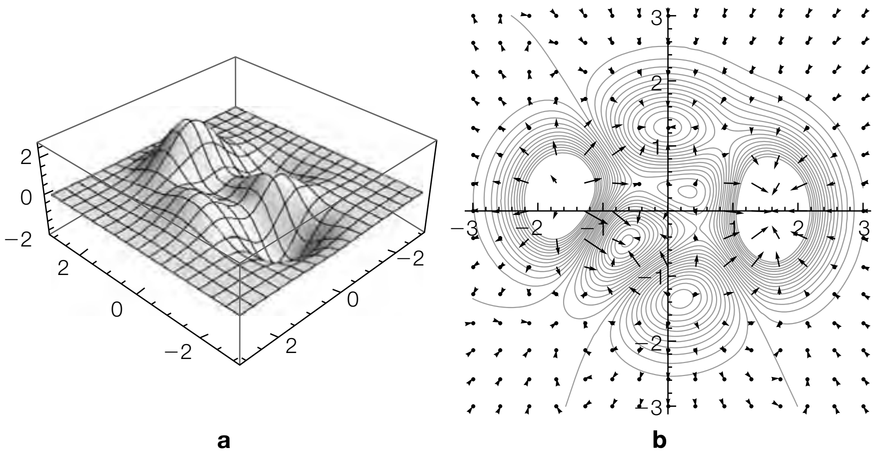



**FIGURE 3.62**

(a) $f(x,y)$ for $-3 \le x \le 3$ and $-3 \le y \le 2$. (b) Contour plot of $f(x,y)$ along with several gradient vectors

```
<< VectorFieldPlots`;
gradfplot = PlotGradientField[f[x, y], {x, -3, 3},
        {y, -3, 3}];
Show[GraphicsRow[{p1, Show[conf, gradfplot]}]]
```

In the result (see Figure 3.62(b)), notice that the gradient is perpendicular to the level curves; the gradient is pointing in the direction of maximal increase of $z = f(x,y)$.

---

### *Classifying Critical Points*

Let $z = f(x,y)$ be a real-valued function of two variables with continuous second-order partial derivatives. A **critical point** of $z = f(x,y)$ is a point $(x_0, y_0)$ in the interior of the domain of $z = f(x,y)$ for which

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

Critical points are classified by the *second derivatives* (or *partials*) *test*.

**Theorem 17 (Second Derivatives Test).** *Let $(x_0, y_0)$ be a critical point of a function $z = f(x,y)$ of two variables and let*

$$d = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - \left[f_{xy}(x_0, y_0)\right]^2. \tag{3.30}$$

1. *If $d > 0$ and $f_{xx}(x_0, y_0) > 0$, then $z = f(x,y)$ has a* ***relative*** *(or* ***local****)* ***minimum*** *at $(x_0, y_0)$.*
2. *If $d > 0$ and $f_{xx}(x_0, y_0) < 0$, then $z = f(x,y)$ has a* ***relative*** *(or* ***local****)* ***maximum*** *at $(x_0, y_0)$.*

3. *If $d < 0$, then $z = f(x,y)$ has a* ***saddle point*** *at $(x_0, y_0)$.*
4. *If $d = 0$, no conclusion can be drawn and $(x_0, y_0)$ is called a* ***degenerate critical point****.*

---

**Example 3.5.5** Find the relative maximum, relative minimum, and saddle points of $f(x,y) = -2x^2 + x^4 + 3y - y^3$.

***Solution*** After defining $f(x,y)$, the critical points are found with Solve and named critpts.

```
f[x_, y_] = -2x^2 + x^4 + 3y - y^3;
critpts = Solve[{D[f[x, y], x]==0, D[f[x, y], y]==0}, {x, y}]
{{x → -1, y → -1}, {x → -1, y → 1}, {x → 0, y → -1},
{x → 0, y → 1}, {x → 1, y → -1}, {x → 1, y → 1}}
```

We then define dfxx. Given $(x_0, y_0)$, dfxx $(x_0, y_0)$ returns the ordered quadruple $x_0$, $y_0$, equation (3.30) evaluated at $(x_0, y_0)$, and $f_{xx}(x_0, y_0)$.

```
dfxx[x0_, y0_] = {x0, y0,
   D[f[x, y], {x, 2}]D[f[x, y], {y, 2}] - D[f[x, y], x, y]^2/.
    {x → x0, y → y0}, D[f[x, y], {x, 2}]/.{x → x0, y → y0}}
{x0, y0, -6 (-4 + 12x0^2) y0, -4 + 12x0^2}
```

For example,

```
dfxx[0, 1]
{0, 1, 24, -4}
```

shows us that a relative maximum occurs at $(0, 1)$. We then use /. (ReplaceAll) to substitute the values in each element of critpts into dfxx.

```
dfxx[x, y]/.critpts//TableForm
 -1  -1   48   8
 -1   1  -48   8
  0  -1  -24  -4
  0   1   24  -4
  1  -1   48   8
  1   1  -48   8
```

From the result, we see that $(0, 1)$ results in a relative maximum, $(0, -1)$ results in a saddle, $(1, 1)$ results in a saddle, $(1, -1)$ results in a relative minimum, $(-1, 1)$ results in a saddle, and $(-1, -1)$ results in a relative minimum. We confirm these results graphically with a three-dimensional plot generated with Plot3D and a contour plot generated with ContourPlot in Figure 3.63.

```
p1 = Plot3D[f[x, y], {x, -3/2, 3/2}, {y, -3/2, 3/2}, PlotPoints → 40];
p2 = ContourPlot[f[x, y], {x, -3/2, 3/2}, {y, -3/2, 3/2},
   PlotPoints → 40, ContourShading → False, Contours → 20];
Show[GraphicsRow[{p1, p2}]]
```

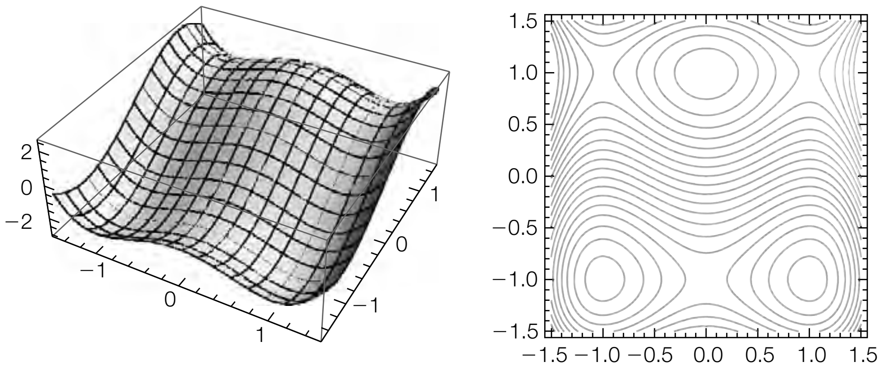



**FIGURE 3.63**

(a) Three-dimensional and (b) contour plots of $f(x, y)$

In the contour plot, notice that near relative extrema, the level curves look like circles, whereas near saddles they look like hyperbolas.

**If the second derivatives test fails, graphical analysis is especially useful.**

**Example 3.5.6** Find the relative maximum, relative minimum, and saddle points of $f(x, y) = x^2 + x^2y^2 + y^4$.

***Solution*** Initially we proceed in the exact same manner as in the previous example: We define $f(x, y)$ and compute the critical points. Several complex solutions are returned, which we ignore.

```
f[x_, y_] = x^2 + x^2y^2 + y^4;
critpts = Solve[{D[f[x, y], x]==0, D[f[x, y], y]==0}, {x, y}]
```

$\{\{x \to 0, y \to 0\}, \{x \to -\sqrt{2}, y \to -i\}, \{x \to -\sqrt{2}, y \to i\}, \{x \to \sqrt{2}, y \to -i\}, \{x \to \sqrt{2}, y \to i\}, \{y \to 0, x \to 0\}, \{y \to 0, x \to 0\}\}$

We then compute the value of (3.30) at the real critical point, and the value of $f_{xx}(x, y)$ at this critical point.

```
dfxx[x0_, y0_] = {x0, y0,
  D[f[x, y], {x, 2}]D[f[x, y], {y, 2}] – D[f[x, y], x, y]^2/.
     {x → x0, y → y0}, D[f[x, y], {x, 2}]/.{x → x0, y → y0}}
```

$\{x0, y0, -16x0^2y0^2 + (2 + 2y0^2)(2x0^2 + 12y0^2), 2 + 2y0^2\}$

```
dfxx[0, 0]
```

$\{0, 0, 0, 2\}$

The result shows us that the second derivatives test fails at $(0, 0)$.

```
p1 = Plot3D[f[x, y], {x, -1, 1}, {y, -1, 1}, BoxRatios → Automatic];
p2 = ContourPlot[f[x, y], {x, -1, 1}, {y, -1, 1}, PlotPoints → 40,
Contours → 20, ContourShading → False];
Show[GraphicsRow[{p1, p2}]]
```

However, the contour plot of $f(x,y)$ near $(0,0)$ indicates that an extreme value occurs at $(0,0)$. The three-dimensional plot shows that $(0,0)$ is a relative minimum. (See Figure 3.64.)

### *Tangent Planes*

Let $z = f(x,y)$ be a real-valued function of two variables. If both $f_x\left(x_0,y_0\right)$ and $f_y\left(x_0,y_0\right)$ exist, then an equation of the plane tangent to the graph of $z = f(x,y)$ at the point $\left(x_0,y_0,f\left(x_0,y_0\right)\right)$ is given by

$$f_x\left(x_0,y_0\right)\left(x-x_0\right)+f_y\left(x_0,y_0\right)\left(y-y_0\right)-\left(z-z_0\right)=0, \tag{3.31}$$

where $z_0 = f\left(x_0,y_0\right)$. Solving for $z$ yields the function (of two variables)

$$z = f_x\left(x_0,y_0\right)\left(x-x_0\right)+f_y\left(x_0,y_0\right)\left(y-y_0\right)+z_0. \tag{3.32}$$

Symmetric equations of the line perpendicular to the surface $z = f(x,y)$ at the point $\left(x_0,y_0,z_0\right)$ are given by

$$\frac{x-x_0}{f_x\left(x_0,y_0\right)}=\frac{y-y_0}{f_y\left(x_0,y_0\right)}=\frac{z-z_0}{-1} \tag{3.33}$$

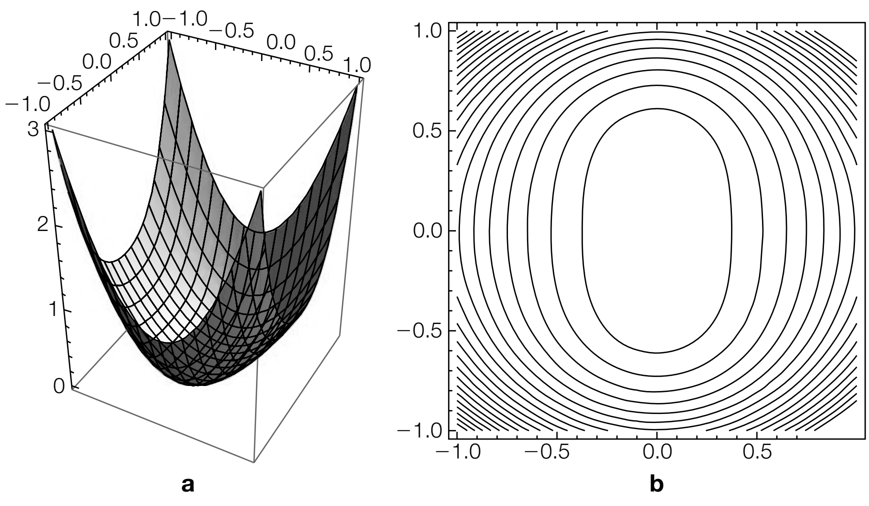



**FIGURE 3.64**

(a) Three-dimensional and (b) contour plots of $f(x,y)$

and parametric equations are

$$\begin{cases} x = x_0 + f_x\left(x_0, y_0\right) t \\ y = y_0 + f_y\left(x_0, y_0\right) t \\ z = z_0 - t. \end{cases} \tag{3.34}$$

The plane tangent to the graph of $z = f(x, y)$ at the point $\left(x_0, y_0, f\left(x_0, y_0\right)\right)$ is the "best" linear approximation of $z = f(x, y)$ near $(x, y) = \left(x_0, y_0\right)$ in the same way as the line tangent to the graph of $y = f(x)$ at the point $\left(x_0, f\left(x_0\right)\right)$ is the "best" linear approximation of $y = f(x)$ near $x = x_0$.

---

**Example 3.5.7** Find an equation of the plane tangent and normal line to the graph of $f(x, y) = 4 - \frac{1}{4}\left(2x^2 + y^2\right)$ at the point $(1, 2, 5/2)$.

***Solution*** We define $f(x, y)$ and compute $f_x(1, 2)$ and $f_y(1, 2)$.

```
f[x_, y_] = 4 − 1/4(2x^2 + y^2);
f[1, 2]
dx = D[f[x, y], x]/.{x → 1, y → 2}
dy = D[f[x, y], y]/.{x → 1, y → 2}
```

$\frac{5}{2}$

−1

−1

Using (3.32), an equation of the tangent plane is $z = -1(x-1) - 1(y-2) + f(1, 2)$. Using (3.34), parametric equations of the normal line are $x = 1 - t$, $y = 2 - t$, $z = f(1, 2) - t$. We confirm the result graphically by graphing $f(x, y)$ together with the tangent plane in p1 using Plot3D. We use ParametricPlot3D to graph the normal line in p2 and then display p1 and p2 together with Show in Figure 3.65.

```
p1 = Plot3D[f[x, y], {x, −1, 3}, {y, 0, 4}];
p2 = Plot3D[dx (x − 1) + dy (y − 2) + f[1, 2], {x, −1, 3}, {y, 0, 4}];
p3 = ParametricPlot3D[{1 + dx t, 2 + dy t, f[1, 2] − t}, {t, −4, 4}];
Show[p1, p2, p3, PlotRange → {{−1, 3}, {0, 4}, {0, 4}},
     BoxRatios → Automatic]
```

Because $z = -1(x-1) - 1(y-2) + f(1, 2)$ is the "best" linear approximation of $f(x, y)$ near $(1, 2)$, the graphs are very similar near $(1, 2)$ as shown in the three-dimensional plot. We also expect the level curves of each near $(1, 2)$ to be similar, which is confirmed with ContourPlot in Figure 3.66.

```
p4 = ContourPlot[f[x, y], {x, 0.75, 1.25}, {y, 1.75, 2.25}];
p5 = ContourPlot[dx(x − 1) + dy( y − 2) + f[1, 2], {x, 0.75, 1.25},
     {y, 1.75, 2.25}];
Show[GraphicsRow[{p4, p5}]]
```

---

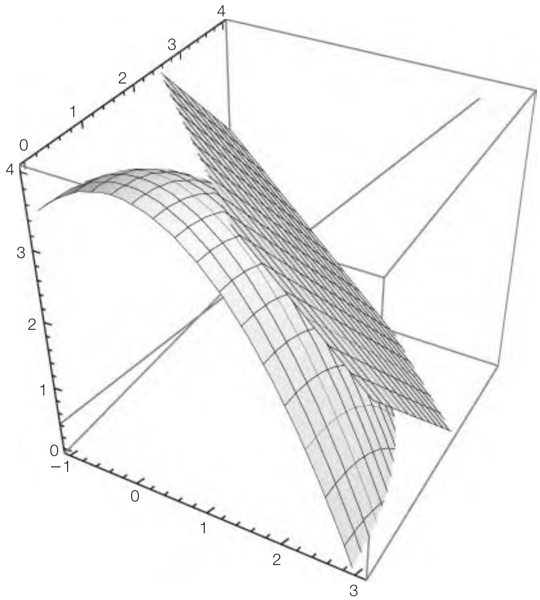



**FIGURE 3.65**

Graph of $f(x,y)$ with a tangent plane and normal line

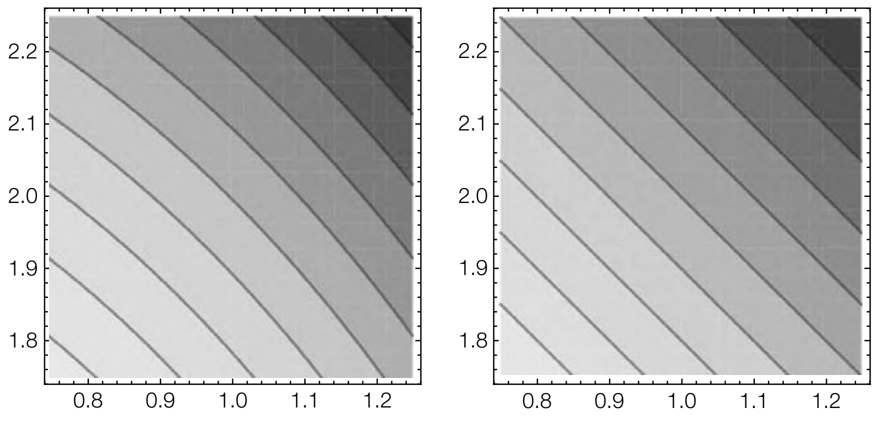



**FIGURE 3.66**

Zooming in near $(1,2)$

## *Lagrange Multipliers*

Certain types of optimization problems can be solved using the method of *Lagrange multipliers* that is based on the following theorem.

**Theorem 18 (Lagrange's Theorem).** *Let $z = f(x,y)$ and $z = g(x,y)$ be real-valued functions with continuous partial derivatives and let $z = f(x,y)$ have an extreme value at a point $(x_0, y_0)$ on the smooth constraint curve $g(x,y) = 0$. If $\nabla g\,(x_0, y_0) \neq 0$, then there is a real number $\lambda$ satisfying*

$$\nabla f\,(x_0, y_0) = \lambda\,\nabla g\,(x_0, y_0)\,. \tag{3.35}$$

Graphically, the points $(x_0, y_0)$ at which the extreme values occur correspond to the points where the level curves of $z = f(x,y)$ are tangent to the graph of $g(x,y) = 0$.

**Example 3.5.8** Find the maximum and minimum values of $f(x,y) = xy$ subject to the constraint $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

***Solution*** For this problem, $f(x,y) = xy$ and $g(x,y) = \frac{1}{4}x^2 + \frac{1}{9}y^2 - 1$. Observe that parametric equations for $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ are $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$. In Figure 3.67(a), we use ParametricPlot3D to parametrically graph $g(x,y) = 0$ and $f(x,y)$ for $x$- and $y$-values on the curve $g(x,y) = 0$ by graphing

$$\begin{cases} x = 2\cos t \\ y = 3\sin t \\ z = 0 \end{cases} \quad \text{and} \quad \begin{cases} x = 2\cos t \\ y = 3\sin t \\ z = x \cdot y = 6\cos t \sin t \end{cases}$$

for $0 \le t \le 2\pi$. Our goal is to find the minimum and maximum values in Figure 3.67(a) and the points at which they occur.

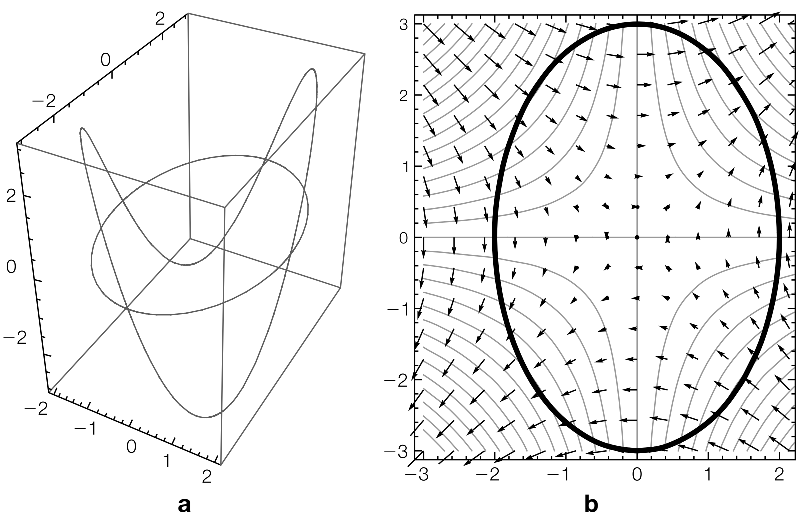



**FIGURE 3.67**

(a) $f(x,y)$ on $g(x,y) = 0$. (b) Level curves of $f(x,y)$ together with $g(x,y) = 0$

```
f[x_, y_] = xy;
g[x_, y_] = x^2/4 + y^2/9 - 1;
s1 = ParametricPlot3D[{2Cos[t], 3Sin[t], 0}, {t, 0, 2Pi}];
s2 = ParametricPlot3D[{2Cos[t], 3Sin[t], 6Cos[t]Sin[t]}, {t, 0, 2Pi}];
plot1 = Show[s1, s2, BoxRatios → Automatic, PlotRange → All];
```

To implement the method of Lagrange multipliers, we compute $f_x(x,y)$, $f_y(x,y)$, $g_x(x,y)$, and $g_y(x,y)$ with D.

```
fx = D[f[x, y], x]
fy = D[f[x, y], y]
gx = D[g[x, y], x]
gy = D[g[x, y], y]
```

y

x

$\frac{x}{2}$

$\frac{2y}{9}$

Solve is used to solve the system of equations (3.35):

$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y) = \lambda g_y(x,y)$$
$$g(x,y) = 0$$

for $x$, $y$, and $\lambda$.

```
vals = Solve[{fx==λgx, fy==λgy, g[x, y]==0}, {x, y, λ}]
```

$\left\{\left\{\lambda \to -3, x \to -\sqrt{2}, y \to \frac{3}{\sqrt{2}}\right\}, \left\{\lambda \to -3, x \to \sqrt{2}, y \to -\frac{3}{\sqrt{2}}\right\}, \left\{\lambda \to 3, x \to -\sqrt{2}, y \to -\frac{3}{\sqrt{2}}\right\}, \left\{\lambda \to 3, x \to \sqrt{2}, y \to \frac{3}{\sqrt{2}}\right\}\right\}$

The corresponding values of $f(x,y)$ are found using ReplaceAll (/.).

```
n1 = {x, y, f[x, y]}/.vals//TableForm
```

| | | |
|---|---|---|
| $-\sqrt{2}$ | $\frac{3}{\sqrt{2}}$ | $-3$ |
| $\sqrt{2}$ | $-\frac{3}{\sqrt{2}}$ | $-3$ |
| $-\sqrt{2}$ | $-\frac{3}{\sqrt{2}}$ | $3$ |
| $\sqrt{2}$ | $\frac{3}{\sqrt{2}}$ | $3$ |

```
N[n1]
```

| | | |
|---|---|---|
| −1.41421 | 2.12132 | −3. |
| 1.41421 | −2.12132 | −3. |
| −1.41421 | −2.12132 | 3. |
| 1.41421 | 2.12132 | 3. |

We conclude that the maximum value $f(x,y)$ subject to the constraint $g(x,y) = 0$ is $3$ and occurs at $\left(\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$ and $\left(-\sqrt{2}, -\frac{3}{2}\sqrt{2}\right)$. The minimum value is $-3$ and occurs at $\left(-\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$ and $\left(\sqrt{2}, -\frac{3}{2}\sqrt{2}\right)$. We graph several level curves of $f(x, y)$ and the graph of $g(x, y) = 0$ with ContourPlot and show the graphs together with Show. The minimum and maximum values of $f(x, y)$ subject to the constraint $g(x, y) = 0$ occur at the points where the level curves of $f(x, y)$ are tangent to the graph of $g(x, y) = 0$ as illustrated in Figure 3.67(b).

```
<< "VectorFieldPlots`"; gradfplot = PlotGradientField[f[x, y], {x, -3, 3},
    {y, -3, 3}];
cp1 = ContourPlot[f[x, y], {x, -3, 3}, {y, -3, 3}, Contours->30,
    ContourShading->False, PlotPoints->40];
cp2 = ContourPlot[g[x, y]==0, {x, -3, 3}, {y, -3, 3},
    ContourStyle->Thickness[0.01],
        ContourShading->False];
plot2 = Show[cp1, cp2, gradfplot];
Show[GraphicsRow[{plot1, plot2}]]
```

Observe that the maximum and minimum values occur where the gradient vectors of $z = f(x, y)$ are parallel to the gradient vectors of $z = g(x, y)$ on the equation $g(x, y) = 0$.

---

## 3.5.3 Iterated Integrals

The Integrate command, used to compute single integrals, is used to compute iterated integrals. The command

```
Integrate[f[x, y], {y, c, d}, {x, a, b}]
```

attempts to compute the iterated integral

$$\int_c^d \int_a^b f(x, y)\, dx\, dy. \tag{3.36}$$

If Mathematica cannot compute the exact value of the integral, it is returned unevaluated, in which case numerical results may be more useful. The iterated integral (3.36) is numerically evaluated with the command N or

```
NIntegrate[f[x, y], {y, c, d}, {x, a, b}]
```

---

**Example 3.5.9** Evaluate each integral: (a) $\int_2^4 \int_1^2 \left(2xy^2 + 3x^2y\right)\, dx\, dy$; (b) $\int_0^2 \int_{y^2}^{2y} \left(3x^2 + y^3\right)\, dx\, dy$; (c) $\int_0^\infty \int_0^\infty xye^{-x^2-y^2}\, dy\, dx$; and (d) $\int_0^\pi \int_0^\pi e^{\sin xy}\, dx\, dy$.

***Solution*** (a) First, we compute $\iint \left(2xy^2 + 3x^2y\right)\, dx\, dy$ with Integrate. Second, we compute $\int_2^4 \int_1^2 \left(2xy^2 + 3x^2y\right)\, dx\, dy$ with Integrate.

```
Integrate[2xy^2 + 3x^2y, y, x]
```

$\frac{1}{6}x^2y^2(3x + 2y)$

**Integrate[2xy^2 + 3x^2y, {y, 2, 4}, {x, 1, 2}]**

98

(b) We illustrate the same commands as in (a), except we are integrating over a nonrectangular region.

**Integrate[3x^2 + y^3, {x, y^2, 2y}]**

$8y^3 + 2y^4 - y^5 - y^6$

**Integrate[3x^2 + y^3, y, {x, y^2, 2y}]**

$2y^4 + \frac{2y^5}{5} - \frac{y^6}{6} - \frac{y^7}{7}$

**Integrate[3x^2 + y^3, {y, 0, 2}, {x, y^2, 2y}]**

$\frac{1664}{105}$

(c) Improper integrals can be handled in the same way as proper integrals.

**Integrate[xyExp[−x^2−y^2], x, y]**

$\frac{1}{4}e^{-x^2-y^2}$

**Integrate[xyExp[−x^2−y^2], {x, 0, Infinity}, {y, 0, Infinity}]**

$\frac{1}{4}$

(d) In this case, Mathematica cannot evaluate the integral exactly so we use NIntegrate to obtain an approximation.

**Integrate[Exp[Sin[xy]], y, x]**

$\int\int e^{\mathrm{Sin}[xy]}dxdy$

**NIntegrate[Exp[Sin[xy]], {y, 0, Pi}, {x, 0, Pi}]**

15.5092

---

### *Area, Volume, and Surface Area*

Typical applications of iterated integrals include determining the area of a planar region, the volume of a region in three-dimensional space, or the surface area of a region in three-dimensional space. The area of the planar region $R$ is given by

$$A = \int\int_R dA. \tag{3.37}$$

If $z = f(x, y)$ has continuous partial derivatives on a closed region $R$, then the surface area of the portion of the surface that projects onto $R$ is given by

$$SA = \int\int_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA. \tag{3.38}$$

If $f(x, y) \geq g(x, y)$ on $R$, the volume of the region between the graphs of $f(x, y)$ and $g(x, y)$ is

$$V = \int\int_R \left(f(x, y) - g(x, y)\right)\, dA. \tag{3.39}$$

**Example 3.5.10** Find the area of the region $R$ bounded by the graphs of $y = 2x^2$ and $y = 1 + x^2$.

***Solution*** We begin by graphing $y = 2x^2$ and $y = 1 + x^2$ with Plot in Figure 3.68. The $x$-coordinates of the intersection points are found with Solve.

```
Plot[Tooltip[{2x^2, 1 + x^2}], {x, -3/2, 3/2}]
Solve[2x^2==1 + x^2]
```

{{x → −1}, {x → 1}}

Using (3.37) and taking advantage of symmetry, the area of $R$ is given by

$$A = \iint_R dA = 2\int_0^1 \int_{2x^2}^{1+x^2} dy\,dx,$$

which we compute with Integrate.

```
2 Integrate[1, {x, 0, 1}, {y, 2x^2, 1 + x^2}]
```

$\frac{4}{3}$

We conclude that the area of $R$ is $4/3$.

---

**If the problem exhibits "circular symmetry," changing to polar coordinates is often useful. If $R = \{(r, \theta) | a \le r \le b, \alpha \le \theta \le \beta\}$, then**

$$\iint_R f(x, y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

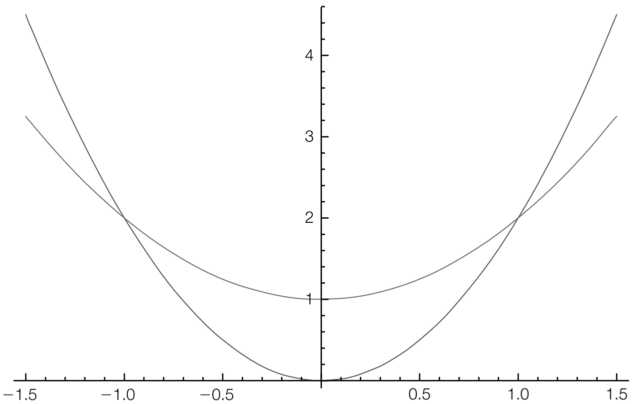



**FIGURE 3.68**

$y = 2x^2$ and $y = 1 + x^2$ for $-3/2 \le x \le 3/2$

**Example 3.5.11** Find the surface area of the portion of

$$f(x,y)=\sqrt{4-x^2-y^2}$$

that lies above the region $R=\{(x,y)\mid x^2+y^2\le 1\}$.

***Solution*** First, observe that the domain of $f(x,y)$ is

$$\left\{(x,y)\,\middle|\,-\sqrt{4-y^2}\le x\le\sqrt{4-y^2},\,-2\le y\le 2\right\}=\{(r,\theta)\,|0\le r\le 2,0\le\theta\le 2\pi\}.$$

Similarly,

$$R=\left\{(x,y)\,\middle|\,-\sqrt{1-y^2}\le x\le\sqrt{1-y^2},\,-1\le y\le 1\right\}=\{(r,\theta)\,|0\le r\le 1,0\le\theta\le 2\pi\}.$$

With this observation, we use ParametricPlot3D to graph $f(x,y)$ in p1 and the portion of the graph of $f(x,y)$ above $R$ in p2 and show the two graphs together with Show. We wish to find the area of the black region in Figure 3.69.

```
f[x_, y_] = Sqrt[4 - x^2 - y^2];
p1 = ParametricPlot3D[{r Cos[t], r Sin[t], f[r Cos[t], r Sin[t]]}, {r, 0, 2},
     {t, 0, 2Pi}, PlotPoints → 45, ColorFunction → "LightTerrain"];
p2 = ParametricPlot3D[{r Cos[t], r Sin[t], f[r Cos[t], r Sin[t]]}, {r, 0, 1},
```

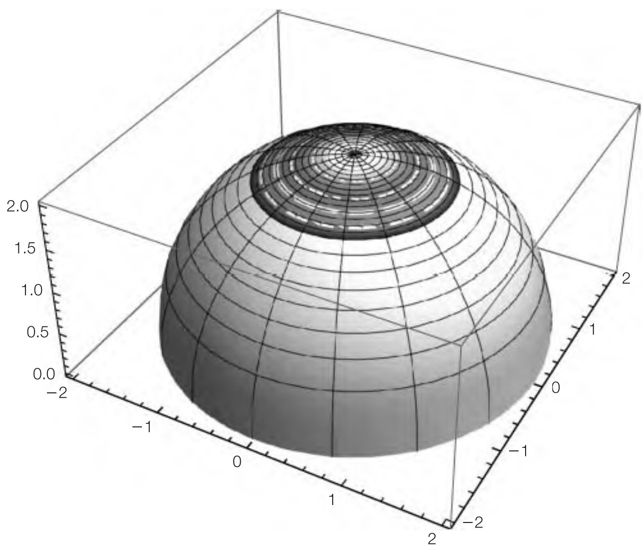



**FIGURE 3.69**

The portion of the graph of $f(x,y)$ above $R$

```
    {t, 0, 2Pi}, PlotPoints → 45, ColorFunction → "DarkTerrain"];
Show[p1, p2, BoxRatios → Automatic]
```

We compute $f_x(x,y)$, $f_y(x,y)$ and $\sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}$ with D and Simplify.

```
fx = D[f[x, y], x]
fy = D[f[x, y], y]
```

$-\frac{x}{\sqrt{4-x^2-y^2}}$

$-\frac{y}{\sqrt{4-x^2-y^2}}$

```
s1 = Simplify[Sqrt[1 + fx^2 + fy^2]]
```

$2\sqrt{-\frac{1}{-4+x^2+y^2}}$

Then, using (3.38), the surface area is given by

$$\begin{aligned} SA &= \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA \\ &= \iint_R \frac{2}{\sqrt{4-x^2-y^2}}\, dA \qquad (3.40) \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\sqrt{4-x^2-y^2}}\, dx\, dy. \end{aligned}$$

However, notice that in polar coordinates,

$$R = \{(r,\theta) | 0 \le r \le 1, 0 \le \theta \le 2\pi\},$$

so in polar coordinates the surface area is given by

$$SA = \int_0^{2\pi} \int_0^1 \frac{2}{\sqrt{4-r^2}}\, r\, dr\, d\theta,$$

```
s2 = Simplify[s1/.{x → r Cos[t], y → r Sin[t]}]
```

$2\sqrt{\frac{1}{4-r^2}}$

which is much easier to evaluate than (3.40). We evaluate the iterated integral with Integrate

```
s3 = Integrate[r s2, {t, 0, 2Pi}, {r, 0, 1}]
```

$-4\left(-2+\sqrt{3}\right)\pi$

```
N[s3]
```

3.36715

and conclude that the surface area is $\left(8-4\sqrt{3}\right)\pi \approx 3.367$.

**Example 3.5.12** Find the volume of the region between the graphs of $z = 4 - x^2 - y^2$ and $z = 2 - x$.

***Solution*** We begin by graphing $z = 4 - x^2 - y^2$ and $z = 2 - x$ together with Plot3D in Figure 3.70(a).

```
p1 = Plot3D[{4 - x^2 - y^2, 2 - x}, {x, -2, 2}, {y, -2, 2},
    PlotRange → {{-2, 2}, {-2, 2}, {-2, 4}}, BoxRatios → Automatic];
```

The region of integration, $R$, is determined by graphing $4 - x^2 - y^2 = 2 - x$ in Figure 3.70(b).

```
p2 = ContourPlot[4 - x^2 - y^2 - (2 - x)==0, {x, -2, 2}, {y, -2, 2},
    PlotPoints → 50, Frame → False, Axes → Automatic,
        AxesOrigin → {0, 0}];
Show[GraphicsRow[{p1, p2}]]
```

Another way to see the situation illustrated in Figure 3.70 is to use RegionPlot3D, which works in the same way as RegionPlot but in three dimensions. Completing the square shows us that

$$\begin{aligned} R &= \left\{ (x, y) \,\middle|\, \left(x - \frac{1}{2}\right)^2 + y^2 \le \frac{9}{4} \right\} \\ &= \left\{ (x, y) \,\middle|\, \frac{1}{2} - \frac{1}{2}\sqrt{9 - 4y^2} \le x \le \frac{1}{2} + \frac{1}{2}\sqrt{9 - 4y^2}, -\frac{3}{2} \le y \le \frac{3}{2} \right\}. \end{aligned}$$

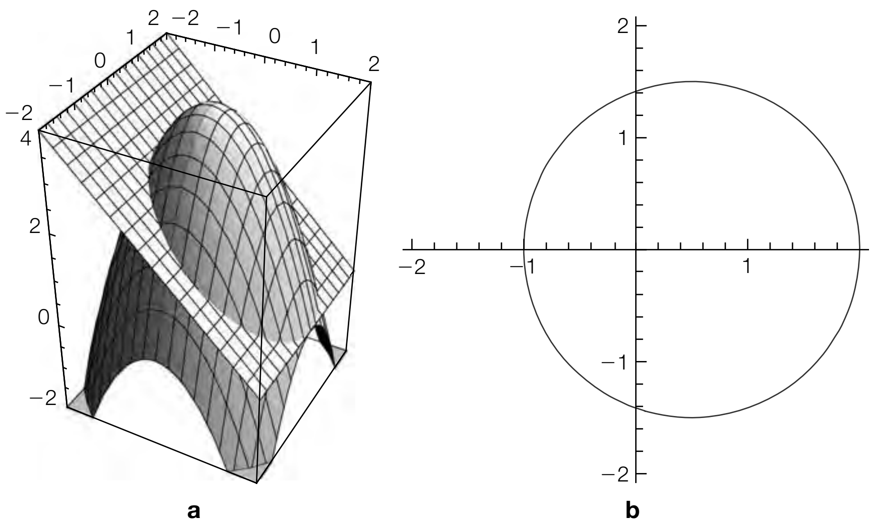
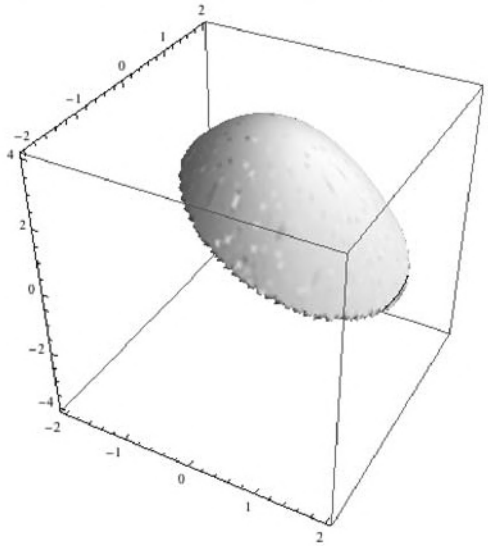



**FIGURE 3.70**

(a) $z = 4 - x^2 - y^2$ and $z = 2 - x$ for $-2 \le x \le 2$ and $-2 \le y \le 2$. (b) Graph of $4 - x^2 - y^2 = 2 - x$

```
RegionPlot3D[2 - x ≤ z ≤ 4 - x^2 - y^2, {x, -2, 2}, {y, -2, 2},
 {z, -4, 4}, PlotPoints → 75, Mesh → False]
```




Thus, using (3.39), the volume of the solid is given by

$$V = \iint_R \left[\left(4 - x^2 - y^2\right) - (2 - x)\right]\, dA$$

$$= \int_{-\frac{3}{2}}^{\frac{3}{2}} \int_{\frac{1}{2}-\frac{1}{2}\sqrt{9-4y^2}}^{\frac{1}{2}+\frac{1}{2}\sqrt{9-4y^2}} \left[\left(4 - x^2 - y^2\right) - (2 - x)\right]\, dx\, dy,$$

which we evaluate with Integrate.

**i1 = Integrate[(4 − x^2 − y^2) − (2 − x), {y, −3/2, 3/2},**
**{x, 1/2 − 1/2Sqrt[9 − 4y^2], 1/2 + 1/2Sqrt[9 − 4y^2]}]**

$\frac{81\pi}{32}$

**N[i1]**

7.95216

We conclude that the volume is $\frac{81}{32}\pi \approx 7.952$.

---

## *Triple Iterated Integrals*

Triple iterated integrals are calculated in the same manner as double iterated integrals.

**Example 3.5.13** Evaluate

$$\int_0^{\pi/4}\int_0^{y}\int_0^{y+z}(x+2z)\sin y\,dx\,dz\,dy.$$

***Solution*** Entering

**i1 = Integrate[(x + 2z)Sin[y], {y, 0, Pi/4}, {z, 0, y}, {x, 0, y + z}]**

$-\frac{17(384-96\pi-12\pi^2+\pi^3)}{384\sqrt{2}}$

calculates the triple integral exactly with Integrate.

An approximation of the exact value is found with N.

**N[i1]**

0.157206

We illustrate how triple integrals can be used to find the volume of a solid when using spherical coordinates.

**Example 3.5.14** Find the volume of the torus with equation in spherical coordinates $\rho = \sin\phi$.

***Solution*** We proceed by graphing the torus with SphericalPlot3D in Figure 3.72 (see Figure 3.71 for the help feature associated with this command).

**SphericalPlot3D[Sin[Phi], {Phi, 0, Pi}, {theta, 0, 2Pi}, PlotPoints → 40]**

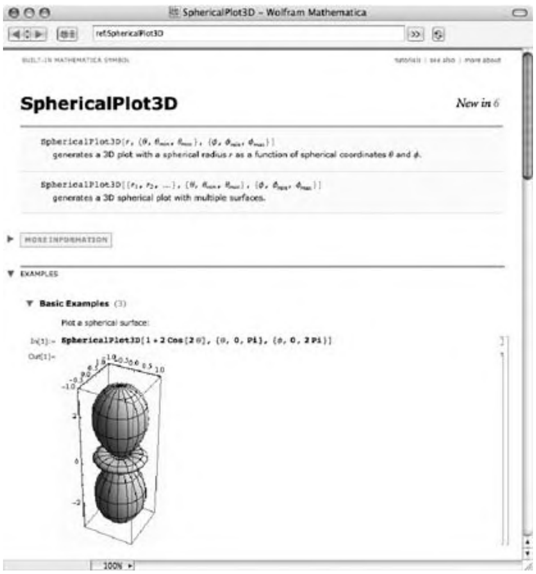



**FIGURE 3.71**

Mathematica's help for SphericalPlot3D

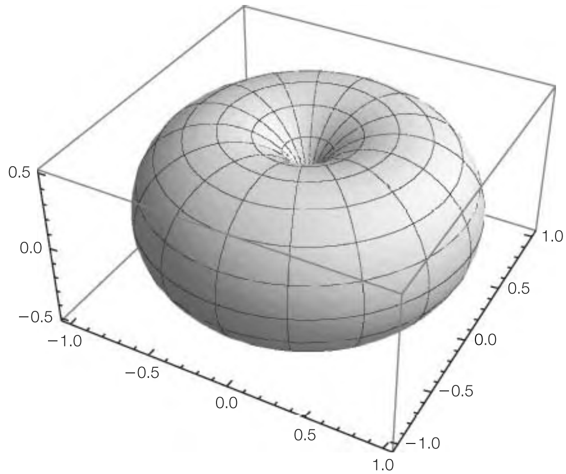



**FIGURE 3.72**

A graph of the torus

In general, the volume of the solid region $D$ is given by

$$V = \iiint_D dV.$$

Thus, the volume of the torus is given by the triple iterated integral

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta,$$

```
i1 = Integrate[rho^2 Sin[phi], {theta, 0, 2Pi},
    {phi, 0, Pi}, {rho, 0, Sin[phi]}]
```

$\frac{\pi^2}{4}$

```
N[i1]
```

2.4674

which we evaluate with Integrate. We conclude that the volume of the torus is $\frac{1}{4}\pi^2 \approx 2.467$.

## 3.6 EXERCISES

**1.** If $\$P$ is compounded $n$ times per year at an annual interest rate of $r$, the value of the account, $A$, after $t$ years is given by

$$A = \left(1 + \frac{r}{n}\right)^{nt}.$$

The formula for continuously compounded interest is obtained by finding the limit of this expression as $t \to \infty$. Find the limit.

2. Let $f(x) = \begin{cases} ax^4 + bx^3 + cx^2 + 8, & \text{if } x \le z \\ ax^3 + bx2 + cx + 4, & \text{if } x > z \end{cases}$.

   **(a)** If $z = 2$, find $a$, $b$, and $c$ so that $f(x)$, $f'(x)$, and $f''(x)$ are continuous for all real numbers.

   **(b)** For what values of $z$, if any, are there no values $a$, $b$, and $c$ so that $f(x)$, $f'(x)$, and $f''(x)$ are continuous for all real numbers?

3. Use Mathematica to generate a *representative* plot of each of the following functions. *Note:* If $x = p/q$ is rational, $p/q$ is assumed to be a reduced fraction.

   **(a)** $f(x) = \begin{cases} \ln p, & \text{if } x = p/q \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

   **(b)** $f(x) = \begin{cases} \cos q, & \text{if } x = p/q \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

   **(c)** $f(x) = \begin{cases} 1/p, & \text{if } x = p/q \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

   **(d)** *Challenge:* Determine the value(s) of $x$, if any, for which each of these functions are continuous.

4. For $f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$, provide a convincing argument that $f^{(n)}(0) = 0$ for all $n$. (Refer to Figure 3.3(b).)

5. Refer to Example 3.1.2. For what values of $x$, if any, is the function not continuous? Not differentiable?

6. (a) Find an equation of the line tangent to the graph of $f(x) = 9 - 4x^2$ at the point $(1, f(1))$. (b) Use Do to generate graphs of $y = f(x)$ and $y = f'(a)(x - a) + f(a)$ for 50 equally spaced values of $a$ between $-3$ and 3. (c) Use Table to create a similar plot for 9 equally spaced values of $a$ between $-3$ and 3 and display the result as a graphics array.

7. Let $f(x) = mx + b$ and $(x_0, y_0)$ be a point *not* on the graph of $f(x)$. Find the point on the graph of $f(x)$ that is closest to $(x_0, y_0)$.

8. If $f(x) = \cos(3x)/(x^2 + 1)$ on $[0, \pi]$, find the value(s) of $c$ that satisfies that conclusion of the mean-value theorem for derivatives. Confirm your results graphically.

9. Sketch $f(x) = x^4 - x^2$. In your plot, label relative and absolute extreme values as well as points of inflection. *Tip:* A good plot indicates both the local and the global behavior of the function.

10. Use Maximize or Minimize to verify each of the results obtained in the examples in **Applied Max/Min Problems**, Section 3.2.6.

**11.** (a) Find $dy/dx$ if $\cos(x + \sin y) = \sin y$. (b) Graph the equation for $-4\pi \leq x \leq 4\pi$ and $-4\pi \leq y \leq 4\pi$. (c) Find a point on the graph at which there are two tangents and then find equations of both tangents. (d) Illustrate that your final result is correct.

**12.** Find the ratio of the volume of the right circular cone of largest volume that can be circumscribed about a sphere of radius $R$ to the volume of the right circular cone of largest volume that can be inscribed in a sphere of radius $R$.

**13.** Plot $f(x) = x(x-1)^{1/3}(x-2)^{2/3}$ without loading the **RealOnly** package. Calculate and then plot $f'(x)$ and $f''(x)$ as well. *Hint:* Use Abs.

**14.** Calculate (a) $\int \frac{1}{\sin^2 x + 2}\,dx$ and (b) $\int x^2 \tan^{-1} x\,dx$. In each case, check that your answer is correct by computing the antiderivative by hand.

**15.** Refer to Figure 3.30. Create a Manipulate object that can be used to illustrate how the lengths of the stayed wires change as $D$, $L_1$, and $L_2$ change. Use $[0, 100]$ for each range. For the initial values set $D = 50$, $L_1 = 20$, and $L_2 = 60$.

**16.** Let $f(x) = ax^2 + c$. For $x = x_0$, let $L(x_0)$ denote the line perpendicular to the tangent at $(x_0, f(x_0))$ and let $d$ denote the length of the line segment formed by the intersection $L(x_0)$ and $f(x)$. Find $x_0$ so that $d$ is minimized. What is the measure of the angle formed by the intersection of the two lines for which $d$ is minimized?

**17.** (a) Define functions simpson, which implements Simpson's rule, and trapezoid, which implements the trapezoidal rule by adjusting the function leftsum (middlesum or rightsum) discussed previously to perform the calculation for the desired method. (b) Let $f(x) = e^{-(x-3)^2 \cos(4(x-3))}$. (i) Graph $y = f(x)$ on the interval $[1, 5]$. Use (ii) Simpson's rule with $n = 4$, (iii) the trapezoidal rule with $n = 4$, and (iv) the midpoint rule with $n = 4$ to approximate $\int_1^5 f(x)\,dx$.

**18.** If $p(x) = \frac{3}{10}x^5 - 3x^4 + 11x^3 - 18x^2 + 12x + 1$ and $q(x) = -4x^3 + 28x^2 - 56x + 32$, find the solutions of $p(x) = q(x)$ using FindRoot. *Challenge:* Use Map together with FindRoot to perform the operation in a single command.

**19.** Let $f(x) = \exp\left(-(x-2)^2 \cos \pi x\right)$ and $g(x) = 4\cos(x-2)$. (a) Find the area of the region bounded by the graphs of the two functions. (b) Find the volume of the solid obtained by revolving the region bounded by the graphs of the two functions about the $x$-axis. (c) Find the volume of the solid obtained by revolving the region bounded by the graphs of the two functions about the $y$-axis. (d) Generate plots illustrating the area and the two solids. *Hint*: Use FindRoot.

**20.** Let $R$ denote the region in the first quadrant bounded by the graphs of $y = x^n$ and $x = y^n$. (a) Find the area of $R$. (b) Find the volume of the solid obtained by revolving $R$ about the $x$-axis. (c) Find the volume of the solid obtained by revolving $R$ about the $y$-axis.

**21.** Calculate (a) $\int_1^n k\,2^{-k}\,dk$, (b) $\lim_{n\to\infty}\int_1^n k\,2^{-k}\,dk$, and (c) $\int_1^\infty k\,2^{-k}\,dk$.

**22.** Show that $\sum_{k=1}^{\infty}\frac{50^k}{k!}$ converges and find its sum.

**23.** Find $\sum_{k=1}^{\infty} x^{3k}$. What is the interval of convergence for this series?

**24.** Evaluate $\sum_{n=1}^{\infty}\frac{3^{n/2}}{5^n}$. Confirm your result by showing that the series converges and finding its sum by hand.

**25.** (a) Plot $(k, a_k)$ for $k = 1, 2, \ldots, 2000$ if $a_k = \frac{\sin k}{k}$. (b) Find $\sum_{k=1}^{\infty}\frac{\sin k}{k}$. *Challenge:* Prove that the infinite series converges. (c) Plot $(k, a_k)$ for $k = 1, 2, \ldots, 2000$ if $a_k = \frac{\sin k}{k^2}$. (d) Prove that the infinite series converges. *Challenge:* Find $\sum_{k=1}^{\infty}\frac{\sin k}{k^2}$. (See Figure 3.73(a).)

**26.** Let $f(x) = \exp\left(-(x-1)^2(x+1)^2\right)$. (a) Graph $f(x)$ together with its 8th degree Taylor polynomial expanded about $x = 1$ on the interval $[-1.75, 1.75]$. (b) What is the interval of convergence for the Taylor series about $x = 1$ for this function? (c) Can you use a Taylor polynomial expanded about $x = 1$ for this function to approximate $f(0)$? Explain. (See Figure 3.73(b).)

**27.** Find the length of the graph of $f(x) = \sin(x + x\sin x)$ from $x = 0$ to $x = 2\pi$.

**28.** Determine $\lim_{(x,y)\to(0,0)} xy/(x^2 + y^2)$. If the limit does not exist, confirm your results graphically.

See Chapter 5 for more discussion regarding curvature.

**29.** *Minimal surfaces* have "zero mean curvature." Minimal surfaces that are parametrically defined by $x = u$, $y = v$, $z = f(u, v)$ satisfy

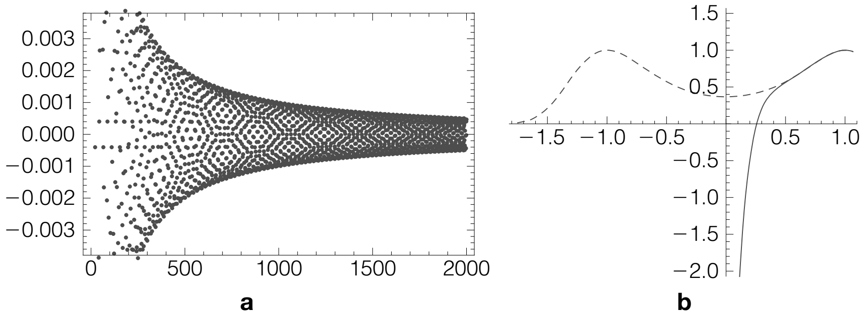



**FIGURE 3.73**

(a) The first 2000 terms of an interesting sequence. (b) What is the radius of the interval of convergence?

**Lagrange's equation**, $\left(1+f_v{}^2\right)f_{uu}-2f_uf_vf_{uv}+\left(1+f_u{}^2\right)f_{vv}=0$. Plot **Enneper's minimal surface**, $x=u-\frac{1}{3}u^3+uv^2$, $y=v-\frac{1}{3}v^3+u^2v$, $z=u^2-v^2$, and show that the equations satisfy Lagrange's equation.

**30.** Let $g(x,y)=\exp\left(-\frac{1}{8}\left(x^2+y^2\right)\left(\cos^2 x+\sin^2 y\right)\right)$. (a) Graph $g(x,y)$ using your favorite color scheme from the **Color Schemes** palette for $-\pi\le x\le\pi$ and $-\pi\le y\le\pi$. (b) Compute and simplify $f_x$, $f_y$, $f_{xy}$, $f_{xx}$, and $f_{yy}$ using D and Simplify. (c) Use Mathematica help to determine the functionality of Derivative and then use Derivative to recalculate the partials.

**31.** Find and classify the critical points of $f(x,y)=-120x^3-30x^4+18x^5+5x^6+30xy^2$. Confirm your results with three-dimensional and contour plots with a gradient plot.

**32.** Find equations of the tangent plane and normal line to $f(x,y)=4\left(x^2+y^2+1\right)^{-1}$ at $(1/2,1,f(1/2,1))$. Confirm your results graphically. (See Figure 3.74(a).)

**33.** Find the minimum and maximum values of $f(x,y)=x^2+4y^3$ subject to $x^2+4y^2=1$. Confirm your results graphically. (See Figure 3.74(b).)

**34.** Evaluate $\int_1^2\int_{1-y}^{\sqrt{y}} xy^2\,dy\,dx$.

**35.** Evaluate $\int_0^{\sqrt{\pi}}\int_0^{\sqrt{\pi}}\cos\left(x^2-y^2\right)dy\,dx$. Determine the meaning of the functions FresnelC and FresnelS.

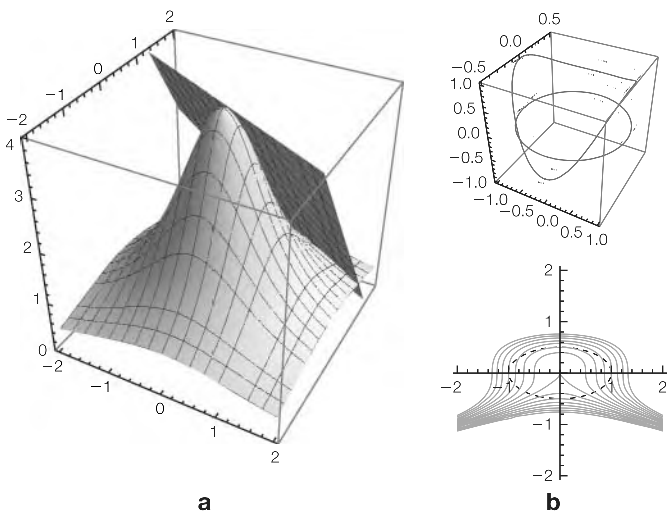


**FIGURE 3.74**

(a) Tangent plane. (b) Lagrange multipliers

CHAPTER

# Introduction to Lists and Tables

# 4

Chapter 4 introduces operations on lists and tables. The examples used to illustrate the various commands in this chapter are taken from calculus, business, dynamical systems, and engineering applications.

## 4.1 LISTS AND LIST OPERATIONS

### 4.1.1 Defining Lists

A **list** of $n$ elements is a Mathematica object of the form

**list = {a1, a2, a3, ... , an}**

The $i$th element of the list is extracted from list with list[[i]] or Part[list,i].

Elements of a list are separated by commas. Lists are always enclosed in braces $\{\ldots\}$ and each element of a list may be (almost any) Mathematica object—even other lists. Because lists are Mathematica objects, they can be named. For easy reference, we will usually name lists.

Lists can be defined in a variety of ways: They may be completely typed in, imported from other programs and text files, or they may be created with either the Table or Array commands. Given a function $f(x)$ and a number $n$, the command

Table and Manipulate have nearly identical syntax. With Manipulate, you can create an interactive dynamic application; Table returns nonadjustable results.

1. Table[f[i],{i,n}] creates the list {f[1], ... ,f[n]};
2. Table[f[i],{i,0,n}] creates the list {f[0], ... ,f[n]};
3. Table[f[i],{i,n,m}] creates the list

   **{f[n], f[n + 1], ... , f[m – 1], f[m]};**

4. Table[f[i],{i,imin,imax,istep}] creates the list

   **{f[imin], f[imin + istep], f[imin + 2*step], ... , f[imax]};**

and

**5.** Array[f,n] creates the list {f[1], ... ,f[n]}.

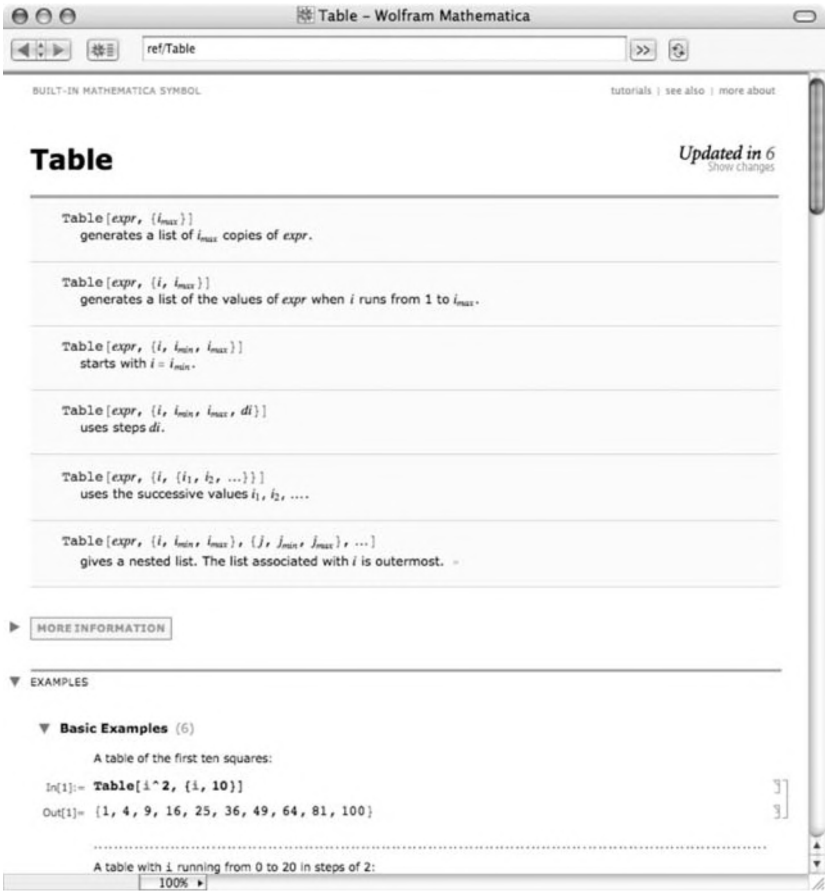
In particular,

**Table[f[x], {x, a, b, (b – a)/(n – 1)}]**

returns a list of $f(x)$ values for $n$ equally spaced values of $x$ between $a$ and $b$;

**Table[{x, f[x]}, {x, a, b, (b – a)/(n – 1)}]**

returns a list of points $(x, f(x))$ for $n$ equally spaced values of $x$ between $a$ and $b$.




In addition to using Table, lists of numbers can be calculated using Range:

**1.** Range[n] generates the list {1,2, . . . ,n};

2. Range[n1,n2] generates the list {n1,n1+1,...,n2-1,n2}; and
3. Range[n1,n2,nstep] generates the list

**{n1, n1 + nstep, n1 + 2*nstep, ... , n2 – nstep, n2}**

---

**Example 4.1.1** Use Mathematica to generate the list {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

***Solution*** Generally, a given list can be constructed in several ways. In fact, each of the following five commands generates the list {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

**{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}**
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

**Table[i, {i, 10}]**
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

**Table[i, {i, 1, 10}]**
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

**Table[$\frac{i}{2}$, {i, 2, 20, 2}]**
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

**Range[10]**
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

---

**Example 4.1.2** Use Mathematica to define listone to be the list of numbers {1, 3/2, 2, 5/2, 3, 7/2, 4}.

***Solution*** In this case, we generate a list and name the result listone. As in Example 4.1.1, we illustrate that listone can be created in several ways.

**listone = {1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4}**
{1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4}

**listone = Table[i, {i, 1, 4, $\frac{1}{2}$}]**
{1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4}

Last, we define $i(n) = \frac{1}{2}n + \frac{1}{2}$ and use Array to create the table listone.

**i[n_] = $\frac{n}{2}$ + $\frac{1}{2}$;**
**listone = Array[i, 7]**
{1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4}

---

**Example 4.1.3** Create a list of the first 25 prime numbers. What is the 15th prime number?

***Solution*** The command Prime[n] yields the $n$th prime number. We use Table to generate a list of the ordered pairs {n,Prime[n]} for $n = 1, 2, 3, \ldots, 25$ and name the resulting list

list. We then use Short to obtain an abbreviated portion of list. Generally, Short returns the first and last few elements of a list. The number of omitted terms between the first few and last few is indicated with <<n>>. In this case, we see that 17 terms are omitted.

```
list = Table[{n, Prime[n]}, {n, 1, 25}];
Short[list]
{{1, 2}, {2, 3}, {3, 5}, {4, 7},
⟨⟨17⟩⟩, {22, 79}, {23, 83}, {24, 89}, {25, 97}}
```

The *i*th element of a list list is extracted from list with list[[i]] or Part[list,i]. From the resulting output, we see that the 15th prime number is 47.

```
list[[15]]
{15, 47}
```

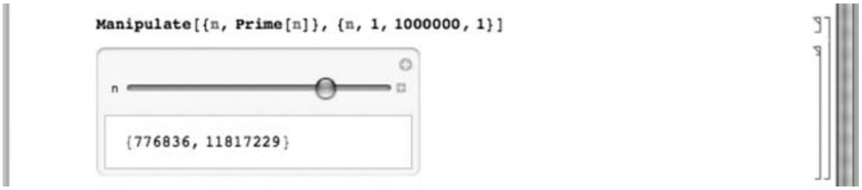
```
Part[list, 15]
{15, 47}
```

---

You can use the Manipulate function in nearly the exact same way as the Table function. With Manipulate, the result is an interactive dynamic object that can be saved as an application that can be run outside of Mathematica. With


we let $i$ and $i/2$ vary continuously for $1 \leq i \leq 10$. By making the stepsize be 1, integer values of $i$ are only allowed.


With the following Manipulate command, you can see $n$ and the $n$th prime number for $1 \leq n \leq 1000000$.




**In addition, we can use Table to generate lists consisting of the same or similar objects.**

---

**Example 4.1.4** (a) Generate a list consisting of five copies of the letter $a$. (b) Generate a list consisting of $10$ random integers between $-10$ and $10$ and then a list of $10$ random real numbers between $-10$ and $10$.

***Solution*** Entering

**Table[a, {5}]**
{a, a, a, a, a}

generates a list consisting of five copies of the letter $a$. For (b), we use the commands RandomInteger and RandomReal to generate the desired lists. Because we are using RandomInteger and RandomReal, your results will certainly differ from those obtained here.

**RandomInteger[{−10, 10}, 10]**
{3, −5, 5, −8, 0, −2, −2, 2, 7, 9}

**RandomReal[{−10, 10}, 10]**
{−3.42641, 4.76027, −3.49249, −9.11795, 3.72502, 7.39518,
−6.84238, −7.85735, 4.94279, −9.4021}

---

**As illustrated previously, Manipulate works in much the same way as Table but allows you to interactively see how adjusting parameters affects a given situation.**

---

**Example 4.1.5** In polar coordinates, the graphs of $r = \sin n\theta$ and $r = \cos n\theta$ are $n$-leaved roses if $n$ is odd and $2n$-leaved roses if $n$ is even. If $n$ is even, the area of the graph enclosed by the $2n$ roses is $A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \pi/2$. If $n$ is odd, the area of the graph enclosed by the $n$ roses is $A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \pi/4$.

To see this with Mathematica, we can use Table. (See Figure 4.1.) (Note that If[condition,f,g] returns f if condition is True and g if it is not.)

**Clear[n, x];**
**t1 = Table[{**

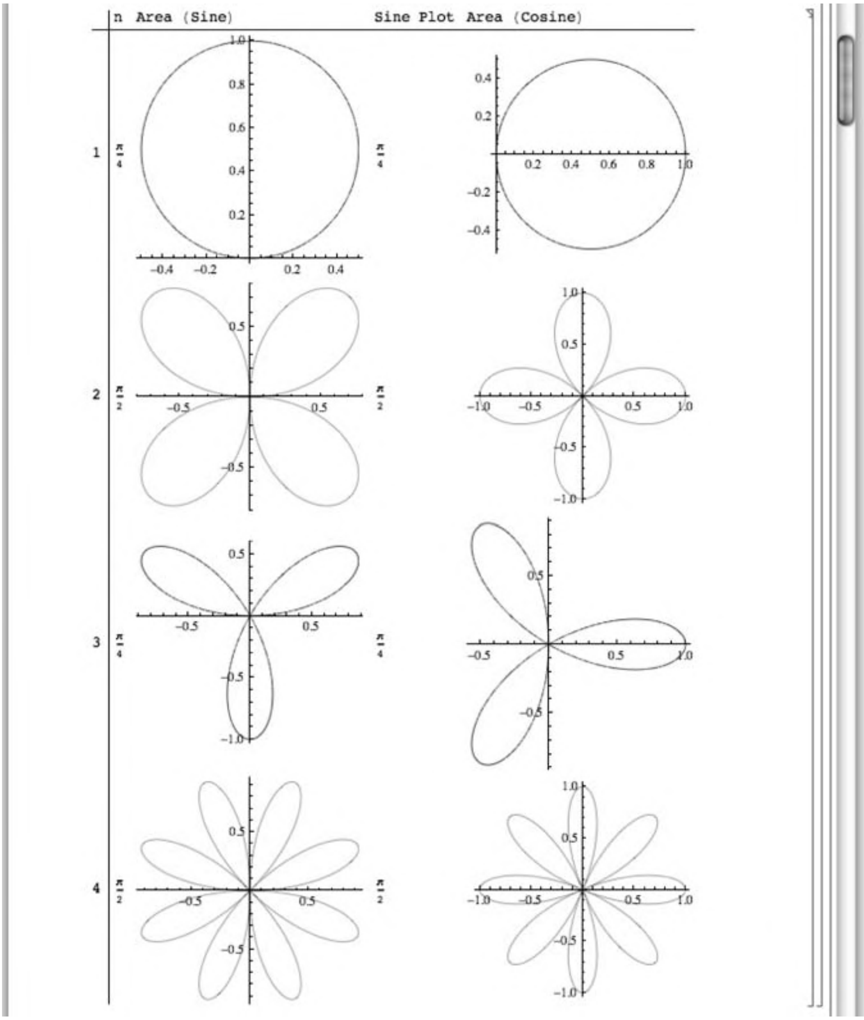



**FIGURE 4.1**

You can use Table to see that the area of the roses depends only on whether $n$ is odd or even

```
If[Mod[n/2, 1] === 0, Integrate[Sin[nx]^2, {x, 0, 2Pi}]/2,
  Integrate[Sin[nx]^2, {x, 0, Pi}]/2],
  PolarPlot[Sin[nx], {x, 0, 2Pi}],
If[Mod[n/2, 1] === 0, Integrate[Cos[nx]^2, {x, 0, 2Pi}]/2,
  Integrate[Cos[nx]^2, {x, 0, Pi}]/2],
  PolarPlot[Cos[nx], {x, 0, 2Pi}]}, {n, 1, 5}];
```

```
TableForm[t1,
  TableHeadings → {Table[n, {n, 1, 5}],
   {"n", "Area(Sine)", "SinePlot", "Area(Cosine)", "CosinePlot"}}]
```

Alternatively, you can use Manipulate. (See Figure 4.2.)

```
Clear[n, x];
Manipulate[{n,
  If[Mod[n/2, 1] === 0, Integrate[Sin[nx]^2, {x, 0, 2Pi}]/2,
   Integrate[Sin[nx]^2, {x, 0, Pi}]/2],
   PolarPlot[Sin[nx], {x, 0, 2Pi}],
  If[Mod[n/2, 1] === 0, Integrate[Cos[nx]^2, {x, 0, 2Pi}]/2,
   Integrate[Cos[nx]^2, {x, 0, Pi}]/2],
   PolarPlot[Cos[nx], {x, 0, 2Pi}]}, {{n, 5}, 1, 100, 1}]
```

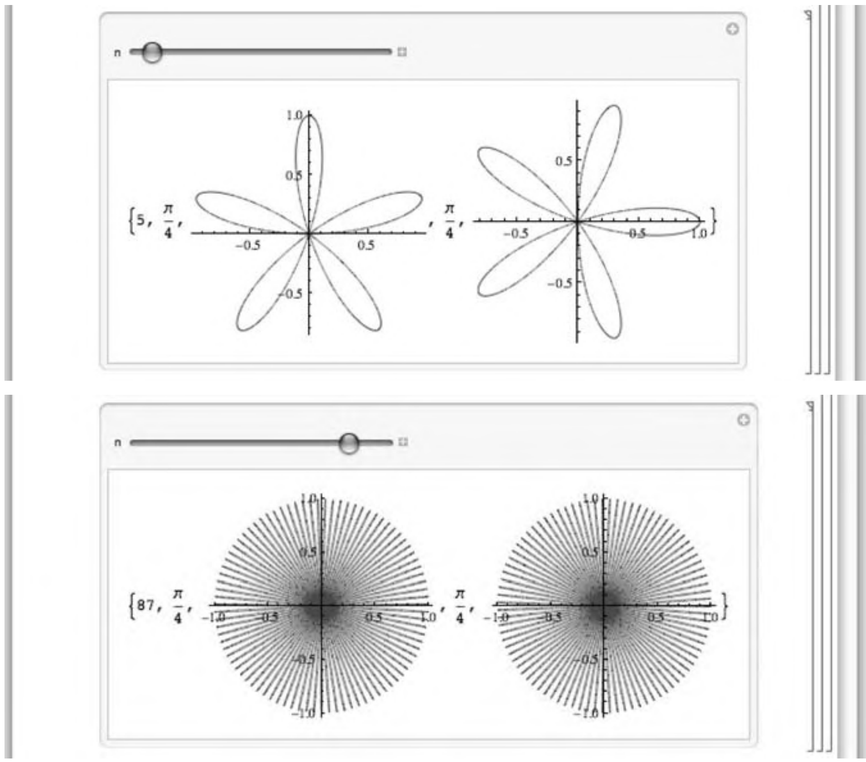


**FIGURE 4.2**

With Manipulate you can see that the area alternates from $\pi/2$ to $\pi/4$ as $n$ alternates from even to odd

## 4.1.2 Plotting Lists of Points

Lists are plotted with ListPlot.

1. ListPlot[{{x1,y1},{x2,y2},...,{xn,yn}}] plots the list of points $\{(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\}$. The size of the points in the resulting plot is controlled with the option PlotStyle->PointSize[w], where $w$ is the fraction of the total width of the graphic. For two-dimensional graphics, the default value is 0.008.
2. ListPlot[{y1,y2,..,yn}] plots the list of points $\{(1,y_1),(2,y_2),\ldots,(n,y_n)\}$.

To connect the consecutive points with line segments, use the option Joined->True.

**Example 4.1.6** Entering

When a semicolon is included at the end of a command, the resulting output is suppressed.

```
t1 = Table[Sin[n], {n, 1, 1000}];
ListPlot[t1]
```

creates a list consisting of $\sin n$ for $n = 1, 2, \ldots, 1000$ and then graphs the list of points $(n, \sin n)$ for $n = 1, 2, \ldots, 1000$. See Figure 4.3.

**Example 4.1.7** **(The Prime Difference Function and the Prime Number Theorem).** In t1, we use Prime and Table to compute a list of the first 25,000 prime numbers.

```
t1 = Table[Prime[n], {n, 1,25000}];
```

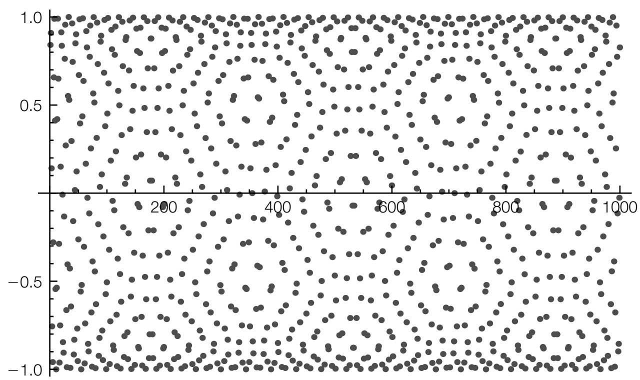



**FIGURE 4.3**

Plot of $(n, \sin n)$ for n = 1, 2, … , 1000

We use Length to verify that t1 has 25,000 elements and Short to see an abbreviated portion of t1.

```
Length[t1]
25000
Short[t1]
{2, 3, 5, 7, 11, 13, 17, 19, ⟨⟨24984⟩⟩, 287047,
287057, 287059, 287087, 287093, 287099, 287107, 287117}
```

You can also use Take to extract elements of lists.

First[list] returns the first element of list; Last[list] returns the last element of list.

1. Take[list,n] returns the first $n$ elements of list;
2. Take[list,-n] returns the last $n$ elements of list; and
3. Take[list,{n,m}] returns the $n$th through $m$th elements of list.

```
Take[t1, 5]
{2, 3, 5, 7, 11}
Take[t1, -5]
{287087, 287093, 287099, 287107, 287117}
Take[t1, {12501, 12505}]
{134059, 134077, 134081, 134087, 134089}
```

Span is new in Mathematica 6 but works in almost the same way as Take.

Working in almost the same way as Take, Span (; ;) selects elements of lists: list[[n;;m returns the $n$ through $m$th elements of list.

---

**Example 4.1.8** Here are the first few terms of sequence A073184,[1] the number of cube free divisors of $n$:

```
ashortlist = {1, 2, 2, 3, 2, 4, 2, 3, 3, 4, 2, 6,
        2, 4, 4, 3, 2, 6, 2, 6, 4, 4, 2, 6,
      3, 4, 3, 6, 2, 8, 2, 3, 4, 4, 4};
```

With ;; (Span), we select the second through eighth elements of ashortlist.

```
ashortlist[[2; ; 8]]
{2, 2, 3, 2, 4, 2, 3}
```

The same results are obtained with Take.

```
Take[ashortlist, {2, 8}]
{2, 2, 3, 2, 4, 2, 3}
```

You can count the number of elements of a list with Length.

```
Length[ashortlist]
35
```

---

[1] Sloane, N. J. A., *The On-Line Encyclopedia of Integer Sequences*, www.research.att.com/njas/sequences, 2007.

With Tally, we count the number of occurrences of each digit in the list. Thus,

**Tally[ashortlist]**
{{1, 1}, {2, 11}, {3, 7}, {4, 10}, {6, 5}, {8, 1}}

shows us that there are eleven 2's, ten 4's, and so on.

However, you can use Table together with Part ([[...]]) to obtain the same results as those obtained with Take or Span.

**Table[t1[[i]], {i, 1, 5}]**
**Table[t1[[i]], {i, 24996, 25000}]**
**Table[t1[[i]], {i, 12501, 12505}]**
{2, 3, 5, 7, 11}
{287087, 287093, 287099, 287107, 287117}
{134059, 134077, 134081, 134087, 134089}

In t2, we compute the difference, $d_n$, between the successive prime numbers in t1. The result is plotted with ListPlot in Figure 4.4.

list[[i]] returns the $i$th element of list so list[[i+1]] – list[[i]] computes the difference between the $(i + 1)$st and $i$th elements of list.

**t2 = Table[t1[[i + 1]] – t1[[i]], {i, 1, Length[t1] – 1}];**
**Short[t2]**
{1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, ⟨⟨24967⟩⟩,
28, 14, 54, 46, 8, 6, 12, 4, 44, 10, 2, 28, 6, 6, 8, 10}
**ListPlot[t2, PlotRange → All]**

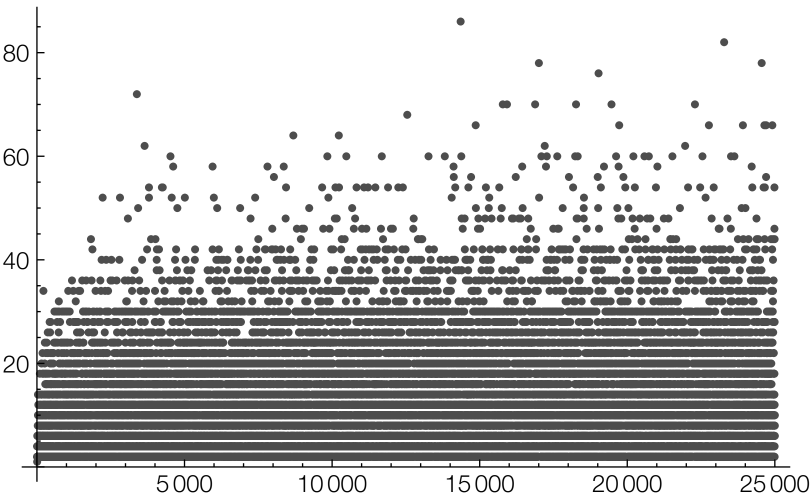



**FIGURE 4.4**

A plot of the difference, $d_n$, between successive prime numbers

Let $\pi(n)$ denote the number of primes less than $n$ and $Li(x)$ denote the **logarithmic integral:**

$$\text{LogIntegral[x]} = Li(x) = \int_0^x \frac{1}{\ln t}\,dt.$$

Remember that p1 is not displayed because a semicolon is included at the end of the Plot command.

We use Plot to graph $Li(x)$ for $1 \le x \le 25{,}000$ in p1.

```
p1 = Plot[LogIntegral[x], {x, 1, 2500}];
```

The **prime number theorem** states that

$$\pi(n) \sim Li(n).$$

(See [20].) In the following, we use Select and Length to define $\pi(n)$. Select[list,criteria] returns the elements of list for which criteria is true. Note that #<n is called a **pure function**: Given an argument #, #<n is true if #<n and false otherwise. The & symbol marks the end of a pure function. Thus, given $n$, Select[t1,#<n&] returns a list of the elements of t1 less than $n$; Select[t1,#<n&]//Length returns the number of elements in the list.

```
smallpi[n_] := Select[t1, # < n&] / /Length
```

For example,

```
smallpi[100]
25
```

shows us that $\pi(100) = 25$. Note that because t1 contains the first $25{,}000$ primes, smallpi[n] is valid for $1 \le n \le N$, where $\pi(N) = 25{,}000$. In t3, we compute $\pi(n)$ for $n = 1, 2, \ldots, 25{,}000$

```
t3 = Table[smallpi[n], {n, 1, 2500}];
Short[t3]
{0, 0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, ⟨⟨2475⟩⟩,
367, 367, 367, 367, 367, 367, 367, 367, 367, 367, 367, 367}
```

and plot the resulting list with ListPlot.

```
p2 = ListPlot[t3, PlotStyle → GrayLevel[0.4]]
```

p1 and p2 are displayed together with Show in Figure 4.5.

```
Show[p1, p2]
```

You can iterate recursively with Table. Both

```
t1 = Table[a[i, j], {j, 2, 10, 2}, {i, 1, 5}]
{{a[1, 2], a[2, 2], a[3, 2], a[4, 2], a[5, 2]},
{a[1, 4], a[2, 4], a[3, 4], a[4, 4], a[5, 4]}, {a[1, 6], a[2, 6], a[3, 6], a[4, 6], a[5, 6]},
{a[1, 8], a[2, 8], a[3, 8], a[4, 8], a[5, 8]}, {a[1, 10], a[2, 10], a[3, 10],
   a[4, 10], a[5, 10]}}
Length[t1]
5
```

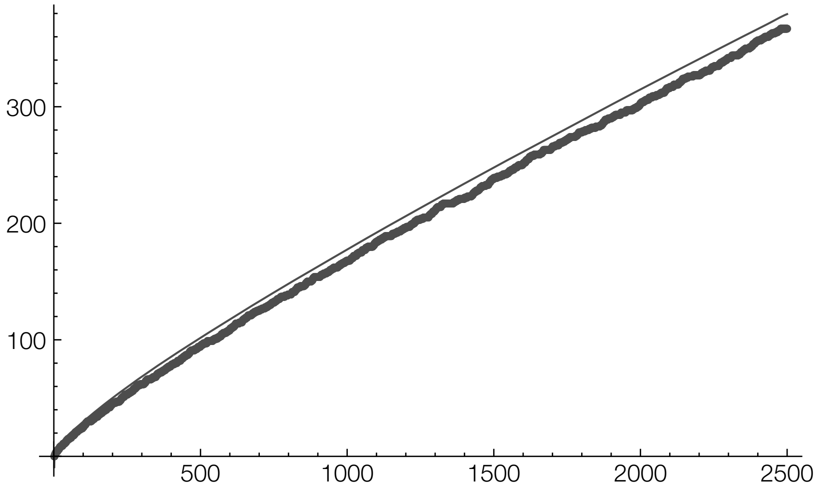



**FIGURE 4.5**

Graphs of $Li(x)$ (in black) and $\pi(n)$ (in gray)

and

```
t2 = Table[Table[a[i, j], {i, 1, 5}], {j, 2, 10, 2}]
{{a[1, 2], a[2, 2], a[3, 2], a[4, 2], a[5, 2]}, {a[1, 4], a[2, 4], a[3, 4], a[4, 4], a[5, 4]},
{a[1, 6], a[2, 6], a[3, 6], a[4, 6], a[5, 6]}, {a[1, 8], a[2, 8], a[3, 8], a[4, 8], a[5, 8]},
{a[1, 10], a[2, 10], a[3, 10], a[4, 10], a[5, 10]}}
```

compute tables of $a_{ij}$. The outermost iterator is evaluated first: In this case, $i$ is followed by $j$ as in t1 and the result is a list of lists. To eliminate the inner lists (that is, the braces), use Flatten. Generally, Flatten[list,n] flattens list (removes braces) to level $n$.

```
Flatten[t1]
{a[1, 2], a[2, 2], a[3, 2], a[4, 2], a[5, 2], a[1, 4], a[2, 4], a[3, 4], a[4, 4],
a[5, 4], a[1, 6], a[2, 6], a[3, 6], a[4, 6], a[5, 6], a[1, 8], a[2, 8], a[3, 8], a[4, 8], a[5, 8],
a[1, 10], a[2, 10], a[3, 10], a[4, 10], a[5, 10]}
```

The observation is especially important when graphing lists of points obtained by iterating Table. For example,

Length[list] returns the number of elements in list.

```
t1 = Table[{Sin[x + y], Cos[x - y]}, {x, 1, 5}, {y, 1, 5}]
{{{Sin[2], 1}, {Sin[3], Cos[1]}, {Sin[4], Cos[2]}, {Sin[5], Cos[3]}, {Sin[6], Cos[4]}},
{{Sin[3], Cos[1]}, {Sin[4], 1}, {Sin[5], Cos[1]}, {Sin[6], Cos[2]}, {Sin[7], Cos[3]}},
{{Sin[4], Cos[2]}, {Sin[5], Cos[1]}, {Sin[6], 1}, {Sin[7], Cos[1]}, {Sin[8], Cos[2]}},
{{Sin[5], Cos[3]}, {Sin[6], Cos[2]}, {Sin[7], Cos[1]}, {Sin[8], 1}, {Sin[9], Cos[1]}},
{{Sin[6], Cos[4]}, {Sin[7], Cos[3]}, {Sin[8], Cos[2]}, {Sin[9], Cos[1]}, {Sin[10], 1}}}
Length[t1]
5
```

is not a list of 25 points: t1 is a list of 5 lists each consisting of 5 points. t1 has two levels. For example, the third element of the second level is

```
t1[[3]]
{{Sin[4], Cos[2]}, {Sin[5], Cos[1]}, {Sin[6], 1}, {Sin[7], Cos[1]}, {Sin[8], Cos[2]}}
```

and the second element of the third level (or the second part of the third part) is

```
t1[[3, 2]]
{Sin[5], Cos[1]}
```

To flatten t2 to level 1, we use Flatten.

```
t2 = Flatten[t1, 1];
```

The resulting list of ordered pairs (in Mathematica, {x,y} corresponds to $(x, y)$) is not displayed because a semicolon is placed at the end of the Flatten command. These are plotted with ListPlot in Figure 4.6(a). We also illustrate the use of the PlotStyle, PlotRange, and AspectRatio options in the ListPlot command.

```
lp1 = ListPlot[t2, PlotStyle → {PointSize[.05], GrayLevel[.5]},
      PlotRange → {{−3/2, 3/2}, {−3/2, 3/2}}, AspectRatio → Automatic];
```

Increasing the number of points further illustrates the use of Flatten. Entering

```
t1 = Table[{Sin[x + y], Cos[x − y]}, {x, 1, 125}, {y, 1, 125}];
Length[t1]
125
```

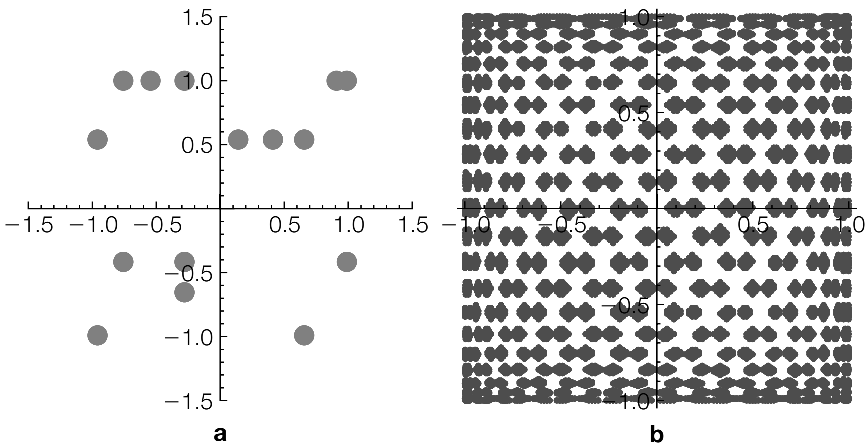



**FIGURE 4.6**

(a) and (b)

results in a very long nested list. t1 has 125 elements, each of which has 125 elements.

An abbreviated version is viewed with Short.

Short[list] yields an abbreviated version of list.

```
Short[t1]
{{{Sin[2], 1}, {Sin[3], Cos[1]}, ⟨⟨121⟩⟩,
{Sin[125], Cos[123]}, {Sin[126], Cos[124]}}, ⟨⟨124⟩⟩}
```

After using Flatten, we see with Length and Short that t2 contains 15,625 points,

```
t2 = Flatten[t1, 1];
Length[t2]
15625
Short[t2]
{{Sin[2], 1}, {Sin[3], Cos[1]}, ⟨⟨15621⟩⟩, {Sin[249], Cos[1]}, {Sin[250], 1}}
```

which are plotted with ListPlot in Figure 4.6(b).

```
lp2 = ListPlot[t2, AspectRatio → Automatic];
Show[GraphicsRow[{lp1, lp2}]]
Show[GraphicsRow[{lp1, lp2}]]
```

**Remark 4.1** Mathematica is very flexible and most calculations can be carried out in more than one way. Depending on how you think, some sequences of calculations may make more sense to you than others, even if they are less efficient than the most efficient way to perform the desired calculations. Often, the difference in time required for Mathematica to perform equivalent—but different—calculations is quite small. For the beginner, we think it is wisest to work with familiar calculations first and then efficiency.

**Example 4.1.9** **(Dynamical Systems).** A sequence of the form $x_{n+1} = f(x_n)$ is called a **dynamical system.**

Sometimes, unusual behavior can be observed when working with dynamical systems. For example, consider the dynamical system with $f(x) = x + 2.5x(1 - x)$ and $x_0 = 1.2$. Note that we define $x_n$ using the form x[n_] := x[n] = ... so that Mathematica "remembers" the functional values it computes and thus avoids recomputing functional values previously computed. This is particularly advantageous when we compute the value of $x_n$ for large values of $n$.

Observe that $x_{n+1} = f(x_n)$ can also be computed with $x_{n+1} = f^n(x_0)$.

```
Clear[f, x]
f[x_] := x + 2.5x(1 − x)
x[n_] := x[n] = f[x[n − 1]]
x[0] = 1.2;
```

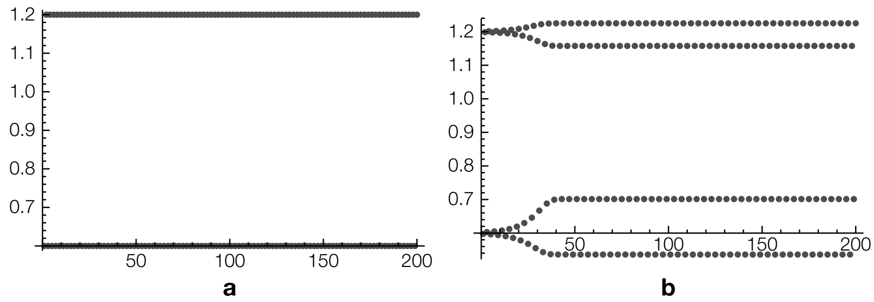



**FIGURE 4.7**

(a) A 2-cycle. (b) A 4-cycle

In Figure 4.7(a), we see that the sequence $x_n$ oscillates between the numbers $0.6$ and $1.2$. We say that the dynamical system has a **2-cycle** because the values of the sequence oscillate between two numbers.

```
tb = Table[x[n], {n, 1, 200}]; ListPlot[tb]
```

In Figure 4.7(b), we see that changing $x_0$ from $1.2$ to $1.201$ results in a 4-cycle.

```
Clear[f, x]
f[x_] := x + 2.5x(1 − x)
x[n_] := x[n] = f[x[n − 1]]
x[0] = 1.201;
tb = Table[x[n], {n, 1, 200}];
ListPlot[tb]
```

The calculations indicate that the behavior of the system can change considerably for small changes in $x_0$. With the following, we adjust the definition of $x$ so that $x$ depends on $x_0 = c$: Given $c$, $x_c(0) = c$.

```
Clear[f, x]
f[x_] := x + 2.5x(1 − x)
x[c_][n_] := x[c][n] = f[x[c][n − 1]]//N
x[c_][0] := c//N;
```

In tb, we create a list of lists of the form $\{x_c(n)|n = 100, \ldots, 150\}$ for $150$ equally spaced values of $c$ between $0$ and $1.5$. Observe that Mathematica issues several error messages. When a Mathematica calculation is larger than the machine's precision, we obtain an Overflow warning. In numerical calculations, we interpret Overflow to correspond to $\infty$.

```
tb = Table[{c, x[c][n]}, {c, 0, 1.5, .01}, {n, 100, 150}];
General::ovfl : Overflow occurred in computation. ⟩⟩
General::ovfl : Overflow occurred in computation. ⟩⟩
General::ovfl : Overflow occurred in computation. ⟩⟩
General::stop : Further output of General::ovfl will be suppressed during this
    calculation. ⟩⟩
```

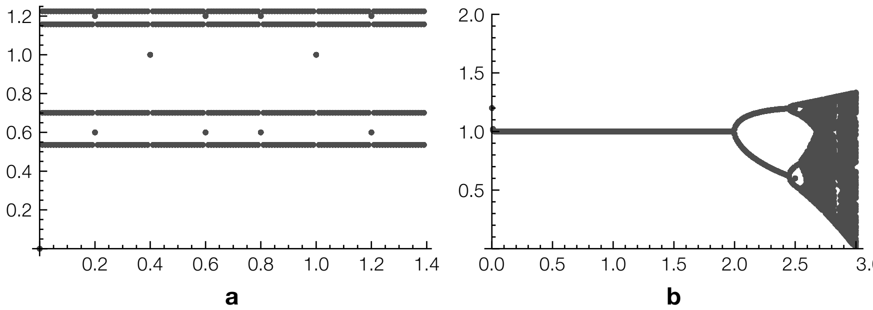



**FIGURE 4.8**

(a) and (b)

Short[expr] prints an abbreviated form of expr.

We ignore the error messages and use Short to view an abbreviated form of tb.

```
Short[tb]
{{{0., 0.}, {0., 0.}, {0., 0.}, ⟨⟨45⟩⟩, {0., 0.}, {0., 0.}, {0., 0.}}, ⟨⟨149⟩⟩, {⟨⟨1⟩⟩}}
```

We then use Flatten to convert tb to a list of points that are plotted with ListPlot in Figure 4.8(a).

```
tb2 = Flatten[tb, 1];
f1 = ListPlot[tb2];
```

Another interesting situation occurs if we fix $x_0$ and let $c$ vary in $f(x) = x + cx(1 - x)$.

With the following, we set $x_0 = 1.2$ and adjust the definition of $f$ so that $f$ depends on $c$: $f(x) = x + cx(1 - x)$.

```
Clear[f, x]
f[c_][x_] := x + cx(1 − x)//N
x[c_][n_] := x[c][n] = f[c][x[c][n − 1]]//N
x[c_][0] := 1.2//N;
```

In tb, we create a list of lists of the form $\{x_c(n)|n = 200, \ldots, 300\}$ for 350 equally spaced values of $c$ between 0 and 3.5. As before, Mathematica issues several error messages, which we ignore and which are not displayed here due to length considerations.

```
tb = Table[{c, x[c][n]}, {c, 0, 3.5, .01}, {n, 200, 300}];
Short[tb]
{{{0., 1.2}, {0., 1.2}, {0., 1.2}, ⟨⟨95⟩⟩, {0., 1.2}, {0., 1.2}, {0., 1.2}}, ⟨⟨350⟩⟩}
```

tb is then converted to a list of points with Flatten and the resulting list is plotted in Figure 4.8(b) with ListPlot. This plot is called a **bifurcation diagram**.

```
tb2 = Flatten[tb, 1];
f2 = ListPlot[tb2, PlotRange → {0, 2}]
Show[GraphicsRow[{f1, f2}]]
```

A function $f$ is **listable** if f[list] and Map[f,list] return the same results.

As indicated previously, elements of lists can be numbers, ordered pairs, functions, and even other lists. You can also use Mathematica to manipulate lists in numerous ways. Most important, the Map function is used to apply a function to a list: Map[f,{x1,x2, . . . ,xn}] returns the list $\{f(x_1), f(x_2), \ldots, f(x_n)\}$. We discuss other operations that can be performed on lists in the following sections.

---

**Example 4.1.10** **(Hermite Polynomials).** The **Hermite polynomials**, $H_n(x)$, satisfy the differential equation $y'' - 2xy' + 2ny = 0$ and the orthogonality relation $\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \delta_{mn}2^n n!\sqrt{\pi}$. The Mathematica command HermiteH[n,x] yields the Hermite polynomial $H_n(x)$. (a) Create a table of the first five Hermite polynomials. (b) Evaluate each Hermite polynomial if $x = 1$. (c) Compute the derivative of each Hermite polynomial in the table. (d) Compute an antiderivative of each Hermite polynomial in the table. (e) Graph the five Hermite polynomials on the interval $[-1, 1]$. (f) Verify that $H_n(x)$ satisfies $y'' - 2xy' + 2ny = 0$ for $n = 1, 2, \ldots, 5$ ($'$ denotes $d/dx$).

***Solution*** We proceed by using HermiteH together with Table to define hermitetable to be the list consisting of the first five Hermite polynomials.

**hermitetable = Table[HermiteH[n, x], {n, 1, 5}]**

$\{2x, -2 + 4x^2, -12x + 8x^3, 12 - 48x^2 + 16x^4, 120x - 160x^3 + 32x^5\}$

We then use ReplaceAll (->) to evaluate each member of hermitetable if $x$ is replaced by 1.

**hermitetable/.x → 1**

{2, 2, −4, −20, −8}

Functions such as D and Integrate are listable. Thus, each of the following commands differentiates each element of hermitetable with respect to $x$. In the second case, we have used a *pure function:* Given an argument #, D[#,x]& differentiates # with respect to $x$. Use the & symbol to indicate the end of a pure function.

**D[hermitetable, x]**

$\{2, 8x, -12 + 24x^2, -96x + 64x^3, 120 - 480x^2 + 160x^4\}$

**Map[D[#, x]&, hermitetable]**

$\{2, 8x, -12 + 24x^2, -96x + 64x^3, 120 - 480x^2 + 160x^4\}$

Similarly, we use Integrate to antidifferentiate each member of hermitetable with respect to $x$. Remember that Mathematica does not automatically include the "$+C$" that we include when we antidifferentiate.

**Integrate[hermitetable, x]**

$\left\{x^2, -2x + \frac{4x^3}{3}, -6x^2 + 2x^4, 12x - 16x^3 + \frac{16x^5}{5}, 60x^2 - 40x^4 + \frac{16x^6}{3}\right\}$

**Map[Integrate[#, x]&, hermitetable]**

$\left\{x^2, -2x + \frac{4x^3}{3}, -6x^2 + 2x^4, 12x - 16x^3 + \frac{16x^5}{5}, 60x^2 - 40x^4 + \frac{16x^6}{3}\right\}$

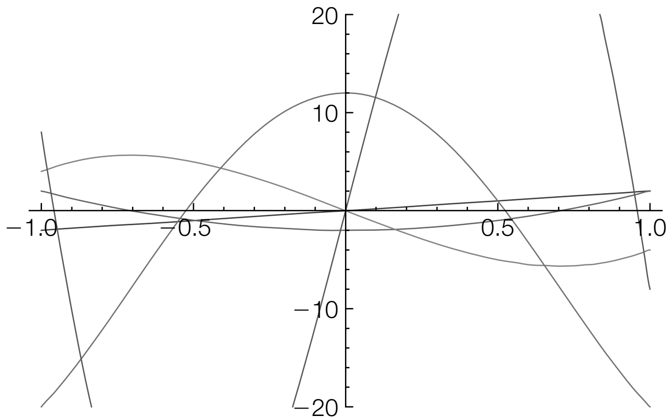



**FIGURE 4.9**

Graphs of $H_1(x)$, $H_2(x)$, $H_3(x)$, $H_4(x)$, and $H_5(x)$

To graph the list hermitetable, we use Plot to plot each function in the set hermitetable on the interval $[-2, 2]$ in Figure 4.9. In this case, we specify that the displayed $y$-values correspond to the interval $[-20, 20]$. Because we apply Tooltip to the set of functions being plotted, you can identify each curve by moving the cursor and placing it over each curve to see which function is being plotted.

```
Plot[Tooltip[hermitetable], {x, -1, 1}, PlotRange → {-20, 20}]
```

hermitetable[[n]] returns the $n$th element of hermitetable, which corresponds to $H_n(x)$. Thus,

```
verifyde =
Table[D[hermitetable[[n]], {x, 2}] - 2x D[hermitetable[[n]], x]+
    2n hermitetable[[n]]//Simplify, {n, 1, 5}]
{0, 0, 0, 0, 0}
```

computes and simplifies $H_n'' - 2xH_n' + 2nH_n$ for $n = 1, 2, \ldots, 5$. We use Table and Integrate to compute $\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx$ for $n = 1, 2, \ldots, 5$ and $m = 1, 2, \ldots, 5$.

```
verifyortho =
Table[Integrate[hermitetable[[n, 2]]hermitetable[[m, 2]]
    Exp[-x^2], {x, -Infinity, Infinity}], {n, 1, 5}, {m, 1, 5}]
```

$\left\{\left\{\frac{\sqrt{\pi}}{2}, 0, 6\sqrt{\pi}, 0, -120\sqrt{\pi}\right\}, \left\{0, 12\sqrt{\pi}, 0, -144\sqrt{\pi}, 0\right\},\right.$
$\left\{6\sqrt{\pi}, 0, 120\sqrt{\pi}, 0, -2400\sqrt{\pi}\right\}, \left\{0, -144\sqrt{\pi}, 0, 1728\sqrt{\pi}, 0\right\},$
$\left.\left\{-120\sqrt{\pi}, 0, -2400\sqrt{\pi}, 0, 48000\sqrt{\pi}\right\}\right\}$

To view a table in traditional row-and-column form use TableForm, as we do here illustrating the use of the TableHeadings option.

```
TableForm[verifyortho,
  TableHeadings → {{"m = 1", "m = 2", "m = 3", "m = 4", "m = 5"},
  {"n = 1", "n = 2", "n = 3", "n = 4", "n = 5"}}]
```

| | n = 1 | n = 2 | n = 3 | n = 4 | n = 5 |
|---|---|---|---|---|---|
| m = 1 | $\frac{\sqrt{\pi}}{2}$ | 0 | $6\sqrt{\pi}$ | 0 | $-120\sqrt{\pi}$ |
| m = 2 | 0 | $12\sqrt{\pi}$ | 0 | $-144\sqrt{\pi}$ | 0 |
| m = 3 | $6\sqrt{\pi}$ | 0 | $120\sqrt{\pi}$ | 0 | $-2400\sqrt{\pi}$ |
| m = 4 | 0 | $-144\sqrt{\pi}$ | 0 | $1728\sqrt{\pi}$ | 0 |
| m = 5 | $-120\sqrt{\pi}$ | 0 | $-2400\sqrt{\pi}$ | 0 | $48000\sqrt{\pi}$ |

Be careful when using TableForm: TableForm[table] is no longer a list and cannot be manipulated like a list.

## 4.2 MANIPULATING LISTS: MORE ON PART AND MAP

Often, Mathematica's output is given to us as a list that we need to use in subsequent calculations. Elements of a list are extracted with Part ([[...]]): list[[i]] returns the $i$th element of list, list[[i,j]] (or list[[i]][[j]]) returns the $j$th element of the $i$th element of list, and so on.

**Example 4.2.1** Let $f(x) = 3x^4 - 8x^3 - 30x^2 + 72x$. Locate and classify the critical points of $y = f(x)$.

***Solution*** We begin by clearing all prior definitions of $f$ and then defining $f$. The critical numbers are found by solving the equation $f'(x) = 0$. The resulting list is named critnums.

```
Clear[f]
f[x_] = 3x^4 - 8x^3 - 30x^2 + 72x;
critnums = Solve[f'[x]==0]
{{x → -2}, {x → 1}, {x → 3}}
```

critnums is actually a list of lists. For example, the number $-2$ is the second part of the first part of the second part of critnums.

```
critnums[[1]]
{x → -2}
critnums[[1, 1]]
x → -2
critnums[[1, 1, 2]]
-2
```

Similarly, the numbers $1$ and $3$ are extracted with critnums[[2,1,2]] and critnums[[3,1,2]], respectively.

**critnums[[2, 1, 2]]**
**critnums[[3, 1, 2]]**
1
3

We locate and classify the points by evaluating $f(x)$ and $f''(x)$ for each of the numbers in critnums. f[x]/.x->a replaces each occurrence of $x$ in $f(x)$ by $a$, so entering

When you plot lists of functions and apply Tooltip to the list being plotted, you can identify each curve by sliding the cursor over the curve. When the cursor is on a curve, the definition of the curve being plotted is displayed.

**{{x, f[x], f''[x]}}/.critnums**
{{−2, −152, 180}, {1, 37, −72}, {3, −27, 120}}

replaces each $x$ in the list $\{x, f(x), f''(x)\}$ by each of the $x$-values in critnums.

By the second derivative test, we conclude that $y = f(x)$ has relative minima at the points $(-2, -152)$ and $(3, -27)$, whereas $f(x)$ has a relative maximum at $(1, 37)$. In fact, because $\lim_{x\to\pm\infty} = \infty$, $-152$ is the absolute minimum value of $f(x)$. These results are confirmed by the graph of $y = f(x)$ in Figure 4.10.

**Plot[Tooltip[{f[x], f'[x], f''[x]}], {x, −4, 4}]**

---

Map is a very powerful and useful function: Map[f,list] creates a list consisting of elements obtained by evaluating f for each element of list, provided that each member of list is an element of the domain of f. Note that if f is **listable**, f[list] produces the same result as Map[f,list].

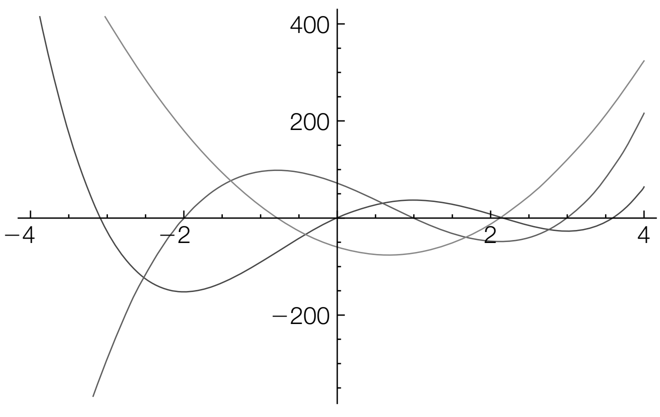



**FIGURE 4.10**

Graph of $f(x) = 3x^4 - 8x^3 - 30x^2 + 72x$, $f'(x)$, and $f''(x)$

**Example 4.2.2** Entering

To determine if f is listable, enter Attributes[f].

```
t1 = Table[n, {n, 1, 100}];
t1b = Partition[t1, 10];
TableForm[t1b]
```

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

computes a list of the first 100 integers and names the result t1. To see t1, we use Partition to partition t1 in 10 element subsets; the results are displayed in a standard row-and-column form with TableForm. We then define $f(x) = x^2$ and use Map to square each number in t1.

```
f[x_] = x^2,
t2 = Map[f, t1];
t2b = Partition[t2, 10];
TableForm[t2b]
```

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |
| 121 | 144 | 169 | 196 | 225 | 256 | 289 | 324 | 361 | 400 |
| 441 | 484 | 529 | 576 | 625 | 676 | 729 | 784 | 841 | 900 |
| 961 | 1024 | 1089 | 1156 | 1225 | 1296 | 1369 | 1444 | 1521 | 1600 |
| 1681 | 1764 | 1849 | 1936 | 2025 | 2116 | 2209 | 2304 | 2401 | 2500 |
| 2601 | 2704 | 2809 | 2916 | 3025 | 3136 | 3249 | 3364 | 3481 | 3600 |
| 3721 | 3844 | 3969 | 4096 | 4225 | 4356 | 4489 | 4624 | 4761 | 4900 |
| 5041 | 5184 | 5329 | 5476 | 5625 | 5776 | 5929 | 6084 | 6241 | 6400 |
| 6561 | 6724 | 6889 | 7056 | 7225 | 7396 | 7569 | 7744 | 7921 | 8100 |
| 8281 | 8464 | 8649 | 8836 | 9025 | 9216 | 9409 | 9604 | 9801 | 10000 |

The same result is accomplished by the pure function that squares its argument. Note how # denotes the argument of the pure function; the & symbol marks the end of the pure function.

```
t3 = Map[#^2&, t1];
t3b = Partition[t3, 10];
TableForm[t3b]
```

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |
| 121 | 144 | 169 | 196 | 225 | 256 | 289 | 324 | 361 | 400 |
| 441 | 484 | 529 | 576 | 625 | 676 | 729 | 784 | 841 | 900 |
| 961 | 1024 | 1089 | 1156 | 1225 | 1296 | 1369 | 1444 | 1521 | 1600 |
| 1681 | 1764 | 1849 | 1936 | 2025 | 2116 | 2209 | 2304 | 2401 | 2500 |
| 2601 | 2704 | 2809 | 2916 | 3025 | 3136 | 3249 | 3364 | 3481 | 3600 |
| 3721 | 3844 | 3969 | 4096 | 4225 | 4356 | 4489 | 4624 | 4761 | 4900 |
| 5041 | 5184 | 5329 | 5476 | 5625 | 5776 | 5929 | 6084 | 6241 | 6400 |
| 6561 | 6724 | 6889 | 7056 | 7225 | 7396 | 7569 | 7744 | 7921 | 8100 |
| 8281 | 8464 | 8649 | 8836 | 9025 | 9216 | 9409 | 9604 | 9801 | 10000 |

On the other hand, entering

```
t1 = Table[{a, b}, {a, 1, 5}, {b, 1, 5}];
Short[t1]
{{{1, 1}, {1, 2}, {1, 3}, {1, 4}, {1, 5}}, ⟨⟨4⟩⟩}
```

is a list (of length 5) of lists (each of length 5). Use **Flatten** to obtain a list of 25 points, which we name **t2**.

```
t2 = Flatten[t1, 1];
Short[t2]
{{1, 1}, {1, 2}, {1, 3}, {1, 4}, ⟨⟨17⟩⟩, {5, 2}, {5, 3}, {5, 4}, {5, 5}}
```

We then use **Map** to apply $f$ to **t2**.

```
f[{x_, y_}] = {{x, y}, x^2 + y^2};
t3 = Map[f, t2];
Short[t3]
{{{1, 1}, 2}, {{1, 2}, 5}, {{1, 3}, 10}, ⟨⟨20⟩⟩, {{5, 4}, 41}, {{5, 5}, 50}}
```

We accomplish the same result with a pure function. Observe how #[[1]] and #[[2]] are used to represent the first and second arguments: Given a list of length 2, the pure function returns the list of ordered pairs consisting of the first element of the list, the second element of the list (as an ordered pair), and the sum of the squares of the first and second elements (of the first ordered pair).

```
t3b = Map[{{#[[1]], #[[2]]}, #[[1]]^2 + #[[2]]^2}&, t2];
Short[t3b]
{{{1, 1}, 2}, {{1, 2}, 5}, {{1, 3}, 10}, ⟨⟨20⟩⟩, {{5, 4}, 41}, {{5, 5}, 50}}
```

**Example 4.2.3** Make a table of the values of the trigonometric functions $y = \sin x$, $y = \cos x$, and $y = \tan x$ for the principal angles.

***Solution*** We first construct a list of the principal angles, which is accomplished by defining t1 to be the list consisting of $n\pi/4$ for $n = 0, 1, \ldots, 8$ and t2 to be the list consisting of $n\pi/6$ for $n = 0, 1, \ldots, 12$. The principal angles are obtained by taking the union of t1 and t2. Union[t1,t2] joins the lists t1 and t2, removes repeated elements, and sorts the results. If we did not wish to remove repeated elements and sort the result, the command Join[t1,t2] concatenates the lists t1 and t2.

```
t1 = Table[nπ/4, {n, 0, 8}];
t2 = Table[nπ/6, {n, 0, 12}];
prinangles = Union[t1, t2]
```

$$\left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}, 2\pi\right\}$$

The **BasicMathInput** palette:

We can also use the symbol ∪, which is obtained by clicking on the  button on the **BasicMathInput** palette to represent Union.

```
prinangles = t1 ∪ t2
```

$$\left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}, 2\pi\right\}$$

Next, we define $f(x)$ to be the function that returns the ordered quadruple $(x, \sin x, \cos x, \tan x)$ and compute the value of $f(x)$ for each number in prinangles with Map naming the resulting table prinvalues. prinvalues is not displayed because a semicolon is included at the end of the command.

```
Clear[f]
f[x_] = {x, Sin[x], Cos[x], Tan[x]};
prinvalues = Map[f, prinangles];
```

Finally, we use TableForm illustrating the use of the TableHeadings option to display prinvalues in row-and-column form; the columns are labeled $x, \sin x, \cos x$, and $\tan x$.

Remember that the result of using TableForm is not a list, so it cannot be manipulated like lists.

```
TableForm[prinvalues,
    TableHeadings → {None, {"x", "sin(x)", "cos(x)", "tan(x)"}}]
```

| x | sin(x) | cos(x) | tan(x) |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | 1 | 0 | ComplexInfinity |
| $\frac{2\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ |
| $\frac{3\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}$ | $-1$ |
| $\frac{5\pi}{6}$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{\sqrt{3}}$ |
| $\pi$ | 0 | $-1$ | 0 |

| | | | |
|---|---|---|---|
| $\frac{7\pi}{6}$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{5\pi}{4}$ | $-\frac{1}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}$ | 1 |
| $\frac{4\pi}{3}$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{3\pi}{2}$ | $-1$ | 0 | ComplexInfinity |
| $\frac{5\pi}{3}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $-\sqrt{3}$ |
| $\frac{7\pi}{4}$ | $-\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $-1$ |
| $\frac{11\pi}{6}$ | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{\sqrt{3}}$ |
| $2\pi$ | 0 | 1 | 0 |

In the table, note that $y = \tan x$ is undefined at odd multiples of $\pi/2$ and Mathematica appropriately returns ComplexInfinity at those values of $x$ for which $y = \tan x$ is undefined.

**Remark 4.2** The result of using TableForm is not a list (or table) and calculations on it using commands such as Map cannot be performed. TableForm helps you see results in a more readable format. To avoid confusion, do not assign the results of using TableForm any name: Adopting this convention avoids any possible attempted manipulation of TableForm objects.

object=name assigns the object object the name name.

We can use Map on any list, including lists of functions and/or other lists.

Lists of functions are graphed with Plot: Plot[listoffunctions,{x,a,b}] graphs the list of functions of $x$, listoffunctions, for $a \le x \le b$. If the command is entered as Plot[Tooltip[listoffunctions],{x,a,b}], you can identify the curves in the plot by moving the cursor over the curves in the graphic.

**Example 4.2.4** **(Bessel Functions).** The **Bessel functions of the first kind,** $J_n(x)$, are nonsingular solutions of $x^2y'' + xy' + \left(x^2 - n^2\right)y = 0$. BesselJ[n,x] returns $J_n(x)$. Graph $J_n(x)$ for $n = 0, 1, 2, \ldots, 8$.

***Solution*** In t1, we use Table and BesselJ to create a list of $J_n(x)$ for $n = 0, 1, 2, \ldots, 8$.

```
t1 = Table[BesselJ[n, x], {n, 0, 8}];
```

We then use Plot to graph each function in t1 in Figure 4.11. You can identify each curve by sliding the cursor over each.

```
Plot[Tooltip[t1], {x, 0, 25}]
```

A different effect is achieved by graphing each function separately. To do so, we define the function pfunc. Given a function of $x$, f, pfunc[f] plots the function for $0 \le x \le 100$. The resulting graphic is not displayed because the option DisplayFunction->Identity is included in the Plot command. We then use Map to apply pfunc to each element of t1. The result is a list of nine graphics objects, which we name t2. A good way to display nine graphics is as a $3 \times 3$ array, so we use Partition to convert t2 from a list of length 9 to a list of lists, each with length

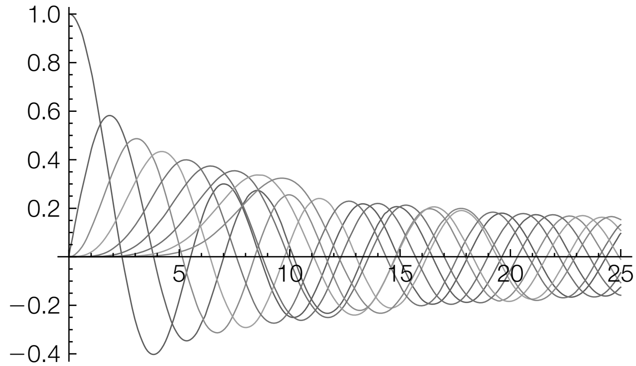



**FIGURE 4.11**

Graphs of $J_n(x)$ for $n = 0, 1, 2, \ldots, 8$

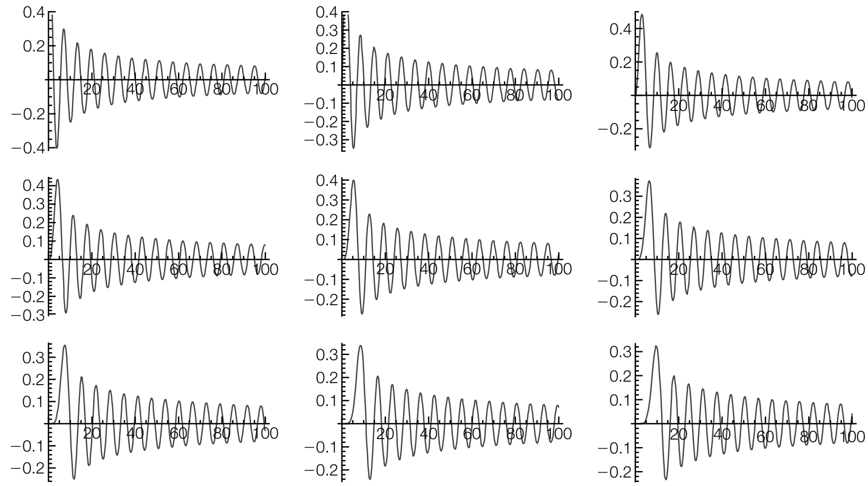



**FIGURE 4.12**

In the first row, from left to right, graphs of $J_0(x)$, $J_1(x)$, and $J_2(x)$; in the second row, from left to right, graphs of $J_3(x)$, $J_4(x)$, and $J_5(x)$; in the third row, from left to right, graphs of $J_6(x)$, $J_7(x)$, and $J_8(x)$

3 – a $3 \times 3$ array. Partition[list,n] returns a list of lists obtained by partitioning list into $n$-element subsets.

Think of Flatten and Partition as inverse functions.

```
pfunc[f_] := Plot[f, {x, 0, 100}];
t2 = Map[pfunc, t1];
t3 = Partition[t2, 3];
```

Instead of defining pfunc, you can use a pure function. The following accomplishes the same result. We display t3 using Show together with GraphicsGrid in Figure 4.12.

```
t2 = (Plot[#1, {x, 0, 100}, DisplayFunction → Identity]&)/@t1;
t3 = Partition[t2, 3];
Show[GraphicsGrid[t3]]
```

---

**Example 4.2.5** **(Dynamical Systems).** Let $f_c(x) = x^2 + c$ and consider the dynamical system given by $x_0 = 0$ and $x_{n+1} = f_c\left(x_n\right)$. Generate a bifurcation diagram of $f_c$.

***Solution*** First, recall that Nest[f,x,n] computes the repeated composition $f^n(x)$. Then, in terms of a composition,

Compare the approach used here with the approach used in Example 4.1.9.

$$x_{n+1} = f_c\left(x_n\right) = f_c{}^n\,(0)\,.$$

We will compute $f_c{}^n\,(0)$ for various values of $c$ and "large" values of $n$ so we begin by defining cvals to be a list of 300 equally spaced values of $c$ between $-2.5$ and $1$.

```
cvals = Table[c, {c, −2.5, 1., 3.5/299}];
```

We then define $f_c(x) = x^2 + c$. For a given value of $c$, f[c] is a function of one variable, $x$, whereas the form f[c_][x_] := ... results in a function of two variables that we think of as an indexed function that might be represented using traditional mathematical notation as $f_c(x)$.

```
Clear[f]
f[c_][x_] := x^2 + c
```

To iterate $f_c$ for various values of $c$, we define $h$. For a given value of $c$, $h(c)$ returns the list of points $\left\{\left(c, f_c{}^{100}(0)\right), \left(c, f_c{}^{101}(0)\right), \ldots, \left(c, f_c{}^{200}(0)\right)\right\}$.

```
h[c_] := {Table[{c, Nest[f[c], 0, n]}, {n, 100, 200}]}
```

We then use Map to apply $h$ to the list cvals. Observe that Mathematica generates several error messages when numerical precision is exceeded. We choose to disregard the error messages.

```
t1 = Map[h, cvals];
```

t1 is a list (of length 300) of lists (each of length 101). To obtain a list of points (or, lists of length 2), we use Flatten. The resulting set of points is plotted with ListPlot in Figure 4.13. Observe that Mathematica again displays several error messages, which are not displayed here for length considerations, that we ignore: Mathematica only plots the points with real coordinates and ignores those containing Overflow[].

```
t2 = Flatten[t1, 2];
ListPlot[t2, AxesLabel → {"c", "xc(n), n = 100 . .200"}]
```

---

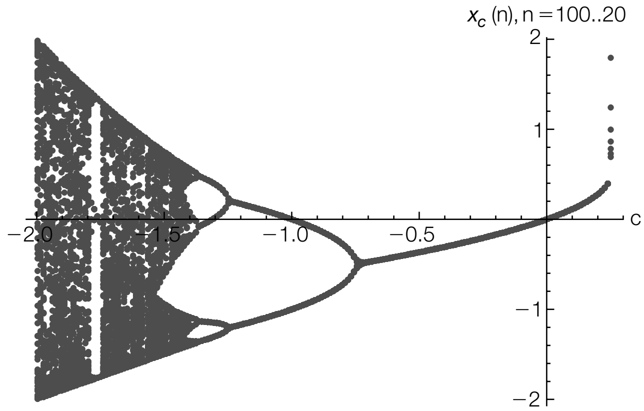



**FIGURE 4.13**

Bifurcation diagram of $f_c$

### 4.2.1 More on Graphing Lists: Graphing Lists of Points Using Graphics Primitives

Include the PlotJoined->True option in a ListPlot command to connect successive points with line segments.

Using *graphics primitives* such as Point and Line gives you even more flexibility. Point[{x,y}] represents a point at $(x, y)$.

**Line[{{x1, y1}, {x2, y2}, ... , {xn, yn}}]**

represents a sequence of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ connected with line segments. A graphics primitive is declared to be a graphics object with Graphics: Show[Graphics[Point[x,y]] displaying the point $(x, y)$. The advantage of using primitives is that each primitive is affected by the options that directly precede it.

---

**Example 4.2.6** Table 4.1 shows the percentage of the U.S. labor force that belonged to unions during certain years. Graph the data represented in the table.

***Solution*** We begin by entering the data represented in the table as dataunion:

```
dataunion = {{30, 11.6}, {35, 13.2}, {40, 26.9}, {45, 35.5},
    {50, 31.5}, {55, 33.2}, {60, 31.4}, {65, 28.4}, {70, 27.3},
        {75, 25.5}, {80, 21.9}, {85, 18.0}, {90, 16.1}};
```

the $x$-coordinate of each point corresponds to the year, where $x$ is the number of years past 1900, and the $y$-coordinate of each point corresponds to the percentage of the U.S. labor force that belonged to unions in the given year. We then use ListPlot to graph the set of points represented in dataunion in lp1, lp2 (illustrating

**Table 4.1** Union membership as a percentage of the labor force

| Year | Percent |
|---|---|
| 1930 | 11.6 |
| 1935 | 13.2 |
| 1940 | 26.9 |
| 1945 | 35.5 |
| 1950 | 31.5 |
| 1955 | 33.2 |
| 1960 | 31.4 |
| 1965 | 28.4 |
| 1970 | 27.3 |
| 1975 | 25.5 |
| 1980 | 21.9 |
| 1985 | 18.0 |
| 1990 | 16.1 |

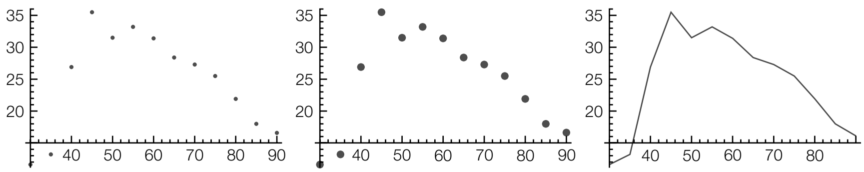


**FIGURE 4.14**

Union membership as a percentage of the labor force

the PlotStyle option), and lp3 (illustrating the PlotJoined option). All three plots are displayed side-by-side in Figure 4.14 using Show together with GraphicsRow.

```
lp1 = ListPlot[dataunion];
lp2 = ListPlot[dataunion, PlotStyle → PointSize[0.03]];
lp3 = ListPlot[dataunion, Joined → True];
Show[GraphicsRow[{lp1, lp2, lp3}]]
```

An alternative to using ListPlot is to use Show, Graphics, and Point to view the data represented in dataunion. In the following command we use Map to apply the function Point to each pair of data in dataunion. The result is not a graphics object and cannot be displayed with Show.

```
datapts1 = Map[Point, dataunion];
Short[datapts1]
{Point[{30, 11.6}], Point[{35, 13.2}], ⟨⟨10⟩⟩, Point[{90, 16.1}]}
```

Next, we use Show and Graphics to declare the set of points Map[Point, dataunion] as graphics objects and name the resulting graphics object dp1. The image is not displayed because a semicolon is included at the end of the command. The PointSize[.03] command specifies that the points be displayed as filled circles of radius 0.03% of the displayed graphics object.

```
dp1 = Show[Graphics[{PointSize[0.03], datapts1},
Axes → Automatic]];
```

The collection of all commands contained within a Graphics command is contained in braces {...}. Each graphics primitive is affected by the options such as PointSize, GrayLevel (or RGBColor) directly preceding it. Thus,

```
datapts2 = ({GrayLevel[RandomReal[]], Point[#1]}&)/@dataunion;
Short[datapts2]
{{GrayLevel[0.827228], Point[{30, 11.6}]}, ⟨⟨11⟩⟩, ⟨⟨1⟩⟩}
```

```
dp2 = Show[Graphics[{PointSize[0.03], datapts2},
    Axes → Automatic]];
```

displays the points in dataunion in various shades of gray in a graphic named dp2, and

```
datapts3 = ({PointSize[RandomReal[{"0.008", "0.1"}]],
    GrayLevel[RandomReal[]], Point[#1]}&)/@dataunion;
dp3 = Show[Graphics[{datapts3}, Axes → Automatic]];
```

shows the points in dataunion in various sizes and in various shades of gray in a graphic named dp3. We connect successive points with line segments

```
connectpts = Graphics[Line[dataunion]];
dp4 = Show[connectpts, dp3, Axes → Automatic];
```

and show all four plots in Figure 4.15 using Show and GraphicsGrid.

```
Show[GraphicsGrid[{{dp1, dp2}, {dp3, dp4}}]]
```

---

**With the speed of today's computers and the power of Mathematica, it is relatively easy to carry out many calculations that required supercomputers and sophisticated programming experience just a few years ago.**

---

**Example 4.2.7** **(Julia Sets).** Plot Julia sets for $f(z) = \lambda \cos z$ if $\lambda = .66i$ and $\lambda = .665i$.

***Solution*** The sets are visualized by plotting the points $(a, b)$ for which $|f^n(a+bi)|$ is *not* large in magnitude so we begin by forming our complex grid. Using Table and Flatten, we define complexpts to be a list of 62,500 points of the form $a+bi$ for 250 equally spaced real values of $a$ between 0 and 8 and 300 equally spaced real values of $b$ between $-4$ and 4 and then $f(z) = .66i \cos z$.

```
complexpts =
Flatten[Table[a + bI, {a, 0., 8., 8/249}, {b, −4., 4., 6/249}], 1];
```

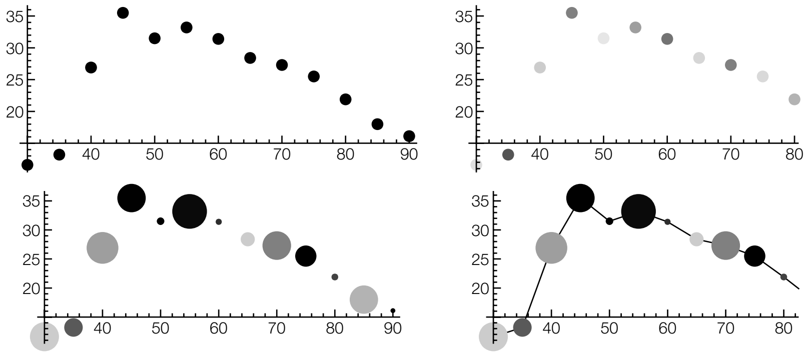


**FIGURE 4.15**

Union membership as a percentage of the labor force

```
Clear[f]
f[z_] = .66ICos[z]
0.66iCos[z]
```

For a given value of $c = a + bi$, $h(c)$ returns the ordered triple consisting of the real part of $c$, the imaginary part of $c$, and the value of $f^{200}(c)$.

```
h[c_] := {Re[c], Im[c], Nest[f, c, 200]}
```

We then use Map to apply $h$ to complexpts. Observe that Mathematica generates several error messages. When machine precision is exceeded, we obtain an Overflow[ ] error message; numerical result smaller than machine precision results in an Underflow[ ] error message. Error messages can be machine specific, so if you do not get any, do not worry. For length considerations, we do not show any that we obtained here.

```
t1 = Map[h, complexpts]//Chop;
```

We use the error messages to our advantage. In t2, we select those elements of t1 for which the third coordinate *is not* Indeterminate, which corresponds to the ordered triples $(a, b, f^n(a+bi))$ for which $|f^n(a+bi)|$ *is not* large in magnitude, whereas in t2b, we select those elements of t1 for which the third coordinate *is* Indeterminate, which corresponds to the ordered triples $(a, b, f^n(a+bi))$ for which $|f^n(a+bi)|$ *is* large in magnitude.

```
t2 = Select[t1, Not[#[[3]] === Indeterminate]&];
t2b = Select[t1, #[[3]] === Indeterminate&];
pt[{x_, y_, z_}] := {x, y}
t3 = Map[pt, t2];
t3b = Map[pt, t2b];
```

which are then graphed with ListPlot and shown side-by-side in Figure 4.16 using Show and GraphicsRow. As expected, the images are inversions of each other.

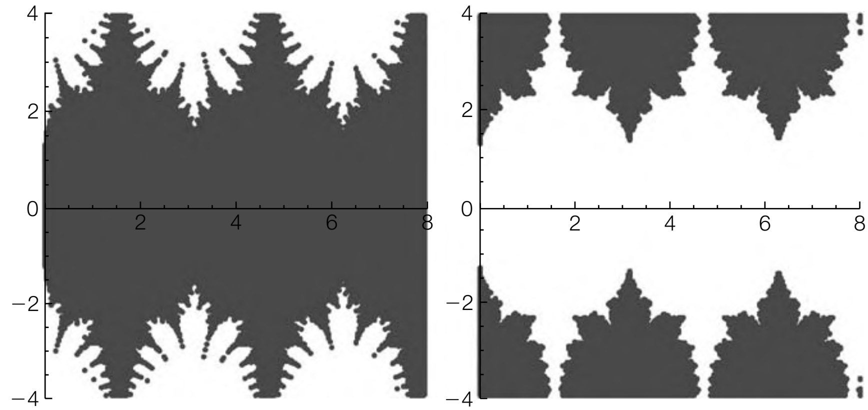



**FIGURE 4.16**

Julia set for $0.66i \cos z$

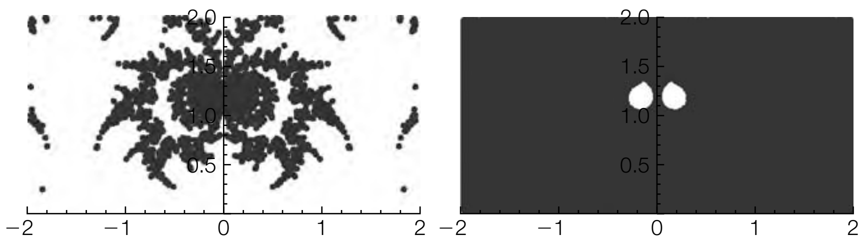



**FIGURE 4.17**

Julia set for $0.665i \cos z$

```
lp1 = ListPlot[t3, PlotRange → {{0, 8}, {−4, 4}},
    AspectRatio → Automatic, DisplayFunction → Identity];
lp2 = ListPlot[t3b, PlotRange → {{0, 8}, {−4, 4}},
    AspectRatio → Automatic, DisplayFunction → Identity];
Show[GraphicsRow[{lp1, lp2}]]
```

We encountered similar error messages as before but we have not included them due to length considerations.

Changing $\lambda$ from $0.66i$ to $0.665i$ results in a surprising difference in the plots. We proceed as before but increase the number of sample points to 120,000. See Figure 4.17.

```
complexpts = Flatten[Table[a + bI, {a, −2., 2., 4/399}, {b, 0., 2., 2/299}], 1]
Clear[f];
f[z_] = .665 I Cos[z]
h[c_] := {Re[c], Im[c], Nest[f, c, 200]}
t1 = Map[h, complexpts] / /Chop;
t2 = Select[t1, Not[#[[3]] === Indeterminate]&];
t2 = Select[t2, Not[#[[3]] === Overflow[]]&];
```

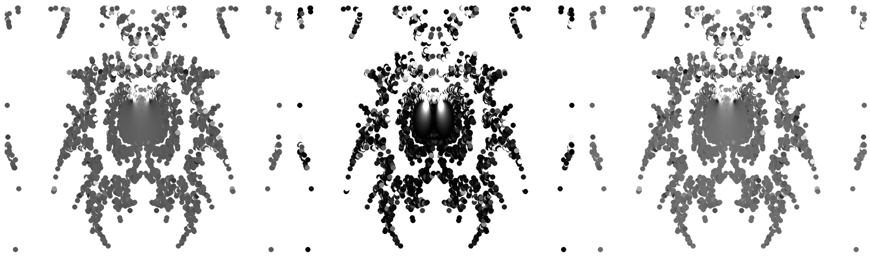


**FIGURE 4.18**

Shaded Julia sets for $0.665i \cos z$

```
t2b = Select[t1, #[[3]] === Indeterminate&];
pt[{x_, y_, z_}] := {x, y}

t3 = Map[pt, t2];
t3b = Map[pt, t2b];

lp1 = ListPlot[t3, PlotRange → {{-2, 2}, {0, 2}}, AspectRatio → Automatic,
     DisplayFunction → Identity];
lp2 = ListPlot[t3b, PlotRange → {{-2, 2}, {0, 2}}, AspectRatio → Automatic,
     DisplayFunction → Identity];
Show[GraphicsRow[{lp1, lp2}]]
```

To see detail, we take advantage of pure functions, Map, and graphics primitives in three different ways. In Figure 4.18, the shading of the point $(a, b)$ is assigned according to the distance of $f^{200}(a + bi)$ from the origin. The color black indicates a distance of zero from the origin; as the distance increases, the shading of the point becomes lighter.

```
t2p = Map[{#[[1]], #[[2]], Min[Abs[#[[3]]], 3]}&, t2];
t2p2 = Map[{GrayLevel[#[[3]]/3], Point[{#[[1]], #[[2]]}]}&,
     t2p];
jp1 = Show[Graphics[t2p2], PlotRange → {{-2, 2}, {0, 2}},
    AspectRatio → 1];

t2p = Map[{#[[1]], #[[2]], Min[Abs[Re[#[[3]]]], .25]}&, t2];
t2p2 = Map[{GrayLevel[#[[3]] / .25], Point[{#[[1]], #[[2]]}]}&,
     t2p];
jp2 = Show[Graphics[t2p2], PlotRange → {{-2, 2}, {0, 2}}, AspectRatio → 1];

t2p = Map[{#[[1]], #[[2]], Min[Abs[Im[#[[3]]]], 2.5]}&, t2];
t2p2 = Map[{GrayLevel[#[[3]]/2.5], Point[{#[[1]], #[[2]]}]}&,
t2p];
jp3 = Show[Graphics[t2p2], PlotRange → {{-2, 2}, {0, 2}}, AspectRatio → 1];

Show[GraphicsRow[{jp1, jp2, jp3}]]
```

### 4.2.2 Miscellaneous List Operations

#### *Other List Operations*

Some other Mathematica commands used with lists include

1. Append[list,element], which appends element to list;
2. AppendTo[list,element], which appends element to list and names the result list;
3. Drop[list,n], which returns the list obtained by dropping the first $n$ elements from list;
4. Drop[list,-n], which returns the list obtained by dropping the last $n$ elements of list;
5. Drop[list,{n,m}], which returns the list obtained by dropping the $n$th through $m$th elements of list;
6. Drop[list,{n}], which returns the list obtained by dropping the $n$th element of list;
7. Prepend[list,element], which prepends element to list; and
8. PrependTo[list,element], which prepends element to list and names the result list.

#### *Alternative Way to Evaluate Lists by Functions*

Abbreviations of several of the commands discussed in this section are summarized in the following table:

| | | |
|---|---|---|
| **/@/@ Apply** | **// (function application)** | **{...}** |
| **@ Map** | **[[...]] Part** | |

## 4.3 OTHER APPLICATIONS

We now present several other applications that we find interesting and that require the manipulation of lists. The examples also illustrate (and combine) many of the techniques that were demonstrated in the previous chapters.

### 4.3.1 Approximating Lists with Functions

Another interesting application of lists is that of curve fitting. The commands

1. Fit[data,functionset,variables] fits the list of data points data using the functions in functionset by the method of least squares. The functions in functionset are functions of the variables listed in variables; and

2. InterpolatingPolynomial[data,x] fits the list of $n$ data points data with an $n-1$ degree polynomial in the variable $x$.

**Example 4.3.1** Define datalist to be the list of numbers consisting of 1.14479, 1.5767, 2.68572, 2.5199, 3.58019, 3.84176, 4.09957, 5.09166, 5.98085, 6.49449, and 6.12113. (a) Find a quadratic approximation of the points in datalist. (b) Find a fourth-degree polynomial approximation of the points in datalist.

***Solution*** The approximating function obtained via the least squares method with Fit is plotted along with the data points in Figure 4.19. Notice that many of the data points are not very close to the approximating function. A better approximation is obtained using a polynomial of higher degree (4).

```
Clear[datalist]
datalist = {1.14479, 1.5767, 2.68572, 2.5199, 3.58019, 3.84176,
    00094.09957, 5.09166, 5.98085, 6.49449, 6.12113};
p1 = ListPlot[datalist];
Clear[y]
y[x_] = Fit[datalist, {1, x, x^2}, x]
0.508266 + 0.608688x − 0.00519281x^2
p2 = Plot[y[x], {x, −1, 11}];
pa = Show[p1, p2];
Clear[y]
y[x_] = Fit[datalist, {1, x, x^2, x^3, x^4}, x]
−0.54133 + 2 . 02744x − 0.532282x^2 + 0.0709201x^3 − 0.00310985x^4
p3 = Plot[y[x], {x, −1, 11}];
pb = Show[p1, p3];
Show[GraphicsRow[{pa, pb}]]
```

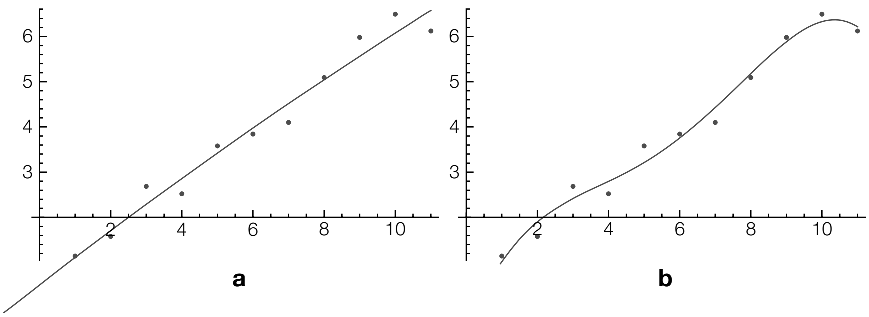



**FIGURE 4.19**

(a) The graph of a quadratic fit shown with the data points. (b) The graph of a quartic fit shown with the data points

**Table 4.2** Petroleum products imported to the United States for certain years

| Year | Percent | Year | Percent |
|---|---|---|---|
| 1973 | 34.8105 | 1983 | 28.3107 |
| 1974 | 35.381 | 1984 | 29.9822 |
| 1975 | 35.8167 | 1985 | 27.2542 |
| 1976 | 40.6048 | 1986 | 33.407 |
| 1977 | 47.0132 | 1987 | 35.4875 |
| 1978 | 42.4577 | 1988 | 38.1126 |
| 1979 | 43.1319 | 1989 | 41.57 |
| 1980 | 37.3182 | 1990 | 42.1533 |
| 1981 | 33.6343 | 1991 | 39.5108 |
| 1982 | 28.0988 | | |

Remember that when a semicolon is placed at the end of the command, the resulting output is *not* displayed by Mathematica.

To check its accuracy, the second approximation is graphed simultaneously with the data points in Figure 4.19(b).

Next, consider a list of data points made up of ordered pairs.

**Example 4.3.2** Table 4.2 shows the average percentage of petroleum products imported to the United States for certain years. (a) Graph the points corresponding to the data in the table and connect the consecutive points with line segments. (b) Use InterpolatingPolynomial to find a function that approximates the data in the table. (c) Find a fourth-degree polynomial approximation of the data in the table. (d) Find a trigonometric approximation of the data in the table.

***Solution*** We begin by defining data to be the set of ordered pairs represented in the table: The $x$-coordinate of each point represents the number of years past 1900, and the $y$-coordinate represents the percentage of petroleum products imported to the United States.

```
data = {{73., 34.8105}, {74., 35.381}, {75., 35.8167},
   {76., 40.6048}, {77., 47.0132}, {78., 42.4577},
   {79., 43.1319}, {80., 37.3182}, {81., 33.6343},
   {82., 28.0988}, {83., 28.3107}, {84., 29.9822},
   {85., 27.2542}, {86., 33.407}, {87., 35.4875},
   {88., 38.1126}, {89., 41.57}, {90., 42.1533}, {91., 39.5108}};
```

We use ListPlot to graph the ordered pairs in data. Note that because the option PlotStyle->PointSize[0.03] is included within the ListPlot command, the points are larger than they would normally be. We also use ListPlot with the option PlotJoined->True to graph the set of points data and connect consecutive points with line segments. Then we use Show to display lp1 and lp2 together in

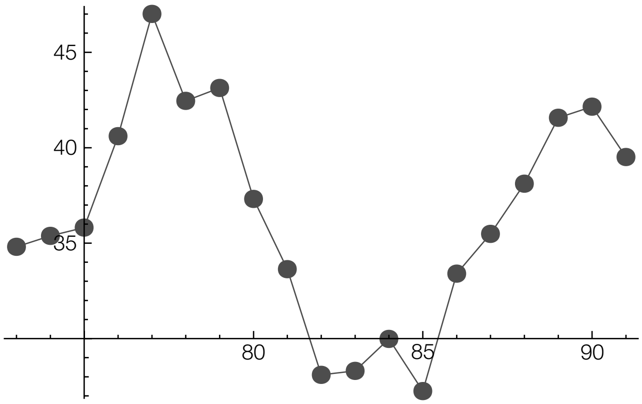



**FIGURE 4.20**

The points in Table 4.2 connected by line segments

Figure 4.20. Note that in the result, the points are easy to distinguish because of their larger size.

```
lp1 = ListPlot[data, PlotStyle → PointSize[0.03]];
lp2 = ListPlot[data, Joined → True];
Show[lp1, lp2]
```

Next, we use InterpolatingPolynomial to find a polynomial approximation, $p$, of the data in the table. Note that the result is lengthy, so Short is used to display an abbreviated form of $p$. We then graph $p$ and show the graph of $p$ along with the data in the table for the years corresponding to 1971 to 1993 in Figure 4.21(a). Although the interpolating polynomial agrees with the data exactly, the interpolating polynomial oscillates wildly.

```
p = InterpolatingPolynomial[data, x];
Short[p, 3]
39.5108 + (0.261128 + (0.111875 + (⟨⟨1⟩⟩)(−82. + x))(−73. + x))(−91. + x)
plotp = Plot[p, {x, 71, 93}];
pa = Show[plotp, lp1, PlotRange → {0, 50}];
```

To find a polynomial that approximates the data but does not oscillate wildly, we use Fit. Again, we graph the fit and display the graph of the fit and the data simultaneously. In this case, the fit does not identically agree with the data but does not oscillate wildly as illustrated in Figure 4.21(b).

```
Clear[p]
p = Fit[data, {1, x, x^2, x^3, x^4}, x]
−198884. + 9597.83x − 173.196x^2 + 1.38539x^3 − 0.00414481x^4
plotp = Plot[p, {x, 71, 93}];
pb = Show[plotp, lp1, PlotRange → {0, 50}]
```

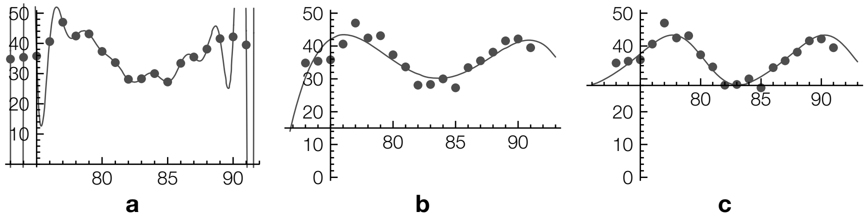



**FIGURE 4.21**

(a) Although interpolating polynomials agree with the data exactly, they may have extreme oscillations, even for relatively small data sets. (b) Although the fit does not agree with the data exactly, the oscillations seen in (a) do not occur. (c) You can use Fit to approximate data by a variety of functions

In addition to curve fitting with polynomials, Mathematica can also fit the data with trigonometric functions. In this case, we use Fit to find an approximation of the data of the form $p = c_1 + c_2 \sin x + c_3 \sin(x/2) + c_4 \cos x + c_5 \cos(x/2)$. As in the previous two cases, we graph the fit and display the graph of the fit and the data simultaneously; the results are shown in Figure 4.21(c).

See texts such as Abell, Braselton, and Rafter's *Statistics with Mathematica* [3] for a more sophisticated discussion of curve fitting and related statistical applications.

```
Clear[p]
p = Fit[data, {1, Sin[x], Sin[x/2], Cos[x], Cos[x/2]}, x]
35.4237 + 4.25768Cos[x/2] − 0.941862Cos[x] + 6.06609Sin[x/2] + 0.0272062Sin[x]
plotp = Plot[p, {x, 71, 93}];
pc = Show[plotp, lp1, PlotRange → {0, 50}];
Show[GraphicsRow[{pa, pb, pc}]]
```

## 4.3.2 Introduction to Fourier Series

Many problems in applied mathematics are solved through the use of Fourier series. Mathematica assists in the computation of these series in several ways. Suppose that $y = f(x)$ is defined on $-p < x < p$. Then the Fourier series for $f(x)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right) \tag{4.1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx \quad n = 1, 2\ldots \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx \quad n = 1, 2\ldots \end{aligned} \tag{4.2}$$

The **$k$th term of the Fourier series** (4.1) is

$$a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}. \tag{4.3}$$

The **$k$th partial sum of the Fourier series** (4.1) is

$$\frac{1}{2}a_0 + \sum_{n=1}^{k}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right). \tag{4.4}$$

It is a well-known theorem that if $y = f(x)$ is a periodic function with period $2p$ and $f'(x)$ is continuous on $[-p, p]$ except at finitely many points, then at each point $x$ the Fourier series for $f(x)$ converges and

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right) = \frac{1}{2}\left(\lim_{z\to x^+} f(z) + \lim_{z\to x^-} f(z)\right).$$

In fact, if the series $\sum_{n=1}^{\infty}\left(|a_n| + |b_n|\right)$ converges, then the Fourier series converges uniformly on $(-\infty, \infty)$.

---

**Example 4.3.3** Let $f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ f(x-2), & x \ge 1 \end{cases}$. Compute and graph the first few partial sums of the Fourier series for $f(x)$.

***Solution*** We begin by clearing all prior definitions of $f$. We then define the piecewise function $f(x)$ and graph $f(x)$ on the interval $[-1, 5]$ in Figure 4.22.

```
Clear[f]
f[x_] := 1/;0 ≤ x < 1
f[x_] := −x/;−1 ≤ x < 0
f[x_] := f[x−2]/;x ≥ 1
graphf = Plot[f[x], {x, −1, 5}]
```

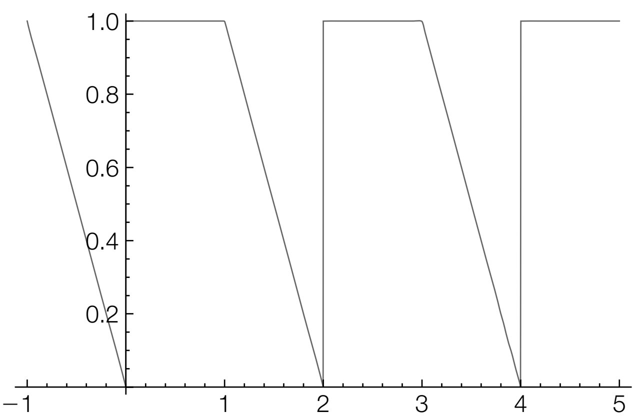



**FIGURE 4.22**

Plot of a few periods of $f(x)$

The Fourier series coefficients are computed with the integral formulas in equation (4.2). Executing the following commands defines $p$ to be $1$, a[0] to be an approximation of the integral $a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$, a[n] to be an approximation of the integral $a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx$, and b[n] to be an approximation of the integral $b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx$.

**Clear[a, b, fs, L]**
**L = 1;**
**a[0] =** $\frac{\text{NIntegrate}[f[x],\{x,-L,L\}]}{2L}$
0.75
**a[n_] :=** $\frac{\text{NIntegrate}\left[f[x]\text{Cos}\left[\frac{n\pi x}{L}\right],\{x,-L,L\}\right]}{L}$
**b[n_] :=** $\frac{\text{NIntegrate}\left[f[x]\text{Sin}\left[\frac{n\pi x}{L}\right],\{x,-L,L\}\right]}{L}$

A table of the coefficients a[i] and b[i] for $i = 1, 2, 3, \ldots, 10$ is generated with Table and named coeffs. Several error messages (which are not displayed here for length considerations) are generated because of the discontinuities, but the resulting approximations are satisfactory for our purposes. The elements in the first column of the table represent the $a_i$'s and those in the second column represents the $b_i$'s. Notice how the elements of the table are extracted using double brackets with coeffs.

**coeffs = Table[{a[i], b[i]}, {i, 1, 10}];**
**TableForm[coeffs]**

| | |
|---|---|
| −0.202642 | 0.31831 |
| 0. | 0 . 159155 |
| −0.0225158 | 0.106103 |
| 0. | 0 . 0795775 |
| −0.00810569 | 0.063662 |
| 0. | 0 . 0530516 |
| −0.00413556 | 0.0454728 |
| 0. | 0 . 0397887 |
| −0.00250176 | 0.0353678 |
| −1.0668549377257364′*^−16 | 0.031831 |

The first element of the list is extracted with coeffs[[1]].

**coeffs[[1]]**
{−0.202642, 0.31831}

The first element of the second element of coeffs and the second element of the third element of coeffs are extracted with coeffs[[2,1]] and coeffs[[3,2]], respectively.

**coeffs[[2, 1]]**
0.
**coeffs[[3, 2]]**
0.106103

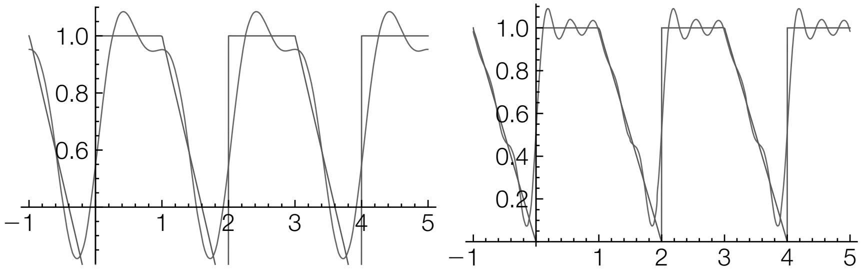



**FIGURE 4.23**

The first few terms of a Fourier series for a periodic function plotted with the function

After the coefficients are calculated, the $n$th partial sum of the Fourier series is obtained with Sum. The $k$th term of the Fourier series, $a_k \cos(k\pi x) + b_k \sin(k\pi x)$, is defined in fs. Hence, the $n$th partial sum of the series is given by

$$a_0 + \sum_{k=1}^{n} \left[a_k \cos(k\pi x) + b_k \sin(k\pi x)\right] = \texttt{a[0]} + \sum_{k=1}^{n} \texttt{fs[k, x]},$$

which is defined in fourier using Sum. We illustrate the use of fourier by finding fourier[2,x] and fourier[3,x].

```
fs[k_, x_] := coeffs[[k, 1]]Cos[kπx] + coeffs[[k, 2]]Sin[kπx]
fourier[n_, x_] := a[0] + ∑_{k=1}^n fs[k, x]
fourier[2, x]
0.75 − 0.202642Cos[πx] + 0.Cos[2πx] + 0.31831Sin[πx] + 0.159155Sin[2πx]
fourier[3, x]
0.75 − 0.202642Cos[πx] + 0.Cos[2πx] − 0.0225158 Cos[3πx] + 0.31831Sin[πx]
+0.159155Sin[2πx] + 0.106103Sin[3πx]
```

To see how the Fourier series approximates the periodic function, we plot the function simultaneously with the Fourier approximation for $n = 2$ and $n = 5$. The results are displayed together using GraphicsArray in Figure 4.23.

```
graphtwo = Plot[fourier[2, x], {x, −1, 5}, PlotStyle → GrayLevel[0 . 4]];
bothtwo = Show[graphtwo, graphf];
graphfive = Plot[fourier[5, x], {x, −1, 5}, PlotStyle → GrayLevel[0 . 4]];
bothfive = Show[graphfive, graphf];
Show[GraphicsRow[{bothtwo, bothfive}]]
```

## *Application: The One-Dimensional Heat Equation*

A typical problem in applied mathematics that involves the use of Fourier series is that of the **one-dimensional heat equation**. The boundary value problem that describes the temperature in a uniform rod with insulated surface is

$$\begin{aligned} k\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t},\ 0 < x < a,\ t > 0, \\ u(0,t) &= T_0,\ t > 0, \\ u(a,t) &= T_a,\ t > 0, \text{ and} \\ u(x,0) &= f(x),\ 0 < x < a. \end{aligned} \tag{4.5}$$

In this case, the rod has "fixed end temperatures" at $x = 0$, and $x = a$ and $f(x)$ is the initial temperature distribution. The solution to the problem is

$$u(x,t) = \underbrace{T_0 + \frac{1}{a}\left(T_a - T_0\right)x}_{v(x)} + \sum_{n=1}^{\infty} b_n \sin\left(\lambda_n x\right) e^{-\lambda_n^2 kt}, \tag{4.6}$$

where

$$\lambda_n = n\pi/a \qquad \text{and} \qquad b_n = \frac{2}{a}\int_0^a \left(f(x) - v(x)\right)\sin\frac{n\pi x}{a}\,dx,$$

and is obtained through separation of variables techniques. The coefficient $b_n$ in the solution equation (4.6) is the Fourier series coefficient $b_n$ of the function $f(x) - v(x)$, where $v(x)$ is the **steady-state temperature**.

---

**Example 4.3.4**

Solve $\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ 0 < x < 1,\ t > 0, \\ u(0,t) = 10,\ u(1,t) = 10,\ t > 0, \\ u(x,0) = 10 + 20\sin^2 \pi x. \end{cases}$

***Solution*** In this case, $a = 1$ and $k = 1$. The fixed end temperatures are $T_0 = T_a = 10$, and the initial heat distribution is $f(x) = 10 + 20\sin^2 \pi x$. The steady-state temperature is $v(x) = 10$. The function $f(x)$ is defined and plotted in Figure 4.24. Also, the steady-state temperature, $v(x)$, and the eigenvalue are defined. Finally, Integrate is used to define a function that will be used to calculate the coefficients of the solution.

```
Clear[f]
f[x_]:= 10 + 20Sin[πx]^2
Plot[f[x]; {x, 0, 1}, PlotRange → {0, 30}]
v[x_]:= 10
lambda[n_]:= nπ/4
b[n_]:= b[n] = ∫_0^4(f[x] v[x])Sin{[nπx/4]}dx
```

Notice that b[n] is defined using the form b[n_]:= b[n] = ... so that Mathematica "remembers" the values of b[n] computed and thus avoids recomputing previously computed values. In the following table, we compute exact and approximate values of b[1],...,b[10].

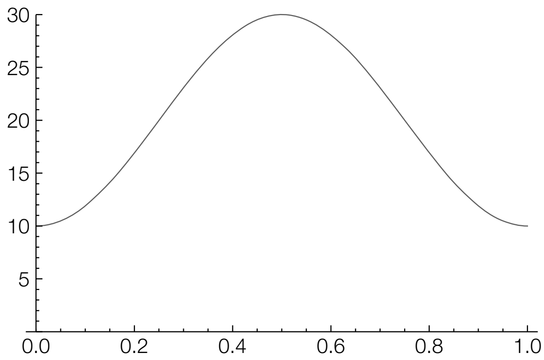



**FIGURE 4.24**

Graph of $f(x) = 10 + 20\sin^2 \pi x$

**Table[{n, b[n], b[n]//N}, {n, 1, 10}]//TableForm**

| | | |
|---|---|---|
| 1 | $\frac{5120}{63\pi}$ | 25.869 |
| 2 | 0 | 0. |
| 3 | $\frac{1024}{33\pi}$ | 9.87725 |
| 4 | 0 | 0. |
| 5 | $\frac{1024}{39\pi}$ | 8.35767 |
| 6 | 0 | 0. |
| 7 | $\frac{1024}{21\pi}$ | 15.5214 |
| 8 | 0 | 0. |
| 9 | $-\frac{5120}{153\pi}$ | −10.6519 |
| 10 | 0 | 0. |

Let $S_m = b_m \sin\left(\lambda_m x\right) e^{-\lambda_m^2 t}$. Then, the desired solution, $u(x,t)$, is given by

$$u(x,t) = v(x) + \sum_{m=1}^{\infty} S_m.$$

Let $u(x,t,n) = v(x) + \sum_{m=1}^{n} S_m$. Notice that $u(x,t,n) = u(x,t,n-1) + S_n$. Consequently, approximations of the solution to the heat equation are obtained recursively taking advantage of Mathematica's ability to compute recursively. The solution is first defined for $n = 1$ by u[x,t,1]. Subsequent partial sums, u[x,t,n], are obtained by adding the $n$th term of the series, $S_n$, to u[x,t,n-1].

```
u[x_, t_, 1] := v[x] + b[1]Sin[lambda[1]x]Exp[−lambda[1]^2t]
u[x_, t_, n_] := u[x, t, n − 1] + b[n]Sin[lambda[n]x]Exp[−lambda[n]^2t]
```

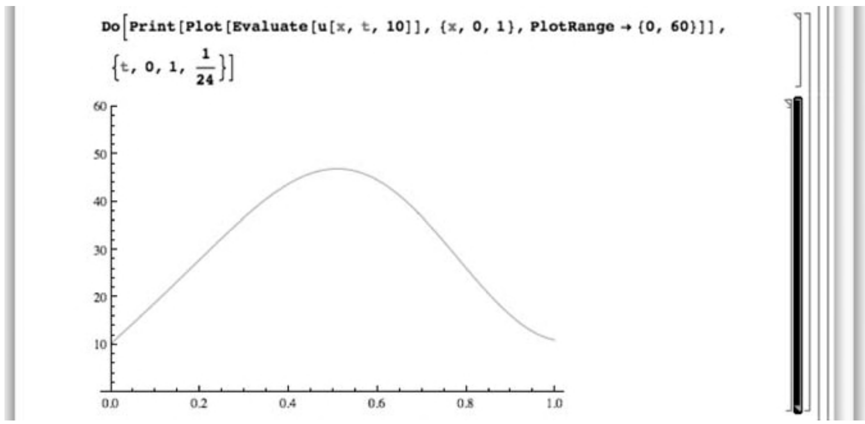
By defining the solution in this manner, a table can be created that includes the partial sums of the solution. In the following table, we compute the first, fourth, and seventh partial sums of the solution to the problem.

```
Table[u[x, t, n], {n, 1, 7, 3}];
```

To generate graphics that can be animated, we use a **Do** loop. The 10th partial sum of the solution is plotted for $t = 0$ to $t = 1$ using a step-size in $t$ of $1/24$. Remember that u[x,t,n] is determined with a **Table** command, so **Evaluate** must be used in the **Do** command so that Mathematica first computes the solution $u$ and then evaluates $u$ at the particular values of $x$. Otherwise, $u$ is recalculated for each value of $x$. The plots of the solution obtained can be animated as indicated in the following screen shot.

```
Do[Print[Plot[Evaluate[u[x, t, 10]], {x, 0, 1}, PlotRange → {0, 60}]],
    {t, 0, 1, 1/24}]
```

Alternatively, we may generate several graphics and display the resulting set of graphics as a **GraphicsArray**. We plot the $10$th partial sum of the solution for $t = 0$ to $t = 1$ using a step-size of $1/15$. The resulting 16 graphs are named **graphs**, which are then partitioned into four element subsets with **Partition** and named **toshow**. We then use **Show** and **GraphicsGrid** to display **toshow** in Figure 4.25.




```
graphs = Table[Plot[Evaluate[u[x, t, 10]], {x, 0, 1}, Ticks → None,
    PlotRange → {0, 60}, DisplayFunction → Identity], {t, 0, 1, 1/15}];
toshow = Partition[graphs, 4];
Show[GraphicsGrid[toshow]]
```

---

Fourier series and generalized Fourier series arise in too many applications to list. Examples using them illustrate Mathematica's power to manipulate lists, symbolics, and graphics.

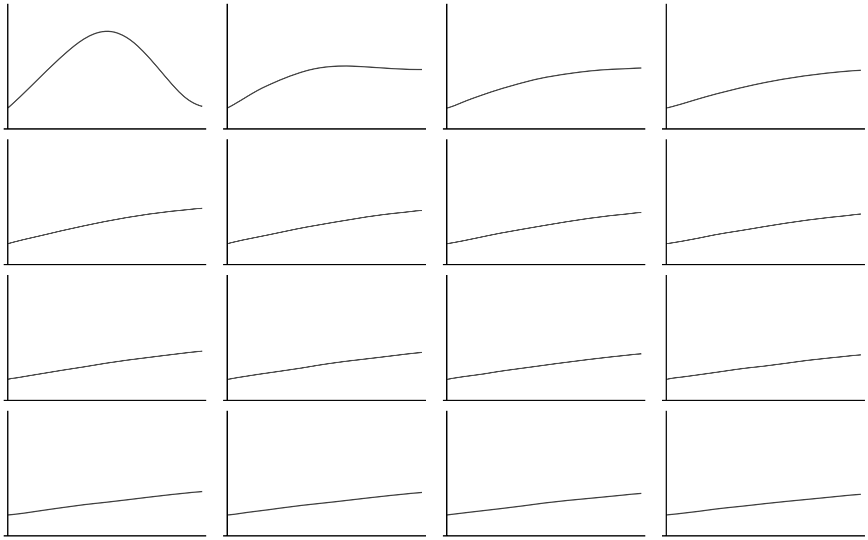


**FIGURE 4.25**

Temperature distribution in a uniform rod with insulated surface

For a classic approach to the subject, see Graff's *Wave Motion in Elastic Solids*, [10].

### *Application: The Wave Equation on a Circular Plate*

The vibrations of a circular plate satisfy the equation

$$D\,\nabla^4 w(r,\theta,t) + \rho h \frac{\partial^2 w(r,\theta,t)}{\partial t^2} = q(r,\theta,t), \tag{4.7}$$

where $\nabla^4 w = \nabla^2 \nabla^2 w$ and $\nabla^2$ is the **Laplacian in polar coordinates**, which is defined by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Assuming no forcing so that $q(r,\theta,t) = 0$ and $w(r,\theta,t) = W(r,\theta)e^{-i\omega t}$, equation (4.7) can be written as

$$\nabla^4 W(r,\theta) - \beta^4 W(r,\theta) = 0, \qquad \beta^4 = \omega^2 \rho h/D. \tag{4.8}$$

For a clamped plate, the boundary conditions are $W(a,\theta) = \partial W(a,\theta)/\partial r = 0$, and after *much work* (see [10]) the **normal modes** are found to be

$$W_{nm}(r,\theta) = \left[J_n\left(\beta_{nm} r\right) - \frac{J_n\left(\beta_{nm} a\right)}{I_n\left(\beta_{nm} a\right)} I_n\left(\beta_{nm} r\right)\right]\begin{pmatrix}\sin n\theta\\ \cos n\theta\end{pmatrix}. \tag{4.9}$$

In equation (4.9), $\beta_{nm} = \lambda_{nm}/a$, where $\lambda_{nm}$ is the $m$th solution of

$$I_n(x)J_n{}'(x) - J_n(x)I_n{}'(x) = 0, \tag{4.10}$$

Out[136]=

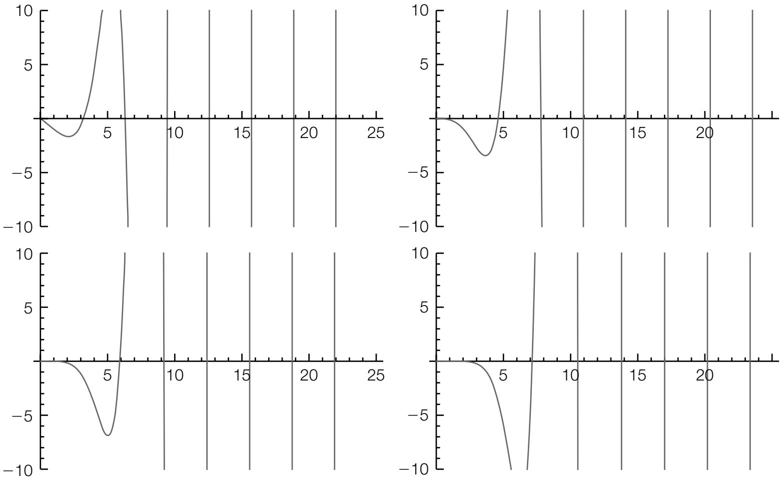



**FIGURE 4.26**

Plot of $I_n(x)J_n{}'(x) - J_n(x)I_n{}'(x)$ for $n = 0$ and $1$ in the first row and $n = 2$ and $3$ in the second row

See Example 4.2.4.

where $J_n(x)$ is the Bessel function of the first kind of order $n$ and $I_n(x)$ is the **modified Bessel function of the first kind** of order $n$, related to $J_n(x)$ by $i^n I_n(x) = J_n(ix)$.

The Mathematica command BesselI[n,x] returns $I_n(x)$.

---

**Example 4.3.5** Graph the first few normal modes of the clamped circular plate.

***Solution*** We must determine the value of $\lambda_{nm}$ for several values of $n$ and $m$, so we begin by defining eqn[n][x] to be $I_n(x)J_n{}'(x) - J_n(x)I_n{}'(x)$. The $m$th solution of equation (4.10) corresponds to the $m$th zero of the graph of eqn[n][x], so we graph eqn[n][x] for $n = 0,\ 1,\ 2,$ and $3$ with Plot in Figure 4.26.

```
eqn[n_][x_]:=BesselI[n, x]D[BesselJ[n, x], x] – BesselJ[n, x]D[BesselI[n, x], x]
```

The result of the Table and Plot command is a list of length four, which is verified with Length[p1].

```
p1 = Table[Plot[Evaluate[eqn[n][x]], {x, 0, 25}, PlotRange → { – 10, 10}], {n, 0, 3}];
```

so we use Partition to create a $2 \times 2$ array of graphics that is displayed using Show and GraphicsGrid.

```
p2 = Show[GraphicsGrid[Partition[p1, 2]]]
```

To determine $\lambda_{nm}$, we use FindRoot. Recall that to use FindRoot to solve an equation, an initial approximation of the solution must be given. For example,

```
l01 = FindRoot[eqn[0][x] == 0, {x, 3.04}]
{x → 3.19622}
```

approximates $\lambda_{01}$, the first solution of equation (4.10) if $n = 0$. However, the result of FindRoot is a list. The specific value of the solution is the second part of the first part of the list, lambda01, extracted from the list with Part ([[...]]).

```
l01[[1, 2]]
3.19622
```

Thus,

We use the graphs in Figure 4.26 to obtain initial approximations of each solution.

```
λ0s = Map[FindRoot[eqn[0][x] == 0, {x, #}][[1, 2]]&,
    {3.04, 6.2, 9.36, 12.5, 15.7}]
{3.19622, 6.30644, 9.4395, 12.5771, 15.7164}
```

approximates the first five solutions of equation (4.10) if $n = 0$ and then returns the specific value of each solution. We use the same steps to approximate the first five solutions of equation (4.10) if $n = 1$, $2$, and $3$.

```
λ1s = Map[FindRoot[eqn[1][x] == 0, {x, #}][[1, 2]]&,
    {4.59, 7.75, 10.9, 14.1, 17.2}]
{4.6109, 7.79927, 10.9581, 14.1086, 17.2557}
λ2s = Map[FindRoot[eqn[2][x] == 0, {x, #}][[1, 2]]&,
    {5.78, 9.19, 12.4, 15.5, 18.7}]
{5.90568, 9.19688, 12.4022, 15.5795, 18.744}
λ3s = Map[FindRoot[eqn[3][x] == 0, {x, #}][[1, 2]]&,
    {7.14, 10.5, 13.8, 17, 20.2}]
{7.14353, 10.5367, 13.7951, 17.0053, 20.1923}
```

All four lists are combined together in λs.

```
λs = {λ0s, λ1s, λ2s, λ3s};
Short[λs]
{{3.19622, 6.30644, ⟨⟨18⟩⟩, 12.5771, 15.7164}, ⟨⟨2⟩⟩, {⟨⟨1⟩⟩}}
```

For $n = 0$, $1$, $2$, and $3$ and $m = 1$, $2$, $3$, $4$, and $5$, $\lambda_{nm}$ is the $m$th part of the $(n + 1)$st part of λs.

Observe that the value of $a$ does not affect the shape of the graphs of the normal modes, so we use $a = 1$ and then define $\beta_{nm}$.

```
a = 1;
β[n_, m_] := λs[[n + 1, m]]/a
```

ws is defined to be the sine part of equation (4.9)

```
ws[n_, m_][r_, θ_]:=
(BesselJ[n, β[n, m]r] - BesselJ[n, β[n, m]a]/BesselI[n, β[n, m]a]
    BesselI[n, β[n, m]r])Sin[nθ]
```

and wc to be the cosine part.

```
wc[n_, m_][r_, θ_]:=
(BesselJ[n, β[n, m]r] - BesselJ[n, β[n, m]a]/BesselI[n, β[n, m]a]
    BesselI[n, β[n, m]r])Cos[nθ]
```

We use ParametricPlot3D to plot ws and wc. For example,

```
ParametricPlot3D[{rCos[θ], rSin[θ], ws[3, 4][r, θ]}, {r, 0, 1}, {θ, -Pi, Pi},
    PlotPoints → 60]
```

graphs the sine part of $W_{34}(r, \theta)$ shown in Figure 4.27. We use Table together with ParametricPlot3D followed by Show and GraphicsGrid to graph the sine part of $W_{nm}(r, \theta)$ for $n = 0, 1, 2,$ and $3$ and $m = 1, 2, 3,$ and $4$ shown in Figure 4.28.

```
ms = Table[ParametricPlot3D[{rCos[θ], rSin[θ], ws[n, m][r, θ]},
        {r, 0, 1}, {θ, -Pi, Pi},
      DisplayFunction → Identity, PlotPoints → 30, BoxRatios → {1, 1, 1}],
        {n, 0, 3}, {m, 1, 4}];
Show[GraphicsGrid[ms]]
```

Out[164]=

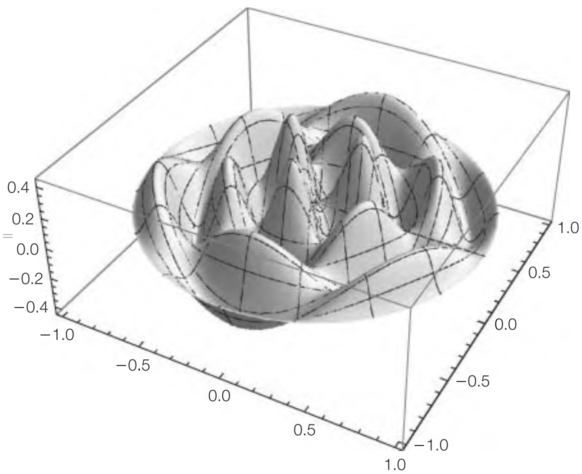



**FIGURE 4.27**

The sine part of $W_{34}(r, \theta)$

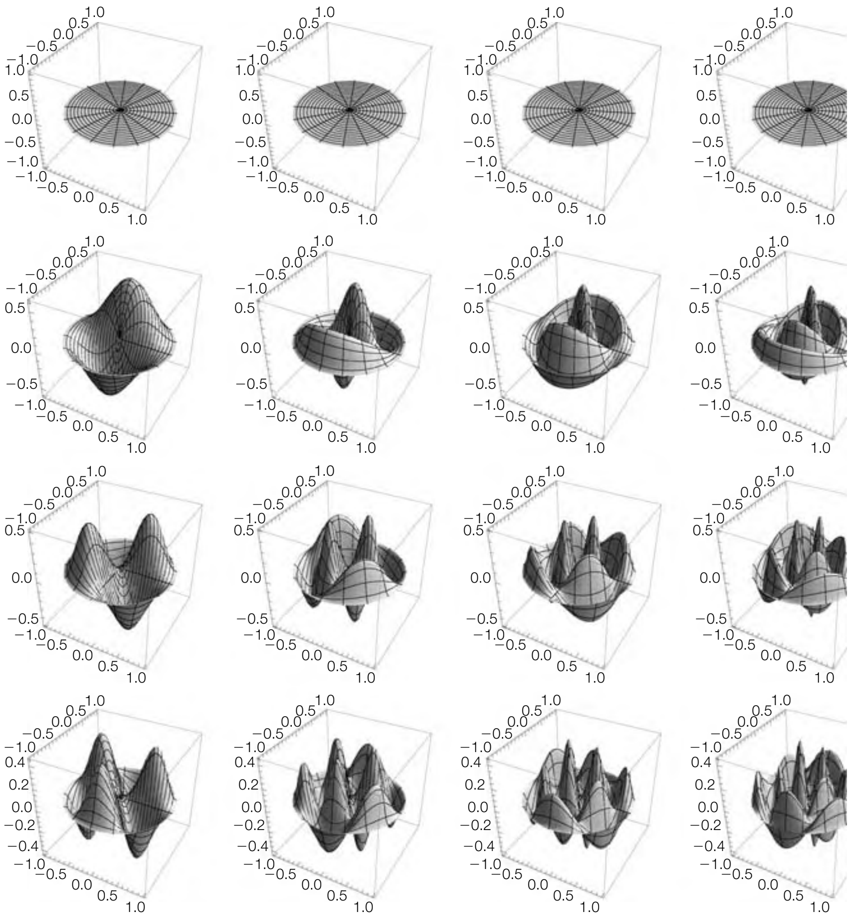


**FIGURE 4.28**

The sine part of $W_{nm}(r, \theta)$: $n = 0$ in row 1, $n = 1$ in row 2, $n = 2$ in row 3, and $n = 3$ in row 4 ($m = 1$ to $4$ from left to right in each row)

Identical steps are followed to graph the cosine part shown in Figure 4.29.

```
mc = Table[ParametricPlot3D[{rCos[θ], rSin[θ], wc[n, m][r, θ]},
        {r, 0, 1}, {θ, -Pi, Pi},
     DisplayFunction → Identity, PlotPoints → 30, BoxRatios → {1, 1, 1}],
        {n, 0, 3}, {m, 1, 4}];
Show[GraphicsGrid[mc]]
```

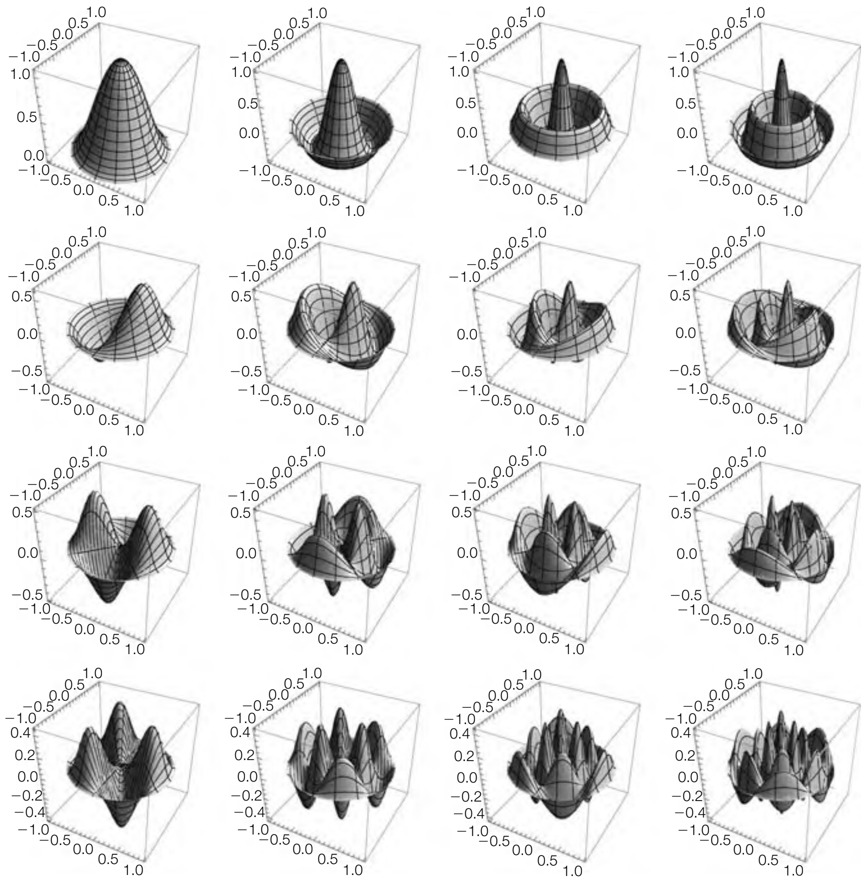



**FIGURE 4.29**

The cosine part of $W_{nm}(r, \theta)$: $n = 0$ in row 1, $n = 1$ in row 2, $n = 2$ in row 3, and $n = 3$ in row 4 ($m = 1$ to 4 from left to right in each row)

See references such as Barnsley's *Fractals Everywhere* [4] or Devaney and Keen's *Chaos and Fractals* [6] for detailed discussions regarding many of the topics briefly described in this section.

## 4.3.3 The Mandelbrot Set and Julia Sets

In Examples 4.1.9, 4.2.5, and 4.2.7 we illustrated several techniques for plotting bifurcation diagrams and Julia sets.

$f_c(x) = x^2 + c$ is the special case of $p = 2$ for $f_{p,c}(x) = x^p + c$.

Let $f_c(x) = x^2 + c$. In Example 4.2.5, we generated the $c$-values when plotting the bifurcation diagram of $f_c$. Depending on how you think, some approaches may be easier to understand than others. With the exception of very serious calculations, the differences in the time needed to carry out the computations may be minimal, so we encourage you to follow the approach that you understand. Learn new techniques as needed.

**Example 4.3.6 (Dynamical Systems).** For example, entering

Compare the approach here with the approach used in Example 4.2.5.

```
Clear[f, h]
f[c_][x_] := x ^ 2 + c//N;
```

defines $f_c(x) = x^2 + c$, so

```
Nest[f[-1], x, 3]
-1. + (-1. + (-1. + x^2)^2)^2
```

computes $f_{-1}{}^3(x) = (f_{-1} \circ f_{-1} \circ f_{-1})(x)$ and

```
Table[Nest[f[1/4], 0, n], {n, 101, 200}]//Short
{0.490693, 0.490779, <<96>>, 0.495148, 0.495171}
```

returns a list of $f_{1/4}{}^n(0)$ for $n = 101, 102, \ldots, 200$. Thus,

```
lgtable = Table[{c, Nest[f[c], 0, n]},
{c, -2, 1/4, 9/(4 * 299)}, {n, 101, 200}];
Length[lgtable]
300
```

returns a list of lists of $f_c{}^n(0)$ for $n = 101, 102, \ldots, 200$ for 300 equally spaced values of $c$ between $-2$ and $1$. The list lgtable is converted to a list of points with Flatten and plotted with ListPlot. See Figure 4.30 and compare this result to the result obtained in Example 4.2.5.

```
toplot = Flatten[lgtable, 1];
ListPlot[toplot]
```

For a given complex number $c$, the **Julia set**, $J_c$, of $f_c(x) = x^2 + c$ is the set of complex numbers, $z = a + bi$, $a, b$ real, for which the sequence $z$, $f_c(z) = z^2 + c$,

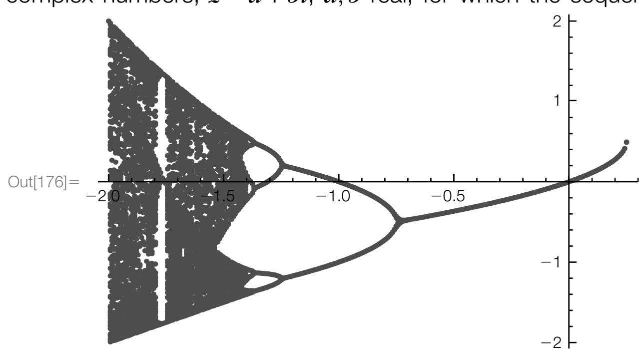



**FIGURE 4.30**

Another bifurcation diagram for $f_c$

$f_c\left(f_c(z)\right) = \left(z^2+c\right)^2 + c, \ldots, f_c{}^n(z), \ldots,$ does *not* tend to $\infty$ as $n \to \infty$:

We use the notation $f^n(x)$ to represent the composition $\underbrace{(f \circ f \circ \cdots \circ f)}_{n}(x)$.

$$J_c = \left\{ z \in \mathbb{C} | z, z^2 + c \ \left(z^2+c\right)^2 + c, \ldots \nrightarrow \infty \right\}.$$

Using a dynamical system, setting $z = z_0$ and computing $z_{n+1} = f_c\left(z_n\right)$ for large $n$ can help us determine if $z$ is an element of $J_c$. In terms of a composition, computing $f_c{}^n(z)$ for large $n$ can help us determine if $z$ is an element of $J_c$.

**Example 4.3.7** **(Julia Sets).** Plot the Julia set of $f_c(x) = x^2 + c$ if $c = -0.122561 + 0.744862i$.

***Solution*** After defining $f_c(x) = x^2 + c$, we use Table together with Nest to compute ordered triples of the form $\left(x, y, f_{-0.122561+0.744862i}{}^{200}(x+iy)\right)$ for 150 equally spaced values of $x$ between $-3/2$ and $3/2$ and 150 equally spaced values of $y$ between $-3/2$ and $3/2$.

As before, all error messages have been deleted.

You do not need to redefine $f_c(x)$ if you have already defined it during your current Mathematica session.

```
Clear[f, h]
f[c_][x_] := x ^ 2 + c//N;
g1 = Table[{x, y, Nest[f[-0 . 12256117 + .74486177I], x + Iy, 200]},
{x, -3/2, 3/2, 3/149}, {y, -3/2, 3/2, 3/149}];
g2 = Flatten[g1, 1];
```

We remove those elements of g2 for which the third coordinate is Overflow[ ] with Select,

```
g3 = Select[g2, Not[#[[3]] === Overflow[ ]]&];
```

extract a list of the first two coordinates, $(x, y)$, from the elements of g3,

```
g4 = Map[{#[[1]], #[[2]]}&, g3];
```

and plot the resulting list of points in Figure 4.31 using ListPlot.

```
lp1 = ListPlot[g4, PlotRange → {{-3/2, 3/2}, {-3/2, 3/2}},
      AxesLabel → {"x", "y"}, AspectRatio → Automatic]
```

We can invert the image as well with the following commands. In the end result, we show the Julia set and its inverted image in Figure 4.32

```
g3b = Select[g2, #[[3]]===Overflow[ ]&];
g4b = Map[{#[[1]], #[[2]]}&, g3b];
lp2 = ListPlot[g4b, PlotRange → {{-3/2, 3/2}, {-3/2, 3/2}},
AxesLabel → {"x", "y"}, AspectRatio → Automatic,
      DisplayFunction → Identity];
j1 = Show[GraphicsRow[{lp1, lp2}]]
```

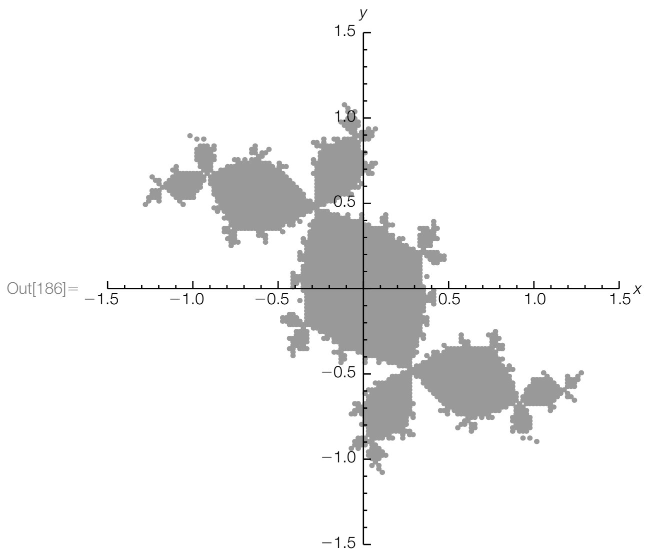



**FIGURE 4.31**

Filled Julia set for $f_c$

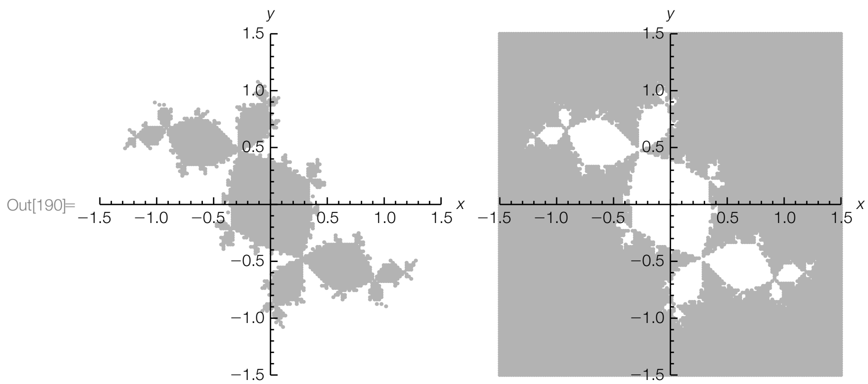



**FIGURE 4.32**

Filled Julia set for $f_c$ on the left; the inverted set on the right

Of course, one can consider functions other than $f_c(x) = x^2 + c$ as well as rearrange the order in which we carry out the computations. You have even greater control over your graphics if you use graphics primitives such as Point.

**Example 4.3.8** **(Julia Sets).** Plot the Julia set for $f_c(z) = z^2 - cz$ if $c = 0.737369 + 0.67549i$.

***Solution*** We initially proceed as in Example 4.3.7.

As before, all error messages have been deleted.

```
Clear[f, h]
f[c_][x_] := x ^ 2 – cx//N;
g1 = Table[{x, y, Nest[f[0 . 737369 + 0 . 67549I], x + Iy, 200]},
{x, –3/2, 3/2, 3/149}, {y, –3/2, 3/2, 3/149}];
g2 = Flatten[g1, 1];
g3 = Select[g2, Not[#[[3]] === Overflow[]]&];
```

After removing the points that result in an Overflow[ ] error message, we code the remaining ones according to their distance from the origin.

```
h[{x_, y_, z_}] := {x, y, Min[Abs[z], 0.5]}
g4 = Map[h, g3];

g5 = Table[{PointSize[0.005], GrayLevel[g4[[i, 3]]/0.5],
      Point[{g4[[i, 1]], g4[[i, 2]]}]}, {i, 1, Length[g4]}];
```

The results are shown in Figure 4.33.

```
lp1 = ListPlot[g4, PlotRange → {{–3/2, 3/2}, {–3/2, 3/2}},
      AxesLabel → {"x", "y"}, AspectRatio → Automatic]
Show[Graphics[g5], PlotRange– >{{–1.2, 1.75}, {–0.7, 1.4}},
      AspectRatio → Automatic]
```

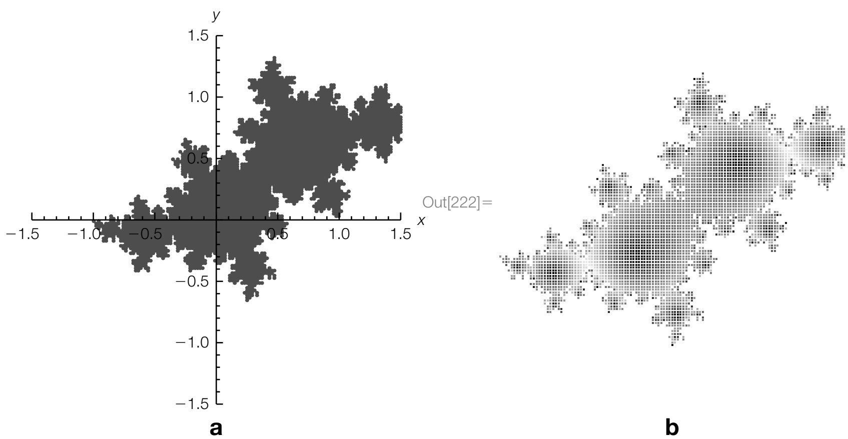



**FIGURE 4.33**

(a) The Julia set. (b) The lightest points $(a, b)$ are the ones for which $\left|f_{0.737369+0.67549i}{}^{200}(z)\right|$ is the largest

**Example 4.3.9 (The Ikeda Map).** The **Ikeda map** is defined by

$$\mathbf{F}(x, y) = \left\langle \gamma + \beta \left( x \cos \tau - y \sin \tau \right), \beta \left( x \sin \tau + y \cos \tau \right) \right\rangle, \tag{4.11}$$

where $\tau = \mu - \alpha / \left(1 + x^2 + y^2\right)$. If $\beta = 0.9$, $\mu = 0.4$, and $\alpha = 4.0$, plot the *basins of attraction* for $F$ if $\gamma = 0.92$ and $\gamma = 1.0$.

***Solution*** The *basins of attraction* for $F$ are the set of points $(x, y)$ for which $\|\mathbf{F}^n(x, y)\| \nrightarrow \infty$ as $n \to \infty$.

After defining `f[γ][x, y]` to be equation (4.11) and then $\beta = 0.9$, $\mu = 0.4$, and $\alpha = 4.0$, we use Table followed by Flatten to define pts to be the list of 40,000 ordered pairs $(x, y)$ for 200 equally spaced values of $x$ between $-2.3$ and $1.3$ and 200 equally spaced values of $y$ between $-2.8$ and $0.8$.

```
f[γ_][{x_, y_}] := {γ + β(xCos[μ − α/(1 + x^2 + y^2)] − y
      Sin[μ − α/(1 + x^2 + y^2)]), β (xSin[μ − α/(1 + x^2 + y^2)] + y
      Cos[μ − α/(1 + x^2 + y^2)])}
 β = 0.9; μ = 0.4; α = 4.0;
pts = Flatten[Table[{x, y}, {x, −2.3, 1.3, 3.6/199}, {y, −2.8, 0.8, 3.6/199}], 1];
```

In l1, we use Map to compute $\left(x, y, \mathbf{F}_{.92}{}^{200}(x, y)\right)$ for each $(x, y)$ in pts. In pts2, we use the graphics primitive Point and shade the points according to the maximum value of $\|\mathbf{F}^{200}(x, y)\|$—those $(x, y)$ for which $\mathbf{F}^{200}(x, y)$ is closest to the origin are darkest; the point $(x, y)$ is shaded lighter as the distance of $\mathbf{F}^{200}(x, y)$ from the origin increases. (See Figure 4.34(a).)

```
l1 = Map[{#[[1]], #[[2]], Nest[f[.92], {#[[1]], #[[2]]}, 200]}&, pts];
g[{x_, y_, z_}] := {x, y, Sqrt[z[[1]]^2 + z[[2]]^2]};
l2 = Map[g, l1];
maxl2 = Table[l2[[i, 3]], {i, 1, Length[l2]}]//Max
4.33321
pts2 = Table[{GrayLevel[l2[[i, 3]]/(maxl2)], Point[{l2[[i, 1]], l2[[i, 2]]}]},
      {i, 1, Length[l2]}];
ik1 = Show[Graphics[pts2], AspectRatio → 1];
```

For $\gamma = 1.0$, we proceed in the same way. The final results are shown in Figure 4.34(b).

```
l1 = Map[{#[[1]], #[[2]], Nest[f[1.0], {#[[1]], #[[2]]}, 200]}&, pts];
l2 = Map[g, l1];
maxl2 = Table[l2[[i, 3]], {i, 1, Length[l2]}]//Max
4.48421
pts2 = Table[{GrayLevel[l2[[i, 3]]/maxl2], Point[{l2[[i, 1]], l2[[i, 2]]}]},
      {i, 1, Length[l2]}];
ik2 = Show[Graphics[pts2], AspectRatio → 1]
Show[GraphicsRow[{ik1, ik2}]]
```


**FIGURE 4.34**

Basins of attraction for **F** if (a) $\gamma = 0.92$ and (b) $\gamma = 1.0$

The **Mandelbrot set**, $M$, is the set of complex numbers, $z = a + bi, a, b$ real, for which the sequence $z, f_z(z) = z^2 + z, f_z\left(f_z(z)\right) = \left(z^2 + z\right)^2 + z, \ldots, f_z^{\,n}(z), \ldots$, does *not* tend to $\infty$ as $n \to \infty$:

$$M = \left\{ z \in \mathbf{C} | z, z^2 + z\ \left(z^2 + z\right)^2 + z, \ldots \nrightarrow \infty \right\}.$$

Using a dynamical system, setting $z = z_0$ and computing $z_{n+1} = f_{z_0}\left(z_n\right)$ for large $n$ can help us determine if $z$ is an element of $M$. In terms of a composition, computing $f_z^{\,n}(z)$ for large $n$ can help us determine if $z$ is an element of $M$.

**Example 4.3.10** **(Mandelbrot Set).** Plot the Mandelbrot set.

***Solution*** We proceed as in Example 4.3.7 except that instead of iterating $f_c(z)$ for fixed $c$, we iterate $f_z(z)$.

As before, all error messages have been deleted.

As with the previous examples, all Overflow[...] messages have been deleted.

```
Clear[f, h]
f[c_][x_] := x ^ 2 + c//N;
g1 = Table[{x, y, Nest[f[x + Iy], x + Iy, 200]},
{x, −3/2, 1, 5/(2 ∗ 149)}, {y, −1, 1, 2/149}];
g2 = Flatten[g1, 1];
Take[g2, 5]
g3 = Select[g2, Not[#[[3]]===Overflow[ ]]&];
g4 = Map[{#[[1]], #[[2]]}&, g3];
```

The following gives us the image in Figure 4.35(a).

```
lp1 = ListPlot[g4, PlotRange → {{−3/2, 1}, {−1, 1}}, Axes → None,
AspectRatio → Automatic, PlotStyle → PointSize[.005]];
```

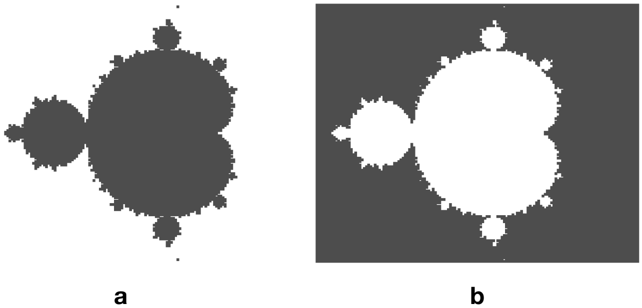


**FIGURE 4.35**

Two different views of the Mandelbrot set: in (a), the black points $(a, b)$ are the points for which ${f_{a+bi}}^{200}(a+bi)$ is finite; in (b), the black points $(a, b)$ are the ones for which ${f_{a+bi}}^{200}(a+bi)$ is not finite

To invert the image, we use the following to obtain the result in Figure 4.35(b).

```
g3b = Select[g2, #[[3]]===Overflow[ ]&];
g4b = Map[{#[[1]], #[[2]]}&, g3b];
lp2 = ListPlot[g4b, PlotRange → {{−3/2, 1}, {−1, 1}},
      Axes → None, AspectRatio → Automatic, PlotStyle → PointSize[.005]];
m1 = Show[GraphicsRow[{lp1, lp2}]]
```

---

In Example 4.3.10, the Mandelbrot set is obtained (or, more precisely, approximated) by repeatedly composing $f_z(z)$ for a grid of $z$-values and then deleting those for which the values exceed machine precision. Those values greater than $MaxNumber result in an Overflow[ ] message; computations with Overflow[ ] result in an Indeterminate message.

We can generalize by considering exponents other than 2 by letting $f_{p,c} = x^p + c$. The **generalized Mandelbrot set**, $M_p$, is the set of complex numbers, $z = a + bi, a, b$ real, for which the sequence $z, f_{p,z}(z) = z^p + z, f_{p,z}\left(f_{p,z}(z)\right) = \left(z^p + z\right)^p + z, \ldots, {f_{p,z}}^n(z), \ldots$, does *not* tend to $\infty$ as $n \to \infty$:

$$M_p = \left\{z \in \mathbf{C} | z, z^p + z \left(z^p + z\right)^p + z, \ldots \nrightarrow \infty\right\}.$$

Using a dynamical system, setting $z = z_0$ and computing $z_{n+1} = f_p\left(z_n\right)$ for large $n$ can help us determine if $z$ is an element of $M_p$. In terms of a composition, computing ${f_p}^n(z)$ for large $n$ can help us determine if $z$ is an element of $M_p$.

**Example 4.3.11** As with the previous examples, all error messages have been omitted.

**(Generalized Mandelbrot Set).** After defining $f_{p,c} = x^p + c$, we use Table, Abs, and Nest to compute a list of ordered triples of the form $\left(x, y, \left|f_{p,x+iy}{}^{100}(x+iy)\right|\right)$ for $p$-values from $1.625$ to $2.625$ spaced by equal values of $1/8$ and 200 values of $x(y)$ values equally spaced between $-2$ and $2$, resulting in 40,000 sample points of the form $x + iy$.

```
Clear[f, p]
f[p_, c_][x_]:=x ^ p + c//N;
g1 =
Map[Table[{x, y, Abs[Nest[f[2, x + Iy], x + Iy, #]]}//N,
      {x, -1.5, 1., 5/(2 * 199)}, {y, -1., 1., 2/199}]&, {5, 10, 15, 25, 50, 100}];
g2 = Map[Flatten[#, 1]&, g1];
```

Next, we extract those points for which the third coordinate is Indeterminate with Select; ordered pairs of the first two coordinates are obtained in g4. The resulting list of points is plotted with ListPlot in Figure 4.36.

```
g3 = Table[Select[g2[[i]], Not[#[[3]]===Overflow[]]&], {i, 1, Length[g2]}];
h[{x_, y_, z_}]:={x, y};
g4 = Map[h, g3, {2}];
t1 = Table[ListPlot[g4[[i]], PlotRange → {{-3/2, 1}, {-1, 1}},
      AspectRatio → Automatic, DisplayFunction → Identity], {i, 1, 6}];
Show[GraphicsGrid[Partition[t1, 3]]]
```

More detail is observed if you use the graphics primitive Point as shown in Figure 4.37. In this case, those points $(x, y)$ for which $|f_{p,x+iy}{}^{100}\ (x+iy)|$ is small are shaded according to a darker GrayLevel than those points for which $\left|f_{p,x+iy}{}^{100}(x+iy)\right|$ is large.

```
h2[{x_, y_, z_}]:={GrayLevel[Min[{z, 1}]], Point[{x, y}]};
g5 = Map[h2, g3, {2}];
t1 = Table[Show[Graphics[g5[[i]]], PlotRange → {{-3/2, 1}, {-1, 1}},
      AspectRatio → Automatic, DisplayFunction → Identity], {i, 1, 6}];
Show[GraphicsGrid[Partition[t1, 3]]]
```

Throughout these examples, we have typically computed the iteration $f^n(z)$ for "large" $n$, such as values of $n$ between 100 and 200. To indicate why we have selected those values of $n$, we revisit the Mandelbrot set plotted in Example 4.3.10.

**Example 4.3.12** As before, all error messages have been deleted.

**(Mandelbrot Set).** We proceed in essentially the same way as in the previous examples. After defining $f_{p,c} = x^p + c$,

```
Clear[f, p]
f[p_, c_][x_]:=x ^ p + c//N;
```

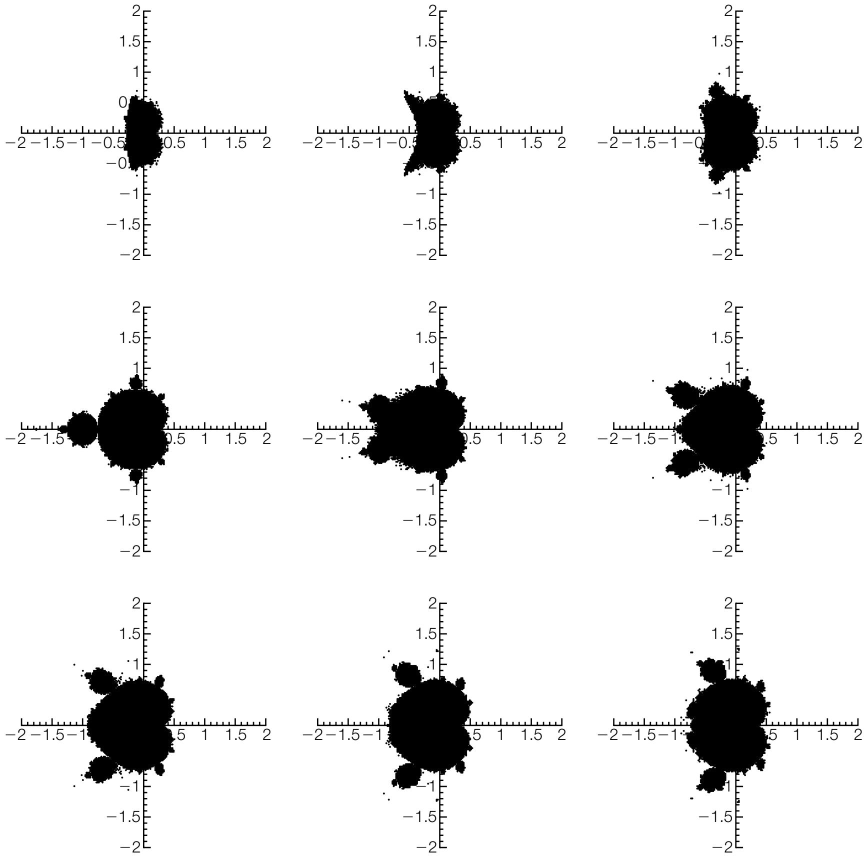


**FIGURE 4.36**

The generalized Mandelbrot set for nine equally spaced values of $p$ between $1.625$ and $2.625$

we use **Table** followed by **Map** to create a nested list. For each $n = 5, 10, 15, 25, 50$, and $100$, a nested list is formed for 200 equally spaced values of $y$ between $-1$ and $1$ and then 200 equally spaced values of $x$ between $-1.5$ and $1$. At the bottom level of each nested list, the elements are of the form $\left(x, y, \left|f_{2,x+iy}{}^{n}(x+iy)\right|\right)$.

```
g1 =
Map[Table[{x, y, Abs[Nest[f[2, x + Iy], x + Iy, #]]}//N,
   {x, −1.5, 1., 5/(2 ∗ 199)}, {y, −1., 1., 2/199}]&,
      {5, 10, 15, 25, 50, 100}];
```

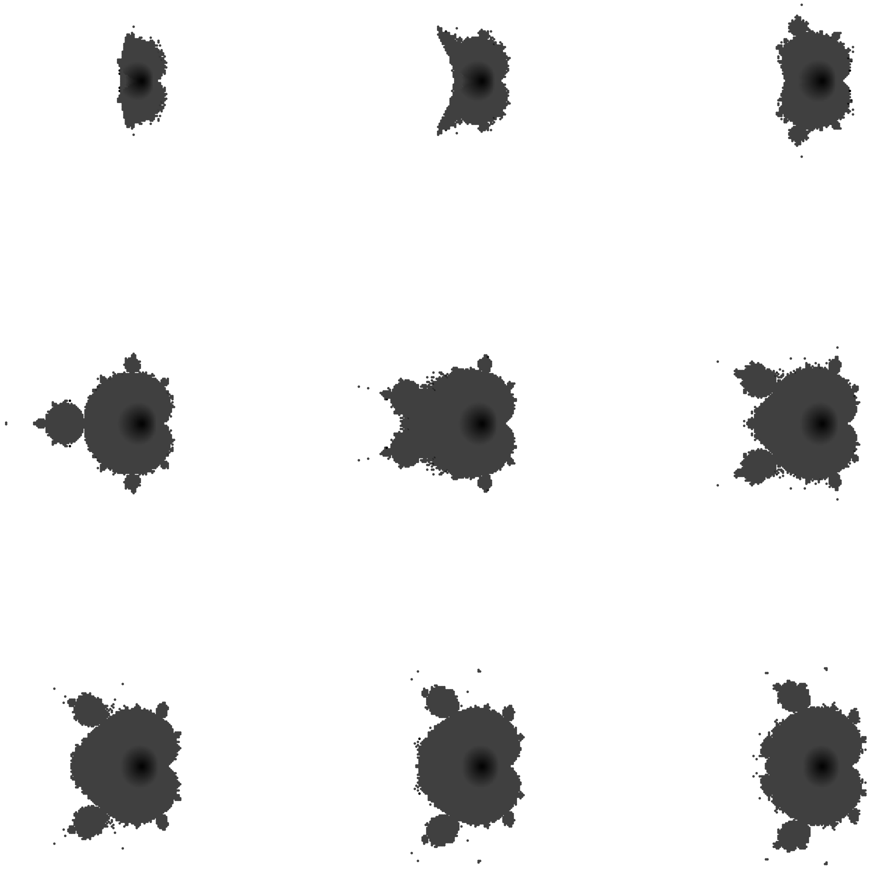


**FIGURE 4.37**

The generalized Mandelbrot set for nine equally spaced values of $p$ between $1.625$ and $2.625$—the points $(x, y)$ for which $\left| f_{p,x+iy}{}^{100}(x+iy) \right|$ is large are shaded lighter than those for which $\left| f_{p,x+iy}{}^{100}(x+iy) \right|$ is small

For each value of $n$, the corresponding list of ordered triples $\left(x, y, \left| f_{2,x+iy}{}^{n}(x+iy) \right|\right)$ is obtained using Flatten.

```
g2 = Map[Flatten[#, 1]&, g1];
```

We then remove those points for which the third coordinate, $\left| f_{2,x+iy}{}^{n}(x+iy) \right|$, is Overflow[] (corresponding to $\infty$),

```
g3 = Table[Select[g2[[i]], Not[#[[3]]===Overflow[]]&], {i, 1, Length[g2]}];
```

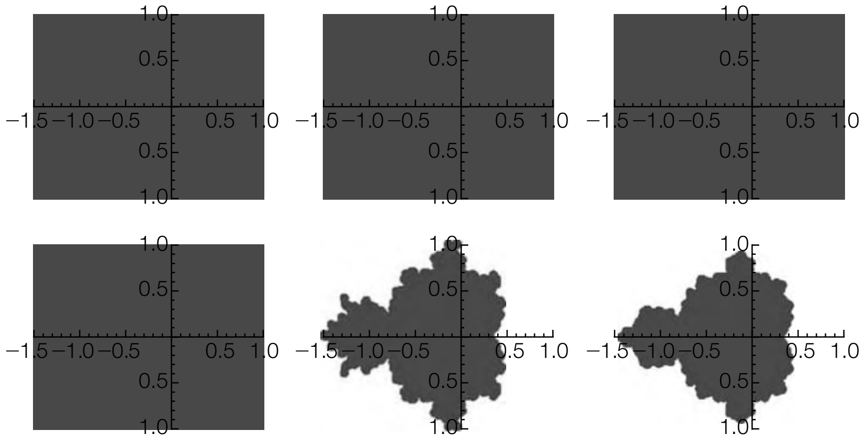



**FIGURE 4.38**

Without shading the points, the effects of iteration are difficult to see until the number of iterations is "large"

extract $(x, y)$ from the remaining ordered triples,

```
h[{x_, y_, z_}] := {x, y};
g4 = Map[h, g3, {2}];
```

Fundamentally, we generated the previous plots by exceeding Mathematica's numerical precision.

and graph the resulting sets of points using ListPlot in Figure 4.38. As shown in Figure 4.38, we see that Mathematica's numerical precision (and consequently decent plots) is obtained when $n = 50$ or $n = 100$.

```
t1 = Table[ListPlot[g4[[i]], PlotRange → {{− 3/2, 1}, {− 1, 1}},
    AspectRatio → Automatic, DisplayFunction → Identity], {i, 1, 6}];
        Show[GraphicsGrid[Partition[t1, 3]]]
```

If instead, we use graphics primitives such as Point and then shade each point $(x, y)$ according to $\left|f_{2,x+iy}{}^{n}(x+iy)\right|$, detail emerges quickly, as shown in Figure 4.39.

```
h2[{x_, y_, z_}]:={GrayLevel[Min[{z, 1}]], Point[{x, y}]};
g5 = Map[h2, g3, {2}];
t1 = Table[Show[Graphics[g5[[i]]], PlotRange → {{− 3/2, 1}, {− 1, 1}},
    AspectRatio → Automatic, DisplayFunction → Identity], {i, 1, 6}];
        Show[GraphicsGrid[Partition[t1, 3]]]
```

Thus, Figures 4.38 and 4.39 indicate that for examples such as these illustrated here, similar results could have been accomplished using far smaller values of $n$ than $n = 100$ or $n = 200$. With fast machines, the differences in the time needed to perform the calculations is minimal; $n = 100$ and $n = 200$ appear to be a "safe" large value of $n$ for well-studied examples such as these.

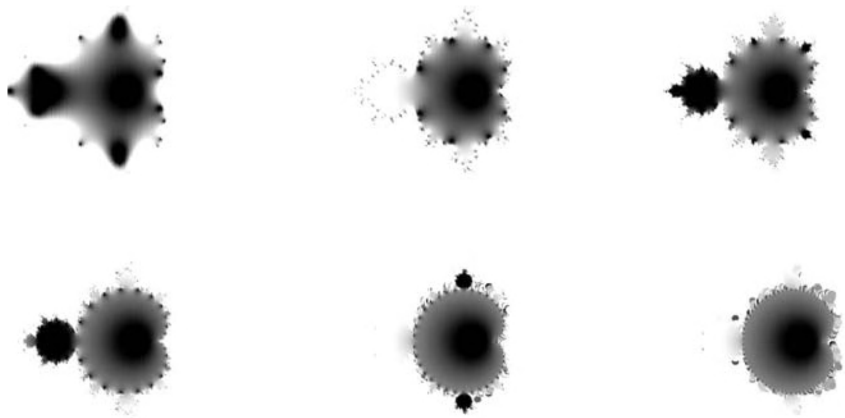


**FIGURE 4.39**

Using graphics primitives and shading, we see that we can use a relatively small number of iterations to visualize the Mandelbrot set

## 4.4 EXERCISES

1. Use Mathematica help to determine the functionality of Chop.
2. Define zeros to be the list of numbers 2.4048, 5.5201, 8.6537, 11.792, 14.931, 18.071, 21.212, 24.352. Use [[...]], Part, First, Last, and/or Take to extract the following from the list zeros. (a) The first and last elements, (b) the fourth through sixth elements, (c) the first three elements, and the last two elements. (d) Use Position to determine if and/or where 18.071 occurs in the list.
3. The **Fibonacci sequence** is defined by $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$. (a) Define a Fibonacci function $f_n = f(n)$ that "remembers" the values computed. (b) Use your Fibonacci function to compute the first 15 values of the Fibonacci sequence. (c) Check that your results are correct by using Table and Fibonacci to compute the first 15 values of the Fibonacci sequence.
4. **(Mathematics of Finance)**

   **(a) (Compound Interest)** A common problem in economics is the determination of the amount of interest earned from an investment. If $P$ dollars are invested for $t$ years at an annual interest rate of $r$% compounded $m$ times per year, the **compound amount**, $A(t)$, at time $t$ is given by

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}.$$

If $P$ dollars are invested for $t$ years at an annual interest rate of $r$% compounded continuously, the compound amount, $A(t)$, at time $t$ is given by $A(t) = Pe^{rnt}$.

Suppose $12,500 is invested at an annual rate of 7% compounded daily. How much money has accumulated and how much interest has been earned at the end of each 5-year period for $t = 0, 5, 10, 15, 20, 25, 30$? How much money has accumulated if interest is compounded continuously instead of daily?

**(b)** Suppose $10,000 is invested at an interest rate of 12% compounded daily. Create a table consisting of the total value of the investment and the interest earned at the end of 0, 5, 10, 15, 20, and 25 years.

What is the total value and interest earned on an investment of $15,000 invested at an interest rate of 15% compounded daily at the end of 0, 10, 20, and 30 years?

**(c)** **(Future Value)** If $R$ dollars are deposited at the end of each period for $n$ periods in an annuity that earns interest at a rate of $j$% per period, the **future value** of the annuity is

$$S_{\text{future}} = R\frac{(1+j)^n - 1}{j}.$$

Define a function future that calculates the future value of an annuity. Compute the future value of an annuity where $250 is deposited at the end of each month for 60 months at a rate of 7% per year. Make a table of the future values of the annuity where $150 is deposited at the end of each month for $12t$ months at a rate of 8% per year for $t = 1, 5, 9, 13, \ldots, 21, 25$.

**(d)** **(Annuity Due)** If $R$ dollars are deposited at the beginning of each period for $n$ periods with an interest rate of $j$% per period, the **annuity due** is

$$S_{\text{due}} = R\left[\frac{(1+j)^{n+1} - 1}{j} - 1\right].$$

Define a function due that computes the annuity due. Use due to (a) compute the annuity due of $500 deposited at the beginning of each month at an annual rate of 12% compounded monthly for 3 years, and (b) calculate the annuity due of $100k deposited at the beginning of each month at an annual rate of 9% compounded monthly for 10 years for $k = 1, 2, 3, \ldots, 10$.

Compare the annuity due on a $100k monthly investment at an annual rate of 8% compounded monthly for $t = 5, 10, 15, 20$ and $k = 1, 2, 3, 4, 5$.

**(e)** **(Present Value)** Another type of problem deals with determining the amount of money that must be invested in order to ensure a particular return on the investment over a certain period of time. The **present value**, $P$, of an annuity of $n$ payments of $R$ dollars each at the end of consecutive interest periods with interest compounded at a rate of $j$% per period is

$$P = R\frac{1-(1+j)^{-n}}{j}.$$

Define a function present to compute the present value of an annuity. (a) Find the amount of money that would have to be invested at 7 1/2% compounded annually to provide an ordinary annuity income of $45,000 per year for 40 years; and (b) find the amount of money that would have to be invested at 8% compounded annually to provide an ordinary annuity income of $\$20,000 + \$5000k$ per year for 35 years for $k = 0, 1, 2, 3, 4$, and 5 years.

**(f) (Deferred Annuities)** The present value of a **deferred annuity** of $R$ dollars per period for $n$ periods deferred for $k$ periods with an interest rate of $j$ per period is

$$P_{def} = R\left[\frac{1-(1+j)^{-(n+k)}}{j} - \frac{1-(1+j)^{-k}}{j}\right].$$

Define a function def[r, n, k, j] to compute the value of a deferred annuity where $r$ equals the amount of the deferred annuity, $n$ equals the number of years in which the annuity is received, $k$ equals the number of years in which the lump sum investment is made, and $j$ equals the rate of interest. Use def to compute the lump sum that would have to be invested for 30 years at a rate of 15% compounded annually to provide an ordinary annuity income of $35,000 per year for 35 years. How much money would have to be invested at the ages of 25, 35, 45, 55, and 65 at a rate of 8 1/2% compounded annually to provide an ordinary annuity income of $30,000 per year for 40 years beginning at age 65?

**(g) (Amortization)** A loan is **amortized** if both the principal and interest are paid by a sequence of equal periodic payments. A loan of $P$ dollars at interest rate $j$ per period may be amortized in $n$ equal periodic payments of $R$ dollars made at the end of each period, where

$$R = \frac{Pj}{1-(1+j)^{-n}}.$$

What is the monthly payment necessary to amortize a loan of $75,000 with an interest rate of 9.5% compounded monthly over 20 years?

What is the monthly payment necessary to amortize a loan of \$80,000 at an annual rate of $j$% in 20 years for $j = 8, 8.5, 9, 9.5, 10$, and 10.5?

In many cases, the amount paid toward the principal of the loan and the total amount that remains to be paid after a certain payment need to be computed.

What is the unpaid balance of the principal at the end of the fifth year of a loan of \$60,000 with an annual interest rate of 8% scheduled to be amortized with monthly payments over a period of 10 years? What is the total interest paid immediately after the 60th payment?

**(h)** What is the total interest paid on a loan of \$60,000 with an interest rate of 8% compounded monthly amortized over a period of 10 years (120 months) immediately after the 60th payment?

**(i)** What is the monthly payment necessary to amortize a loan of \$45,000 with an interest rate of 7% compounded monthly over a period of 15 years (180 months)? What is the total principal and interest paid after 0, 3, 6, 9, 12, and 15 years?

**(j)** Suppose that a loan of \$45,000 with interest rate of 7% compounded monthly is amortized over a period of 15 years (180 months). What is the principal and interest paid during each of the first 5 years of the loan?

**(k)** *Challenge:* Suppose a retiree has \$1,200,000. If she can invest this sum at 7%, compounded annually, what level payment can she withdraw annually for a period of 40 years?

**(l)** *Challenge:* Suppose an investor begins investing at a rate of $d$ dollars per year at an annual rate of $j$%. Each year the investor increases the amount invested by $i$%. How much has the investor accumulated after $m$ years?

**(m)** Another interesting investment problem is discussed in the following exercise. In this case, Mathematica is useful in solving a recurrence equation that occurs in the problem. The command

```
RSolve[{equations},a[n],n]
```

attempts to solve the recurrence equations `equations` for the variable `a[n]` with no dependence on `a[j]`, $j \leq n - 1$.

I am 50 years old and I have \$500,000 that I can invest at a rate of 7% annually. Furthermore, I wish to receive a payment of \$50,000 the first year. Future annual payments should include cost-of-living adjustments at a rate of 3% annually. Is \$500,000 enough to guarantee this amount of annual income if I live to be 80 years old?

**(n)** A 30-year mortgage of \$80,000 with an annual interest rate of 8.125% requires monthly payments of approximately \$600 (\$7200 annually) to amortize the loan in 30 years. However, using annuitytable, show that if the amount of the payments is increased by 3% each year, the 30-year mortgage is amortized in 17 years.

**5.** Define list to be a list of the first 100 positive integers. (a) Find the sum of the first 100 positive integers using Apply together with Plus. (b) Find the product of the first 100 positive integers using Apply and Product. (c) Describe the functionality of Apply. What is an abbreviated form?

**6.** Use RealDigits to find the first 101 digits in the decimal expansion of $\pi$. Use Table together with Count to determine the number of occurrences of each digit (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9). *Challenge:* Repeat the exercise for a greater number of digits. Can you make a reasonable conclusion about the occurrence of each digit in the decimal expansion of $\pi$?

**7.** Recall that a sequence of the form $x_{n+1} = f(x_n)$ is called a **dynamical system**.

**(a)** Using $f(x) = x^2$ with $x_1 = a$, determine if $x_{n+1} = f(x_n)$ has a limit if $a = 1, a = 1.05$, and $a = 0.95$.

This dynamical system is said to have a **fixed point** at $x$ if $f(x) = x$.

To find the fixed points $x_{n+1} = f(x_n)$ with $f(x) = x^2$, we solve $x^2 = x$ or $x^2 - x = 0$ with solutions and $x = 0$ and $x = 1$. In simple terms, a fixed point is called **stable** if a sequence that starts close to the fixed point has the fixed point as a limit. Otherwise, the fixed point is called **unstable**.

**(b)** Would you classify $x = 1$ as stable or unstable? Would you classify $x = 0$ as stable or unstable? Briefly explain.

**(c)** Consider $x_{n+1} = f(x_n)$ with $f(x) = 2x(1 - x)$.

**i.** Find the two fixed points.
**ii.** Let $x_1 = 0.25$. Does the sequence $x_{n+1} = f(x_n)$ converge in this case? If so, what is the limit?
**iii.** Let $x_1 = 0.75$. Does the sequence $x_{n+1} = f(x_n)$ converge in this case? If so, what is the limit?
**iv.** Select any value of $x_1$ between 0 and 1. Does this choice affect the limit?
**v.** Classify the two fixed points as stable or unstable.

**(d)** Sometimes, unusual behavior can be observed when working with dynamical systems. For example, consider the dynamical system with $f(x) = x + 2.5x(1 - x)$ and $x_1 = 1.2$. We see that the sequence oscillates between 0.6 and 1.2. We say that the dynamical system has a **2-cycle** because the values of the sequence oscillate between two numbers.

**(e)** Describe the behavior of $x_{n+1} = f(x_n)$ if $f(x) = x + 2.5x(1 - x)$ and $x_1 = 1.201$. Do you see a cycle? If so, how many numbers. What are these numbers? Does a small change in the initial value of the sequence affect the resulting values of the sequence based on the results of this problem and the previous example?

**(f)** Describe the behavior of $x_{n+1} = f(x_n)$ if $f(x) = x + 2.5x(1 - x)$ and $x_1 = 1.3$. Do you see a cycle? If so, how many numbers. What are these numbers?

**(g)** Describe the behavior of $x_{n+1} = f(x_n)$ if $f(x) = x + 2.5x(1 - x)$ and $x_1 = 1.2$. If the values do not seem to approach a single value or a cycle of several values, we say that the dynamical system is chaotic. Does this system appear to be chaotic?

In addition to your explanations, turn in the graphs obtained with plot for each problem.

**8.** Plot the Julia set for $f(z) = .36e^z$. *Hint:* Use the rectangle $a + bi$ for $0 \le a \le 5$ and $-2.5 \le b \le 2.5$.

CHAPTER

# Matrices and Vectors: Topics from Linear Algebra and Vector Calculus

# 5

Chapter 5 discusses operations on matrices and vectors, including topics from linear algebra, linear programming, and vector calculus.

## 5.1 NESTED LISTS: INTRODUCTION TO MATRICES, VECTORS, AND MATRIX OPERATIONS

### 5.1.1 Defining Nested Lists, Matrices, and Vectors

In Mathematica, a **matrix** is a list of lists where each list represents a row of the matrix. Therefore, the $m \times n$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

is entered with

```
A = {{a11, a12, . . . , a1n}, {a21, a22, . . . , a2n}, . . . , {am1, am2, . . . amn}}
```

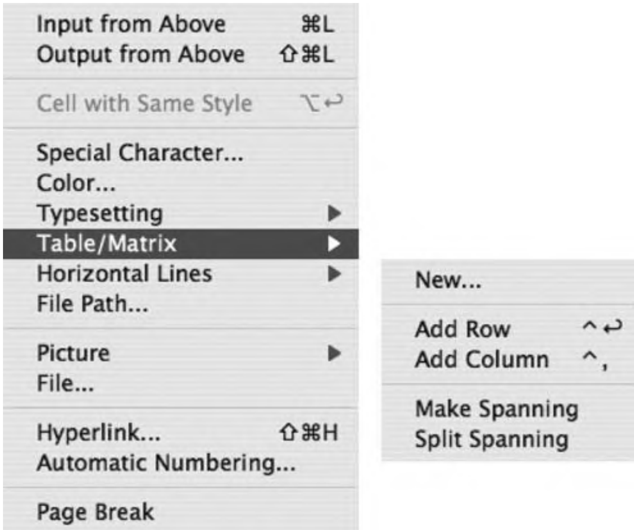
For example, to use Mathematica to define m to be the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, enter the command

```
m = {{a11, a12}, {a21, a22}}
```

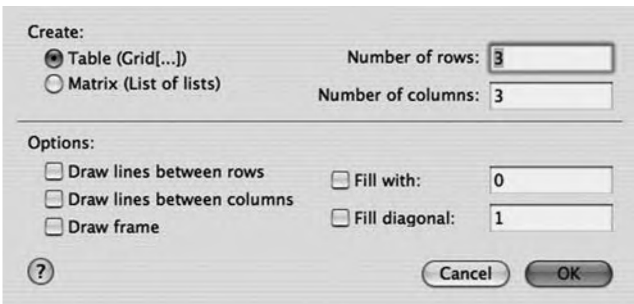
The command m=Array[a,{2,2}] produces a result equivalent to this. Once a matrix A has been entered, it can be viewed in the traditional row-and-column form using the command MatrixForm[A]. You can quickly construct 2 × 2 matrices by clicking on the button from the **BasicMathInput** palette, which is accessed by going to **Palettes** followed by **BasicMathInput**.

As when using TableForm, the result of using MatrixForm is no longer a list that can be manipulated using Mathematica commands. Use MatrixForm to view a matrix in traditional row-and-column form. Do not attempt to perform matrix operations on a MatrixForm object.

Alternatively, you can construct matrices of any dimension by going to the Mathematica menu under **Input** and selecting **Create Table/Matrix/ Palette...**




The resulting pop-up window allows you to create tables, matrices, and palettes. To create a matrix, select **Matrix**, enter the number of rows and columns of the matrix, and select any other options. Pressing the **OK** button places the desired matrix at the position of the cursor in the Mathematica notebook.



**Example 5.1.1** Use Mathematica to define the matrices $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix}$.

***Solution*** In this case, both Table[$a_{i,j}$, {i, 1, 3},{j, 1, 3}] and Array[a,{3,3}] produce equivalent results when we define **matrixa** to be the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The commands MatrixForm or TableForm are used to display the results in traditional matrix form.

**Clear[a, b, matrixa, matrixb]**
**matrixa = Table[$a_{i,j}$, {i, 1, 3}, { j, 1, 3}]**

$\{\{a_{1,1}, a_{1,2}, a_{1,3}\}, \{a_{2,1}, a_{2,2}, a_{2,3}\}, \{a_{3,1}, a_{3,2}, a_{3,3}\}\}$

**MatrixForm[matrixa]**

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

**matrixa = Array[a, {3, 3}]**

{{a[1, 1], a[1, 2], a[1, 3]},
{a[2, 1], a[2, 2], a[2, 3]}, {a[3, 1], a[3, 2], a[3, 3]}}

**MatrixForm[matrixa]**

$$\begin{pmatrix} a[1,1] & a[1,2] & a[1,3] \\ a[2,1] & a[2,2] & a[2,3] \\ a[3,1] & a[3,2] & a[3,3] \end{pmatrix}$$

We may also use Mathematica to define non-square matrices.

**matrixb = Array[b, {2, 4}]**

{{b[1, 1], b[1, 2], b[1, 3], b[1, 4]},{b[2, 1], b[2, 2], b[2, 3], b[2, 4]}}

**MatrixForm[matrixb]**

$$\begin{pmatrix} b[1,1] & b[1,2] & b[1,3] & b[1,4] \\ b[2,1] & b[2,2] & b[2,3] & b[2,4] \end{pmatrix}$$

Equivalent results would have been obtained by entering Table[$b_{i,j}$,{i,1,2}, {j,1,4}].

More generally, the commands Table[f[i, j],{i, imax},{j, jmax}] and Array [f, {imax, jmax}] yield nested lists corresponding to the imax × jmax matrix

$$\begin{pmatrix} f(1,1) & f(1,2) & \cdots & f(1,\mathrm{jmax}) \\ f(2,1) & f(2,2) & \cdots & f(2,\mathrm{jmax}) \\ \vdots & \vdots & \vdots & \vdots \\ f(\mathrm{imax},1) & f(\mathrm{imax},2) & \cdots & f(\mathrm{imax},\mathrm{jmax}) \end{pmatrix}.$$

Table[f[i,j],{i,imin,imax,istep},{j,jmin,jmax,jstep}] returns the list of lists

```
{{f[imin, jmin], f[imin, jmin + jstep],. . . , f[imin, jmax]},
   {f[imin + istep, jmin],. . . ,f[imin + istep, jmax]},
      . . . ,{f[imax, jmin],. . . ,f[imax, jmax]}}
```

and the command

```
Table[f[i, j, k, . . . ],{i, imin, imax, istep}, {j, jmin, jmax, jstep},
{k, kmin, kmax, kstep},. . . ]
```

calculates a nested list; the list associated with $i$ is outermost. If istep is omitted, the step size is one.

**Example 5.1.2** Define $\mathbf{C}$ to be the $3 \times 4$ matrix $(c_{ij})$, where $c_{ij}$, the entry in the $i$th row and $j$th column of $\mathbf{C}$, is the numerical value of $\cos(j^2 - i^2)\sin(i^2 - j^2)$.

***Solution*** After clearing all prior definitions of c, if any, we define c[i,j] to be the numerical value of $\cos(j^2 - i^2)\sin(i^2 - j^2)$ and then use Array to compute the $3 \times 4$ matrix matrixc.

```
Clear[c,matrixc]
c[i_, j_] = N[Cos[j^2 - i^2]Sin[i^2 - j^2]]
Cos[i^2 - 1. j^2] Sin[i^2 - 1. j^2]

matrixc = Array[c, {3, 4}]
{{0., 0.139708, 0.143952, 0.494016},
{-0.139708, 0., 0.272011, 0.452789},
{-0.143952, -0.272011, 0., -0.495304}}

MatrixForm[matrixc]
```

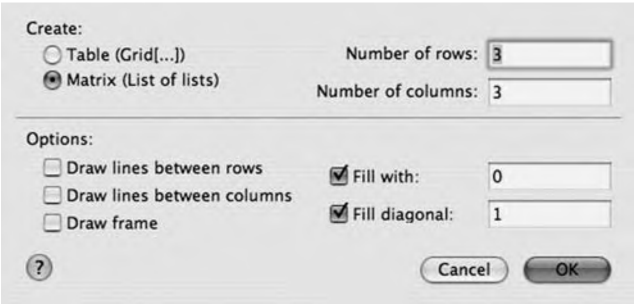
$$\begin{pmatrix} 0. & 0.139708 & 0.143952 & 0.494016 \\ -0.139708 & 0. & 0.272011 & 0.452789 \\ -0.143952 & -0.272011 & 0. & -0.495304 \end{pmatrix}$$

**Example 5.1.3** Define the matrix $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

***Solution*** The matrix $\mathbf{I}_3$ is the $3 \times 3$ **identity matrix**. Generally, the $n \times n$ matrix with 1's on the diagonal and 0's elsewhere is the $n \times n$ identity matrix. The command IdentityMatrix[n] returns the $n \times n$ identity matrix.

```
IdentityMatrix[3]
{{1,0,0}, {0,1,0}, {0,0,1}}
```

The same result is obtained by going to **Insert** under the Mathematica menu and selecting **Table/Matrix/** followed by **New. . .** . We then check **Matrix**, **Fill with**: 0 and **Fill diagonal**: 1.




Pressing the **OK** button inserts the $3 \times 3$ identity matrix at the location of the cursor.

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

---

With Mathematica, you do not need to distinguish between row and column vectors. Provided that computations are well-defined, Mathematica carries them out correctly. Mathematica warns of any ambiguities when they (rarely) occur.

In Mathematica, a **vector** is a list of numbers and, thus, is entered in the same manner as lists. For example, to use Mathematica to define the row vector vectorv to be $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$, enter vectorv={v1,v2,v3}. Similarly, to define the column vector vectorv to be $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, enter vectorv={v1,v2,v3} or vectorv={{v1},{v2},{v3}}.

Generally, with Mathematica you do not need to distinguish between row and column vectors: Mathematica usually performs computations with vectors and matrices correctly as long as the computations are well-defined.

---

**Example 5.1.4** Define the vector $\mathbf{w} = \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}$, vectorv to be the vector $\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}$ and zerovec to be the vector $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

***Solution*** To define **w**, we enter

```
w = {-4, -5, 2}
{-4, -5, 2}
```

or

```
w = {{-4}, {-5}, {2}};
MatrixForm[w]
```

$$\begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}$$

To define vectorv, we use Array.

```
vectorv = Array[v, 4]
{v[1], v[2], v[3], v[4]}
```

Equivalent results would have been obtained by entering Table[$v_i$,{i,1,4}]. To define zerovec, we use Table.

```
zerovec = Table[0, {5}]
{0, 0, 0, 0, 0}
```

The same result is obtained by going to **Insert** under the Mathematica menu and selecting **Table/Matrix** to create the zero vector.

```
(0 0 0 0 0)
{{0, 0, 0, 0, 0}}
```

---

## 5.1.2 Extracting Elements of Matrices

For the $2 \times 2$ matrix m = {{$a_{1,1}$, $a_{1,2}$}, {$a_{2,1}$, $a_{2,2}$}} defined previously, m[[1]] yields the first element of matrix m which is the list {$a_{1,1}$, $a_{1,2}$} or the first row of m; m[[2,1]] yields the first element of the second element of matrix m which is $a_{2,1}$. In general, if m is an $i \times j$ matrix, m[[i,j]] or Part[m,i,j] returns the unique element in the $i$th row and $j$th column of m. Specifically, m[[i,j]] yields the $j$th part of the $i$th part of m; list[[i]] or Part[list,i] yields the $i$th part of list; list[[i,j]] or Part[list,i,j] yields the $j$th part of the $i$th part of list, and so on.

---

**Example 5.1.5** Define mb to be the matrix $\begin{pmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{pmatrix}$. (a) Extract the third row of mb. (b) Extract the element in the first row and third column of mb. (c) Display mb in traditional matrix form.

***Solution*** We begin by defining mb. mb[[i,j]] yields the (unique) number in the *i* th row and *j* th column of mb. Observe how various components of mb (rows and elements) can be extracted and how mb is placed in MatrixForm.

```
mb = {{10, -6, -9}, {6, -5, -7}, {-10, 9, 12}};
MatrixForm[mb]
```

$$\begin{pmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{pmatrix}$$

```
mb[[3]]
{-10, 9, 12}
```

```
mb[[1, 3]]
-9
```

---

If m is a matrix, the *i*th row of m is extracted with m[[i]]. The command Transpose[m] yields the transpose of the matrix m, the matrix obtained by interchanging the rows and columns of m. We extract columns of m by computing Transpose[m] and then using Part to extract rows from the transpose. Namely, if m is a matrix, Transpose[m][[i]] extracts the *i*th row from the transpose of m which is the same as the *i*th column of m.

Alternatively, if **A** is $n \times m$ (rows × columns), the *i*th column of **A** is the vector that consists of the *i*th part of each row of the matrix, so given an *i*-value Table[A[[j,i]],{j,1,n}] returns the *i*th column of **A**.

---

**Example 5.1.6** Extract the second and third columns from $\mathbf{A} = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 1 & -3 \\ 2 & -4 & 1 \end{pmatrix}$.

***Solution*** We first define matrixa and then use Transpose to compute the transpose of matrixa, naming the result ta, and then displaying ta in MatrixForm.

```
matrixa = {{0, -2, 2}, {-1, 1, -3}, {2, -4, 1}};
MatrixForm[matrixa]
```

$$\begin{pmatrix} 0 & -2 & 2 \\ -1 & 1 & -3 \\ 2 & -4 & 1 \end{pmatrix}$$

```
ta = Transpose[matrixa]
MatrixForm[ta]
{{0, -1, 2}, {-2, 1, -4}, {2, -3, 1}}
```

$$\begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & -4 \\ 2 & -3 & 1 \end{pmatrix}$$

Next, we extract the second column of **matrixa** using **Transpose** together with **Part** ([[...]]). Because we have already defined **ta** to be the transpose of **matrixa**, entering **ta[[2]]** would produce the same result.

**Transpose[matrixa][[2]]**

{−2, 1, −4}

To extract the third column, we take advantage of the fact that we have already defined **ta** to be the transpose of **matrixa**. Entering **Transpose[matrixa][[3]]** produces the same result.

**ta[[3]]**

{2, −3, 1}

You can also use **Take** to extract elements of lists and matrices. Entering

**Take[matrixa, 2]**
**Take[matrixa, 2]//MatrixForm**

{{0, −2, 2}, {−1, 1, −3}}

$$\begin{pmatrix} 0 & -2 & 2 \\ -1 & 1 & -3 \end{pmatrix}$$

returns the first two rows of **matrixa** because the first two parts of **matrixa** are the lists corresponding to those rows. Similarly,

**Take[matrixa, {2}]**
**Take[matrixa, {2}]//MatrixForm**

{{−1, 1, −3}}

$$\begin{pmatrix} -1 & 1 & -3 \end{pmatrix}$$

returns the second row, whereas

**Take[matrixa, {2, 3}]**
**Take[matrixa, {2, 3}]//MatrixForm**

{{−1, 1, −3}, {2, −4, 1}}

$$\begin{pmatrix} -1 & 1 & -3 \\ 2 & -4 & 1 \end{pmatrix}$$

returns the second and third rows.

---

The example illustrates that Take[list,n] returns the first $n$ elements of list; Take[list,{n}] returns the $n$th element of list; Take[list,{n1, n2, ...}] returns the $n_1$st, $n_2$nd,... elements of list; and so on.

## 5.1.3 Basic Computations with Matrices

Mathematica performs all of the usual operations on matrices. Matrix addition ($\mathbf{A}+\mathbf{B}$), scalar multiplication ($k\mathbf{A}$), matrix multiplication (when defined) ($\mathbf{AB}$), and combinations of these operations are all possible. The **transpose** of $\mathbf{A}$, $\mathbf{A}^t$, is obtained by interchanging the rows and columns of $\mathbf{A}$ and is computed with the command Transpose[A]. If $\mathbf{A}$ is a square matrix, the determinant of $\mathbf{A}$ is obtained with Det[A].

If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices satisfying $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ matrix with 1's on the diagonal and 0's elsewhere (the $n \times n$ identity matrix), $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$ and is denoted by $\mathbf{A}^{-1}$. If the inverse of a matrix $\mathbf{A}$ exists, the inverse is found with Inverse[A]. Thus, assuming that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse ($ad - bc \neq 0$), the inverse is

**Inverse[{{a, b}, {c, d}}]**

$$\left\{\left\{\frac{d}{-bc+ad}, -\frac{b}{-bc+ad}\right\}, \left\{-\frac{c}{-bc+ad}, \frac{a}{-bc+ad}\right\}\right\}$$

---

**Example 5.1.7** Let $\mathbf{A} = \begin{pmatrix} 3 & -4 & 5 \\ 8 & 0 & -3 \\ 5 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{pmatrix}$. Compute (a) $\mathbf{A}+\mathbf{B}$; (b) $\mathbf{B} - 4\mathbf{A}$; (c) the inverse of $\mathbf{AB}$; (d) the transpose of $(\mathbf{A} - 2\mathbf{B})\mathbf{B}$; and (e) $\det \mathbf{A} = |\mathbf{A}|$.

***Solution*** We enter ma (corresponding to $\mathbf{A}$) and mb (corresponding to $\mathbf{B}$) as nested lists, where each element corresponds to a row of the matrix. We suppress the output by ending each command with a semicolon.

**ma = {{3, −4, 5}, {8, 0, −3}, {5, 2, 1}};**
**mb = {{10, −6, −9}, {6, −5, −7}, {−10, 9, 12}};**

Entering

**ma + mb//MatrixForm**

$$\begin{pmatrix} 13 & -10 & -4 \\ 14 & -5 & -10 \\ -5 & 11 & 13 \end{pmatrix}$$

adds matrix ma to mb and expresses the result in traditional matrix form. Entering

**mb − 4ma//MatrixForm**

$$\begin{pmatrix} -2 & 10 & -29 \\ -26 & -5 & 5 \\ -30 & 1 & 8 \end{pmatrix}$$

subtracts four times matrix ma from mb and expresses the result in traditional matrix form. Entering

**Inverse[ma · mb]//MatrixForm**

$$\begin{pmatrix} \frac{59}{380} & \frac{53}{190} & -\frac{167}{380} \\ -\frac{223}{570} & -\frac{92}{95} & \frac{979}{570} \\ \frac{49}{114} & \frac{18}{19} & -\frac{187}{114} \end{pmatrix}$$

computes the inverse of the matrix product $\mathbf{AB}$. Similarly, entering

Matrix products, when defined, are computed by placing a period (.) between the matrices being multiplied. Note that a period is also used to compute the dot product of two vectors, when the dot product is defined.

**Transpose[(ma − 2mb) · mb]//MatrixForm**

$$\begin{pmatrix} -352 & -90 & 384 \\ 269 & 73 & -277 \\ 373 & 98 & -389 \end{pmatrix}$$

computes the transpose of $(\mathbf{A} - 2\mathbf{B})\mathbf{B}$, and entering

**Det[ma]**

190

computes the determinant of $\mathbf{A}$.

---

**Example 5.1.8** Compute $\mathbf{AB}$ and $\mathbf{BA}$ if $\mathbf{A} = \begin{pmatrix} -1 & -5 & -5 & -4 \\ -3 & 5 & 3 & -2 \\ -4 & 4 & 2 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ -4 & 3 \\ 4 & -4 \\ -5 & -3 \end{pmatrix}$.

***Solution*** Because $\mathbf{A}$ is a $3 \times 4$ matrix and $\mathbf{B}$ is a $4 \times 2$ matrix, $\mathbf{AB}$ is defined and is a $3 \times 2$ matrix. We define matrixa and matrixb with the following commands.

Remember that you can also define matrices by going to **Insert** under the Mathematica menu and selecting **Table/Matrix**. After entering the desired number of rows and columns and pressing the **OK** button, a matrix template is placed at the location of the cursor that you can fill in.

**matrixa =** $\begin{pmatrix} -1 & -5 & -5 & -4 \\ -3 & 5 & 3 & -2 \\ -4 & 4 & 2 & -3 \end{pmatrix}$**;**

**matrixb =** $\begin{pmatrix} 1 & -2 \\ -4 & 3 \\ 4 & -4 \\ -5 & -3 \end{pmatrix}$**;**

We then compute the product, naming the result ab, and display ab in MatrixForm.

**ab = matrixa.matrixb;**
**MatrixForm[ab]**

$$\begin{pmatrix} 19 & 19 \\ -1 & 15 \\ 3 & 21 \end{pmatrix}$$

However, the matrix product $\mathbf{BA}$ is not defined and Mathematica produces error messages when we attempt to compute it.

```
matrixa = ( -1 -5 -5 -4
            -3  5  3 -2
            -4  4  2 -3 );

matrixb = (  1 -2
            -4  3
             4 -4
            -5 -3 );

ab = matrixa.matrixb;
MatrixForm[ab]

( 19 19
  -1 15
   3 21 )

matrixb.matrixa

Dot::dotsh : Tensors {{1, -2}, {-4, 3}, {4, -4}, {-5, -3}}
   and {{-1, -5, -5, -4}, {-3, 5, 3, -2}, {-4, 4, 2, -3}}
   have incompatible shapes. >>

{{1, -2}, {-4, 3}, {4, -4}, {-5, -3}}.

{{-1, -5, -5, -4}, {-3, 5, 3, -2}, {-4, 4, 2, -3}}
```

---

Special attention must be given to the notation that must be used in taking the product of a square matrix with itself. The following example illustrates how Mathematica interprets the expression (matrixb)^n. The command (matrixb)^n raises each element of the matrix matrixb to the $n$th power. The command MatrixPower is used to compute powers of matrices.

---

**Example 5.1.9** Let $\mathbf{B} = \begin{pmatrix} -2 & 3 & 4 & 0 \\ -2 & 0 & 1 & 3 \\ -1 & 4 & -6 & 5 \\ 4 & 8 & 11 & -4 \end{pmatrix}$. (a) Compute $\mathbf{B}^2$ and $\mathbf{B}^3$. (b) Cube each entry of $\mathbf{B}$.

***Solution*** After defining $\mathbf{B}$, we compute $\mathbf{B}^2$. The same results would have been obtained by entering MatrixPower[matrixb,2].

```
matrixb = {{-2, 3, 4, 0}, {-2, 0, 1, 3}, {-1, 4, -6, 5},
{4, 8, 11, -4}};
MatrixForm[matrixb . matrixb]
```

$$\begin{pmatrix} -6 & 10 & -29 & 29 \\ 15 & 22 & 19 & -7 \\ 20 & 13 & 91 & -38 \\ -51 & 24 & -86 & 95 \end{pmatrix}$$

Next, we use MatrixPower to compute $\mathbf{B}^3$. The same results would be obtained by entering matrixb.matrixb.matrixb.

**MatrixForm[MatrixPower[matrixb, 3]]**

$$\begin{pmatrix} 137 & 98 & 479 & -231 \\ -121 & 65 & -109 & 189 \\ -309 & 120 & -871 & 646 \\ 520 & 263 & 1381 & -738 \end{pmatrix}$$

Last, we cube each entry of $\mathbf{B}$ with ^.

**MatrixForm[matrixb^3]**

$$\begin{pmatrix} -8 & 27 & 64 & 0 \\ -8 & 0 & 1 & 27 \\ -1 & 64 & -216 & 125 \\ 64 & 512 & 1331 & -64 \end{pmatrix}$$

---

If $|\mathbf{A}| \neq 0$, the inverse of $\mathbf{A}$ can be computed using the formula

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^a, \tag{5.1}$$

The **cofactor matrix**, $\mathbf{A}^c$, of $\mathbf{A}$ is the matrix obtained by replacing each element of $\mathbf{A}$ by its cofactor.

where $\mathbf{A}^a$ is the *transpose of the cofactor matrix*.

If $\mathbf{A}$ has an inverse, reducing the matrix $(\mathbf{A}|\mathbf{I})$ to reduced row echelon form results in $(\mathbf{I}|\mathbf{A}^{-1})$. This method is often easier to implement than (5.1).

---

**Example 5.1.10** Calculate $\mathbf{A}^{-1}$ if $\mathbf{A} = \begin{pmatrix} 2 & -2 & 1 \\ 0 & -2 & 2 \\ -2 & -1 & -1 \end{pmatrix}$.

***Solution*** After defining $\mathbf{A}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we compute $|\mathbf{A}| = 12$, so $\mathbf{A}^{-1}$ exists.

**capa = {{2, −2, 1}, {0, −2, 2}, {−2, −1, −1}}**
**i3 = IdentityMatrix[3]**

{{2, −2, 1}, {0, −2, 2}, {−2, −1, −1}}
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

**Det[capa]**

12

Join[a,b,n] concatenates lists a and b at level n. For matrices the level one objects (capa[[i]]) are the rows; the level two objects (capa[[i,j]]) are the entries. Thus, Join[capa,i3] returns the matrix $\begin{pmatrix} \mathbf{A} \\ \mathbf{I} \end{pmatrix}$, whereas Join[capa,i3,2] forms the matrix $(\mathbf{A}|\mathbf{I})$.

**ai3 = Join[capa, i3, 2]**

{{2, −2, 1, 1, 0, 0}, {0, −2, 2, 0, 1, 0},
{−2, −1, −1, 0, 0, 1}}

**MatrixForm[ai3]**

$$\begin{pmatrix} 2 & -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

RowReduce[A] reduces **A** to **reduced row echelon form**.

We then use RowReduce to reduce $(\mathbf{A}|\mathbf{I})$ to row echelon form.

**rrai3 = RowReduce[ai3]**

$\left\{\left\{1, 0, 0, \frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}\right\}, \left\{0, 1, 0, -\frac{1}{3}, 0, -\frac{1}{3}\right\}, \left\{0, 0, 1, -\frac{1}{3}, \frac{1}{2}, -\frac{1}{3}\right\}\right\}$

**MatrixForm[rrai3]**

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{4} & -\frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}$$

The result indicates that $\mathbf{A}^{-1} = \begin{pmatrix} 1/3 & -1/4 & -1/6 \\ -1/3 & 0 & -1/3 \\ -1/3 & 1/2 & -1/3 \end{pmatrix}$.

## 5.1.4 Basic Computations with Vectors

### *Basic Operations on Vectors*

**Computations with vectors are performed in the same way as computations with matrices.**

**Example 5.1.11** Let $\mathbf{v} = \begin{pmatrix} 0 \\ 5 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -2 \end{pmatrix}$. (a) Calculate $\mathbf{v} - 2\mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$. (b) Find a unit vector with the same direction as $\mathbf{v}$ and a unit vector with the same direction as $\mathbf{w}$.

***Solution*** We begin by defining $\mathbf{v}$ and $\mathbf{w}$ and then compute $\mathbf{v} - 2\mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$.

**v = {0, 5, 1, 2};**
**w = {3, 0, 4, −2};**
**v − 2w**

{−6, 5, −7, 6}

**v · w**

0

The **norm** of the vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

The command Norm[v] returns the norm of the vector $\mathbf{v}$.
If $k$ is a scalar, the direction of $k\mathbf{v}$ is the same as the direction of $\mathbf{v}$. Thus, if $\mathbf{v}$ is a nonzero vector, the vector $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ has the same direction as $\mathbf{v}$ and because $\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$, $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector. First, we compute $\|\mathbf{v}\|$ with Norm. We then compute $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$, calling the result uv, and $\frac{1}{\|\mathbf{w}\|}\mathbf{w}$. The results correspond to unit vectors with the same direction as $\mathbf{v}$ and $\mathbf{w}$, respectively.

**Norm[v]**

$\sqrt{30}$

**uv =** $\frac{\mathbf{v}}{\mathbf{Norm[v]}}$

$\left\{0, \sqrt{\frac{5}{6}}, \frac{1}{\sqrt{30}}, \sqrt{\frac{2}{15}}\right\}$

**Norm[uv]**

1

$\frac{\mathbf{w}}{\mathbf{Norm[w]}}$

$\left\{\frac{3}{\sqrt{29}}, 0, \frac{4}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\right\}$

---

### *Basic Operations on Vectors in 3-Space*

Vector calculus is discussed in Section 5.5.

We review the elementary properties of vectors in 3-space. Let

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

and

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space.

**1.** u and v are **equal** if and only if their components are equal:

$$\mathbf{u} = \mathbf{v} \Leftrightarrow u_1 = v_1, u_2 = v_2, \text{ and } u_3 = v_3.$$

In space, the **standard unit vectors** are $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$. With the exception of the cross product, the vector operations discussed here are performed in the same way for vectors in the plane as they are in space. In the plane, the **standard unit vectors** are $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.

A **unit vector** is a vector with length 1.

2. The **length** (or **norm**) of $\mathbf{u}$ is

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

3. If $c$ is a scalar (number),

$$c\mathbf{u} = \langle cu_1, cu_2, cu_3\rangle .$$

4. The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is defined to be the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle .$$

5. If $\mathbf{u} \neq \mathbf{0}$, a unit vector with the same direction as $\mathbf{u}$ is

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{u_1^2 + u_2^2 + u_3^2}}\langle u_1, u_2, u_3\rangle .$$

6. $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is a scalar $c$ so that $\mathbf{u} = c\mathbf{v}$.
7. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

If $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Consequently, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
8. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \left(u_2 v_3 - u_3 v_2\right)\mathbf{i} - \left(u_1 v_3 - u_3 v_1\right)\mathbf{j} + \left(u_1 v_2 - u_2 v_1\right)\mathbf{k}.\end{aligned}$$

You should verify that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. Hence, $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Topics from linear algebra (including determinants) are discussed in more detail in the next sections. For now, we illustrate several of the basic operations listed previously: u.v and Dot[u,v] compute $\mathbf{u} \cdot \mathbf{v}$; Cross[u,v] computes $\mathbf{u} \times \mathbf{v}$.

**Example 5.1.12** Let $\mathbf{u} = \langle 3, 4, 1 \rangle$ and $\mathbf{v} = \langle -4, 3, -2 \rangle$. Calculate (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{u} \times \mathbf{v}$, (c) $\|\mathbf{u}\|$, and (d) $\|\mathbf{v}\|$. (e) Find the angle between $\mathbf{u}$ and $\mathbf{v}$. (f) Find unit vectors with the same direction as $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$.

***Solution*** We begin by defining $\mathbf{u} = \langle 3, 4, 1 \rangle$ and $\mathbf{v} = \langle -4, 3, -2 \rangle$. Notice that to define $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ with Mathematica, we use the form

**u = {u1, u2, u3}**

Similarly, to define $\mathbf{u} = \langle u_1, u_2 \rangle$, we use the form u={u1,u2}.

We illustrate the use of Dot and Cross to calculate (a)–(d).

```
u = {3, 4, 1};
v = {-4, 3, -2};
udv = Dot[u, v]
-2
u·v
-2
ucv = Cross[u, v]
{-11, 2, 25}
nu = Norm[u]
√26
nv = Sqrt[v·v]
√29
```

We use the formula $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$ to find the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$.

**ArcCos[u·v/(nu nv)]**
**N[%]**

$\text{ArcCos}\left[-\sqrt{\frac{2}{377}}\right]$

1.6437

Unit vectors with the same direction as $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ are found next.

**normu = u/nu**
**normv = v/nv**

$\left\{\frac{3}{\sqrt{26}}, 2\sqrt{\frac{2}{13}}, \frac{1}{\sqrt{26}}\right\}$

$\left\{-\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\right\}$

**nucrossv = ucv/Norm[ucv]**

$\left\{-\frac{11}{5\sqrt{30}}, \frac{\sqrt{\frac{2}{15}}}{5}, \sqrt{\frac{5}{6}}\right\}$

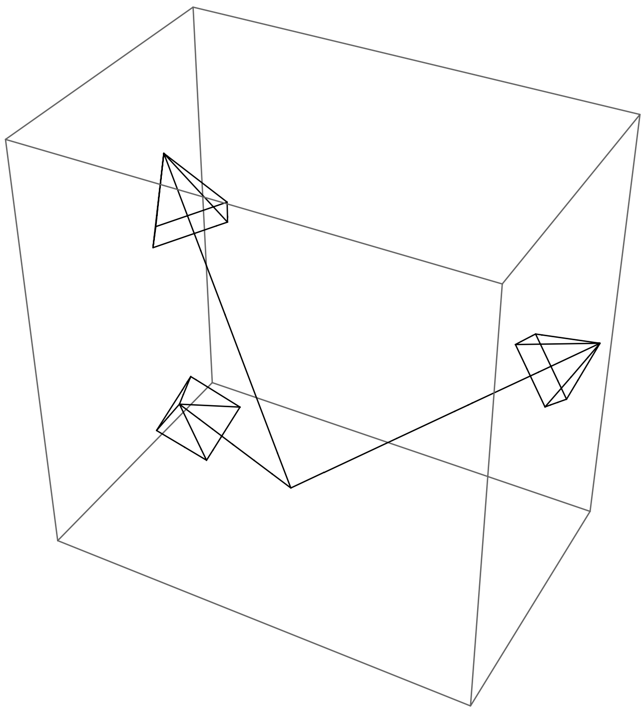


**FIGURE 5.1**

Orthogonal vectors

We can graphically confirm that these three vectors are orthogonal by graphing all three vectors with the **ListVectorFieldPlot3D** function, which is contained in the VectorFieldPlots package. After loading the **ListVectorFieldPlot3D** package, the command

**ListVectorFieldPlot3D[listofvectors]**

graphs the list of vectors listofvectors. Each element of listofvectors is of the form {{u1,u2,u3},{v1,v2,v3}}, where $(u_1, u_2, u_3)$ and $(v_1, v_2, v_3)$ are the initial and terminal points of each vector. We show the vectors in Figure 5.1.

```
Needs["VectorFieldPlots"]
ListVectorFieldPlot3D[{{{0, 0, 0}, normu},
   {{0, 0, 0}, normv}, {{0, 0, 0}, nucrossv}},
      VectorHeads → True]
```

In the plot, the vectors do appear to be orthogonal as expected.

With the exception of the cross product, the calculations described previously can also be performed on vectors in the plane.

**Example 5.1.13** If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, the **projection** of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}.$$

Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ if $\mathbf{u} = \langle -1, 4\rangle$ and $\mathbf{v} = \langle 2, 6\rangle$.

***Solution*** First, we define $\mathbf{u} = \langle -1, 4\rangle$ and $\mathbf{v} = \langle 2, 6\rangle$ and then compute $\text{proj}_{\mathbf{v}}\mathbf{u}$.

```
u = {-1, 4};
v = {2, 6};
projvu = u·v v/v·v
```

$\left\{\frac{11}{10}, \frac{33}{10}\right\}$

Next, we graph $\mathbf{u}$, $\mathbf{v}$, and $\text{proj}_{\mathbf{v}}\mathbf{u}$ together using Arrow, Show, and GraphicsRow in Figure 5.2.

**?Arrow**

Info3403594915-4587150

Arrow[{{$x_1$, $y_1$}, {$x_2$, $y_2$}}] is a graphics primitive which represents an arrow from ($x_1$, $y_1$) to ($x_2$, $y_2$).
Arrow[{$pt_1$, $pt_2$}, $s$] represents an arrow with its ends set back from $pt_1$ and $pt_2$ by a distance $s$.
Arrow[{$pt_1$, $pt_2$}, {$s_1$, $s_2$}] sets back by $s_1$ from $pt_1$ and $s_2$ from $pt_2$. ≫

```
p1 = Show[Graphics[{Arrowheads[Medium], Arrow[{{0, 0}, u}],
        Arrow[{{0, 0}, v}],
        Thickness[.05], Arrow[{{0, 0}, projvu}]}],
        Axes → Automatic, AspectRatio → Automatic];
p2 = Show[Graphics[{Arrowheads[Medium], Arrow[{{0, 0}, u}],
```

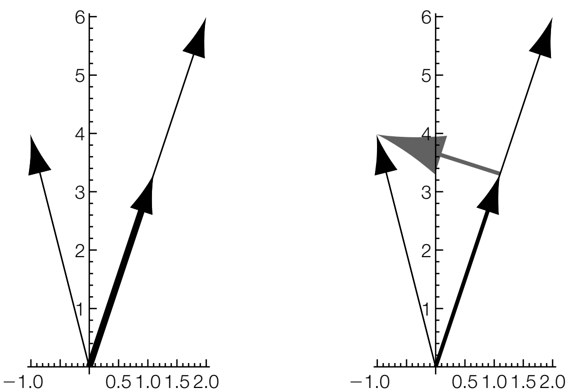



**FIGURE 5.2**

Projection of a vector

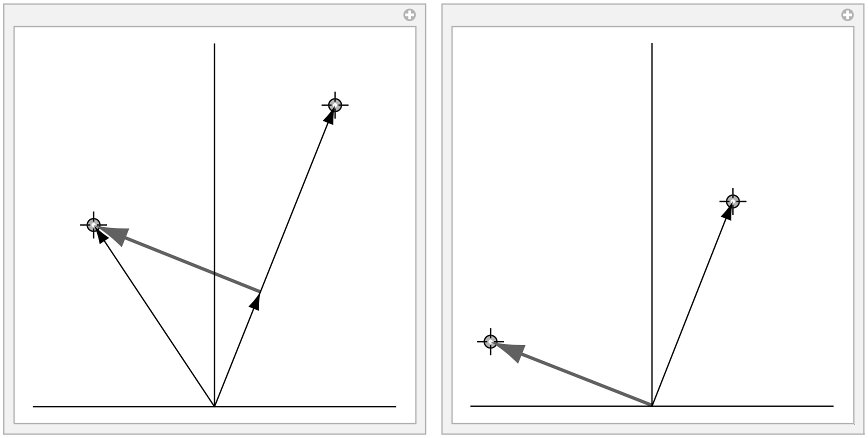


**FIGURE 5.3**

Using Manipulate to visualize the projection of one vector onto another

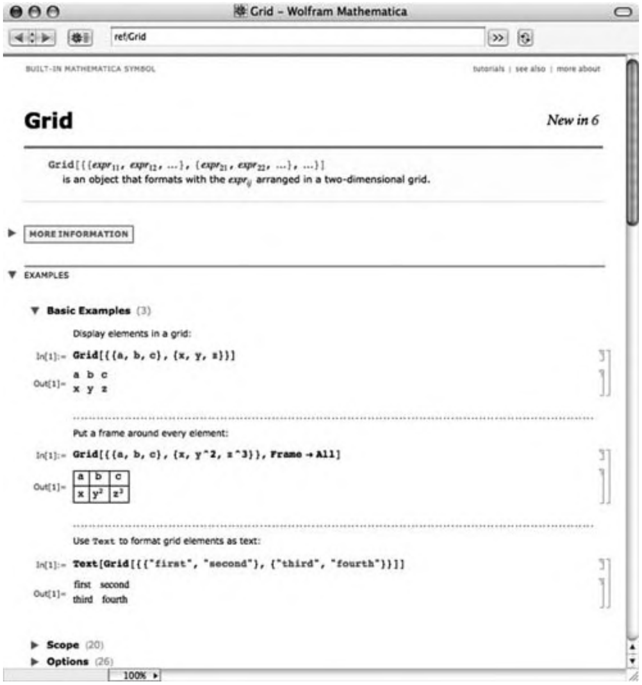
```
Arrow[{{0, 0}, v}],
        Thickness[.03], Arrow[{{0, 0}, projvu}], GrayLevel[.4],
        Arrowheads[Large], Arrow[{projvu, u}]}],
        Axes → Automatic, AspectRatio → Automatic];
Show[GraphicsRow[{p1, p2}]]
```

In the graph, notice that $\mathbf{u} = \text{proj}_{\mathbf{v}}\mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u})$ and the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is perpendicular to $\mathbf{v}$.

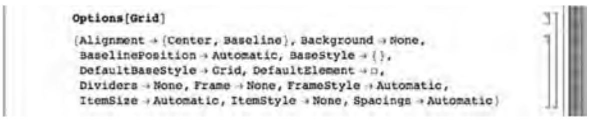
With the following, we use Manipulate to generalize the example. See Figure 5.3.

```
Clear[u, v, projvu, p1, p2];
Manipulate[
projvu = u.vv/v.v;
Show[Graphics[{Arrowheads[Medium], Arrow[{{0, 0}, u}],
        Arrow[{{0, 0}, v}],
        Thickness[.005], Arrow[{{0, 0}, projvu}], GrayLevel[.4],
        Arrowheads[Large], Arrow[{projvu, u}]}],
        Axes → Automatic, PlotRange → {{-3, 3}, {0, 6}},
        AspectRatio → Automatic, Ticks → None], {{u, {-2, 3}}, Locator},
{{v, {2, 5}}, Locator}]
```

If you only need to display a two-dimensional array in row-and-column form, it is easier to use Grid rather than Table together with TableForm or MatrixForm.




For a list of all the options associated with Grid, enter Options[Grid].




Thus,

**p0 = Grid[{{a, b, c}, {d, e}, {f}}, Frame → All]**

creates a basic grid. The first row consists of the entries *a*, *b*, and *c*; the second row *d* and *e*; and the third row *f*. See Figure 5.4.

You can create quite complex arrays with Grid. For example, elements of grids can be any Mathematica object, including grids.

In the following, we use ExampleData to generate several typical Mathematica objects.

StringTake[string,n] returns the first *n* characters of the string string.

**p1 = ExampleData[{"AerialImage", "Earth"}];**
**p2 = StringTake[ExampleData[{"Text", "GettysburgAddress"}], 100];**
**p3 = ExampleData[{"Geometry3D", "KleinBottle"}];**
**p4 = ExampleData[{"Texture", "Bubbles3"}]; .1in**

| | | |
|---|---|---|
| a | b | c |
| d | e | |
| f | | |

**FIGURE 5.4**

A basic grid

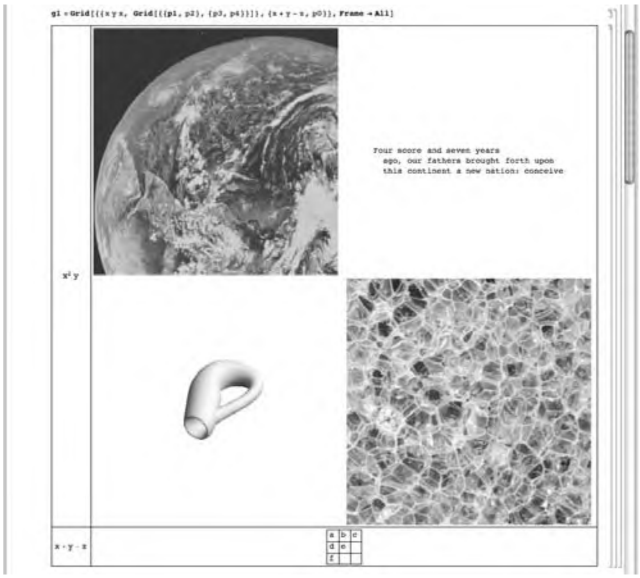



**FIGURE 5.5**

Very basic grids can appear to be quite complicated

Using our first grid, the previous data, and a few more strings, we create a more sophisticated grid in Figure 5.5.

**g1 = Grid[{{xyx, Grid[{{p1, p2}, {p3, p4}}]}, {x + y − z, p0}}, Frame → All]**

## 5.2 LINEAR SYSTEMS OF EQUATIONS

### 5.2.1 Calculating Solutions of Linear Systems of Equations

To solve the system of linear equations $\mathbf{A}x = b$, where $\mathbf{A}$ is the coefficient matrix, b is the known vector, and x is the unknown vector, we often proceed as follows: If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}\mathbf{A}^{-1}x = \mathbf{A}^{-1}b$ so $x = \mathbf{A}^{-1}b$.

Mathematica offers several commands for solving systems of linear equations, however, that do not depend on the computation of the inverse of **A**. The command

**Solve[{eqn1, eqn2, ..., eqnm}, {var1, var2, ..., varn}]**

solves an $m \times n$ system of linear equations ($m$ equations and $n$ unknown variables). Note that both the equations and the variables are entered as lists. If one wishes to solve for all variables that appear in a system, the command Solve[{eqn1, eqn2, ..., eqnn}] attempts to solve eqn1, eqn2, ..., eqnn for all variables that appear in them. (Remember that a double equals sign (==) must be placed between the left- and right-hand sides of each equation.)

---

**Example 5.2.1** Solve the matrix equation $\begin{pmatrix} 3 & 0 & 2 \\ -3 & 2 & 2 \\ 2 & -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

***Solution*** The solution is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ -3 & 2 & 2 \\ 2 & -3 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$. We proceed by defining matrixa and b and then using Inverse to calculate Inverse[matrixa].b, naming the resulting output {x,y,z}.

```
matrixa = {{3, 0, 2}, {-3, 2, 2}, {2, -3, 3}};
b = {3, -1, 4};
{x, y, z} = Inverse[matrixa].b
```

$\left\{\frac{13}{23}, -\frac{7}{23}, \frac{15}{23}\right\}$

We verify that the result is the desired solution by calculating matrixa.{x,y,z}. Because the result of this procedure is $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, we conclude that the solution to the system is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13/23 \\ -7/23 \\ 15/23 \end{pmatrix}$.

```
matrixa.{x, y, z}
```

{3, −1, 4}

We note that this matrix equation is equivalent to the system of equations

$$\begin{aligned} 3x + 2z &= 3 \\ -3x + 2y + 2z &= -1, \\ 2x - 3y + 3z &= 4 \end{aligned}$$

which we are able to solve with Solve. (Note that Thread[{f1, f2, ...} = {g1, g2, ...}] returns the system of equations {f1==g1, f2==g2, ...}.)

```
Clear[x, y, z]
sys = Thread[matrixa . {x, y, z}=={3, -1, 4}]
```

{3x + 2z==3, −3x + 2y + 2z==−1, 2x−3y + 3z==4}

```
Solve[sys]
```

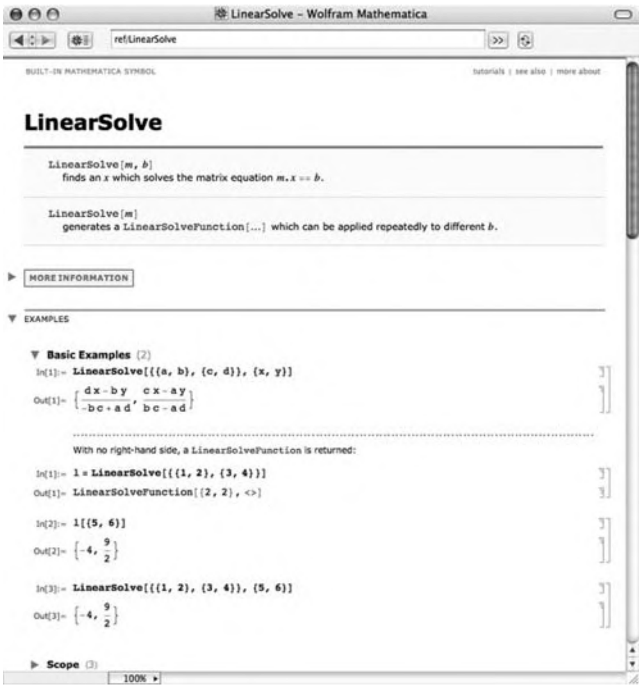
$\{\{x \to \frac{13}{23}, z \to \frac{15}{23}, y \to -\frac{7}{23}\}\}$

---

In addition to using Solve to solve a system of linear equations, the command

**LinearSolve[A,b]**

calculates the solution vector x of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$. LinearSolve generally solves a system more quickly than does Solve, as we see from the comments in the **Documentation Center**.




---

**Example 5.2.2** Solve the system $\begin{cases} x - 2y + z = -4 \\ 3x + 2y - z = 8 \\ -x + 3y + 5z = 0 \end{cases}$ for $x$, $y$, and $z$.

***Solution*** In this case, entering either

```
Solve[{x-2y + z==-4, 3x + 2y-z==8, -x + 3y + 5z==0}]
```

or

```
Solve[{x-2y + z, 3x + 2y-z, -x + 3y + 5z}=={-4, 8, 0}]
```

gives the same result.

```
Solve[{x-2y + z==-4, 3x + 2y-z==8, -x + 3y + 5z==0}, {x, y, z}]
{{x → 1, y → 2, z → -1}}
```

Another way to solve systems of equations is based on the matrix form of the system of equations, $\mathbf{Ax} = \mathbf{b}$. This system of equations is equivalent to the matrix equation

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & -1 \\ -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}.$$

The matrix of coefficients in the previous example is entered as **matrixa** along with the vector of right-hand side values **vectorb**. After defining the vector of variables, **vectorx**, the system $\mathbf{Ax} = \mathbf{b}$ is solved explicitly with the command **Solve**.

```
matrixa = {{1, -2, 1}, {3, 2, -1}, {-1, 3, 5}};
vectorb = {-4, 8, 0};
vectorx = {x1, y1, z1};
Solve[matrixa.vectorx==vectorb, vectorx]
{{x1 → 1, y1 → 2, z1 → -1}}

LinearSolve[matrixa, vectorb]
{1, 2, -1}
```

---

**Example 5.2.3** Solve the system $\begin{cases} 2x - 4y + z = -1 \\ 3x + y - 2z = 3 \\ -5x + y - 2z = 4 \end{cases}$. Verify that the result returned satisfies the system.

***Solution*** To solve the system using **Solve**, we define **eqs** to be the set of three equations to be solved and **vars** to be the variables $x$, $y$, and $z$ and then use **Solve** to solve the set of equations **eqs** for the variables in **vars**. The resulting output is named **sols**.

```
eqs = {2x-4y + z==-1, 3x + y-2z==3, -5x + y-2z==4}; vars = {x, y, z};
sols = Solve[eqs, vars]
```

$\{\{x \to -\frac{1}{8}, y \to -\frac{15}{56}, z \to -\frac{51}{28}\}\}$

To verify that the result given in **sols** is the desired solution, we replace each occurrence of $x$, $y$, and $z$ in **eqs** by the values found in **sols** using ReplaceAll (/.). Because the result indicates each of the three equations is satisfied, we conclude that the values given in **sols** are the components of the desired solution.

**eqs/.sols**

{{True, True, True}}

To solve the system using **LinearSolve**, we note that the system is equivalent to the matrix equation $\begin{pmatrix} 2 & -4 & 1 \\ 3 & 1 & -2 \\ -5 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$, define **matrixa** and **vectorb**, and use **LinearSolve** to solve this matrix equation.

**matrixa = {{2, −4, 1}, {3, 1, −2}, {−5, 1, −2}};**
**vectorb = {−1, 3, 4};**
**solvector = LinearSolve[matrixa, vectorb]**

$\{-\frac{1}{8}, -\frac{15}{56}, -\frac{51}{28}\}$

To verify that the results are correct, we compute **matrixa.solvector**. Because the result is $\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$, we conclude that the solution to the system is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/8 \\ -15/36 \\ -51/28 \end{pmatrix}$.

**matrixa.solvector**

{−1, 3, 4}

The command LinearSolve[A] returns a function that when given a vector $\mathbf{b}$ solves the equation $\mathbf{Ax} = \mathbf{b}$: LinearSolve[A][b] returns $\mathbf{x}$.

**LinearSolve[matrixa]**

LinearSolveFunction[{3, 3}, <>]

**LinearSolve[matrixa][{−1, 3, 4}]**

$\{-\frac{1}{8}, -\frac{15}{56}, -\frac{51}{28}\}$

---

Enter indexed variables such $x_1, x_2, \ldots, x_n$ as x[1], x[2], ..., x[n]. If you need to include the entire list, Table[x[i],{i,1,n}] usually produces the desired result(s).

**Example 5.2.4** Solve the system of equations

$$\begin{cases} 4x_1 + 5x_2 - 5x_3 - 8x_4 - 2x_5 = 5 \\ 7x_1 + 2x_2 - 10x_3 - x_4 - 6x_5 = -4 \\ 6x_1 + 2x_2 + 10x_3 - 10x_4 + 7x_5 = -7. \\ -8x_1 - x_2 - 4x_3 + 3x_5 = 5 \\ 8x_1 - 7x_2 - 3x_3 + 10x_4 + 5x_5 = 7 \end{cases}$$

***Solution*** We solve the system in two ways. First, we use Solve to solve the system. Note that in this case, we enter the equations in the form

**set of left−hand sides==set of right−hand sides**

```
Solve[{4x[1] + 5x[2] - 5x[3] - 8x[4] - 2x[5],
    7x[1] + 2x[2] - 10x[3] - x[4] - 6x[5],
        6x[1] + 2x[2] + 10x[3] - 10x[4] + 7x[5],
        -8x[1] - x[2] - 4x[3] + 3x[5],
        8x[1] - 7x[2] - 3x[3] + 10x[4] + 5x[5]}=={5, -4, -7, 5, 7}]
```

$\left\{\left\{x[1] \to \frac{1245}{6626}, x[2] \to \frac{113174}{9939}, x[3] \to -\frac{7457}{9939}, x[4] \to \frac{38523}{6626}, x[5] \to \frac{49327}{9939}\right\}\right\}$

We also use LinearSolve after defining matrixa and t2. As expected, in each case, the results are the same.

```
Clear[matrixa]
matrixa = {{4, 5, -5, -8, -2}, {7, 2, -10, -1, -6}, {6, 2, 10, -10, 7},
{-8, -1, -4, 0, 3}, {8, -7, -3, 10, 5}};
t2 = {5, -4, -7, 5, 7};
LinearSolve[matrixa, t2]
```

$\left\{\frac{1245}{6626}, \frac{113174}{9939}, -\frac{7457}{9939}, \frac{38523}{6626}, \frac{49327}{9939}\right\}$

## 5.2.2 Gauss–Jordan Elimination

Given the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

the $m \times n$ matrix $\mathbf{A}$ is called the **coefficient matrix** for the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, and the $m \times (n+1)$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

is called the **augmented** (or **associated**) **matrix** for the matrix equation. We may enter the augmented matrix associated with a linear system of equations directly or we can use commands such as Join to help us construct the augmented matrix. For example, if $\mathbf{A}$ and $\mathbf{B}$ are rectangular matrices that have the same number of columns, Join[A,B] returns $\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}$. On the other hand, if $\mathbf{A}$ and $\mathbf{B}$ are rectangular matrices that have the same number of rows, Join[A,B,2] returns the concatenated matrix $\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}$.

---

**Example 5.2.5** Solve the system $\begin{cases} -2x + y - 2x = 4 \\ 2x - 4y - 2z = -4 \\ x - 4y - 2z = 3 \end{cases}$ using Gauss–Jordan elimination.

***Solution*** The system is equivalent to the matrix equation

$$\begin{pmatrix} -2 & 1 & -2 \\ 2 & -4 & -2 \\ 1 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix}.$$

The augmented matrix associated with this system is

$$\begin{pmatrix} -2 & 1 & -2 & 4 \\ 2 & -4 & -2 & -4 \\ 1 & -4 & -2 & 3 \end{pmatrix},$$

which we construct using the command **Join**.

```
matrixa = {{-2, 1, -2}, {2, -4, -2}, {1, -4, -2}};
b = {{4}, {-4}, {3}};
augm = Join[matrixa, b, 2];
MatrixForm[augm]
```

$$\begin{pmatrix} -2 & 1 & -2 & 4 \\ 2 & -4 & -2 & -4 \\ 1 & -4 & -2 & 3 \end{pmatrix}$$

We calculate the solution by row-reducing **augm** using **RowReduce**. Generally, RowReduce[A] reduces $\mathbf{A}$ to **reduced row echelon form**.

```
RowReduce[augm]//MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

From this result, we see that the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ -4 \\ 3 \end{pmatrix}.$$

We verify this by replacing each occurrence of $x$, $y$, and $z$ on the left-hand side of the equations by $-7$, $-4$, and $3$, respectively, and noting that the components of the result are equal to the right-hand side of each equation.

```
Clear[x, y, z]
{-2x + y - 2z, 2x - 4y - 2z, x - 4y - 2z}/.{x → -7, y → -4, z → 3}
{4, -4, 3}
```

---

In the following example, we carry out the steps of the row reduction process.

---

**Example 5.2.6** Solve

$$\begin{aligned} -3x + 2y - 2z &= -10 \\ 3x - y + 2z &= 7 \\ 2x - y + z &= 6. \end{aligned}$$

***Solution*** The associated matrix is $\mathbf{A} = \begin{pmatrix} -3 & 2 & -2 & -10 \\ 3 & -1 & 2 & 7 \\ 2 & -1 & 1 & 6 \end{pmatrix}$, defined in capa and then displayed in traditional row-and-column form with MatrixForm.

```
Clear[capa]
capa = {{-3, 2, -2, -10}, {3, -1, 2, 7}, {2, -1, 1, 6}};
MatrixForm[capa]
```

$$\begin{pmatrix} -3 & 2 & -2 & -10 \\ 3 & -1 & 2 & 7 \\ 2 & -1 & 1 & 6 \end{pmatrix}$$

We eliminate methodically. First, we multiply row 1 by $-1/3$ so that the first entry in the first column is $1$.

```
capa = {-1/3capa[[1]], capa[[2]], capa[[3]]}
```

$$\left\{\left\{1, -\frac{2}{3}, \frac{2}{3}, \frac{10}{3}\right\}, \{3, -1, 2, 7\}, \{2, -1, 1, 6\}\right\}$$

We now eliminate below. First, we multiply row 1 by $-3$ and add it to row 2 and then we multiply row 1 by $-2$ and add it to row 3.

```
capa = {capa[[1]], -3capa[[1]] + capa[[2]],
        -2capa[[1]] + capa[[3]]}
```

$$\left\{\left\{1, -\frac{2}{3}, \frac{2}{3}, \frac{10}{3}\right\}, \{0, 1, 0, -3\}, \left\{0, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right\}\right\}$$

Observe that the first nonzero entry in the second row is $1$. We eliminate below this entry by adding $-1/3$ times row 2 to row 3.

```
capa = {capa[[1]], capa[[2]],
-1/3capa[[2]] + capa[[3]]}
```

```
{{1, -2/3, 2/3, 10/3}, {0, 1, 0, -3}, {0, 0, -1/3, 1/3}}
```

We multiply the third row by $-3$ so that the first nonzero entry is 1.

```
capa = {capa[[1]], capa[[2]], -3capa[[3]]}
```

```
{{1, -2/3, 2/3, 10/3}, {0, 1, 0, -3}, {0, 0, 1, -1}}
```

This matrix is equivalent to the system

$$\begin{aligned} x - \frac{2}{3}y + \frac{2}{3}z &= \frac{10}{3} \\ y &= -3 \\ z &= -1, \end{aligned}$$

which shows us that the solution is $x = 2$, $y = -3$, $z = -1$.

Working backwards confirms this. Multiplying row 2 by $2/3$ and adding to row 1 and then multiplying row 3 by $-2/3$ and adding to row 1 results in

```
capa = {2/3capa[[2]] + capa[[1]], capa[[2]],
        capa[[3]]}
capa = {-2/3capa[[3]] + capa[[1]], capa[[2]],
        capa[[3]]}
MatrixForm[capa]
```

```
{{1, 0, 2/3, 4/3}, {0, 1, 0, -3}, {0, 0, 1, -1}}
          {{1, 0, 0, 2}, {0, 1, 0, -3}, {0, 0, 1, -1}}
```

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

which is equivalent to the system $x = 2$, $y = -3$, $z = -1$.

Equivalent results are obtained with RowReduce.

```
capa = {{-3, 2, -2, -10}, {3, -1, 2, 7}, {2, -1, 1, 6}};
capa = RowReduce[capa]
MatrixForm[capa]
```

```
{{1, 0, 0, 2}, {0, 1, 0, -3}, {0, 0, 1, -1}}
```

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Finally, we confirm the result directly with Solve.

```
Solve[{-3x + 2y - 2z==-10,
       3x - y + 2z==7, 2x - y + z==6}]
```

```
{{x → 2, y → -3, z → -1}}
```

**Example 5.2.7** Solve

$$-3x_1 + 2x_2 + 5x_3 = -12$$
$$3x_1 - x_2 - 4x_3 = 9$$
$$2x_1 - x_2 - 3x_3 = 7.$$

***Solution*** The associated matrix is $\mathbf{A} = \begin{pmatrix} -3 & 2 & 5 & -12 \\ 3 & -1 & -4 & 9 \\ 2 & -1 & -3 & 7 \end{pmatrix}$, which is reduced to row echelon form with RowReduce.

```
capa = {{-3, 2, 5, -12}, {3, -1, -4, 9}, {2, -1, -3, 7}};
rrcapa = RowReduce[capa];
MatrixForm[rrcapa]
```

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The result shows that the original system is equivalent to

$$\begin{aligned} x_1 - x_3 &= 2 \\ x_2 + x_3 &= -3 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 &= 2 + x_3 \\ x_2 &= -3 - x_3 \end{aligned}$$

so $x_3$ is *free*. That is, for any real number $t$, a solution to the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2+t \\ -3-t \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

The system has infinitely many solutions.

Equivalent results are obtained with Solve.

```
In[15]:= Solve[{-3 x1 + 2 x2 + 5 x3 == -12, 3 x1 - x2 - 4 x3 == 9,
            2 x1 - x2 - 3 x3 == 7}]

         Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[15]= {{x1 -> 2 + x3, x2 -> -3 - x3}}
```

---

**Example 5.2.8** Solve

$$-3x_1 + 2x_2 + 5x_3 = -14$$
$$3x_1 - x_2 - 4x_3 = 11$$
$$2x_1 - x_2 - 3x_3 = 8.$$

***Solution*** The associated matrix is $\mathbf{A} = \begin{pmatrix} -3 & 2 & 5 & -14 \\ 3 & -1 & -4 & 11 \\ 2 & -1 & -3 & 8 \end{pmatrix}$, which is reduced to row echelon form with RowReduce.

```
Clear[x]
capa = {{-3, 2, 5, -14}, {3, -1, -4, 11}, {2, -1, -3, 8}};
rrcapa = RowReduce[capa];
MatrixForm[rrcapa]
```

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The result shows that the original system is equivalent to

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 1. \end{aligned}$$

Of course, $0$ is not equal to $1$: The last equation is false. The system has no solutions.

We check the calculation with Solve. In this case, the results indicate that Solve cannot find any solutions to the system.

```
Solve[{-3x[1] + 2x[2] + 5x[3]==-14,
3x[1]-x[2]-4x[3]==11, 2x[1]-x[2]-3x[3]==8}]
{}
```

Generally, if Mathematica returns nothing, the result means either that there is no solution or that Mathematica cannot solve the problem. In such a situation, we must always check using another method.

---

**Example 5.2.9** The **nullspace** of $\mathbf{A}$ is the set of solutions to the system of equations $\mathbf{Ax} = \mathbf{0}$. Find the nullspace of $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 & 1 & -2 \\ 3 & 3 & 1 & 2 & -1 \\ 2 & 2 & 1 & 1 & -1 \\ -1 & -1 & 0 & -1 & 0 \\ 5 & 4 & 2 & 2 & -3 \end{pmatrix}$.

***Solution*** Observe that row reducing $(\mathbf{A}|\mathbf{0})$ is equivalent to row reducing $\mathbf{A}$. After defining $\mathbf{A}$, we use RowReduce to row reduce $\mathbf{A}$.

```
capa = {{3, 2, 1, 1, -2}, {3, 3, 1, 2, -1},
{2, 2, 1, 1, -1}, {-1, -1, 0, -1, 0},
```

**{5, 4, 2, 2, −3}};**
**RowReduce[capa]//MatrixForm**

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The result indicates that the solutions of $\mathbf{Ax} = \mathbf{0}$ are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} t \\ -s-t \\ s+t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

where $s$ and $t$ are any real numbers. The dimension of the nullspace, the **nullity**, is 2; a basis for the nullspace is

$$\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

You can use the command NullSpace[A] to find a basis of the nullspace of a matrix $\mathbf{A}$ directly.

**NullSpace[capa]**
{{1, −1, 1, 0, 1}, {0, −1, 1, 1, 0}}

$\mathbf{A}$ is **singular** because $|\mathbf{A}| = 0$.

**Det[capa]**
0

Do *not* use LinearSolve on singular matrices,
because the results returned may not be (completely) correct.

```
Don't use LinearSolve on non-singular matrices:

LinearSolve[capa]

LinearSolve::sing1 :
  The matrix {{3, 2, 1, 1, -2}, {3, 3, 1, 2, -1}, {2, 2, 1, 1, -1}, {-1, -1, 0, -1, 0}, {5, 4, 2, 2, -3}}
      is singular so a factorization will not be saved.

LinearSolveFunction[{5, 5}, <>]

Det[capa]

0

LinearSolve[capa, {0, 0, 0, 0, 0}]

{0, 0, 0, 0, 0}
```

```
LinearSolve[capa, {0, 0, 0, 0, 0}]
{0, 0, 0, 0, 0}
```

## 5.3 SELECTED TOPICS FROM LINEAR ALGEBRA

### 5.3.1 Fundamental Subspaces Associated with Matrices

Let $\mathbf{A} = (a_{ij})$ be an $n \times m$ matrix with entry $a_{ij}$ in the $i$th row and $j$th column. The **row space** of $\mathbf{A}$, row($\mathbf{A}$), is the spanning set of the rows of $\mathbf{A}$; the **column space** of $\mathbf{A}$, col($\mathbf{A}$), is the spanning set of the columns of $\mathbf{A}$. If $\mathbf{A}$ is any matrix, then the dimension of the column space of $\mathbf{A}$ is equal to the dimension of the row space of $\mathbf{A}$. The dimension of the row space (column space) of a matrix $\mathbf{A}$ is called the **rank** of $\mathbf{A}$. The **nullspace** of $\mathbf{A}$ is the set of solutions to the system of equations $\mathbf{Ax} = \mathbf{0}$. The nullspace of $\mathbf{A}$ is a subspace and its dimension is called the **nullity** of $\mathbf{A}$. The rank of $\mathbf{A}$ is equal to the number of nonzero rows in the row echelon form of $\mathbf{A}$, and the nullity of $\mathbf{A}$ is equal to the number of zero rows in the row echelon form of $\mathbf{A}$. Thus, if $\mathbf{A}$ is a square matrix, the sum of the rank of $\mathbf{A}$ and the nullity of $\mathbf{A}$ is equal to the number of rows (columns) of $\mathbf{A}$.

1. NullSpace[A] returns a list of vectors that form a basis for the nullspace (or kernel) of the matrix $\mathbf{A}$.
2. RowReduce[A] yields the reduced row echelon form of the matrix $\mathbf{A}$.

**Example 5.3.1** Place the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 2 & 0 & -1 \\ -2 & 2 & 0 & 0 & -2 \\ 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 2 & 2 \\ 1 & -2 & 2 & -2 & 0 \end{pmatrix}$$

in reduced row echelon form. What is the rank of $\mathbf{A}$? Find a basis for the nullspace of $\mathbf{A}$.

***Solution*** We begin by defining the matrix matrixa. Then, RowReduce is used to place matrixa in reduced row echelon form.

```
capa = {{-1, -1, 2, 0, -1}, {-2, 2, 0, 0, -2},
{2, -1, -1, 0, 1}, {-1, -1, 1, 2, 2},
{1, -2, 2, -2, 0}};
RowReduce[capa]//MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Because the row-reduced form of matrixa contains four nonzero rows, the rank of $\mathbf{A}$ is $4$ and thus the nullity is $1$. We obtain a basis for the nullspace with NullSpace.

```
NullSpace[capa]
{{2, 2, 2, 1, 0}}
```

As expected, because the nullity is $1$, a basis for the nullspace contains one vector.

---

**Example 5.3.2** Find a basis for the column space of

$$\mathbf{B} = \begin{pmatrix} 1 & -2 & 2 & 1 & -2 \\ 1 & 1 & 2 & -2 & -2 \\ 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -2 & 0 \\ -2 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

***Solution*** A basis for the column space of $\mathbf{B}$ is the same as a basis for the row space of the transpose of $\mathbf{B}$. We begin by defining matrixb and then using Transpose to compute the transpose of matrixb, naming the resulting output tb.

```
matrixb = {{1, -2, 2, 1, -2}, {1, 1, 2, -2, -2},
   {1, 0, 0, 2, -1}, {0, 0, 0, -2, 0},
     {-2, 1, 0, 1, 2}};
tb = Transpose[matrixb]
{{1, 1, 1, 0, -2}, {-2, 1, 0, 0, 1},
   {2, 2, 0, 0, 0}, {1, -2, 2, -2, 1}, {-2, -2, -1, 0, 2}}
```

Next, we use RowReduce to row reduce tb and name the result rrtb. A basis for the column space consists of the first four elements of rrtb. We also use Transpose to show that the first four elements of rrtb are the same as the first four columns of the transpose of rrtb. Thus, the $j$th column of a matrix $\mathbf{A}$ can be extracted from $\mathbf{A}$ with Transpose [A][[j]].

```
rrtb = RowReduce[tb];
Transpose[rrtb]//MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & -2 & -3 & 0 \end{pmatrix}$$

We extract the first four elements of rrtb with Take. The results correspond to a basis for the column space of **B**.

```
Take[rrtb, 4]
```

$\{\{1, 0, 0, 0, -\frac{1}{3}\}, \{0, 1, 0, 0, \frac{1}{3}\}, \{0, 0, 1, 0, -2\}, \{0, 0, 0, 1, -3\}\}$

---

## 5.3.2 The Gram–Schmidt Process

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **orthonormal** means that $\|\mathbf{v}_i\| = 1$ for all values of $i$ and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$. Given a set of linearly independent vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, the set of all linear combinations of the elements of $S$, $V = \operatorname{span} S$, is a vector space. Note that if $S$ is an orthonormal set and $\mathbf{u} \in \operatorname{span} S$, then $\mathbf{u} = (\mathbf{u} \cdot \mathbf{v}_1)\,\mathbf{v}_1 + (\mathbf{u} \cdot \mathbf{v}_2)\,\mathbf{v}_2 + \cdots + (\mathbf{u} \cdot \mathbf{v}_n)\,\mathbf{v}_n$. Thus, we may easily express $\mathbf{u}$ as a linear combination of the vectors in $S$. Consequently, if we are given any vector space, $V$, it is frequently convenient to be able to find an orthonormal basis of $V$. We may use the **Gram–Schmidt process** to find an orthonormal basis of the vector space $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

We summarize the algorithm of the Gram–Schmidt process so that given a set of $n$ linearly independent vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, where $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, we can construct a set of orthonormal vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ so that $V = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$.

**1.** Let $\mathbf{u}_1 = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$;

**2.** Compute $\operatorname{proj}_{\{\mathbf{u}_1\}}\mathbf{v}_2 = (\mathbf{u}_1 \cdot \mathbf{v}_2)\,\mathbf{u}_1$, $\mathbf{v}_2 - \operatorname{proj}_{\{\mathbf{u}_1\}}\mathbf{v}_2$, and let

$$\mathbf{u}_2 = \frac{1}{\left\| \mathbf{v}_2 - \operatorname{proj}_{\{\mathbf{u}_1\}}\mathbf{v}_2 \right\|}\left(\mathbf{v}_2 - \operatorname{proj}_{\{\mathbf{u}_1\}}\mathbf{v}_2\right).$$

Then, $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$;

**3.** Generally, for $3 \leq i \leq n$, compute

$$\operatorname{proj}_{\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}}\mathbf{v}_i = (\mathbf{u}_1 \cdot \mathbf{v}_i)\,\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{v}_i)\,\mathbf{u}_2 + \ldots + (\mathbf{u}_{i-1} \cdot \mathbf{v}_i)\,\mathbf{u}_{i-1},$$

$\mathbf{v}_i - \text{proj}_{\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}}\mathbf{v}_i$, and let

$$\mathbf{u}_1 = \frac{1}{\left\| \text{proj}_{\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}}\mathbf{v}_i \right\|}\left(\text{proj}_{\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}}\mathbf{v}_i\right).$$

Then, span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_i\}$ = span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_i\}$ and

$$\text{span}\,\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\} = \text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\};$$

and

4. Because span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ = span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal set, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis of $V$.

The Gram–Schmidt procedure is well-suited to computer arithmetic. The following code performs each step of the Gram–Schmidt process on a set of $n$ linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$. At the completion of each step of the procedure, gramschmidt[vecs] prints the list of vectors corresponding to $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$ and returns the list of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. Note how comments are inserted into the code using (*...*).

```
gramschmidt[vecs_] := Module[{n, proj, u, capw},
(*n represents the number of vectors in
the listvecs*)
n = Length[vecs];
(*proj[v, capw] computes the projection
of v onto capw*)
proj[v_, capw_] :=
    Sum_{i=1}^{Length[capw]} capw[[i]].v capw[[i]];
u[1] = vecs[[1]] / Sqrt[vecs[[1]].vecs[[1]]];
capw = {};
u[i_] := u[i] = Module[{stepone},

stepone = vecs[[i]] - proj[vecs[[i]], capw];
Together[stepone / Sqrt[stepone.stepone]]];
Do[
    u[i];
    AppendTo[capw, u[i]];
    Print[Join[capw, Drop[vecs, i]]], {i, 1, n - 1}];
    u[n];
    AppendTo[capw, u[n]]]
```

**Example 5.3.3** Use the Gram–Schmidt process to transform the basis $S = \left\{ \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \right\}$ of $\mathbf{R}^3$ into an orthonormal basis.

***Solution*** We proceed by defining v1, v2, and v3 to be the vectors in the basis $S$ and using gramschmidt[{v1,v2,v3}] to find an orthonormal basis.

```
v1 = {-2, -1, -2};
v2 = {0, -1, 2};
v3 = {1, 3, -2};
gramschmidt[{v1, v2, v3}]
```

$\{\{-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\}, \{0, -1, 2\}, \{1, 3, -2\}\}$

$\{\{-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\}, \{-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\}, \{1, 3, -2\}\}$

$\{\{-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\}, \{-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\}, \{-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\}\}$

On the first line of output, the result $\{\mathbf{u}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is given; $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_3\}$ appears on the second line; $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ follows on the third.

**Example 5.3.4** Compute an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{pmatrix} 2 \\ 4 \\ 4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 1 \\ -3 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 4 \\ 4 \\ -1 \end{pmatrix}$. Also, verify that the basis vectors are orthogonal and have norm $1$.

***Solution*** With gramschmidt, we compute the orthonormal basis vectors. Note that Mathematica names oset the last result returned by gramschmidt. The orthogonality of these vectors is then verified. Notice that Together is used to simplify the result in the case of oset[[2]].oset[[3]]. The norm of each vector is then found to be $1$.

```
oset = gramschmidt[{{2, 4, 4, 1}, {-4, 1, -3, 2}, {1, 4, 4, -1}}]
```

$\left\{\left\{\frac{2}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{1}{\sqrt{37}}\right\}, \{-4, 1, -3, 2\}, \{1, 4, 4, -1\}\right\}$

$\left\{\left\{\frac{2}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{1}{\sqrt{37}}\right\},\right.$
$\left.\left\{-60\sqrt{\frac{2}{16909}}, \frac{93}{\sqrt{33818}}, -\frac{55}{\sqrt{33818}}, 44\sqrt{\frac{2}{16909}}\right\}, \{1, 4, 4, -1\}\right\}$

$\left\{\left\{\frac{2}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{1}{\sqrt{37}}\right\},\right.$
$\left\{-60\sqrt{\frac{2}{16909}}, \frac{93}{\sqrt{33818}}, -\frac{55}{\sqrt{33818}}, 44\sqrt{\frac{2}{16909}}\right\},$
$\left.\left\{-\frac{449}{\sqrt{934565}}, \frac{268}{\sqrt{934565}}, \frac{156}{\sqrt{934565}}, -\frac{798}{\sqrt{934565}}\right\}\right\}$

The three vectors are extracted with oset using oset[[1]], oset[[2]], and oset[[3]].

```
oset[[1]].oset[[2]]
oset[[1]].oset[[3]]
oset[[2]].oset[[3]]
0
0
0
```

```
Sqrt[oset[[1]].oset[[1]]]
Sqrt[oset[[2]].oset[[2]]]
Sqrt[oset[[3]].oset[[3]]]
1
1
1
```

---

Mathematica contains functions that perform most of the operations discussed here.

1. Orthogonalize[{v1, v2, ...},Method->GramSchmidt] returns an orthonormal set of vectors given the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Note that this command does not illustrate each step of the Gram–Schmidt procedure as the gramschmidt function defined previously.
2. Normalize[v] returns $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ given the nonzero vector $\mathbf{v}$.
3. Projection[v1,v2] returns the projection of $\mathbf{v}_1$ onto $\mathbf{v}_2$: $\text{proj}_{\mathbf{v}_2}\mathbf{v}_1 = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$.

Thus,

```
Orthogonalize[{{2, 4, 4, 1}, {-4, 1, -3, 2}, {1, 4, 4, -1}},
 Method → "GramSchmidt"]
```

$$\left\{\left\{\frac{2}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{1}{\sqrt{37}}\right\}, \left\{-60\sqrt{\frac{2}{16909}}, \frac{93}{\sqrt{33818}}, -\frac{55}{\sqrt{33818}}, 44\sqrt{\frac{2}{16909}}\right\}, \left\{-\frac{449}{\sqrt{934565}}, \frac{268}{\sqrt{934565}}, \frac{156}{\sqrt{934565}}, -\frac{798}{\sqrt{934565}}\right\}\right\}$$

returns an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{pmatrix}2\\4\\4\\1\end{pmatrix}$, $\begin{pmatrix}-4\\1\\-3\\2\end{pmatrix}$, and $\begin{pmatrix}1\\4\\4\\-1\end{pmatrix}$. The command

```
Normalize[{2, 4, 4, 1}]
```

$\left\{\frac{2}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{4}{\sqrt{37}}, \frac{1}{\sqrt{37}}\right\}$

finds a unit vector with the same direction as the vector $\mathbf{v} = \begin{pmatrix}2\\4\\4\\1\end{pmatrix}$. Entering

```
Projection[{2, 4, 4, 1}, {-4, 1, -3, 2}]
```

$\left\{\frac{28}{15}, -\frac{7}{15}, \frac{7}{5}, -\frac{14}{15}\right\}$

finds the projection of $\mathbf{v} = \begin{pmatrix}2\\4\\4\\1\end{pmatrix}$ onto $\mathbf{w} = \begin{pmatrix}-4\\1\\-3\\2\end{pmatrix}$.

### 5.3.3 Linear Transformations

A function $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ is a **linear transformation** means that $T$ satisfies the properties $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{u})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ and all real numbers $c$. Let $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ be a linear transformation and suppose $T(\mathbf{e}_1) = \mathbf{v}_1$, $T(\mathbf{e}_2) = \mathbf{v}_2, \ldots, T(\mathbf{e}_n) = \mathbf{v}_n$, where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ represents the standard basis of $\mathbf{R}^n$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are (column) vectors in $\mathbf{R}^m$. The **associated matrix** of $T$ is the $m \times n$ matrix $\mathbf{A} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}$:

$$\text{If } \mathbf{x} = \begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}, \quad T(\mathbf{x}) = T\left(\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\right) = \mathbf{A}\mathbf{x} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}.$$

Moreover, if $\mathbf{A}$ is any $m \times n$ matrix, then $\mathbf{A}$ is the associated matrix of the linear transformation defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$. In fact, a linear transformation $T$ is completely determined by its action on any basis.

The **kernel** of the linear transformation $T$, $\ker(T)$, is the set of all vectors $\mathbf{x}$ in $\mathbf{R}^n$ such that $T(\mathbf{x}) = \mathbf{0}$: $\ker(T) = \{x \in \mathbf{R}^n | T(\mathbf{x}) = \mathbf{0}\}$. The kernel of $T$ is a subspace of $\mathbf{R}^n$. Because $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$, $\ker(T) = \{x \in \mathbf{R}^n | T(\mathbf{x}) = \mathbf{0}\} = \{x \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{0}\}$ so the kernel of $T$ is the same as the nullspace of $\mathbf{A}$.

**Example 5.3.5** Let $T : \mathbf{R}^5 \longrightarrow \mathbf{R}^3$ be the linear transformation defined by $T(\mathbf{x}) = \begin{pmatrix} 0 & -3 & -1 & -3 & -1 \\ -3 & 3 & -3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 2 \end{pmatrix}\mathbf{x}$. (a) Calculate a basis for the kernel of the linear transformation. (b) Determine which of the vectors $\begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 3 \end{pmatrix}$ is in the kernel of $T$.

***Solution*** We begin by defining **matrixa** to be the matrix $\mathbf{A} = \begin{pmatrix} 0 & -3 & -1 & -3 & -1 \\ -3 & 3 & -3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 2 \end{pmatrix}$ and then defining **t**. A basis for the kernel of $T$ is the same as a basis for the nullspace of $\mathbf{A}$ found with **NullSpace**.

```
Clear[t, x, matrixa]
matrixa = {{0, -3, -1, -3, -1}, {-3, 3, -3, -3, -1},
        {2, 2, -1, 1, 2}};
t[x_] = matrixa.x;
NullSpace[matrixa]
{{-2, -1, 0, 0, 3}, {-6, -8, -15, 13, 0}}
```

Because $\begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \\ -6 \end{pmatrix}$ is a linear combination of the vectors that form a basis for the kernel, $\begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \\ -6 \end{pmatrix}$ is in the kernel, whereas $\begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 3 \end{pmatrix}$ is not. These results are verified by evaluating **t** for each vector.

```
t[{4, 2, 0, 0, -6}]
{0, 0, 0}
t[{1, 2, -1, -2, 3}]
{-2, 9, 11}
```

## *Application: Rotations*

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector in $\mathbf{R}^2$ and $\theta$ an angle. Then, there are numbers $r$ and $\phi$ given by $r = \sqrt{{x_1}^2 + {x_2}^2}$ and $\phi = \tan^{-1}(x_2/x_1)$ so that $x_1 = r\cos\phi$

and $x_2 = r\sin\phi$. When we rotate $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$ through the angle $\theta$, we obtain the vector $\mathbf{x}' = \begin{pmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{pmatrix}$. Using the trigonometric identities $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$ and $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$, we rewrite

$$\mathbf{x}' = \begin{pmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{pmatrix} = \begin{pmatrix} r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ r\sin\theta\cos\phi + r\sin\phi\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus, the vector $\mathbf{x}'$ is obtained from $\mathbf{x}$ by computing $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{x}$. Generally, if $\theta$ represents an angle, the linear transformation $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ defined by $T(\mathbf{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{x}$ is called the **rotation of $\mathbf{R}^2$ through the angle** $\theta$. We write code to rotate a polygon through an angle $\theta$. The procedure rotate uses a list of $n$ points and the rotation matrix defined in r to produce a new list of points that are joined using the Line graphics directive. Entering

```
Line[{{x1, y1}, {x2, y2}, . . ., {xn, yn}}]
```

represents the graphics primitive for a line in two dimensions that connects the points listed in {{x1, y1}, {x2, y2}, ..., {xn, yn}}. Entering

```
Show[Graphics[Line[{{x1, y1}, {x2, y2}, . . ., {xn, yn}}]]]
```

displays the line. This rotation can be determined for one value of $\theta$. However, a more interesting result is obtained by creating a list of rotations for a sequence of angles and then displaying the graphics objects. This is done for $\theta = 0$ to $\theta = \pi/2$ using increments of $\pi/16$. Hence, a list of nine graphs is given for the square with vertices $(-1, 1)$, $(1, 1)$, $(1, -1)$, and $(-1, -1)$ and displayed in Figure 5.6.

```
r[θ_] = ( Cos[θ]  -Sin[θ]
          Sin[θ]   Cos[θ] );

rotate[pts_, angle_] := Module[{newpts},
newpts = Table[r[angle].pts[[i]], {i, 1, Length[pts]}];
newpts = AppendTo[newpts, newpts[[1]]];
figure = Line[newpts];
Show[Graphics[figure], AspectRatio → 1,
    PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}},
        DisplayFunction → Identity]]
```

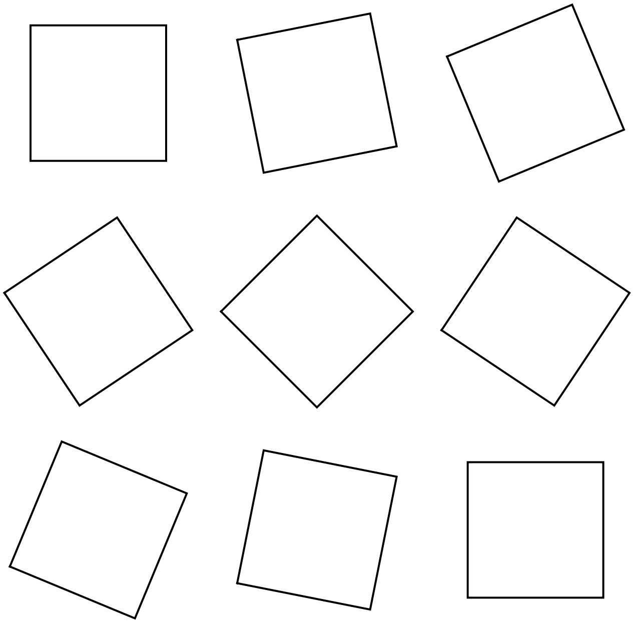


**FIGURE 5.6**

A rotated square

```
graphs = Table[rotate[{{-1, 1}, {1, 1}, {1, -1}, {-1, -1}}, t],
    {t, 0, π/2, π/16}]; array = Partition[graphs, 3];
Show[GraphicsGrid[array]]
```

## 5.3.4 Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be an $n \times n$ matrix. $\lambda$ is an **eigenvalue** of $\mathbf{A}$ if there is a *nonzero* vector, $\mathbf{v}$, called an **eigenvector**, satisfying $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. We find the eigenvalues of $\mathbf{A}$ by solving the **characteristic polynomial** $|\mathbf{A} - \lambda\mathbf{I}| = 0$ for $\lambda$. Once we find the eigenvalues, the corresponding eigenvectors are found by solving $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$.

If $\mathbf{A}$ is $n \times n$, Eigenvalues[A] finds the eigenvalues of $\mathbf{A}$, Eigenvectors[A] finds the eigenvectors, and Eigensystem[A] finds the eigenvalues and corresponding eigenvectors. CharacteristicPolynomial[A,lambda] finds the characteristic polynomial of $\mathbf{A}$ as a function of $\lambda$.

**Example 5.3.6** Find the eigenvalues and corresponding eigenvectors for each of the following matrices: (a) $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$, (c) $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, and (d) $\mathbf{A} = \begin{pmatrix} -1/4 & 2 \\ -8 & -1/4 \end{pmatrix}$.

***Solution*** (a) We begin by finding the eigenvalues. Solving

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 5 = 0$$

gives us $\lambda_1 = -5$ and $\lambda_2 = -1$.

Observe that the same results are obtained using CharacteristicPolynomial and Eigenvalues.

```
capa = {{-3, 2}, {2, -3}};
CharacteristicPolynomial[capa, λ]
5 + 6λ + λ^2

e1 = Eigenvalues[capa]
{-5, -1}
```

We now find the corresponding eigenvectors. Let $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ be an eigenvector corresponding to $\lambda_1$; then

$$(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$$

$$\left[\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} - (-5)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which row reduces to

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That is, $x_1 + y_1 = 0$ or $x_1 = -y_1$. Hence, for any value of $y_1 \neq 0$,

$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_1 \end{pmatrix} = y_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_1$. Of course, this represents infinitely many vectors. However, they are all linearly dependent. Choosing $y_1 = 1$ yields $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Note that you might have chosen $y_1 = -1$ and obtained $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. However, both of our results are "correct" because these vectors are linearly dependent.

Similarly, letting $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be an eigenvector corresponding to $\lambda_2$, we solve $(\mathbf{A} - \lambda_2 \mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $x_2 - y_2 = 0$ or $x_2 = y_2$. Hence, for any value of $y_2 \neq 0$,

$$\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_2 \end{pmatrix} = y_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_2$. Choosing $y_2 = 1$ yields $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We confirm these results using RowReduce.

```
i2 = IdentityMatrix[2];
ev1 = capa - e1[[1]] i2
{{2, 2}, {2, 2}}

RowReduce[ev1]
{{1, 1}, {0, 0}}

ev2 = capa - e1[[2]] i2;
RowReduce[ev2]
{{1, -1}, {0, 0}}
```

We obtain the same results using Eigenvectors and Eigensystem.

```
Eigenvectors[capa]
Eigensystem[capa]
{{-1, 1}, {1, 1}}
{{-5, -1}, {{-1, 1}, {1, 1}}}
```

(b) In this case, we see that $\lambda = 2$ has multiplicity $2$. There is only one linearly independent eigenvector, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, corresponding to $\lambda$.

```
capa = {{1, -1}, {1, 3}};
Factor[CharacteristicPolynomial[capa, λ]]
Eigenvectors[capa]
Eigensystem[capa]
(-2 + λ)^2
{{-1, 1}, {0, 0}}
{{2, 2}, {{-1, 1}, {0, 0}}}
```

(c) The eigenvalue $\lambda_1 = 2$ has corresponding eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The eigenvalue $\lambda_{2,3} = -1$ has multiplicity $2$. In this case, there are two linearly independent eigenvectors corresponding to this eigenvalue: $\mathbf{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{v_3} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

```
capa = {{0, 1, 1}, {1, 0, 1}, {1, 1, 0}};
Factor[CharacteristicPolynomial[capa, λ]]
```

```
Eigenvectors[capa]
Eigensystem[capa]
```

$-(-2+\lambda)(1+\lambda)^2$

{{1, 1, 1}, {−1, 0, 1}, {−1, 1, 0}}

{{2, −1, −1}, {{1, 1, 1}, {−1, 0, 1}, {−1, 1, 0}}}

(d) In this case, the eigenvalues $\lambda_{1,2} = -\frac{1}{4} \pm 4i$ are complex conjugates. We see that the eigenvectors $\mathbf{v}_{1,2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$ are complex conjugates as well.

```
capa = {{-1/4, 2}, {-8, -1/4}};
Factor[CharacteristicPolynomial[capa, λ],
   GaussianIntegers → True]
Eigenvectors[capa]
Eigensystem[capa]
```

$\frac{1}{16}((1-16i)+4\lambda)((1+16i)+4\lambda)$

$\{\{-\frac{i}{2}, 1\}, \{\frac{i}{2}, 1\}\}$

$\{\{-\frac{1}{4}+4i, -\frac{1}{4}-4i\}, \{\{-\frac{i}{2}, 1\}, \{\frac{i}{2}, 1\}\}\}$

---

## 5.3.5 Jordan Canonical Form

Let $\mathbf{N}_k = (n_{ij}) = \begin{cases} 1, & j = i+1 \\ 0, & \text{otherwise} \end{cases}$ represent a $k \times k$ matrix with the indicated elements. The $k \times k$ **Jordan block matrix** is given by $\mathbf{B}(\lambda) = \lambda\mathbf{I} + \mathbf{N}_k$, where $\lambda$ is a constant:

$$\mathbf{N}_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B}(\lambda) = \lambda\mathbf{I} + \mathbf{N}_k = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Hence, $\mathbf{B}(\lambda)$ can be defined as $\mathbf{B}(\lambda) = (b_{ij}) = \begin{cases} \lambda, & i = j \\ 1, & j = i+1 \\ 0, & \text{otherwise} \end{cases}$. A **Jordan matrix** has the form

$$\mathbf{J} = \begin{pmatrix} \mathbf{B}_1(\lambda) & 0 & \cdots & 0 \\ 0 & \mathbf{B}_2(\lambda) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathbf{B}_n(\lambda) \end{pmatrix},$$

where the entries $\mathbf{B}_j(\lambda)$, $j = 1, 2, \ldots, n$ represent Jordan block matrices.

Suppose that $\mathbf{A}$ is an $n \times n$ matrix. Then there is an invertible $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{C}^{-1}\mathbf{AC} = \mathbf{J}$, where $\mathbf{J}$ is a Jordan matrix with the eigenvalues of $\mathbf{A}$ as diagonal elements. The matrix $\mathbf{J}$ is called the **Jordan canonical form** of $\mathbf{J}$. The command

**JordanDecomposition[m]**

yields a list of matrices {s,j} such that m = s.j.Inverse[s] and j is the Jordan canonical form of the matrix m.

For a given matrix $\mathbf{A}$, the unique monic polynomial $q$ of least degree satisfying $q(\mathbf{A}) = 0$ is called the **minimal polynomial of** $\mathbf{A}$. Let $p$ denote the characteristic polynomial of $\mathbf{A}$. Because $p(\mathbf{A}) = 0$, it follows that $q$ divides $p$. We can use the Jordan canonical form of a matrix to determine its minimal polynomial.

---

**Example 5.3.7** Find the Jordan canonical form, $\mathbf{J}_\mathbf{A}$, of $\mathbf{A} = \begin{pmatrix} 2 & 9 & -9 \\ 0 & 8 & -6 \\ 0 & 9 & -7 \end{pmatrix}$.

***Solution*** After defining matrixa, we use JordanDecomposition to find the Jordan canonical form of a and name the resulting output ja.

```
matrixa = {{2, 9, -9}, {0, 8, -6}, {0, 9, -7}};
ja = JordanDecomposition[matrixa]
{{{3, 0, 1}, {2, 1, 0}, {3, 1, 0}},
   {{-1, 0, 0}, {0, 2, 0}, {0, 0, 2}}}
```

The Jordan matrix corresponds to the second element of ja extracted with ja[[2]] and displayed in MatrixForm.

```
ja[[2]]//MatrixForm
```

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We also verify that the matrices ja[[1]] and ja[[2]] satisfy

**matrixa = ja[[1]].ja[[2]].Inverse[ja[[1]]].**

```
ja[[1]].ja[[2]].Inverse[ja[[1]]]
{{2, 9, -9}, {0, 8, -6}, {0, 9, -7}}
```

Next, we use CharacteristicPolynomial to find the characteristic polynomial of matrixa and then verify that matrixa satisfies its characteristic polynomial.

```
p = CharacteristicPolynomial[matrixa, x]
```

$-4 + 3x^2 - x^3$

```
-4IdentityMatrix[3] + 3MatrixPower[matrixa, 2] -
    MatrixPower[matrixa, 3]
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

From the Jordan form, we see that the minimal polynomial of $\mathbf{A}$ is $(x+1)(x-2)$. We define the minimal polynomial to be q and then verify that matrixa satisfies its minimal polynomial.

```
q = Expand[(x + 1)(x - 2)]
```
$-2 - x + x^2$

```
-2 IdentityMatrix[3] - matrixa + MatrixPower[matrixa, 2]
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

As expected, q divides p.

```
Cancel[p/q]
```
$2 - x$

---

**Example 5.3.8** If $\mathbf{A} = \begin{pmatrix} 3 & 8 & 6 & -1 \\ -3 & 2 & 0 & 3 \\ 3 & -3 & -1 & -3 \\ 4 & 8 & 6 & -2 \end{pmatrix}$, find the characteristic and minimal polynomials of $\mathbf{A}$.

***Solution*** As in the previous example, we first define matrixa and then use JordanDecomposition to find the Jordan canonical form of $\mathbf{A}$.

```
matrixa = {{3, 8, 6, -1}, {-3, 2, 0, 3}, {3, -3, -1, -3},
    {4, 8, 6, -2}}; ja = JordanDecomposition[matrixa]
```
$\{\{\{3, -1, 1, 0\}, \{-1, -1, 0, \frac{1}{2}\}, \{0, 2, 0, -\frac{1}{2}\}, \{4, 0, 1, 0\}\}, \{\{-1, 0, 0, 0\}, \{0, -1, 0, 0\}, \{0, 0, 2, 1\}, \{0, 0, 0, 2\}\}\}$

The Jordan canonical form of $\mathbf{A}$ is the second element of ja, extracted with ja[[2]] and displayed in MatrixForm.

```
ja[[2]]//MatrixForm
```
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

From this result, we see that the minimal polynomial of $\mathbf{A}$ is $(x+1)(x-2)^2$. We define q to be the minimal polynomial of $\mathbf{A}$ and then verify that matrixa satisfies q.

```
q = Expand[(x-2)^2(x + 1)]
4-3x^2 + x^3

4 IdentityMatrix[4]-3MatrixPower[matrixa, 2] +
   MatrixPower[matrixa, 3]
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

The characteristic polynomial is obtained next and named p. As expected, q divides p, verified with Cancel.

```
p = CharacteristicPolynomial[matrixa, x]
4 + 4x-3x^2-2x^3 + x^4

Cancel[p/q]
1 + x
```

---

## 5.3.6 The QR Method

The **conjugate transpose** (or **Hermitian adjoint matrix**) of the $m \times n$ complex matrix $\mathbf{A}$ which is denoted by $\mathbf{A}^*$ is the transpose of the complex conjugate of $\mathbf{A}$. Symbolically, we have $\mathbf{A}^* = (\overline{\mathbf{A}})^t$. A complex matrix $\mathbf{A}$ is **unitary** if $\mathbf{A}^* = \mathbf{A}^{-1}$. Given a matrix $\mathbf{A}$, there is a unitary matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{R}$ such that $\mathbf{A} = \mathbf{QR}$. The product matrix $\mathbf{QR}$ is called the **QR factorization of** $\mathbf{A}$. The command

**QRDecomposition[N[m]]**

determines the QR decomposition of the matrix m by returning the list {q,r}, where q is an orthogonal matrix, r is an upper triangular matrix, and m=Transpose[q].r.

---

**Example 5.3.9** Find the QR factorization of the matrix $\mathbf{A} = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

***Solution*** We define matrixa and then use QRDecomposition to find the QR decomposition of matrixa, naming the resulting output qrm.

```
matrixa = {{4, -1, 1}, {-1, 4, 1}, {1, 1, 4}};
qrm = QRDecomposition[N[matrixa]]
{{{-0.942809, 0.235702, -0.235702},
   {-0.142134, -0.92387, -0.355335},
   {-0.301511, -0.301511, 0.904534}},
      {{-4.24264, 1.64992, -1.64992},
      {0., -3.90868, -2.48734}, {0., 0., 3.01511}}}
```

The first matrix in qrm is extracted with qrm[[1]] and the second with qrm[[2]].

```
qrm[[1]]//MatrixForm
```

$$\begin{pmatrix} -0.942809 & 0.235702 & -0.235702 \\ -0.142134 & -0.92387 & -0.355335 \\ -0.301511 & -0.301511 & 0.904534 \end{pmatrix}$$

```
qrm[[2]]//MatrixForm
```

$$\begin{pmatrix} -4.24264 & 1.64992 & -1.64992 \\ 0. & -3.90868 & -2.48734 \\ 0. & 0. & 3.01511 \end{pmatrix}$$

We verify that the results returned are the QR decomposition of $\mathbf{A}$.

```
Transpose[qrm[[1]]].qrm[[2]]//MatrixForm
```

$$\begin{pmatrix} 4. & -1. & 1. \\ -1. & 4. & 1. \\ 1. & 1. & 4. \end{pmatrix}$$

---

One of the most efficient and most widely used methods for numerically calculating the eigenvalues of a matrix is the QR method. Given a matrix $\mathbf{A}$, there is a Hermitian matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{R}$ such that $\mathbf{A} = \mathbf{QR}$. If we define a sequence of matrices $\mathbf{A}_1 = \mathbf{A}$, factored as $\mathbf{A}_1 = \mathbf{Q}_1\mathbf{R}_1$; $\mathbf{A}_2 = \mathbf{R}_1\mathbf{Q}_1$, factored as $\mathbf{A}_2 = \mathbf{R}_2\mathbf{Q}_2$; $\mathbf{A}_3 = \mathbf{R}_2\mathbf{Q}_2$, factored as $\mathbf{A}_2 = \mathbf{R}_3\mathbf{Q}_3$; and in general, $\mathbf{A}_k = \mathbf{R}_{k+1}\mathbf{Q}_{k+1}$, $k = 1, 2, \ldots$, then the sequence $\{\mathbf{A}_n\}$ converges to a triangular matrix with the eigenvalues of $\mathbf{A}$ along the diagonal or to a nearly triangular matrix from which the eigenvalues of $\mathbf{A}$ can be calculated rather easily.

---

**Example 5.3.10** Consider the $3 \times 3$ matrix $\mathbf{A} = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. Approximate the eigenvalues of $\mathbf{A}$ with the QR method.

***Solution*** We define the sequence a and qr recursively. We define a using the form a[n_] := a[n] = ... and qr using the form qr[n_] := qr[n] = ... so that Mathematica "remembers" the values of a and qr computed, and thus Mathematica avoids recomputing values previously computed. This is of particular advantage when computing a[n] and qr[n] for large values of $n$.

```
matrixa = {{4, -1, 1}, {-1, 4, 1}, {1, 1, 4}};
a[1] = N[matrixa];
qr[1] = QRDecomposition[a[1]];

a[n_] := a[n] = qr[n-1][[2]].Transpose[qr[n-1][[1]]];
qr[n_] := qr[n] = QRDecomposition[a[n]];
```

We illustrate a[n] and qr[n] by computing qr[9] and a[10]. Note that computing a[10] requires the computation of qr[9]. From the results, we suspect that the eigenvalues of $\mathbf{A}$ are $5$ and $2$.

```
qr[9];
a[10]//MatrixForm
```

$$\begin{pmatrix} 5. & -1.7853841942690367`*^-7 & -0.000556091 \\ -1.7853841949997794`*^-7 & 5. & -0.000963178 \\ -0.000556091 & -0.000963178 & 2. \end{pmatrix}$$

Next, we compute a[n] for $n = 5$, $10$, and $15$, displaying the result in TableForm. We obtain further evidence that the eigenvalues of $\mathbf{A}$ are $5$ and $2$.

```
Table[a[n]//MatrixForm, {n, 5, 15, 5}]//TableForm
```

| | | |
|---|---|---|
| 4.99902 | −0.001701 | 0.0542614 |
| −0.001701 | 4.99706 | 0.0939219 |
| 0.0542614 | 0.0939219 | 2.00393 |
| 5. | −1.7853841942690367`*^ − 7 | −0.000556091 |
| −1.7853841949997794`*^ − 7 | 5. | −0.000963178 |
| −0.000556091 | −0.000963178 | 2. |
| 5. | −1.872117829091164`*^ − 11 | 5.694375936943897`*^ − 6 |
| −1.8721251221839952`*^ − 11 | 5. | 9.862948440894718`*^ − 6 |
| 5.69437593740387`*^ − 6 | 9.862948440910032`*^ − 6 | 2. |

We verify that the eigenvalues of $\mathbf{A}$ are indeed $5$ and $2$ with Eigenvalues.

```
Eigenvalues[matrixa]
{5, 5, 2}
```

## 5.4 MAXIMA AND MINIMA USING LINEAR PROGRAMMING

### 5.4.1 The Standard Form of a Linear Programming Problem

We call the linear programming problem of the following form the **standard form** of the linear programming problem:

$$\text{Minimize } Z = \underbrace{c_1x_1 + c_2x_2 + \cdots + c_nx_n}_{\text{function}}, \text{ subject to the restrictions}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{cases} \tag{5.2}$$

and $x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$.

The command

**Minimize[{function, inequalities}, {variables}]**

solves the standard form of the linear programming problem. Similarly, the command

**Maximize[{function, inequalities}, {variables}]**

solves the linear programming problem: Maximize $Z = \underbrace{c_1x_1 + c_2x_2 + \cdots + c_nx_n}_{\text{function}}$,

subject to the restrictions

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{cases}$$

and $x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$.

---

**Example 5.4.1** Maximize $Z(x_1, x_2, x_3) = 4x_1 - 3x_2 + 2x_3$ subject to the constraints $3x_1 - 5x_2 + 2x_3 \le 60$, $x_1 - x_2 + 2x_3 \le 10$, $x_1 + x_2 - x_3 \le 20$, and $x_1$, $x_2$, $x_3$ all nonnegative.

***Solution*** In order to solve a linear programming problem with Mathematica, the variables {x1,x2,x3} and objective function z[x1,x2,x3] are first defined. In an effort to limit the amount of typing required to complete the problem, the set of inequalities is assigned the name **ineqs** while the set of variables is called **vars**. The symbol "<=", obtained by typing the "<" key and then the "=" key, represents "less than or equal to" and is used in **ineqs**. Hence, the maximization problem is solved with the command

**Maximize[{z[x1, x2, x3], ineqs}, vars].**

```
Clear[x1, x2, x3, z, ineqs, vars]
vars = {x1, x2, x3};
z[x1_, x2_, x3_] = 4x1 − 3x2 + 2x3;
ineqs = {3x1 − 5x2 + x3 ≤ 60, x1 − x2 + 2x3 ≤ 10, x1 + x2 − x3 ≤ 20,
    x1 ≥ 0, x2 ≥ 0, x3 ≥ 0};
Maximize[{z[x1, x2, x3], ineqs}, vars]
{45, {x1 → 15, x2 → 5, x3 → 0}}
```

The solution gives the maximum value of z subject to the given constraints as well as the values of x1, x2, and x3 that maximize z. Thus, we see that the maximum value of $Z$ is 45 if $x_1 = 15$, $x_2 = 5$, and $x_3 = 0$.

---

We demonstrate the use of Minimize in the following example.

**Example 5.4.2** Minimize $Z(x, y, z) = 4x - 3y + 2z$ subject to the constraints $3x - 5y + z \le 60$, $x - y + 2z \le 10$, $x + y - z \le 20$, and $x$, $y$, $z$ all nonnegative.

***Solution*** After clearing all previously used names of functions and variable values, the variables, objective function, and set of constraints for this problem are defined and entered as they were in the first example. By using Minimize, the minimum value of the objective function is obtained as well as the variable values that give this minimum.

```
Clear[x1, x2, x3, z, ineqs, vars]
vars = {x1, x2, x3};
z[x1_, x2_, x3_] = 4x1 - 3x2 + 2x3;
ineqs = {3x1 - 5x2 + x3 ≤ 60, x1 - x2 + 2x3 ≤ 10, x1 + x2 - x3 ≤ 20,
        x1 ≥ 0, x2 ≥ 0, x3 ≥ 0};
Minimize[{z[x1, x2, x3], ineqs}, vars]
{-90, {x1 → 0, x2 → 50, x3 → 30}}
```

We conclude that the minimum value is $-90$ and occurs if $x_1 = 0$, $x_2 = 50$, and $x_3 = 30$.

## 5.4.2 The Dual Problem

Given the standard form of the linear programming problem in equations (5.4.1), the **dual problem** is as follows: "Maximize $Y = \sum_{i=1}^{m} b_i y_y$ subject to the constraints $\sum_{i=1}^{m} a_{ij} y_i \le c_{ij}$ for $j = 1, 2, \ldots, n$ and $y_i \ge 0$ for $i = 1, 2, \ldots, m$." Similarly, for the problem, "Maximize $Z = \sum_{j=1}^{n} c_j x_j$ subject to the constraints $\sum_{j=1}^{n} a_{ij} x_j \le b_j$ for $i = 1, 2, \ldots, m$ and $x_j \ge 0$ for $j = 1, 2, \ldots, n$," the dual problem is as follows: "Minimize $Y = \sum_{i=1}^{m} b_i y_i$ subject to the constraints $\sum_{i=1}^{m} a_{ij} y_i \ge c_j$ for $j = 1, 2, \ldots, n$ and $y_i \ge 0$ for $i = 1, 2, \ldots, m$."

**Example 5.4.3** Maximize $Z = 6x + 8y$ subject to the constraints $5x + 2y \le 20$, $x + 2y \le 10$, $x \ge 0$, and $y \ge 0$. State the dual problem and find its solution.

***Solution*** First, the original (or *primal*) problem is solved. The objective function for this problem is represented by zx. Finally, the set of inequalities for the primal is defined to be ineqsx. Using the command

```
Maximize[{zx, ineqsx}, {x[1], x[2]}]
```

the maximum value of zx is found to be 45.

```
Clear[zx, zy, x, y, valsx, valsy, ineqsx, ineqsy]
zx = 6x[1] + 8x[2];
```

```
ineqsx = {5x[1] + 2x[2] ≤ 20, x[1] + 2x[2] ≤ 10, x[1] ≥ 0,
      x[2] ≥ 0};
Maximize[{zx, ineqsx}, {x[1], x[2]}]
{45, {x[1] → 5/2, x[2] → 15/4}}
```

Because in this problem we have $c_1 = 6$, $c_2 = 8$, $b_1 = 20$, and $b_2 = 10$, the dual problem is as follows: Minimize $Z = 20y_1 + 10y_2$ subject to the constraints $5y_1 + y_2 \geq 6$, $2y_1 + 2y_2 \geq 8$, $y_1 \geq 0$, and $y_2 \geq 0$. The dual is solved in a similar manner by defining the objective function zy and the collection of inequalities ineqsy. The minimum value obtained by zy subject to the constraints ineqsy is 45, which agrees with the result of the primal and is found with

**Minimize[{zy, ineqsy}, {y[1], y[2]}]**

```
zy = 20y[1] + 10y[2];
ineqsy = {5y[1] + y[2] ≥ 6, 2y[1] + 2y[2] ≥ 8,
      y[1] ≥ 0, y[2] ≥ 0};
Minimize[{zy, ineqsy}, {y[1], y[2]}]
{45, {y[1] → 1/2, y[2] → 7/2}}
```

---

Of course, linear programming models can involve numerous variables. Consider the following: Given the standard form linear programming problem in equations (5.4.1), let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}$, and $\mathbf{A}$ denote the $m \times n$ matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$. Then the standard form of the linear programming problem is equivalent to finding the vector $\mathbf{x}$ that maximizes $Z = \mathbf{c} \cdot \mathbf{x}$ subject to the restrictions $\mathbf{Ax} \geq \mathbf{b}$ and $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$. The dual problem is "Minimize $Y = \mathbf{y} \cdot \mathbf{b}$ where $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots y_m \end{pmatrix}$ subject to the restrictions $\mathbf{yA} \leq \mathbf{c}$ (componentwise) and $y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0$."
The command

**LinearProgramming[c, A, b]**

finds the vector x that minimizes the quantity Z=c.x subject to the restrictions A.x>=b and x>=0. LinearProgramming does not yield the minimum value of Z as did Minimize and Maximize, and the value must be determined from the resulting vector.

**Example 5.4.4** Maximize $Z = 5x_1 - 7x_2 + 7x_3 + 5x_4 + 6x_5$ subject to the constraints $2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \geq 10$, $6x_1 + 5x_2 + 4x_3 + x_4 + 4x_5 \geq 30$, $-3x_1 - 2x_2 - 3x_3 - 4x_4 \geq -5$, $-x_1 - x_2 - x_4 \geq -10$, and $x_1 \geq 0$ for $i = 1, 2, 3, 4$, and $5$. State the dual problem. What is its solution?

***Solution*** For this problem, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 10 \\ 30 \\ -5 \\ -10 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 5 & -7 & 7 & 5 & 6 \end{pmatrix}$, and $\mathbf{A} = \begin{pmatrix} 2 & 3 & 3 & 2 & 2 \\ 6 & 5 & 4 & 1 & 4 \\ -3 & -2 & -3 & -4 & 0 \\ -1 & -1 & 0 & -1 & 0 \end{pmatrix}$. First, the vectors $\mathbf{c}$ and $\mathbf{b}$ are entered and then matrix $\mathbf{A}$ is entered and named matrixa.

```
Clear[matrixa, x, y, c, b]
c = {5, -7, 7, 5, 6}; b = {10, 30, -5, -10};
matrixa = {{2, 3, 3, 2, 2}, {6, 5, 4, 1, 4},
      {-3, -2, -3, -4, 0}, {-1, -1, 0, -1, 0}};
```

Next, we use Array[x,5] to create the list of five elements {x[1], x[2], ..., x[5]} named xvec. The command Table[x[i], {i,1,5}] returns the same list. These variables must be defined before attempting to solve this linear programming problem.

```
xvec = Array[x, 5]
{x[1], x[2], x[3], x[4], x[5]}
```

After entering the objective function coefficients with the vector $\mathbf{c}$, the matrix of coefficients from the inequalities with matrixa, and the right-hand side values found in $\mathbf{b}$, the problem is solved with

**LinearProgramming[c, matrixa, b]**

The solution is called xvec. Hence, the maximum value of the objective function is obtained by evaluating the objective function at the variable values that yield a maximum. Because these values are found in xvec, the maximum is determined with the dot product of the vector $\mathbf{c}$ and the vector xvec. (Recall that this product is entered as c.xvec.) This value is found to be $35/4$.

**xvec = LinearProgramming[c, matrixa, b]**

$\{0, \frac{5}{2}, 0, 0, \frac{35}{8}\}$

**c.xvec**

$\frac{35}{4}$

Because the dual of the problem is “Minimize the number Y=y.b subject to the restrictions y.A<c and y>0,” we use Mathematica to calculate y.b and y.A. A list of the dual variables {y[1],y[2],y[3],y[4]} is created with Array[y,4]. This list includes four elements because there are four constraints in the original problem. The objective function of the dual problem is therefore found with **yvec.b**, and the left-hand sides of the set of inequalities are given with **yvec.matrixa**.

```
yvec = Array[y, 4]
{y[1], y[2], y[3], y[4]}
yvec.b
10y[1] + 30y[2] − 5y[3] − 10y[4]
yvec.matrixa
{2y[1] + 6y[2] − 3y[3] − y[4],
3y[1] + 5y[2] − 2y[3] − y[4],
3y[1] + 4y[2] − 3y[3],
2y[1] + y[2] − 4y[3] − y[4],
2y[1] + 4y[2]}
```

Hence, we may state the dual problem as follows:

Minimize $Y = 10y_1 + 30y_2 - 5y_3 - 10y_4$ subject to the constraints

$$\begin{cases} 2y_1 + 6y_2 - 3y_3 - y_4 \le 5 \\ 3y_1 + 5y_2 - 2y_3 - y_4 \le -7 \\ 3y_1 + 4y_2 - 3y_3 \le 7 \\ 2y_1 + y_2 - 4y_3 - y_4 \le 5 \\ 2y_1 + 4y_2 \le 6 \end{cases}$$

and $y_i \ge 0$ for $i = 1, 2, 3,$ and $4$.

---

### *Application: A Transportation Problem*

**A certain company has two factories, F1 and F2, each producing two products, P1 and P2, that are to be shipped to three distribution centers, D1, D2, and D3. The following table illustrates the cost associated with shipping each product from the factory to the distribution center, the minimum number of each product each distribution center needs, and the maximum output of each factory. How much of each product should be shipped from each plant to each distribution center to minimize the total shipping costs?**

***Solution*** Let $x_1$ denote the number of units of P1 shipped from F1 to D1; $x_2$ the number of units of P2 shipped from F1 to D1; $x_3$ the number of units of P1 shipped from F1 to D2; $x_4$ the number of units of P2 shipped from F1 to D2; $x_5$ the number of units of P1 shipped from F1 to D3; $x_6$ the number of units of P2 shipped from F1 to D3; $x_7$ the number of units of P1 shipped from F2 to D1; $x_8$ the number of units of P2 shipped from F2 to D1; $x_9$ the number of units of P1 shipped from F2 to D2; $x_{10}$ the number of units of P2 shipped from F2 to D2; $x_{11}$ the number

| | F1/P1 | F1/P2 | F2/P1 | F2/P2 | Minimum |
|---|---|---|---|---|---|
| **D1/P1** | $0.75 | | $0.80 | | 500 |
| **D1/P2** | | $0.50 | | $0.40 | 400 |
| **D2/P1** | $1.00 | | $0.90 | | 300 |
| **D2/P2** | | $0.75 | | $1.20 | 500 |
| **D3/P1** | $0.90 | | $0.85 | | 700 |
| **D3/P2** | | $0.80 | | $0.95 | 300 |
| **Maximum output** | 1000 | 400 | 800 | 900 | |

of units of P1 shipped from F2 to D3; and $x_{12}$ the number of units of P2 shipped from F2 to D3.

Then, it is necessary to minimize the number

$$Z = .75x_1 + .5x_2 + x_3 + .75x_4 + .9x_5 + .8x_6 + .8x_7 \\ + .4x_8 + .9x_9 + 1.2x_{10} + .85x_{11} + .95x_{12}$$

subject to the constraints $x_1 + x_3 + x_5 \leq 1000$, $x_2 + x_4 + x_6 \leq 400$, $x_7 + x_9 + x_{11} \leq 800$, $x_8 + x_{10} + x_{12} \leq 900$, $x_1 + x_7 \geq 500$, $x_3 + x_9 \geq 500$, $x_5 + x_{11} \geq 700$, $x_2 + x_8 \geq 400$, $x_4 + x_{10} \geq 500$, $x_6 + x_{12} \geq 300$, and $x_i$ nonnegative for $i = 1, 2, \ldots, 12$. In order to solve this linear programming problem, the objective function which computes the total cost, the 12 variables, and the set of inequalities must be entered. The coefficients of the objective function are given in the vector **c**. Using the command Array[x,12] illustrated in the previous example to define the list of 12 variables {x[1], x[2], ..., x[12]}, the objective function is given by the product z=xvec.c, where xvec is the name assigned to the list of variables.

```
Clear[xvec, z, constraints, vars, c]
c = {0.75, 0.5, 1, 0.75, 0.9, 0.8, 0.8, 0.4, 0.9, 1.2,
   0.85, 0.95};
xvec = Array[x, 12]

{x[1], x[2], x[3], x[4], x[5], x[6],
     x[7], x[8], x[9], x[10], x[11], x[12]}

z = xvec.c

0.75x[1] + 0.5x[2] + x[3] + 0.75x[4] +
   0.9x[5] + 0.8x[6] + 0.8x[7] + 0.4x[8] +
     0.9x[9] + 1.2x[10] + 0.85x[11] + 0.95x[12]
```

The set of constraints are then entered and named **constraints** for easier use. Therefore, the minimum cost and the value of each variable that yields this minimum cost are found with the command

Minimize[{z,constraints},xvec]

```
constraints = {x[1] + x[3] + x[5] ≤ 1000, x[2] + x[4] + x[6] ≤ 400,
   x[7] + x[9] + x[11] ≤ 800, x[8] + x[10] + x[12] ≤ 900,
   x[1] + x[7] ≥ 500, x[3] + x[9] ≥ 300, x[5] + x[11] ≥ 700,
   x[2] + x[8] ≥ 400, x[4] + x[10] > 500, x[6] + x[12] > 300,
      x[1] ≥ 0, x[2] ≥ 0, x[3] ≥ 0, x[4] ≥ 0,
      x[5] ≥ 0, x[6] ≥ 0, x[7] ≥ 0, x[8] ≥ 0,
      x[9] ≥ 0, x[10] ≥ 0, x[11] ≥ 0, x[12] ≥ 0};
values = Minimize[{z, constraints}, xvec]
{2115., {x[1] → 500., x[2] → 0., x[3] → 0., x[4] → 400.,
   x[5] → 200., x[6] → 0., x[7] → 0., x[8] → 400.,
      x[9] → 300., x[10] → 100., x[11] → 500., x[12] → 300.}}
```

Notice that **values** is a list consisting of two elements: the minimum value of the cost function, $2115$, and the list of the variable values {x[1]->500, x[2]->0, ...}. Hence, the minimum cost is obtained with the command values[[1]] and the list of variable values that yield the minimum cost is extracted with values[[2]].

```
values[[1]]
2115.
values[[2]]
{x[1] → 500., x[2] → 0., x[3] → 0., x[4] → 400.,
   x[5] → 200., x[6] → 0., x[7] → 0., x[8] → 400.,
      x[9] → 300., x[10] → 100., x[11] → 500., x[12] → 300.}
```

Using these extraction techniques, the number of units produced by each factory can be computed. Because $x_1$ denotes the number of units of P1 shipped from F1 to D1, $x_3$ the number of units of P1 shipped from F1 to D2, and $x_5$ the number of units of P1 shipped from F1 to D3, the total number of units of Product 1 produced by Factory 1 is given by the command x[1] + x[3] + x[5] /. values[[2]], which evaluates this sum at the values of x[1], x[3], and x[5] given in the list values[[2]].

```
x[1] + x[3] + x[5]/.values[[2]]
700.
```

Also, the number of units of Products 1 and 2 received by each distribution center can be computed. The command x[3] + x[9] //values[[2]] gives the total amount of P1 received at D2 because x[3] = amount of P1 received by D2 from F1 and x[9] = amount of P1 received by D2 from F2. Notice that this amount is the minimum number of units (300) of P1 requested by D2.

```
x[3] + x[9]/.values[[2]]
300.
```

The number of units of each product that each factory produces can be calculated, and the amount of P1 and P2 received at each distribution center is calculated in a similar manner.

```
{x[1] + x[3] + x[5], x[2] + x[4] + x[6], x[7] + x[9] + x[11],
   x[8] + x[10] + x[12], x[1] + x[7], x[3] + x[9],
   x[5] + x[11], x[2] + x[8],
      x[4] + x[10], x[6] + x[12]}/.values[[2]]//
      TableForm
700.
400.
800.
800.
500.
300.
700.
400.
500.
300.
```

From these results, we see that F1 produces 700 units of P1, F1 produces 400 units of P2, F2 produces 800 units of P1, F2 produces 800 units of P2, and each distribution center receives exactly the minimum number of each product it requests.

## 5.5 SELECTED TOPICS FROM VECTOR CALCULUS

### 5.5.1 Vector-Valued Functions

Basic operations on two- and three-dimensional vectors are discussed in Section 5.1.4.

We now turn our attention to vector-valued functions. In particular, we consider vector-valued functions of the following forms.

Plane curves: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ (5.3)

Space curves: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ (5.4)

Parametric surfaces: $\mathbf{r}(s, t) = x(s, t)\mathbf{i} + y(s, t)\mathbf{j} + z(s, t)\mathbf{k}$ (5.5)

Vector fields in the plane: $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ (5.6)

Vector fields in space: $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ (5.7)

For the vector-valued functions (5.3) and (5.4), differentiation and integration are carried out term-by-term, provided that all the terms are differentiable and integrable. Suppose that $C$ is a smooth curve defined by $\mathbf{r}(t)$, $a \le t \le b$.

In 2-space, $\mathbf{i} = <1, 0>$ and $\mathbf{j} = <0, 1>$. In 3-space, $\mathbf{i} = <1, 0, 0>$, $\mathbf{j} = <0, 1, 0>$, and $\mathbf{k} = <0, 0, 1>$.

1. If $\mathbf{r}'(t) \neq 0$, the **unit tangent vector**, $\mathbf{T}(t)$, is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.
2. If $\mathbf{T}'(t) \neq 0$, the **principal unit normal vector**, $\mathbf{N}(t)$, is $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.
3. The **arc length function**, $s(t)$, is $s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du$. In particular, the length of $C$ on the interval $[a, b]$ is $\int_a^b \|\mathbf{r}'(t)\|\, dt$.
4. The **curvature**, $\kappa$, of $C$ is

It is a good exercise to show that the curvature of a circle of radius $r$ is $1/r$.

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3},$$

where $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{r}''(t)$.

---

**Example 5.5.1** **(Folium of Descartes)**. Consider the **folium of Descartes**,

$$\mathbf{r}(t) = \frac{3at}{1+t^3}\mathbf{i} + \frac{3at^2}{1+t^3}\mathbf{j}$$

for $t \neq -1$, if $a = 1$. (a) Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, and $\int \mathbf{r}(t)\, dt$. (b) Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$. (c) Find the curvature, $\kappa$. (d) Find the length of the loop of the folium.

***Solution*** (a) After defining $\mathbf{r}(t)$,

```
r[t_] = {3a t/(1 + t^3), 3a t^2/(1 + t^3)};
a = 1;
```

we compute $\mathbf{r}'(t)$ and $\int \mathbf{r}(t)\, dt$ with ', '' and Integrate, respectively. We name $\mathbf{r}'(t)$ dr, $\mathbf{r}''(t)$ dr2, and $\int \mathbf{r}(t)\, dt$ ir.

```
dr = Simplify[r'[t]]
dr2 = Simplify[r''[t]]
ir = Integrate[r[t], t]
```

$$\left\{\frac{3-6t^3}{(1+t^3)^2}, -\frac{3t(-2+t^3)}{(1+t^3)^2}\right\}$$

$$\left\{\frac{18t^2(-2+t^3)}{(1+t^3)^3}, \frac{6(1-7t^3+t^6)}{(1+t^3)^3}\right\}$$

$$\left\{3\left(\frac{\text{ArcTan}\left[\frac{-1+2t}{\sqrt{3}}\right]}{\sqrt{3}} - \frac{1}{3}\text{Log}[1+t] + \frac{1}{6}\text{Log}\left[1-t+t^2\right]\right),\right.$$

$$\left.\text{Log}\left[1+t^3\right]\right\}$$

(b) Mathematica does not automatically make assumptions regarding the value of $t$, so it does not algebraically simplify $\|\mathbf{r}'(t)\|$ as we might typically do unless we use PowerExpand

PowerExpand [Sqrt[x^2]] returns $x$.

**nr = PowerExpand[Sqrt[dr.dr]//Simplify]**

$$\frac{3\sqrt{1+4t^2-4t^3-4t^5+4t^6+t^8}}{\left(1+t^3\right)^2}$$

The unit tangent vector, $\mathbf{T}(t)$, is formed in ut.

**ut = dr/nr//Simplify**

$$\left\{\frac{1-2t^3}{\sqrt{1+4t^2-4t^3-4t^5+4t^6+t^8}},\ -\frac{t\left(-2+t^3\right)}{\sqrt{1+4t^2-4t^3-4t^5+4t^6+t^8}}\right\}$$

We perform the same steps to compute the unit normal vector, $\mathbf{N}(t)$. In particular, note that dutb = $\|\mathbf{T}'(t)\|$.

**dut = D[ut, t]//Simplify**

$$\left\{\frac{2t\left(-2-3t^3+t^9\right)}{\left(1+4t^2-4t^3-4t^5+4t^6+t^8\right)^{3/2}},\ \frac{2-6t^6-4t^9}{\left(1+4t^2-4t^3-4t^5+4t^6+t^8\right)^{3/2}}\right\}$$

**duta = dut . dut//Simplify**

$$\frac{4\left(1+t^3\right)^4}{\left(1+4t^2-4t^3-4t^5+4t^6+t^8\right)^2}$$

**dutb = PowerExpand[Sqrt[duta]]**

$$\frac{2\left(1+t^3\right)^2}{1+4t^2-4t^3-4t^5+4t^6+t^8}$$

**nt = dut/dutb//Simplify**

$$\left\{\frac{t\left(-2+t^3\right)}{\sqrt{1+4t^2-4t^3-4t^5+4t^6+t^8}},\ \frac{1-2t^3}{\sqrt{1+4t^2-4t^3-4t^5+4t^6+t^8}}\right\}$$

(c) We use the formula $\kappa = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$ to determine the curvature in curvature.

**curvature = Simplify[dutb/nr]**

$$\frac{2\left(1+t^3\right)^4}{3\left(1+4t^2-4t^3-4t^5+4t^6+t^8\right)^{3/2}}$$

We graphically illustrate the unit tangent and normal vectors at $\mathbf{r}(1) = \langle 3/2, 3/2\rangle$. First, we compute the unit tangent and normal vectors if $t = 1$ using /. (ReplaceAll).

**ut1 = ut/.t → 1**
**nt1 = nt/.t → 1**

$\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$
$\left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$

We then compute the curvature if $t = 1$ in smallk. The center of the osculating circle at $\mathbf{r}(1)$ is found in x0 and y0.

The radius of the osculating circle is $1/\kappa$; the position vector of the center is $\mathbf{r} + \frac{1}{\kappa}\mathbf{N}$

```
smallk = curvature/.t → 1
N[smallk]
x0 = (r[t] + 1 / curvaturent/.t → 1)[[1]]
y0 = (r[t] + 1 / curvaturent/.t → 1)[[2]]
```

$\frac{8\sqrt{2}}{3}$
3.77124
$\frac{21}{16}$
$\frac{21}{16}$

Graphics[Circle[{x0, y0}, r]] is a two-dimensional graphics object that represents a circle of radius $r$ centered at the point $(x_0, y_0)$. Use Show to display the graph.

We now graph $\mathbf{r}(t)$ with ParametricPlot. The unit tangent and normal vectors at $\mathbf{r}(1)$ are graphed with Arrow in a1 and a2. The osculating circle at $\mathbf{r}(1)$ is graphed with Circle in c1. All four graphs are displayed together with Show in Figure 5.7.

```
p1 = ParametricPlot[r[t], {t, −100, 100},
     PlotRange → {{−2, 3}, {−2, 3}}, AspectRatio → 1];
```

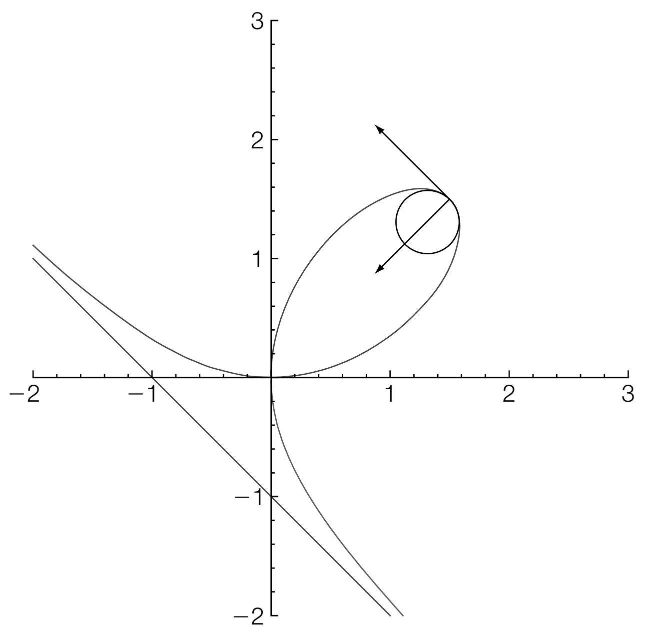



**FIGURE 5.7**

The folium with an osculating circle

```
p2 = Graphics[{Circle[{x0, y0}, 1/smallk],
    Arrow[{r[1], r[1] + ut1}], Arrow[{r[1], r[1] + nt1}]}];
Show[p1, p2]
```

(d) The loop is formed by graphing $\mathbf{r}(t)$ for $t \geq 0$. Hence, the length of the loop is given by the improper integral $\int_0^\infty \|\mathbf{r}(t)\|\, dt$, which we compute with NIntegrate.

```
NIntegrate[nr,{t,0,Infinity}]
4.91749
```

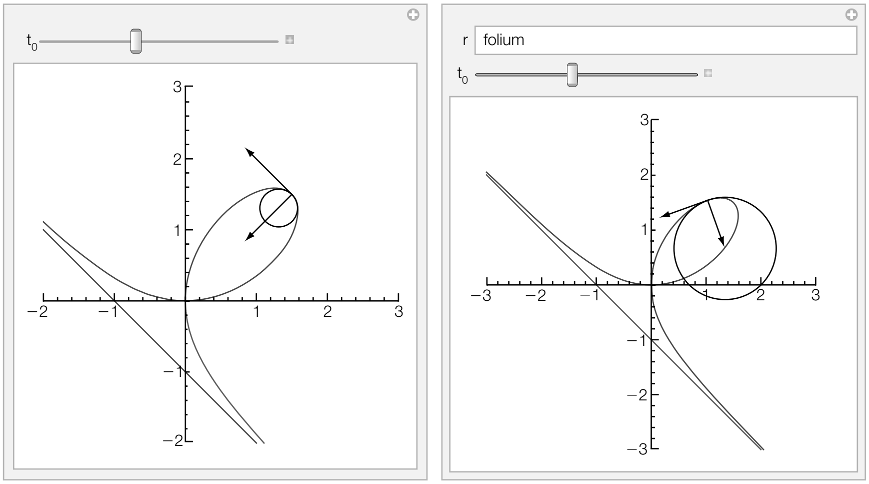
In the example, we computed the curvature at $t = 1$. Of course, we could choose other $t$ values. With Manipulate,

```
Manipulate[
r[t_] = {3t/(1 + t^3), 3t^2/(1 + t^3)};
dr = Simplify[r'[t]];
dr2 = Simplify[r''[t]];
ir = Integrate[r[t], t];
nr = PowerExpand[Sqrt[dr . dr]//Simplify];
ut = dr/nr//Simplify;
dut = D[ut, t]//Simplify;
duta = dut.dut//Simplify;
dutb = PowerExpand[Sqrt[duta]];
nt = dut/dutb//Simplify;
curvature = Simplify[dutb/nr];
ut1 = ut/.t → t0;
nt1 = nt/.t → t0;
smallk = curvature/.t → t0;
x0 = (r[t] + 1/curvaturent/.t → t0) [[1]];
y0 = (r[t] + 1/curvaturent/.t → t0) [[2]];
p1 = ParametricPlot[r[t], {t, −10, 10},
     PlotRange → {{−2, 3}, {−2, 3}}, AspectRatio → 1, PlotPoints → 200];
p2 = Graphics[{Circle[{x0, y0}, 1/smallk],
     Arrow[{r[t0], r[t0] + ut1}], Arrow[{r[t0], r[t0] + nt1}]}];
Show[p1, p2], {{t0, 1}, −5, 10}]
```

we can see the osculating circle at various values of $y_0 \neq -1$. See Figure 5.8(a).

Of course, this particular choice of using the folium to illustrate the procedure could be modified as well. With

```
Manipulate[
folium[t_] = {3t/(1 + t^3), 3t^2/(1 + t^3)};
cycloid[t_] = {1/(2Pi)(t − Sin[t]), (1 − Cos[t])/(2Pi)};
rose[t_] = {3/2Cos[2t]Cos[t], 3/2Cos[2t]Sin[t]};
squiggle[t_] = {Cos[t] − Sin[2t], Sin[2t] + Cos[5t]};
cornu[t_] = {2.5FresnelC[t], 2.5FresnelS[t]};
lissajous[t_] = {2Cos[t], Sin[2t]};
```




a b

**FIGURE 5.8**

(a) Using Manipulate to see the osculating circle at various values of $t_0$. (b) The osculating circle for various $\mathbf{r}(t)$ and $t_0$

```
evolute[t_] = {Cos[t]^3, 2Sin[t]^3};
dr = Simplify[r′[t]];
dr2 = Simplify[r″[t]];
ir = Integrate[r[t], t];
nr = PowerExpand[Sqrt[dr . dr]//Simplify];
ut = dr/nr//Simplify;
dut = D[ut, t]//Simplify;
duta = dut . dut//Simplify;
dutb = PowerExpand[Sqrt[duta]];
nt = dut/dutb//Simplify;
curvature = Simplify[dutb/nr];
ut1 = ut/.t → t0;
nt1 = nt/.t → t0;
smallk = curvature/.t → t0;
x0 = (r[t] + 1/curvaturent/.t → t0) [[1]];
y0 = (r[t] + 1/curvaturent/.t → t0) [[2]];
p1 = ParametricPlot[r[t], {t, −10, 10},
     PlotRange → {{−3, 3}, {−3, 3}}, AspectRatio → 1, PlotPoints → 200];
p2 = Graphics[{Circle[{x0, y0}, 1/smallk],
       Arrow [{r [t0] , r [t0] + ut1}] , Arrow [{r [t0] , r [t0] + nt1}] }] ;
Show[p1, p2], {{r, folium},
      {folium, cycloid, rose, squiggle, cornu, lissajous, evolute}},
         {{t0, 3/2}, −5, 10}]
```

we allow not only $t_0$ but also $\mathbf{r}(t)$ to vary. Note that the resulting Manipulate object is quite slow on all except the fastest computers. See Figure 5.8(b).

Recall that the **gradient** of $z = f(x, y)$ is the vector-valued function $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$. Similarly, we define the **gradient** of $w = f(x, y, z)$ to be

$$\nabla f(x, y, z) = \langle f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}. \tag{5.8}$$

A vector field $\mathbf{F}$ is **conservative** if there is a function $f$, called a **potential function**, satisfying $\nabla f = \mathbf{F}$. In the special case that $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The **divergence** of the vector field $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is the scalar field

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{5.9}$$

The Div command, which is contained in the VectorAnalysis package, can be used to find the divergence of a vector field:

**Div[{P(x,y,z),Q(x,y,z),R(x,y,z)},Cartesian[x,y,z]]**

computes the divergence of $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$. The **laplacian** of the scalar field $w = f(x, y, z)$ is defined to be

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f. \tag{5.10}$$

In the same way that Div computes the divergence of a vector field, Laplacian, which is also contained in the VectorAnalysis package, computes the laplacian of a scalar field.

The **curl** of the vector field $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is

$$\begin{aligned} \operatorname{curl} \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y, z) & Q(x, y, z) & R(x, y, z) \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}. \end{aligned} \tag{5.11}$$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$, $\mathbf{F}$ is conservative if and only if $\operatorname{curl} \mathbf{F}(x, y, z) = \mathbf{0}$, in which case $\mathbf{F}$ is said to be **irrotational**.

**Example 5.5.2** Determine if

$$\mathbf{F}(x,y) = \left(1-2x^2\right) y e^{-x^2-y^2}\mathbf{i} + \left(1-2y^2\right) x e^{-x^2-y^2}\mathbf{j}$$

is conservative. If $\mathbf{F}$ is conservative, find a potential function for $\mathbf{F}$.

***Solution*** We define $P(x,y) = \left(1-2x^2\right) y e^{-x^2-y^2}$ and $Q(x,y) = \left(1-2y^2\right) x e^{-x^2-y^2}$. Then we use D and Simplify to see that $P_y(x,y) = Q_x(x,y)$. Hence, $\mathbf{F}$ is conservative.

```
p[x_, y_] = (1-2x^2)y Exp[-x^2-y^2];
q[x_, y_] = (1-2y^2)x Exp[-x^2-y^2];
Simplify[D[p[x, y], y]]
Simplify[D[q[x, y], x]]
```

$e^{-x^2-y^2}\left(-1+2x^2\right)\left(-1+2y^2\right)$
$e^{-x^2-y^2}\left(-1+2x^2\right)\left(-1+2y^2\right)$

We use Integrate to find $f$ satisfying $\nabla f = \mathbf{F}$.

```
i1 = Integrate[p[x, y], x] + g[y]
```

$e^{-x^2-y^2}xy + g[y]$

```
Solve[D[i1, y]==q[x, y], g'[y]]
```

$\{\{g'[y] \to 0\}\}$

Therefore, $g(y) = C$, where $C$ is an arbitrary constant. Letting $C = 0$ gives us the following potential function.

```
f = i1/.g[y]->0
```

$e^{-x^2-y^2}xy$

Remember that the vectors $\mathbf{F}$ are perpendicular to the level curves of $f$. To see this, we normalize $\mathbf{F}$ in uv.

```
uv = {p[x, y], q[x, y]}/
Sqrt[{p[x, y], q[x, y]} . {p[x, y], q[x, y]}]//Simplify
```

$$\left\{\frac{e^{-x^2-y^2}\left(y-2x^2y\right)}{\sqrt{e^{-2\left(x^2+y^2\right)}\left(y^2+4x^4y^2+x^2\left(1-8y^2+4y^4\right)\right)}}, \frac{e^{-x^2-y^2}\left(x-2xy^2\right)}{\sqrt{e^{-2\left(x^2+y^2\right)}\left(y^2+4x^4y^2+x^2\left(1-8y^2+4y^4\right)\right)}}\right\}$$

We then graph several level curves of $f$ in cp1 and cp2 with ContourPlot and several vectors of uv with VectorFieldPlot, which is contained in the VectorFieldPlots package, in fp. We show the graphs together with Show in Figure 5.9.

```
<<"VectorFieldPlots`"
cp1 = ContourPlot[f, {x, -3/2, 3/2}, {y, -3/2, 3/2}, Contours -> 15,
    ContourShading -> False, PlotPoints -> 60];
```

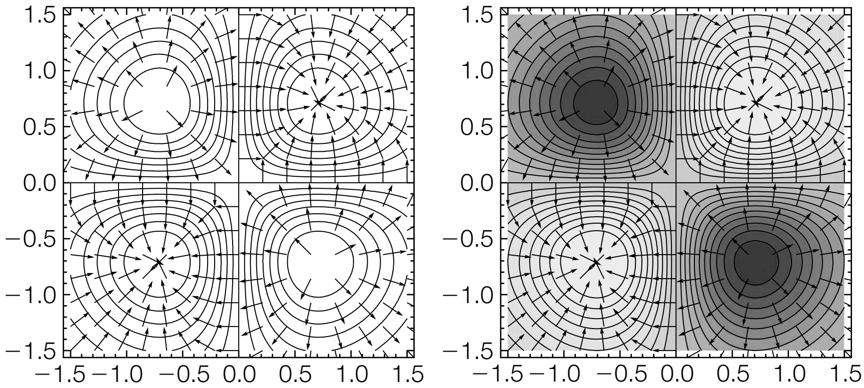



**FIGURE 5.9**

Two different views illustrating that the vectors $\mathbf{F}$ are perpendicular to the level curves of $f$

```
cp2 = ContourPlot[f, {x, -3/2, 3/2}, {y, -3/2, 3/2}, Contours → 20,
        PlotPoints → 60];
fp = VectorFieldPlot[uv, {x, -3/2, 3/2}, {y, -3/2, 3/2}];
Show[GraphicsRow[{Show[cp1, fp], Show[cp2, fp]}]]
```

Note that we can use GradientFieldPlot3D, which is contained in the VectorFieldPlots package, to graph several vectors of $\nabla f$. However, the vectors are scaled and it can be difficult to see that the vectors are perpendicular to the level curves of $f$. The advantage of proceeding this way is that by graphing unit vectors, it is easier to see that the vectors are perpendicular to the level curves of $f$ in the resulting plot.

**Example 5.5.3** (a) Show that

$$\mathbf{F}(x, y, z) = -10xy^2\mathbf{i} + \left(3z^3 - 10x^2y\right)\mathbf{j} + 9yz^2\mathbf{k}$$

is irrotational. (b) Find $f$ satisfying $\nabla f = \mathbf{F}$. (c) Compute div $\mathbf{F}$ and $\nabla^2 f$.

***Solution*** (a) After defining $\mathbf{F}(x, y, z)$, we use Curl, which is contained in the VectorAnalysis package, to see that curl $\mathbf{F}(x, y, z) = \mathbf{0}$.

```
Needs["VectorAnalysis`"]
Clear[f, x, y, z]
f[x_, y_, z_] = {-10xy^2, 3z^3 - 10x^2y, 9yz^2}
```

$\left\{-10xy^2, -10x^2y + 3z^3, 9yz^2\right\}$

```
Curl[f[x, y, z]]
```

{0, 0, 0}

(b) We then use Integrate to find $w = f(x, y, z)$ satisfying $\nabla f = \mathbf{F}$.

```
i1 = Integrate[f[x, y, z][[1]], x] + g[y, z]
```

$-5x^2y^2 + g[y, z]$

```
i2 = D[i1, y]
```

$-10x^2y + g^{(1,0)}[y, z]$

```
Solve[i2==f[x, y, z][[2]], g^(1,0)[y, z]]
```

$\{\{g^{(1,0)}[y, z] \to 3z^3\}\}$

```
i3 = Integrate[3z^3, y] + h[z]
```

$3yz^3 + h[z]$

```
i4 = i1/.g[y, z]->i3
```

$-5x^2y^2 + 3yz^3 + h[z]$

```
Solve[D[i4, z]==f[x, y, z][[3]]]
```

$\{\{h'[z] \to 0\}\}$

With $h(z) = C$ and $C = 0$, we have $f(x, y, z) = -5x^2y^2 + 3yz^3$.

```
lf = -5x^2y^2 + 3yz^3;
```

$\nabla f$ is orthogonal to the level surfaces of $f$. To illustrate this, we use ContourPlot3D to graph several level surfaces of $w = f(x, y, z)$ for $-10 \le x \le 10$, $-10 \le y \le 10$, and $-10 \le z \le 10$ in pf. We then use GradientFieldPlot3D, which is contained in the VectorFieldPlots package, to graph several vectors in the gradient field of $f$ over the same domain in gradf. The two plots are shown together with Show in Figure 5.10. In the plot, notice that the vectors appear to be perpendicular to the surface.

```
pf1=ContourPlot3D[lf ==-5, {x,-10,10}, {y,-10,10},
    {z,-10,10}, PlotPoints → 40];
pf2=ContourPlot3D[lf ==10,{x,-10,10}, {y,-10,10},
{z,-10,10}, PlotPoints → 40, Mesh → None,
    ContourStyle → Directive[Red,Opacity[0.8],
    Specularity[White,10]]];
pf3=ContourPlot3D[lf==100, {x,-10,10}, {y,-10,10},
{z,-10,10}, Mesh → None,
    ContourStyle → Directive[Red,Opacity[0.5]],
    PlotPoints → 40];
pf4=ContourPlot3D[lf, {x,-10,10}, {y,-10,10}, {z,-10,10},
PlotPoints → 50, Mesh → None,
    ContourStyle → Directive[Purple,Opacity[0.3],
    Specularity[White,30]]];
Needs["VectorFieldPlots`"]
gf = GradientFieldPlot3D[lf, {x, -10, 10}, {y, -10, 10},
```

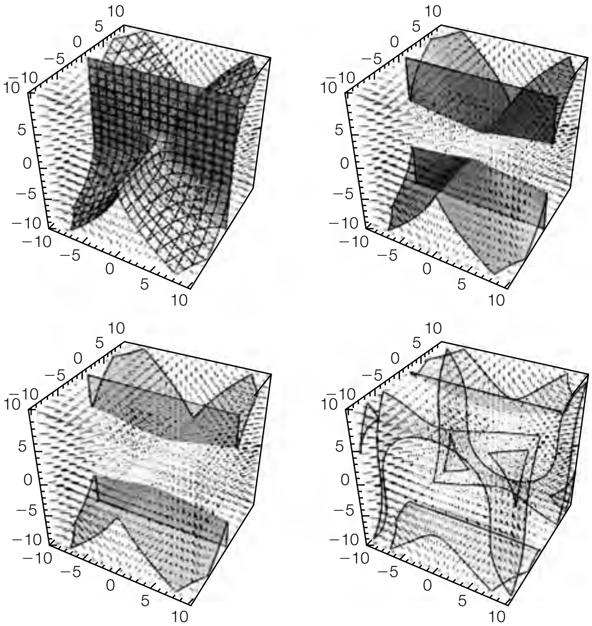


**FIGURE 5.10**

$\nabla f$ is orthogonal to the level surfaces of $f$

```
    {z, -10, 10}, PlotPoints -> 15];
Show[GraphicsGrid[{{Show[pf1, gf], Show[pf2, gf]},
    {Show[pf3, gf], Show[pf4, gf]}}]]
```

For (c), we take advantage of Div and Laplacian. As expected, the results are the same.

## 5.5.2 Line Integrals

If $\mathbf{F}$ is continuous on the smooth curve $C$ with parametrization $\mathbf{r}(t)$, $a \le t \le b$, the **line integral** of $\mathbf{F}$ on $C$ is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt. \tag{5.12}$$

If $\mathbf{F}$ is conservative and $C$ is piecewise smooth, line integrals can be evaluated using the *fundamental theorem of line integrals*.

**Theorem 19 (Fundamental Theorem of Line Integrals).** *If* $\mathbf{F}$ *is conservative and the curve* $C$ *defined by* $\mathbf{r}(t)$, $a \le t \le b$ *is piecewise smooth,*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)), \tag{5.13}$$

*where* $\mathbf{F} = \nabla f$.

**Example 5.5.4** Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \left(e^{-y} - ye^{-x}\right)\mathbf{i} + \left(e^{-x} - xe^{-y}\right)\mathbf{j}$ and $C$ is defined by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \ln(2t/\pi)\,\mathbf{j}$, $\pi/2 \le t \le 4\pi$.

***Solution*** We see that $\mathbf{F}$ is conservative with D and find that $f(x, y) = xe^{-y} + ye^{-x}$ satisfies $\nabla f = \mathbf{F}$ with Integrate.

```
f[x_,y_]={Exp[-y]-y Exp[-x],Exp[-x]-x Exp[-y]};
r[t_]={Cos[t],Log[2t/Pi]};
D[f[x,y][[1]],y]//Simplify
D[f[x,y][[2]],x]//Simplify
```

$-e^{-x}-e^{-y}$
$-e^{-x}-e^{-y}$

```
lf=Integrate[f[x,y][[1]],x]
```

$e^{-y}x + e^{-x}y$

Hence, using (5.13),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left(xe^{-y} + ye^{-x}\right)\Big]_{x=0,y=0}^{x=1,y=\ln 8} = \frac{3\ln 2}{e} + \frac{1}{8} \approx 0.890.$$

```
xr[t_]=Cos[t];
yr[t_]=Log[2t/Pi];
{xr[Pi/2], yr[Pi/2]}
{xr[4Pi], yr[4Pi]}
```

{0,0}
{1,Log[8]}

```
Simplify[lf/.{x->1, y->Log[8]}]
N[%]
```

$\frac{1}{8} + \frac{\text{Log}[8]}{e}$
0.889984

---

If $C$ is a piecewise smooth simple closed curve and $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives, *Green's theorem* relates the line integral $\oint_C \left(P(x, y)\,dx + Q(x, y)\,dy\right)$ to a double integral.

We assume that the symbol $\oint$ means to evaluate the integral in the positive (or counterclockwise) direction.

**Theorem 1 (Green's Theorem).** *Let C be a piecewise smooth simple closed curve in the plane and R the region bounded by C. If P(x, y) and Q(x, y) have continuous partial derivatives on R,*

$$\oint_C \left(P(x, y)\,dx + Q(x, y)\,dy\right) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{5.14}$$

**Example 5.5.5** Evaluate

$$\oint_C \left(e^{-x} - \sin y\right)\, dx + \left(\cos x - e^{-y}\right)\, dy,$$

where $C$ is the boundary of the region between $y = x^2$ and $x = y^2$.

***Solution*** After defining $P(x, y) = e^{-x} - \sin y$ and $Q(x, y) = \cos x - e^{-y}$, we use Plot to determine the region $R$ bounded by $C$ in Figure 5.11.

```
p[x_,y_] = Exp[-x] -Sin[y];
q[x_,y_] = Cos[x] -Exp[-y];
Plot[{x^2, Sqrt[x]}, {x, 0, 1.1},
    PlotStyle->{GrayLevel[0],GrayLevel[.3]},
        AspectRatio->Automatic]
```

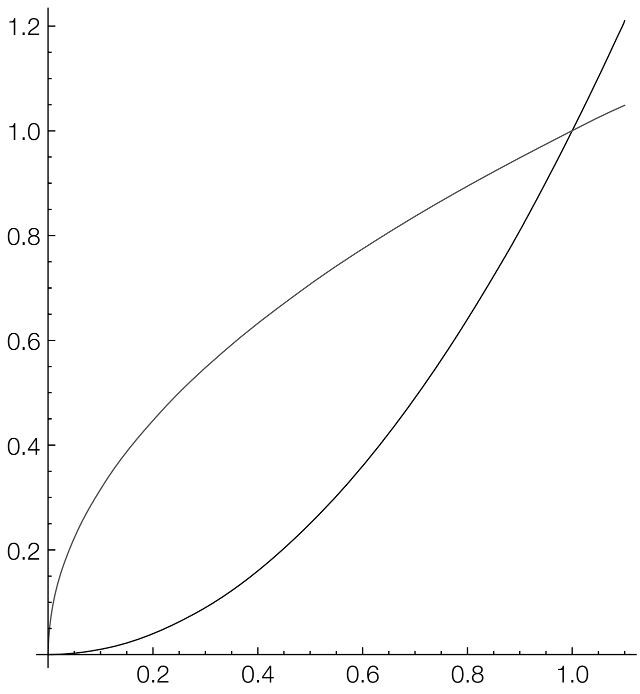



**FIGURE 5.11**

$y = x^2$ and $y = \sqrt{x},\ 0 \le x \le 1$

Using equation (5.14),

$$\begin{aligned}\oint_C \left(e^{-x} - \sin y\right)\, dx + \left(\cos x - e^{-y}\right)\, dy &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \iint_R \left(\cos y - \sin x\right) dA \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} \left(\cos y - \sin x\right)\, dy\, dx,\end{aligned}$$

```
dqdp = Simplify[D[q[x, y], x] – D[p[x, y], y]]
Cos[y] – Sin[x]
```

which we evaluate with Integrate.

```
Integrate[dqdp, {x, 0, 1}, {y, x^2, Sqrt[x]}]
N[%]
```

$\frac{1}{2}\left(-4-\sqrt{2\pi}\left(\text{FresnelC}\left[\sqrt{\frac{2}{\pi}}\right]+\text{FresnelS}\left[\sqrt{\frac{2}{\pi}}\right]\right)+8\text{Sin}[1]\right)$

0.151091

Notice that the result is given in terms of the FresnelS and FresnelC functions, which are defined by

$$\text{FresnelS[x]} = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt \quad \text{and} \quad \text{FresnelC[x]} = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt.$$

A more meaningful approximation is obtained with N. We conclude that

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \left(\cos y - \sin x\right)\, dy\, dx \approx 0.151.$$

---

## 5.5.3 Surface Integrals

Let $S$ be the graph of $z = f(x, y)$ ($y = h(x, z)$, $x = k(y, z)$) and let $R_{xy}$ ($R_{xz}$, $R_{yz}$) be the projection of $S$ onto the $xy$ ($xz$, $yz$) plane. Then,

$$\iint_S g(x, y, z)\, dS = \iint_{R_{xy}} g\left(x, y, f(x, y)\right) \sqrt{\left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2 + 1}\, dA \tag{5.15}$$

$$= \iint_{R_{xz}} g\left(x, h(x, z), z\right) \sqrt{\left[h_x(x, z)\right]^2 + \left[h_z(x, z)\right]^2 + 1}\, dA \tag{5.16}$$

$$= \iint_{R_{yz}} g\left(k(y, z), y, z\right) \sqrt{\left[k_y(y, z)\right]^2 + \left[k_z(y, z)\right]^2 + 1}\, dA. \tag{5.17}$$

If $S$ is defined parametrically by

$$\mathbf{r}(s, t) = x(s, t)\mathbf{i} + y(s, t)\mathbf{j} + z(s, t)\mathbf{k}, \quad (s, t) \in R,$$

the formula

$$\iint_S g(x,y,z)\,dS = \iint_R g\left(\mathbf{r}(s,t)\right)\|\mathbf{r}_s \times \mathbf{r}_t\|\,dA, \tag{5.18}$$

where

$$\mathbf{r}_s = \frac{\partial x}{\partial s}\mathbf{i} + \frac{\partial y}{\partial s}\mathbf{j} + \frac{\partial z}{\partial s}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_t = \frac{\partial x}{\partial t}\mathbf{i} + \frac{\partial y}{\partial t}\mathbf{j} + \frac{\partial z}{\partial t}\mathbf{k},$$

is also useful.

**Theorem 2 (The Divergence Theorem).** *Let $Q$ be any domain with the property that each line through any interior point of the domain cuts the boundary in exactly two points, and such that the boundary $S$ is a piecewise smooth closed, oriented surface with unit normal* $\mathbf{n}$*. If* $\mathbf{F}$ *is a vector field that has continuous partial derivatives on $Q$, then*

For our purposes, a surface is **oriented** if it has two distinct sides.

$$\iiint_Q \nabla \cdot \mathbf{F}\,dV = \iiint_Q div\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS. \tag{5.19}$$

In (5.19), $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ is called the **outward flux** of the vector field $\mathbf{F}$ across the surface $S$. If $S$ is a portion of the level curve $g(x,y) = C$ for some $g$, then a unit normal vector $\mathbf{n}$ may be taken to be either

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} \qquad \text{or} \qquad \mathbf{n} = -\frac{\nabla g}{\|\nabla g\|}.$$

If $S$ is defined parametrically by

$$\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}, \quad (s,t) \in R,$$

a unit normal vector to the surface is $\mathbf{n} = \dfrac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|}$ and (5.19) becomes $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\left(\mathbf{r}_s \times \mathbf{r}_t\right)\,dA$.

**Example 5.5.6** Find the outward flux of the vector field

$$\mathbf{F}(x,y,z) = \left(xz + xyz^2\right)\mathbf{i} + \left(xy + x^2yz\right)\mathbf{j} + \left(yz + xy^2z\right)\mathbf{k}$$

through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

***Solution*** By the divergence theorem,

$$\iint_{\text{cube surface}} \mathbf{F}\cdot\mathbf{n}\,dA = \iiint_{\text{cube interior}} \nabla\cdot\mathbf{F}\,dV.$$

Div is contained in the VectorAnalysis package. You do not need to reload the VectorAnalysis package if you have already loaded it during your *current* Mathematica session.

Hence, without the divergence theorem, calculating the outward flux would require six separate integrals, corresponding to the six faces of the cube. After defining **F**, we compute $\nabla \cdot \mathbf{F}$ with Div.

```
Needs["VectorAnalysis`"]
f[x_, y_, z_] = {xz + xyz^2, xy + x^2yz, yz + xy^2z};
divf = Div[f[x, y, z], Cartesian[x, y, z]]
```

$x + y + xy^2 + z + x^2z + yz^2$

The outward flux is then given by

$$\iiint_{\text{cube interior}} \nabla \cdot \mathbf{F}\, dV = \int_0^1 \int_0^1 \int_0^1 \nabla \cdot \mathbf{F}\, dz\, dy\, dx = 2,$$

which we compute with Integrate.

```
Integrate[divf, {z, 0, 1}, {y, 0, 1}, {x, 0, 1}]
```

2

---

**Theorem 3 (Stokes' Theorem).** *Let S be an oriented surface with finite surface area, unit normal* **n**, *and boundary C. Let* **F** *be a continuous vector field defined on S such that the components of* **F** *have continuous partial derivatives at each nonboundary point of S. Then,*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS. \tag{5.20}$$

In other words, the surface integral of the normal component of the curl of **F** taken over $S$ equals the line integral of the tangential component of the field taken over $C$. In particular, if $\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$, then

$$\int_C \left(P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz\right) = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS.$$

---

**Example 5.5.7** Verify Stokes' theorem for the vector field

$$\mathbf{F}(x, y, z) = \left(x^2 - y\right)\mathbf{i} + \left(y^2 - z\right)\mathbf{j} + \left(x + z^2\right)\mathbf{k}$$

and $S$ the portion of the paraboloid $z = f(x, y) = 9 - \left(x^2 + y^2\right)$, $z \geq 0$.

***Solution*** After loading the VectorAnalysis package, we define **F** and $f$. The curl of **F** is computed with Curl in curlF.

```
Needs["VectorAnalysis`"]
capf[x_, y_, z_] = {x^2 - y, y^2 - z, x + z^2};
f[x_, y_] = 9 - (x^2 + y^2);
curlcapf = Curl[capf[x, y, z], Cartesian[x, y, z]]
```

{1, −1, 1}

Next, we define the function $h(x,y,z) = z - f(x,y)$. A normal vector to the surface is given by $\nabla h$. A unit normal vector, $\mathbf{n}$, is then given by $\mathbf{n} = \dfrac{\nabla h}{\|\nabla h\|}$, which is computed in **un**.

```
h[x_, y_, z_] = z - f[x, y]
normtosurf = Grad[h[x, y, z], Cartesian[x, y, z]]
-9 + x^2 + y^2 + z
{2x, 2y, 1}
```

```
un = Simplify[normtosurf/Sqrt[normtosurf . normtosurf]]
```

$\left\{\frac{2x}{\sqrt{1+4x^2+4y^2}}, \frac{2y}{\sqrt{1+4x^2+4y^2}}, \frac{1}{\sqrt{1+4x^2+4y^2}}\right\}$

The dot product curl $\mathbf{F}\cdot\mathbf{n}$ is computed in **g**.

```
g = Simplify[curlcapf . un]
```

$\frac{1+2x-2y}{\sqrt{1+4x^2+4y^2}}$

Using the surface integral evaluation formula (5.15),

In this example, $R$, the projection of $f(x,y)$ onto the $xy$-plane, is the region bounded by the graph of the circle $x^2+y^2=9$.

$$\begin{aligned}\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS &= \iint_R g\left(x,y,f(x,y)\right)\sqrt{\left[f_x(x,y)\right]^2+\left[f_y(x,y)\right]^2+1}\, dA \\ &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} g\left(x,y,f(x,y)\right)\sqrt{\left[f_x(x,y)\right]^2+\left[f_y(x,y)\right]^2+1}\, dy\, dx \\ &= 9\pi,\end{aligned}$$

which we compute with **Integrate**.

```
tointegrate =
Simplify[(g/.z->f[x, y]) *
Sqrt[D[f[x, y], x]^2 + D[f[x, y], y]^2 + 1]]
1 + 2x - 2y
```

```
i1 = Integrate[tointegrate, {x, -3, 3},
 {y, -Sqrt[9 x^2], Sqrt[9 - x^2]}]
9π
```

To verify Stokes' theorem, we must compute the associated line integral. Notice that the boundary of $z = f(x,y) = 9 - (x^2+y^2)$, $z = 0$, is the circle $x^2+y^2=9$ with parametrization $x = 3\cos t$, $y = 3\sin t$, $z = 0$, $0 \le t \le 2\pi$. This parametrization is substituted into $\mathbf{F}(x,y,z)$ and named **pvf**.

```
pvf = capf[3Cos[t], 3Sin[t], 0]
{9Cos[t]^2 - 3Sin[t], 9Sin[t]^2, 3Cos[t]}
```

To evaluate the line integral along the circle, we next define the parametrization of the circle and calculate $d\mathbf{r}$. The dot product of **pvf** and **dr** represents the integrand of the line integral.

```
r[t_] = {3Cos[t], 3Sin[t], 0};
dr = r'[t]
{-3Sin[t], 3Cos[t], 0}
tointegrate = pvf . dr;
```

As before with $x$ and $y$, we instruct Mathematica to assume that $t$ is real, compute the dot product of **pvf** and **dr**, and evaluate the line integral with **Integrate**.

```
Integrate[tointegrate, {t, 0, 2Pi}]
9π
```

As expected, the result is $9\pi$.

---

## 5.5.4 A Note on Nonorientability

See "When is a surface *not* orientable?" by Braselton, Abell, and Braselton [5] for a detailed discussion regarding the examples in this section.

Suppose that $S$ is the surface determined by

$$\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}, \quad (s,t) \in R$$

and let

$$\mathbf{n} = \frac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|} \quad \text{or} \quad \mathbf{n} = -\frac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|}, \tag{5.21}$$

where

$$\mathbf{r}_s = \frac{\partial x}{\partial s}\mathbf{i} + \frac{\partial y}{\partial s}\mathbf{j} + \frac{\partial z}{\partial s}\mathbf{k} \quad \text{and} \quad \mathbf{r}_t = \frac{\partial x}{\partial t}\mathbf{i} + \frac{\partial y}{\partial t}\mathbf{j} + \frac{\partial z}{\partial t}\mathbf{k},$$

if $\|\mathbf{r}_s \times \mathbf{r}_t\| \neq 0$. If $\mathbf{n}$ is defined, $\mathbf{n}$ is orthogonal (or perpendicular) to $S$. We state three familiar definitions of *orientable*.

- $S$ is **orientable** if $S$ has a unit normal vector field, $\mathbf{n}$, that varies continuously between any two points $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$ on $S$. (See [7].)
- $S$ is **orientable** if $S$ has a continuous unit normal vector field, $\mathbf{n}$. (See [7] and [19].)
- $S$ is **orientable** if a unit vector $\mathbf{n}$ can be defined at every nonboundary point of $S$ in such a way that the normal vectors vary continuously over the surface $S$. (See [14].)

A path is **order preserving** if our chosen orientation is preserved as we move along the path.

Thus, a surface such as a torus is orientable.

**Example 5.5.8 (The Torus).** Using the standard parametrization of the torus, we use ParametricPlot3D to plot the torus if $c = 3$ and $a = 1$ in Figure 5.12.

Also see Example 2.3.18.

```
Clear[r, c, a, x, y, z, r]
c = 3;
a = 1;
x[s_, t_] = (c + aCos[s])Cos[t];
y[s_, t_] = (c + aCos[s])Sin[t];
z[s_, t_] = aSin[s];
r[s_, t_] = {x[s, t], y[s, t], z[s, t]};
threedp1t = ParametricPlot3D[r[s, t], {s, -Pi, Pi},
    {t, -Pi, Pi}, PlotPoints->{30, 30}, AspectRatio->1,
    PlotRange->{{-4, 4}, {-4, 4}, {-1, 1}},
        BoxRatios->{4, 4, 1}, AxesLabel->{"x", "y", "z"},
        ColorFunction → "FruitPunchColors", PlotStyle → Opacity[.3]]
```

To plot a normal vector field on the torus, we compute $\frac{\partial}{\partial s}\mathbf{r}(s,t)$,

```
rs = D[r[s, t], s]
{-Cos[t]Sin[s], -Sin[s]Sin[t], Cos[s]}
```

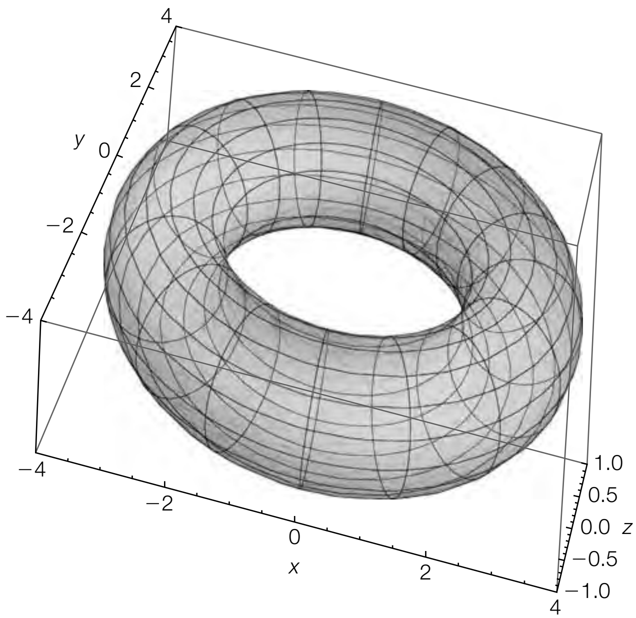



**FIGURE 5.12**

A torus

and $\frac{\partial}{\partial t}\mathbf{r}(s,t)$.

```
rt = D[r[s, t], t]
{-(3 + Cos[s])Sin[t], (3 + Cos[s])Cos[t], 0}
```

The cross product $\frac{\partial}{\partial s}\mathbf{r}(s,t) \times \frac{\partial}{\partial t}$ is formed in **rscrossrt**.

```
rscrossrt = Cross[rs, rt]//Simplify
{-Cos[s](3 + Cos[s])Cos[t],
   -Cos[s](3 + Cos[s])Sin[t], -(3 + Cos[s])Sin[s]}

Sqrt[rscrossrt . rscrossrt]//FullSimplify
```

$\sqrt{(3+\text{Cos}[s])^2}$

Using equation (5.24), we define un: Given $s$ and $t$, un[s,t] returns a unit normal to the torus.

```
Clear[un]
un[s_, t_] =
   -rscrossrt/Sqrt[rscrossrt .
     rscrossrt]//PowerExpand//FullSimplify
```

$$\left\{\frac{\text{Cos}[s](3+\text{Cos}[s])\text{Cos}[t]}{\sqrt{(3+\text{Cos}[s])^2}}, \frac{\text{Cos}[s](3+\text{Cos}[s])\text{Sin}[t]}{\sqrt{(3+\text{Cos}[s])^2}}, \frac{(3+\text{Cos}[s])\text{Sin}[s]}{\sqrt{(3+\text{Cos}[s])^2}}\right\}$$

```
Map[PowerExpand, un[s, t]]
{Cos[s]Cos[t], Cos[s]Sin[t], Sin[s]}

r[s, t]
{(3 + Cos[s])Cos[t],
   (3 + Cos[s])Sin[t], Sin[s]}

un[s, t]
```

$$\left\{\frac{\text{Cos}[s](3+\text{Cos}[s])\text{Cos}[t]}{\sqrt{(3+\text{Cos}[s])^2}}, \frac{\text{Cos}[s](3+\text{Cos}[s])\text{Sin}[t]}{\sqrt{(3+\text{Cos}[s])^2}}, \frac{(3+\text{Cos}[s])\text{Sin}[s]}{\sqrt{(3+\text{Cos}[s])^2}}\right\}$$

To plot the normal vector field on the torus, we take advantage of the command ListVectorFieldPlot3D, which is contained in the **VectorFieldPlots** package. See Figure 5.13.

```
<<"VectorFieldPlots`"
Clear[vecs]
vecs = Flatten[Table[{r[s, t], un[s, t]},
      {s, -Pi, Pi, 2Pi/14}, {t, -Pi, Pi, 2Pi/29}], 1];
pp2 = ListVectorFieldPlot3D[vecs, VectorHeads → True]
Show[threedp1t, pp2, AspectRatio->1,
```

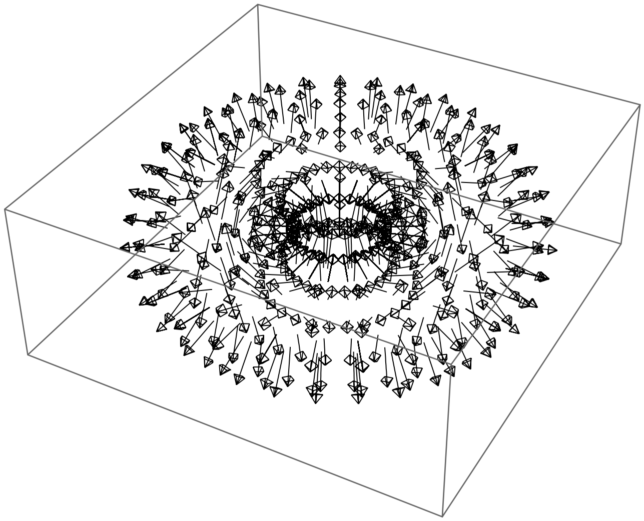


**FIGURE 5.13**

Unit normal vector field on a torus

```
PlotRange->{{-5,5},{-5,5},{-2,2}},
BoxRatios->{4,4,1},AxesLabel->{"x","y","z"}]
```

We use Show (illustrating the use of the ViewPoint option) together with GraphicsArray to see the vector field on the torus together from various angles in Figure 5.14. Regardless of the viewing angle, the figure looks the same; the torus is orientable.

```
g1 = Show[threedp1t,pp2,AspectRatio->1,
    PlotRange->{{-5,5},{-5,5},{-2,2}},
        BoxRatios->{4,4,1},AxesLabel->{"x","y","z"},
            ViewPoint->{2.729, -0.000,2.000}];
g2 = Show[threedp1t,pp2,AspectRatio->1,
    PlotRange->{{-5,5},{-5,5},{-2,2}},
        BoxRatios->{4,4,1},AxesLabel->{"x","y","z"},
            ViewPoint->{1.365, -2.364,2.000}];
g3 = Show[threedp1t,pp2,AspectRatio->1,
    PlotRange->{{-5,5},{-5,5},{-2,2}},
        BoxRatios->{4,4,1},AxesLabel->{"x","y","z"},
            ViewPoint->{-1.365, -2.364,2.000}];
g4 = Show[threedp1t,pp2,AspectRatio->1,
    PlotRange->{{-5,5},{-5,5},{-2,2}},
        BoxRatios->{4,4,1},AxesLabel->{"x","y","z"},
            ViewPoint->{-2.729,0.000,2.000}];
g5 = Show[threedp1t,pp2,AspectRatio->1,
    PlotRange->{{-5,5},{-5,5},{-2,2}},
```

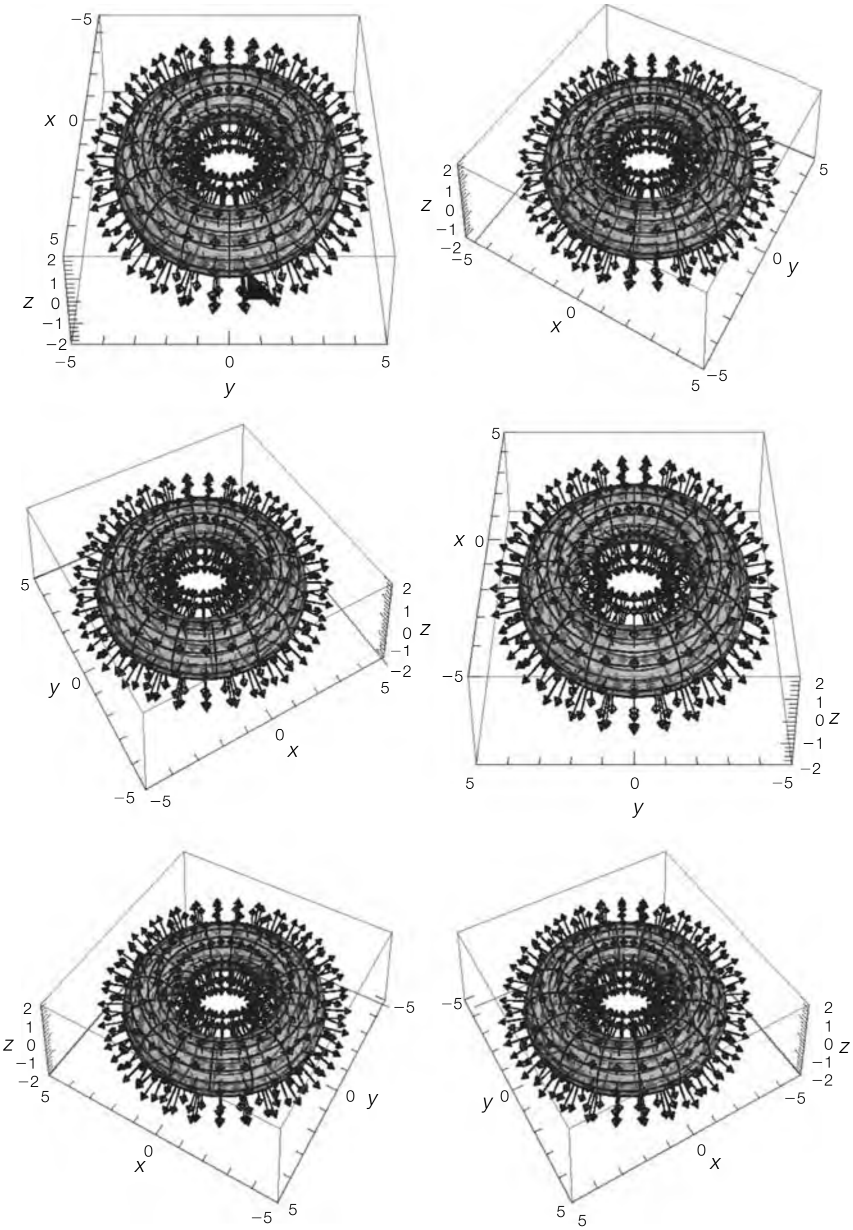


**FIGURE 5.14**

The torus is orientable

```
        BoxRatios->{4, 4, 1}, AxesLabel->{"x", "y", "z"},
            ViewPoint->{-1.365, 2.364, 2.000}];
g6 = Show[threedp1t, pp2, AspectRatio->1,
    PlotRange->{{-5, 5}, {-5, 5}, {-2, 2}},
        BoxRatios->{4, 4, 1}, AxesLabel->{"x", "y", "z"},
            ViewPoint->{1.365, 2.364, 2.000}];
Show[GraphicsGrid[{{g1, g2}, {g3, g4}, {g5, g6}}]]
```

If a 2-manifold, $S$, has an **order reversing path** (or **not order preserving path**), $S$ is **nonorientable** (or **not orientable**).

Determining whether a given surface $S$ is orientable or not may be a difficult problem.

**Example 5.5.9 (The Möbius Strip).** The *Möbius strip* is frequently cited as an example of a nonorientable surface with boundary: It has one side and is physically easy to construct by hand by half twisting and taping (or pasting) together the ends of a piece of paper (for example, see [5], [7], [14], and [19]). A parametrization of the Möbius strip is $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$, $-1 \le s \le 1$, $-\pi \le t \le \pi$, where

$$x = \left[c + s\cos\left(\frac{1}{2}t\right)\right]\cos t, \quad y = \left[c + s\cos\left(\frac{1}{2}t\right)\right]\sin t, \text{ and}$$

$$z = s\sin\left(\frac{1}{2}t\right), \tag{5.22}$$

and we assume that $c > 1$. In Figure 5.15, we graph the Möbius strip using $c = 3$.

```
c = 3;
x[s_, t_] = (c + sCos[t/2])Cos[t];
y[s_, t_] = (c + sCos[t/2])Sin[t];
z[s_, t_] = sSin[t/2];
r[s_, t_] = {x[s, t], y[s, t], z[s, t]};
threedp1 = ParametricPlot3D[r[s, t], {s, -1, 1},
    {t, -Pi, Pi}, PlotPoints->{30, 30},
    AspectRatio->1, PlotRange →
        {{-4, 4}, {-4, 4}, {-1, 1}}, BoxRatios->{4, 4, 1},
        AxesLabel->{"x", "y", "z"}, ColorFunction → "NeonColors",
        Mesh → False, PlotStyle → Opacity[.8]]
```

Although it is relatively easy to see in the plot that the Möbius strip has only one side, the fact that a unit vector, $\mathbf{n}$, normal to the Möbius strip at a point $P$ reverses its direction as $\mathbf{n}$ moves around the strip to $P$ is not obvious to the novice.

With Mathematica, we compute $\|\mathbf{r}_s \times \mathbf{r}_t\|$ and $\mathbf{n} = \frac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|}$.

```
rs = D[r[s,t],s]
```

$\left\{\text{Cos}\left[\frac{t}{2}\right]\text{Cos[t]}, \text{Cos}\left[\frac{t}{2}\right]\text{Sin[t]}, \text{Sin}\left[\frac{t}{2}\right]\right\}$

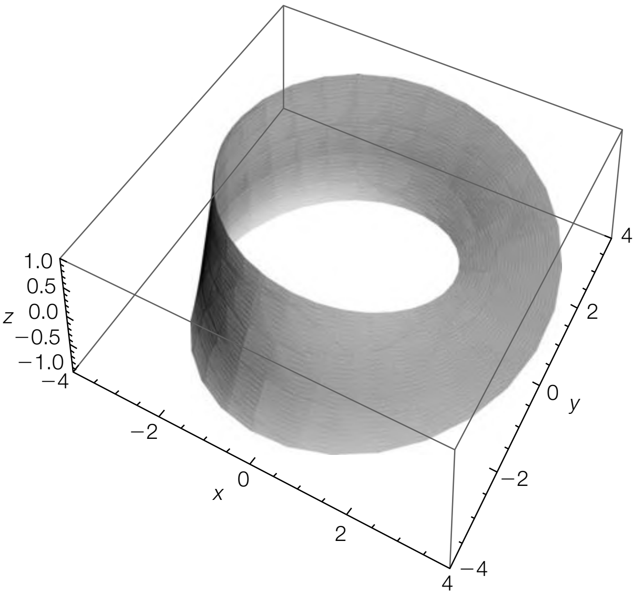



**FIGURE 5.15**

Parametric plot of equations (5.22) if $c = 3$

**rt = D[r[s, t], t]**

$$\left\{-\frac{1}{2}\text{sCos[t]Sin}\left[\frac{t}{2}\right]-\left(3+\text{sCos}\left[\frac{t}{2}\right]\right)\text{Sin[t]},\ \left(3+\text{sCos}\left[\frac{t}{2}\right]\right)\text{Cos[t]}-\frac{1}{2}\text{sSin}\left[\frac{t}{2}\right]\text{Sin[t]},\ \frac{1}{2}\text{sCos}\left[\frac{t}{2}\right]\right\}$$

**rscrossrt = Cross[rs, rt]//Simplify**

$$\left\{-\frac{1}{2}\left(-\text{sCos}\left[\frac{t}{2}\right]+6\text{Cos[t]}+\text{sCos}\left[\frac{3t}{2}\right]\right)\text{Sin}\left[\frac{t}{2}\right],\ \frac{1}{4}\left(-\text{s}-6\text{Cos}\left[\frac{t}{2}\right]-2\text{sCos[t]}+6\text{Cos}\left[\frac{3t}{2}\right]+\text{sCos[2t]}\right),\ \text{Cos}\left[\frac{t}{2}\right]\left(3+\text{sCos}\left[\frac{t}{2}\right]\right)\right\}$$

**Sqrt[rscrossrt . rscrossrt]//FullSimplify**

$$\sqrt{9+\frac{3\text{s}^2}{4}+6\text{sCos}\left[\frac{t}{2}\right]+\frac{1}{2}\text{s}^2\text{Cos[t]}}$$

**Clear[un]**

**un[s_, t_] =**
**rscrossrt/Sqrt[rscrossrt . rscrossrt]//FullSimplify**

$$\left\{\frac{\text{sSin[t]}-\text{Cos[t]}\left(6\text{Sin}\left[\frac{t}{2}\right]+\text{sSin[t]}\right)}{\sqrt{36+3\text{s}^2+24\text{sCos}\left[\frac{t}{2}\right]+2\text{s}^2\text{Cos[t]}}},\right.$$

$$-\frac{3\text{Cos}\left[\frac{t}{2}\right]-3\text{Cos}\left[\frac{3t}{2}\right]+s\left(\text{Cos}[t]+\text{Sin}[t]^2\right)}{\sqrt{36+3s^2+24s\text{Cos}\left[\frac{t}{2}\right]+2s^2\text{Cos}[t]}},$$

$$\left.\frac{s+6\text{Cos}\left[\frac{t}{2}\right]+s\text{Cos}[t]}{\sqrt{36+3s^2+24s\text{Cos}\left[\frac{t}{2}\right]+2s^2\text{Cos}[t]}}\right\}$$

Consider the path $C$ given by $\mathbf{r}(0,t)$, $-\pi \le t \le \pi$ that begins and ends at $\langle -3,0,0\rangle$. On $C$, $\mathbf{n}(0,t)$ is given by

**un[0, t]**

$\left\{-\text{Cos}[t]\text{Sin}\left[\frac{t}{2}\right], \frac{1}{6}\left(-3\text{Cos}\left[\frac{t}{2}\right]+3\text{Cos}\left[\frac{3t}{2}\right]\right), \text{Cos}\left[\frac{t}{2}\right]\right\}$

At $t=-\pi$, $\mathbf{n}(0,-\pi)=\langle 1,0,0\rangle$, whereas at $t=\pi$, $\mathbf{n}(0,\pi)=\langle -1,0,0\rangle$.

```
r[0, -Pi]
r[0, Pi]
{-3, 0, 0}
{±3, 0, 0}
```

As $\mathbf{n}$ moves along $C$ from $\mathbf{r}(0,-\pi)$ to $\mathbf{r}(0,\pi)$, the orientation of $\mathbf{n}$ reverses, as shown in Figure 5.16.

```
l1 = Table[r[0, t], {t, -Pi, Pi, 2Pi/179}];
threedp2 = Show[Graphics3D[{Thickness[.02],
  GrayLevel[.6], Line[l1]}], Axes->Automatic,
  PlotRange->{{-4, 4}, {-4, 4}, {-1, 1}},
    BoxRatios->{4, 4, 1}, AspectRatio->1];
<< "VectorFieldPlots`";
```

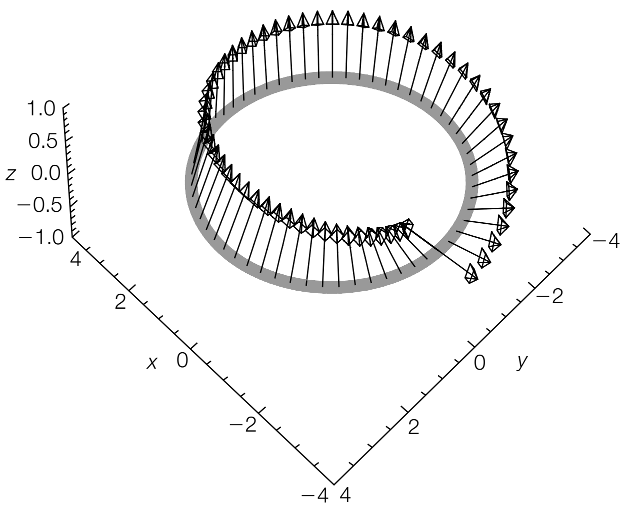



**FIGURE 5.16**

Parametric plot of equations (5.22) if $c=3$

```
vecs = Table[{r[0, t], un[0, t]}, {t, −π, π, 2π/59}];
pp2 = ListVectorFieldPlot3D[vecs, VectorHeads → True];
Show[threedp2, pp2, ViewPoint →
    {−2.093, 2.124, 1.600}, AxesLabel −>{"x", "y", "z"},
        Boxed −>False, DisplayFunction −>$DisplayFunction]
```

Several different views of Figure 5.16 on the Möbius strip shown in Figure 5.15 are shown in Figure 5.17. $C$ is an orientation reversing path and we can conclude that the Möbius strip is not orientable.

An animation is particularly striking.

```
g1 = Show[threedp1, threedp2, pp2,
    ViewPoint −>{2.729, −0.000, 2.000},
      AxesLabel −>{"x", "y", "z"}, Boxed −>False];
g2 = Show[threedp1, threedp2, pp2,
    ViewPoint −>{1.365, −2.364, 2.000},
      AxesLabel −>{"x", "y", "z"}, Boxed −>False];
g3 = Show[threedp1, threedp2, pp2,
    ViewPoint −>{−1.365, −2.364, 2.000},
      AxesLabel −>{"x", "y", "z"}, Boxed −>False];
g4 = Show[threedp1, threedp2, pp2,
    ViewPoint −>{−2.729, 0.000, 2.000},
      AxesLabel −>{"x", "y", "z"}, Boxed −>False];
g5 = Show[threedp1, threedp2, pp2,
    ViewPoint −>{−1.365, 2.364, 2.000},
      AxesLabel −>{"x", "y", "z"}, Boxed −>False];
g6 = Show[threedp1, threedp2, pp2,
    ViewPoint −>{1.365, 2.364, 2.000},
      AxesLabel −>{"x", "y", "z"}, Boxed −>False];
Show[GraphicsGrid[{{g1, g2}, {g3, g4}, {g5, g6}}]]
```

---

**Example 5.5.10** The *Klein bottle* is an interesting surface with neither an inside nor an outside, which indicates to us that it is not orientable. In Figure 5.18(a) we show the "usual" *immersion* of the Klein bottle. Although the Klein bottle does not intersect itself, it is not possible to visualize it in Euclidean 3-space without it doing so. Visualizations of 2-manifolds such as the Klein bottle's "usual" rendering in Euclidean 3-space are called *immersions*. (See [11] for a nontechnical discussion of immersions.)

```
r = 4(1 − 1/2Cos[u]);
x1[u_, v_] = 6(1 + Sin[u])Cos[u] + rCos[u]Cos[v];
x2[u_, v_] = 6(1 + Sin[u])Cos[u] + rCos[v + Pi];
y1[u_, v_] = 16Sin[u] + rSin[u]Cos[v];
y2[u_, v_] = 16Sin[u];
z[u_, v_] = rSin[v];
kb1a = ParametricPlot3D[{x1[s, t], y1[s, t], z[s, t]},
    {s, 0, Pi}, {t, 0, 2Pi}, PlotPoints −>{30, 30},
```

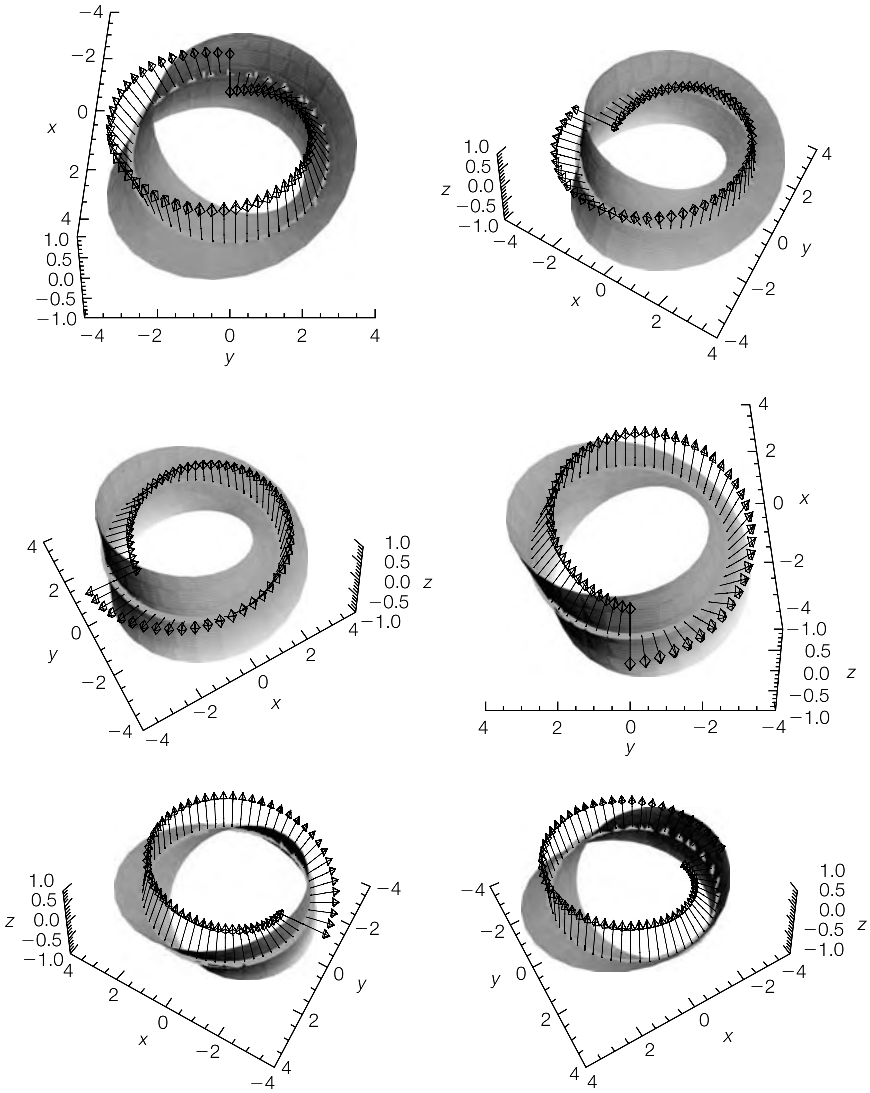



**FIGURE 5.17**

Different views of a Möbius strip with an orientation reversing path

```
    AspectRatio->1, AxesLabel->{"x", "y", "z"},
      Mesh → False, PlotStyle → Opacity[.8]];
kb1b = ParametricPlot3D[{x1[s, t], y1[s, t], z[s, t]},
    {s, Pi, 2Pi}, {t, 0, 2Pi}, PlotPoints->{30, 30},
    AspectRatio->1, AxesLabel->{"x", "y", "z"},
      Mesh → False, PlotStyle → Opacity[.8]]
kb1 = Show[kb1a, kb1b, PlotRange → {{-20, 20}, {-20, 20}, {-20, 20}}]
```

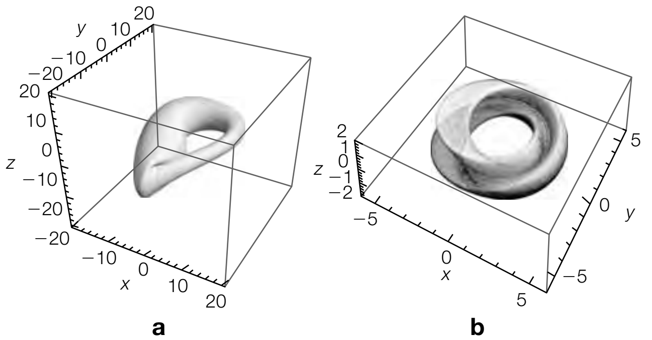



**FIGURE 5.18**

Two different immersions of the Klein bottle: (a) the "usual" immersion; (b) the figure-8 immersion

Figure 5.18(b) shows the *figure-8* immersion of the Klein bottle. Notice that it is not easy to see that the Klein bottle has neither an inside nor an outside in Figure 5.14.

```
a = 3;
x[u_, v_] = (a + Cos[u/2]Sin[v] - Sin[u/2]Sin[2v])Cos[u];
y[u_, v_] = (a + Cos[u/2]Sin[v] - Sin[u/2]Sin[2v])Sin[u];
z[u_, v_] = Sin[u/2]Sin[v] + Cos[u/2]Sin[2v];
r[u_, v_] = {x[u, v], y[u, v], z[u, v]};

ParametricPlot3D[r[t, t], {t, 0, 2Pi}]

kb2 = ParametricPlot3D[r[s, t], {s, -Pi, Pi}, {t, -Pi, Pi},
   PlotPoints->{30, 30}, AspectRatio->1,
   AxesLabel->{"x", "y", "z"},
     PlotRange->{{-6, 6}, {-6, 6}, {-2, 2}}, BoxRatios->{4, 4, 1},
     ColorFunction → "SunsetColors", Mesh → False,
       PlotStyle → Opacity[.4]]
Show[GraphicsRow[{kb1, kb2}]]
```

In fact, to many readers it may not be clear whether the Klein bottle is orientable or nonorientable, especially when we compare the graph to the graphs of the Möbius strip and torus in the previous examples.

A parametrization of the figure-8 immersion of the Klein bottle (see [20]) is $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$, $-\pi \le s \le \pi$, $-\pi \le t \le \pi$, where

$$x = \left[c + \cos\left(\frac{1}{2}s\right)\sin t - \sin\left(\frac{1}{2}s\right)\sin 2t\right]\cos s,$$

$$y = \left[c + \cos\left(\frac{1}{2}s\right)\sin t - \sin\left(\frac{1}{2}s\right)\sin 2t\right]\sin s, \tag{5.23}$$

and

$$z = \sin\left(\frac{1}{2}s\right)\sin t + \cos\left(\frac{1}{2}s\right)\sin 2t.$$

The plot in Figure 5.18(b) uses equation (5.23) if $c = 3$.

Using (5.21), let

$$\mathbf{n} = \frac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|}.$$

Let $C$ be the path given by

$$\mathbf{r}(t,t) = x(t,t)\mathbf{i} + y(t,t)\mathbf{j} + z(t,t)\mathbf{k}, \quad -\pi \le t \le \pi \tag{5.24}$$

that begins and ends at $\mathbf{r}(-\pi,-\pi) = \mathbf{r}(\pi,\pi) = \langle -3,0,0\rangle$ and where the components are given by (5.23). The components of $\mathbf{r}$ and $\mathbf{n}$ are computed with Mathematica. The final calculations are quite lengthy, so we suppress the output of the last few by placing a semicolon (;) at the end of those commands.

```
rs = D[r[s, t], s]//Simplify
{ - 1/2 Cos[s](Sin[s/2]Sin[t] + Cos[s/2]Sin[2t]) +
    Sin[s](-3 - Cos[s/2]Sin[t] + Sin[s/2]Sin[2t]),
    - 1/2 Sin[s](Sin[s/2]Sin[t] + Cos[s/2]Sin[2t]) +
      Cos[s](3 + Cos[s/2]Sin[t] - Sin[s/2]Sin[2t]),
      1/2 (Cos[s/2] - 2Cos[t]Sin[s/2])Sin[t]}

rt = D[r[s, t], t]//Simplify
{Cos[s](Cos[s/2]Cos[t] - 2Cos[2t]Sin[s/2]),
   (Cos[s/2]Cos[t] - 2Cos[2t]Sin[s/2])Sin[s],
     2Cos[s/2]Cos[2t] + Cos[t]Sin[s/2]}

rscrossrt = Cross[rs, rt];
normcross = Sqrt[rscrossrt . rscrossrt];
Clear[un]
un[s_, t_] = -rscrossrt / Sqrt[rscrossrt . rscrossrt];
```

At $t=-\pi$, $\mathbf{n}(-\pi,-\pi) = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\rangle$, whereas at $t=\pi$, $\mathbf{n}(\pi,\pi) = \left\langle -\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \right\rangle$ so as $\mathbf{n}$ moves along $C$ from $\mathbf{r}(-\pi,-\pi)$ to $\mathbf{r}(\pi,\pi)$, the orientation of $\mathbf{n}$ reverses. Several different views of the orientation reversing path on the Klein bottle shown in Figure 5.18(b) are shown in Figure 5.19.

```
l1 = Table[r[s, s], {s, -Pi, Pi, 2Pi/179}];
threedp2 = Show[Graphics3D[{Thickness[.02],
    GrayLevel[.6], Line[l1]}], Axes->Automatic,
      PlotRange->{{-4, 4}, {-4, 4}, {-4, 4}},
        BoxRatios->{4, 4, 1}, AspectRatio->1];
<< "VectorFieldPlots`";
vecs = Table[{r[s, s], un[s, s]}, {s, -π, π, 2π/59}];
```

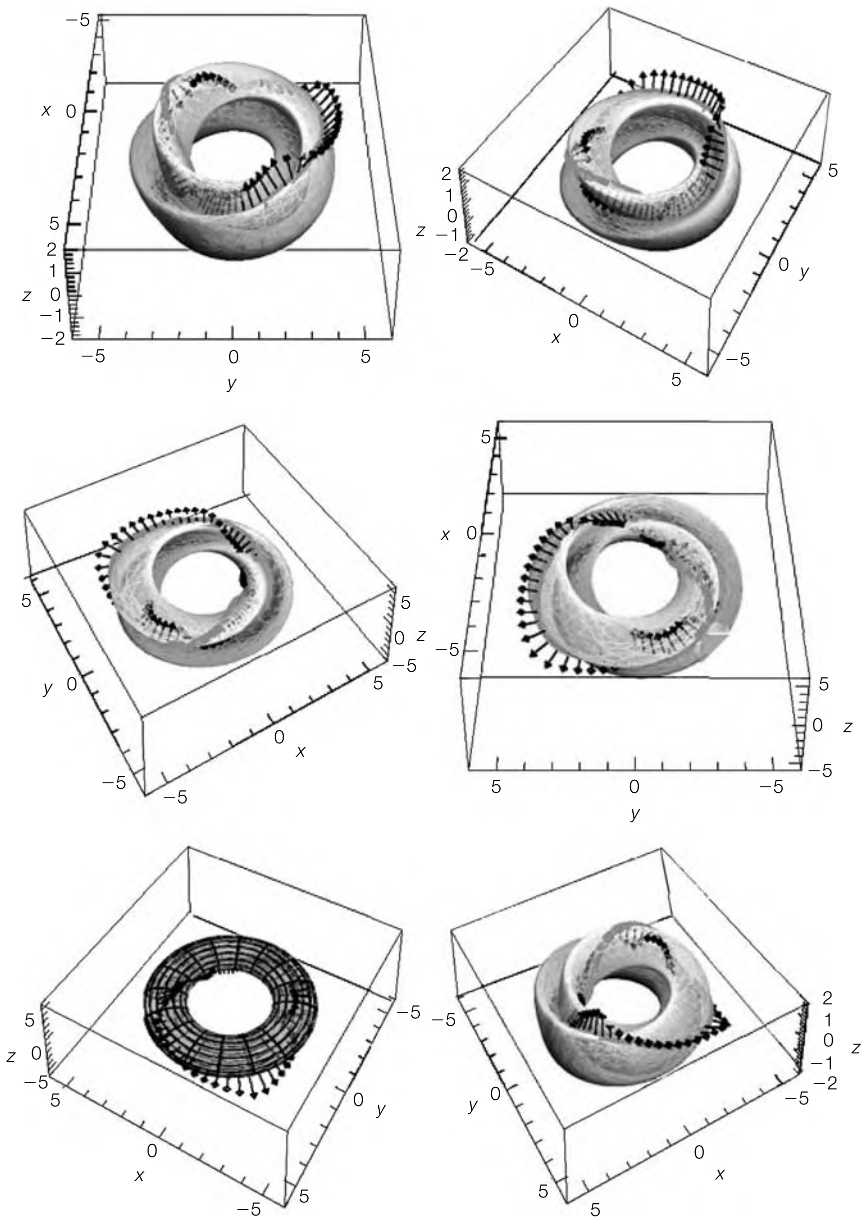



**FIGURE 5.19**

Different views of the figure-8 immersion of the Klein bottle with an orientation reversing path

```
pp2 = ListVectorFieldPlot3D[vecs, VectorHeads → True];
pp3 = Show[threedp2, pp2,
    AxesLabel->{"x", "y", "z"},
        Boxed->False, PlotRange → {{-5, 5},
            {-5, 5}, {-5, 5}}]
g1 = Show[kb2, threedp2, pp2, AspectRatio->1,
    PlotRange->{{-6, 6}, {-6, 6}, {-2, 2}},
        BoxRatios->{4, 4, 1}, AxesLabel->{"x", "y", "z"},
            ViewPoint->{2.729, -0.000, 2.000}]
g2 = Show[kb2, threedp2, pp2,
    AspectRatio->1,
    PlotRange->{{-6, 6}, {-6, 6}, {-2, 2}},
        BoxRatios->{4, 4, 1},
        AxesLabel->{"x", "y", "z"},
            ViewPoint->{1.365, -2.364, 2.000}]
g3 = Show[kb2, threedp2, pp2, AspectRatio->1,
    PlotRange->{{-6, 6}, {-6, 6}, {-6, 6}},
        BoxRatios->{4, 4, 1}, AxesLabel->{"x", "y", "z"},
            ViewPoint->{-1.365, -2.364, 2.000}]
g4 = Show[kb2, threedp2, pp2, AspectRatio->1,
    PlotRange->{{-6, 6}, {-6, 6}, {-6, 6}},
        BoxRatios->{4, 4, 1}, AxesLabel->{"x", "y", "z"},
            ViewPoint->{-2.729, 0.000, 2.000}]
g5 = Show[threedp1t, pp2, AspectRatio->1,
    PlotRange->{{-6, 6}, {-6, 6}, {-6, 6}},
        BoxRatios->{4, 4, 1}, AxesLabel->{"x", "y", "z"},
            ViewPoint->{-1.365, 2.364, 2.000}]
g6 = Show[kb2, pp3, AspectRatio->1,
    PlotRange->{{-6, 6}, {-6, 6}, {-2, 2}},
        BoxRatios->{4, 4, 1}, AxesLabel->{"x", "y", "z"},
            ViewPoint->{1.365, 2.364, 2.000}]
Show[GraphicsGrid[{{g1, g2}, {g3, g4}, {g5, g6}}]]
```

$C$ is an orientation reversing path and we can conclude that the Klein bottle is not orientable.

---

These concepts are presented *beautifully* and extensively for the Mathematica user in *Modern Differential Geometry of Curves and Surfaces with Mathematica*, third edition, by Alfred Gray, Elsa Abbena, and Simon Salamon. Our treatment just touches on a few of the topics discussed by Gray et al and updates some of their wonderful and elegant work to Mathematica 6.

### 5.5.5 More on Tangents, Normals, and Curvature in $\mathcal{R}^3$

Previously, we discussed the unit tangent and normal vectors and curvature for a vector-valued function $\gamma : (a, b) \to \mathcal{R}^2$. These concepts can be extended to curves and surfaces in space.

For $\gamma : (a, b) \to \mathcal{R}^3$, the **Frenet frame field** is the ordered triple $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$, where $\mathrm{T}$ is the **unit tangent vector field**, $\mathrm{N}$ is the **unit**

**normal vector field**, and B is the **unit binormal vector field**. Each of these vectors has norm 1 and each is orthogonal to the other (the dot product of one with another is 0) and the ***Frenet formulas*** are satisfied: $\mathbf{T}' = \kappa\mathbf{N}$, $\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$, $\mathbf{B}' = -\tau\mathbf{N}$. $\tau$ is the **torsion** of the curve $\gamma$; $\kappa$ is the curvature. For the curve $\gamma : (a,b) \to \mathcal{R}^3$, formulas for these quantities are given by

For many good reasons, sometimes the "Frenet formulas" are also called the "Frenet-Serret formulas."

$$\mathbf{T} = \frac{\gamma'}{\|\gamma'\|}, \qquad \mathbf{N} = \mathbf{B} \times \mathbf{T}, \qquad \mathbf{B} = \frac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|},$$
$$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}, \qquad \tau = \frac{\gamma' \times \gamma'' \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2}. \tag{5.25}$$

We adjust Gray's routines slightly for Mathematica 6. Here is the unit tangent vector:

```
tangent[γ_][t_]:=D[γ[tt], tt]/FullSimplify[Norm[D[γ[tt], tt]],
    Assumptions → tt ∈ Reals]/.tt → t
```

Similarly, the binormal is defined with

```
binormal[γ_][t_]:=FullSimplify[
  Cross[D[γ[tt], tt], D[γ[tt], {tt, 2}]]]/
    FullSimplify[Norm[Cross[D[γ[tt], tt], D[γ[tt], {tt, 2}]]],
      Assumptions → tt ∈ Reals]/.tt → t
```

so the unit normal is defined with

```
normal[γ_][t_]:=Cross[binormal[γ][t], tangent[γ][t]];
```

Notice how we use Assumptions to instruct Mathematica to assume that the domain of $\gamma$ consists of real numbers. In the same manner, we define the curvature and torsion.

```
curve2[γ_][t_]:=Simplify[Norm[Cross[D[γ[tt], tt],
  D[γ[tt], {tt, 2}]]]/
    Norm[D[γ[tt], tt]]^3,
      Assumptions → tt ∈ Reals]/.tt → t;

torsion2[γ_][t_]:=Simplify[Cross[D[γ[tt], tt],
  D[γ[tt], {tt, 2}]] . D[γ[tt], {tt, 3}]/
    Norm[Cross[D[γ[tt], tt], D[γ[tt], {tt, 2}]]]^2,
      Assumptions → tt ∈ Reals]/.tt → t;
```

In even the simplest situations, these calculations are quite complicated. Graphically seeing the results may be more meaningful that the explicit formulas.

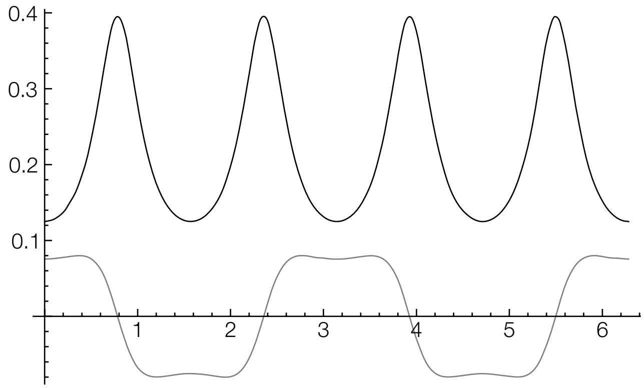



**FIGURE 5.20**

The curvature and torsion for a spherical spiral

**Example 5.5.11** Consider the spherical spiral given by $\gamma(t) = \langle 8\cos 3t\ \cos 2t, 8\sin 3t\ \cos 2t, 8\sin 2t\rangle$. The curvature and torsion for the curve are graphed with Plot and shown in Figure 5.20.

```
<< VectorFieldPlots`;
γ[t_] = {8Cos[3t]Cos[2t], 8Sin[3t]Cos[2t], 8Sin[2t]}
{8Cos[2t]Cos[3t], 8Cos[2t]Sin[3t], 8Sin[2t]}
Plot[Tooltip[{curve2[γ][t], torsion2[γ][t]}], {t, 0, 2Pi},
    PlotStyle → {Black, Gray}]
```

We now compute **T**, **B**, and **N**. For length considerations, we display an abbreviated portion of **B** with Short.

```
tangent[γ][t]
binormal[γ][t]
normal[γ][t]//Short
```

$$\left\{\frac{-16\text{Cos}[3t]\text{Sin}[2t]-24\text{Cos}[2t]\text{Sin}[3t]}{4\sqrt{34+18\text{Cos}[4t]}},\ \frac{24\text{Cos}[2t]\text{Cos}[3t]-16\text{Sin}[2t]\text{Sin}[3t]}{4\sqrt{34+18\text{Cos}[4t]}},\ \frac{4\text{Cos}[2t]}{\sqrt{34+18\text{Cos}[4t]}}\right\}$$

$$\left\{\frac{2(-3\text{Sin}[t]+34\text{Sin}[3t]+15\text{Sin}[7t])}{\sqrt{21886+12456\text{Cos}[4t]+810\text{Cos}[8t]}},\ -\frac{2(3\text{Cos}[t]+34\text{Cos}[3t]+15\text{Cos}[7t])}{\sqrt{21886+12456\text{Cos}[4t]+810\text{Cos}[8t]}},\ \frac{6(21+5\text{Cos}[4t])}{\sqrt{21886+12456\text{Cos}[4t]+810\text{Cos}[8t]}}\right\}$$

$$\left\{ \langle\langle 7\rangle\rangle + \frac{\langle\langle 1\rangle\rangle}{\langle\langle 1\rangle\rangle} + \frac{120\langle\langle 2\rangle\rangle \mathrm{Sin}[3t]}{\sqrt{34 + 18\mathrm{Cos}[\langle\langle 1\rangle\rangle]}\sqrt{1}}, \langle\langle 1\rangle\rangle, \langle\langle 1\rangle\rangle \right\}$$

It is difficult to see how these complicated formulas relate to this spherical spiral. To help us understand what they mean, we first plot the spiral with ParametricPlot3D. See Figure 5.21(a).

```
p1 = ParametricPlot3D[γ[t], {t, 0, 2Pi},
   PlotRange → {{−8.5, 8.5}, {−8.5, 8.5}, {−8.5, 8.5}},
      PlotStyle → {{Gray, Thick}}]
```

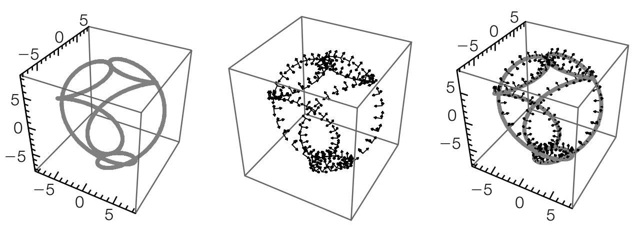
Next, we use Table to compute lists of two ordered triples. For each list, the first ordered triple consists of $\gamma(t)$ and the second the value of $\mathbf{T}(\gamma(t))$ ($\mathbf{B}(\gamma(t))$, $\mathbf{N}(\gamma(t))$). These ordered triples that correspond to vectors are plotted with ListVectorFieldPlot3D, which is contained in the **VectorFieldPlots** package, in Figure 5.21(b).

```
ts = Table[{γ[t], tangent[γ][t]}//N, {t, 0, 2Pi, 2Pi/99}];
bs = Table[{γ[t], binormal[γ][t]}//N, {t, 0, 2Pi, 2Pi/99}];
ns = Table[{γ[t], normal[γ][t]}//N, {t, 0, 2Pi, 2Pi/99}];

ysplot = ListVectorFieldPlot3D[ts, VectorHeads → True];
bsplot = ListVectorFieldPlot3D[bs, VectorHeads → True];
nsplot = ListVectorFieldPlot3D[ns, VectorHeads → True];
p2 = Show[ysplot, bsplot, nsplot]
```

For a good view of p1 and p2, display them together with Show. See Figure 5.21(c).

```
Show[p1, p2]Show[GraphicsRow[{p1, p2, Show[p1, p2]}]]
```

The previous example illustrates that capturing the depth of three-dimensional curves by projections into two dimensions can be difficult. Sometimes taking advantage of three-dimensional surface plots can help.



a b c

**FIGURE 5.21**

(a) The spherical spiral. (b) Various $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ for the spherical spiral. (c) The spherical spiral with various $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ shown together

For a basic space curve, tubecurve places a "tube" of radius $r$ around the space curve.

**Clear[tubecurve]**
**tubecurve[γ_][r_][t_, θ_] = γ[t] +**
**r(Cos[θ]normal[γ][t] + Sin[θ]binormal[γ][t])**

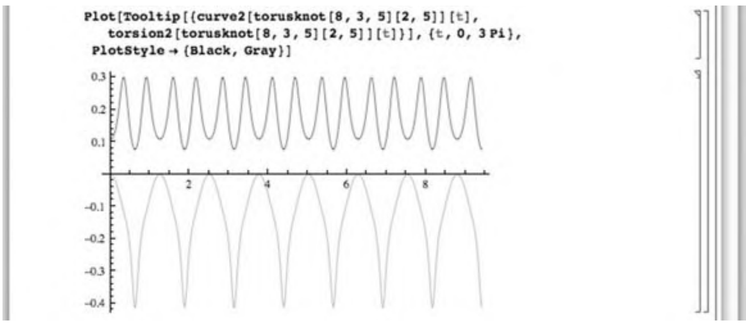
$$r\left(\mathrm{Cos}[\theta]\frac{\gamma'[t]\times\gamma''[t]}{\mathrm{Norm}[\gamma'[t]\times\gamma''[t]]}\times\frac{\gamma'[t]}{\mathrm{Norm}[\gamma'[t]]}+\frac{\gamma'[t]\times\gamma''[t]\mathrm{Sin}[\theta]}{\mathrm{Norm}[\gamma'[t]\times\gamma''[t]]}\right)+\gamma[t]$$

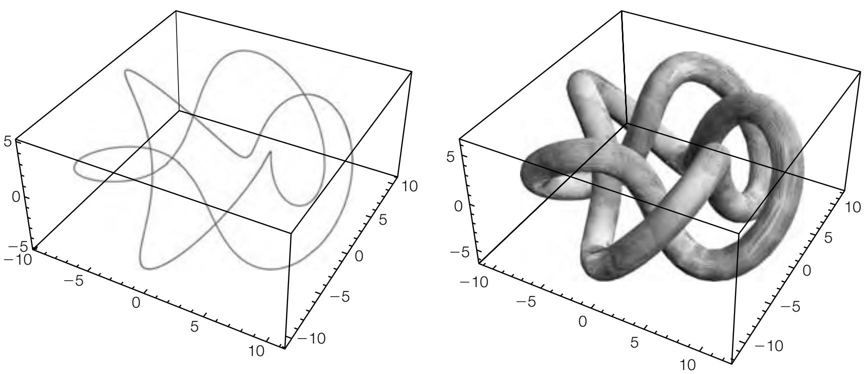
The results displayed in the text are in black-and-white and do not reflect the stunning color images generated by these commands.

To illustrate the utility, we redefine torusknot that was presented in Chapter 2.

**torusknot[a_, b_, c_][p_, q_][t_]:=**
**{(a + b Cos[q t])Cos[p t], (a + b Cos[q t])Sin[p t],**
**c Sin[qt]}**

**Example 5.5.12** For the knot torusknot}[8,3,5][2,5] we plot the curvature and torsion with Plot.



a b

**FIGURE 5.22**

(a) A basic plot of a curve in 3-space. (b) Placing a "tube" around the curve

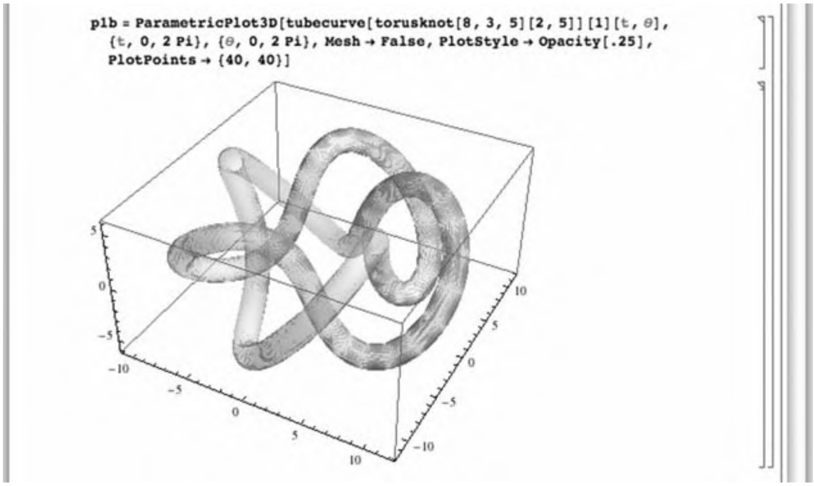
We generate a basic plot of this torus knot in 3-space with ParametricPlot3D. See Figure 5.22(a).

**ParametricPlot3D[torusknot[8, 3, 5][2, 5][t], {t, 0, 3Pi}]**

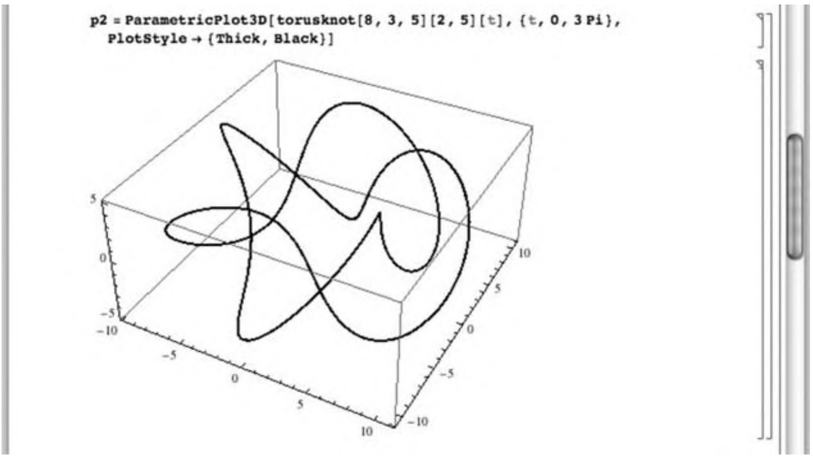
Using tubeplot, we place a "tube" around the knot. See Figure 5.22(b).

**p1 = ParametricPlot3D[tubecurve[torusknot[8, 3, 5][2, 5]][1 . 3][t, θ], {t, 0, 2Pi}, {θ, 0, 2Pi}, Mesh → False, PlotStyle → Opacity[.5], PlotPoints → {40, 40}]**

A more interesting graphic is obtained by placing a transparent tube around the curve

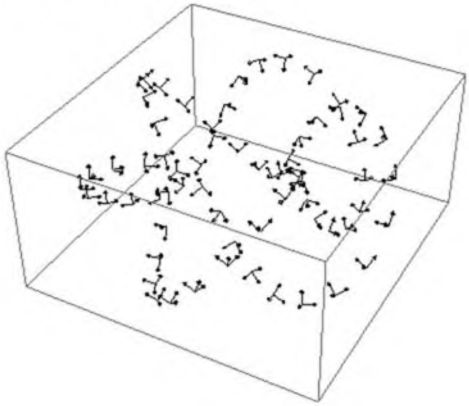



and then creating a thicker version of the curve.



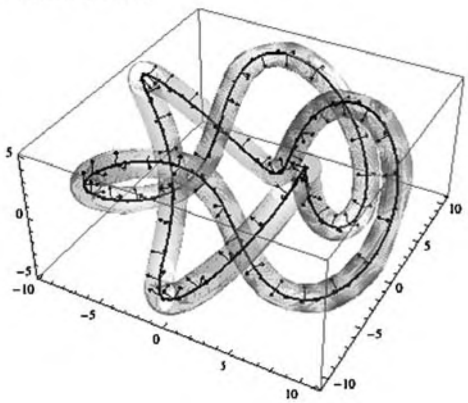

As before, we use tangent, normal, and binormal to create a vector field on the curve.

```
ts = Table[{torusknot[8, 3, 5][2, 5][t],
     tangent[torusknot[8, 3, 5][2, 5]][t]} // N, {t, 0, 3 Pi, 3 Pi / 99}];
bs = Table[{torusknot[8, 3, 5][2, 5][t],
     binormal[torusknot[8, 3, 5][2, 5]][t]} // N, {t, 0, 3 Pi, 3 Pi / 99}];
ns = Table[{torusknot[8, 3, 5][2, 5][t],
     normal[torusknot[8, 3, 5][2, 5]][t]} // N, {t, 0, 3 Pi, 3 Pi / 99}];

ysplot = ListVectorFieldPlot3D[ts, VectorHeads → True];
bsplot = ListVectorFieldPlot3D[bs, VectorHeads → True];
nsplot = ListVectorFieldPlot3D[ns, VectorHeads → True];
p3 = Show[ysplot, bsplot, nsplot]
```



A striking graph is generated by showing the three graphs together.

```
Show[p2, p1b, p3]
```



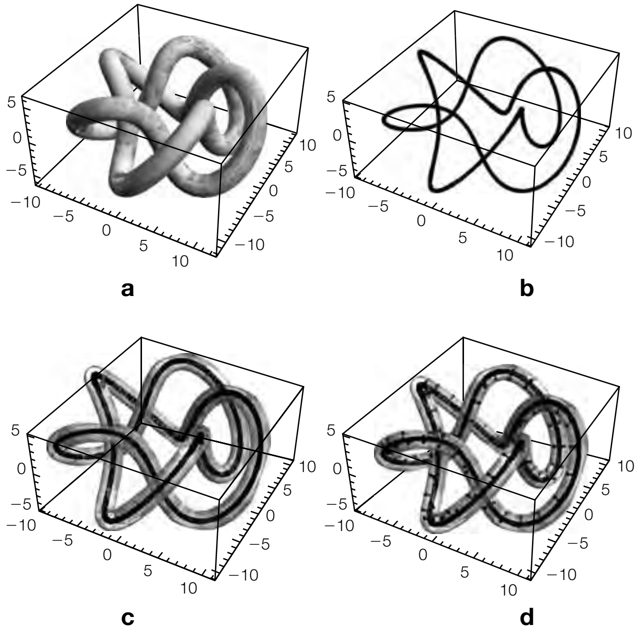



**FIGURE 5.23**

(a) A “tubed” knot. (b) A thick knot. (c) A knot within a tube around it. (d) A knot within a tube illustrating the Frenet field

Alternatively, display the results as an array with GraphicsGrid. See Figure 5.23.

```
Show[GraphicsGrid[{{p1, p2}, {Show[p2, p1b], Show[p2, p1b, p3]}}]]
```

**Example 5.5.13** The **Trefoil knot** is the special case of torusknot[8,3,5][2,3]][t]. We use Plot to graph its curvature and torsion in Figure 5.24. Because we have used Tooltip, you can identify each plot by moving the cursor over the curve in Figure 5.24.

```
Plot[Tooltip[{curve2[torusknot[8, 3, 5][2, 3]][t],
   torsion2[torusknot[8, 3, 5][2, 3]][t]}], {t, 0, 2Pi}, PlotRange → All]
```

Next, we generate a thickened version of the Trefoil knot.

```
p1 = ParametricPlot3D[torusknot[8, 3, 5][2, 3][t], {t, 0, 2Pi},
    PlotStyle → {Black, Thick}]
```

Three different tube plots of the Trefoil knot are generated. In p2, the result is a basic plot. In p2b, the plot is shaded accoring to the Rainbow color gradient.

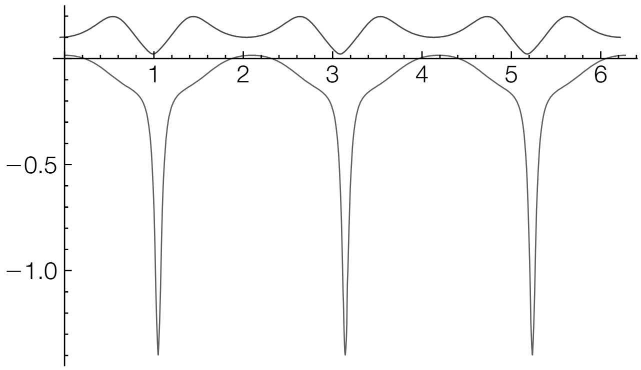



**FIGURE 5.24**

The curvature and torsion for the Trefoil knot


**FIGURE 5.25**

(a) The Trefoil knot with a tube around it. (b) Changing the color of the tube. (c) Coloring the knot according to its curvature

In p2c, the plot is shaded according to the knots curvature. The knot together with the three surfaces are shown in Figure 5.25.

```
p2 = ParametricPlot3D[tubecurve[torusknot[8, 3, 5][2, 3]][1.3][t, θ],
  {t, 0, 2Pi}, {θ, 0, 2Pi}, Mesh → False, PlotStyle → Opacity[.5],
    PlotPoints → {40, 40}]
p2b = ParametricPlot3D[tubecurve[torusknot[8, 3, 5][2, 3]][1.3][t, θ],
  {t, 0, 2Pi}, {θ, 0, 2Pi}, Mesh → False, PlotStyle → Opacity[.5],
    PlotPoints → {40, 40}, ColorFunction –>ColorData[“Rainbow”]]
p2c = ParametricPlot3D[tubecurve[torusknot[8, 3, 5][2, 3]][1.3][t, θ],
  {t, 0, 2Pi}, {θ, 0, 2Pi}, Mesh → False, PlotStyle → Opacity[.5],
    PlotPoints → {40, 40}, ColorFunction →
      (ColorData[“BrightBands”][curve2[torusknot[8, 3, 5][2, 3]][#1]]&)]
ba1 = Show[p2, p1, Boxed → False, Axes → None]
ba2 = Show[p2b, p1, Boxed → False, Axes → None]
ba3 = Show[p2c, p1, Boxed → False, Axes → None]
Show[GraphicsRow[{ba1, ba2, ba3}]]
```

For surfaces in $\mathcal{R}^3$, extending and stating these definitions precisely becomes even more complicated. First, define the **vector triple product** (xyz), where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, and $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, by $(\mathbf{xyz}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$. We assume that $\gamma = \gamma(u, v)$ is a vector-valued function with domain contained in a "nice" region $U \subset \mathcal{R}^2$ and range in $\mathcal{R}^3$. The **Gaussian curvature**, $\mathcal{K}$, and the **mean curvature**, $\mathcal{H}$, under reasonable conditions, are given by the formulas

$$\mathcal{K} = \frac{\left(\gamma_{uu}\gamma_u\gamma_v\right)\left(\gamma_{vv}\gamma_u\gamma_v\right) - \left(\gamma_{uv}\gamma_u\gamma_v\right)^2}{\left(\|\gamma_u\|^2\|\gamma_v\|^2 - \left(\gamma_u \cdot \gamma_v\right)^2\right)^2}$$

and (5.26)

$$\mathcal{H} = \frac{\left(\gamma_{uu}\gamma_u\gamma_v\right)\|\gamma_v\|^2 - 2\left(\gamma_{uv}\gamma_u\gamma_v\right)\left(\gamma_u \cdot \gamma_v\right) + \left(\gamma_{vv}\gamma_u\gamma_v\right)\|\gamma_u\|^2}{2\left(\|\gamma_u\|^2\|\gamma_v\|^2 - \left(\gamma_u \cdot \gamma_v\right)^2\right)^{3/2}}.$$

For the parametrically defined surface $\gamma = \gamma(u, v)$, the **unit normal field**, U, is $\mathbf{U} = \dfrac{\gamma_u \times \gamma_v}{\|\gamma_u \times \gamma_v\|}$. Observe that the expressions that result from explicitly computing U, $\mathcal{K}$, and $\mathcal{H}$ are almost always so complicated that they are impossible to understand.

After defining vtp to return the vector triple product of three vectors, we define gaussianc and meanc to compute $\mathcal{K}$ and $\mathcal{H}$ for a parametrically defined surface $\gamma(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$.

```
vtp[x_, y_, z_]:=Det[{{x[[1]], x[[2]], x[[3]]},
{y[[1]], y[[2]], y[[3]]}, {z[[1]], z[[2]], z[[3]]}}]

gaussianc[γ_][u_, v_]:=
Module[{lu, lv, vtp},
vtp[x_, y_, z_]:=Det[{{x[[1]], x[[2]], x[[3]]},
   {y[[1]], y[[2]], y[[3]]}, {z[[1]], z[[2]], z[[3]]}}];
(vtp[D[γ[lu, lv], {lu, 2}], D[γ[lu, lv], lu], D[γ[lu, lv], lv]]
   vtp[D[γ[lu, lv], {lv, 2}], D[γ[lu, lv], lu], D[γ[lu, lv], lv]] –
      vtp[D[γ[lu, lv], lu, lv], D[γ[lu, lv], lu], D[γ[lu, lv], lv]] ^ 2) /
   (Norm[D[γ[lu, lv], lu]] ^ 2Norm[D[γ[lu, lv], lv]] ^ 2 –
(D[γ[lu, lv], lu] . D[γ[lu, lv], lv]) ^ 2) ^ 2/.
      {lu → u, lv → v}//PowerExpand//Simplify
         ]
```

```
meanc[γ_][u_, v_]:=
Module[{lu, lv, vtp},
vtp[x_, y_, z_]:=Det[{{x[[1]], x[[2]], x[[3]]},
   {y[[1]], y[[2]], y[[3]]}, {z[[1]], z[[2]], z[[3]]}}];
(vtp[D[γ[lu, lv], {lu, 2}], D[γ[lu, lv], lu], D[γ[lu, lv], lv]]
   Norm[D[γ[lu, lv], lv]] ^ 2 −
      2vtp[D[γ[lu, lv], lu, lv], D[γ[lu, lv], lu], D[γ[lu, lv], lv]]
   (D[γ[lu, lv], lu] . D[γ[lu, lv], lv]) +
vtp[D[γ[lu, lv], {lv, 2}], D[γ[lu, lv], lu], D[γ[lu, lv], lv]]
   Norm[D[γ[lu, lv], lu]] ^ 2) /
      (2(Norm[D[γ[lu, lv], lu]] ^ 2Norm[D[γ[lu, lv], lv]] ^ 2 −
   (D[γ[lu, lv], lu] . D[γ[lu, lv], lv]) ^ 2) ^ (3/2))/.
{lu → u, lv → v}//PowerExpand//Simplify
      ]
```

**Example 5.5.14** We illustrate the commands with the torus, first discussed in Chapter 2, and ParametricPlot3D. For convenience, we redefine torus.

```
torus[a_, b_, c_][p_, q_][u_, v_]:={(a + bCos[u])
     Cos[v], (a + b Cos[u])Sin[v], c Sin[u]}
```

In pp1, we generate a basic plot of the torus. The shading is changed in pp2. In pp3 the surface is shaded according to its Gaussian curvature, whereas in pp4 the surface is shaded according to its mean curvature. All four plots are shown together in Figure 5.26.

```
pp1 = ParametricPlot3D[Evaluate[torus[8, 3, 5][2, 5][u, v]], {u, 0, 2Pi},
     {v, 0, 2Pi}, PlotPoints → 60]
pp2 = ParametricPlot3D[torus[8, 3, 5][2, 5][u, v],
{u, 0, 2Pi}, {v, 0, 2Pi}, Mesh → False, PlotStyle → Opacity[.75],
PlotPoints → {25, 25}, ColorFunction →
ColorData["MintColors"]]
pp3 = ParametricPlot3D[torus[8, 3, 5][2, 5][u, v],
{u, 0, 2Pi}, {v, 0, 2Pi}, Mesh → False, PlotStyle → Opacity[.5],
PlotPoints → {25, 25}, ColorFunction →
(ColorData["MintColors"][gaussianc[torus[8, 3, 5][2, 5]]
    [#1, #2]//N//Chop]&)]
pp4 = ParametricPlot3D[torus[8, 3, 5][2, 5][u, v],
   {u, 0, 2Pi}, {v, 0, 2Pi}, Mesh → False, PlotStyle → Opacity[.5],
      PlotPoints → {25, 25}, ColorFunction →
      (ColorData["MintColors"][meanc[torus[8, 3, 5][2, 5]]
            [#1, #2]//N//Chop]&)]
Show[GraphicsGrid[{{pp1, pp2}, {pp3, pp4}}]]
```

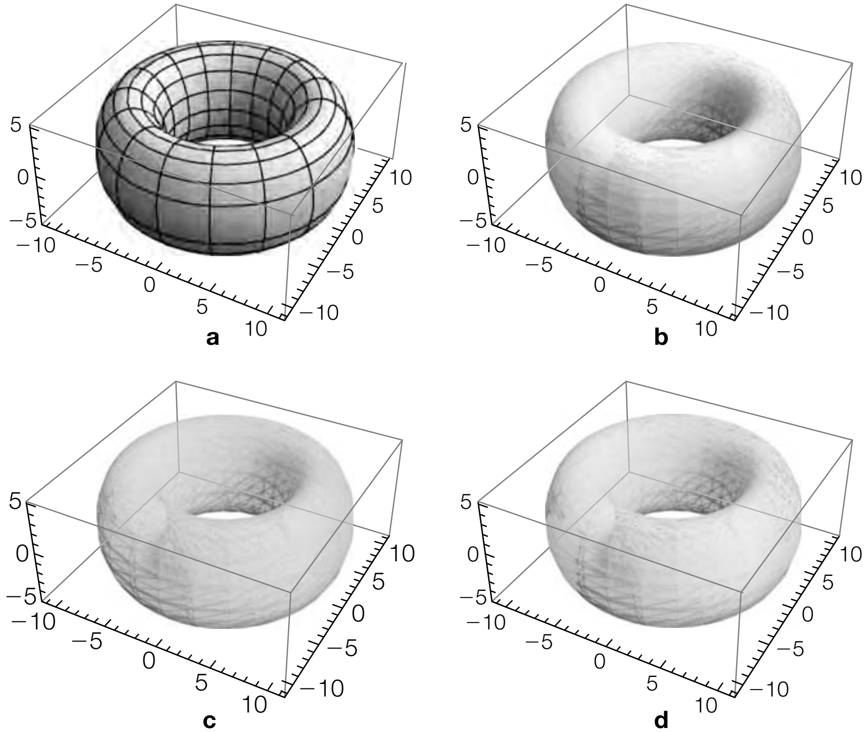



**FIGURE 5.26**

(a) A basic torus. (b) Changing the coloring of the torus. (c) Shading according to Gaussian curvature. (d) Shading according to mean curvature

## 5.6 MATRICES AND GRAPHICS

Mathematica contains several functions that allow you to represent matrices graphically. These commands are analogous to the corresponding ones for dealing with lists (such as ListPlot) or functions (such as Plot, Plot3D, and ContourPlot).

1. MatrixPlot[A] generates a grid with the same dimensions as **A**. The cells are shaded according to the entries of **A**. The default is in color.
2. ArrayPlot[A] generates a grid with the same dimensions as **A**. The cells are shaded according to the entries of **A**. The default is in black and white.
3. ListContourPlot[A] generates a contour plot using the entries of **A** as the height values.
4. ReliefPlot[A] generates a relief plot using the entries of **A** as the height values.

Because the figures in the text are in black and white, refer to the CD that accompanies the text to see the images in color.

Observe that ArrayPlot and MatrixPlot are virtually interchangeable. However, the entries of ArrayPlot need not be numbers. If Mathematica cannot determine how to shade a cell, the default is to shade it in a dark maroon color. Although these functions generate graphics that depend on the entries of the matrix, loosely speaking we will use phrases such as "we use MatrixPlot to plot $\mathbf{A}$" and "we use ArrayPlot to graph $\mathbf{A}$" to describe the graphic that results from applying one of these functions to an array.

For example, consider the arrays $\mathbf{A} = \begin{pmatrix} 1 & 0 & .3 \\ .4 & .5 & .1 \\ .2 & .3 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ .1 & .2 & .3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} .3 & .4 & .5 \end{pmatrix} & \begin{pmatrix} .1 & .2 & .3 \end{pmatrix} \end{pmatrix}$.

In the first command, Mathematica shades all the cells according to its GrayLevel value. However, in the second and third commands, Mathematica cannot shade the cells in the second row and all the cells, respectively, because ordered triples cannot be evaluated by GrayLevel. However, RGBColor evaluates ordered triples so Mathematica shades the cells in Figure 5.27(c) according to their RGBColor value.

```
ap1 = ArrayPlot[{{1, 0, .3}, {.4, .5, .1}, {.2, .3, 0}}];
ap2 = ArrayPlot[{{1, 0}, {{.3, .4, .5}, {.1, .2, .3}}}];
ap3 = ArrayPlot[{{{1, 0, 0}, {0, 1, 0}}, {{.3, .4, .5},
      {.1, .2, .3}}}];
ap4 = ArrayPlot[{{{1, 0, 0}, {0, 1, 0}}, {{.3, .4, .5},
     {.1, .2, .3}}},
        ColorFunction → RGBColor];
Show[GraphicsRow[{ap1, ap2, ap3, ap4}]]
```

MatrixPlot is unable to graphically represent $\mathbf{B}$ or $\mathbf{C}$. However, coloring is automatic with MatrixPlot. See Figure 5.28.

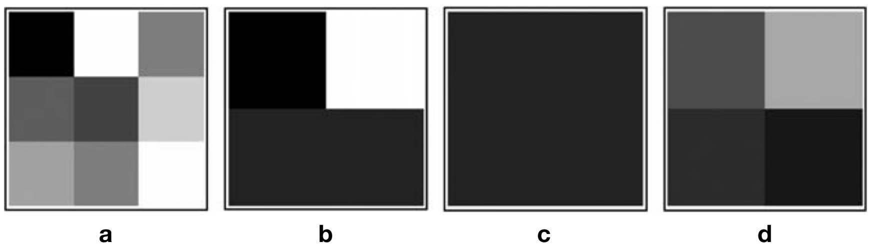



**FIGURE 5.27**

(a) Mathematica shades all cells according to their heights. (b) Mathematica does not know how to shade the cells in the second row. (c) Mathematica cannot shade any of the cells. (d) Mathematica shades all four cells using RGBColor

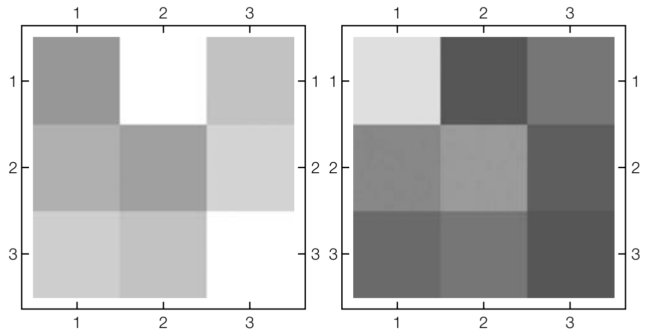



**FIGURE 5.28**

By default, MatrixPlot uses a color scheme. Use ColorFunction to change the colors

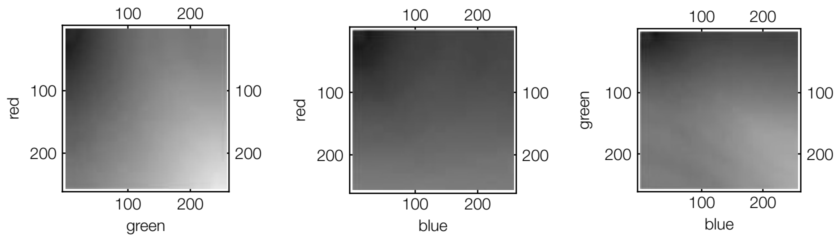



**FIGURE 5.29**

A comparison of how red, green, and blue affect RGBColor[r,g,b]

```
mp1 = MatrixPlot[{{1, 0, .3}, {.4, .5, .1}, {.2, .3, 0}}];
mp2 = MatrixPlot[{{1, 0, .3}, {.4, .5, .1}, {.2, .3, 0}},
    ColorFunction → "PlumColors"];
Show[GraphicsRow[{mp1, mp2}]]
```

In these calculations, t1 is a 256 × 256 array for which each entry is an ordered triple. In the first t1, the ordered triple has the form $(r, g, 0)$, in the second the form $(r, 0, b)$, and so on.

If you need to adjust the color of a graphic, usually you can use the **ColorSchemes** palette to select an appropriate gradient or color function. In other situations, you might wish to create your own using Blend. To use Blend, you might need to know how various RGBColors or CMYKColors vary as the variables affecting the color change.

ArrayPlot can help us see the variability in the colors. With the following, we see how RGBColor[r,g,b] affects color for $b = 0$, $g = 0$, and then $r = 0$. The results are shown together in Figure 5.29. The figure can help us select appropriate values to generate our own color blending function using Blend rather than relying on Mathematica's built-in color schemes and gradients.

```
t1 = Table[{r, g, 0}//N, {r, 0, 255}, {g, 0, 255}];
redgreen = ArrayPlot[t1, Axes → Automatic, AxesOrigin → {0, 0},
```

```
    FrameTicks → Automatic, FrameLabel → {red, green},
        LabelStyle → Medium, ColorFunction → RGBColor];
t2 = Table[{r, 0, b}//N, {r, 0, 255}, {b, 0, 255}];
redblue = ArrayPlot[t2, Axes → Automatic, AxesOrigin → {0, 0},
    FrameTicks → Automatic, FrameLabel → {red, blue},
        LabelStyle → Medium, ColorFunction → RGBColor];
t2 = Table[{0, g, b}//N, {g, 0, 255}, {b, 0, 255}];
greenblue = ArrayPlot[t2, Axes → Automatic, AxesOrigin → {0, 0},
    FrameTicks → Automatic, FrameLabel → {green, blue},
        LabelStyle → Medium, ColorFunction → RGBColor];
s1 = Show[GraphicsRow[{redgreen, redblue, greenblue}]]
```

We modify the calculation slightly to see how CMYKColor varies as we adjust two parameters. Keep in mind that each t2 is a $256 \times 256$ array. Each entry of t2 is an ordered quadruple, which is illustrated in the first calculation, in which we use Part to take the fifth element of the eighth part of t2. (See Figure 5.30.)

```
t2 = Table[{c, m, 0, 0}//N, {c, 0, 255}, {m, 0, 255}];
t2[[8, 5]]
{7., 4., 0., 0 . }
cmplot = ArrayPlot[t2, Axes → Automatic, AxesOrigin → {0, 0},
    FrameTicks → Automatic, FrameLabel → {c, m},
        LabelStyle → Medium, ColorFunction → CMYKColor];
t2 = Table[{c, 0, y, 0}//N, {c, 0, 255}, {y, 0, 255}];
```

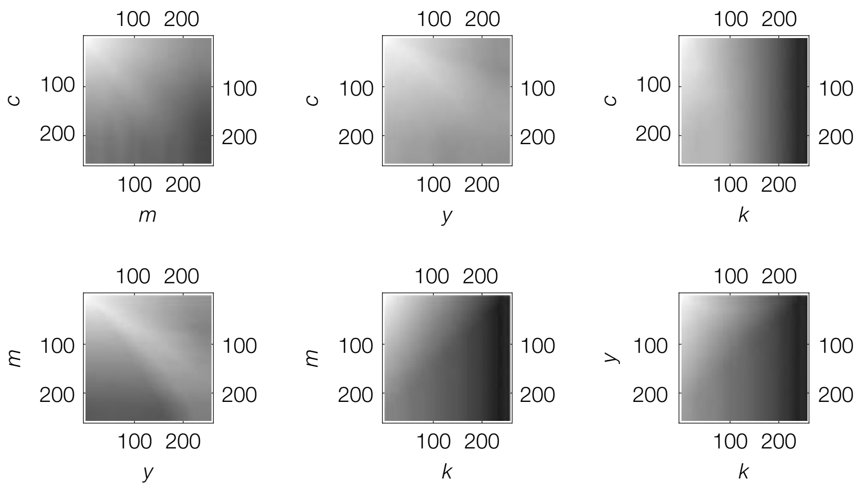



**FIGURE 5.30**

A comparison of how $c$, $m$, $y$, and $k$ affect CMYKColor[c,m,y,k]

```
cyplot = ArrayPlot[t2, Axes → Automatic, AxesOrigin → {0, 0},
    FrameTicks → Automatic, FrameLabel → {c, y},
        LabelStyle → Medium, ColorFunction → CMYKColor];
t2 = Table[{c, 0, 0, k}//N, {c, 0, 255}, {k, 0, 255}];
ckplot = ArrayPlot[t2, Axes → Automatic, AxesOrigin → {0, 0},
    FrameTicks → Automatic, FrameLabel → {c, k},
        LabelStyle → Medium, ColorFunction → CMYKColor];
t2 = Table[{0, m, y, 0}//N, {m, 0, 255}, {y, 0, 255}];
myplot = ArrayPlot[t2, Axes → Automatic, AxesOrigin → {0, 0},
    FrameTicks → Automatic, FrameLabel → {m, y},
        LabelStyle → Medium, ColorFunction → CMYKColor];
t2 = Table[{0, m, 0, k}//N, {m, 0, 255}, {k, 0, 255}];
mkplot = ArrayPlot[t2, Axes → Automatic, AxesOrigin → {0, 0},
    FrameTicks → Automatic, FrameLabel → {m, k},
        LabelStyle → Medium, ColorFunction → CMYKColor];
t2 = Table[{0, 0, y, k}//N, {y, 0, 255}, {k, 0, 255}];
ykplot = ArrayPlot[t2, Axes → Automatic, AxesOrigin → {0, 0},
    FrameTicks → Automatic, FrameLabel → {y, k},
        LabelStyle → Medium, ColorFunction → CMYKColor];
Show[GraphicsGrid[{{cmplot, cyplot, ckplot}, {myplot, mkplot, ykplot}}]]
```

You can load files into Mathematica with Import. Generally, the underlying structure of the loaded file is relatively easy to understand. Be careful when you import data into Mathematica. We recommend that you use ExampleData to investigate your routines before finalizing them. Although importing external files into Mathematica is easy, understanding the underlying structure of the imported data may take some time but may be necessary to produce the results you desire.

We illustrate a few of the subtle differences that can be encountered with several color and black-and-white gifs and jpegs.

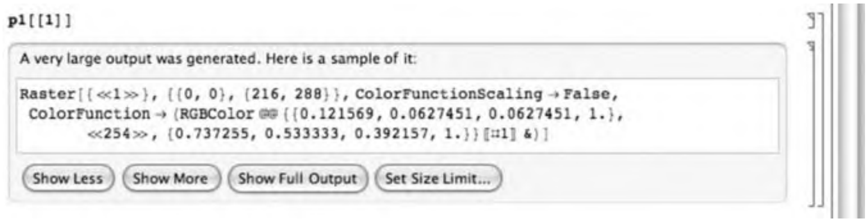
Using Import, we import a graphic of the primary author into Mathematica. The result is shown in Figure 5.31(a).

```
p1 = Import["martha01sa . gif"]
```

With Length, we see that p1 has four elements. We can examine the entries with Part.

```
Length[p1]
4
```

With p1[[1]], we select the first part of p1. When large output is the result of a calculation, Mathematica warns you before displaying it. (Note that Short[p1[[1]]] returns a similar result.)




The result helps us understand the structure of p1 and p1[[1]], which is another list. The second element of p1 tells us the size of the image.

**p1[[2]]**
ImageSize → {216, 288}

The third and fourth elements tell us how the image is to be plotted—its plot range and background.

**p1[[3]]**
PlotRange → {{0, 216}, {0, 288}}
**p1[[4]]**
Background → None

The data determining the image is contained in p1[[1,1]], which is a 288 × 216 matrix. To see so, first enter p1[[1,1]]


and then click on **Show More** twice.
To determine the dimensions of the matrix use Length.
We now use ArrayPlot to graph p1[[1,1]]. ArrayPlot goes from up (first row) to down (last row), so our initial image (Figure 5.31(b)) is a reverse of the original.

**g1a = ArrayPlot[p1[[1, 1]]]**

We use Reverse to correct the situation (Figure 5.31(c)). Generally, Reverse[{a1, a2, ..., an}] returns the list {an, ..., a2, a1}; the reverse of the original list.

**g1b = ArrayPlot[Reverse[p1[[1, 1]]]]**

```
p1[[1, 1]]

A very large output was generated. Here is a sample of it:

{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
  1, 1, 2, 2, 58, 58, 58, 120, 120, 120, 172, 172, 136, 136, 59, 59, 59, 59, 59,
  59, 59, 136, 256, 136, 136, 136, 179, 172, 172, 176, 132, 10, 7, 45, 193, 132,
  170, 125, 132, 42, 24, 7, 168, 168, 165, 165, 191, 210, 163, 57, 147, 144, 138,
  6, 175, 175, 175, 6, 133, 191, 191, 145, 147, 57, 147, 191, 191, 191, 191, 191,
  145, 193, 170, 176, 132, 165, 165, 6, 6, 139, 139, 189, 145, 145, 145, 145, 8,
  8, 8, 8, 8, 8, 8, 8, 57, 191, 191, 191, 191, 191, 191, 191, 191, 191, 191, 191,
  155, 155, 76, 151, 62, 62, 94, 175, 6, 154, 147, 147, 191, 138, 165, 165, 165,
  7, 165, 165, 134, 120, 172, 179, 136, 136, 136, 136, 178, 178, 183, 178, 178,
  202, 136, 136, 172, 172, 172, 173, 134, 58, 58, 58, 2, 3, 1, 1, 1, 1, 1, 1, 1,
  1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
 <<286>>, {3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 22, 24, 22, 3, 22, 22, 3, 22, 36, 3,
  22, 22, 24, 24, 24, 22, 3, 22, 39, 39, 24, 24, 12, 24, 24, 24, 39, 7, 39, 39,
  12, 24, 7, 39, 24, 24, 7, 24, 24, 7, 24, 39, 7, 24, 7, 24, 24, 24, 24, 7, 7,
  24, 7, 7, 7, 10, 7, 7, 24, 7, 28, 7, 28, 7, 7, 7, 7, 24, 24, 10, 41, 7, 7, 10,
  28, 7, 7, 7, 7, 7, 7, 7, 28, 28, 28, 7, 7, 7, 7, 7, 3, 7, 7, 7, 7, 7, 7, 7, 7,
  7, 7, 9, 9, 9, 3, 9, 28, 3, 9, 9, 3, 9, 9, 3, 3, 3, 3, 28, 3, 3, 30, 30, 30,
  30, 3, 3, 30, 30, 30, 3, 30, 30, 30, 30, 30, 30, 30, 30, 32, 30, 32, 32, 3,
  3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21,
  21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 16, 16, 16, 16, 16, 16,
  16, 16, 16, 16, 16, 16, 16, 16, 16, 1, 1, 1, 1, 1, 16, 1, 1, 1, 1, 1, 1, 1}}

Show Less   Show More   Show Full Output   Set Size Limit...
```

```
Length[p1[[1, 1]]]
Length[p1[[1, 1, 1]]]

288

216
```

Now that we have oriented the image in the way we want, we appy a color function to it by applying one of the built-in color gradients to the list. We choose Pastel.

**g2a = ArrayPlot[Reverse[p1[[1, 1]]], ColorFunction → "Pastel"]**

The results are shown side-by-side in Figure 5.31. Printed on a color printer, the results are amazing.

**Show[GraphicsRow[{p1, g1a, g1b, g2a}]]**

**a** **b** **c** **d**

**FIGURE 5.31**

(a) The original. (b) Applying ArrayPlot to the original data points. (c) Reorienting the image. (d) Applying a color function to the data points

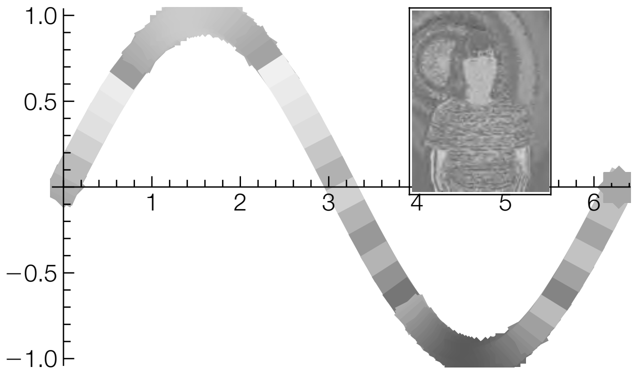



**FIGURE 5.32**

Use Inset to place one graphic within another

Now that we understand how to manipulate the gif image, we can be creative. In the following, the image is scaled so that the width of the image is 70 pixels (because of ImageSize->70). We then display the small image with another graphic. Using Inset, we put Martha next to a sine graph that is plotted using the same coloring gradient. See Figure 5.32.

```
g1 = ArrayPlot[p1[[1, 1]], ColorFunction → "BrightBands",
      ImageSize → 70];
p2 = Plot[Sin[x], {x, 0, 2Pi}, Epilog –>Inset[g1, {3Pi/2, 1/2}],
   ColorFunction → "BrightBands", PlotStyle → Thickness[.05]]
```

An alternative way to visualize the data is to use ListContourPlot. To ensure that the aspect ratio of the original image is preserved, include the AspectRatio->Automatic option in the ListContourPlot command. (See Figure 5.33)

```
glp1 = ListContourPlot[p1[[1, 1]], AspectRatio → Automatic];
g1p2 = ListContourPlot[p1[[1, 1]],ColorFunction →  "Pastel",
   AspectRatio → Automatic]
g1p3 = ListContourPlot[p1[[1, 1]],ColorFunction →  "GrayTones",
   AspectRatio → Automatic]
Show[GraphicsRow[{glp1, g1p2, g1p3}]]
```

The underlying format of each structure (jpeg, gif, etc.) is different. With

```
p1 = Import["jim01a.jpg"];
Length[p1]
```

3

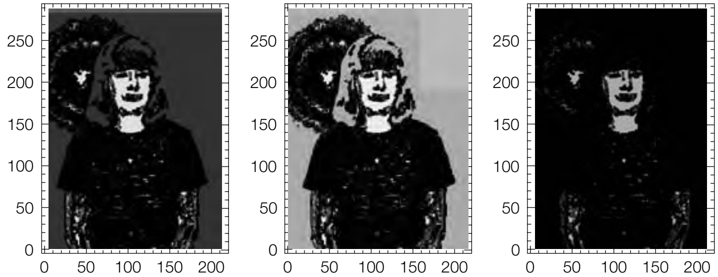



**FIGURE 5.33**

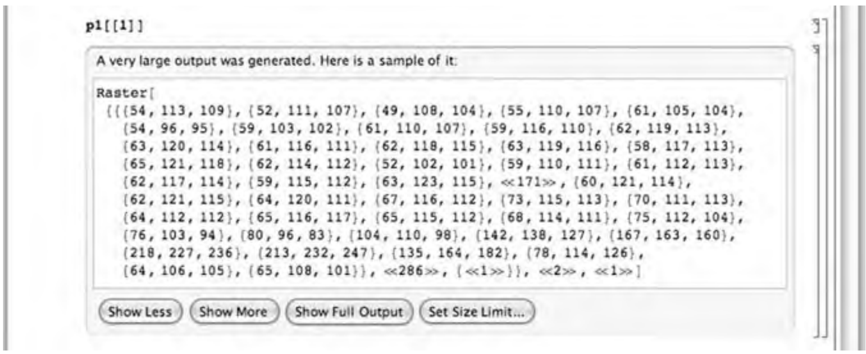
Using ListContourPlot rather than ArrayPlot

we import a jpeg of the second author of the text into Mathematica and name the result p1. Length shows us that p1 is a list with three elements. (See Figure 5.34)

The first part of p1, obtained with p1[[1]], is quite long.

```
p1[[1]]
A very large output was generated. Here is a sample of it:
Raster[{«1»}, {{0, 0}, {216, 288}}, {0, 255}, ColorFunction → RGBColor]
Show Less   Show More   Show Full Output   Set Size Limit...
```

By clicking on **Show More** we see that p1[[1]] is an array of ordered triples.



```
p1[[1]]
A very large output was generated. Here is a sample of it:
Raster[
 {{{54, 113, 109}, {52, 111, 107}, {49, 108, 104}, {55, 110, 107}, {61, 105, 104},
   {54, 96, 95}, {59, 103, 102}, {61, 110, 107}, {59, 116, 110}, {62, 119, 113},
   {63, 120, 114}, {61, 116, 111}, {62, 118, 115}, {63, 119, 116}, {58, 117, 113},
   {65, 121, 118}, {62, 114, 112}, {52, 102, 101}, {59, 110, 111}, {61, 112, 113},
   {62, 117, 114}, {59, 115, 112}, {63, 123, 115}, «171», {60, 121, 114},
   {62, 121, 115}, {64, 120, 111}, {67, 116, 112}, {73, 115, 113}, {70, 111, 113},
   {64, 112, 112}, {65, 116, 117}, {65, 115, 112}, {68, 114, 111}, {75, 112, 104},
   {76, 103, 94}, {80, 96, 83}, {104, 110, 98}, {142, 138, 127}, {167, 163, 160},
   {218, 227, 236}, {213, 232, 247}, {135, 164, 182}, {78, 114, 126},
   {64, 106, 105}, {65, 108, 101}}, «286», {«1»}}, «2», «1»]
Show Less   Show More   Show Full Output   Set Size Limit...
```

The other parts of p1 are not as long. They specify the style of the image.

**p1[[1, 2]]**

{{0, 0}, {216, 288}}

**p1[[1, 3]]**

{0, 255}

**p1[[1, 4]]**

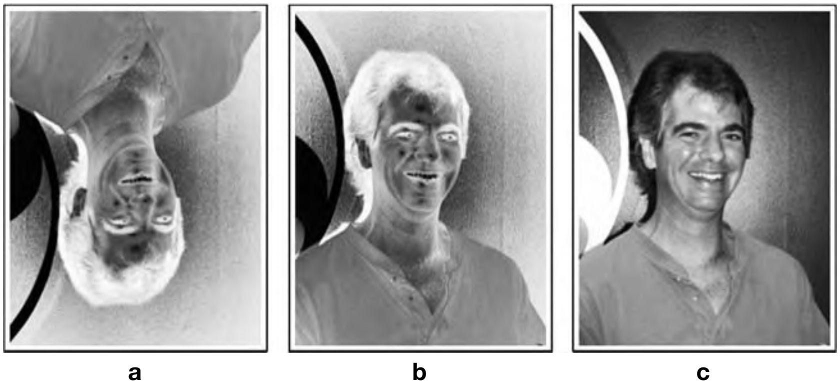



**FIGURE 5.34**

(a) An upside down scary Jim. (b) Correct side up but still scary. (c) Color applied but a bit less scary

```
ColorFunction → RGBColor
p1[[2]]
ImageSize → {216, 288}
p1[[3]]
PlotRange → {{0, 216}, {0, 288}}
```

For the color jpeg, the first element of the first component is the data array that determines the image.

```
Short[p1[[1,1]]]
{{{54, 113, 109}, {52, 111, 107}, {49, 108, 104}, {55, 110, 107},
    ⟨⟨209⟩⟩, {78, 114, 126}, {64, 106, 105}, {65, 108, 101}}, ⟨⟨287⟩⟩}
```

However, as before, the image generated with ArrayPlot is upside down—and the coloring is off.

```
j1a = ArrayPlot[p1[[1, 1]]]
```

To invert the image, we use Reverse. As stated previously, Reverse[list] reverses the entries of list.

```
j1b = ArrayPlot[Reverse[p1[[1, 1]]]]
```

To correct the color, we tell Mathematica to use the RGBColor function.

```
j1c = ArrayPlot[Reverse[p1[[1, 1]]], ColorFunction → RGBColor]
Show[GraphicsRow[{ j1a, j1b, j1c}]]
```

To apply your own color function, you need to manipulate the data. For this image, viewing it as a matrix, it has 288 rows and 216 columns.

```
Length[p1[[1,1]]]
288
Length[p1[[1,1,1]]]
216
```

Each entry of the matrix p1[[1,1]] is an ordered triple. To apply a color function to the ordered triple, we can proceed in a variety of ways. One approach is to convert the matrix to a list of ordered triples.

```
p2 = Flatten[p1[[1, 1]], 1];
Short[p2]
{{54, 113, 109}, {52, 111, 107}, {49, 108, 104}, {55, 110, 107},
⟨⟨62200⟩⟩, {63, 67, 96}, {41, 45, 74}, {40, 45, 74}, {37, 42, 71}}
```

p2 is a list of ordered triples. Our function, *h*, adds the last two elements of each triple and divides by two. We apply *h* to p2 with Map and name the result p3. We convert p3 back to a 288 × 216 array using Partition. We use ArrayPlot to visualize the result. In this case, the gray level used to shade each cell is scaled by the corresponding entry of p4.

```
h[{x_, y_, z_}] = (y + z)/2;
p3 = Map[h, p2];
p4 = Partition[p3, 216];
j2a = ArrayPlot[Reverse[p4]];
```

The built-in color gradients (refer to the **ColorSchemes** palette) are functions of a single variable. Thus,

```
j2b = ArrayPlot[Reverse[p4],ColorFunction → "SolarColors"]
```

applies the SolarColors function to the array (Figure 5.35(b)) whereas,

```
j2c = ArrayPlot[Reverse[p4],ColorFunction → "DarkBands"]
```

applies DarkBands to the array (Figure 5.35(c)).

```
Show[GraphicsRow[{j2a, j2b, j2c}]]
```

The structure of a black-and-white jpeg differs from that of a color one. To see so, we import a *very old* picture of the second author of this text, and name the result p1. With Length, we see that p1 has three parts

```
Length[p1]
3
```

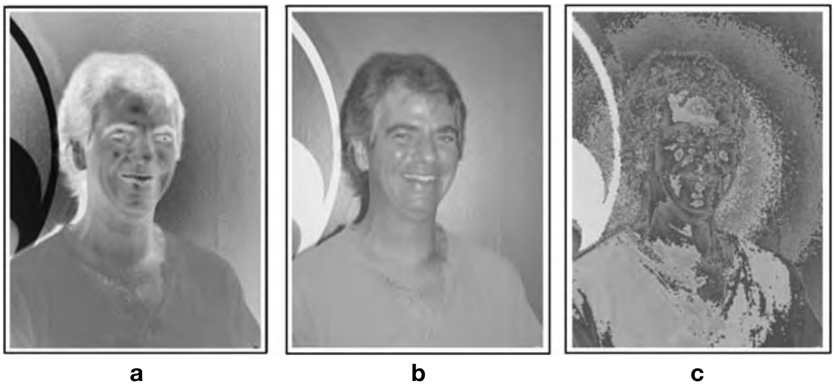



**FIGURE 5.35**

Manipulating a color jpeg with ColorFunction

```
p1 = Import["littlejim.jpg"];
Show[p1, ImageSize → Small]
```

p1[[1]] is also a list.

```
p1[[1, 1]]
```

A very large output was generated. Here is a sample of it:

```
{{244, 193, 189, 174, 182, 173, 182, 173, 171, 164, 169, 169, 162, 169,
  174, 160, 166, 168, 164, 176, 174, 171, 157, 160, 169, 156, 161,
  176, 155, 160, 165, 162, 162, <<362>>, 66, 62, 61, 62, 66, 77, 82,
  85, 100, 108, 110, 128, 139, 149, 150, 151, 153, 157, 162, 162, 160,
  137, 149, 124, 50, 88, 109, 132, 144, 155, 183, 197, 207}, <<599>>}
```

Show Less | Show More | Show Full Output | Set Size Limit...

The data defining the graphic is contained in the list p1[[1,1]], which is a 600 × 428 array/matrix. The other parts of p1 describe the remaining parts of the graphic.

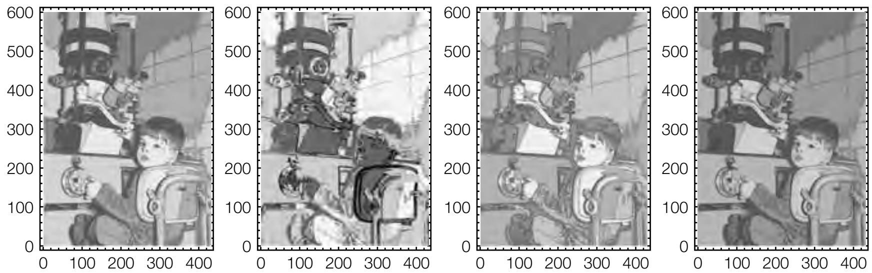


**FIGURE 5.36**

Using ListContourPlot along with various options to graphically represent a matrix

```
p1[[1,1]]
p1[[2]]
ImageSize → {428, 600}
p1[[3]]
PlotRange → {{0, 428}, {0, 600}}
```

In Figure 5.36, we illustrate the use of ListContourPlot along with various options.

```
g1 = ListContourPlot[p1[[1, 1]], AspectRatio → Automatic]
g2 = ListContourPlot[p1[[1, 1]],
     AspectRatio → Automatic,ColorFunction → "ThermometerColors"]
g3 = ListContourPlot[p1[[1, 1]], AspectRatio → Automatic,
     ColorFunction → "DarkBands"]
g4 = ListContourPlot[p1[[1, 1]], AspectRatio → Automatic,
     ColorFunction → "SolarColors"]
Show[GraphicsGrid[{{g1, g2}, {g3, g4}}]]
```

Figure 5.37 shows variations obtained with ReliefPlot and ListContourPlot.

```
g2 = ReliefPlot[p1[[1, 1]], AspectRatio → Automatic,
     ColorFunction → "GrayTones",FrameTicks → None];
g3 = ListContourPlot[p1[[1, 1]], AspectRatio → Automatic,
     ColorFunction → "GrayTones",FrameTicks → None];
g4 = ListContourPlot[p1[[1, 1]], AspectRatio → Automatic,
  ContourStyle → Black, ContourShading → False,
     FrameTicks → None];
Show[GraphicsRow[{g2, g3, g4}]]
```

ReliefPlot can help add insight to images, especially when they have geographical or biological meaning. For example, this jpeg shows the beginning of a biological process of a cell.

```
p1 = Import["071105fertx2c.jpg"];
Show[p1, ImageSize → Small]
```

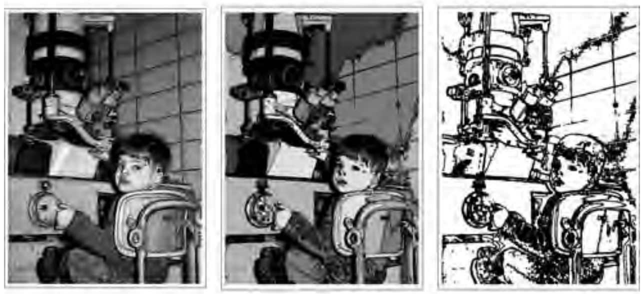
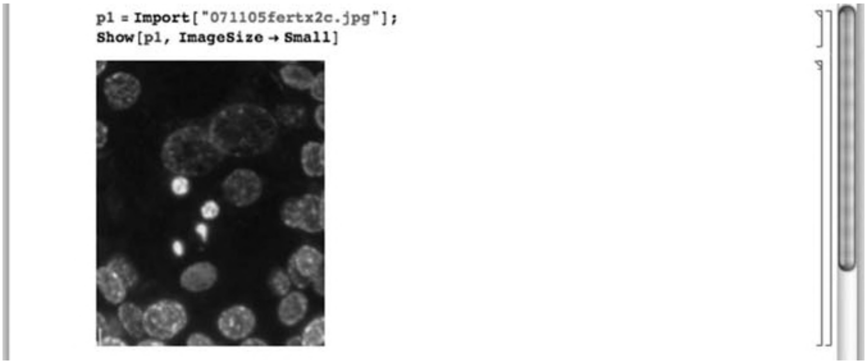


**FIGURE 5.37**

Manipulating an image with ListContourPlot and ReliefPlot




With Length, we see that p1[[1,1]] is a 500 × 400 array.

**Length[p1[[1,1]]]**
500
**Length[p1[[1,1,1]]]**
400

Viewing p1[[1,1]] as a 500 × 400 array, each entry is 1 × 3 array/vector. To easily apply a function, *f*, that assigns a number to each ordered triple, we use Flatten to convert the nested list/array p1[[1,1]] to a list of ordered triples in p2.

**p2 = Flatten[p1[[1, 1]], 1];**
**Short[p2]**
**Length[p2]**
{{194, 215, 158}, {189, 208, 144},
⟨⟨199996⟩⟩, {137, 66, 46}, {139, 68, 48}}
200000

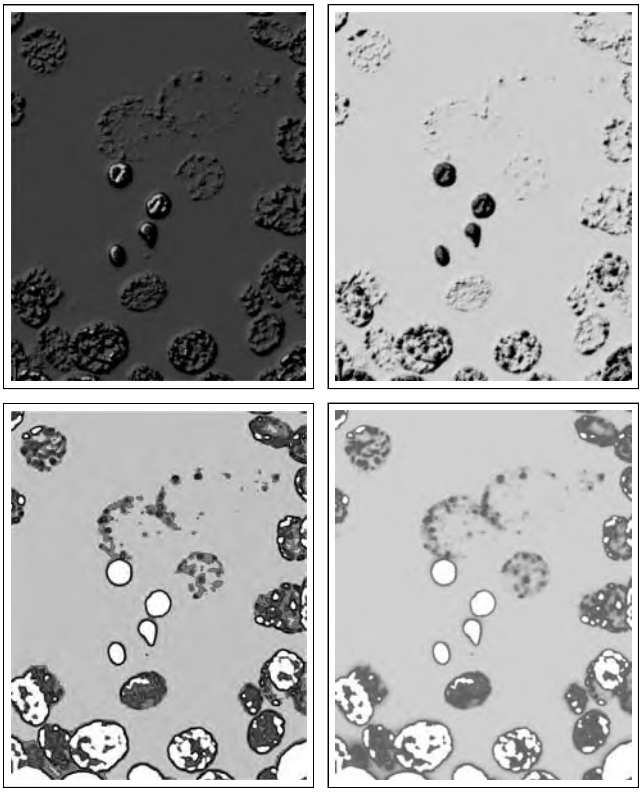


FIGURE 5.38

Using ReliefPlot, ListContourPlot, and ListDensityPlot along with various options to graphically represent a matrix

To apply our own color function to this data set, we convert the ordered triples to some other form. For illustrative purposes, we convert each ordered triple $(x, y, z)$ in p2 to the number $x + y^2$. The result is converted back to a $500 \times 400$ array, with Partition in p3.

```
f[y_]:=y[[1]] + y[[2]]^2
p3 = Partition[Map[f, p2], 400];
Length[p3]
500
```

We then use ReliefPlot, ListContourPlot, and ListDensityPlot along with various options to graph the result in Figure 5.38.

```
g1 = ReliefPlot[p3, AspectRatio → Automatic,
    ColorFunction → "DarkRainbow"];
```

```
g2 = ReliefPlot[p3, AspectRatio → Automatic,
      ColorFunction → "NeonColors",Ticks → None,
        Axes → None, FrameTicks → None]
g3 = ListContourPlot[p3, AspectRatio → Automatic,
       ColorFunction → "NeonColors",Ticks → None,
        Axes → None, FrameTicks → None]
g4 = ListDensityPlot[p3, AspectRatio → Automatic,
      ColorFunction → "NeonColors", Ticks → None,
        Axes → None, FrameTicks → None]
```

## 5.7 EXERCISES

**1.** Solve $-3y-z-3w=-1,\ -3x+3y-3z-3w=-1,\ 2x+2y-z+\ w=2.$

**2.** Find the eigenvalues and eigenvectors of each matrix. Verify that your results are correct. (a) $\mathbf{B}=\begin{pmatrix}0&4\\2&-2\end{pmatrix}$, (b) $\mathbf{A}=\begin{pmatrix}3&5&-4\\-5&6&3\\-3&2&-2\end{pmatrix}$, and (c) $\mathbf{A}=\begin{pmatrix}5&2/3&1&-4/3&-4&-4/3\\0&-1/6&-2&-1/6&7&23/6\\-1/2&-1/4&5/2&-1/4&1&3/4\\4&1/2&0&1/2&-3&-1/2\\0&-1&0&0&4&1\\-1&1/6&-1&1/6&2&19/6\end{pmatrix}$ *Comment:* In some cases, numerical results (use N) may be more meaningful than the exact ones.

**3.** For each of the following matrices, find the eigenvalues, eigenvectors, characteristic polynomial, and minimal polynomial:

(a) $\mathbf{A}=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$, (b) $\mathbf{B}=\begin{pmatrix}-3&-2&-2&-4\\2&1&2&4\\2&2&1&4\\-1&-1&-1&-3\end{pmatrix}$,

(c) $\mathbf{C}=\begin{pmatrix}-3&-1&5&-5\\-6&-9&10&-15\\6&8&-11&15\\8&10&-14&19\end{pmatrix}$, and (d) $\mathbf{D}=\begin{pmatrix}-3&0&1&-1\\-1&-1&0&-1\\-3&1&1&-1\\0&1&0&-1\end{pmatrix}$.

**4.** Let $\mathbf{J}_n(\lambda)$ denote the $n\times n$ matrix with $\lambda$'s down the diagonal, 0's below, and 1's to the right of each $\lambda$ (for example, $\mathbf{J}_3(\lambda)=\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}$), and let $\mathbf{J}=\begin{pmatrix}\mathbf{J}_n(\lambda_1)&0\\0&\mathbf{J}_m(\lambda_2)\end{pmatrix}$ denote the $(n+m)\times(n+m)$

matrix with "blocks" $\mathbf{J}_n$ and $\mathbf{J}_{\mathrm{m}}$ and 0's elsewhere (for example, $\begin{pmatrix} \mathbf{J}_2(\lambda_1) & 0 \\ 0 & \mathbf{J}_3(\lambda_2) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix}$). Find the eigenvalues, eigenvectors, characteristic polynomial, and minimal polynomial of $\mathbf{J}$. Illustrate your results with examples.

**5.** Find the unit normals to $w = \cos(4x^2 + 9y^2)$. Illustrate the result graphically.

**6.** Evaluate $\oint_C \left(e^{\sqrt{y}} + x\right) dx + \left(2y + \cos x\right) dy$, where $C$ is the boundary of the region between $y = x^2$ and $x = y^2$.

**7.** Find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = \left(xy + x^2yz\right) \mathbf{i} + \left(yz + xy^2z\right) \mathbf{j} + \left(xz + xyz^2\right) \mathbf{k}$$

through the surface of the cube cut from the first octant by the planes $x = 2$, $y = 2$, and $z = 2$.

**8.** Verify Stokes' theorem for the vector field

$$\mathbf{F}(x, y, z) = \left(y^2 - z\right) \mathbf{i} + \left(x + z^2\right) \mathbf{j} + \left(x^2 - y\right) \mathbf{k}$$

and $S$ the portion of the paraboloid $z = f(x, y) = 4 - (x^2 + y^2)$, $z \geq 0$.

**9.** The **Fibonacci numbers** satisfy the recurrence relation $F_n = F_{n-2} + F_{n-1}$, where $F_1 = F_2 = 1$. Provide a convincing argument that

$$\begin{vmatrix} F_{n+1} & F_{n+2} & \cdots & F_{n+k} \\ F_{n+k+1} & F_{n+k+2} & \cdots & F_{n+2k} \\ \vdots & \vdots & \vdots & \vdots \\ F_{n+k(k-1)+1} & F_{n+k(k-1)+2} & \cdots & F_{n+k^2} \end{vmatrix} = 0.$$

*Suggestion:* Use Fibonacci.

**10.** The **Boy surface** has parametrization

$$x(s, t) = \frac{\sqrt{2}\cos^2 t \cos 2s + \cos s \sin 2t}{2 - \sqrt{2}\sin 3s \sin 2t}$$

$$y(s, t) = \frac{\sqrt{2}\cos^2 t \cos 2s + \cos s \sin 2t}{2 - \sqrt{2}\sin 3s \sin 2t}$$

$$z(s, t) = \frac{3\cos^2 t}{2 - \sqrt{2}\sin 3s \sin 2t}$$

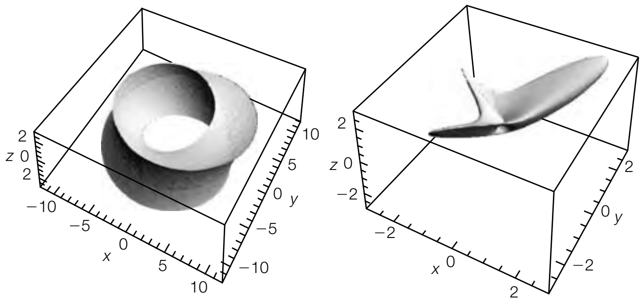



**FIGURE 5.39**

On the left, umbilic torus; on the right Boy surface

and the **umbilic torus** has parametrization

$$x(s,t) = (7 + \cos(s/3 - 2t) + 2\cos(s/3 + t))\sin s$$
$$y(s,t) = (7 + \cos(s/3 - 2t) + 2\cos(s/3 + t))\cos s$$
$$z(s,t) = \sin(s/3 - 2t) + 2\sin(s/3 + t).$$

See Figure 5.39. Determine if either of these surfaces is orientable.

**11.** Using 0's for dots and 1's for dashes and omitting spaces and punctuation, the following phrases are translated to Morse code as follows: "S.O.S" becomes 0, 0, 0, 1, 1, 1, 0, 0, 0; "Save our souls" becomes 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0; "Mathematica is terrific" becomes 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0; "Can I borrow the car" becomes 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0; "Are aliens on earth" becomes 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0; and "Work harder" becomes 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0. Represent this array graphically using (at least) three different methods. *Challenge:* Write a function symboltomorse that converts strings of letters to Morse code. Translate your favorite five quotes into Morse code and represent the result graphically.

**12.** A given curvature function determines a plane curve: The curve $C$ parametrized by arc length with curvature $\kappa(s)$ has parametrization $\mathbf{r}(s) = \langle x(s), y(s)\rangle$, where

$$\begin{cases} dx/ds = \cos\theta \\ dy/ds = \sin\theta \\ d\theta/ds = \kappa \\ x(a) = c,\ y(a) = d,\ \theta(0) = \theta_0 \end{cases} . \tag{5.27}$$

You can often use NDSolve to solve system (5.27). Plot the curve $C$ for which $\kappa(s) = e^{-s} + e^{s}$ for $-5 \le s \le 5$ if $x(0) = y(0) = \theta(0) = 0$. *Hint:*

```
In[4]:= Clear[curvek, κ];
        curvek[k_, ss_: {s, -15, 15}, opts___] :=
         Module[{numsol},
          numsol = NDSolve[{x'[s] == Cos[θ[s]], y'[s] == Sin[θ[s]], θ'[s] == k,
             x[0] == 0, y[0] == 0, θ[0] == 0}, {x[s], y[s], θ[s]}, ss];
          ParametricPlot[Evaluate[{x[s], y[s]} /. numsol], ss, opts,
           AspectRatio → Automatic]
         ]
```

Repeat the exercise given the following curvature functions (all for $-40 \le s \le 40$): $\kappa(s) = s + \sin s$, $\kappa(s) = sJ_1(s)$, $\kappa(s) = sJ_2(s)$, $\kappa(s) = s \sin \sin(s)$, $\kappa(s) = s \sin^2 \sin(s)$, and $\kappa(s) = |s \sin \sin(s)|$. See Figure 5.40.

**13.** Consider placing a tube around a curve (refer to tubeplot) but letting the radius change:

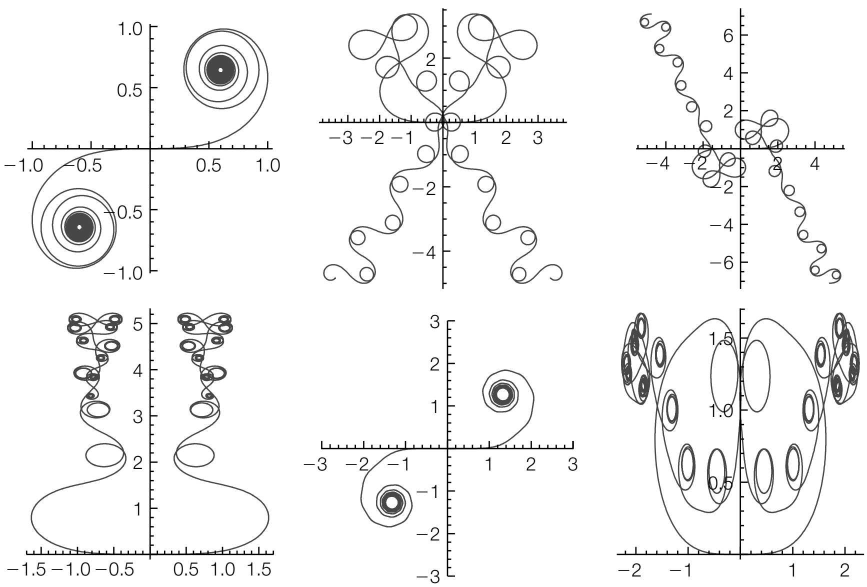



**FIGURE 5.40**

You can generate stunning curves by specifying a curvature function

**seashell[γ_][r_][t_, θ_]:=γ[t] +**
**rt(Cos[θ]normal[γ][t] + Sin[θ]binormal[γ][t])**

(a) Use seashell with $\gamma(t) = \langle t, t\cos 2t, t\sin 2t\rangle$ to create Figure 5.41.
(b) Illustrate the curve with its Frenet frame field.

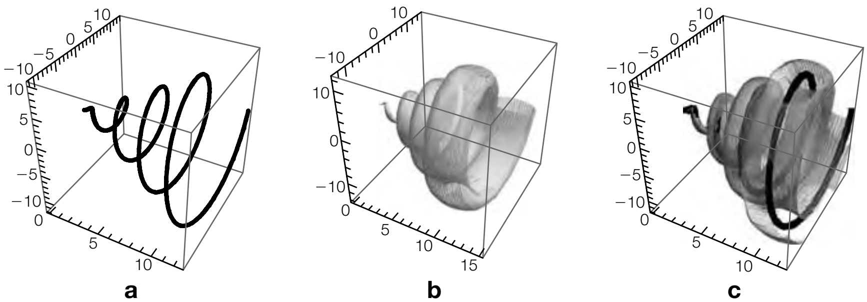



**FIGURE 5.41**

(a) A winding curve. (b) A seashell. (c) A winding curve in a seashell

CHAPTER

# Applications Related to Ordinary and Partial Differential Equations

# 6

For more detailed discussions regarding Mathematica and differential equations, see references such as Abell and Braselton's *Differential Equations with Mathematica* [1].

Chapter 6 discusses Mathematica's differential equations commands. The examples used to illustrate the various commands are similar to examples routinely done in a one- or two-semester differential equations course.

## 6.1 FIRST-ORDER DIFFERENTIAL EQUATIONS

### 6.1.1 Separable Equations

Because they are solved by integrating, separable differential equations are usually the first introduced in the introductory differential equations course.

**Definition 1 (Separable Differential Equation).** *A differential equation of the form*

$$f(y)\,dy = g(t)\,dt \tag{6.1}$$

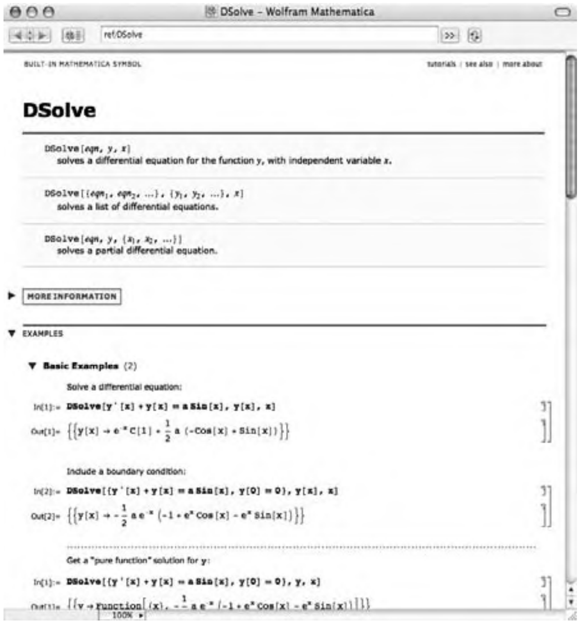
*is called a first-order* ***separable differential equation****.*

We solve separable differential equations by integrating.

**Remark 6.1** The command

**DSolve[y'[t]==f[t, y[t]], y[t], t]**

attempts to solve $y' = dy/dt = f(t, y)$ for $y$.




**Example 6.1.1** Solve each of the following equations: (a) $y' - y^2 \sin t = 0$; (b) $y' = \alpha y\left(1 - \frac{1}{K}y\right)$, $K$, $\alpha > 0$ constant.

***Solution*** (a) The equation is separable so we separate and then integrate:

$$\frac{1}{y^2}dy = \sin t\,dt$$
$$\int \frac{1}{y^2}dy = \int \sin t\,dt$$
$$-\frac{1}{y} = -\cos t + C$$
$$y = \frac{1}{\cos t + C}.$$

We check our result with DSolve.

```
sola = DSolve[y'[t] - y[t]^2Sin[t]==0, y[t], t]
```

$$\left\{\left\{y[t] \to \frac{1}{-C[1] + Cos[t]}\right\}\right\}$$

Observe that the result is given as a list. The formula for the solution is the second part of the first part of the first part of sola.

```
sola[[1, 1, 2]]
```

$$\frac{1}{-C[1] + Cos[t]}$$

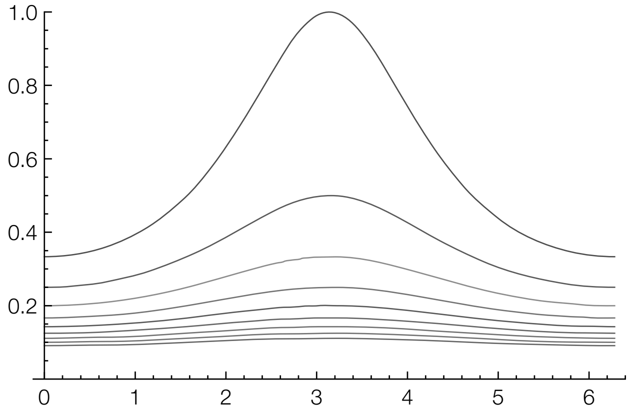



**FIGURE 6.1**

Several solutions of $y' - y^2 \sin t = 0$

We then graph the solution for various values of $C$ with Plot in Figure 6.1.

```
toplota = Table[sola[[1, 1, 2]]/.C[1] → −i, {i, 2, 10}]
```

$\left\{\frac{1}{2+\text{Cos}[t]}, \frac{1}{3+\text{Cos}[t]}, \frac{1}{4+\text{Cos}[t]}, \frac{1}{5+\text{Cos}[t]}, \frac{1}{6+\text{Cos}[t]}, \frac{1}{7+\text{Cos}[t]}, \frac{1}{8+\text{Cos}[t]}, \frac{1}{9+\text{Cos}[t]}, \frac{1}{10+\text{Cos}[t]}\right\}$

```
Plot[Tooltip[toplota], {t, 0, 2Pi}, PlotRange → {0, 1},
    AxesOrigin → {0, 0}]
```

expression /. x->y replaces all occurrences of $x$ in expression by $y$. Table[a[k],{k,n,m}] generates the list $a_n, a_{n+1}, \ldots, a_{m-1}, a_m$.

To graph the list of functions {list} for $a \le x \le b$, enter Plot[list,{x,a,b}].

(b) After separating variables, we use partial fractions to integrate:

$$
\begin{aligned}
y' &= \alpha y \left(1 - \frac{1}{K} y\right) \\
\frac{1}{\alpha y \left(1 - \frac{1}{K} y\right)} dy &= dt \\
\frac{1}{\alpha}\left(\frac{1}{y} + \frac{1}{K-y}\right) &= dt \\
\frac{1}{\alpha}\left(\ln|y| - \ln|K-y|\right) &= C_1 + t \\
\frac{y}{K-y} &= Ce^{\alpha t} \\
y &= \frac{CKe^{\alpha t}}{Ce^{\alpha t} - 1}.
\end{aligned}
$$

We check the calculations with Mathematica. First, we use Apart to find the partial fraction decomposition of $\dfrac{1}{\alpha y \left(1 - \frac{1}{K} y\right)}$.

```
s1 = Apart[1/(α y(1 − 1/k y)), y]
```

$\frac{1}{y\alpha} - \frac{1}{(-k+y)\alpha}$

Then, we use **Integrate** to check the integration.

```
s2 = Integrate[s1, y]
```

$k\left(\frac{\text{Log}[y]}{k\alpha} - \frac{\text{Log}[-k+y]}{k\alpha}\right)$

Last, we use **Solve** to solve $\frac{1}{\alpha}\left(\ln|y| - \ln|K-y|\right) = ct$ for $y$.

```
Solve[s2==c + t, y]
```

$\left\{\left\{y \to \frac{e^{c\alpha+t\alpha}k}{-1+e^{c\alpha+t\alpha}}\right\}\right\}$

We can use **DSolve** to find a general solution of the equation

```
solb = DSolve[y'[t]==αy[t](1 – 1/ky[t]), y[t], t]
```

$\left\{\left\{y[t] \to \frac{e^{t\alpha+kC[1]}k}{-1+e^{t\alpha+kC[1]}}\right\}\right\}$

as well as find the solution that satisfies the initial condition $y(0) = y_0$, although Mathematica generates several error messages because inverse functions are being used so the resulting solution set may not be complete.

```
solc = DSolve[{y'[t]==αy[t](1 – 1/ky[t]), y[0]==y0}, y[t], t]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.⟩⟩

$$\left\{\left\{y[t] \to \frac{e^{t\alpha}ky0}{k - y0 + e^{t\alpha}y0}\right\}\right\}$$

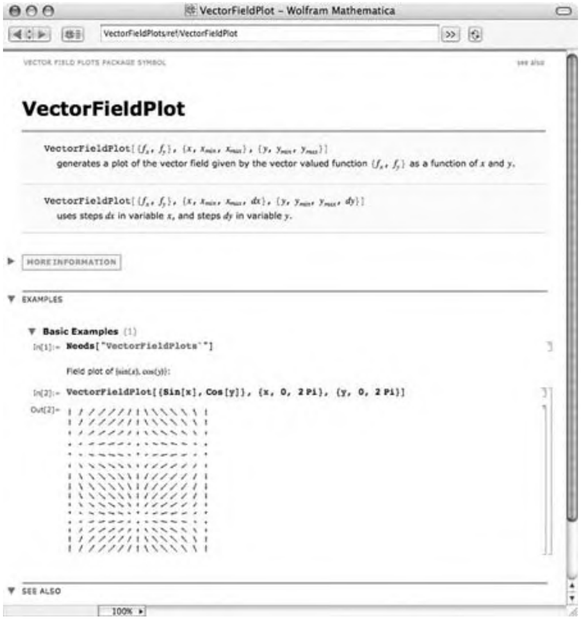
The equation $y' = \alpha y\left(1 - \frac{1}{K}y\right)$ is called the **logistic equation** (or **Verhulst equation**) and is used to model the size of a population that is not allowed to grow in an unbounded manner. Assuming that $y(0) > 0$, then all solutions of the equation have the property that $\lim_{t\to\infty} y(t) = K$.

To see this, we set $\alpha = K = 1$ and use VectorFieldPlot, which is contained in the **VectorFieldPlots** package, to graph the direction field associated with the equation in Figure 6.2.

```
Needs["VectorFieldPlots`"]
pvf1 = Show[VectorFieldPlot[{1, y(1 – y)}, {t, 0, 5}, {y, 0, 5/2},
        ScaleFunction → (1&)], Axes → Automatic, AxesOrigin → {0, 0}];
```

The property is more easily seen when we graph various solutions along with the direction field as done next in Figure 6.2.

```
toplot = Table[solc[[1, 1, 2]]/.{α → 1, k → 1, y0 → i/5}, {i, 1, 12}];
sols = Plot[toplot, {t, 0, 5}, PlotStyle → GrayLevel[0], PlotRange → All];
pvf2 = Show[pvf1, sols];
Show[GraphicsColumn[{pvf1, pvf2}]]
```



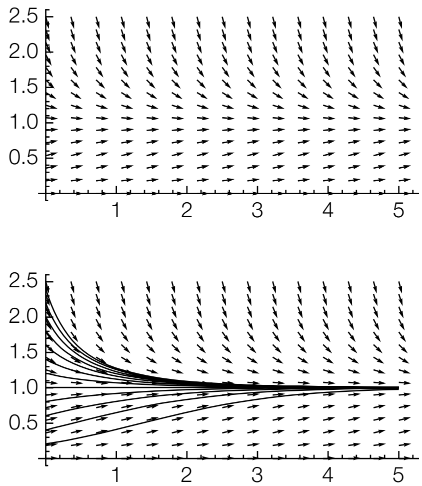




**FIGURE 6.2**

(Top) A typical direction field for the logistic equation. (Bottom) A typical direction field for the logistic equation along with several solutions

**When Mathematica encounters inverse functions, it might choose the incorrect *branch* to form a continuous solution to an initial-value problem.**

---

**Example 6.1.2** Solve $dy/dt = \sin t \cos y$, $y(1) = 3$.

***Solution*** When we use DSolve to solve the equation and the initial-value problem, Mathematica warns us that inverse functions are being used.

```
sol = DSolve[y'[t] == Sin[t] Cos[y[t]], y[t], t]
Solve::ifun : Inverse functions are being used by Solve, so
    some solutions may not be found; use Reduce for complete solution information. >>
{{y[t] → 2 ArcTan[Tanh[1/4 (C[1] - 2 Cos[t])]]}}

sol1 = DSolve[{y'[t] == Sin[t] Cos[y[t]], y[1] == 3}, y[t], t]
Solve::ifun : Inverse functions are being used by Solve, so
    some solutions may not be found; use Reduce for complete solution information. >>
Solve::ifun : Inverse functions are being used by Solve, so
    some solutions may not be found; use Reduce for complete solution information. >>
{{y[t] → 2 ArcTan[Tanh[1/4 (2 (2 ArcTanh[Tan[3/2]] + Cos[1]) - 2 Cos[t])]]}}
```

From the direction field, we see that the solution satisfying $y(1) = 3$ is continuous for (at least) $0 \le t \le 4\pi$. However, the explicit solution returned by DSolve is not the solution that is continuous on $[0, 4\pi]$. See Figure 6.3(a).

```
Needs["VectorFieldPlots`"]
pvf1 = Show[VectorFieldPlot[{1, Sin[t]Cos[y]}, {t, 0, 4Pi}, {y, -2Pi, 2Pi},
      ScaleFunction → (1&), PlotPoints → 25], Axes → Automatic,
         AxesOrigin → {0, 0}];
psol1 = Plot[y[t]/.sol1, {t, 0, 4Pi},
      PlotStyle → {{GrayLevel[.5], Thickness[.01]}}];
discont = Show[pvf1, psol1]
```

To see the continuous solution, we use NDSolve to generate a numerical solution to the initial value problem. If possible,

NDSolve is discussed in more detail later in the chapter.

NDSolve[{y'[t]==f[t,y[t]],y[t0]=y0},y[t],{t,a,b}]

attempts to numerically solve $y' = f(t, y)$, $y(t_0) = y_0$ for $a \le t \le b$.

```
sol2 = NDSolve[{y'[t]==Sin[t]Cos[y[t]], y[1]==3}, y[t], {t, 0, 4Pi}]
{{y[t] → InterpolatingFunction[{{0., 12.5664}}, <>][t]}}
```

In Figure 6.3(b), we see that the result returned by NDSolve is continuous on $[0, 4\pi]$.

```
psol2 = Plot[y[t]/.sol2, {t, 0, 4Pi},
      PlotStyle → {{GrayLevel[.5], Thickness[.01]}}];
cont = Show[pvf1, psol2]
```

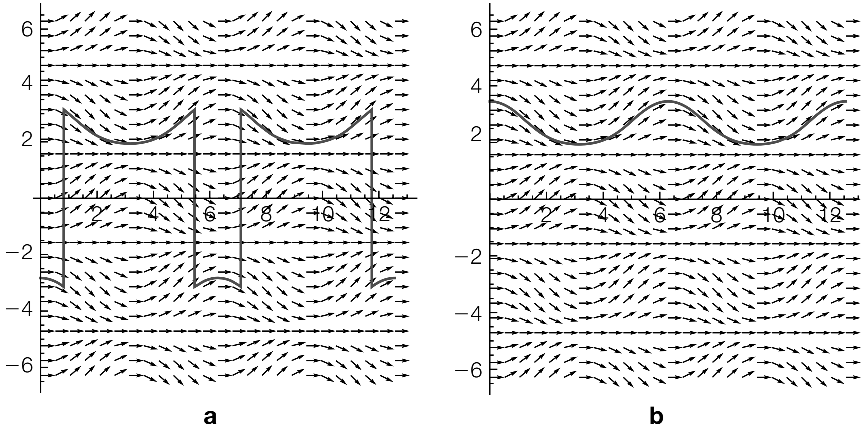



**FIGURE 6.3**

(a) The solution returned by DSolve is discontinuous on $[0, 4\pi]$. (b) We use NDSolve to find the continuous solution of the initial-value problem

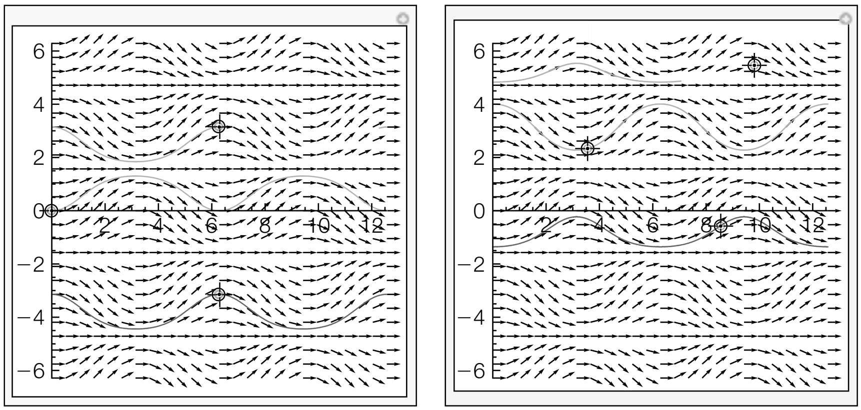



**FIGURE 6.4**

Visualizing how changing $t_0$ and $y_0$ affects the solution that satisfies $y(t_0) = y_0$

With Manipulate, you can see how varying the initial conditions affects the solution. See Figure 6.4. When you drag the locator points, the solution changes accordingly.

```
Manipulate[
sol1 = NDSolve[{y′[t]==Sin[t]Cos[y[t]], y[pt[[1, 1]]]==pt[[1, 2]]},
      y[t], {t, 0, 4Pi}];
psol1 = Plot[y[t]/.sol1, {t, 0, 4Pi},
      PlotStyle → {{GrayLevel[.7], Thickness[.01]}}];
```

```
sol2 = NDSolve[{y'[t]==Sin[t]Cos[y[t]], y[pt[[2, 1]]]==pt[[2, 2]]},
       y[t], {t, 0, 4Pi}];
psol2 = Plot[y[t]/.sol2, {t, 0, 4Pi},
       PlotStyle → {{GrayLevel[.7], Dashing[{0.02}], Thickness[.01]}}];
sol3 = NDSolve[{y'[t]==Sin[t]Cos[y[t]], y[pt[[3, 1]]]==pt[[3, 2]]}, y[t],
          {t, 0, 4Pi}];
psol3 = Plot[y[t]/.sol3, {t, 0, 4Pi},
       PlotStyle → {{GrayLevel[.4], Thickness[.01]}}];
initialpt = Graphics[Point[{pt}], PlotRange → {{0, 4Pi}, { - 2Pi, 2Pi}}];
Show[pvf1, psol1, psol2, psol3, initialpt, Axes → Automatic,
       PlotRange → {{0, 4Pi}, { - 2Pi, 2Pi}}, AspectRatio → Automatic],
{{pt, {{0, 0}, {2Pi, Pi}, {2Pi, - Pi}}}, Locator}
]
```

## 6.1.2 Linear Equations

**Definition 2 (First-Order Linear Equation).** *A differential equation of the form*

$$a_1(t)\frac{dy}{dt} + a_0(t)y = f(t), \tag{6.2}$$

*where $a_1(t)$ is not identically the zero function, is a first-order **linear differential equation**.*

Assuming that $a_1(t)$ is not identically the zero function, dividing equation (6.2) by $a_1(t)$ gives us the **standard form** of the first-order linear equation:

$$\frac{dy}{dt} + p(t)y = q(t). \tag{6.3}$$

If $q(t)$ is identically the zero function, we say that the equation is **homogeneous**. The **corresponding homogeneous equation** of equation (6.3) is

$$\frac{dy}{dt} + p(t)y = 0. \tag{6.4}$$

Observe that equation (6.4) is separable:

$$\begin{aligned} \frac{dy}{dt} + p(t)y &= 0 \\ \frac{1}{y}dy &= -p(t)\,dt \\ \ln|y| &= -\int p(t)\,dt + C \\ y &= Ce^{-\int p(t)\,dt}. \end{aligned}$$

A **particular solution** is a specific solution to the equation that does not contain any arbitrary constants.

Notice that any constant multiple of a solution to a linear homogeneous equation is also a solution. Now suppose that $y$ is any solution of equation (6.3) and $y_p$ is a particular solution of equation (6.3). Then,

$$\begin{aligned}(y - y_p)' + p(t)(y - y_p) &= y' + p(t)y - (y_p' + p(t)y_p) \\ &= q(t) - q(t) = 0.\end{aligned}$$

Thus, $y - y_p$ is a solution to the corresponding homogeneous equation of equation (6.3). Hence,

$$\begin{aligned}y - y_p &= Ce^{-\int p(t)\,dt} \\ y &= Ce^{-\int p(t)\,dt} + y_p \\ y &= y_h + y_p,\end{aligned}$$

where $y_h = Ce^{-\int p(t)\,dt}$. That is, a general solution of equation (6.3) is $y = y_h + y_p$, where $y_p$ is a particular solution to the nonhomogeneous equation and $y_h$ is a general solution to the corresponding homogeneous equation. Thus, to solve equation (6.3), we need to first find a general solution to the corresponding homogeneous equation, $y_h$, which we can accomplish through separation of variables, and then find a particular solution, $y_p$, to the nonhomogeneous equation.

If $y_h$ is a solution to the corresponding homogeneous equation of equation (6.3), then for any constant $C$, $Cy_h$ is also a solution to the corresponding homogeneous equation. Therefore, it is impossible to find a particular solution to equation (6.3) of this form. Instead, we search for a particular solution of the form $y_p = u(t)y_h$, where $u(t)$ is *not* a constant function. Assuming that a particular solution, $y_p$, to equation (6.3) has the form $y_p = u(t)y_h$, differentiating gives us $y_p' = u'y_h + uy_h'$ and substituting into equation (6.3) results in

$$y_p' + p(t)y_p = u'y_h + uy_h' + p(t)uy_h = q(t).$$

$y_h$ is a solution to the corresponding homogeneous equation, so $y_h' + p(t)y_h = 0$.

Because $uy_h' + p(t)uy_h = u\left[y_h' + p(t)y_h\right] = u \cdot 0 = 0$, we obtain

$$\begin{aligned}u'y_h &= q(t) \\ u' &= \frac{1}{y_h}q(t) \\ u' &= e^{\int p(t)\,dt}q(t) \\ u &= \int e^{\int p(t)\,dt}q(t)\,dt\end{aligned}$$

so

$$y_p = u(t)y_h = Ce^{-\int p(t)\,dt}\int e^{\int p(t)\,dt}q(t)\,dt.$$

Because we can include an arbitrary constant of integration when evaluating $\int e^{\int p(t)\,dt} q(t)\,dt$, it follows that we can write a general solution of equation (6.3) as

$$y = e^{-\int p(t)\,dt} \int e^{\int p(t)\,dt} q(t)\,dt. \tag{6.5}$$

Alternatively, multiplying equation (6.3) by the **integrating factor** $\mu(t) = e^{\int p(t)\,dt}$ gives us the same result:

$$\begin{aligned}
e^{\int p(t)\,dt}\frac{dy}{dt} + p(t)e^{\int p(t)\,dt}y &= q(t)e^{\int p(t)\,dt} \\
\frac{d}{dt}\left(e^{\int p(t)\,dt}y\right) &= q(t)e^{\int p(t)\,dt} \\
e^{\int p(t)\,dt}y &= \int q(t)e^{\int p(t)\,dt}dt \\
y &= e^{-\int p(t)\,dt}\int q(t)e^{\int p(t)\,dt}dt.
\end{aligned}$$

Thus, first-order linear equations can always be solved, although the resulting integrals may be difficult or impossible to evaluate exactly.

Mathematica is able to solve the general form of the first-order equation, the initial-value problem $y' + p(t)y = q(t)$, $y(0) = y_0$,

```
DSolve[y'[t] + p[t] y[t] == q[t], y[t], t]
{{y[t] → e^(∫_1^t -p[K[1]] dK[1]) C[1] +
    e^(∫_1^t -p[K[1]] dK[1]) ∫_1^t e^(-∫_1^K[2] -p[K[1]] dK[1]) q[K[2]] dK[2]}}
```

```
DSolve[{y'[t] + p[t] y[t] == q[t], y[0] == y0}, y[t], t]
{{y[t] → -e^(-∫_1^0 -p[K[1]] dK[1] + ∫_1^t -p[K[1]] dK[1])
    (-y0 + e^(∫_1^0 -p[K[1]] dK[1]) ∫_1^0 e^(-∫_1^K[2] -p[K[1]] dK[1]) q[K[2]] dK[2] -
      e^(∫_1^0 -p[K[1]] dK[1]) ∫_1^t e^(-∫_1^K[2] -p[K[1]] dK[1]) q[K[2]] dK[2])}}
```

as well as the corresponding homogeneous equation,

```
DSolve[y'[t] + p[t] y[t] == 0, y[t], t]
{{y[t] → e^(∫_1^t -p[K[1]] dK[1]) C[1]}}
```

```
DSolve[{y'[t] + p[t] y[t] == 0, y[0] == y0}, y[t], t]
{{y[t] → e^(-∫_1^0 -p[K[1]] dK[1] + ∫_1^t -p[K[1]] dK[1]) y0}}
```

although the results contain unevaluated integrals.

**Example 6.1.3** **(Exponential Growth).** Let $y = y(t)$ denote the size of a population at time $t$. If $y$ grows at a rate proportional to the amount present, $y$ satisfies

$$\frac{dy}{dt} = \alpha y, \tag{6.6}$$

where $\alpha$ is the **growth constant**. If $y(0) = y_0$, using equation (6.5) results in $y = y_0 e^{\alpha t}$. We use DSolve to confirm this result.

```
DSolve[{y'[t]==α y[t], y[0]==y0}, y[t], t]
{{y[t] → e^(tα) y0}}
```

**Example 6.1.4** Solve each of the following equations: (a) $dy/dt = k\left(y - y_s\right)$, $y(0) = y_0$, $k$ and $y_s$ constant; (b) $y' - 2ty = t$ (c) $ty' - y = 4t\cos 4t - \sin 4t$.

***Solution*** (a) By hand, we rewrite the equation and obtain $y' - ky = -ky_s$. A general solution of the corresponding homogeneous equation $y' - ky = 0$ is $y_h = e^{kt}$. Because $k$ and $-ky_s$ are constants, we suppose that a particular solution of the nonhomogeneous equation, $y_p$, has the form $y_p = A$, where $A$ is a constant.

$dy/dt = k\left(y - y_s\right)$ models *Newtons law of cooling:* The rate at which the temperature, $y(t)$, changes in a heating/cooling body is proportional to the difference between the temperature of the body and the constant temperature, $y_s$, of the surroundings.

Assuming that $y_p = A$, we have $y_p' = 0$, and substitution into the nonhomogeneous equation gives us

$$y_p{}' - ky_p = -KA = -ky_s \qquad \text{so} \qquad A = y_s.$$

Thus, a general solution is $y = y_h + y_p = Ce^{kt} + y_s$. Applying the initial condition $y(0) = y_0$ results in $y = y_s + (y_0 - y_s)e^{kt}$.

We obtain the same result with DSolve. We graph the solution satisfying $y(0) = 75$ assuming that $k = -1/2$ and $y_s = 300$ in Figure 6.5. Notice that $y(t) \to y_s$ as $t \to \infty$.

This will turn out to be a lucky guess. If there is not a solution of this form, we would not find one of this form.

```
sola = DSolve[{y'[t]==k(y[t]-ys), y[0]==y0}, y[t], t]
{{y[t] → e^(kt) y0 + ys - e^(kt) ys}}
Plot[y[t]/.sola/.{k → -1/2, ys → 300, y0 → 75}, {t, 0, 10}]
```

(b) The equation is in standard form and we identify $p(t) = -2t$. Then, the integrating factor is $\mu(t) = e^{\int p(t)\,dt} = e^{-t^2}$. Multiplying the equation by the integrating factor, $\mu(t)$, results in

$$e^{-t^2}(y' - 2ty) = te^{-t^2} \qquad \text{or} \qquad \frac{d}{dt}\left(ye^{-t^2}\right) = te^{-t^2}.$$

Integrating gives us $ye^{-t^2} = -\frac{1}{2}e^{-t^2} + C$ or $y = -\frac{1}{2} + Ce^{t^2}$. We confirm the result with DSolve.

```
DSolve[y'[t] - 2ty[t]==t, y[t], t]
{{y[t] → -1/2 + e^(t^2) C[1]}}
```

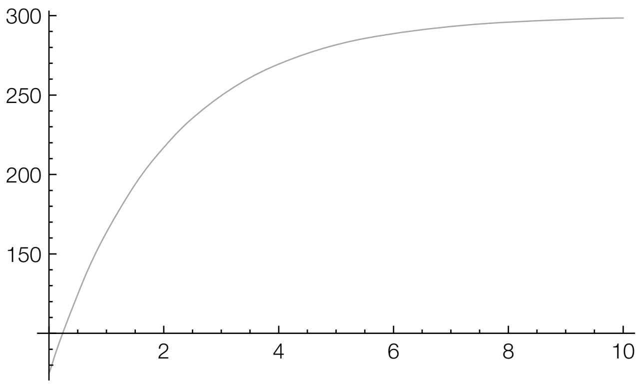



**FIGURE 6.5**

The temperature of the body approaches the temperature of its surroundings

(c) In standard form, the equation is $y' - y/t = (4t\cos 4t - \sin 4t)/t$ so $p(t) = -1/t$. The integrating factor is $\mu(t) = e^{\int p(t)\,dt} = e^{-\ln t} = 1/t$, and multiplying the equation by the integrating factor and then integrating gives us

$$\begin{aligned}\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y &= \frac{1}{t^2}(4t\cos 4t - \sin 4t)\\ \frac{d}{dt}\left(\frac{1}{t}y\right) &= \frac{1}{t^2}(4t\cos 4t - \sin 4t)\\ \frac{1}{t}y &= \frac{\sin 4t}{t} + C\\ y &= \sin 4t + Ct,\end{aligned}$$

where we use the Integrate function to evaluate $\int \frac{1}{t^2}(4t\cos 4t - \sin 4t)\,dt = \frac{\sin 4t}{t} + C$.

```
Integrate[(4tCos[4t] - Sin[4t])/t^2, t]
Sin[4t]
-------
   t
```

We confirm this result with DSolve.

```
sol = DSolve[y'[t] - y[t]/t==
      (4tCos[4t] - Sin[4t])/t, y[t], t]
{{y[t] → tC[1] + Sin[4t]}}
```

In the general solution, observe that *every* solution satisfies $y(0) = 0$. That is, the initial-value problem

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1}{t^2}(4t\cos 4t - \sin 4t), \quad y(0) = 0$$

has infinitely many solutions. We see this in the plot of several solutions that is generated with Plot in Figure 6.6.

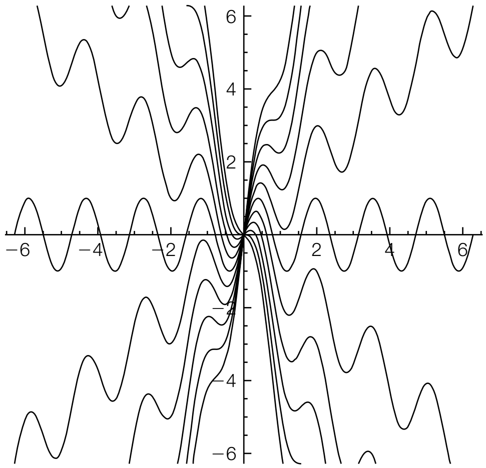



**FIGURE 6.6**

Every solution satisfies $y(0) = 0$

```
toplot = Table[sol/.C[1] → i, {i, −5, 5}];
Plot[y[t]/.toplot, {t, −2Pi, 2Pi}, PlotRange → {−2Pi, 2Pi},
    PlotStyle → GrayLevel[0], AspectRatio → Automatic]
```

### *Application: Free-Falling Bodies*

The motion of objects can be determined through the solution of first-order initial-value problems. We begin by explaining some of the theory that is needed to set up the differential equation that models the situation.

> ***Newton's Second Law of Motion:*** *The rate at which the momentum of a body changes with respect to time is equal to the resultant force acting on the body.*

Because the body's momentum is defined as the product of its mass and velocity, this statement is modeled as

$$\frac{d}{dt}(mv) = F,$$

where $m$ and $v$ represent the body's mass and velocity, respectively, and $F$ is the sum of the forces (the resultant force) acting on the body. Because $m$ is constant, differentiation leads to the well-known equation

$$m\frac{dv}{dt} = F.$$

If the body is subjected only to the force due to gravity, then its velocity is determined by solving the differential equation

$$m\frac{dv}{dt} = mg \qquad \text{or} \qquad \frac{dv}{dt} = g,$$

where $g = 32\text{ft/s}^2$ (English system) and $g = 9.8\text{m/s}^2$ (metric system). This differential equation is applicable only when the resistive force due to the medium (such as air resistance) is ignored. If this offsetting resistance is considered, we must discuss all of the forces acting on the object. Mathematically, we write the equation as

$$m\frac{dv}{dt} = \sum \left(\text{forces acting on the object}\right),$$

where the direction of motion is taken to be the positive direction. Because air resistance acts against the object as it falls and $g$ acts in the same direction of the motion, we state the differential equation in the form

$$m\frac{dv}{dt} = mg + \left(-F_R\right) \qquad \text{or} \qquad m\frac{dv}{dt} = mg - F_R,$$

where $F_R$ represents this resistive force. Note that down is assumed to be the positive direction. The resistive force is typically proportional to the body's velocity, $v$, or the square of its velocity, $v^2$. Hence, the differential equation is linear or nonlinear based on the resistance of the medium taken into account.

---

**Example 6.1.5** An object of mass $m = 1$ is dropped from a height of 50 feet above the surface of a small pond. While the object is in the air, the force due to air resistance is $v$. However, when the object is in the pond, it is subjected to a buoyancy force equivalent to $6v$. Determine how much time is required for the object to reach a depth of 25 feet in the pond.

***Solution*** This problem must be broken into two parts: an initial-value problem for the object above the pond and an initial-value problem for the object below the surface of the pond. The initial-value problem above the pond's surface is found to be

$$\begin{cases} dv/dt = 32 - v \\ v(0) = 0. \end{cases}$$

However, to define the initial-value problem to find the velocity of the object beneath the pond's surface, the velocity of the object when it reaches the surface must be known. Hence, the velocity of the object above the surface must be determined by solving the initial-value problem above. The equation $dv/dt = 32 - v$ is separable and solved with DSolve in d1.

```
Clear[v, y]
d1 = DSolve[{v′[t]==32 − v[t], v[0]==0}, v[t], t]
{{v[t] → 32e^−t (−1 + e^t)}}
```

In order to find the velocity when the object hits the pond's surface, we must know the time at which the distance traveled by the object (or the displacement

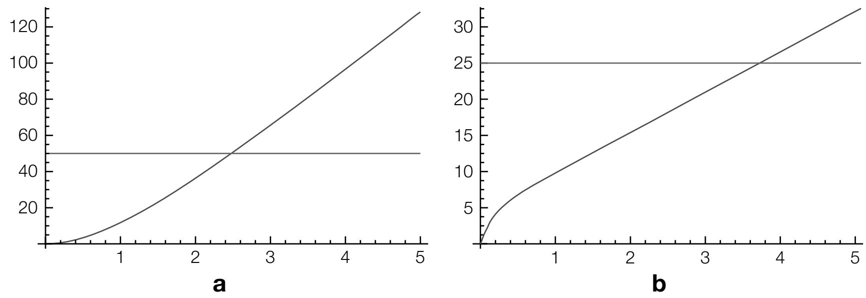



**FIGURE 6.7**

(a) The object has traveled 50 feet when $t \approx 2.5$. (b) After approximately 4 seconds, the object is 25 feet below the surface of the pond

of the object) is 50. Thus, we must find the displacement function, which is done by integrating the velocity function obtaining $s(t) = 32e^{-t} + 32t - 32$.

```
p1 = DSolve[{y'[t]==v[t]/.d1, y[0]==0}, y[t], t]
{{y[t] → 32e^-t (1 - e^t + e^t t)}}
```

The displacement function is graphed with Plot in Figure 6.7(a). The value of $t$ at which the object has traveled 50 feet is needed. This time appears to be approximately $2.5$ seconds.

```
Plot[{y[t]/.p1, 50}, {t, 0, 5}]
```

A more accurate value of the time at which the object hits the surface is found using FindRoot. In this case, we obtain $t \approx 2.47864$. The velocity at this time is then determined by substitution into the velocity function resulting in $v(2.47864) \approx 29.3166$. Note that this value is the initial velocity of the object when it hits the surface of the pond.

```
t1 = FindRoot[Evaluate[y[t]/.p1]==50, {t, 2.5}]
{t → 2.47864}
v1 = d1/.t1
{{v[2.47864] → 29.3166}}
```

Thus, the initial-value problem that determines the velocity of the object beneath the surface of the pond is given by

$$\begin{cases} dv/dt = 32 - 6v \\ v(0) = 29.3166. \end{cases}$$

The solution of this initial-value problem is $v(t) = \frac{16}{3} + 23.9833e^{-t}$, and integrating to obtain the displacement function (the initial displacement is $0$) we obtain $s(t) = 3.99722 - 3.99722e^{-6t} + \frac{16}{3}t$. These steps are carried out in d2 and p2.

```
d2 = DSolve[{v'[t]==32-6v[t], v[0]==v1[[1, 1, 2]]}, v[t], t]
```

$\{\{v[t] \to e^{-6t}(23.9832 + 5.33333e^{6t})\}\}$

```
p2 = DSolve[{y'[t]==v[t]/.d2, y[0]==0}, y[t], t]
```

$\{\{y[t] \to e^{-6.t}(-3.99721 + 3.99721e^{6.t} + 5.33333e^{6.t}t)\}\}$

This displacement function is then plotted in Figure 6.7(b) to determine when the object is 25 feet beneath the surface of the pond. This time appears to be near 4 seconds.

```
Plot[{y[t]/.p2, 25}, {t, 0, 5}]
```

A more accurate approximation of the time at which the object is 25 feet beneath the pond's surface is obtained with FindRoot. In this case, we obtain $t \approx 3.93802$. Finally, the time required for the object to reach the pond's surface is added to the time needed for it to travel 25 feet beneath the surface to see that approximately 6.41667 seconds are required for the object to travel from a height of 50 feet above the pond to a depth of 25 feet below the surface.

```
t2 = FindRoot[Evaluate[y[t]/.p2]==25, {t, 4}]
```

$\{t \to 3.93802\}$

```
t1[[1, 2]] + t2[[1, 2]]
```

6.41667

---

## 6.1.3 Nonlinear Equations

**Mathematica can solve a variety of nonlinear first-order equations that are typically encountered in the introductory differential equations course.**

---

**Example 6.1.6** Solve each: (a) $(\cos x + 2xe^y)\,dx + (\sin y + x^2e^y - 1)\,dy = 0$; (b) $(y^2 + 2xy)\,dx - x^2dy = 0$.

***Solution*** (a) Notice that $(\cos x + 2xe^y)\,dx + (\sin y + x^2e^y - 1)\,dy = 0$ can be written as $dy/dx = -(\cos x + 2xe^y)/(\sin x + x^2e^y - 1)$. The equation is an example of an *exact equation*. A theorem tells us that the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is **exact** if and only if $\partial M/\partial y = \partial N/\partial x$.

```
m = Cos[x] + 2x Exp[y];
n = Sin[y] + x^2 Exp[y] - 1;
D[m, y]
D[n, x]
```

$2e^y x$

$2e^y x$

We solve exact equations by integrating. Let $F(x,y)=C$ satisfy $(y\cos x+2xe^{y})\,dx+\left(\sin y+x^2e^y-1\right)dy=0$. Then,

$$F(x,y)=\int\left(\cos x+2xe^{y}\right)dx=\sin x+x^2e^y+g(y),$$

where $g(y)$ is a function of $y$.

```
f1 = Integrate[m, x]
e^y x^2 + Sin[x]
```

We next find that $g'(y)=\sin y-1$ so $g(y)=-\cos y-y$. Hence, a general solution of the equation is

$$\sin x+x^2e^y-\cos y-y=C.$$

```
f2 = D[f1, y]
e^y x^2
f3 = Solve[f2 + c==n, c]
{{c → -1 + Sin[y]}}
Integrate[f3[[1, 1, 2]], y]
-y - Cos[y]
```

We confirm this result with DSolve. Notice that Mathematica warns us that it cannot solve for $y$ explicitly and returns the same implicit solution obtained by us.

```
mf = m/.y → y[x];
nf = n/.y → y[x];
sol = DSolve[mf + nf y'[x]==0, y[x], x]
Solve::tdep : The equations appear to involve the variables to be solved for in
an essentially non-algebraic way.⟫
    Solve [e^y[x] x^2 - Cos[y[x]] + Sin[x] - y[x]==C[1], y[x]]
```

Graphs of several solutions using the values of $C$ generated in cvals are graphed with ContourPlot in Figure 6.8.

```
sol2 = sol[[1, 1]]/.y[x] → y
e^y x^2 - y - Cos[y] + Sin[x]

cvals = Table[sol2/.{x → -3Pi/2, y → i}, {i, 0, 6Pi, 6Pi/24}];
ContourPlot[sol2, {x, -3Pi, 3Pi}, {y, 0, 6Pi}, Contours → cvals,
    ContourShading → False, Axes → Automatic, Frame → False,
        AxesOrigin → {0, 0}, ContourStyle → GrayLevel[0]]
```

(b) We can write $\left(y^2+2xy\right)dx-x^2dy=0$ as $dy/dx=\left(y^2+2xy\right)/x^2$. A first-order equation is **homogeneous** if it can be written in the form $dy/dx=F\left(y/x\right)$. Homogeneous equations are reduced to separable equations with either the substitution $y=ux$ or $x=vy$. In this case, we have that $dy/dx=(y/x)^2+2(y/x)$, so the equation is homogeneous.

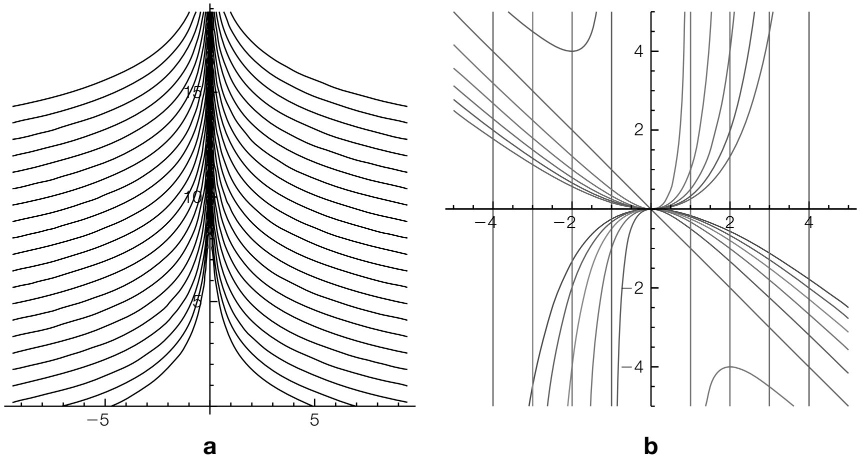



**FIGURE 6.8**

(a) Graphs of several solutions of $(\cos x + 2xe^y)\,dx + (\sin y + x^2 e^y - 1)\,dy = 0$. (b) Graphs of several solutions of $(y^2 + 2xy)\,dx - x^2 dy = 0$

Let $y = ux$. Then, $dy = u\,dx + x\,du$. Substituting into $(y^2 + 2xy)\,dx - x^2 dy = 0$ and separating gives us

$$\begin{aligned}(y^2 + 2xy)\,dx - x^2 dy &= 0\\ (u^2x^2 + 2ux^2)\,dx - x^2(u\,dx + x\,du) &= 0\\ (u^2 + 2u)\,dx - (u\,dx + x\,du) &= 0\\ (u^2 + u)\,dx &= x\,du\\ \frac{1}{u(u+1)}du &= \frac{1}{x}dx.\end{aligned}$$

Integrating the left- and right-hand sides of this equation with **Integrate**,

```
Integrate[1/(u(u + 1)), u]
Log[u] – Log[1 + u]
Integrate[1/x, x]
Log[x]
```

exponentiating, resubstituting $u = y/x$, and solving for $y$ gives us

$$\begin{aligned}\ln|u| - \ln|u+1| &= \ln|x| + C\\ \frac{u}{u+1} &= Cx\\ \frac{\frac{y}{x}}{\frac{y}{x}+1} &= Cx\\ y &= \frac{Cx^2}{1 - Cx}.\end{aligned}$$

**sol1 = Solve[(y/x)/(y/x + 1)==cx, y]**

$$\left\{\left\{y \to -\frac{cx^2}{-1+cx}\right\}\right\}$$

We confirm this result with **DSolve** and then graph several solutions with **Plot** in Figure 6.8(b).

**sol2 = DSolve[y[x]^2 + 2x y[x] − x^2 y′[x]==0, y[x], x]**

$$\left\{\left\{y[x] \to -\frac{x^2}{x - C[1]}\right\}\right\}$$

**toplot = Table[sol2[[1, 1, 2]]/.C[1] → i, {i, −5, 5}];**
**Plot[Tooltip[toplot], {x, −5, 5}, PlotRange → {−5, 5},**
**AspectRatio → Automatic]**

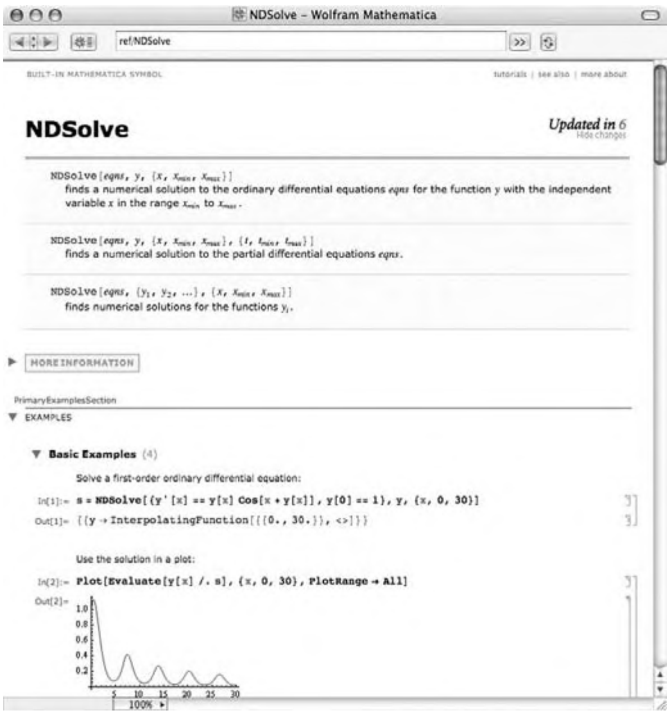
---

## 6.1.4 Numerical Methods

If numerical results are desired, use NDSolve:

**NDSolve[{y′[t]==f[t, y[t]], y[t0]==y0}, y[t], {t, a, b}]**

attempts to generate a numerical solution of $dy/dt = f(t, y)$, $y(t_0) = y_0$, valid for $a \le t \le b$.



**Example 6.1.7** Consider $dy/dt = \left(t^2 - y^2\right)\sin y$, $y(0) = -1$. (a) Determine $y(1)$. (b) Graph $y(t)$, $-1 \le t \le 10$.

***Solution*** We first remark that DSolve can neither exactly solve the differential equation $y' = \left(t^2 - y^2\right)\sin y$ nor find the solution that satisfies $y(0) = -1$.

```
sol = DSolve[y'[t] == (t^2 - y[t]^2) Sin[y[t]], y[t], t]
Solve::tdep : The equations appear to involve the variables to be solved for in an essentially non-algebraic way. >>
DSolve[y'[t] == Sin[y[t]] (t^2 - y[t]^2), y[t], t]

sol = DSolve[{y'[t] == (t^2 - y[t]^2) Sin[y[t]], y[0] == -1}, y[t], t]
Solve::tdep : The equations appear to involve the variables to be solved for in an essentially non-algebraic way. >>
DSolve[{y'[t] == Sin[y[t]] (t^2 - y[t]^2), y[0] == -1}, y[t], t]
```

However, we obtain a numerical solution valid for $0 \le t \le 1000$ using the NDSolve function.

```
sol = NDSolve[{y'[t]==(t^2-y[t]^2)Sin[y[t]], y[0]==-1}, y[t],
    {t, -1, 10}]
{{y[t] → InterpolatingFunction[{{-1., 10.}}, <>][t]}}
```

Entering sol /.t->1 evaluates the numerical solution if $t = 1$.

```
sol/.t → 1
{{y[1] → -0.766013}}
```

The result means that $y(1) \approx -0.766$. We use the Plot command to graph the solution for $0 \le t \le 10$ in Figure 6.9.

```
Plot[y[t]/.sol, {t, -1, 10}]
```

---

**Example 6.1.8** **(Logistic Equation with Predation).** Incorporating predation into the **logistic equation**, $y' = \alpha y\left(1 - \frac{1}{K}y\right)$, results in $\frac{dy}{dt} = \alpha y\left(1 - \frac{1}{K}y\right) - P(y)$, where $P(y)$ is a function of $y$ describing the rate of predation. A typical choice for $P$ is $P(y) = ay^2/(b^2 + y^2)$ because $P(0) = 0$ and $P$ is bounded above: $\lim_{t\to\infty} P(y) < \infty$.

**Remark 6.2** Of course, if $\lim_{t\to\infty} y(t) = Y$, then $\lim_{t\to\infty} P(y) = aY^2/(b^2 + Y^2)$. Generally, however, $\lim_{t\to\infty} P(y) \neq a$ because $\lim_{t\to\infty} y(t) \le K \neq \infty$, for some $K \ge 0$, in the predation situation.

If $\alpha = 1$, $a = 5$, and $b = 2$, graph the direction field associated with the equation as well as various solutions if (a) $K = 19$ and (b) $K = 20$.

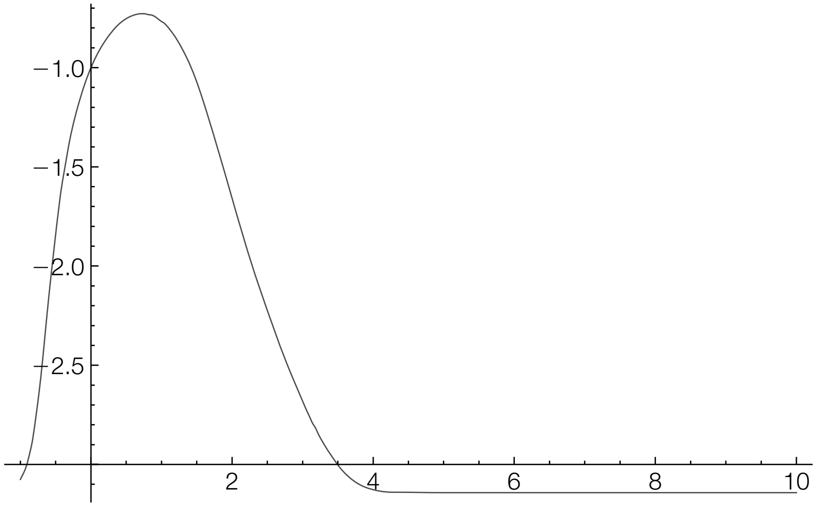



**FIGURE 6.9**

Graph of the solution to $y' = \left(t^2 - y^2\right) \sin y$, $y(0) = -1$

***Solution*** (a) We define eqn[k] to be $\dfrac{dy}{dt} = y\left(1 - \dfrac{1}{K}y\right) - \dfrac{5y^2}{4 + y^2}$.

```
Needs["VectorFieldPlots`"]
eqn[k_] = y′[t]==y[t](1 – y[t]/k) – 5y[t]^2/(4 + y[t]^2);
```

We use VectorPlotField to graph the direction field in Figure 6.10(a) and then the direction field along with the solutions that satisfy $y(0) = 0.5$, $y(0) = 0.2$, and $y(0) = 4$ in Figure 6.10(b).

```
pvf19 = Show[VectorFieldPlot[{1, y(1 – 1/19y) – 5y^2/(4 + y^2)}, {t, 0, 10},
   {y, 0, 6}, ScaleFunction → (1&), PlotPoints → 25],
      Axes → Automatic, AxesOrigin → {0, 0}];
numsols = Map[NDSolve[{eqn[19], y[0]==#}, y[t], {t, 0, 10}]&,
      {0.5, 2, 4}];
solplot = Plot[y[t]/.numsols, {t, 0, 10}, PlotRange → All,
      PlotStyle → {{GrayLevel[.4], Thickness[.01]}}];
Show[GraphicsRow[{pvf19, Show[pvf19, solplot]}]]
```

In the plot, notice that all nontrivial solutions appear to approach an equilibrium solution. We determine the equilibrium solution by solving $y' = 0$

```
eqn[19][[2]]
```

$\left(1 - \frac{y[t]}{19}\right) y[t] - \frac{5y[t]^2}{4 + y[t]^2}$.

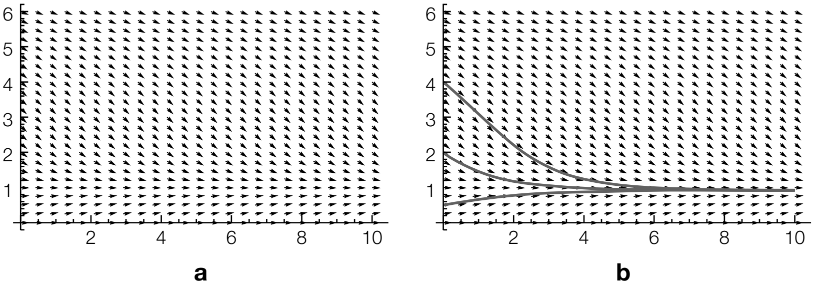



**FIGURE 6.10**

(a) Direction field and (b) direction field with three solutions

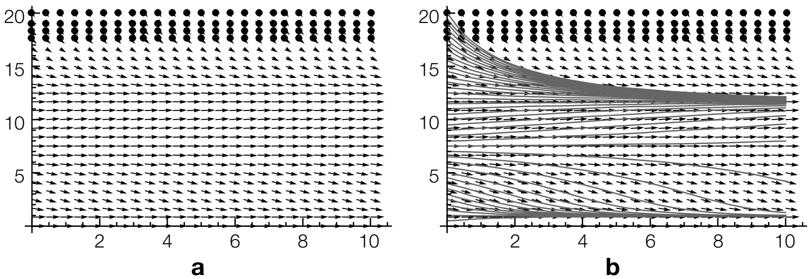



**FIGURE 6.11**

(a) Direction field and (b) direction field with several solutions

```
Solve[eqn[19.][[2]]==0, y[t]]
{{y[t] -> 0.}, {y[t] -> 0.923351}, {y[t] -> 9.03832 − 0.785875 I},
  {y[t] -> 9.03832 + 0.785875 I}}
```

to see that it is $y \approx 0.923$.

(b) We carry out similar steps for (b). First, we graph the direction field with PlotVectorField in Figure 6.11(a).

```
pvf20 = Show[VectorFieldPlot[{1, y(1 − 1/20y) − 5y^2/(4 + y^2)}, {t, 0, 10},
   {y, 0, 20}, ScaleFunction → (1&), PlotPoints → 25],
     Axes → Automatic, AxesOrigin → {0, 0},
     AspectRatio → 1/GoldenRatio];
```

We then use Map together with NDSolve to numerically find the solution satisfying $y(0) = 0.5i$, for $i = 1, 2, \ldots, 40$ and name the resulting list numsols. The functions contained in numsols are graphed with Plot in solplot.

```
numsols = Map[NDSolve[{eqn[20], y[0]==#}, y[t], {t, 0, 10}]&,
      Table[0.5i, {i, 1, 40}]];
   solplot = Plot[y[t]/.numsols, {t, 0, 10}, PlotRange → All,
      PlotStyle → {{GrayLevel[.4], Thickness[.005]}}];
```

Last, we display the direction field along with the solution graphs in **solplot** using **Show** in Figure 6.11(b).

```
Show[GraphicsRow[{pvf20, Show[pvf20, solplot]}]]
```

Notice that there are three nontrivial equilibrium solutions that are found by solving $y' = 0$.

```
Solve[eqn[20.][[2]]==0, y[t], t]
{{y[t] → 0.}, {y[t] → 0.926741}, {y[t] → 7.38645}, {y[t] → 11.6868}}
```

In this case, $y \approx 0.926$ and $y \approx 11.687$ are stable, whereas $y \approx 7.386$ is unstable.

## 6.2 SECOND-ORDER LINEAR EQUATIONS

We now present a concise discussion of second-order linear equations, which are extensively discussed in the introductory differential equations course.

### 6.2.1 Basic Theory

The **general form** of the **second-order linear equation** is

$$a_2(t)\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t), \tag{6.7}$$

where $a_2(t)$ is not identically the zero function.

The **standard form** of the second-order linear equation (6.7) is

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t). \tag{6.8}$$

The **corresponding homogeneous equation** of equation (6.8) is

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{6.9}$$

A **general solution** of equation (6.9) is $y = c_1 y_1 + c_2 y_2$, where

1. $y_1$ and $y_2$ are solutions of equation (6.9), and
2. $y_1$ and $y_2$ are *linearly independent*.

If $y_1$ and $y_2$ are solutions of equation (6.9), then $y_1$ and $y_2$ are **linearly independent** if and only if the **Wronskian**,

$$W(\{y_1, y_2\}) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2, \tag{6.10}$$

is not the zero function. If $y_1$ and $y_2$ are linearly independent solutions of equation (6.9), we call the set $S = \{y_1, y_2\}$ a **fundamental set of solutions** for equation (6.9).

A particular solution, $y_p$, is a solution that does not contain any arbitrary constants.

Let $y$ be a general solution of equation (6.8) and $y_p$ be a particular solution of equation (6.8). It follows that $y - y_p$ is a solution of equation (6.9), so $y - y_p = y_h$, where $y_h$ is a general solution of equation (6.9). Hence, $y = y_h + y_p$. That is, to solve the nonhomogeneous equation, we need a general solution, $y_h$, of the corresponding homogeneous equation and a particular solution, $y_p$, of the nonhomogeneous equation.

### 6.2.2 **Constant Coefficients**

Suppose that the coefficient functions of equation (6.7) are constants: $a_2(t) = a$, $a_1(t) = b$, and $a_0(t) = c$, and that $f(t)$ is identically the zero function. In this case, equation (6.7) becomes

$$ay'' + by' + cy = 0. \tag{6.11}$$

Now suppose that $y = e^{kt}$, $k$ constant, is a solution of equation (6.11). Then, $y' = ke^{kt}$ and $y'' = k^2 e^{kt}$. Substitution into equation (6.11) then gives us

$$\begin{aligned} ay'' + by' + cy &= ak^2e^{kt} + bke^{kt} + ce^{kt} \\ &= e^{kt}\left(ak^2 + bk + c\right) = 0. \end{aligned}$$

Because $e^{kt} \neq 0$, the solutions of equation (6.11) are determined by the solutions of

$$ak^2 + bk + c = 0, \tag{6.12}$$

called the **characteristic equation** of equation (6.11).

**Theorem 1.** *Let $k_1$ and $k_2$ be the solutions of equation* (6.12).

1. *If $k_1 \neq k_2$ are real and distinct, two linearly independent solutions of equation* (6.11) *are $y_1 = e^{k_1 t}$ and $y_2 = e^{k_2 t}$; a general solution of equation* (6.11) *is*

$$y = c_1 e^{k_1 t} + c_2 e^{k_2 t}.$$

2. *If $k_1 = k_2$, two linearly independent solutions of equation* (6.11) *are $y_1 = e^{k_1 t}$ and $y_2 = te^{k_1 t}$; a general solution of equation* (6.11) *is*

$$y = c_1 e^{k_1 t} + c_2 t e^{k_1 t}.$$

3. *If $k_{1,2} = \alpha \pm \beta i$, $\beta \neq 0$, two linearly independent solutions of equation* (6.11) *are $y_1 = e^{\alpha t}\cos\beta t$ and $y_2 = e^{\alpha t}\sin\beta t$; a general solution of equation* (6.11) *is*

$$y = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right).$$

**Example 6.2.1** Solve each of the following equations: (a) $6y'' + y' - 2y = 0$; (b) $y'' + 2y' + y = 0$; (c) $16y'' + 8y' + 145y = 0$.

***Solution*** (a) The characteristic equation is $6k^2 + k - 2 = (3k + 2)(2k - 1) = 0$ with solutions $k = -2/3$ and $k = 1/2$. We check with either Factor or Solve.

**Factor[6k^2 + k - 2]**
(−1 + 2k)(2 + 3k)
**Solve[6k^2 + k - 2==0]**
$\{\{k \to -\frac{2}{3}\}, \{k \to \frac{1}{2}\}\}$

Then, a fundamental set of solutions is $\{e^{-2t/3}, e^{t/2}\}$ and a general solution is

$$y = c_1 e^{-2t/3} + c_2 e^{t/2}.$$

Of course, we obtain the same result with DSolve.

**Clear[y]**
**DSolve[6y''[t] + y'[t] - 2y[t]==0, y[t], t]**
$\{\{y[t] \to e^{-2t/3}C[1] + e^{t/2}C[2]\}\}$

(b) The characteristic equation is $k^2 + 2k + 1 = (k + 1)^2 = 0$ with solution $k = -1$, which has multiplicity two, so a fundamental set of solutions is $\{e^{-t}, te^{-t}\}$ and a general solution is

$$y = c_1 e^{-t} + c_2 te^{-t}.$$

We check the calculation in the same way as in (a).

**Factor[k^2 + 2k + 1]**
**Solve[k^2 + 2k + 1==0]**
**DSolve[y''[t] + 2y'[t] + y[t]==0, y[t], t]**
$(1 + k)^2$
$\{\{k \to -1\}, \{k \to -1\}\}$
$\{\{y[t] \to e^{-t}C[1] + e^{-t}tC[2]\}\}$

(c) The characteristic equation is $16k^2 + 8k + 145 = 0$ with solutions $k_{1,2} = -\frac{1}{4} \pm 3i$, so a fundamental set of solutions is $\left\{e^{-t/4}\cos 3t, e^{-t/4}\sin 3t\right\}$ and a general solution is

$$y = e^{-t/4}\left(c_1 \cos 3t + c_2 \sin 3t\right).$$

The calculation is verified in the same way as in (a) and (b).

**Factor[16k^2 + 8k + 145, GaussianIntegers → True]**
((1 − 12i) + 4k)((1 + 12i) + 4k)

**Solve[16k^2 + 8k + 145==0]**
$\{\{k \to -\frac{1}{4} - 3i\}, \{k \to -\frac{1}{4} + 3i\}\}$

**DSolve[16y″[t] + 8y′[t] + 145y[t]==0, y[t], t]**
$\{\{y[t] \to e^{-t/4}C[2]Cos[3t] + e^{-t/4}C[1]Sin[3t]\}\}$

---

**Example 6.2.2** Solve $64\frac{d^2y}{dt^2} + 16\frac{dy}{dt} + 1025y = 0,\ y(0) = 1,\ \frac{dy}{dt}(0) = 3.$

***Solution*** A general solution of $64y'' + 16y' + 1025y = 0$ is $y = e^{-t/8}\left(c_1 \sin 4t + c_2 \cos 4t\right)$.

**gensol = DSolve[64y″[t] + 16y′[t] + 1025y[t]==0, y[t], t]**
$\{\{y[t] \to e^{-t/8}C[2]Cos[4t] + e^{-t/8}C[1]Sin[4t]\}\}$

Applying $y(0) = 1$ shows us that $c_2 = 1$.

**e1 = gensol[[1, 1, 2]]/.t → 0**
C[2]

Computing $y'$

**D[y[t]/.gensol[[1]], t]**
$4e^{-t/8}C[1]Cos[4t] - \frac{1}{8}e^{-t/8}C[2]Cos[4t] - \frac{1}{8}e^{-t/8}C[1]Sin[4t] - 4e^{-t/8}C[2]Sin[4t]$

and then $y'(0)$, shows us that $-4c_1 - \frac{1}{8}c_2 = 3$.

**e2 = D[y[t]/.gensol[[1]], t]/.t → 0**
$4C[1] - \frac{C[2]}{8}$

Solving for $c_1$ and $c_2$ with **Solve** shows us that $c_1 = -25/32$ and $c_1 = 1$.

**cvals = Solve[{e1==1, e2==3}]**
$\{\{C[1] \to \frac{25}{32}, C[2] \to 1\}\}$

Thus, $y = e^{-t/8}\left(\frac{-25}{32}\sin 4t + \cos 4t\right)$, which we graph with **Plot** in Figure 6.12.

**sol = y[t]/.gensol[[1]]/.cvals[[1]]**
$e^{-t/8}Cos[4t] + \frac{25}{32}e^{-t/8}Sin[4t]$

**Plot[sol, {t, 0, 8Pi}]**

We verify the calculation with **DSolve**.

**DSolve[{64y″[t] + 16y′[t] + 1025y[t]==0, y[0]==1, y′[0]==3}, y[t], t]**
$\{\{y[t] \to \frac{1}{32}e^{-t/8}(32Cos[4t] + 25Sin[4t])\}\}$

---

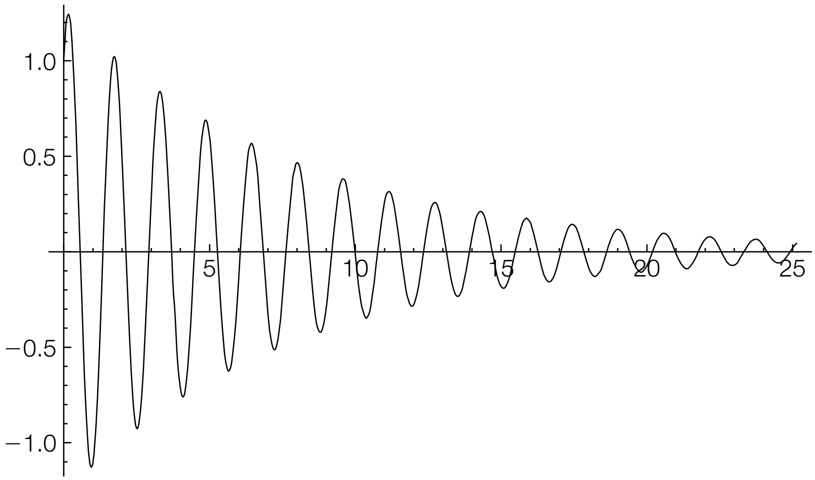



**FIGURE 6.12**

The solution to the initial-value problem tends to 0 as $t \to \infty$

### *Application: Harmonic Motion*

Suppose that a mass is attached to an elastic spring that is suspended from a rigid support such as a ceiling. According to Hooke's law, the spring exerts a restoring force in the upward direction that is proportional to the displacement of the spring.

> ***Hooke's Law:*** *$F = ks$, where $k > 0$ is the constant of proportionality or spring constant, and $s$ is the displacement of the spring.*

Using Hooke's law and assuming that $x(t)$ represents the displacement of the mass from the equilibrium position at time $t$, we obtain the initial-value problem

$$m\frac{d^2x}{dt^2} + kx = 0, \quad x(0) = \alpha, \quad \frac{dx}{dt}(0) = \beta.$$

Note that the initial conditions give the initial displacement and velocity, respectively. This differential equation disregards all retarding forces acting on the motion of the mass and a more realistic model that takes these forces into account is needed. Studies in mechanics reveal that resistive forces due to damping are proportional to a power of the velocity of the motion. Hence, $F_R = a\,dx/dt$ or $F_R = a\,(dx/dt)^3$, where $a > 0$, are typically used to represent the damping force. Then, we have the following initial-value problem assuming that $F_R = a\,dx/dt$:

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0, \quad x(0) = \alpha, \quad \frac{dx}{dt}(0) = \beta.$$

Problems of this type are characterized by the value of $a^2 - 4mk$ as follows:

1. $a^2 - 4mk > 0$. This situation is said to be **overdamped** because the damping coefficient $a$ is large in comparison to the spring constant $k$.
2. $a^2 - 4mk = 0$. This situation is described as **critically damped** because the resulting motion is oscillatory with a slight decrease in the damping coefficient $a$.
3. $a^2 - 4mk < 0$. This situation is called **underdamped** because the damping coefficient $a$ is small in comparison with the spring constant $k$.

---

**Example 6.2.3** Classify the following differential equations as overdamped, underdamped, or critically damped. Also, solve the corresponding initial-value problem using the given initial conditions and investigate the behavior of the solutions.

**(a)** $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$ subject to $x(0) = 0$ and $\frac{dx}{dt}(0) = 1$;

**(b)** $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0$ subject to $x(0) = 1$ and $\frac{dx}{dt}(0) = 1$; and

**(c)** $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 16x = 0$ subject to $x(0) = 0$ and $\frac{dx}{dt}(0) = 1$.

***Solution*** For (a), we identify $m = 1$, $a = 8$, and $k = 16$ so that $a^2 - 4mk = 0$, which means that the differential equation $x'' + 8x' + 16x = 0$ is critically damped. After defining de1, we solve the equation subject to the initial conditions and name the resulting output sol1. We then graph the solution shown in Figure 6.13(a).

```
Clear[de1, x, t]
de1 = x''[t] + 8x'[t] + 16x[t]==0;
sol1 = DSolve [{de1, x[0]==0, x'[0]==1}, x[t], t]
{{x[t] → e^-4t t}}
p1 = Plot[sol1[[1, 1, 2]], {t, 0, 4}]
```

For (b), we proceed in the same manner. We identify $m = 1$, $a = 5$, and $k = 4$ so that $a^2 - 4mk = 9$ and the equation $x'' + 5x' + 4x = 0$ is overdamped. We then

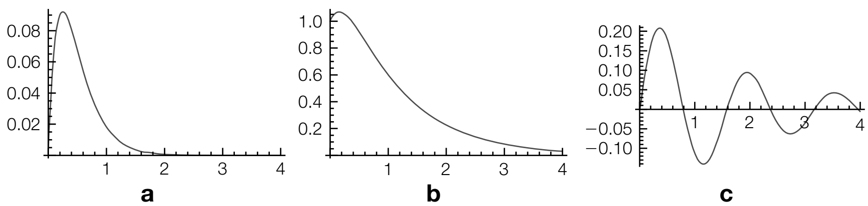



**FIGURE 6.13**

(a) Critically damped motion. (b) Overdamped motion. (c) Underdamped motion define **de2** to be the equation and the solution to the initial-value problem obtained with **DSolve**, **sol2**, and then graph $x(t)$ on the interval $[0, 4]$ in Figure 6.13(b).

```
Clear[de2, x, t]
de2 = x''[t] + 5x'[t] + 4x[t]==0;
sol2 = DSolve [{de2, x[0]==1, x'[0]==1}, x[t], t]
```

$\{\{x[t] \to \frac{1}{3}e^{-4t}(-2+5e^{3t})\}\}$

```
p2 = Plot[sol2[[1, 1, 2]], {t, 0, 4}]
```

For (c), we proceed in the same manner as in (a) and (b) to show that the equation is underdamped because the value of $a^2 - 4mk$ is $-63$. See Figure 6.13(c).

---

You can also use **Manipulate** to help you visualize harmonic motion. With

```
Manipulate[
  sol = DSolve[{mx''[t] + ax'[t] + kx[t]==0, x[0]==0, x'[0]==1}, x[t], t];
  Plot[x[t]/.sol, {t, 0, 5}, PlotRange → {−1/2, 1/2}, AspectRatio → 1],
    {{m, 1}, 0, 5}, {{a, 8}, 0, 15, 1}, {{k, 16}, 0, 20, 1}]
```

we generate a **Manipulate** object that lets us investigate harmonic motion for various values of $m$, $a$, and $k$ if the initial position is zero ($x(0) = 0$) and the initial velocity is one ($x'(0) = 1$). See Figure 6.14. (Note that $m$ is centered at 1, $a$ at 8, and $k$ at 16.)

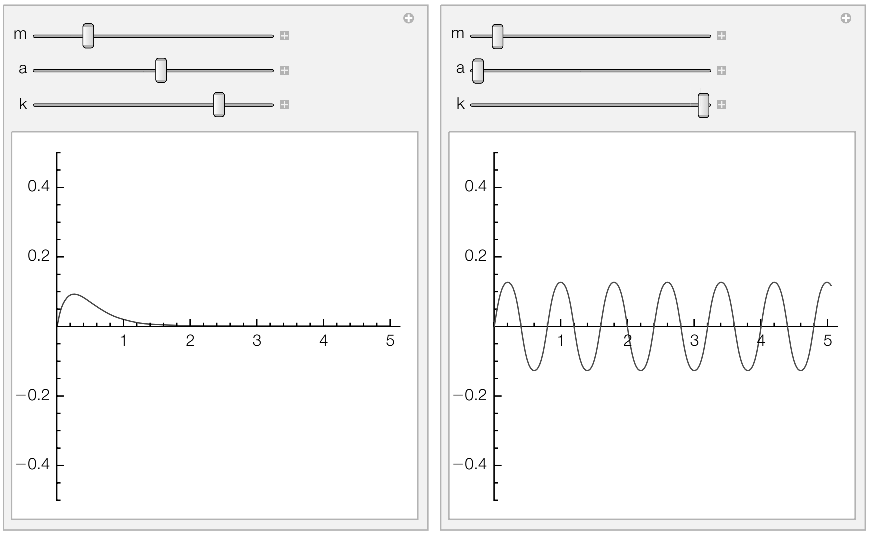



**FIGURE 6.14**

Using **Manipulate** to investigate harmonic motion

## 6.2.3 Undetermined Coefficients

If equation (6.7) has constant coefficients and $f(t)$ is a product of terms $t^n$, $e^{\alpha t}$, $\alpha$ constant, $\cos\beta t$, and/or $\sin\beta t$, $\beta$ constant, ***undetermined coefficients*** can often be used to find a particular solution of equation (6.7). The key to implementing the method is to ***judiciously*** choose the correct form of $y_p$.

Assume that a general solution, $y_h$, of the corresponding homogeneous equation has been found and that each term of $f(t)$ has the form

$$t^n e^{\alpha t} \cos\beta t \qquad \text{or} \qquad t^n e^{\alpha t} \sin\beta t.$$

For ***each*** term of $f(t)$, write down the ***associated set***

$$F = \left\{t^n e^{\alpha t}\cos\beta t, t^n e^{\alpha t}\sin\beta t, t^{n-1}e^{\alpha t}\cos\beta t, t^{n-1}e^{\alpha t}\sin\beta t, \ldots, e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t,\right\}.$$

If any element of $F$ is a solution to the corresponding homogeneous equation, multiply each element of $F$ by $t^m$, where $m$ is the smallest positive integer so that none of the elements of $t^m F$ are solutions to the corresponding homogeneous equation. A particular solution will be a linear combination of the functions in all the $F$'s.

---

**Example 6.2.4** Solve $4\dfrac{d^2y}{dt^2} - y = t - 2 - 5\cos t - e^{-t/2}$.

***Solution*** The corresponding homogeneous equation is $4y'' - y = 0$ with general solution $y_h = c_1 e^{-t/2} + c_2 e^{t/2}$.

**DSolve[4y''[t] – y[t]==0, y[t], t]**
$\{\{y[t] \to e^{t/2}C[1] + e^{-t/2}C[2]\}\}$

A fundamental set of solutions for the corresponding homogeneous equation is $S = \left\{e^{-t/2}, e^{t/2}\right\}$. The associated set of functions for $t - 2$ is $F_1 = \{1, t\}$, the associated set of functions for $-5\cos t$ is $F_2 = \{\cos t, \sin t\}$, and the associated set of functions for $-e^{-t/2}$ is $F_3 = \left\{e^{-t/2}\right\}$. Note that $e^{-t/2}$ is an element of $S$ so we multiply $F_3$ by $t$ resulting in $tF_3 = \left\{te^{-t/2}\right\}$. Then, we search for a particular solution of the form

No element of $F_1$ is contained in $S$ and no element of $F_2$ is contained in $S$.

$$y_p = A + Bt + C\cos t + D\sin t + Ete^{-t/2},$$

where $A$, $B$, $C$, $D$, and $E$ are constants to be determined.

We do not use capital letters so as to avoid any confusion with built-in Mathematica commands.

**yp[t_] = a + bt + c Cos[t] + d Sin[t] + et Exp[– t/2]**
$a + b\,t + e e^{-t/2} t + c\,\mathrm{Cos}[t] + d\,\mathrm{Sin}[t]$

Computing $y_p'$ and $y_p''$

**dyp = yp'[t]**
$b + e\,e^{-t/2} - \frac{1}{2} e\,e^{-t/2} t + d\,\mathrm{Cos}[t] - c\,\mathrm{Sin}[t]$
**d2yp=yp''[t]**
$-e\,e^{-t/2} + \frac{1}{4} e\,e^{-t/2} t - c\,\mathrm{Cos}[t] - d\,\mathrm{Sin}[t]$

and substituting into the nonhomogeneous equation results in

$$-A - Bt - 5C\cos t - 5D\sin t - 4Ee^{-t/2} = t - 2 - 5\cos t - e^{-t/2}.$$

**eqn = 4yp″[t] − yp[t]==t − 2 − 5Cos[t] − Exp[ − t/2]**

$-a-bt-ee^{-t/2}t-c\text{Cos}[t]-d\text{Sin}[t]+4\left(-ee^{-t/2}+\frac{1}{4}ee^{-t/2}t-c\text{Cos}[t]-d\text{Sin}[t]\right)==$
$-2-e^{-t/2}+t-5\text{Cos}[t]$

Equating coefficients results in

$$-A = -2 \qquad -B = 1 \qquad -5C = -5 \qquad -5D = 0 \qquad -4E = -1$$

so $A = 2$, $B = -1$, $C = 1$, $D = 0$, and $E = 1/4$.

**cvals = Solve[{ − a== − 2, − b==1, − 5c== − 5, − 5d==0, − 4e== − 1}]**

$\{\{a \to 2, b \to -1, c \to 1, d \to 0, e \to \frac{1}{4}\}\}$

$y_p$ is then given by $y_p = 2 - t + \cos t + \frac{1}{4}te^{-t/2}$

**yp[t]/.cvals[[1]]**

$2-t+\frac{1}{4}e^{-t/2}t+\text{Cos}[t]$

and a general solution is given by

$$y = y_h + y_p = c_1e^{-t/2} + c_2e^{t/2} + 2 - t + \cos t + \frac{1}{4}te^{-t/2}.$$

Note that $-A - Bt - 5C\cos t - 5D\sin t - 4Ee^{-t/2} = t - 2 - 5\cos t - e^{-t/2}$ is true for *all* values of $t$. Evaluating for five different values of $t$ gives us five equations that we then solve for $A$, $B$, $C$, $D$, and $E$, resulting in the same solutions as already obtained.

**e1 = eqn/.t → 0**

$-a-c+4(-c-e)==-8$

**e2 = eqn/.t → 1;**

**e3 = eqn/.t → 2;**

**e4 = eqn/.t → 3;**

**e5 = eqn/.t → 4;**

**Solve[{e1, e2, e3, e4, e5}]//Simplify**

$\{\{b \to -1, d \to 0, a \to 2, c \to 1, e \to \frac{1}{4}\}\}$

Last, we check our calculation with DSolve and Simplify.

**DSolve[4y″[t] − y[t]==t − 2 − 5Cos[t] − Exp[ − t/2], y[t], t]//Simplify**

$\{\{y[t] \to \frac{1}{4}e^{-t/2}\left(1-4e^{t/2}(-2+t)+t+4e^{t}C[1]+4C[2]+4e^{t/2}\text{Cos}[t]\right)\}\}$

**Example 6.2.5** Solve $y'' + 4y = \cos 2t,\ y(0) = 0,\ y'(0) = 0$.

***Solution*** A general solution of the corresponding homogeneous equation is $y_h = c_1 \cos 2t + c_2 \sin 2t$. For this equation, $F = \{\cos 2t, \sin 2t\}$. Because elements of $F$ are solutions to the corresponding homogeneous equation, we multiply each element of $F$ by $t$ resulting in $tF = \{t\cos 2t,\ t\sin 2t\}$. Therefore, we assume that a particular solution has the form

$$y_p = At\cos 2t + Bt\sin 2t,$$

where $A$ and $B$ are constants to be determined. Proceeding in the same manner as before, we compute $y_p'$ and $y_p''$

```
yp[t_] = a tCos[2t] + b tSin[2t]
yp'[t]
yp''[t]
a t Cos[2t] + b t Sin[2t]
a Cos[2t] + 2b tCos[2t] + b Sin[2t] – 2a t Sin[2t]
4b Cos[2t] – 4a t Cos[2t] – 4a Sin[2t] – 4bt Sin[2t]
```

and then substitute into the nonhomogeneous equation.

```
eqn = yp''[t] + 4yp[t]==Cos[2t]
4b Cos[2t] – 4a t Cos[2t] – 4a Sin[2t] – 4b t Sin[2t] + 4(a t Cos[2t] + b t Sin[2t])
   ==Cos[2t]
```

Equating coefficients readily yields $A = 0$ and $B = 1/4$. Alternatively, remember that $-4A\sin 2t + 4B\cos 2t = \cos 2t$ is true for *all* values of $t$. Evaluating for two values of $t$ and then solving for $A$ and $B$ gives the same result.

```
e1 = eqn/.t → 0
e2 = eqn/.t → 1
cvals = Solve[{e1, e2}]
4b==1
– 4a Cos[2] + 4b Cos[2] – 4a Sin[2] – 4b Sin[2] + 4(a Cos[2] + b Sin[2])==Cos[2]
{{a → 0, b → 1/4}}
```

It follows that $y_p = \frac{1}{4}t\sin 2t$ and $y = c_1\cos 2t + c_2\sin 2t + \frac{1}{4}t\sin 2t$.

```
yp[t]/.cvals[[1]]
1/4 tSin[2t]
y[t_] = c1Cos[2t] + c2Sin[2t] + 1/4tSin[2t]
c1 Cos[2t] + c2 Sin[2t] + 1/4 t Sin[2t]
```

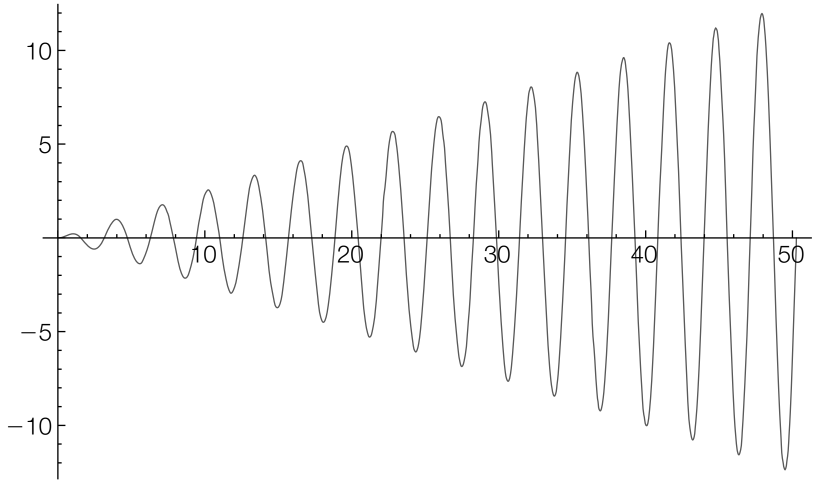



**FIGURE 6.15**

The forcing function causes the solution to become unbounded as $t \to \infty$

Applying the initial conditions after finding $y'$

**y′[t]**
2c2 Cos[2t] + $\frac{1}{2}$t Cos[2t] + $\frac{1}{4}$Sin[2t] − 2c1Sin[2t]
**cvals = Solve[{y[0]==0, y′[0]==0}]**
{{c1 → 0, c2 → 0}}

results in $y = \frac{1}{4} t \sin 2t$, which we graph with Plot in Figure 6.15.

**y[t]/.cvals[[1]]**
$\frac{1}{4}$tSin[2t]
**Plot[y[t]/.cvals, {t, 0, 16Pi}]**

We verify the calculation with DSolve.

**Clear[y]**
**DSolve[{y″[t] + 4y[t]==Cos[2t], y[0]==0, y′[0]==0},**
**y[t], t]//Simplify**
{{y[t] → $\frac{1}{4}$tSin[2t]}}

---

Use Manipulate to help you see how changing parameter values and equations affect a system. With

**Manipulate[**
**sol1 = DSolve[{mx″[t] + ax′[t] + kx[t]==ΓCos[ωt], x[0]==0, x′[0]==0}, x[t], t];**
**sol2 = NDSolve[{mx″[t] + ax′[t] + k Sin[x[t]]==ΓCos[ωt], x[0]==0, x′[0]==0},**
**x[t], {t, 0, 50}];**

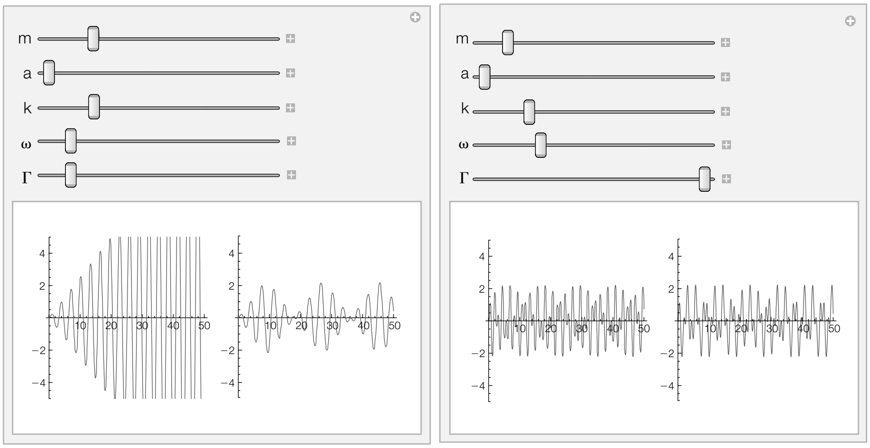


**FIGURE 6.16**

Comparing solutions of nonlinear initial-value problems to their corresponding linear approximations

```
p1 = Plot[x[t]/.sol1, {t, 0, 50}, PlotRange → {−5, 5}, AspectRatio → 1];
p2 = Plot[x[t]/.sol2, {t, 0, 50}, PlotRange → {−5, 5}, AspectRatio → 1];
Show[GraphicsRow[{p1, p2}]], {{m, 1}, 0, 5}, {{a, 0}, 0, 15, 1}, {{k, 4}, 0, 20, 1},
    {{ω, 2}, 0, 20, 1}, {{Γ, 1}, 0, 10, 1}]
```

we can compare the solution of $mx'' + ax' + kx = \Gamma\cos\omega t$, $x(0) = 0$, $x'(0) = 0$ to the solution of $mx'' + ax' + k\sin x = \Gamma\cos\omega t$, $x(0) = 0$, $x'(0) = 0$ for various values of $m$, $a$, $k$, $\omega$, and $\Gamma$. See Figure 6.16.

**Example 6.2.6** **(Hearing Beats and Resonance).** In order to *hear* beats and resonance, we solve the initial-value problem

$$x'' + \omega^2 x = F\cos\beta t, \quad x(0) = \alpha, \quad x'(0) = \beta, \tag{6.13}$$

for each of the following parameter values: (a) $\omega^2 = 6000^2$, $\beta = 5991.62$, $F = 2$; and (b) $\omega^2 = 6000^2$, $\beta = 6000$, $F = 2$.

First, we define the function **sol** which, when given the parameters, solves the initial-value problem (6.13).

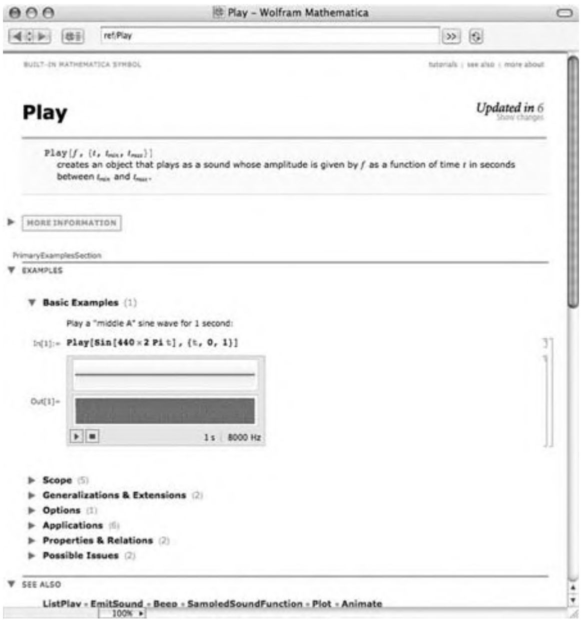
```
Clear[x, t, f, sol]
sol[ω_, β_, f_]:=DSolve[{
x''[t] + ω^2x[t]==fCos[βt], x[0]==0, x'[0]==0},
x[t], t][[1, 1, 2]]
```

Thus, our solution for (a) is obtained by entering

**a = sol[6000, 5991.62, 2]**

−0.0000199025Cos[6000.t] + 0.0000198886Cos[8.38t]Cos[6000.t] +
1.389859474088294`*^−8Cos[6000.t]Cos[11991.6t] +
0.0000198886Sin[8.38t]Sin[6000.t] +
1.389859474088294`*^−8Sin[6000.t]Sin[11991.6t]

To *hear* the function we use **Play** in the same way that we use **Plot** to *see* functions.




The values of **a** correspond to the amplitude of the sound as a function of time. See Figure 6.17(a).

**Play[a, {t, 0, 6}]**

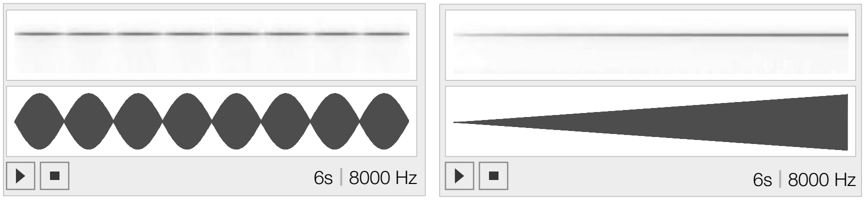
Similarly, the solution for (b) is obtained by entering

**b = sol[6000., 6000., 2]//Chop**

−2.777`*^−8 Cos[6000.t]+2.777`*^−8 Cos[6000.t]$^{3}$+
0.000166667 t Sin[6000.t]+1.3889`*^−8 Sin[6000.t] Sin[12000.t]

We hear resonance with **Play**. See Figure 6.17(b).

**Play[b, {t, 0, 6}]**




a b

**FIGURE 6.17**

Hearing and seeing beats and resonance: (a) Beats (b) Resonance

## 6.2.4 Variation of Parameters

A particular solution, $y_p$, is a solution that does not contain any arbitrary constants.

Let $S = \{y_1, y_2\}$ be a fundamental set of solutions for equation (6.9). To solve the nonhomogeneous equation (6.8), we need to find a particular solution, $y_p$ of equation (6.8). We search for a particular solution of the form

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t), \tag{6.14}$$

Observe that it is pointless to search for solutions of the form $y_p = c_1y_1 + c_2y_2$, where $c_1$ and $c_2$ are constants, because for every choice of $c_1$ and $c_2$, $c_1y_1 + c_2y_2$ is a solution to the corresponding homogeneous equation.

where $u_1$ and $u_2$ are functions of $t$. Differentiating equation (6.14) gives us

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

Assuming that

$$y_1u_1' + y_2u_2' = 0 \tag{6.15}$$

results in $y_p' = u_1y_1' + u_2y_2'$. Computing the second derivative then yields

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

Substituting $y_p$, $y_p'$, and $y_p''$ into equation (6.8) and using the facts that

$$u_1\left(y_1'' + py_1' + qy_1\right) = 0 \quad \text{and} \quad u_2\left(y_2'' + py_2' + qy_2\right) = 0$$

(because $y_1$ and $y_2$ are solutions to the corresponding homogeneous equation) results in

$$\begin{aligned}\frac{d^2y_p}{dt^2} + p(t)\frac{dy_p}{dt} + q(t)y_p &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(t)\left(u_1y_1' + u_2y_2'\right) \\ &\quad + q(t)\left(u_1y_1 + u_2y_2\right) \\ &= y_1'u_1' + y_2'u_2' = f(t). \end{aligned} \tag{6.16}$$

Observe that equation (6.15) and equation (6.16) form a system of two linear equations in the unknowns $u_1'$ and $u_2'$:

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(t). \end{aligned} \tag{6.17}$$

Applying Cramer's Rule gives us

$$u_1{}' = \frac{\begin{vmatrix} 0 & y_2 \\ f(t) & y_2{}' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}} = -\frac{y_2(t)f(t)}{W(S)} \quad \text{and} \quad u_2{}' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1{}' & f(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}} = \frac{y_1(t)f(t)}{W(S)}, \tag{6.18}$$

where $W(S)$ is the Wronskian, $W(S) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}$. After integrating to obtain $u_1$ and $u_2$, we form $y_p$ and then a general solution, $y = y_h + y_p$.

---

**Example 6.2.7** Solve $y'' + 9y = \sec 3t$, $y(0) = 0$, $y'(0) = 0$, $0 \le t < \pi/6$.

***Solution*** The corresponding homogeneous equation is $y'' + 9y = 0$ with general solution $y_h = c_1 \cos 3t + c_2 \sin 3t$. Then, a fundamental set of solutions is $S = \{\cos 3t, \sin 3t\}$ and $W(S) = 3$, as we see using Det, and Simplify.

```
fs = {Cos[3t], Sin[3t]};
wm = {fs, D[fs, t]};
wm//MatrixForm
wd = Det[wm]//Simplify
```

$$\begin{pmatrix} \text{Cos[3t]} & \text{Sin[3t]} \\ -\text{3Sin[3t]} & \text{3Cos[3t]} \end{pmatrix}$$

3

We use equation (6.18) to find $u_1 = \frac{1}{9} \ln \cos 3t$ and $u_2 = \frac{1}{3} t$.

```
u1 = Integrate[-Sin[3t]Sec[3t]/3, t]
u2 = Integrate[Cos[3t]Sec[3t]/3, t]
```

$\frac{1}{9}$Log[Cos[3t]]

$\frac{t}{3}$

Absolute value is not needed in the antiderivatives because we are restricting the domain to $0 \le t < \pi/6$ and $\cos t > 0$ on this interval.

It follows that a particular solution of the nonhomogeneous equation is $y_p = \frac{1}{9} \cos 3t \ln \cos 3t + \frac{1}{3} t \sin 3t$ and a general solution is $y = y_h + y_p = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{9} \cos 3t \ln \cos 3t + \frac{1}{3} t \sin 3t$.

```
yp = u1Cos[3t] + u2Sin[3t]
```

$\frac{1}{9}$Cos[3t]Log[Cos[3t]] + $\frac{1}{3}$tSin[3t]

Identical results are obtained using DSolve.

The negative sign in the output does not affect the result because C[1] is arbitrary.

```
DSolve[y''[t] + 9y[t]==Sec[3t], y[t], t]
```

{{y[t] → C[1]Cos[3t] + C[2]Sin[3t]+
 $\frac{1}{9}$(Cos[3t]Log[Cos[3t]] + 3tSin[3t])}}

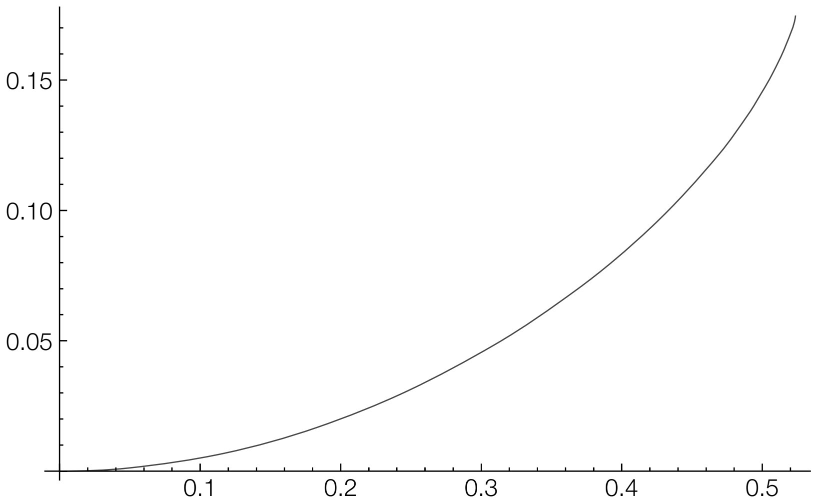



**FIGURE 6.18**

The domain of the solution is $0 \le t < \pi/6$

Applying the initial conditions gives us $c_1 = c_2 = 0$, so we conclude that the solution to the initial-value problem is $y = \frac{1}{9}\cos 3t \ln \cos 3t + \frac{1}{3} t \sin 3t$.

```
sol = DSolve[{y''[t] + 9y[t]==Sec[3t], y[0]==0, y'[0]==0}, y[t], t]
{{y[t] → 1/9(Cos[3t]Log[Cos[3t]] + 3tSin[3t])}}
```

We graph the solution with **Plot** in Figure 6.18.

```
Plot[y[t]/.sol, {t, 0, Pi/6}]
```

# 6.3 HIGHER-ORDER LINEAR EQUATIONS

## 6.3.1 Basic Theory

The **standard form of the *n*th-order linear equation** is

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t). \tag{6.19}$$

The **corresponding homogeneous equation** of equation (6.19) is

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0. \tag{6.20}$$

Let $y_1, y_2, \ldots, y_n$ be $n$ solutions of equation (6.20). The set $S = \{y_1, y_2, \ldots, y_n\}$ is **linearly independent** if and only if the **Wronskian**,

$$W(S) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} & \cdots & y_n^{(3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{6.21}$$

is not identically the zero function. $S$ is **linearly dependent** if $S$ is not linearly independent.

If $y_1, y_2, \ldots, y_n$ are $n$ linearly independent solutions of equation (6.20), we say that $S = \{y_1, y_2, \ldots, y_n\}$ is a **fundamental set** for equation (6.20), and a **general solution** of equation (6.20) is $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n$.

A **general solution** of equation (6.19) is $y = y_h + y_p$, where $y_h$ is a general solution of the corresponding homogeneous equation and $y_p$ is a particular solution of equation (6.19).

## 6.3.2 **Constant Coefficients**

If

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

has real constant coefficients, we assume that $y = e^{kt}$ and find that $k$ satisfies the **characteristic equation**

$$k^n + a_{n-1}k^{n-1} + \cdots + a_1 k + a_0 = 0. \tag{6.22}$$

If a solution $k$ of equation (6.22) has multiplicity $m$, $m$ linearly independent solutions corresponding to $k$ are

$$e^{kt}, te^{kt}, \ldots, t^{m-1}e^{kt}.$$

If a solution $k = \alpha + \beta i$, $\beta \neq 0$, of equation (6.22) has multiplicity $m$, $2m$ linearly independent solutions corresponding to $k = \alpha + \beta i$ (and $k = \alpha - \beta i$) are

$$e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, te^{\alpha t}\cos\beta t, te^{\alpha t}\sin\beta t, \ldots, t^{m-1}e^{\alpha t}\cos\beta t, t^{m-1}e^{\alpha t}\sin\beta t.$$

**Example 6.3.1** Solve $12y''' - 5y'' - 6y' - y = 0$.

***Solution*** The characteristic equation is

$$12k^3 - 5k^2 - 6k - 1 = (k-1)(3k+1)(4k+1) = 0$$

Factor[expression] attempts to factor expression.

with solutions $k_1 = -1/3$, $k_2 = -1/4$, and $k_3 = 1$.

```
Factor[12k^3 - 5k^2 - 6k - 1]
(-1 + k)(1 + 3k)(1 + 4k)
```

Thus, three linearly independent solutions of the equation are $y_1 = e^{-t/3}$, $y_2 = e^{-t/4}$, and $y_3 = e^t$; a general solution is $y = c_1 e^{-t/3} + c_2 e^{-t/4} + c_3 e^t$. We check with DSolve.

```
Clear[y]
DSolve[12y'''[t] - 5y''[t] - 6y'[t] - y[t]==0, y[t], t]
{{y[t] → e^(-t/4)C[1] + e^(-t/3)C[2] + e^t C[3]}}
```

---

**Example 6.3.2** Solve $y''' + 4y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$.

***Solution*** The characteristic equation is $k^3 + 4k = k(k^2 + 4) = 0$ with solutions $k_1 = 0$ and $k_{2,3} = \pm 2i$ that are found with Solve.

Enter ?Solve to obtain basic help regarding the Solve function.

```
Solve[k^3 + 4k==0]
{{k → 0}, {k → -2i}, {k → 2i}}
```

Three linearly independent solutions of the equation are $y_1 = 1$, $y_2 = \cos 2t$, and $y_3 = \sin 2t$. A general solution is $y = c_1 + c_2 \sin 2t + c_3 \cos 2t$.

```
gensol = DSolve[y'''[t] + 4y'[t]==0, y[t], t]
{{y[t] → C[3] - 1/2 C[2]Cos[2t] + 1/2 C[1]Sin[2t]}}
```

Application of the initial conditions shows us that $c_1 = -1/4$, $c_2 = 1/2$, and $c_3 = 1/4$, so the solution to the initial-value problem is $y = -\frac{1}{4} + \frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t$. We verify the computation with DSolve and graph the result with Plot in Figure 6.19.

```
e1 = y[t]/.gensol[[1]]/.t → 0
-C[2]/2 + C[3]
e2 = D[y[t]/.gensol[[1]], t]/.t → 0
e3 = D[y[t]/.gensol[[1]], {t, 2}]/.t → 0
C[1]
2C[2]

cvals = Solve[{e1==0, e2==1, e3==-1}]
{{C[1] → 1, C[2] → -1/2, C[3] → -1/4}}
Clear[y]
partsol = DSolve[{y'''[t] + 4y'[t]==0, y[0]==0,
        y'[0]==1, y''[0]==-1}, y[t], t]
{{y[t] → 1/4(-1 + Cos[2t] + 2Sin[2t])}}
Plot[y[t]/.partsol, {t, 0, 2Pi}, AspectRatio → Automatic]
```

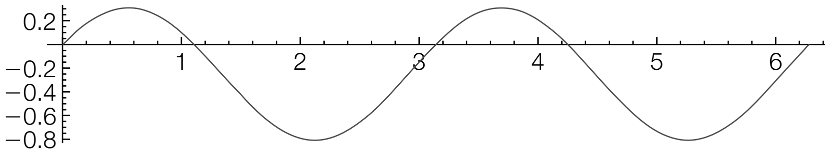



**FIGURE 6.19**

Graph of $y = -\frac{1}{4} + \frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t$

**Example 6.3.3** Find a differential equation with general solution $y = c_1 e^{-2t/3} + c_2 t e^{-2t/3} + c_3 t^2 e^{-2t/3} + c_4 \cos t + c_5 \sin t + c_6 t \cos t + c_7 t \sin t + c_8 t^2 \cos t + c_9 t^2 \sin t$.

***Solution*** A linear homogeneous differential equation with constant coefficients that has this general solution has fundamental set of solutions

$$S = \left\{ e^{-2t/3}, te^{-2t/3}, t^2 e^{-2t/3}, \cos t, \sin t, t\cos t, t\sin t, t^2\cos t, t^2 \sin t \right\}.$$

Hence, in the characteristic equation $k = -2/3$ has multiplicity 3, whereas $k = \pm i$ has multiplicity 3. The characteristic equation is

$$27\left(k+\frac{2}{3}\right)^3 (k-i)^3(k+i)^3 = k^9 + 2k^8 + \frac{13}{3}k^7 + \frac{170}{27}k^6 + 7k^5 + \frac{62}{9}k^4 + 5k^3 + \frac{26}{9}k^2 + \frac{4}{3}k + \frac{8}{27},$$

where we use Mathematica to compute the multiplication with Expand.

```
Expand[27(k + 2/3)^3(k^2 + 1)^3]
8 + 36k + 78k^2 + 135k^3 + 186k^4 + 189k^5+
   170k^6 + 117k^7 + 54k^8 + 27k^9
```

Thus, a differential equation obtained after dividing by 27 with the indicated general solution is

$$\frac{d^9y}{dt^9} + 2\frac{d^8y}{dt^8} + \frac{13}{3}\frac{d^7y}{dt^7} + \frac{170}{27}\frac{d^6y}{dt^6} + 7\frac{d^5y}{dt^5} + \frac{62}{9}\frac{d^4y}{dt^4} + 5\frac{d^3y}{dt^3} + \frac{26}{9}\frac{d^2y}{dt^2} + \frac{4}{3}\frac{dy}{dt} + \frac{8}{27}y = 0.$$

## 6.3.3 **Undetermined Coefficients**

For higher-order linear equations with constant coefficients, the method of undetermined coefficients is the same as for second-order equations discussed in Section 6.2.3, provided that the forcing function involves the terms discussed in Section 6.2.3.

**Example 6.3.4** Solve $\frac{d^3y}{dt^3} + \frac{2}{3}\frac{d^2y}{dt^2} + \frac{145}{9}\frac{dy}{dt} = e^{-t}$, $y(0) = 1$, $\frac{dy}{dt}(0) = 2$, $\frac{d^2y}{dt^2}(0) = -1$.

***Solution*** The corresponding homogeneous equation, $y''' + \frac{2}{3}y'' + \frac{145}{9}y' = 0$, has general solution $y_h = c_1 + \left(c_2 \sin 4t + c_3 \cos 4t\right) e^{-t/3}$ and a fundamental set of solutions for the corresponding homogeneous equation is $S = \left\{1, e^{-t/3}\cos 4t, e^{-t/3}\sin 4t\right\}$.

**DSolve[y'''[t] + 2/3y''[t] + 145/9y'[t]==0,**
**y[t], t]**

$\{\{y[t] \to C[3] - \frac{3}{145}e^{-t/3}$
$((12C[1] + C[2])Cos[4t] + (C[1] - 12C[2])Sin[4t])\}\}$

For $e^{-t}$, the associated set of functions is $F = \left\{e^{-t}\right\}$. Because no element of $F$ is an element of $S$, we assume that $y_p = Ae^{-t}$, where $A$ is a constant to be determined. After defining $y_p$, we compute the necessary derivatives

**Clear[yp]**
**yp[t_] = aExp[-t];**
**yp'[t]**
**yp''[t]**
**yp'''[t]**

$-ae^{-t}$
$ae^{-t}$
$-ae^{-t}$

and substitute into the nonhomogeneous equation.

**eqn = yp'''[t] + 2/3yp''[t] + 145/9yp'[t]==Exp[-t]**

$-\frac{148}{9}ae^{-t}==e^{-t}$

Equating coefficients and solving for $A$ gives us $A = -9/148$ so $y_p = -\frac{9}{148}e^{-t}$ and a general solution is $y = y_h + y_p$.

**Remark 6.3** SolveAlways[equation,variable] attempts to solve an equation so that it is true for all values of variable.

**SolveAlways[eqn, t]**

$\{\{a \to -\frac{9}{148}\}\}$

We verify the result with DSolve.

**gensol = DSolve[y'''[t] + 2/3y''[t] +**
**145/9y'[t]==Exp[-t], y[t], t]//FullSimplify**

$\left\{\left\{y[t] \to -\frac{9e^{-t}}{148} + C[3] - \frac{3}{145}e^{-t/3}((12C[1] + C[2])Cos[4t]+\right.\right.$
$(C[1] - 12C[2])Sin[4t])\}\}$

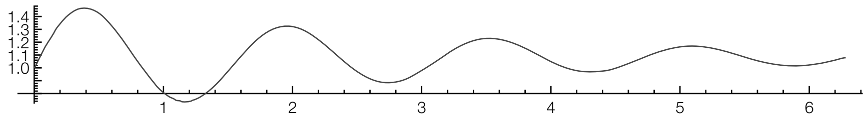



**FIGURE 6.20**

The solution of the equation that satisfies $y(0) = 1$, $y'(0) = 2$, and $y''(0) = -1$

To obtain a real-valued solution, we use ComplexExpand. If you are using a version of Mathematica older than version 6, you might receive a complex valued function rather than the real-valued function that we obtained. In those cases, ComplexExpand can help you rewrite your complex solution as a real-valued solution.

To apply the initial conditions, we compute $y(0) = 1$, $y'(0) = 2$, and $y''(0) = -1$ and solve for $c_1$, $c_2$, and $c_3$. The solution of the initial-value problem is obtained by substituting these values into the general solution, and then we graph the result with Plot in Figure 6.20.

```
initsol = DSolve[{y'''[t] + 2/3y''[t] +
145/9y'[t]==Exp[-t], y[0]==1, y'[0]==2, y''[0]==-1},
y[t], t]//FullSimplify
```

$$\left\{\left\{y[t] \to \frac{e^{-t}\left(-2610 + 46472e^{t} + e^{2t/3}(-942\mathrm{Cos}[4t] + 20729\mathrm{Sin}[4t])\right)}{42920}\right\}\right\}$$

```
Plot[y[t]/.initsol, {t, 0, 2Pi}, AspectRatio → Automatic]
```

**Example 6.3.5** Solve

$$\frac{d^8y}{dt^8} + \frac{7}{2}\frac{d^7y}{dt^7} + \frac{73}{2}\frac{d^6y}{dt^6} + \frac{229}{2}\frac{d^5y}{dt^5} + \frac{801}{2}\frac{d^4y}{dt^4} + 976\frac{d^3y}{dt^3} + 1168\frac{d^2y}{dt^2} + 640\frac{dy}{dt} + 128y = te^{-t} + \sin 4t + t.$$

***Solution*** Solving the characteristic equation

```
Solve[k^8 + 7/2k^7 + 73/2k^6 + 229/2k^5 +
    801/2k^4 + 976k^3 + 1168k^2 + 640k + 128==0]
```

$$\{\{k \to -1\}, \{k \to -1\}, \{k \to -1\}, \{k \to -\tfrac{1}{2}\}, \{k \to -4i\}, \{k \to -4i\}, \{k \to 4i\}, \{k \to 4i\}\}$$

shows us that the solutions are $k_1 = -1/2$, $k_2 = -1$ with multiplicity 3, and $k_{3,4} = \pm 4i$, each with multiplicity 2. A fundamental set of solutions for the corresponding homogeneous equation is

$$S = \left\{e^{-t/2}, e^{-t}, te^{-t}, t^2e^{-t}, \cos 4t, t\cos 4t, \sin 4t, t\sin 4t\right\}.$$

A general solution of the corresponding homogeneous equation is

$$y_h = c_1 e^{-t/2} + \left(c_2 + c_3 t + c_4 t^2\right) e^{-t} + \left(c_5 + c_7 t\right) \sin 4t + \left(c_6 + c_8 t\right) \cos 4t.$$

```
gensol = DSolve[D[y[t], {t, 8}] + 7/2D[y[t], {t, 7}] +
   73/2D[y[t], {t, 6}] + 229/2D[y[t], {t, 5}] +
     801/2D[y[t], {t, 4}] + 976y'''[t] + 1168y''[t] +
     640y'[t] + 128y[t]==0, y[t], t]
```

```
{{y[t] → e^(-t/2)C[5] + e^(-t)C[6] + e^(-t)tC[7] + e^(-t)t^2C[8]+
      C[1]Cos[4t] + tC[2]Cos[4t] + C[3]Sin[4t] + tC[4]Sin[4t]}}
```

The associated set of functions for $te^{-t}$ is $F_1 = \{e^{-t}, te^{-t}\}$. We multiply $F_1$ by $t^n$, where $n$ is the smallest nonnegative integer so that no element of $t^n F_1$ is an element of $S$: $t^3 F_1 = \{t^3 e^{-t}, t^4 e^{-t}\}$. The associated set of functions for $\sin 4t$ is $F_2 = \{\cos 4t, \sin 4t\}$. We multiply $F_2$ by $t^n$, where $n$ is the smallest nonnegative integer so that no element of $t^n F_2$ is an element of $S$: $t^2 F_2 = \{t^2 \cos 4t, t^2 \sin 4t\}$. The associated set of functions for $t$ is $F_3 = \{1, t\}$. No element of $F_3$ is an element of $S$.

Thus, we search for a particular solution of the form

$$y_p = A_1 t^3 e^{-t} + A_2 t^4 e^{-t} + A_3 t^2 \cos 4t + A_4 t^2 \sin 4t + A_5 + A_6 t,$$

where the $A_i$ are constants to be determined.

After defining $y_p$,

```
yp[t_] = a[1]t^3Exp[-t] + a[2]t^4Exp[-t] +
   a[3]t^2Cos[4t] + a[4]t^2Sin[4t] + a[5] + a[6]t;
```

we substitute into the nonhomogeneous equation, naming the result **eqn**. At this point we can either equate coefficients and solve for $A_i$ or use the fact that **eqn** is true for *all* values of $t$.

```
eqn = D[yp[t], {t, 8}] + 7/2D[yp[t], {t, 7}] +
   73/2D[yp[t], {t, 6}] + 229/2D[yp[t], {t, 5}] +
     801/2D[yp[t], {t, 4}] + 976yp'''[t] + 1168yp''[t] +
     640yp'[t] + 128yp[t]==tExp[-t] + Sin[4t] + t//Simplify
```

```
e^(-t) (-867a[1] + 7752a[2] - 3468ta[2] + 128e^t a[5]+
   640e^t a[6] + 128e^t ta[6] - 64e^t (369a[3] - 428a[4])Cos[4t] -
     64e^t (428a[3] + 369a[4])Sin[4t]) ==t + e^(-t)t + Sin[4t]
```

We substitute in six values of $t$

```
sysofeqs = Table[eqn/.t → n//N, {n, 0, 5}];
```

and then solve for $A_i$.

```
coeffs = Solve[sysofeqs, {a[1.], a[2.], a[3.], a[4.], a[5.], a[6.]}]
{{a[1.] → -0.00257819, a[2.] → -0.000288351, a[3.] → -0.0000209413,
      a[4.] → -0.0000180545, a[5.] → -0.0390625, a[6.] → 0.0078125}}
```

$y_p$ is obtained by substituting the values for $A_i$ into $y_p$ and a general solution is $y = y_h + y_p$. DSolve is able to find an exact solution.

```
gensol = DSolve[D[y[t], {t, 8}] + 7/2D[y[t], {t, 7}] +
   73/2D[y[t], {t, 6}] + 229/2D[y[t], {t, 5}] +
     801/2D[y[t], {t, 4}] + 976y‴[t] + 1168y″[t] +
     640y′[t] + 128y[t]==tExp[−t] + Sin[4t] + t, y[t], t]//Simplify
```

```
{{y[t] → 1/407272223623424000 e^-t
  (4394000 (72412707e^t(−5 + t) + 9268826496e^(t/2)C[5] − 32(35097672+
    746776t^3 + 83521t^4 − 289650828C[6] − 204t(−86016 + 1419857C[7]) −
     3468t^2(−1270 + 83521C[8]))) −
    204e^t (−9041976373 + 4180789600t^2 − 199643253056000C[1] −
    4420t(−1568449 + 45168156800C[2]))Cos[4t] −
    51e^t (−13794625331 + 14417863200t^2 − 798573012224000C[3] −
    2263040t(20406 + 352876225C[4]))Sin[4t])}}
```

### *Variation of Parameters*

In the same way as with second-order equations, we assume that a particular solution of the $n$th-order linear equation (6.19) has the form $y_p = u_1(t)y_1 + u_2(t)y_2 + \cdots + u_n(t)y_n$, where $S = \{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions to the corresponding homogeneous equation (6.20). With the assumptions

$$\begin{aligned} y_p{}' &= y_1u_1{}' + y_2u_2{}' + \cdots + y_nu_n{}' = 0 \\ y_p{}'' &= y_1{}'u_1{}' + y_2{}'u_2{}' + \cdots + y_n{}'u_n{}' = 0 \\ &\vdots \\ y_p{}^{(n-1)} &= y_1{}^{(n-2)}u_1{}' + y_2{}^{(n-2)}u_2{}' + \cdots + y_n{}^{(n-2)}u_n{}' = 0, \end{aligned} \tag{6.23}$$

we obtain the equation

$$y_1{}^{(n-1)}u_1{}' + y_2{}^{(n-1)}u_2{}' + \cdots + y_n{}^{(n-1)}u_n{}' = f(t). \tag{6.24}$$

Equations (6.23) and (6.24) form a system of $n$ linear equations in the unknowns $u_1{}', u_2{}', \ldots, u_n{}'$. Applying Cramer's rule,

$$u_i{}' = \frac{W_i(S)}{W(S)}, \tag{6.25}$$

where $W(S)$ is given by equation (6.21) and $W_i(S)$ is the determinant of the matrix obtained by replacing the $i$th column of

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1{}' & y_2{}' & \cdots & y_n{}' \\ \vdots & \vdots & \cdots & \vdots \\ y_1{}^{(n-1)} & y_2{}^{(n-1)} & \cdots & y_n{}^{(n-1)} \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}.$$

**Example 6.3.6** Solve $y^{(3)} + 4y' = \sec 2t$.

***Solution*** A general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2 \cos 2t + c_3 \sin 2t$; a fundamental set is $S = \{1, \cos 2t, \sin 2t\}$ with Wronskian $W(S) = 8$.

```
yh = DSolve[y'''[t] + 4y'[t]==0, y[t], t]
{{y[t] → C[3] − 1/2 C[2]Cos[2t] + 1/2 C[1]Sin[2t]}}

s = {1, Cos[2t], Sin[2t]};
ws = {s, D[s, t], D[s, {t, 2}]};
MatrixForm[ws]
( 1    Cos[2t]     Sin[2t]  )
( 0   −2Sin[2t]   2Cos[2t]  )
( 0   −4Cos[2t]  −4Sin[2t]  )

dws = Simplify[Det[ws]]
8
```

Using variation of parameters to find a particular solution of the nonhomogeneous equation, we let $y_1 = 1$, $y_2 = \cos 2t$, and $y_3 = \sin 2t$ and assume that a particular solution has the form $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$. Using the variation of parameters formula, we obtain

$$u_1' = \frac{1}{8}\begin{vmatrix} 0 & \cos 2t & \sin 2t \\ 0 & -2\sin 2t & 2\cos 2t \\ \sec 2t & -4\cos 2t & -4\sin 2t \end{vmatrix} = \frac{1}{4}\sec 2t \quad \text{so} \quad u_1 = \frac{1}{8}\ln|\sec 2t + \tan 2t|,$$

$$u_2' = \frac{1}{8}\begin{vmatrix} 1 & 0 & \sin 2t \\ 0 & 0 & 2\cos 2t \\ 0 & \sec 2t & -4\sin 2t \end{vmatrix} = -\frac{1}{4} \quad \text{so} \quad u_2 = -\frac{1}{4}t,$$

and

$$u_3' = \frac{1}{8}\begin{vmatrix} 1 & \cos 2t & 0 \\ 0 & -2\sin 2t & 0 \\ 0 & -4\cos 2t & \sec 2t \end{vmatrix} = -\frac{1}{2}\tan 2t \quad \text{so} \quad u_3 = \frac{1}{8}\ln|\cos 2t|,$$

where we use **Det** and **Integrate** to evaluate the determinants and integrals. In the case of $u_1$, the output given by Mathematica looks different than the result we obtained by hand, but differentiating the difference between the two results yields 0, so the results obtained by hand and with Mathematica are the same.

```
u1p = 1/8Det[{{0, Cos[2t], Sin[2t]}, {0, −2Sin[2t], 2Cos[2t]},
        {Sec[2t], −4Cos[2t], −4Sin[2t]}}]//Simplify

1/4 Sec[2t]

u1 = Integrate[u1p, t]
1/4 ArcTanh[Tan[t]]

s1 = D[u1 − 1/8Log[Sec[2t] + Tan[2t]], t]
Sec[t]^2/(4(1 − Tan[t]^2)) − (2Sec[2t]^2 + 2Sec[2t]Tan[2t])/(8(Sec[2t] + Tan[2t]))
```

```
Simplify[s1]
0
u2p = 1/8Det[{{1, 0, Sin[2t]}, {0, 0, 2Cos[2t]},
      {0, Sec[2t], -4Sin[2t]}}]//Simplify
```
$-\frac{1}{4}$
```
u2 = Integrate[u2p, t]
```
$-\frac{t}{4}$
```
u3p = 1/8Det[{{1, Cos[2t], 0}, {0, -2Sin[2t], 0},
      {0, -4Cos[2t], Sec[2t]}}]//Simplify
```
$-\frac{1}{4}$Tan[2t]
```
u3 = Integrate[u3p, t]
```
$\frac{1}{8}$Log[Cos[2t]]

Thus, a particular solution of the nonhomogeneous equation is

$$y_p = \frac{1}{8}\ln|\sec 2t + \tan 2t| - \frac{1}{4}t\cos 2t + \frac{1}{8}\ln|\cos 2t|\sin 2t$$

and a general solution is $y = y_h + y_p$. We verify that the calculations using DSolve return an equivalent solution.

```
gensol = DSolve[y'''[t] + 4y'[t]==Sec[2t], y[t], t]//Simplify
```
$\{\{$y[t] → $\frac{1}{8}$(2ArcTanh[Tan[t]] + 8C[3] − 8C[2]Cos[t]$^2$ − 2tCos[2t] + 4C[1]Sin[2t] + Log[Cos[2t]]Sin[2t])$\}\}$

---

## 6.3.4 Laplace Transform Methods

The *method of Laplace transforms* can be useful when the forcing function is piecewise-defined or periodic.

**Definition 3 (Laplace Transform and Inverse Laplace Transform).** *Let $y = f(t)$ be a function defined on the interval $[0, \infty)$. The* ***Laplace transform*** *is the function (of s)*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt, \tag{6.26}$$

*provided the improper integral exists. $f(t)$ is the* ***inverse Laplace transform*** *of $F(s)$ means that $\mathcal{L}\{f(t)\} = F(s)$ and we write $\mathcal{L}^{-1}\{F(s)\} = f(t)$.*

1. LaplaceTransform[f[t],t,s] computes $\mathcal{L}\{f(t)\} = F(s)$.
2. InverseLaplaceTransform[F[s],t,s] computes $\mathcal{L}^{-1}\{F(s)\} = f(t)$.
3. UnitStep[t] returns $\mathcal{U}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$

Typically, when we use Laplace transforms to solve a differential equation for a function $y(t)$, we will compute the Laplace transform of each term of the differential equation, solve the resulting algebraic equation for the Laplace transform of $y(t)$, $\mathcal{L}\{y(t)\} = Y(s)$, and, finally, determine $y(t)$ by computing the inverse Laplace transform of $Y(s)$, $\mathcal{L}^{-1}\{Y(s)\} = y(t)$.

---

**Example 6.3.7** Let $y = f(t)$ be defined recursively by $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \end{cases}$ and $f(t) = f(t-2)$ if $t \ge 2$. Solve $y'' + 4y' + 20y = f(t)$.

***Solution*** We begin by defining and graphing $y = f(t)$ for $0 \le t \le 5$ in Figure 6.21.

```
Clear[f, g, u, y1, y2, sol]
f[t_]:=1/;0 ≤ t < 1
f[t_]:= -1/;1 ≤ t ≤ 2
f[t_]:=f[t-2]/;t > 2
Plot[f[t], {t, 0, 5}, Ticks → {Automatic, {-2, -1, 0, 1, 2}},
    PlotRange → {-2, 2}]
```

We then define lhs to be the left-hand side of the equation $y'' + 4y' + 20y = f(t)$,

```
Clear[y, x, lhs, stepone, steptwo]
lhs = y''[t] + 4y'[t] + 20y[t];
```

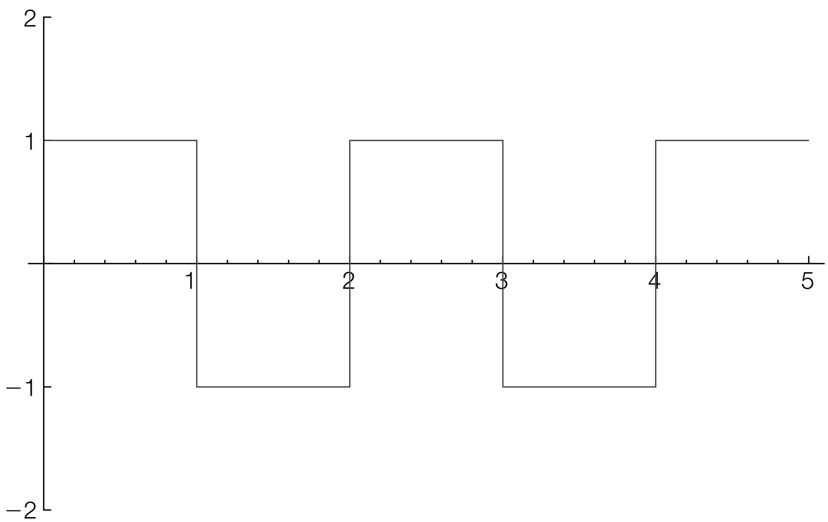



**FIGURE 6.21**

Plot of $f(t)$ for $0 \le t \le 5$

and compute the Laplace transform of **lhs** with **LaplaceTransform**, naming the result **stepone**.

**stepone = LaplaceTransform[lhs, t, s]**

20LaplaceTransform[y[t], t, s] + s$^2$LaplaceTransform[y[t], t, s] +
(4(s LaplaceTransform[y[t], t, s] − y[0]) − sy[0] − y′[0]

Let **lr** denote the Laplace transform of the right-hand side of the equation, $f(t)$. We now solve the equation $20$**ly** $+ 4s$**ly** $+ s^2$**ly**$-4y(0) - sy(0) - y'(0) =$ **lr** for **ly** and name the resulting output **steptwo**.

**steptwo = Solve[stepone==lr, LaplaceTransform[y[t], t, s]]**

$\left\{\left\{\text{LaplaceTransform[y[t], t, s]} \to \frac{\text{lr} + 4\text{y[0]} + \text{sy[0]} + \text{y}'\text{[0]}}{20 + 4\text{s} + \text{s}^2}\right\}\right\}$

**stepthree = ExpandNumerator[steptwo[[1, 1, 2]], 1r]**

$\frac{\text{lr} + 4\text{y[0]} + \text{sy[0]} + \text{y}'\text{[0]}}{20 + 4\text{s} + \text{s}^2}$

To find $y(t)$, we must compute the inverse Laplace transform of $\mathcal{L}\{y(t)\}$; the formula for which is explicitly obtained from **steptwo** with **steptwo[[1,1,2]]**. First, we rewrite : $\mathcal{L}\{y(t)\}$. Then,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{f(t)\}}{s^2+4s+20} + \frac{4y(0)+sy(0)+y'(0)}{s^2+4s+20}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{f(t)\}}{s^2+4s+20}\right\} + \mathcal{L}^{-1}\left\{\frac{4y(0)+sy(0)+y'(0)}{s^2+4s+20}\right\}. \end{aligned}$$

Completing the square yields $s^2 + 4s + 20 = (s+2)^2 + 16$. Because

$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin bt \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt,$$

the inverse Laplace transform of

$$\frac{4y(0)+sy(0)+y'(0)}{s^2+4s+20} = y(0)\frac{s+2}{(s+2)^2+4^2} + \frac{y'(0)+2y(0)}{4}\frac{4}{(s+2)^2+4^2}$$

is

$$y(0)e^{-2t}\cos 4t + \frac{y'(0)+2y(0)}{4}e^{-2t}\sin 4t,$$

which is defined as $y_1(t)$. We perform these steps with Mathematica by first using **InverseLaplaceTransform** to calculate $\mathcal{L}^{-1}\left\{\frac{4y(0)+sy(0)+y'(0)}{s^2+4s+20}\right\}$, naming the result **stepfour**.

**stepfour = InverseLaplaceTransform** $\left[-\frac{-4\text{y[0]} - \text{sy[0]} - \text{y}'\text{[0]}}{20 + 4\text{s} + \text{s}^2}, \text{s}, \text{t}\right]$

$-\frac{1}{8}\text{ie}^{(-2-4\text{i})\text{t}}\left((-2+4\text{i})\text{y[0]} + (2+4\text{i})\text{e}^{8\text{it}}\text{y[0]} - \text{y}'\text{[0]} + \text{e}^{8\text{it}}\text{y}'\text{[0]}\right)$

To see that this is a real-valued function, we use ComplexExpand together with Simplify.

**stepfive = ComplexExpand[stepfour]//Simplify**

$\frac{1}{4}e^{-2t}\left(4\text{Cos}[4t]y[0]+\text{Sin}[4t]\left(2y[0]+y'[0]\right)\right)$

If the result in **stepfive** is given in terms of real and imaginary parts of $y(0)$ and $y'(0)$, because $y'(0)$ is assumed to be a real number, the imaginary part of $y'(0)$ is $0$; the real part of $y'(0)$ is $y'(0)$.

**y1[t_] = stepfive/.{Im[y′[0]] → 0, Re[y′[0]] → y′[0]}//Simplify**

$\frac{1}{4}e^{-2t}\left(4\text{Cos}[4t]y[0]+\text{Sin}[4t]\left(2y[0]+y'[0]\right)\right)$

To compute the inverse Laplace transform of $\frac{\mathcal{L}\{f(t)\}}{s^2+4s+20}$, we begin by computing lr $=\mathcal{L}\{f(t)\}$. Let $\mathcal{U}_a(t)=\begin{cases}1, & t\ge a\\ 0, & t<a\end{cases}$. Then, $\mathcal{U}_a(t)=\mathcal{U}(t-a)=$ UnitStep[t-a].

The periodic function $f(t)=\begin{cases}1, & 0\le t<1\\ -1, & 1\le t<2\end{cases}$ and $f(t)=f(t-2)$ if $t\ge 2$ can be written in terms of step functions as

$$\begin{aligned} f(t) &= \mathcal{U}_0(t)-2\mathcal{U}_1(t)+2\mathcal{U}_2(t)-2\mathcal{U}_3(t)+2\mathcal{U}_4(t)-\dots\\ &= \mathcal{U}(t)-2\mathcal{U}(t-1)+2\mathcal{U}(t-2)-2\mathcal{U}(t-3)+2\mathcal{U}(t-4)-\dots\\ &= \mathcal{U}(t)+2\sum_{n=1}^{\infty}(-1)^n\mathcal{U}(t-n). \end{aligned}$$

The Laplace transform of $\mathcal{U}_a(t)=\mathcal{U}(t-a)$ is $\frac{1}{s}e^{-as}$ and the Laplace transform of $f(t)\mathcal{U}_a(t)=f(t)\mathcal{U}(t-a)$ is $e^{-as}F(s)$, where $F(s)$ is the Laplace transform of $f(t)$. Then,

$$\begin{aligned} \text{lr} &= \frac{1}{s}-\frac{2}{s}e^{-s}+\frac{2}{s}e^{-2s}-\frac{2}{s}e^{-3s}+\cdots\\ &= \frac{1}{s}\left(1-2e^{-s}+2e^{-2s}-2e^{-3s}+\cdots\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\text{lr}}{s^2+4s+20} &= \frac{1}{s\left(s^2+4s+20\right)}\left(1-2e^{-s}+2e^{-2s}-2e^{-3s}+\cdots\right)\\ &= \frac{1}{s\left(s^2+4s+20\right)}+2\sum_{n=1}^{\infty}(-1)^n\frac{e^{-ns}}{s\left(s^2+4s+20\right)}. \end{aligned}$$

Because $\frac{1}{s^2+4s+20}=\frac{1}{4}\frac{1}{(s+2)^2+4^2}$, $\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+4s+20\right)}\right\}=\int_0^t\frac{1}{4}e^{-2\alpha}\sin 4\alpha\,d\alpha$, computed and defined to be the function $g(t)$.

**g[t_] = $\int_0^t\frac{1}{4}$Exp[−2α]Sin[4α] dα**

$\frac{1}{40}\left(2-e^{-2t}(2\text{Cos}[4t]+\text{Sin}[4t])\right)$

Alternatively, we can use InverseLaplaceTransform to obtain the same result.

```
g[t_] = InverseLaplaceTransform[1/(s(s^2+4s+20)), s, t] //ExpToTrig//
    Simplify
```

$\frac{1}{80}(4 + (2\mathrm{Cos}[4t] + \mathrm{Sin}[4t])(-2\mathrm{Cosh}[2t] + 2\mathrm{Sinh}[2t]))$

Then, $\mathcal{L}^{-1}\left\{2(-1)^n \frac{e^{-ns}}{s\left(s^2+4s+20\right)}\right\} = 2(-1)^n g(t-n)\mathcal{U}(t-n)$ and the inverse Laplace transform of

$$\frac{1}{s\left(s^2+4s+20\right)} + 2\sum_{n=1}^{\infty}(-1)^n \frac{e^{-ns}}{s\left(s^2+4s+20\right)}$$

is

$$y_2(t) = g(t) + 2\sum_{n=1}^{\infty}(-1)^n g(t-n)\mathcal{U}(t-n).$$

It then follows that

$$\begin{aligned} y(t) &= y_1(t) + y_2(t) \\ &= y(0)e^{-2t}\cos 4t + \frac{y'(0)+2y(0)}{4}e^{-2t}\sin 4t + 2\sum_{n=1}^{\infty}(-1)^n g(t-n)\mathcal{U}(t-n), \end{aligned}$$

where $g(t) = \frac{1}{20} - \frac{1}{20}e^{-2t}\cos 4t - \frac{1}{40}e^{-2t}\sin 4t$.

To graph the solution for various initial conditions on the interval $[0,5]$, we define $y_2(t) = g(t) + 2\sum_{n=1}^{5}(-1)^n g(t-n)\mathcal{U}(t-n)$, sol, and inits. (Note that we can graph the solution for various initial conditions on the interval $[0,m]$ by defining $y_2(t) = g(t) + 2\sum_{n=1}^{m}(-1)^n g(t-n)\mathcal{U}(t-n)$.)

```
y2[t_]:=g[t] + 2 Sum[(-1)^n g[t-n]UnitStep[t-n], {n,1,5}]
Clear[sol]
sol[t_]:=y1[t] + y2[t]
inits = {-1/2, 0, 1/2};
```

We then create a table of graphs of sol[t] on the interval $[0,5]$ corresponding to replacing $y(0)$ and $y'(0)$ by the values $-1/2$, $0$, and $1/2$ and then displaying the resulting graphics array in Figure 6.22.

```
graphs = Table[Plot[sol[t]/.{y[0] → inits[[i]], y'[0] → inits[[j]]},
    {t, 0, 5}, DisplayFunction → Identity], {i, 1, 3}, {j, 1, 3}];
Show[GraphicsGrid[graphs]]
```

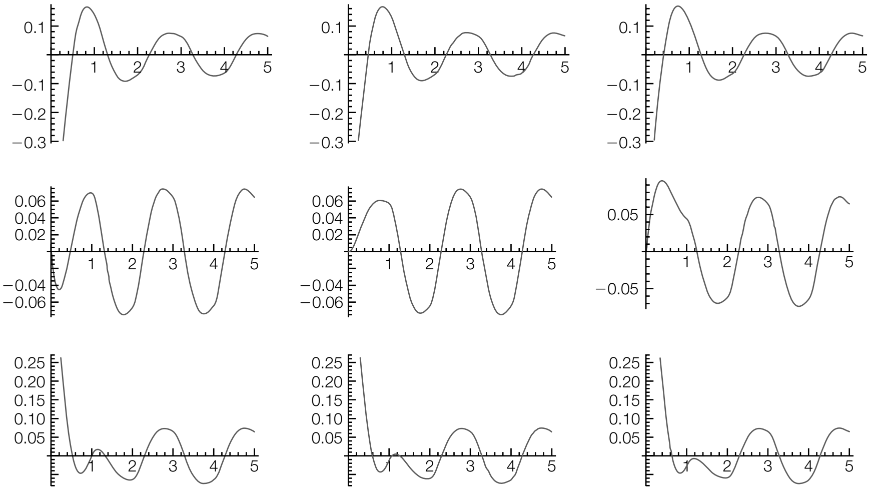


**FIGURE 6.22**

Solutions to a differential equation with a piecewise-defined periodic forcing function

### *Application: The Convolution Theorem*

Sometimes we are required to determine the inverse Laplace transform of a product of two functions. Just as in differential and integral calculus when the derivative and integral of a product of two functions did not produce the product of the derivatives and integrals, respectively, neither does the inverse Laplace transform of the product yield the product of the inverse Laplace transforms. *The convolution theorem* tells us how to compute the inverse Laplace transform of a product of two functions.

**Theorem 2 (The Convolution Theorem).** *Suppose that $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and both are of exponential order. Furthermore, suppose that the Laplace transform of $f(t)$ is $F(s)$ and that of $g(t)$ is $G(s)$. Then,*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\left(f*g\right)(t)\right\}\right\} = \int_0^t f(t-\nu)g(\nu)\,d\nu. \tag{6.27}$$

Note that $\left(f*g\right)(t) = \int_0^t f(t-\nu)g(\nu)\,d\nu$ is called the **convolution integral**.

**Example 6.3.8** **(L–R–C Circuits).** The initial-value problem used to determine the charge $q(t)$ on the capacitor in an L–R–C circuit is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = f(t), \quad Q(0) = 0, \quad \frac{dQ}{dt}(0) = 0,$$

where $L$ denotes inductance, $dQ/dt = I$, $I(t)$ current, $R$ resistance, $C$ capacitance, and $E(t)$ voltage supply. Because $dQ/dt = I$, this differential equation can be represented as

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(u)\,du = E(t).$$

Note also that the initial condition $Q(0) = 0$ is satisfied because $Q(0) = \frac{1}{C}\int_0^0 I(u)\,du = 0$. The condition $dQ/dt(0) = 0$ is replaced by $I(0) = 0$. (a) Solve this *integro-differential equation*, an equation that involves a derivative as well as an integral of the unknown function, by using the convolution theorem. (b) Consider this example with constant values $L = C = R = 1$ and $E(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2 \end{cases}$. Determine $I(t)$ and graph the solution.

***Solution*** We proceed as in the case of a differential equation by taking the Laplace transform of both sides of the equation. The convolution theorem, equation (6.27), is used in determining the Laplace transform of the integral with

$$\mathcal{L}\left\{\int_0^t I(u)\,du\right\} = \mathcal{L}\{1 * I(t)\} = \mathcal{L}\{1\}\,\mathcal{L}\{I(t)\} = \frac{1}{s}\mathcal{L}\{I(t)\}.$$

Therefore, application of the Laplace transform yields

$$Ls\mathcal{L}\{I(t)\} - LI(0) + R\mathcal{L}\{I(t)\} + \frac{1}{C}\frac{1}{s}\mathcal{L}\{I(t)\} = \mathcal{L}\{E(t)\}.$$

Because $I(0) = 0$, we have $Ls\mathcal{L}\{I(t)\} + R\mathcal{L}\{I(t)\} + \frac{1}{C}\frac{1}{s}\mathcal{L}\{I(t)\} = \mathcal{L}\{E(t)\}$. Simplifying and solving for $\mathcal{L}\{I(t)\}$ results in $\mathcal{L}\{I(t)\} = \dfrac{Cs\mathcal{L}\{E(t)\}}{LCs^2 + RCs + 1}$

```
Clear[i]
LaplaceTransform[l i'[t] + r i[t], t, s]
r LaplaceTransform[i[t], t, s] +
l(−i[0] + s LaplaceTransform[i[t], t, s])
Solve[l s lapi + rlapi + lapi/(cs)==lape, lapi]
{{lapi → clapes/(1+crs+cls^2)}}
```

We use lowercase letters to avoid any possible ambiguity with built-in Mathematica functions, such as E and I.

so that $I(t) = \mathcal{L}^{-1}\left\{\dfrac{Cs\mathcal{L}\{E(t)\}}{LCs^2 + RCs + 1}\right\}$. In the **Solve** command we use **lapi** to denote $\mathcal{L}\{I(t)\}$ and **lape** to denote $\mathcal{L}\{E(t)\}$. For (b), we note that $E(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2 \end{cases}$ can be written as $E(t) = \sin t\,(\mathcal{U}(t) - \mathcal{U}(t - \pi/2))$. We define and plot the forcing function $E(t)$ on the interval $[0, \pi]$ in Figure 6.23(a).

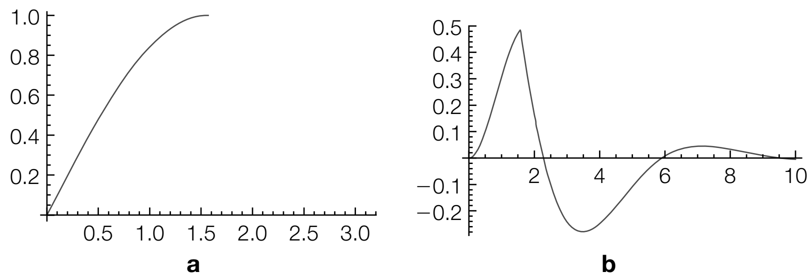



**FIGURE 6.23**

(a) Plot of $E(t) = \sin t\left(\mathcal{U}(t) - \mathcal{U}(t-\pi/2)\right)$. (b) $I(t)$ (in black) and $E(t)$ (in gray)

**e[t_]:=Sin[t] (UnitStep[t] − UnitStep [t − $\frac{\pi}{2}$])**
**p1 = Plot[e[t], {t, 0, π}]**

Next, we compute the Laplace transform of $\mathcal{L}\{E(t)\}$ with LaplaceTransform. We call this result lcape.

**lcape = LaplaceTransform[e[t], t, s]**

$$\frac{1}{1+s^2} - \frac{e^{-\frac{\pi s}{2}}s}{1+s^2}$$

Using the general formula obtained for the Laplace transform of $I(t)$, we note that the denominator of this expression is given by $s^2+s+1$, which is entered as denom. Hence, the Laplace transform of $I(t)$, called lcapi, is given by the ratio s lcape/denom.

**denom = s² + s + 1;**
**lcapi = s lcape/denom;**
**lcapi = Simplify[lcapi]**

$$\frac{s - e^{-\frac{\pi s}{2}}s^2}{1+s+2s^2+s^3+s^4}$$

We determine $I(t)$ with InverseLaplaceTransform. Note that HeavisideTheta[x] is defined by $\theta(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$.

**i[t_] = InverseLaplaceTransform[lcapi, s, t]**

$$\text{Sin}[t] - \text{HeavisideTheta}\left[-\tfrac{\pi}{2}+t\right]\left(-\tfrac{1}{3}e^{\frac{1}{4}(\pi-2t)}\left(3\text{Cos}\left[\tfrac{1}{4}\sqrt{3}(\pi-2t)\right] + \sqrt{3}\text{Sin}\left[\tfrac{1}{4}\sqrt{3}(\pi-2t)\right]\right) + \text{Sin}[t]\right) - \frac{2e^{-t/2}\text{Sin}\left[\frac{\sqrt{3}t}{2}\right]}{\sqrt{3}}$$

This solution is plotted in p2 (in black) and displayed with the forcing function (in gray) in Figure 6.23(b). Notice the effect that the forcing function has on the solution to the differential equation.

```
p2 = Plot[i[t], {t, 0, 10}, DisplayFunction → Identity];
Show[p1, p2, PlotRange → All, DisplayFunction → $DisplayFunction]
Show[GraphicsRow[{p1, p2}]]
```

In this case, we see that we can use DSolve to solve the initial-value problem

$$Q'' + Q' + Q = E(t), \quad Q(0) = 0, \quad Q'(0) = 0$$

as well. However, the result is very lengthy, so only a portion is displayed here using Short.

```
sol = DSolve [{q''[t] + q'[t] + q[t]==e[t], q[0]==0, q'[0]==0}, q[t], t] ;
Short[sol]
```

$$\left\{\left\{\text{q[t]} \to \frac{e^{-t/2}\langle\langle 1\rangle\rangle}{3\left(\langle\langle 1\rangle\rangle^2+\langle\langle 1\rangle\rangle^2\right)} + \text{UnitStep}\left[\frac{\pi}{2}-t\right]\,\langle\langle\langle 1\rangle\rangle\rangle\right\}\right\}$$

We see that this result is a real-valued function using ComplexExpand followed by Simplify.

```
q[t_] = ComplexExpand[sol[[1, 1, 2]]]//Simplify
```

$$\begin{cases} \frac{1}{3}e^{-t}\sqrt{e^t}\left(3\text{Cos}\left[\frac{\sqrt{3}t}{2}\right] + \sqrt{3}\left(-2e^{\pi/4}\text{Sin}\left[\frac{1}{4}\sqrt{3}(\pi-2t)\right] + \text{Sin}\left[\frac{\sqrt{3}t}{2}\right]\right)\right) & 2t > \pi \\ \frac{1}{3}e^{-t}\left(-3e^t\text{Cos}[t] + \sqrt{e^t}\left(3\text{Cos}\left[\frac{\sqrt{3}t}{2}\right] + \sqrt{3}\text{Sin}\left[\frac{\sqrt{3}t}{2}\right]\right)\right) & 0 \le t \le \frac{\pi}{2} \end{cases}$$

We use this result to graph $Q(t)$ and $I(t) = Q'(t)$ in Figure 6.24.

```
Plot[{q[t], q'[t]}, {t, 0, 10},
   PlotStyle → {GrayLevel[0], GrayLevel[0.5]}]
```

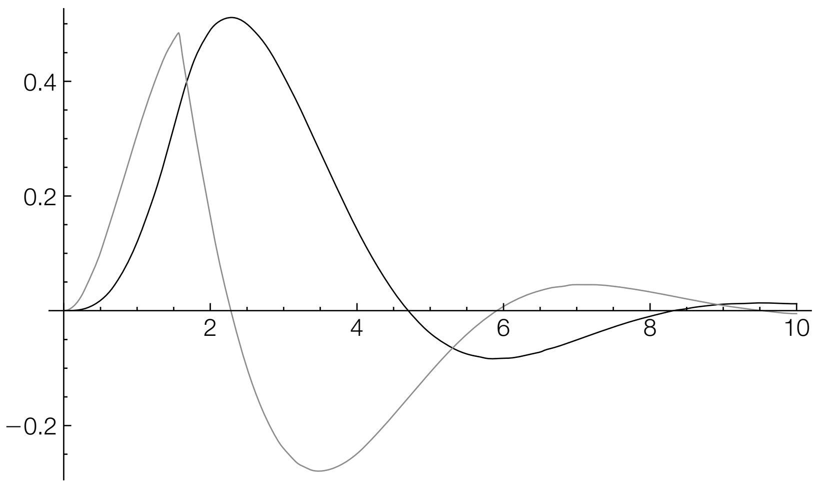



**FIGURE 6.24**

$Q(t)$ (in black) and $I(t) = Q'(t)$ (in gray)

### *Application: The Dirac Delta Function*

Let $\delta\left(t-t_0\right)$ denote the (generalized) function with the two properties

1. $\delta\left(t-t_0\right)=0$ if $t\neq t_0$ and
2. $\int_{-\infty}^{\infty}\delta\left(t-t_0\right)\,dt=1$

which is called the **Dirac delta function** and is quite useful in the definition of impulse forcing functions that arise in some differential equations. The Laplace transform of $\delta\left(t-t_0\right)$ is $\mathcal{L}\left\{\delta\left(t-t_0\right)\right\}=e^{-st_0}$. The Mathematica function DiracDelta represents the $\delta$ distribution.

```
LaplaceTransform[DiracDelta[t–t0], t, s]
```
$e^{-st0}$HeavisideTheta[t0]

---

**Example 6.3.9** Solve $\begin{cases} x''+x'+x=\delta(t)+\mathcal{U}(t-2\pi) \\ x(0)=0,\ x'(0)=0 \end{cases}.$

***Solution*** We define eq to be the equation $x''+x'+x=\delta(t)+\mathcal{U}(t-2\pi)$ and then use LaplaceTransform to compute the Laplace transform of eq, naming the resulting output leq. The symbol LaplaceTransform [x[t],t,s] represents the Laplace transform of x[t]. We then apply the initial conditions $x(0)=0$ and $x'(0)=0$ to leq and name the resulting output ics.

```
Clear[x, eq]
eq = x''[t] + x'[t] + x[t]==DiracDelta[t] + UnitStep[t–2π];
leq = LaplaceTransform[eq, t, s]
```
LaplaceTransform[x[t], t, s] + s LaplaceTransform[x[t], t, s] +
$s^2$LaplaceTransform[x[t], t, s] – x[0] – sx[0] – x′[0]==1 + $\frac{e^{-2\pi s}}{s}$

```
ics = leq/.{x[0] → 0, x'[0] → 0}
```
LaplaceTransform[x[t], t, s] + s LaplaceTransform[x[t], t, s] +
$s^2$LaplaceTransform[x[t], t, s]==1 + $\frac{e^{-2\pi s}}{s}$

Next, we use Solve to solve the equation ics for the Laplace transform of $x(t)$. The expression for the Laplace transform is extracted from lapx with lapx[[1,1,2]].

```
lapx = Solve[ics, LaplaceTransform[x[t], t, s]]
```
$\left\{\left\{\text{LaplaceTransform[x[t], t, s]} \to \frac{e^{-2\pi s}\left(1+e^{2\pi s}s\right)}{s\left(1+s+s^2\right)}\right\}\right\}$

To find $x(t)$, we must compute the inverse Laplace transform of the Laplace transform of $\mathcal{L}\{x(t)\}$ obtained in lapx. We use InverseLaplaceTransform to compute the inverse Laplace transform of lapx[[1,1,2]] and name the resulting function x[t].

```
x[t_] = InverseLaplaceTransform[lapx[[1, 1, 2]], s, t]
```
$\frac{2e^{-t/2}\text{Sin}\left[\frac{\sqrt{3}t}{2}\right]}{\sqrt{3}}+\frac{1}{3}$HeavisideTheta[$-2\pi+t$]

$$\left(3-e^{\pi-\frac{t}{2}}\left(3\text{Cos}\left[\frac{1}{2}\sqrt{3}(-2\pi+t)\right]+\sqrt{3}\text{Sin}\left[\frac{1}{2}\sqrt{3}(-2\pi+t)\right]\right)\right)$$

If necessary, to see that this is a real-valued function, we use ComplexExpand followed by Simplify. If needed, we see that the result is a real-valued function using ComplexExpand followed by Simplify.

```
x[t_] = ComplexExpand[x[t]]//Simplify
```

$$\frac{1}{3}e^{-t}\left(2\sqrt{3}\sqrt{e^t}\text{Sin}\left[\frac{\sqrt{3}t}{2}\right]+e^{t/2}\text{HeavisideTheta}[-2\pi+t]\right.$$
$$\left.\left(3e^{t/2}-3e^{\pi}\text{Cos}\left[\frac{1}{2}\sqrt{3}(-2\pi+t)\right]-\sqrt{3}e^{\pi}\text{Sin}\left[\frac{1}{2}\sqrt{3}(-2\pi+t)\right]\right)\right)$$

We use Plot to graph the solution on the interval $[0, 8\pi]$ in Figure 6.25.

```
Plot[x[t], {t, 0, 8π}]
```

Finally, we note that DSolve is able to solve the initial-value problem directly as well. The result is very lengthy, so only an abbreviated portion is displayed here using Short.

```
Clear[x]
sol = DSolve[{x''[t] + x'[t] + x[t]==DiracDelta[t] + UnitStep[t − 2π],
x[0]==0, x'[0]==0}, x[t], t];
Short[sol, 2]
```

$$\left\{\left\{x[t]\to\frac{e^{-t/2}(\langle\langle 1\rangle\rangle)}{3\left(\text{Cos}\left[\sqrt{3}\pi\right]^2+\text{Sin}[\langle\langle 1\rangle\rangle]^2\right)}+\right.\right.$$
$$\left.\left.\left(-\frac{2\langle\langle 1\rangle\rangle\langle\langle 1\rangle\rangle\text{Sin}\left[\frac{\langle\langle 1\rangle\rangle}{2}\right]}{\sqrt{3}}-\frac{e^{\langle\langle 1\rangle\rangle}\langle\langle 1\rangle\rangle}{3(\langle\langle 1\rangle\rangle)}\right)\langle\langle 8\rangle\rangle\;[\langle\langle 1\rangle\rangle]\right\}\right\}$$

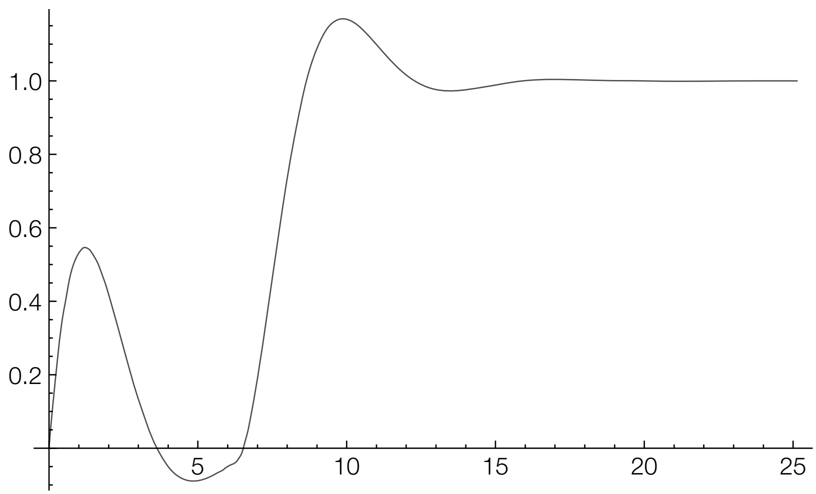



**FIGURE 6.25**

Plot of $x(t)$ on the interval $[0, 8\pi]$

### 6.3.5 Nonlinear Higher-Order Equations

Generally, rigorous results regarding nonlinear equations are very difficult to obtain. In some cases, analysis is best carried out numerically and/or graphically. In other situations, rewriting the equation as a system can be of benefit, which is discussed in the next section. (See Examples 6.4.5, 6.4.6, and 6.4.8.)

## 6.4 SYSTEMS OF EQUATIONS

### 6.4.1 Linear Systems

We now consider first-order linear systems of differential equations:

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{F}(t), \tag{6.28}$$

where

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \dots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

#### *Homogeneous Linear Systems*

The corresponding homogeneous system of equation (6.28) is

$$\mathbf{X}' = \mathbf{A}\mathbf{X}. \tag{6.29}$$

A **particular solution** to a system of ordinary differential equations is a set of functions that satisfy the system but do not contain any arbitrary constants. That is, a particular solution to a system is a set of specific functions, *containing no arbitrary constants*, that satisfy the system.

In the same way as with the previously discussed linear equations, a **general solution** of equation (6.28) is $\mathbf{X} = \mathbf{X}_h + \mathbf{X}_p$, where $\mathbf{X}_h$ is a *general solution* of equation (6.29) and $\mathbf{X}_p$ is a *particular solution* of the nonhomogeneous system equation (6.28).

If $\mathbf{\Phi}_1, \mathbf{\Phi}_2, \dots, \mathbf{\Phi}_n$ are $n$ linearly independent solutions of equation (6.29), a **general solution** of equation (6.29) is

$$\mathbf{X} = c_1\mathbf{\Phi}_1 + c_2\mathbf{\Phi}_2 + \cdots + c_n\mathbf{\Phi}_n = \begin{pmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \cdots & \mathbf{\Phi}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{\Phi}\mathbf{C},$$

where

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \dots & \mathbf{\Phi}_n \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

$\mathbf{\Phi}$ is called a **fundamental matrix** for equation (6.29). If $\mathbf{\Phi}$ is a fundamental matrix for equation (6.29), $\mathbf{\Phi}' = \mathbf{A\Phi}$ or $\mathbf{\Phi}' - \mathbf{A\Phi} = \mathbf{0}$.

### $\mathbf{A}(t)$ constant

Suppose that $\mathbf{A}(t) = \mathbf{A}$ has constant real entries. Let $\lambda$ be an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$. Then, $\mathbf{v}e^{\lambda t}$ is a solution of $\mathbf{X}' = \mathbf{AX}$.

If $\lambda = \alpha + \beta i$, $\beta \neq 0$, is an eigenvalue of $\mathbf{A}$ and has corresponding eigenvector $\mathbf{v} = \mathbf{a} + \mathbf{b}i$, two linearly independent solutions of $\mathbf{X}' = \mathbf{AX}$ are

$$e^{\alpha t}\left(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t\right) \quad \text{and} \quad e^{\alpha t}\left(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t\right). \tag{6.30}$$

**Example 6.4.1** Solve each of the following systems:

(a) $\mathbf{X}' = \begin{pmatrix} -1/2 & -1/3 \\ -1/3 & -1/2 \end{pmatrix}\mathbf{X}$; (b) $\begin{cases} x' = \frac{1}{2}y \\ y' = -\frac{1}{8}x \end{cases}$; (c) $\begin{cases} dx/dt = -\frac{1}{4}x + 2y \\ dy/dt = -8x - \frac{1}{4}y. \end{cases}$

***Solution*** (a) With Eigensystem, we see that the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} -1/2 & -1/3 \\ -1/3 & -1/2 \end{pmatrix}$ are $\lambda_1 = -1/6$ and $\lambda_2 = -5/6$ and $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively.

```
capa = {{-1/2, -1/3}, {-1/3, -1/2}};
Eigensystem[capa]
```

$\left\{\left\{-\frac{5}{6}, -\frac{1}{6}\right\}, \{\{1, 1\}, \{-1, 1\}\}\right\}$

Then $\mathbf{X}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/6}$ and $\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t/6}$ are two linearly independent solutions of the system, so a general solution is $\mathbf{X} = \begin{pmatrix} -e^{-t/6} & e^{-5t/6} \\ e^{-t/6} & e^{-5t/6} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$; a fundamental matrix is $\mathbf{\Phi} = \begin{pmatrix} -e^{-t/6} & e^{-5t/6} \\ e^{-t/6} & e^{-5t/6} \end{pmatrix}$.

We use DSolve to find a general solution of the system by entering

```
Clear[x, y]
gensol = DSolve[{x'[t]==-1/2x[t]-1/3y[t],
y'[t]==-1/3x[t]-1/2y[t]}, {x[t], y[t]}, t]
```

$\left\{\left\{x[t] \to \frac{1}{2}e^{-5t/6}\left(1 + e^{2t/3}\right)C[1] - \frac{1}{2}e^{-5t/6}\left(-1 + e^{2t/3}\right)C[2],\right.\right.$
$\left.\left. y[t] \to -\frac{1}{2}e^{-5t/6}\left(-1 + e^{2t/3}\right)C[1] + \frac{1}{2}e^{-5t/6}\left(1 + e^{2t/3}\right)C[2]\right\}\right\}$

We graph the direction field with VectorFieldPlot, which is contained in the VectorFieldPlots, in Figure 6.26.

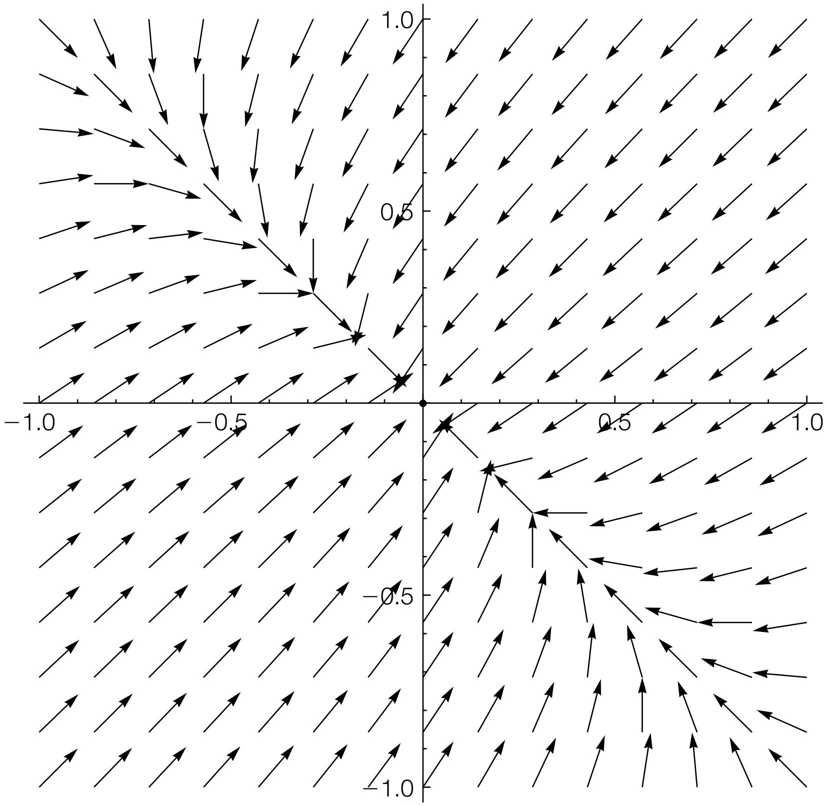



**FIGURE 6.26**

Direction field for $\mathbf{X}' = \mathbf{AX}$

**Remark 6.4** After you have loaded the VectorFieldPlots package,

**VectorFieldPlot[{f[x, y], g[x, y]}, {x, a, b}, {y, c, d}]**

generates a basic direction field for the system $\{x' = f(x, y),\ y' = g(x, y)\}$ for $a \le x \le b$ and $c \le y \le d$.

```
<< "VectorFieldPlots`"
pvf = Show[VectorFieldPlot[{-1/2x - 1/3y, -1/3x - 1/2y},
   {x, -1, 1}, {y, -1, 1}, ScaleFunction → (1&)],
     Axes → Automatic]
```

Several solutions are also graphed with ParametricPlot and shown together with the direction field in Figure 6.27. To do so, we first solve the system if $x(0) = x_0$ and $y(0) = y_0$.

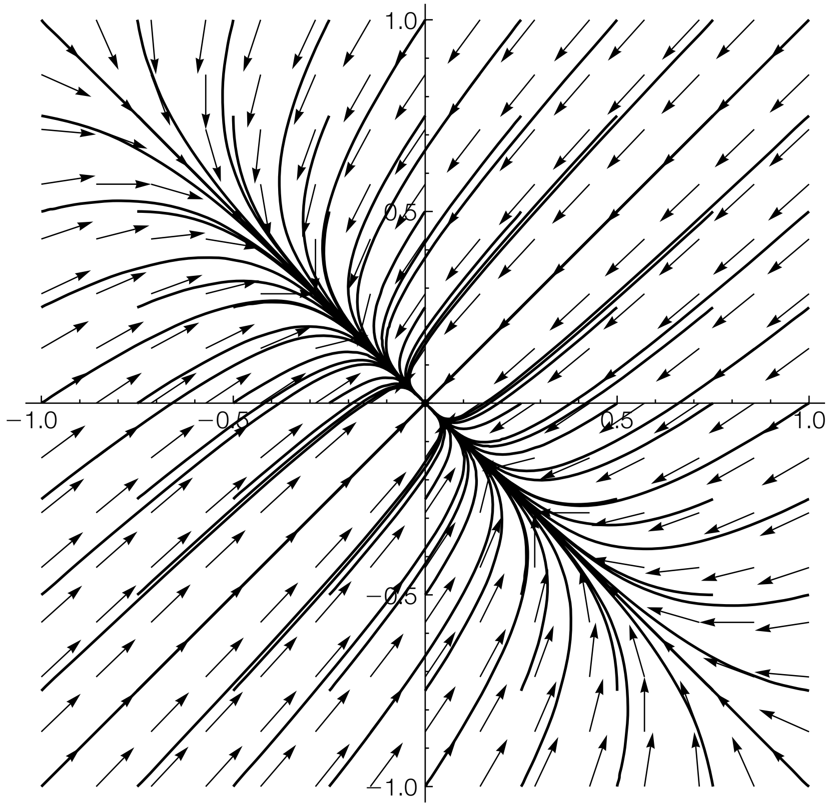



**FIGURE 6.27**

Direction field for $\mathbf{X}' = \mathbf{AX}$ along with various solution curves

```
initsol = DSolve[{x'[t]== - 1/2x[t] - 1/3y[t],
   y'[t]== - 1/3x[t] - 1/2y[t], x[0]==x0, y[0]==y0}, {x[t], y[t]}, t]
```

$$\left\{\left\{x[t] \to \tfrac{1}{2}e^{-5t/6}\left(x0 + e^{2t/3}x0 + y0 - e^{2t/3}y0\right),\right.\right.$$
$$\left.\left. y[t] \to -\tfrac{1}{2}e^{-5t/6}\left(-x0 + e^{2t/3}x0 - y0 - e^{2t/3}y0\right)\right\}\right\}$$

Given an ordered pair, **solplot** parametrically graphs the solution satisfying $x(0) = x_0$ and $y(0) = y_0$ for $0 \le t \le 15$.

```
solplot[pair_]:=
ParametricPlot[
   Evaluate[{x[t], y[t]}/.initsol/.{x0 → pair[[1]], y0 → pair[[2]]}],
      {t, 0, 15}, PlotStyle → {{Black, Thickness[.005]}}]
```

We then define a list of ordered pairs with Table followed by Flatten

```
Clear[i, j]
orderedpairs = Flatten[Table[{i, j}, {i, -1, 1, 1/4}, {j, -1, 1, 1/4}], 1];
Short[orderedpairs]
```

$\{\{-1,-1\},\{-1,-\frac{3}{4}\},\{-1,-\frac{1}{2}\},\langle\langle 75\rangle\rangle,\{1,\frac{1}{2}\},\{1,\frac{3}{4}\},\{1,1\}\}$

and use Map to apply solplot to orderedpairs.

```
toshow = Map[solplot, orderedpairs];
```

The resulting list of graphics objects is displayed together with Show. See Figure 6.27.

```
Show[toshow, pvf, PlotRange → {{-1, 1}, {-1, 1}}]
```

(b) In matrix form the system is equivalent to the system $\mathbf{X}' = \begin{pmatrix} 0 & 1/2 \\ -1/8 & 0 \end{pmatrix}\mathbf{X}$. As in (a), we use Eigensystem to see that the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 0 & 1/2 \\ -1/8 & 0 \end{pmatrix}$ are $\lambda_{1,2} = 0 \pm \frac{1}{4}i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} i$.

```
capa = {{0, 1/2}, {-1/8, 0}};
Eigensystem[capa]
```

$\{\{\frac{i}{4},-\frac{i}{4}\},\{\{-2i,1\},\{2i,1\}\}\}$

Two linearly independent solutions are then $\mathbf{X}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos\frac{1}{4}t - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\sin\frac{1}{4}t = \begin{pmatrix} \cos\frac{1}{4}t \\ -\frac{1}{2}\sin\frac{1}{4}t \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin\frac{1}{4}t + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\cos\frac{1}{4}t = \begin{pmatrix} \sin\frac{1}{4}t \\ \frac{1}{2}\cos\frac{1}{4}t \end{pmatrix}$, and a general solution is $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = \begin{pmatrix} \cos\frac{1}{4}t & \sin\frac{1}{4}t \\ -\frac{1}{2}\sin\frac{1}{4}t & \frac{1}{2}\cos\frac{1}{4}t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ or $x = c_1\cos\frac{1}{4}t + c_2\sin\frac{1}{4}t$ and $y = -c_1\frac{1}{2}\sin\frac{1}{4}t + \frac{1}{2}c_2\cos\frac{1}{4}t$.

As before, we use DSolve to find a general solution.

```
Clear[x, y]
gensol = DSolve[{x'[t]==1/2y[t], y'[t]==-1/8x[t]},
{x[t], y[t]}, t]
```

$\{\{x[t]\to C[1]\mathrm{Cos}[\frac{t}{4}]+2C[2]\mathrm{Sin}[\frac{t}{4}],$

$y[t]\to C[2]\mathrm{Cos}[\frac{t}{4}]-\frac{1}{2}C[1]\mathrm{Sin}[\frac{t}{4}]\}\}$

Initial-value problems for systems are solved in the same way as for other equations. For example, entering

```
partsol = DSolve[{x'[t]==1/2y[t], y'[t]==-1/8x[t],
```

```
    x[0]==1, y[0]==-1},
        {x[t], y[t]}, t]
```

$\{\{x[t] \to \text{Cos}\left[\frac{t}{4}\right] - 2\text{Sin}\left[\frac{t}{4}\right], y[t] \to \frac{1}{2}\left(-2\text{Cos}\left[\frac{t}{4}\right] - \text{Sin}\left[\frac{t}{4}\right]\right)\}\}$

finds the solution that satisfies $x(0) = 1$ and $y(0) = -1$.

We graph $x(t)$ and $y(t)$ together as well as parametrically with Plot and ParametricPlot, respectively, in Figure 6.28.

```
p1 = Plot[{x[t], y[t]}/.partsol, {t, 0, 8Pi}];
p2 = ParametricPlot[{x[t], y[t]}/.partsol, {t, 0, 8Pi},
        AspectRatio → Automatic];
Show[GraphicsRow[{p1, p2}]]
```

We can also use VectorFieldPlot and ParametricPlot to graph the direction field and/or various solutions as we do next with Manipulate. In this case, Manipulate is used to graph the solution passing through the locator point. As you move the point with the cursor, the corresponding solution is plotted. See Figure 6.29.

```
Clear[pvf, sol, p1];
Manipulate[
<< "VectorFieldPlots`";
Module[{pvf, sol, p1},
pvf = Show[VectorFieldPlot[{1/2y, -1/8x},
    {x, -2, 2}, {y, -1, 1}, ScaleFunction → (1&)],
        Axes → Automatic];
sol = DSolve[{x'[t]==1/2y[t], y'[t]==-1/8x[t],
    x[0]==pt[[1]], y[0]==pt[[2]]},
        {x[t], y[t]}, t];
p1 = ParametricPlot[{x[t], y[t]}/.sol, {t, 0, 8Pi},
      PlotStyle → Thickness[.01]];
Show[p1, pvf, PlotRange → {{-2, 2}, {-1, 1}},
      AspectRatio → 1]], {{pt, {1, .5}}, Locator}]
```

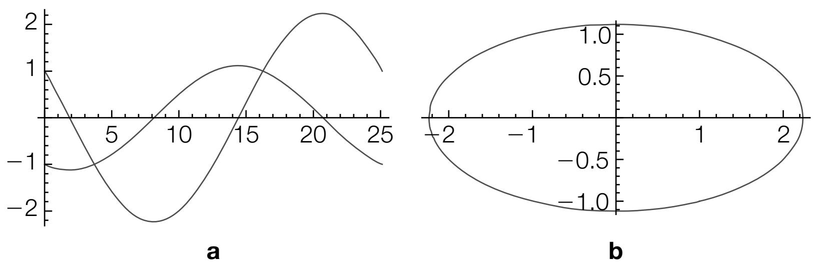



**FIGURE 6.28**

(a) Graph of $x(t)$ and $y(t)$. (b) Parametric plot of $x(t)$ versus $y(t)$

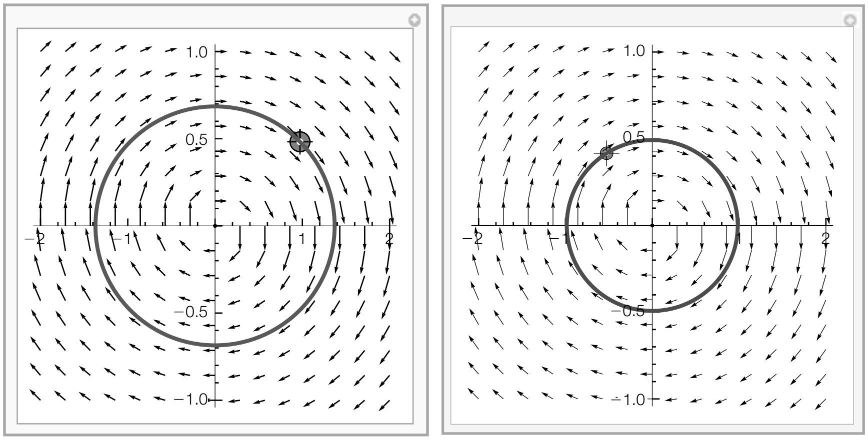



**FIGURE 6.29**

With **Manipulate**, notice that all nontrivial solutions are periodic

(c) In matrix form, the system is equivalent to the system $\mathbf{X}' = \begin{pmatrix} -\frac{1}{4} & 2 \\ -8 & -\frac{1}{4} \end{pmatrix}\mathbf{X}$. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} -\frac{1}{4} & 2 \\ -8 & -\frac{1}{4} \end{pmatrix}$ are found to be $\lambda_{1,2} = -\frac{1}{4} \pm 4i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$ with **Eigensystem**.

```
capa = {{-1/4, 2}, {-8, -1/4}};
Eigensystem[capa]
```

$\left\{\left\{-\frac{1}{4}+4i, -\frac{1}{4}-4i\right\}, \left\{\left\{-\frac{i}{2}, 1\right\}, \left\{\frac{i}{2}, 1\right\}\right\}\right\}$

A general solution is then

$$\begin{aligned} \mathbf{X} &= c_1\mathbf{X}_1 + c_2\mathbf{X}_2 \\ &= c_1 e^{-t/4}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin 4t\right) + c_2 e^{-t/4}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 4t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos 4t\right) \\ &= e^{-t/4}\left[c_1\begin{pmatrix} \cos 4t \\ -2\sin 4t \end{pmatrix} + c_2\begin{pmatrix} \sin 4t \\ 2\cos 4t \end{pmatrix}\right] = e^{-t/4}\begin{pmatrix} \cos 4t & \sin 4t \\ -2\sin 4t & 2\cos 4t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

or $x = e^{-t/4}\left(c_1\cos 4t + c_2\sin 4t\right)$ and $y = e^{-t/4}\left(2c_2\cos 4t - 2c_1\sin 4t\right)$. We confirm this result using **DSolve**.

```
gensol = DSolve[{x'[t]==-1/4x[t] + 2y[t],
    y'[t]==-8x[t] - 1/4y[t]}, {x[t], y[t]}, t]
```

$\{\{x[t] \to e^{-t/4}C[1]Cos[4t] + \frac{1}{2}e^{-t/4}C[2]Sin[4t],$
$y[t] \to e^{-t/4}C[2]Cos[4t] - 2e^{-t/4}C[1]Sin[4t]\}\}$

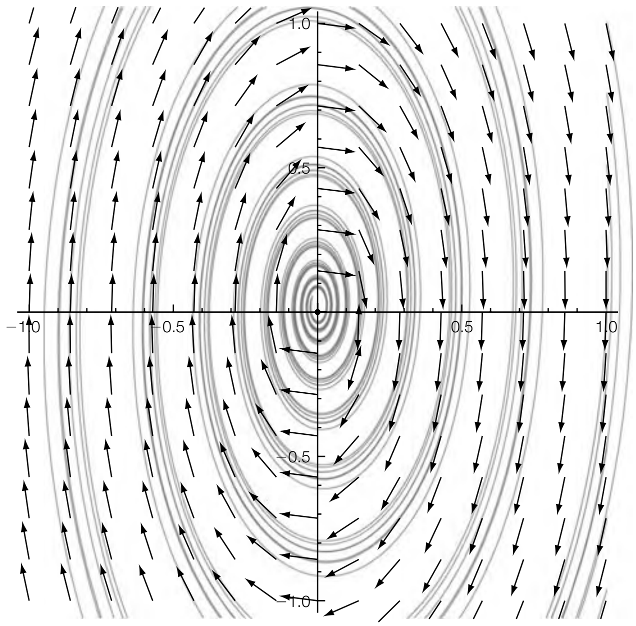



**FIGURE 6.30**

Various solutions and direction field associated with the system

We use VectorFieldPlot and ParametricPlot to graph the direction field associated with the system along with various solutions in Figure 6.30.

```
Clear[x, y]
initsol = DSolve[{x′[t]== − 1/4x[t] + 2y[t],
    y′[t]== − 8x[t] − 1/4y[t], x[0]==x0, y[0]==y0},
        {x[t], y[t]}, t]
```

$\{\{x[t] \to \frac{1}{2}e^{-t/4}(2x0Cos[4t] + y0Sin[4t]),$

$y[t] \to e^{-t/4}(y0Cos[4t] - 2x0Sin[4t])\}\}$

```
t1 = Table[ParametricPlot[{x[t], y[t]}/.initsol/.
    {x0 → 1, y0 → i}, {t, 0, 15}, PlotStyle → Gray],
        {i, − 1, 1, 2/8}];
pvf = Show[VectorFieldPlot[{ − 1/4x + 2y, − 8x − 1/4y},
    {x, − 1, 1}, {y, − 1, 1}, ScaleFunction → (1&)],
        Axes → Automatic];
        Show[t1, pvf, PlotRange → {{ − 1, 1}, { − 1, 1}},
        AspectRatio → Automatic]
```

Last, we illustrate how to solve an initial-value problem and graph the resulting solutions by finding the solution that satisfies the initial conditions $x(0) = 100$ and $y(0) = 10$ and then graphing the results with Plot and ParametricPlot in Figure 6.31.

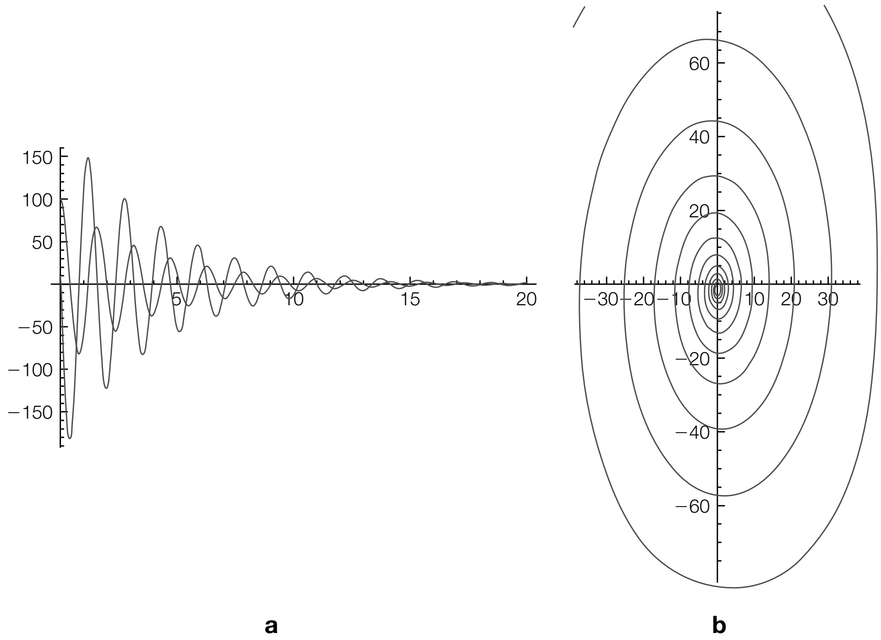



**FIGURE 6.31**

(a) Graph of $x(t)$ and $y(t)$. (b) Parametric plot of $x(t)$ versus $y(t)$ (for help with Show and GraphicsRow use the **Documentation Center**)

```
Clear[x, y]
partsol = DSolve[{x′[t]== − 1/4x[t] + 2y[t],
    y′[t]== − 8x[t] − 1/4y[t], x[0]==100, y[0]==10},
        {x[t], y[t]}, t]
```

$\{\{x[t] \to 5e^{-t/4}(20Cos[4t] + Sin[4t]), y[t] \to 10e^{-t/4}(Cos[4t] - 20Sin[4t])\}\}$

```
p1 = Plot[{x[t], y[t]}/.partsol, {t, 0, 20}, PlotRange → All];
p2 = ParametricPlot[{x[t], y[t]}/.partsol, {t, 0, 20},
    AspectRatio → Automatic];
        Show[GraphicsRow[{p1, p2}]]
```

## *Application: The Double Pendulum*

The motion of a double pendulum is modeled by the system of differential equations

$$\begin{cases} (m_1 + m_2)\, l_1{}^2 \dfrac{d^2\theta_1}{dt^2} + m_2 l_1 l_2 \dfrac{d^2\theta_2}{dt^2} + (m_1 + m_2)\, l_1 g \theta_1 = 0 \\ m_2 l_2{}^2 \dfrac{d^2\theta_2}{dt^2} + m_2 l_1 l_2 \dfrac{d^2\theta_1}{dt^2} + m_2 l_2 g \theta_2 = 0 \end{cases}$$

using the approximation $\sin\theta \approx \theta$ for small displacements. $\theta_1$ represents the displacement of the upper pendulum and $\theta_2$ that of the lower pendulum. Also, $m_1$ and $m_2$ represent the mass attached to the upper and lower pendulums, respectively, whereas the length of each is given by $l_1$ and $l_2$.

**Example 6.4.2** Suppose that $m_1 = 3$, $m_2 = 1$, and each pendulum has length $16$. If $\theta_1(0) = 1$, $\theta_1{}'(0) = 0$, $\theta_2(0) = -1$, and $\theta_2{}'(0) = 0$, solve the double pendulum problem using $g = 32$. Plot the solution.

***Solution*** In this case, the system to be solved is

$$\begin{cases} 4 \cdot 16^2 \dfrac{d^2\theta_1}{dt^2} + 16^2 \dfrac{d^2\theta_2}{dt^2} + 4 \cdot 16 \cdot 32\theta_1 = 0 \\ 16^2 \dfrac{d^2\theta_2}{dt^2} + 16^2 \dfrac{d^2\theta_1}{dt^2} + 16 \cdot 32\theta_2 = 0, \end{cases}$$

which we simplify to obtain

$$\begin{cases} 4\dfrac{d^2\theta_1}{dt^2} + \dfrac{d^2\theta_2}{dt^2} + 8\theta_1 = 0 \\ \dfrac{d^2\theta_2}{dt^2} + \dfrac{d^2\theta_1}{dt^2} + 2\theta_2 = 0. \end{cases}$$

In the following code, we let $x(t)$ and $y(t)$ represent $\theta_1(t)$ and $\theta_2(t)$, respectively. First, we use **DSolve** to solve the initial-value problem.

```
sol = DSolve[{4x''[t] + y''[t] + 8x[t]==0, x''[t] + y''[t] + 2y[t]==0,
     x[0]==1, x'[0]==1, y[0]==0, y'[0]== - 1}, {x[t], y[t]}, t]
```

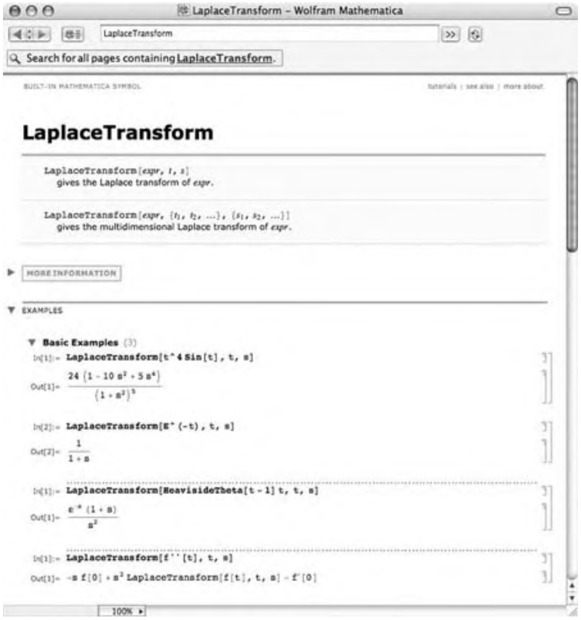
$\left\{\left\{x[t] \to \frac{1}{8}\left(4\text{Cos}[2t] + 4\text{Cos}\left[\frac{2t}{\sqrt{3}}\right] + 3\text{Sin}[2t] + \sqrt{3}\text{Sin}\left[\frac{2t}{\sqrt{3}}\right]\right),\right.\right.$
$\left.\left. y[t] \to \frac{1}{4}\left(-4\text{Cos}[2t] + 4\text{Cos}\left[\frac{2t}{\sqrt{3}}\right] - 3\text{Sin}[2t] + \sqrt{3}\text{Sin}\left[\frac{2t}{\sqrt{3}}\right]\right)\right\}\right\}$

The **Laplace transform** of $y = f(t)$ is $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$.

To solve the initial-value problem using traditional methods, we use the *method of Laplace transforms*. To do so, we define **sys** to be the system of equations and use **LaplaceTransform** to compute the Laplace transform of each equation.

```
step1 = LaplaceTransform[sys, t, s]
```

```
{8LaplaceTransform[x[t], t, s] +
s^2LaplaceTransform[y[t], t, s] - sy[0] +
  4 (s^2LaplaceTransform[x[t], t, s] - sx[0] - x'[0]) - y'[0]==0,
  s^2LaplaceTransform[x[t], t, s] +
  2LaplaceTransform[y[t], t, s] +
  s^2LaplaceTransform[y[t], t, s] - sx[0] -
  sy[0] - x'[0] - y'[0]==0}
```




Next, we apply the initial conditions and solve the resulting system of equations for $\mathcal{L}\{\theta_1(t)\} = X(s)$ and $\mathcal{L}\{\theta_2(t)\} = Y(s)$.

**step2 = step1/.{x[0] –> 1, x′[0] –> 1, y[0] –> 0, y′[0] –> –1}**

{1 + 8LaplaceTransform[x[t], t, s] +
4 (–1 – s + s²LaplaceTransform[x[t], t, s]) +
s²LaplaceTransform[y[t], t, s]==0,
–s + s²LaplaceTransform[x[t], t, s] +
2LaplaceTransform[y[t], t, s] +
s²LaplaceTransform[y[t], t, s]==0}

**step3 =**
**Solve[step2, {LaplaceTransform[x[t], t, s],**
**LaplaceTransform[y[t], t, s]}]**

$$\left\{\left\{\text{LaplaceTransform[x[t], t, s]} \to -\frac{-6-8s-3s^2-3s^3}{16+16s^2+3s^4},\ \text{LaplaceTransform[y[t], t, s]} \to -\frac{-8s+3s^2}{16+16s^2+3s^4}\right\}\right\}$$

InverseLaplaceTransform is then used to find $\theta_1(t)$ and $\theta_2(t)$.

$f(t)$ is the **inverse Laplace transform** of $F(s)$ if $\mathcal{L}\{f(t)\} = F(s)$; we write $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

**x[t_] = InverseLaplaceTransform** $\left[-\frac{-6-8s-3s^2-3s^3}{16+16s^2+3s^4}, s, t\right]$

$$\frac{1}{8}\left(4\text{Cos}[2t] + 4\text{Cos}\left[\frac{2t}{\sqrt{3}}\right] + 3\text{Sin}[2t] + \sqrt{3}\text{Sin}\left[\frac{2t}{\sqrt{3}}\right]\right)$$

```
y[t_] = InverseLaplaceTransform[-(-8s + 3s^2)/(16 + 16s^2 + 3s^4), s, t]
```

$\frac{1}{4}\left(-4\text{Cos}[2t] + 4\text{Cos}\left[\frac{2t}{\sqrt{3}}\right] - 3\text{Sin}[2t] + \sqrt{3}\text{Sin}\left[\frac{2t}{\sqrt{3}}\right]\right)$

These two functions are graphed together in Figure 6.32(a) and parametrically in Figure 6.32(b).

```
p1 = Plot[{x[t], y[t]}, {t, 0, 20},
    PlotStyle -> {GrayLevel[0], GrayLevel[0.5]}]
p2 = ParametricPlot[{x[t], y[t]}, {t, 0, 20},
    PlotRange -> {{-5/2, 5/2}, {-5/2, 5/2}}, AspectRatio -> 1]
Show[GraphicsRow[{p1, p2}]]
```

We can illustrate the motion of the pendulum as follows. First, we define the function pen2.

```
Clear[pen2]
pen2[t_, len1_, len2_]:=Module[{pt1, pt2},
pt1 = {len1 Cos[3π/2 + x[t]], len1 Sin[3π/2 + x[t]]};
pt2 = {len1 Cos[3π/2 + x[t]] + len2 Cos[3π/2 + y[t]],
   len1Sin[3π/2 + x[t]] + len2Sin[3π/2 + y[t]]};

Show[Graphics[{Line[{{0, 0}, pt1}], PointSize[.05], Point[pt1],
  Line[{pt1, pt2}], PointSize[.05], Point[pt2]}], Axes → Automatic,
    Ticks → None, AxesStyle → GrayLevel[.5],
    PlotRange → {{-32, 32}, {-34, 0}},
    DisplayFunction → Identity]]
```

Next, we define tvals to be a list of 16 evenly spaced numbers between 0 and 10. Map is then used to apply pen2 to the list of numbers in tvals. The resulting

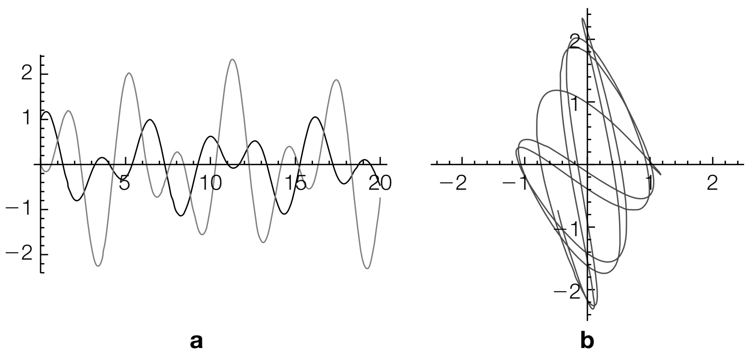



**FIGURE 6.32**

(a) $\theta_1(t)$ (in black) and $\theta_2(t)$ (in gray) as functions of $t$. (b) Parametric plot of $\theta_1(t)$ versus $\theta_2(t)$

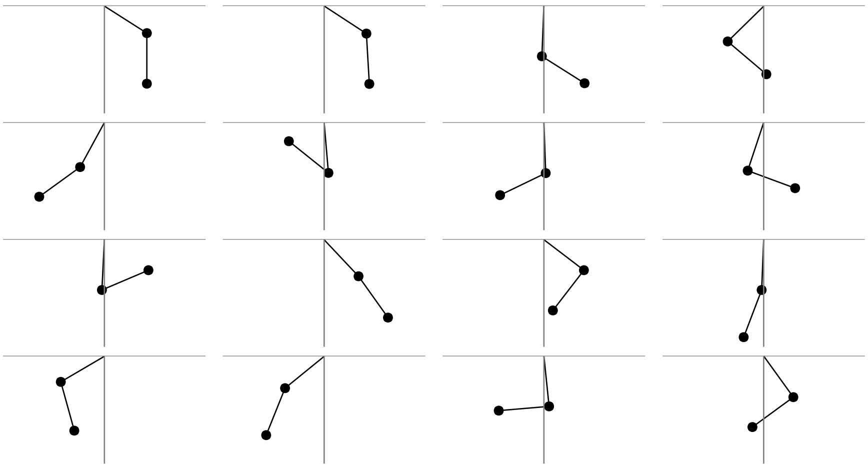


**FIGURE 6.33**

The double pendulum for 16 equally spaced values of $t$ between 0 and 10

set of graphics is partitioned into four element subsets and displayed using Show and GraphicsGrid in Figure 6.33.

```
tvals = Table[t, {t, 0, 10, 10/15}];
graphs = Map[pen2[#, 16, 16]&, tvals];
toshow = Partition[graphs, 4];
Show[GraphicsGrid[toshow]]
```

If the option DisplayFunction->Identity is omitted from the definition of pen2, we can use a Do loop together with Print to generate a set of graphics that can then be animated.

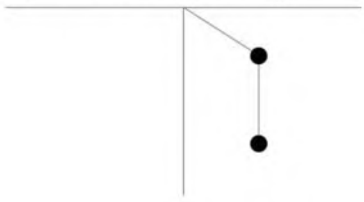
```
Clear[pen2]
pen2[t_, len1_, len2_]:=Module[{pt1, pt2},
pt1 = {len1Cos[3π/2 + x[t]], len1Sin[3π/2 + x[t]]};
pt2 = {len1Cos[3π/2 + x[t]] + len2Cos[3π/2 + y[t]],
    len1Sin[3π/2 + x[t]] + len2Sin[3π/2 + y[t]]};
Show[Graphics[{Line[{{0, 0}, pt1}], PointSize[.05], Point[pt1],
   Line[{pt1, pt2}], PointSize[.05], Point[pt2]}], Axes → Automatic,
   Ticks → None, AxesStyle → GrayLevel[.5],
   PlotRange → {{-32, 32}, {-34, 0}}]]
```

We show one frame from the animation that results from the Do loop

```
Do[Print[pen2[t, 16, 16]], {t, 0, 10, 10/59}]
```

in the following screen shot.

```
Clear[pen2]
pen2[t_, len1_, len2_] := Module[{pt1, pt2},
      pt1 = {len1 Cos[3π/2 + x[t]], len1 Sin[3π/2 + x[t]]};
      pt2 = {len1 Cos[3π/2 + x[t]] + len2 Cos[3π/2 + y[t]],
   len1 Sin[3π/2 + x[t]] + len2 Sin[3π/2 + y[t]]};

   Show[Graphics[{Line[{{0, 0}, pt1}], PointSize[.05],
      Point[pt1], Line[{pt1, pt2}], PointSize[.05],
      Point[pt2]}], Axes → Automatic, Ticks → None,
    AxesStyle → GrayLevel[.5], PlotRange → {{-32, 32}, {-34, 0}}]]

Do[Print[pen2[t, 16, 16]], {t, 0, 10, 10/59}]
```



Alternatively, you can use Manipulate

```
Clear[pen2]
Manipulate[
x[t_] = 1/8 (4Cos[2t] + 4Cos[2t/√3] + 3Sin[2t] + √3Sin[2t/√3]);
y[t_] = 1/4 (−4Cos[2t] + 4Cos[2t/√3] − 3Sin[2t] + √3Sin[2t/√3]);
pen2[t_, len1_, len2_]:=Module[{pt1, pt2},
   pt1 = {len1Cos[3π/2 + x[t]], len1Sin[3π/2 + x[t]]};
   pt2 = {len1Cos[3π/2 + x[t]] + len2Cos[3π/2 + y[t]],
   len1Sin[3π/2 + x[t]] + len2Sin[3π/2 + y[t]]};

   Show[Graphics[{Line[{{0, 0}, pt1}], PointSize[.05], Point[pt1],
      Line[{pt1, pt2}], PointSize[.05], Point[pt2]}], Axes → Automatic,
      Ticks → None, AxesStyle → GrayLevel[.5], AspectRatio → 1,
      PlotRange → {{−32, 32}, {−34, 0}}]];
pen2[t, 16, 16], {{t, 0}, 0, 6}]
```

to generate a nearly identical animation as shown in Figure 6.34.

## 6.4.2 Nonhomogeneous Linear Systems

Generally, the method of undetermined coefficients is difficult to implement for nonhomogeneous linear systems because the choice for the particular solution must be very carefully made. Variation of parameters is implemented in much the same way as for first-order linear equations.

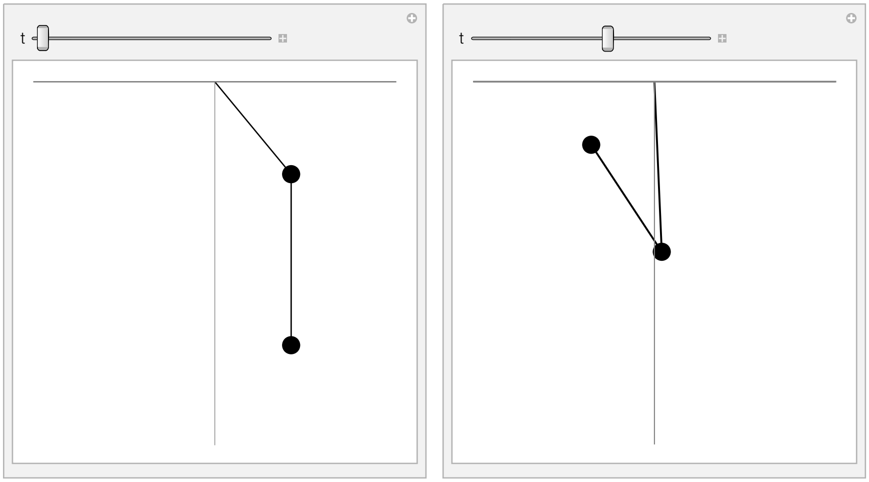



**FIGURE 6.34**

Two frames from a Manipulate animation of a double pendulum

Let $\mathbf{X}_h$ be a general solution to the corresponding homogeneous system of equation (6.28), $\mathbf{X}$ a general solution of equation (6.28), and $\mathbf{X}_p$ a particular solution of equation (6.28). It then follows that $\mathbf{X} - \mathbf{X}_p$ is a solution to the corresponding homogeneous system so $\mathbf{X} - \mathbf{X}_p = \mathbf{X}_h$ and, consequently, $\mathbf{X} = \mathbf{X}_h + \mathbf{X}_p$. A particular solution of equation (6.28) is found in much the same way as with first-order linear equations. Let $\mathbf{\Phi}$ be a fundamental matrix for the corresponding homogeneous system. We assume that a particular solution has the form $\mathbf{X}_p = \mathbf{\Phi}\mathbf{U}(t)$. Differentiating $\mathbf{X}_p$ gives us

$$\mathbf{X}_p{}' = \mathbf{\Phi}'\mathbf{U} + \mathbf{\Phi}\mathbf{U}'.$$

Substituting into equation (6.28) results in

$$\begin{aligned}
\mathbf{\Phi}'\mathbf{U} + \mathbf{\Phi}\mathbf{U}' &= \mathbf{A}\mathbf{\Phi}\mathbf{U} + \mathbf{F} \\
\mathbf{\Phi}\mathbf{U}' &= \mathbf{F} \\
\mathbf{U}' &= \mathbf{\Phi}^{-1}\mathbf{F} \\
\mathbf{U} &= \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt,
\end{aligned}$$

where we have used the fact that $\mathbf{\Phi}'\mathbf{U} - \mathbf{A}\mathbf{\Phi}\mathbf{U} = \left(\mathbf{\Phi}' - \mathbf{A}\mathbf{\Phi}\right)\mathbf{U} = \mathbf{0}$. It follows that

$$\mathbf{X}_p = \mathbf{\Phi}\int \mathbf{\Phi}^{-1}\mathbf{F}\,dt. \tag{6.31}$$

A general solution is then

$$\begin{aligned}
\mathbf{X} &= \mathbf{X}_h + \mathbf{X}_p \\
&= \mathbf{\Phi}\mathbf{C} + \mathbf{\Phi}\int \mathbf{\Phi}^{-1}\mathbf{F}\,dt
\end{aligned}$$

$$= \Phi\left(\mathbf{C} + \int \Phi^{-1}\mathbf{F}\,dt\right) = \Phi\int \Phi^{-1}\mathbf{F}\,dt,$$

where we have incorporated the constant vector $\mathbf{C}$ into the indefinite integral $\int \Phi^{-1}\mathbf{F}\,dt$.

**Example 6.4.3** Solve the initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}\mathbf{X} - \begin{pmatrix} t\cos 3t \\ t\sin t + t\cos 3t \end{pmatrix}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

**Remark 6.5** In traditional form, the system is equivalent to

$$\begin{cases} x' = x - y - t\cos 3t \\ y' = 10x - y - t\sin t - t\cos 3t, \end{cases} \qquad x(0) = 1,\ y(0) = -1.$$

***Solution*** The corresponding homogeneous system is $\mathbf{X}_h' = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}\mathbf{X}_h$. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}$ are $\lambda_{1,2} = \pm 3i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 10 \end{pmatrix} \pm \begin{pmatrix} -3 \\ 0 \end{pmatrix} i$, respectively.

```
capa = {{1, -1},{10, -1}};
Eigensystem[capa]
```

$\{\{3i, -3i\}, \{\{\frac{1}{10} + \frac{3i}{10}, 1\}, \{\frac{1}{10} - \frac{3i}{10}, 1\}\}\}$

A fundamental matrix is $\Phi = \begin{pmatrix} \sin 3t & \cos 3t \\ \sin 3t - 3\cos 3t & \cos 3t + 3\sin 3t \end{pmatrix}$ with inverse $\Phi^{-1} = \begin{pmatrix} \frac{1}{3}\cos 3t + \sin 3t & -\frac{1}{3}\cos 3t \\ -\frac{1}{3}\sin 3t + \cos 3t & \frac{1}{3}\sin 3t \end{pmatrix}$.

```
fm = {{Sin[3t], Sin[3t] - 3Cos[3t]},{Cos[3t], Cos[3t] + 3Sin[3t]}};
fminv = Inverse[fm]//Simplify
```

$\{\{\frac{1}{3}\text{Cos[3t]} + \text{Sin[3t]}, \text{Cos[3t]} - \frac{1}{3}\text{Sin[3t]}\}, \{-\frac{1}{3}\text{Cos[3t]}, \frac{1}{3}\text{Sin[3t]}\}\}$

We now compute $\Phi^{-1}\mathbf{F}(t)$

```
ft = {-tCos[3t], -tSin[t] - tCos[3t]};
step1 = fminv.ft
```

$\{(-\text{tCos[3t]} - \text{tSin[t]})\,(\text{Cos[3t]} - \frac{1}{3}\text{Sin[3t]}) -$
$\quad \text{tCos[3t]}\,(\frac{1}{3}\text{Cos[3t]} + \text{Sin[3t]}),$
$\quad \frac{1}{3}\text{tCos[3t]}^2 + \frac{1}{3}(-\text{tCos[3t]} - \text{tSin[t]})\text{Sin[3t]}\}$

and $\int \Phi^{-1}\mathbf{F}(t)\,dt$.

**step2 = Integrate[step1, t]**

$\{-\frac{t^2}{3}+\frac{1}{24}$Cos[2t] $-\frac{1}{4}$tCos[2t] $-\frac{1}{96}$Cos[4t] $+\frac{1}{8}$tCos[4t] $-$
$\frac{1}{54}$Cos[6t] $+\frac{1}{18}$tCos[6t] $+\frac{1}{8}$Sin[2t] $+\frac{1}{12}$tSin[2t] $-$
$\frac{1}{32}$Sin[4t] $-\frac{1}{24}$tSin[4t] $-\frac{1}{108}$Sin[6t] $-\frac{1}{9}$tSin[6t],
$\frac{t^2}{12}-\frac{1}{24}$Cos[2t] $+\frac{1}{96}$Cos[4t] $+\frac{1}{216}$Cos[6t] $+\frac{1}{36}$tCos[6t] $-$
$\frac{1}{12}$tSin[2t] $+\frac{1}{24}$tSin[4t] $-\frac{1}{216}$Sin[6t] $+\frac{1}{36}$tSin[6t]$\}$

A general solution of the nonhomogeneous system is then $\mathbf{\Phi}\left(\int \mathbf{\Phi}^{-1}\mathbf{F}(t)\,dt+\mathbf{C}\right)$.

**Simplify[fm . step2]**

{$\frac{1}{288}$(27Cos[t] –
4((1 + 6t + 18t$^2$)Cos[3t] + 27tSin[t] + (– 1 + 6t + 18t$^2$)Sin[3t])),
$\frac{1}{288}$(– 36tCos[t] – 4(1 – 6t + 18t$^2$)Cos[3t] –
45Sin[t] – 4Sin[3t] – 24tSin[3t] + 72t$^2$Sin[3t])}

It is easiest to use DSolve to solve the initial-value problem directly as we do next.

**check = DSolve[{x′[t]==x[t] – y[t] – tCos[3t], y′[t]==**
**10x[t] – y[t] – tSin[t] – tCos[3t], x[0]==1, y[0]== – 1},**
**{x[t], y[t]}, t]**

{{x[t] → $\frac{1}{288}$
(301Cos[3t] – 72t$^2$Cos[3t] – 12Cos[2t]Cos[3t]
+ 3Cos[3t]Cos[4t] – 4Cos[3t]Cos[6t] – 24tCos[3t]Sin[2t]
+ 192Sin[3t] + 24tCos[2t]Sin[3t] – 12tCos[4t]Sin[3t]
+ 24tCos[6t]Sin[3t] – 12Sin[2t]Sin[3t] + 12tCos[3t]Sin[4t]
+ 3Sin[3t]Sin[4t] – 24tCos[3t]Sin[6t] – 4Sin[3t]Sin[6t]),
y[t] → $\frac{1}{288}$(– 275Cos[3t] – 72t$^2$Cos[3t]
– 12Cos[2t]Cos[3t] – 72tCos[2t]Cos[3t]
+ 3Cos[3t]Cos[4t] + 36tCos[3t]Cos[4t] – 4Cos[3t]Cos[6t]
– 72tCos[3t]Cos[6t] + 36Cos[3t]Sin[2t] – 24tCos[3t]Sin[2t]
+ 1095Sin[3t] – 216t$^2$Sin[3t] – 36Cos[2t]Sin[3t]
+ 24tCos[2t]Sin[3t] + 9Cos[4t]Sin[3t] – 12tCos[4t]Sin[3t]
– 12Cos[6t]Sin[3t] + 24tCos[6t]Sin[3t] – 12Sin[2t]Sin[3t]
– 72tSin[2t]Sin[3t] – 9Cos[3t]Sin[4t] + 12tCos[3t]Sin[4t]
+ 3Sin[3t]Sin[4t] + 36tSin[3t]Sin[4t] + 12Cos[3t]Sin[6t]
– 24tCos[3t]Sin[6t] – 4Sin[3t]Sin[6t] – 72tSin[3t]Sin[6t])}}

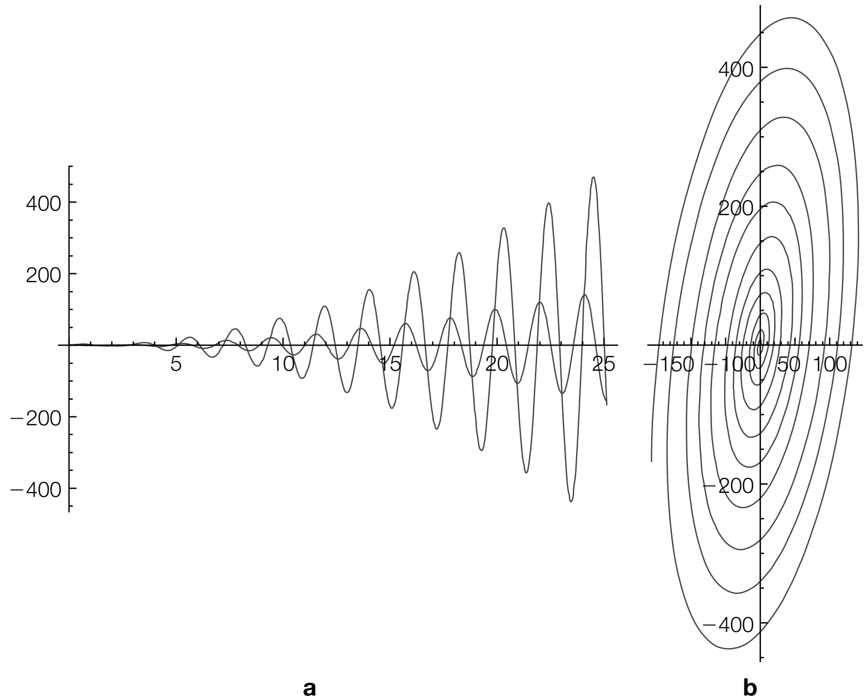



**FIGURE 6.35**

(a) Graph of $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of $x(t)$ versus $y(t)$

The solutions are graphed with Plot and ParametricPlot in Figure 6.35.

```
p1 = Plot[{x[t], y[t]}/.check, {t, 0, 8π}, PlotRange → All];
p2 = ParametricPlot[Evaluate[{x[t], y[t]}/.check], {t, 0, 8π},
    AspectRatio → Automatic];
Show[GraphicsRow[{p1, p2}]]
```

In the case that $\mathbf{A}$ is constant, $\mathbf{X}' = \mathbf{AX}$ is called an *autonomous system* and the only *equilibrium* (*rest point*) solution is the zero solution: $\mathbf{X} = \mathbf{0}$. The stability of the solution is determined by the eigenvalues of $\mathbf{A}$. If all the eigenvalues of $\mathbf{A}$ have negative real part, then $\mathbf{X} = \mathbf{0}$ is **globally asymptotically stable** because $\lim_{t\to 0} \mathbf{X}(t) = \mathbf{0}$ for *all* solutions. If not all the eigenvalues of $\mathbf{A}$ have negative real part, then $\mathbf{X} = \mathbf{0}$ is unstable.

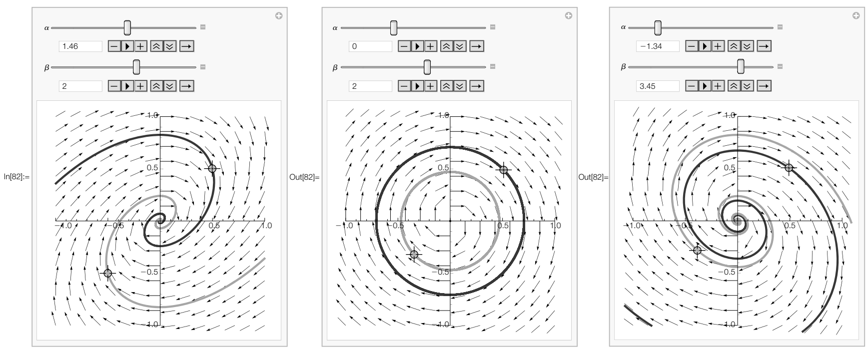
For the $2 \times 2$ system, $\mathbf{X}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{X}$ or, equivalently, $x' = ax + by$, $y' = cx + dy$, the stability of $(0, 0)$ is easily seen by examining the direction field for the system. If all vectors lead to the origin, it is stable; if they do not, it is not.

---

**Example 6.4.4** The eigenvalues of $\begin{pmatrix} -\alpha & \beta \\ -\beta & 0 \end{pmatrix}$ are $\lambda_{1,2} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b^2}\right)$. (See the exercises.) Thus, $(0, 0)$ is globally asymptotically stable for the system $\mathbf{X}' = \begin{pmatrix} -\alpha & \beta \\ -\beta & 0 \end{pmatrix}\mathbf{X}$.

With Manipulate, you can investigate the various situations. In the following, we can vary $\alpha$ and $\beta$ and then plot the solution passing through each locator point. Several results are shown in Figure 6.36.

```
Manipulate[
Needs["VectorFieldPlots`"];
sol1 = DSolve[{x'[t]==−αx[t] + βy[t], y'[t]==−βx[t],
    x[0]==init1[[1]], y[0]==init1[[2]]}, {x[t], y[t]}, t];
sol2 = DSolve[{x'[t]==−αx[t] + βy[t], y'[t]==−βx[t], x[0]==init2[[1]],
    y[0]==init2[[2]]}, {x[t], y[t]}, t];
psol1 = ParametricPlot[{x[t], y[t]}/.sol1, {t, −20, 20},
    PlotStyle → {{GrayLevel[.3], Thickness[.01]}}, PlotPoints → 200];
psol2 = ParametricPlot[{x[t], y[t]}/.sol2, {t, −20, 20},
    PlotStyle → {{GrayLevel[.6], Thickness[.01]}}, PlotPoints → 200];
p1 = Show[{VectorFieldPlot[{−αx + βy, −βx}, {x, −1, 1}, {y, −1, 1},
    ScaleFunction → (1&)], psol1, psol2},
Axes → Automatic, AxesOrigin → {0, 0}, PlotRange → {{−1, 1}, {−1, 1}},
    AspectRatio → Automatic],
{{α, 1}, −2.5, 5}, {{β, 2}, −2 . 5, 5}, {{init1, {.5, .5}}, Locator},
    {{init2, {−.5, −.5}}, Locator}]
```




a b c

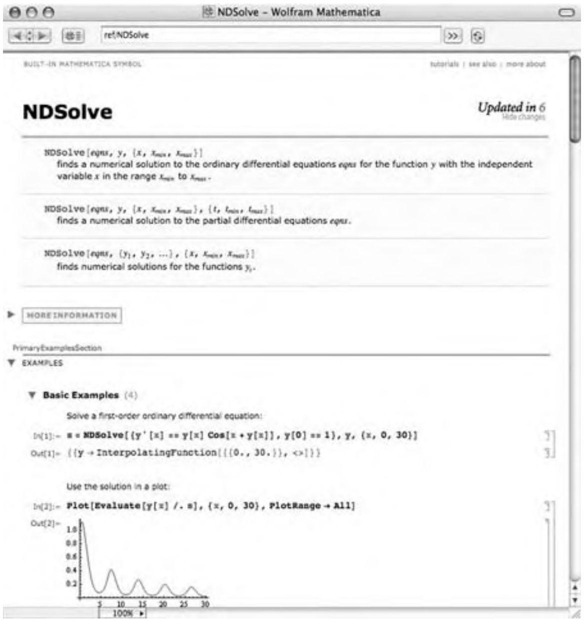
**FIGURE 6.36**

As we vary $\alpha$ and $\beta$ and change the initial conditions, the system behaves differently. (a) A stable spiral. (b) A center. (c) An unstable spiral

## 6.4.3 **Nonlinear Systems**

Nonlinear systems of differential equations arise in numerous situations. Rigorous analysis of the behavior of solutions to nonlinear systems is usually very difficult, if not impossible.

To generate numerical solutions of equations, use NDSolve.




**Example 6.4.5** **(Van der Pol's equation). Van der Pol's equation,** $x'' + \mu\left(x^2 - 1\right)x' + x = 0$ can be written as the system

Also see Example 6.4.8.

$$
\begin{aligned}
x' &= y \\
y' &= -x - \mu\left(x^2 - 1\right)y.
\end{aligned}
\tag{6.32}
$$

If $\mu = 2/3$, $x(0) = 1$, and $y(0) = 0$, (a) find $x(1)$ and $y(1)$. (b) Graph the solution that satisfies these initial conditions.

***Solution*** We use NDSolve together to solve equation (6.32) with $\mu = 2/3$ subject to $x(0) = 1$ and $y(0) = 0$. We name the resulting numerical solution numsol.

```
numsol = NDSolve[{x'[t]==y[t], y'[t]==-x[t]-2/3(x[t]^2-1)y[t], x[0]==1,
    y[0]==0}, {x[t], y[t]}, {t, 0, 30}]
{{x[t] → InterpolatingFunction[{{0., 30.}}, <>][t],
```

We evaluate **numsol** if t = 1 to see that x(1) ≈ 0.5128 and y(1) ≈ −0.9692.

y[t] → InterpolatingFunction[{{0., 30.}}, <>][t]}}

**{x[t], y[t]}/.numsol/.t->1**

{{0.512848, −0.969204}}

**Plot**, **ParametricPlot**, and **ParametricPlot3D** are used to graph $x(t)$ and $y(t)$ together in Figure 6.37(a); a three-dimensional plot, $(t, x(t), y(t))$, is shown in Figure 6.37(b); a parametric plot is shown in Figure 6.37(c); and the limit

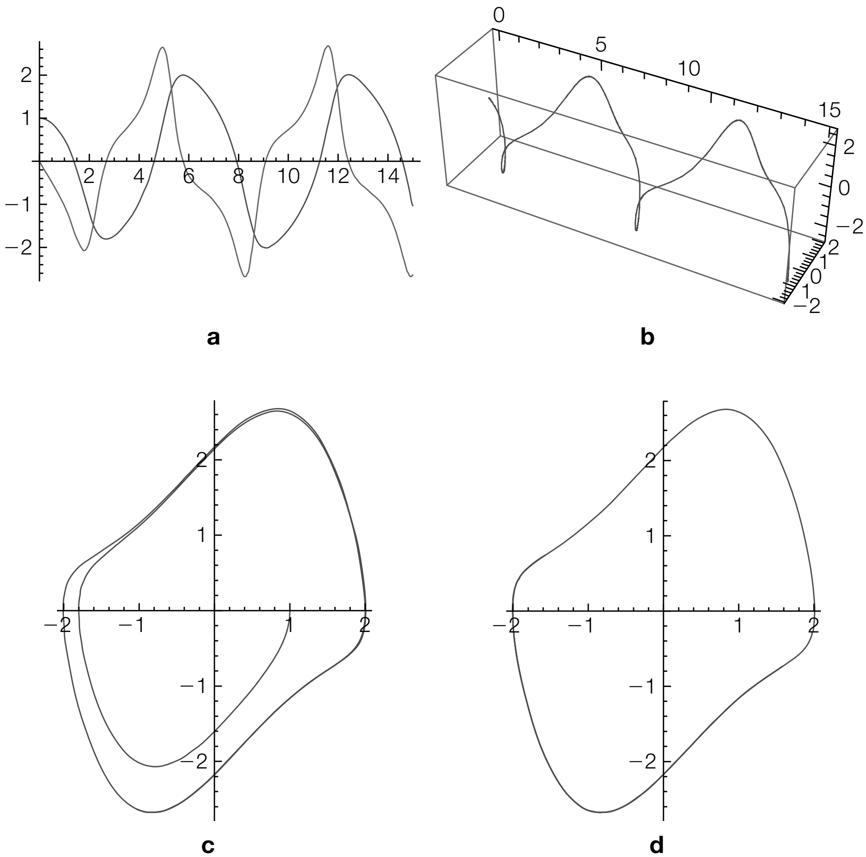



**FIGURE 6.37**

(a) $x(t)$ and $y(t)$. (b) A three-dimensional plot. (c) $x(t)$ versus $y(t)$. (d) $x(t)$ versus $y(t)$ for $20 \le t \le 30$

cycle is shown more clearly in Figure 6.37(d) by graphing the solution for $20 \le t \le 30$.

```
p1 = Plot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 15}];
p2 = ParametricPlot3D[Evaluate[{t, x[t], y[t]}/.numsol], {t, 0, 15}];
p3 = ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 15},
    AspectRatio → Automatic];
p4 = ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 20, 30},
    AspectRatio → Automatic];
Show[GraphicsGrid[{{p1, p2}, {p3, p4}}]]
```

To consider other $\mu$ values, decide on a $\mu$ range, combine the previous commands, replace $2/3$ with $\mu$, and use Manipulate. See Figure 6.38.

To avoid conflicts with the variables in the Manipulate, consider quitting Mathematica, restarting, and then entering the Manipulate command in a new notebook.

```
Manipulate[
numsol = NDSolve[{x′[t]==y[t], y′[t]== − x[t] − μ(x[t]^2 − 1)y[t],
        x[0]==1, y[0]==0}, {x[t], y[t]}, {t, 0, 30}];
p1 = Plot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 15},
    PlotRange → All, AspectRatio → 1];
p2 = ParametricPlot3D[Evaluate[{t, x[t], y[t]}/.numsol], {t, 0, 15},
    BoxRatios → {4, 1, 1}];
p3 = ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 15},
    AspectRatio → 1, PlotRange → All];
p4 = ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 20, 30},
    AspectRatio → 1, PlotRange → All];
Show[GraphicsGrid[{{p1, p2}, {p3, p4}}]], {{μ, 1}, 0, 3}]
```

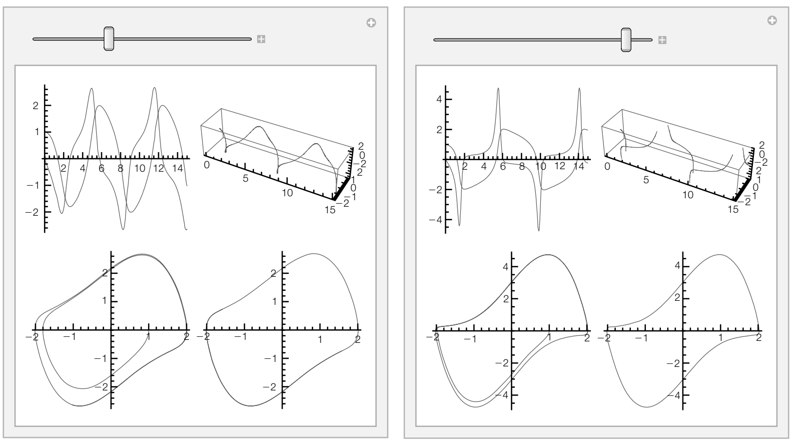


**FIGURE 6.38**

Plots of solutions of Van der Pol's equation for various values of $\mu$

### *Linearization*

An **autonomous system** does not explicitly depend on the independent variable, $t$. That is, if you write the system omitting all arguments, the independent variable (typically $t$) does not appear.

Consider the autonomous system of the form

$$\begin{aligned} x_1{}' &= f_1\left(x_1, x_2, \ldots, x_n\right) \\ x_2{}' &= f_2\left(x_1, x_2, \ldots, x_n\right) \\ &\vdots \\ x_n{}' &= f_n\left(x_1, x_2, \ldots, x_n\right). \end{aligned} \tag{6.33}$$

An **equilibrium** (or **rest**) **point**, $E = \left(x_1{}^*, x_2{}^*, \ldots, x_n{}^*\right)$, of equation (6.33) is a solution of the system

$$\begin{aligned} f_1\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ f_2\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ &\vdots \\ f_n\left(x_1, x_2, \ldots, x_n\right) &= 0. \end{aligned} \tag{6.34}$$

The **Jacobian** of equation (6.33) is

$$\mathbf{J}\left(x_1, x_2, \ldots, x_n\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

The rest point, $E$, is **locally stable** if and only if all the eigenvalues of $\mathbf{J}(E)$ have negative real part. If $E$ is not locally stable, $E$ is **unstable**.

**Example 6.4.6** **(Duffing's Equation).** Consider the forced **pendulum equation** with damping,

Establishing global stability of an equilibrium point for a nonlinear system is *significantly* more difficult than establishing global stability of an equilibrium point ($E = (0, 0)$) for a linear autonomous system.

$$x'' + kx' + \omega \sin x = F(t). \tag{6.35}$$

Recall the Maclaurin series for $\sin x$: $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$. Using $\sin x \approx x$, equation (6.35) reduces to the linear equation $x'' + kx' + \omega x = F(t)$.

On the other hand, using the approximation $\sin x \approx x - \frac{1}{6}x^3$, we obtain $x'' + kx' + \omega\left(x - \frac{1}{6}x^3\right) = F(t)$. Adjusting the coefficients of $x$ and $x^3$ and assuming that $F(t) = F\cos\omega t$ gives us **Duffing's equation**:

$$x'' + kx' + cx + \epsilon x^3 = F\cos\omega t, \tag{6.36}$$

where $k$ and $c$ are positive constants.

Let $y = x'$. Then, $y' = x'' = F\cos\omega t - kx' - cx - \epsilon x^3 = F\cos\omega t - ky - cx - \epsilon x^3$ and we can write equation (6.36) as the system

$$\begin{aligned} x' &= y \\ y' &= F\cos\omega t - ky - cx - \epsilon x^3. \end{aligned} \tag{6.37}$$

Assuming that $F = 0$ results in the autonomous system

$$\begin{aligned} x' &= y \\ y' &= -cx - \epsilon x^3 - ky. \end{aligned} \tag{6.38}$$

The rest points of system equation (6.38) are found by solving

$$\begin{aligned} x' &= 0 \\ y' &= -cx - \epsilon x^3 - ky, = 0, \end{aligned}$$

resulting in $E_0 = (0, 0)$.

```
Solve[{y==0, -cx-ϵx^3-ky==0}, {x, y}]
```

$\left\{\{y \to 0, x \to 0\}, \left\{y \to 0, x \to -\frac{i\sqrt{c}}{\sqrt{\epsilon}}\right\}, \left\{y \to 0, x \to \frac{i\sqrt{c}}{\sqrt{\epsilon}}\right\}\right\}$

We find the Jacobian of equation (6.38) in **s1**, evaluate the Jacobian at $E_0$,

```
s1 = {{0, 1}, {-c-3ϵx^2, -k}};
s2 = s1/.x->0
```

$\{\{0, 1\}, \{-c, -k\}\}$

and then compute the eigenvalues with **Eigenvalues**.

```
s3 = Eigenvalues[s2]
```

$\left\{\frac{1}{2}\left(-k-\sqrt{-4c+k^2}\right), \frac{1}{2}\left(-k+\sqrt{-4c+k^2}\right)\right\}$

Because $k$ and $c$ are positive, $k^2 - 4c < k^2$, so the real part of each eigenvalue is always negative if $k^2 - 4c \neq 0$. Thus, $E_0$ is locally stable.

For the autonomous system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y), \end{aligned} \tag{6.39}$$

**Bendixson's theorem** states that if $f_x(x, y) + g_y(x, y)$ is a continuous function that is either always positive or always negative in a particular region $R$ of the plane, then system (6.39) has no limit cycles in $R$. For equation (6.38), we have

$$\frac{d}{dx}(y) + \frac{d}{dy}\left(-cx - \epsilon x^3 - ky\right) = -k,$$

which is always negative. Hence, equation (6.38) has no limit cycles and it follows that $E_0$ is globally, asymptotically stable.

```
D[y, x] + D[-cx - ϵx^3 - ky, y]
-k
```

We use VectorFieldPlot and ParametricPlot to illustrate two situations that occur. In Figure 6.39(a), we use $c = 1$, $\epsilon = 1/2$, and $k = 3$. In this case, $E_0$ is a *stable node*. On the other hand, in Figure 6.39(b), we use $c = 10$, $\epsilon = 1/2$, and $k = 3$. In this case, $E_0$ is a *stable spiral*.

```
Needs["VectorFieldPlots`"];
pvf1 = VectorFieldPlots`VectorFieldPlot[{y, -x - x^3/2 - 3y}, {x, -2.5, 2.5},
        {y, -2.5, 2.5}];
numgraph[init_, c_, opts___]:=Module[{numsol},
    numsol = NDSolve[{x'[t]==y[t], y'[t]== - cx[t] - 1/2x[t]^3 - 3y[t],
        x[0]==init[[1]], y[0]==init[[2]]}, {x[t], y[t]}, {t, 0, 10}];
    ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 10},
        opts, DisplayFunction->Identity]]
i1 = Table[numgraph[{2.5, i}, 1], {i, -2.5, 2.5, 1/2}];
i2 = Table[numgraph[{-2.5, i}, 1], {i, -2.5, 2.5, 1/2}];
i3 = Table[numgraph[{i, 2.5}, 1], {i, -2.5, 2.5, 1/2}];
i4 = Table[numgraph[{i, -2.5}, 1], {i, -2.5, 2.5, 1/2}];
c1 = Show[i1, i2, i3, i4, pvf1, PlotRange → {{-2.5, 2.5}, {-2.5, 2.5}},
        AspectRatio → Automatic]
pvf2 = VectorFieldPlots`VectorFieldPlot[{y, -10x - x^3/2 - 3y}, {x, -2.5, 2.5},
        {y, -2.5, 2.5}];
i1 = Table[numgraph[{2.5, i}, 10], {i, -2.5, 2.5, 1/2}];
```

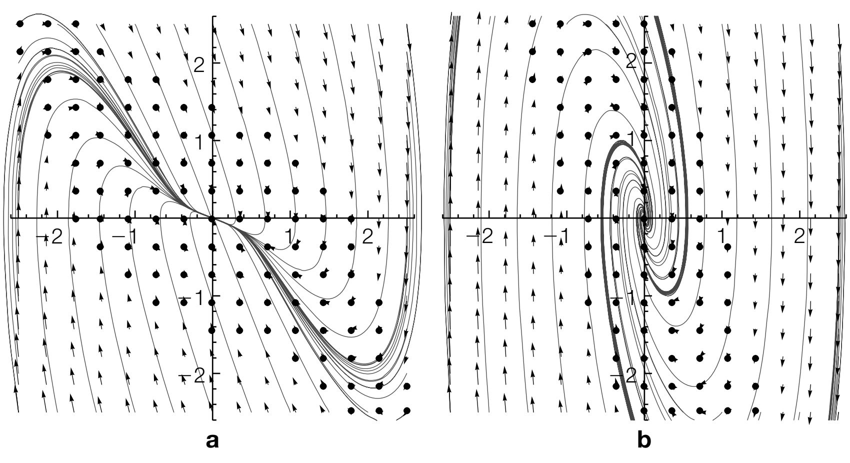


**FIGURE 6.39**

(a) The origin is a stable node. (b) The origin is a stable spiral

```
i2 = Table[numgraph[{−2.5, i}, 10], {i, −2.5, 2.5, 1/2}];
i3 = Table[numgraph[{i, 2.5}, 10], {i, −2.5, 2.5, 1/2}];
i4 = Table[numgraph[{i, −2.5}, 10], {i, −2.5, 2.5, 1/2}];
c2 = Show[i1, i2, i3, i4, pvf2, PlotRange → {{−2.5, 2.5}, {−2.5, 2.5}},
        AspectRatio → Automatic]
Show[GraphicsRow[{c1, c2}]]
```

To experiment with different parameter values, use Manipulate. In the following, we investigate how varying $c$ from $0$ to $10$ affects the solutions of Duffing's equation. See Figure 6.40.

```
Clear[pvf, i1, i2, i3, i4];
Manipulate[
Needs["VectorFieldPlots`"];
numgraph[init_, c_, opts___]:=Module[{numsol},
numsol = NDSolve[{x′[t]==y[t], y′[t]==−cx[t]−1/2x[t]^3−3y[t],
        x[0]==init[[1]], y[0]==init[[2]]}, {x[t], y[t]}, {t, 0, 10}];
ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 10},
        opts, DisplayFunction->Identity]];
pvf = VectorFieldPlots`VectorFieldPlot[{y, −cx − x^3/2 − 3y}, {x, −2.5, 2.5},
        {y, −2.5, 2.5}];
i1 = Table[numgraph[{2.5, i}, c], {i, −2.5, 2.5, 1/2}];
i2 = Table[numgraph[{−2.5, i}, c], {i, −2.5, 2.5, 1/2}];
i3 = Table[numgraph[{i, 2.5}, c], {i, −2.5, 2.5, 1/2}];
i4 = Table[numgraph[{i, −2.5}, c], {i, −2.5, 2.5, 1/2}];
```

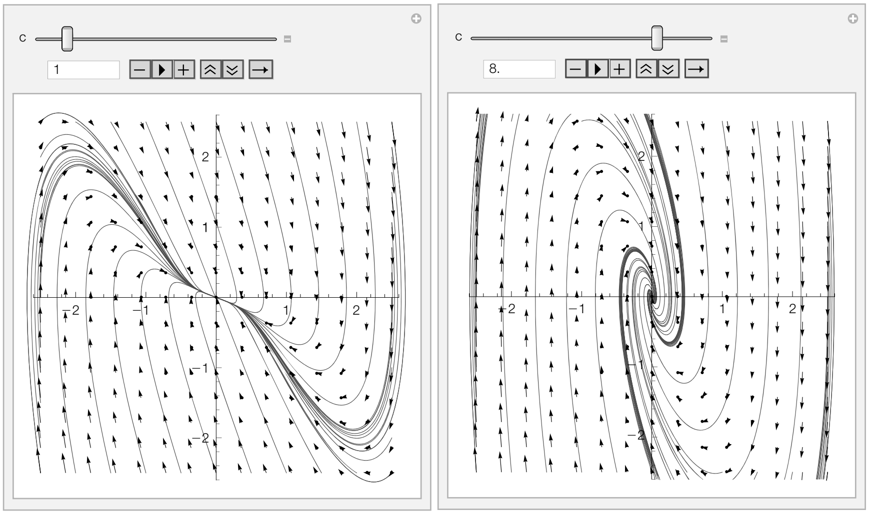



**FIGURE 6.40**

Allowing $c$ to vary in Duffing's equation

```
Show[i1, i2, i3, i4, pvf, PlotRange → {{–2.5, 2.5}, {–2.5, 2.5}},
    AspectRatio → Automatic], {{c, 1}, 0, 10}]
```

**Example 6.4.7** **(Predator–Prey).** One form of the **predator–prey** is

There are *many* other predator–prey models.

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = dxy - cy,$$

where $a$, $b$, $c$, and $d$ are positive constants. $x$ represents the size of the prey population at time $t$, whereas $y$ represents the size of the predator population at time $t$. We use Solve to calculate the rest points. In this case, there is one boundary rest point, $E_0 = (0, 0)$, and one interior rest point, $E_1 = (c/d, a/b)$.

```
rps = Solve[{ax – bxy==0, dxy – cy==0}, {x, y}]
```

$\{\{x \to 0, y \to 0\}, \{x \to \frac{c}{d}, y \to \frac{a}{b}\}\}$

The Jacobian is then found using D.

```
jac = {{D[ax – bxy, x], D[ax – bxy, y]}, {D[dxy – cy, x], D[dxy – cy, y]}};
MatrixForm[jac]
```

$$\begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$$

$E_0$ is unstable because one eigenvalue of $\mathbf{J}(E_0)$ is positive. For the linearized system, $E_1$ is a center because the eigenvalues of $\mathbf{J}(E_1)$ are complex conjugates.

```
Eigenvalues[jac/.rps[[2]]]
```

$\{-i\sqrt{a}\sqrt{c}, i\sqrt{a}\sqrt{c}\}$

In fact, $E_1$ is a center for the nonlinear system as illustrated in Figure 6.41, where we have used $a = 1$, $b = 2$, $c = 2$, and $d = 1$. Notice that there are multiple limit cycles around $E_1 = (1/2, 1/2)$.

```
Needs["VectorFieldPlots`"];
pvf = VectorFieldPlot[{x – 2xy, 2xy – y}, {x, 0, 2},
   {y, 0, 2}, PlotPoints → 30];
numgraph[init_, opts___]:=Module[{numsol},
numsol = NDSolve[{x'[t]==x[t] – 2x[t]y[t], y'[t]==2x[t]y[t] – y[t],
    x[0]==init[[1]], y[0]==init[[2]]}, {x[t], y[t]}, {t, 0, 50}];
ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 10},
   PlotStyle → {{Thickness[.01]}},
    opts]]
i1 = Table[numgraph[{i, i}], {i, 3/20, 1/2, 1/20}];
Show[i1, pvf, DisplayFunction->$DisplayFunction, PlotRange->{{0, 2}, {0, 2}},
    AspectRatio->Automatic]
```

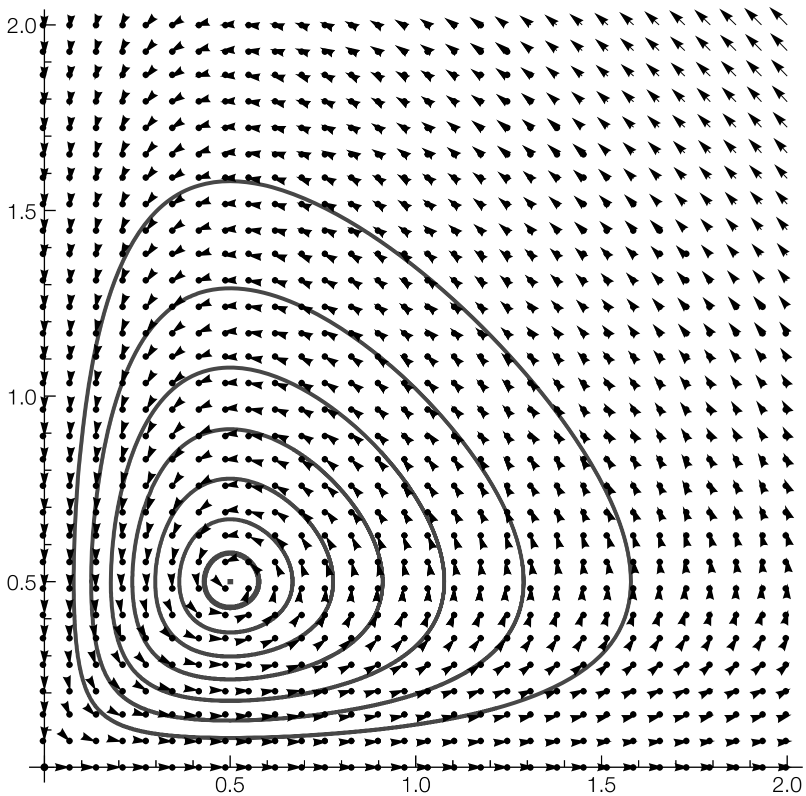



**FIGURE 6.41**

Multiple limit cycles about the interior rest point

As illustrated previously, if you want to play around with the system, use Manipulate. In this case, we allow $a$, $b$, $c$, and $d$ to vary. The solution plotted is the one that passes through the locator point. See Figure 6.42.

```
Manipulate[
Needs["VectorFieldPlots`"];
pvf = VectorFieldPlot[{ax – bxy, dxy – cy}, {x, 0, 5},
        {y, 0, 5}, PlotPoints → 20];
numsol = NDSolve[{x′[t]==ax[t] – bx[t]y[t], y′[t]==dx[t]y[t] – cy[t],
        x[0]==init[[1]], y[0]==init[[2]]}, {x[t], y[t]}, {t, 0, 25}];
p1 = ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 25},
        PlotStyle → {{Thickness[.01]}}];
Show[p1, pvf, PlotRange->{{0, 5}, {0, 5}}, AspectRatio → 1,
    AxesOrigin → {0, 0}],
        {{a, 1}, 0, 5}, {{b, 2}, 0, 5}, {{c, 1}, 0, 5}, {{d, 2}, 0, 5},
    {{init, {1, 1}}, Locator}]
```

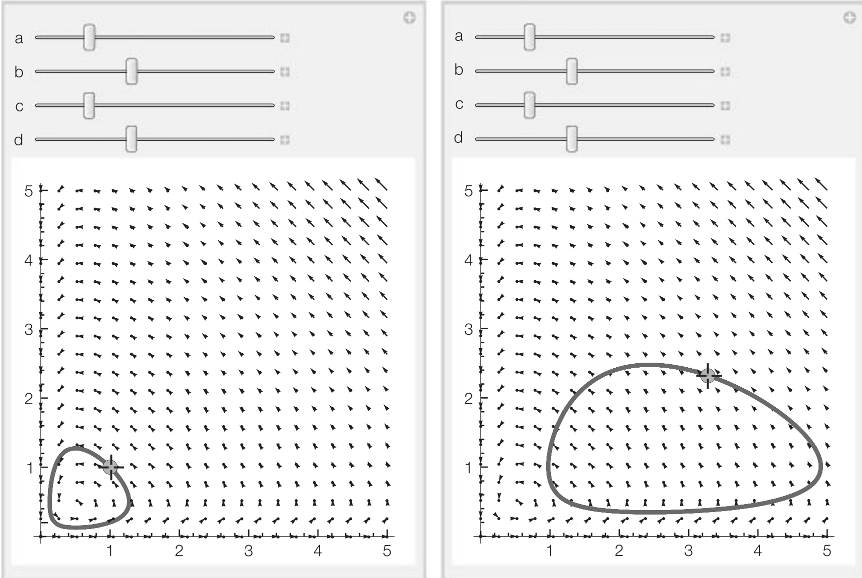



**FIGURE 6.42**

Multiple limit cycles about the interior rest point

In this model, a stable interior rest state is not possible.

The complexity of the behavior of solutions to the system increases based on the assumptions made. Typical assumptions include adding satiation terms for the predator ($y$) and/or limiting the growth of the prey ($x$). The **standard predator–prey equations of Kolmogorov type**,

$$x' = \alpha x\left(1 - \frac{1}{K}x\right) - \frac{mxy}{a+x}$$
$$y' = y\left(\frac{mx}{a+x} - s\right), \tag{6.40}$$

incorporate both of these assumptions.

We use **Solve** to find the three rest points of system (6.40). Let $E_0 = (0,0)$ and $E_1 = (k,0)$ denote the two boundary rest points, and let $E_2$ represent the interior rest point.

**rps = Solve[{αx(1 – 1/kx) – mxy/(a + x)==0, y(mx/(a + x) – s)==0}, {x, y}]**

$$\left\{\{x \to 0, y \to 0\}, \{y \to 0, x \to k\}, \left\{y \to \frac{(akm - a^2s - aks)\alpha}{k(m-s)^2}, x \to -\frac{as}{-m+s}\right\}\right\}$$

The Jacobian, $\mathbf{J}$, is calculated next in s1.

**s1 = {{D[αx(1 – 1/kx) – mxy/(a + x), x], D[αx(1 – 1/kx) – mxy/(a + x), y]},**
**{D[y(mx/(a + x) – s), x], D[y(mx/(a + x) – s), y]}};**
**MatrixForm[s1]**

$$\begin{pmatrix} \frac{mxy}{(a+x)^2} - \frac{my}{a+x} - \frac{x\alpha}{k} + \left(1 - \frac{x}{k}\right)\alpha & -\frac{mx}{a+x} \\ \left(-\frac{mx}{(a+x)^2} + \frac{m}{a+x}\right)y & -s + \frac{mx}{a+x} \end{pmatrix}$$

Because $\mathbf{J}(E_0)$ has one positive eigenvalue, $E_0$ is unstable.

```
e0 = s1/.rps[[1]];
MatrixForm[e0]
eigs0 = Eigenvalues[e0]
```

$$\begin{pmatrix} \alpha & 0 \\ 0 & -s \end{pmatrix}$$

$\{-s, \alpha\}$

The stability of $E_1$ is determined by the sign of $m - s - am/(a+k)$.

```
e1 = s1/.rps[[2]];
MatrixForm[e1]
eigs1 = Eigenvalues[e1]
```

$$\begin{pmatrix} -\alpha & -\frac{km}{a+k} \\ 0 & \frac{km}{a+k} - s \end{pmatrix}$$

$\left\{\frac{km}{a+k} - s, -\alpha\right\}$

The eigenvalues of $\mathbf{J}(E_2)$ are quite complicated.

```
e2 = s1/.rps[[3]];
MatrixForm[e2]
eigs2 = Eigenvalues[e2]
```

$$\begin{pmatrix} \frac{as\alpha}{k(-m+s)} - \frac{ams(akm - a^2s - aks)\alpha}{k(m-s)^2(-m+s)\left(a - \frac{as}{-m+s}\right)^2} - \frac{m(akm - a^2s - aks)\alpha}{k(m-s)^2\left(a - \frac{as}{-m+s}\right)} + \left(1 + \frac{as}{k(-m+s)}\right)\alpha & \frac{ams}{(-m+s)\left(a - \frac{as}{-m+s}\right)} \\ \frac{(akm - a^2s - aks)\left(\frac{ams}{(-m+s)\left(a - \frac{as}{-m+s}\right)^2} + \frac{m}{a - \frac{as}{-m+s}}\right)\alpha}{k(m-s)^2} & -s - \frac{ams}{(-m+s)\left(a - \frac{as}{-m+s}\right)} \end{pmatrix}$$

$$\left\{ \frac{1}{2(-km^2+kms)} \left(ams\alpha - kms\alpha + as^2\alpha + ks^2\alpha - \sqrt{\left(\left(-ams\alpha + kms\alpha - as^2\alpha - ks^2\alpha\right)^2 - 4\left(-km^2 + kms\right)\left(-km^2s\alpha + ams^2\alpha + 2kms^2\alpha - as^3\alpha - ks^3\alpha\right)\right)}\right), \frac{1}{2(-km^2+kms)} \left(ams\alpha - kms\alpha + as^2\alpha + ks^2\alpha + \sqrt{\left(\left(-ams\alpha + kms\alpha - as^2\alpha - ks^2\alpha\right)^2 - 4\left(-km^2 + kms\right)\left(-km^2s\alpha + ams^2\alpha + 2kms^2\alpha - as^3\alpha - ks^3\alpha\right)\right)}\right) \right\}$$

Instead of using the eigenvalues, we compute the characteristic polynomial of $\mathbf{J}(E_2)$, $p(\lambda) = c_2\lambda^2 + c_1\lambda + c_0$, and examine the coefficients. Notice that $c_2$ is always positive.

```
cpe2 = CharacteristicPolynomial[e2,λ]//Simplify
```

$$\frac{as\alpha(m(-s+\lambda) + s(s+\lambda)) + k(m-s)\left(-s\alpha(s+\lambda) + m\left(s\alpha + \lambda^2\right)\right)}{km(m-s)}$$

**c0 = cpe2/.λ->0//Simplify**

$\frac{s(k(m-s)-as)\alpha}{km}$

**c1 = Coefficient[cpe2,λ]//Simplify**

$\frac{s(k(-m+s)+a(m+s))\alpha}{km(m-s)}$

**c2 = Coefficient[cpe2,λ^2]//Simplify**

1

On the other hand, $c_0$ and $m-s-am/(a+k)$ have the same sign because

**c0/eigs1[[1]]//Simplify**

$\frac{(a+k)s\alpha}{km}$

is always positive. In particular, if $m-s-am/(a+k)<0$, $E_1$ is stable. Because $c_0$ is negative, by Descartes' rule of signs, it follows that $p(\lambda)$ will have one positive root and hence $E_2$ will be unstable.

On the other hand, if $m-s-am/(a+k)>0$ so that $E_1$ is unstable, $E_2$ may be either stable or unstable. To illustrate these two possibilities, let $\alpha = K = m = 1$ and $a = 1/10$. We recalculate.

**α = 1; k = 1; m = 1; a = 1/10;**
**rps = Solve[{αx(1 – 1/kx) – mxy/(a + x)==0, y(mx/(a + x) – s)==0}, {x, y}]**

$\left\{\{x\to 0, y\to 0\}, \{y\to 0, x\to 1\}, \left\{y\to \frac{10-11s}{100(-1+s)^2}, x\to -\frac{s}{10(-1+s)}\right\}\right\}$

**s1 = {{D[αx(1 – 1/kx) – mxy/(a + x), x], D[αx(1 – 1/kx) – mxy/(a + x), y]},**
**{D[y(mx/(a + x) – s), x], D[y(mx/(a + x) – s), y]}};**
**MatrixForm[s1]**

$$\begin{pmatrix} 1-2x+\frac{xy}{\left(\frac{1}{10}+x\right)^2}-\frac{y}{\frac{1}{10}+x} & -\frac{x}{\frac{1}{10}+x} \\ \left(-\frac{x}{\left(\frac{1}{10}+x\right)^2}+\frac{1}{\frac{1}{10}+x}\right)y & -s+\frac{x}{\frac{1}{10}+x} \end{pmatrix}$$

**e2 = s1/.rps[[3]];**
**cpe2 = CharacteristicPolynomial[e2,λ]//Simplify**

$\frac{-11s^3+s^2(21-11\lambda)-10\lambda^2+s\left(-10+9\lambda+10\lambda^2\right)}{10(-1+s)}$

**c0 = cpe2/.λ->0//Simplify**

$s-\frac{11s^2}{10}$

**c1 = Coefficient[cpe2,λ]//Simplify**

$\frac{(9-11s)s}{10(-1+s)}$

**c2 = Coefficient[cpe2,λ^2]//Simplify**

1

Using **Reduce**, we see that

1. $c_0$, $c_1$, and $c_2$ are positive if $9/11 < s < 10/11$, and
2. $c_0$ and $c_2$ are positive and $c_1$ is negative if $0 < s < 9/11$.

**Reduce[c0 > 0&&c1 > 0, s]**

$\frac{9}{11} < s < \frac{10}{11}$

**Reduce[c0 > 0&&c1 < 0, s]**

$0 < s < \frac{9}{11}$

In the first situation, $E_2$ is stable; in the second, $E_2$ is unstable.
Using $s = 19/22$, we graph the direction field associated with the system as well as various solutions in Figure 6.43(a). In the plot, notice that all nontrivial solutions approach $E_2 \approx (0.63, 0.27)$; $E_2$ is stable—a situation that cannot occur with the standard predator–prey equations.

**rps/.s->19/22//N**

{{x → 0., y → 0.}, {y → 0., x → 1.}, {y → 0.268889, x → 0.633333}}

```
Needs["VectorFieldPlots`"]
Clear[pvf, numgraph, i1, i2]
pvf = VectorFieldPlot[{αx(1 - x/k) - mxy/(a+x), y(mx/(a+x) - 19/22)}, {x, 0, 1},
        {y, 0, 1}, PlotPoints → 30];
numgraph[init_, s_, opts_]:=Module[{numsol},
numsol = NDSolve[{x'[t]==αx[t](1 - 1/kx[t]) - mx[t]y[t]/(a + x[t]),
    y'[t]==y[t](mx[t]/(a + x[t]) - s), x[0]==init[[1]], y[0]==init[[2]]},
        {x[t], y[t]}, {t, 0, 50}];
ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 50},
        PlotStyle → Thickness[.01], opts]]
i1 = Table[numgraph[{1, i}, 19/22], {i, 0, 1, 1/10}];
i2 = Table[numgraph[{i, 1}, 19/22], {i, 0, 1, 1/10}];
Show[i1, i2, pvf, PlotRange->{{0, 1}, {0, 1}}, AspectRatio->Automatic]
```

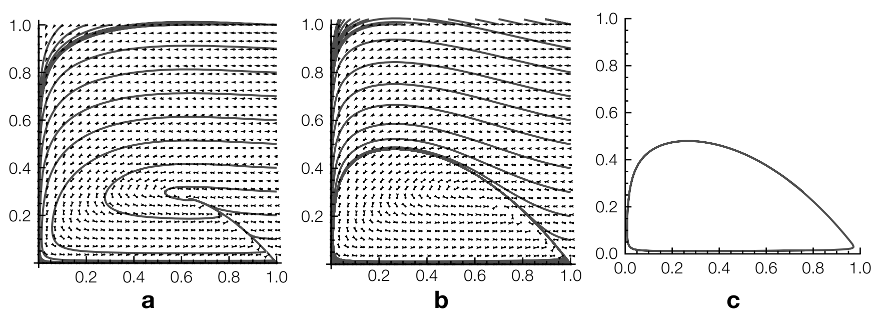



**FIGURE 6.43**

(a) $s = 19/22$. (b) $s = 8/11$. (c) A better view of the limit cycle without the direction field

On the other hand, using $s = 8/11$ (so that $E_2$ is unstable) in Figure 6.43(b), we see that all nontrivial solutions appear to approach a limit cycle.

```
rps/.s->8/11//N
{{x → 0., y → 0.}, {y → 0., x → 1.}, {y → 0.268889, x → 0.266667}}
```

```
i1 = Table[numgraph[{1, i}, 8/11], {i, 0, 1, 1/10}];
i2 = Table[numgraph[{i, 1}, 8/11], {i, 0, 1, 1/10}];
p1 = Show[i1, i2, pvf, PlotRange->{{0, 1}, {0, 1}}, AspectRatio->Automatic]
```

The limit cycle is shown more clearly in Figure 6.43(c).

```
numgraph[{ . 759, .262}, 8/11, PlotRange->{{0, 1}, {0, 1}},
    AspectRatio->Automatic]
```

As we have seen in similar situations, these commands can be collected into a single Manipulate command to investigate the situation. See Figure 6.44.

```
Clear[pvf, numgraph, i1, i2, α, k, m]
Manipulate[
Needs["VectorFieldPlots`"];
pvf = VectorFieldPlot [{αx (1 − x/k) − mxy/(a+x), y (mx/(a+x) − s) }, {x, 0, 1},
{y, 0, 1}, PlotPoints → 20] ;
numgraph[init_, s_]:=Module[{numsol},
```

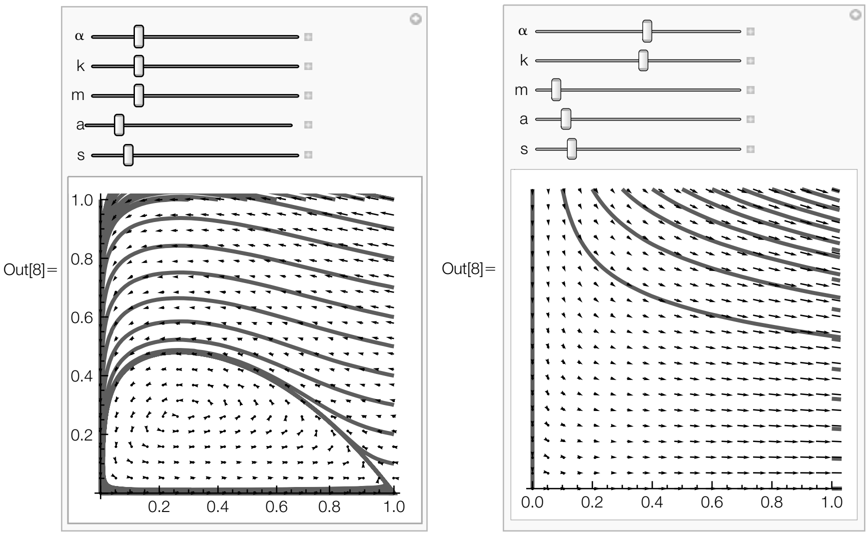



**FIGURE 6.44**

Using Manipulate to investigate the standard predator–prey equations of Komogorov type

```
        numsol = NDSolve[{x'[t]==αx[t](1 – 1/kx[t]) – mx[t]y[t]/(a + x[t]),
            y'[t]==y[t](mx[t]/(a + x[t]) – s), x[0]==init[[1]],
            y[0]==init[[2]]},
            {x[t], y[t]}, {t, 0, 50}];
    ParametricPlot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 50},
        PlotPoints → 200,
        PlotStyle → Thickness[.01]]];
    i1 = Table[numgraph[{1, i}, s], {i, 0, 1, 1/10}];
    i2 = Table[numgraph[{i, 1}, s], {i, 0, 1, 1/10}];
    Show[i1, i2, pvf, PlotRange->{{0, 1}, {0, 1}}, AspectRatio->Automatic],
        {{α, 1}, 0, 5}, {{k, 1}, 0, 5}, {{m, 1}, 0, 5}, {{a, 1/10}, 0, 1},
            {{s, 8/11}, 0, 5}]
```

**Example 6.4.8** **(Van der Pol's equation).** In Example 6.4.5, we saw that **Van der Pol's equation**, $x'' + \mu\left(x^2 - 1\right)x' + x = 0$, is equivalent to the system $\begin{cases} x' = y \\ y' = \mu\left(1 - x^2\right)y - x \end{cases}$. Classify the equilibrium points, use NDSolve to approximate the solutions to this nonlinear system, and plot the phase plane.

Also see Example 6.4.5.

***Solution*** We find the equilibrium points by solving $\begin{cases} y = 0 \\ \mu\left(1 - x^2\right)y - x = 0 \end{cases}$. From the first equation, we see that $y = 0$. Then, substitution of $y = 0$ into the second equation yields $x = 0$. Therefore, the only equilibrium point is $(0, 0)$. The Jacobian matrix for this system is

$$\mathbf{J}(x, y) = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & -\mu\left(x^2 - 1\right) \end{pmatrix}.$$

The eigenvalues of $\mathbf{J}(0, 0)$ are $\lambda_{1,2} = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4}\right)$.

```
Clear[f, g]
f[x_, y_] = y;
g[x_, y_] = −x − μ (x^2 − 1) y;
jac = ( D[f[x, y], x]   D[f[x, y], y]
        D[g[x, y], x]   D[g[x, y], y] );
jac/.{x−>0, y−>0}//Eigenvalues
```

$\left\{\frac{1}{2}\left(\mu - \sqrt{-4 + \mu^2}\right), \frac{1}{2}\left(\mu + \sqrt{-4 + \mu^2}\right)\right\}$

Notice that if $\mu > 2$, then both eigenvalues are positive and real. Hence, we classify $(0, 0)$ as an **unstable node**. On the other hand, if $0 < \mu < 2$, then the eigenvalues are a complex conjugate pair with a positive real part. Hence, $(0, 0)$ is an **unstable spiral**. (We omit the case $\mu = 2$ because the eigenvalues are repeated.)

We now show several curves in the phase plane that begin at various points for various values of $\mu$. First, we define the function sol, which given $\mu$, $x_0$, and $y_0$, generates a numerical solution to the initial-value problem

$$\begin{cases} x' = y \\ y' = \mu\left(1 - x^2\right)y - x \\ x(0) = x_0\,, y(0) = y_0, \end{cases}$$

and then parametrically graphs the result for $0 \le t \le 20$.

```
Clear[sol]
sol[μ_, {x0_, y0_}, opts_]:=
Module[{eqone, eqtwo, solt}, eqone = x′[t]==y[t];
        eqtwo = y′[t]==μ (1 − x[t]^2) y[t] − x[t];
        solt = NDSolve[{eqone, eqtwo, x[0]==x0, y[0]==y0}, {x[t], y[t]},
            {t, 0, 20}];
        ParametricPlot[{x[t], y[t]}/.solt, {t, 0, 20}, opts]]
```

We then use Table and Union to generate a list of ordered pairs initconds that will correspond to the initial conditions in the initial-value problem.

```
initconds1 = Table[{0.1Cos[t], 0.1Sin[t]}, {t, 0, 2π, 2π/9}];
initconds2 = Table[{−5, i}, {i, −5, 5, 10/9}];
initconds3 = Table[{5, i}, {i, −5, 5, 10/9}];
initconds4 = Table[{i, 5}, {i, −5, 5, 10/9}];
initconds5 = Table[{i, −5}, {i, −5, 5, 10/9}];
initconds = initconds1 ∪ initconds2 ∪ initconds3 ∪ initconds4 ∪ initconds5;
```

Last, we use Map to apply sol to the list of ordered pairs in initconds for $\mu = 1/2$.

```
somegraphs1 = Map[sol[1/2, #, DisplayFunction->Identity]&, initconds];
phase1 = Show[somegraphs1, PlotRange → {{−5, 5}, {−5, 5}},
    AspectRatio → 1, Ticks → {{−4, 4}, {−4, 4}}]
```

Similarly, we use Map to apply sol to the list of ordered pairs in initconds for $\mu = 1$, $3/2$, and $3$.

```
somegraphs2 = Map[sol[1, #, DisplayFunction->Identity]&, initconds];
phase2 = Show[somegraphs2, PlotRange → {{−5, 5}, {−5, 5}},
    AspectRatio → 1, Ticks → {{−4, 4}, {−4, 4}}]
somegraphs3 = Map[sol[3/2, #, DisplayFunction->Identity]&, initconds];
phase3 = Show[somegraphs3, PlotRange → {{−5, 5}, {−5, 5}},
    AspectRatio → 1, Ticks → {{−4, 4}, {−4, 4}}]
somegraphs4 = Map[sol[3, #, DisplayFunction->Identity]&, initconds];
phase4 = Show[somegraphs3, PlotRange → {{−5, 5}, {−5, 5}},
    AspectRatio → 1, Ticks → {{−4, 4}, {−4, 4}}]
```

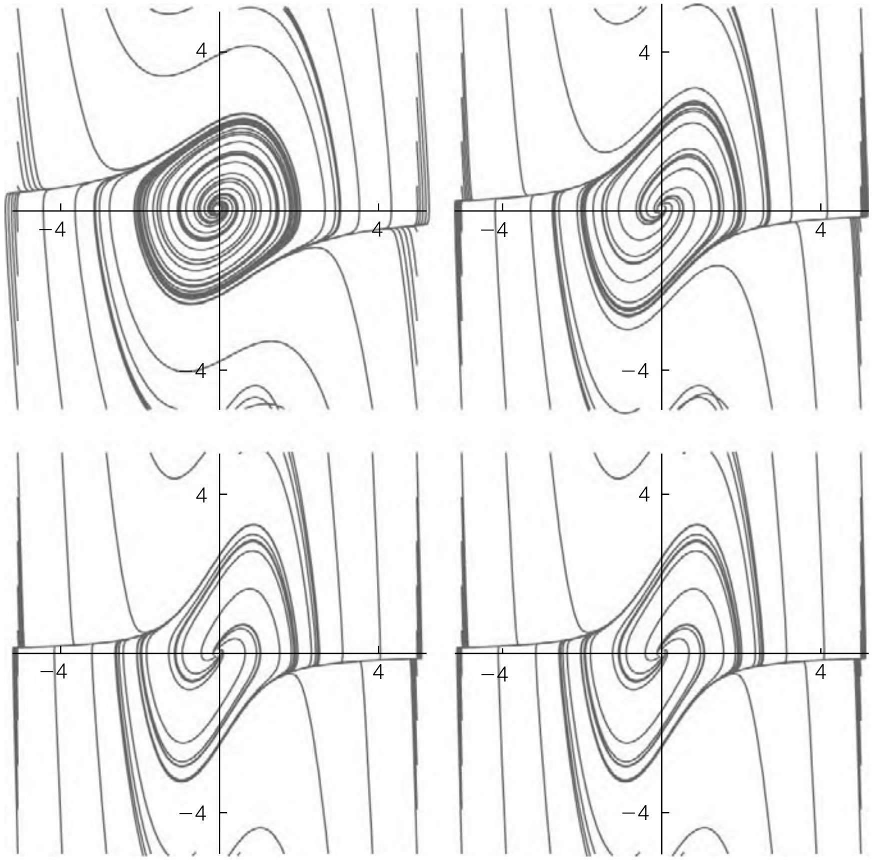



**FIGURE 6.45**

Solutions to the Van der Pol equation for various values of $\mu$

All four graphs are shown together in Figure 6.45. In each figure, we see that all of the curves approach a curve called a *limit cycle*. Physically, the fact that the system has a limit cycle indicates that for all oscillations, the motion eventually becomes periodic, which is represented by a closed curve in the phase plane.

```
Show[GraphicsGrid[{{phase1, phase2}, {phase3, phase4}}]]
```

On the other hand, in Figure 6.46 we graph the solutions that satisfy the initial conditions $x(0) = 1$ and $y(0) = 0$ parametrically and individually for various values of $\mu$. Notice that for small values of $\mu$ the system more closely approximates that of the harmonic oscillator because the damping coefficient is small. The curves are more circular than those for larger values of $\mu$.

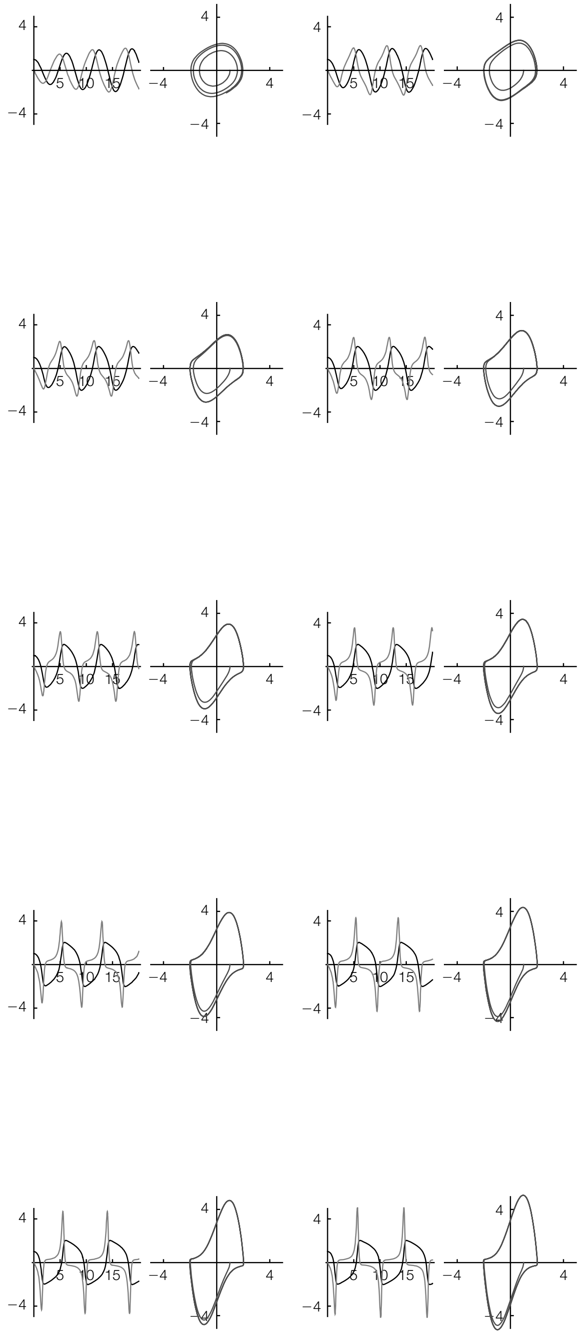



**FIGURE 6.46**

The solutions to the Van der Pol equation satisfying $x(0) = 1$ and $y(0) = 0$ individually ($x$ in black and $y$ in gray) for various values of $\mu$

```
Clear[x, y, t, s]
graph[μ_]:=Module[{numsol, pp, pxy},
        numsol = NDSolve[{x′[t]==y[t], y′[t]==μ(1 - x[t]^2)y[t] - x[t],
            x[0]==1, y[0]==0}, {x[t], y[t]}, {t, 0, 20}];
        pp = ParametricPlot[{x[t], y[t]}/.numsol, {t, 0, 20},
            PlotRange → {{-5, 5}, {-5, 5}}, AspectRatio → 1,
            Ticks → {{-4, 4}, {-4, 4}}, DisplayFunction → Identity];
        pxy = Plot[Evaluate[{x[t], y[t]}/.numsol], {t, 0, 20},
          PlotStyle → {GrayLevel[0], GrayLevel["0.5"]}, PlotRange → {-5, 5},
        AspectRatio → 1, Ticks → {{5, 10, 15}, {-4, 4}},
            DisplayFunction → Identity];
        GraphicsRow[{pxy, pp}]]
graphs = Table[graph[i], {i, 0 . 25, 3, 2 . 75/9}];
toshow = Partition[graphs, 2];
Show[GraphicsGrid[toshow]]
```

An alternative to comparing the graphics together is to use Manipulate to create an animation of how the $\mu$ values affect the solutions of the equation. See Figure 6.47.

```
Manipulate[
sol[μ_, {x0_, y0_}, opts_]:=
        Module[{eqone, eqtwo, solt}, eqone = x′[t]==y[t];
        eqtwo = y′[t]==μ(1 - x[t]^2)y[t] - x[t];
solt = NDSolve[{eqone, eqtwo, x[0]==x0, y[0]==y0}, {x[t], y[t]}, {t, 0, 20}];
```

Out[1]=

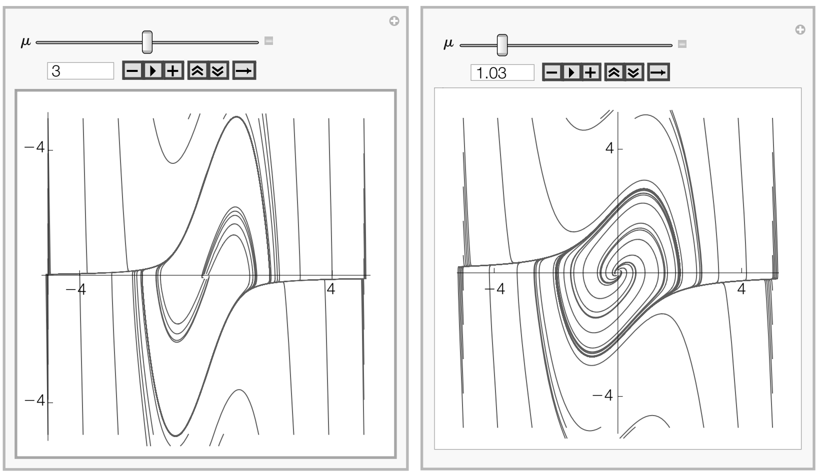



**FIGURE 6.47**

Varying $\mu$ in the Van der Pol equation with Manipulate

```
        ParametricPlot[{x[t], y[t]}/.solt, {t, 0, 20}, PlotPoints → 200]];
initconds1 = Table[{0.1Cos[t], 0.1Sin[t]}, {t, 0, 2π, 2π/9}];
initconds2 = Table[{−5, i}, {i, −5, 5, 10/9}];
initconds3 = Table[{5, i}, {i, −5, 5, 10/9}];
initconds4 = Table[{i, 5}, {i, −5, 5, 10/9}];
initconds5 = Table[{i, −5}, {i, −5, 5, 10/9}];
initconds = initconds1 ∪ initconds2 ∪ initconds3 ∪ initconds4 ∪ initconds5;
somegraphs1 = Map[sol[μ, #, DisplayFunction->Identity]&, initconds];
phase1 = Show[somegraphs1, PlotRange → {{−5, 5}, {−5, 5}},
        AspectRatio → 1, Ticks → {{−4, 4}, {−4, 4}}], {{μ, 3}, 0, 6}]
```

Although linearization can help you determine local behavior near rest points, the long-term behavior of solutions to nonlinear systems can be quite complicated, even for deceptively simple looking systems.

**Example 6.4.9** **(Lorenz Equations).** The **Lorenz equations** are

$$\begin{cases} dx/dt = a(y - x) \\ dy/dt = bx - y - xz \\ dz/dt = xy - cz \end{cases}.$$

Graph the solutions to the Lorenz equations if $a = 7$, $b = 27.2$, and $c = 3$ if the initial conditions are $x(0) = 3$, $y(0) = 4$, and $z(0) = 2$.

See texts such as Jordan and Smith's *Nonlinear Ordinary Differential Equations* [23] for discussions of ways to analyze systems such as the Rössler attractor and the Lorenz equations.

***Solution*** So that you can experiment with different parameters and initial conditions, we use Manipulate to solve the Lorenz system using initial conditions $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$ for $950 \le t \le 1000$; generate parametric plots of $x$ versus $y$, $y$ versus $z$, $x$ versus $z$, and $x$ versus $y$ versus $z$; and display the four resulting plots as a graphics array.

Because the behavior of solutions can be quite intricate, we include the option MaxSteps->Infinity in the NDSolve command to help Mathematica capture the oscillatory behavior in the long-term solution. See Figure 6.48.

On the other hand, if you define lorenzsol separately,

```
Clear[x, y, z, lorenzsol]
lorenzsol[a_, b_, c_][{x0_, y0_, z0_}, ts_:{t, 0, 1000},
        opts_]:=Module[{numsol},
numsol =
        NDSolve[{x′[t]==−ax[t] + ay[t],
            y′[t]==bx[t] − y[t] − x[t]z[t],
            z′[t]==x[t]y[t] − cz[t], x[0]==x0,
            y[0]==y0, z[0]==z0}, {x[t], y[t], z[t]},
                ts, MaxSteps → Infinity]
        ]
```

### The Lorenz Equations

```
Manipulate[
 Clear[x, y, z, lorenzsol];
 lorenzsol[a_, b_, c_][{x0_, y0_, z0_}, ts_: {t, 0, 1000},
   opts___] := Module[{numsol},
   numsol =
    NDSolve[{x'[t] == -a x[t] + a y[t],
      y'[t] == b x[t] - y[t] - x[t] z[t],
      z'[t] == x[t] y[t] - c z[t], x[0] == x0,
      y[0] == y0, z[0] == z0}, {x[t], y[t], z[t]},
     ts, MaxSteps → Infinity]];
n1 = lorenzsol[a, b, c][{x0, y0, z0}];
p1a = ParametricPlot[Evaluate[{x[t], y[t]} /. n1],
   {t, 900, 950}, PlotPoints → 1000, AspectRatio → 1,
   AxesLabel → {"x", "y"}];
 p1b = ParametricPlot[Evaluate[{x[t], z[t]} /. n1],
   {t, 900, 950}, PlotPoints → 1000, AspectRatio → 1,
   AxesLabel → {"x", "z"}];
 p1c = ParametricPlot[Evaluate[{y[t], z[t]} /. n1],
   {t, 900, 950}, PlotPoints → 1000, AspectRatio → 1,
   AxesLabel → {"y", "z"}];
 p1d = ParametricPlot3D[
   Evaluate[{x[t], y[t], z[t]} /. n1], {t, 950, 1000},
   PlotPoints → 3000, BoxRatios → {1, 1, 1},
   AxesLabel → {"x", "y", "z"}];
 Show[GraphicsGrid[{{p1a, p1b}, {p1c, p1d}}]], {{a, 7}, 5, 9},
 {{b, 27.2}, 25, 30}, {{c, 3}, 1, 5}, {{x0, 3}, 0, 5}, {{y0, 4}, 0, 5},
 {{z0, 2}, 0, 5}]
```

Out[1]=

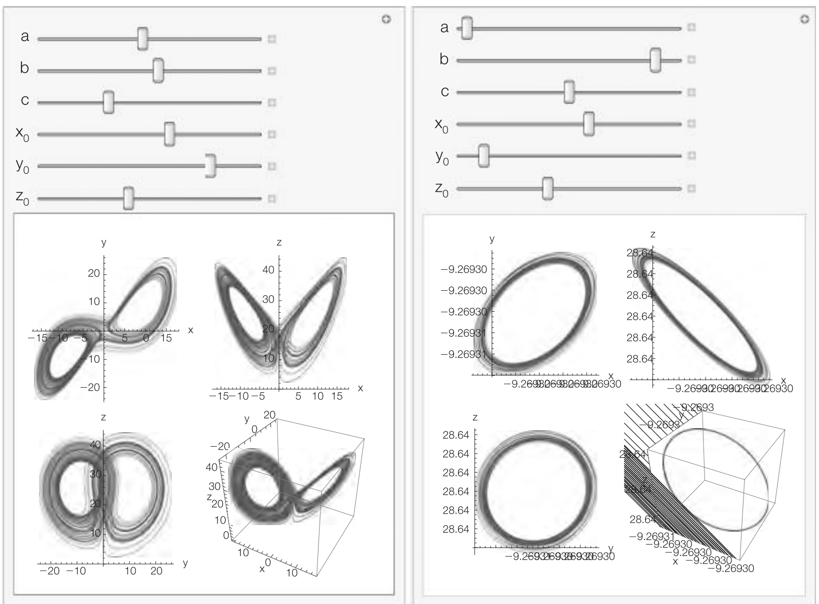



**FIGURE 6.48**

Comparing a chaotic solution to a non-chaotic solution with Manipulate

we can then use lorenzplot to generate a numerical solution for our parameter values and initial conditions.

**n2 = lorenzsol[7, 27.2, 3][{3, 4, 2}];**

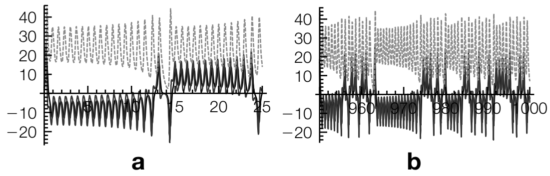



**FIGURE 6.49**

(a) Plots of $x(t)$ (in black), $y(t)$ (in gray), and $z(t)$ (dashed) for $0 \le t \le 25$. (b) Plots of $x(t)$ (in black), $y(t)$ (in gray), and $z(t)$ (dashed) for $950 \le t \le 1000$

We generate a short-term plot of the solution in Figure 6.49(a) and a long-term plot in Figure 6.49(b).

```
pp1 = Plot[Evaluate[{x[t], y[t], z[t]}/.n2], {t, 0, 25},
        PlotStyle → {GrayLevel[0], GrayLevel[.3], Dashing[{0.01}]},
            PlotPoints → 1000];

pp2 = Plot[Evaluate[{x[t], y[t], z[t]}/.n2], {t, 950, 1000},
        PlotStyle → {GrayLevel[0], GrayLevel[.3], Dashing[{0.01}]},
            PlotPoints → 1000];

Show[GraphicsRow[{pp1, pp2}]]
```

## 6.5 SOME PARTIAL DIFFERENTIAL EQUATIONS

### 6.5.1 The One-Dimensional Wave Equation

Suppose that we pluck a string (such as a guitar or violin string) of length $p$ and constant mass density that is fixed at each end. A question that we might ask is: What is the position of the string at a particular instance of time? We answer this question by modeling the physical situation with a partial differential equation, namely the wave equation in one spatial variable:

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{or} \qquad c^2 u_{xx} = u_{tt}. \tag{6.41}$$

In equation (6.41), $c^2 = T/\rho$, where $T$ is the tension of the string and $\rho$ is the constant mass of the string per unit length. The solution $u(x, t)$ represents the displacement of the string from the $x$-axis at time $t$. To determine $u$, we must describe the boundary and initial conditions that model the physical situation. At the ends of the string, the displacement from the $x$-axis is fixed at zero, so we use the homogeneous boundary conditions $u(0, t) = u(p, t) = 0$

for $t > 0$. The motion of the string also depends on the displacement and the velocity at each point of the string at $t = 0$. If the initial displacement is given by $f(x)$ and the initial velocity by $g(x)$, we have the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ for $0 \le x \le p$. Therefore, we determine the displacement of the string with the initial boundary value problem

$$\begin{cases} c^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \; 0 < x < p, \; t > 0 \\ u(0, t) = u(p, t) = 0, \; t > 0 \\ u(x, 0) = f(x), \; u_t(x, 0) = g(x), \; 0 < x < p. \end{cases} \tag{6.42}$$

This problem is solved through separation of variables by assuming that $u(x, t) = X(x)T(t)$. Substitution into equation (6.41) yields

$\lambda$ is a constant.

$$c^2 X''T = XT'' \qquad \text{or} \qquad \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda,$$

so we obtain the two second-order ordinary differential equations $X'' + \lambda X = 0$ and $T'' + c^2 \lambda T = 0$. At this point, we solve the equation that involves the homogeneous boundary conditions. The boundary conditions in terms of $u(x, t) = X(x)T(t)$ are $u(0, t) = X(0)T(t) = 0$ and $u(p, t) = X(p)T(t) = 0$, so we have $X(0) = 0$ and $X(p) = 0$. Therefore, we determine $X(x)$ by solving the *eigenvalue problem*

$$\begin{cases} X'' + \lambda X = 0, \; 0 < x < p \\ X(0) = X(p) = 0. \end{cases}$$

The eigenvalues of this problem are $\lambda_n = \left(n\pi/p\right)^2$, $n = 1,3,\ldots$ with corresponding eigenfunctions $X_n(x) = \sin\left(n\pi x/p\right)^2$, $n = 1,3,\ldots$. Next, we solve the equation $T'' + c^2 \lambda_n T = 0$. A general solution is

$$T_n(t) = a_n \cos\left(c\sqrt{\lambda_n} t\right) + b_n \sin\left(c\sqrt{\lambda_n} t\right) = a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p},$$

where the coefficients $a_n$ and $b_n$ must be determined. Putting this information together, we obtain

$$u_n(x, t) = \left(a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p},$$

so by the principle of superposition, we have

$$u(x, t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p}.$$

Applying the initial displacement $u(x, 0) = f(x)$ yields

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{p} = f(x),$$

so $a_n$ is the *Fourier sine series coefficient* for $f(x)$, which is given by

$$a_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx, \quad n = 1, 2, \ldots.$$

In order to determine $b_n$, we must use the initial velocity. Therefore, we compute

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left(-a_n\frac{cn\pi}{p}\sin\frac{cn\pi t}{p} + b_n\frac{cn\pi}{p}\cos\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p}.$$

Then,

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n\frac{cn\pi}{p}\sin\frac{n\pi x}{p} = g(x)$$

so $b_n\frac{cn\pi}{p}$ represents the Fourier sine series coefficient for $g(x)$, which means that

$$b_n = \frac{p}{cn\pi}\int_0^p g(x)\sin\frac{n\pi x}{p}\,dx, \quad n = 1, 2, \ldots.$$

---

**Example 6.5.1** Solve $\begin{cases} u_{xx} = u_{tt},\ 0 < x < 1,\ t > 0 \\ u(0,t) = u(1,t) = 0,\ t > 0 \\ u(x,0) = \sin \pi x,\ u_t(x,0) = 3x + 1,\ 0 < x < 1. \end{cases}$

***Solution*** The initial displacement and velocity functions are defined first.

```
f[x_] = Sin[πx];
g[x_] = 3x + 1;
```

Next, the functions to determine the coefficients $a_n$ and $b_n$ in the series approximation of the solution $u(x, t)$ are defined. Here, $p = c = 1$.

**$a_1 = 2\int_0^1 f[x]Sin[\pi x]\,dx$**

1

**$a_{n\_} = 2\int_0^1 f[x]Sin[n\pi x]dx$**

$\frac{2Sin[n\pi]}{\pi - n^2\pi}$

**$b_{n\_} = \frac{2\int_0^1 g[x]Sin[n\pi x]dx}{n\pi}$//Simplify**

$\frac{2n\pi - 8n\pi Cos[n\pi] + 6Sin[n\pi]}{n^3\pi^3}$

Because $n$ represents an integer, these results indicate that $a_n = 0$ for all $n \geq 2$, which we confirm with Simplify together with the Assumptions by instructing Mathematica to assume that $n$ is an integer.

**Simplify$\left[\frac{2Sin[n\pi]}{\pi - n^2\pi}\right.$, Assumptions → Element[n, Integers]$\left.\right]$**

0

```
Simplify[(2nπ - 8nπCos[nπ]+6Sin[nπ])/(n^3π^3),
   Assumptions → Element[n, Integers]]
```

$\frac{2-8(-1)^n}{n^2\pi^2}$

We use Table to calculate the first 10 values of $b_n$.

```
Table[{n, b_n, b_n//N}, {n, 1, 10}] //TableForm
```

| | | |
|---|---|---|
| 1 | $\frac{10}{\pi^2}$ | 1.01321 |
| 2 | $-\frac{3}{2\pi^2}$ | −0.151982 |
| 3 | $\frac{10}{9\pi^2}$ | 0.112579 |
| 4 | $-\frac{3}{8\pi^2}$ | −0.0379954 |
| 5 | $\frac{2}{5\pi^2}$ | 0.0405285 |
| 6 | $-\frac{1}{6\pi^2}$ | −0.0168869 |
| 7 | $\frac{10}{49\pi^2}$ | 0.0206778 |
| 8 | $-\frac{3}{32\pi^2}$ | −0.00949886 |
| 9 | $\frac{10}{81\pi^2}$ | 0.0125088 |
| 10 | $-\frac{3}{50\pi^2}$ | −0.00607927 |

Notice that we define uapprox[n] so that Mathematica "remembers" the terms uapprox that are computed. That is, Mathematica does not need to recompute uapprox[n−1] to compute uapprox[n] provided that uapprox[n−1] has already been computed.

The function u defined next computes the $n$th term in the series expansion. Thus, uapprox determines the approximation of order $k$ by summing the first $k$ terms of the expansion, as illustrated with approx[10].

```
Clear[u, uapprox]
u[n_] = b_nSin[nπt]Sin[nπx];
uapprox[k_]:=uapprox[k] = uapprox[k-1] + u[k];
uapprox[0] = Cos[πt]Sin[πx];
uapprox[10]
```

$$\text{Cos}[\pi t]\text{Sin}[\pi x]+\frac{10\text{Sin}[\pi t]\text{Sin}[\pi x]}{\pi^2}-\frac{3\text{Sin}[2\pi t]\text{Sin}[2\pi x]}{2\pi^2}+\frac{10\text{Sin}[3\pi t]\text{Sin}[3\pi x]}{9\pi^2}-\frac{3\text{Sin}[4\pi t]\text{Sin}[4\pi x]}{8\pi^2}+\frac{2\text{Sin}[5\pi t]\text{Sin}[5\pi x]}{5\pi^2}-\frac{\text{Sin}[6\pi t]\text{Sin}[6\pi x]}{6\pi^2}+\frac{10\text{Sin}[7\pi t]\text{Sin}[7\pi x]}{49\pi^2}-\frac{3\text{Sin}[8\pi t]\text{Sin}[8\pi x]}{32\pi^2}+\frac{10\text{Sin}[9\pi t]\text{Sin}[9\pi x]}{81\pi^2}-\frac{3\text{Sin}[10\pi t]\text{Sin}[10\pi x]}{50\pi^2}$$

To illustrate the motion of the string, we graph uapprox[10], the 10th partial sum of the series, on the interval $[0, 1]$ for $16$ equally spaced values of $t$ between $0$ and $2$ in Figure 6.50.

```
somegraphs = Table[Plot[Evaluate[uapprox[10]], {x, 0, 1},
      DisplayFunction → Identity, PlotRange → {-3/2, 3/2},
      Ticks → {{0, 1}, {-1, 1}}], {t, 0, 2, 2/15}] ;
toshow = Partition[somegraphs, 4];
Show[GraphicsGrid[toshow]]
```

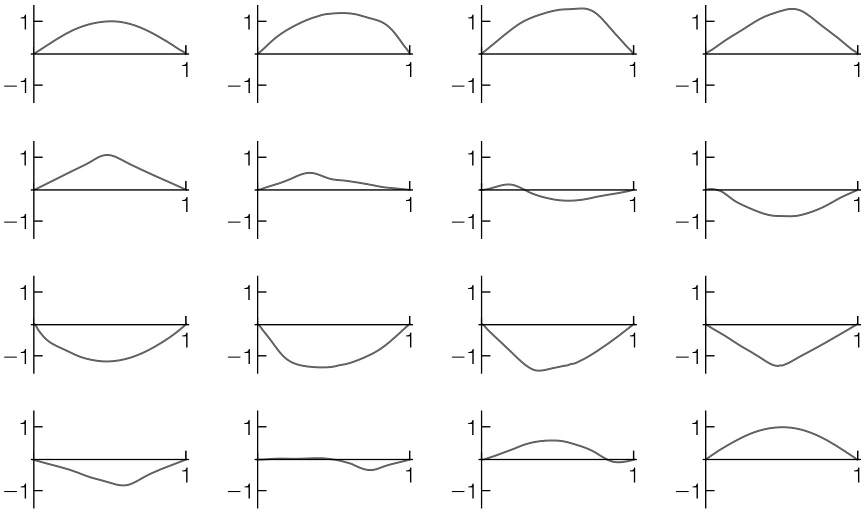
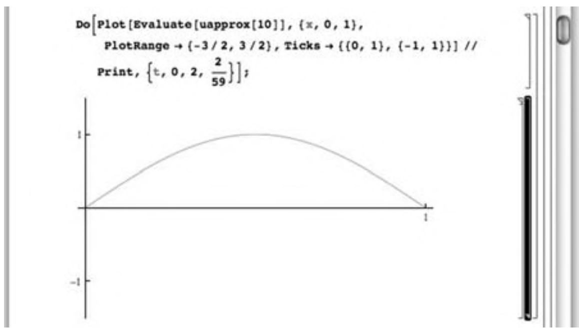


**FIGURE 6.50**

The motion of the string for 16 equally spaced values of $t$ between 0 and 2

If instead we wished to see the motion of the string, we can use a Do loop together with Print to generate many graphs and animate the result. We show a frame from the resulting animation.

```
Do[Plot[Evaluate[uapprox[10]], {x, 0, 1},
      PlotRange → {−3/2, 3/2}, Ticks → {{0, 1}, {−1, 1}}]//
       Print, {t, 0, 2, 2/59}] ;
```




Finally, we remark that DSolve can find **D'Alembert's solution** to the wave equation.

```
Clear[u, c]
DSolve[c^2 D[u[x, t], {x, 2}]==D[u[x, t], {t, 2}],
u[x, t], {x, t}]
```

$$\left\{\left\{u[x,t]\to C[1]\left[t-\frac{\sqrt{c^2}x}{c^2}\right]+C[2]\left[t+\frac{\sqrt{c^2}x}{c^2}\right]\right\}\right\}$$

**DSolve** $\left[c^2\partial_{\{x,2\}}u[x,t]==\partial_{\{t,2\}}u[x,t],u[x,t],\{x,t\}\right]$

$$\left\{\left\{u[x,t]\to C[1]\left[t-\frac{\sqrt{c^2}x}{c^2}\right]+C[2]\left[t+\frac{\sqrt{c^2}x}{c^2}\right]\right\}\right\}$$

---

## 6.5.2 The Two-Dimensional Wave Equation

One of the more interesting problems involving two spatial dimensions ($x$ and $y$) is the wave equation. The two-dimensional wave equation in a circular region that is radially symmetric (not dependent on $\theta$) with boundary and initial conditions is expressed in polar coordinates as

$$\begin{cases} c^2\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)=\dfrac{\partial^2 u}{\partial t^2},\ 0<r<\rho,\ t>0 \\ u(\rho,t)=0,\ |u(0,t)|<\infty,\ t>0 \\ u(r,0)=f(r),\ \dfrac{\partial u}{\partial t}(r,0)=g(r),\ 0<r<\rho. \end{cases}$$

Notice that the boundary condition $u(\rho,t)=0$ indicates that $u$ is fixed at zero around the boundary; the condition $|u(0,t)|<\infty$ indicates that the solution is bounded at the center of the circular region. Like the wave equation discussed previously, this problem is typically solved through separation of variables by assuming a solution of the form $u(r,t)=F(r)G(t)$. Applying separation of variables yields the solution

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_n\cos ck_nt+B_n\sin ck_nt\right)J_0\left(k_nr\right),$$

$\alpha_n$ represents the $n$th zero of the Bessel function of the first kind of order zero.

where $\lambda_n=c\alpha_n/\rho$, and the coefficients $A_n$ and $B_n$ are found through application of the initial displacement and velocity functions. With

$$u(r,0)=\sum_{n=1}^{\infty}A_nJ_0\left(k_nr\right)=f(r)$$

and the orthogonality conditions of the Bessel functions, we find that

$$A_n=\frac{\int_0^\rho rf(r)J_0\left(k_nr\right)dr}{\int_0^\rho r\left[J_0\left(k_nr\right)\right]^2dr}=\frac{2}{\left[J_1\left(\alpha_n\right)\right]^2}\int_0^\rho rf(r)J_0\left(k_nr\right)dr,\ n=1,2,\ldots.$$

Similarly, because

$$\frac{\partial u}{\partial t}(r,0)=\sum_{n=1}^{\infty}\left(-ck_nA_n\sin ck_nt+ck_nB_n\cos ck_nt\right)J_0\left(k_nr\right),$$

we have

$$u_t(r,0) = \sum_{n=1}^{\infty} ck_n B_n J_0\left(k_n r\right) = g(r).$$

Therefore,

$$B_n = \frac{\int_0^\rho rg(r)J_0\left(k_n r\right)\,dr}{ck_n \int_0^\rho r\left[J_0\left(k_n r\right)\right]^2 dr} = \frac{2}{ck_n\left[J_1\left(\alpha_n\right)\right]^2}\int_0^\rho rg(r)J_0\left(k_n r\right)\,dr,\ n = 1, 2, \ldots.$$

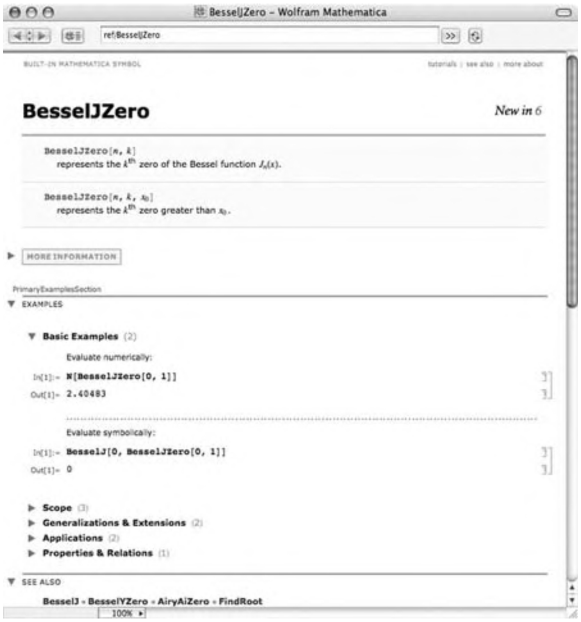
As a practical matter, in nearly all cases, these formulas are difficult to evaluate.

---

**Example 6.5.2**

$$\text{Solve}\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} = \dfrac{\partial^2 u}{\partial t^2},\ 0 < r < 1,\ t > 0 \\ u(1,t) = 0,\ |u(0,t)| < \infty,\ t > 0 \\ u(r,0) = r(r-1),\ \dfrac{\partial u}{\partial t}(r,0) = \sin \pi r,\ 0 < r < 1. \end{cases}$$

***Solution*** In this case, $\rho = 1$, $f(r) = r(r-1)$, and $g(r) = \sin \pi r$. The command BesselJZero[n,k] represents the $k$th zero of the Bessel function $J_k(x)$. To obtain an approximation of the number, use N.



```
αn_:=Evaluate[BesselJZero[0, n]//N]
```

Next, we define the constants $\rho$ and $c$ and the functions $f(r) = r(r-1)$, $g(r) = \sin \pi r$, and $k_n = \alpha_n/\rho$.

```
c = 1;
ρ = 1;
f[r_] = r(r − 1);
g[r_] = Sin[πr];
kn_:=kn = αn/ρ;
```

The formulas for the coefficients $A_n$ and $B_n$ are then defined so that an approximate solution may be determined. (We use lowercase letters to avoid any possible ambiguity with built-in Mathematica functions.) Note that we use NIntegrate to approximate the coefficients and avoid the difficulties in integration associated with the presence of the Bessel function of order zero.

```
an_:=an = (2NIntegrate [rf[r]BesselJ [0, knr] , {r, 0, ρ}])/
          BesselJ [1, αn]^2;
bn_:=bn = (2NIntegrate [rg[r]BesselJ [0, knr] , {r, 0, ρ}])/
          (c kn BesselJ [1, αn]^2)
```

We now compute the first 10 values of $A_n$ and $B_n$. Because a and b are defined using the form an_:=ax=… and bn_:=bx=…, Mathematica remembers these values for later use.

```
Table [{n, an, bn}, {n, 1, 10}] //TableForm
```

| | | |
|---|---|---|
| 1 | −0.323503 | 0.52118 |
| 2 | 0.208466 | −0.145776 |
| 3 | 0.00763767 | −0.0134216 |
| 4 | 0.0383536 | −0.00832269 |
| 5 | 0.00534454 | −0.00250503 |
| 6 | 0.0150378 | −0.00208315 |
| 7 | 0.00334937 | −0.000882012 |
| 8 | 0.00786698 | −0.000814719 |
| 9 | 0.00225748 | −0.000410202 |
| 10 | 0.00479521 | −0.000399219 |

The $n$th term of the series solution is defined in u. Then, an approximate solution is obtained in uapprox by summing the first 10 terms of u.

```
u[n_, r_, t_]:=(anCos [ckn t] + bnSin [ckn t]) BesselJ [0, knr] ;
uapprox[r_, t_] = Σ_{n=1}^{10} u[n, r, t];
```

We graph uapprox for several values of $t$ in Figure 6.51.

```
somegraphs =
Table[ParametricPlot3D[{rCos[θ], rSin[θ], uapprox[r, t]},
```

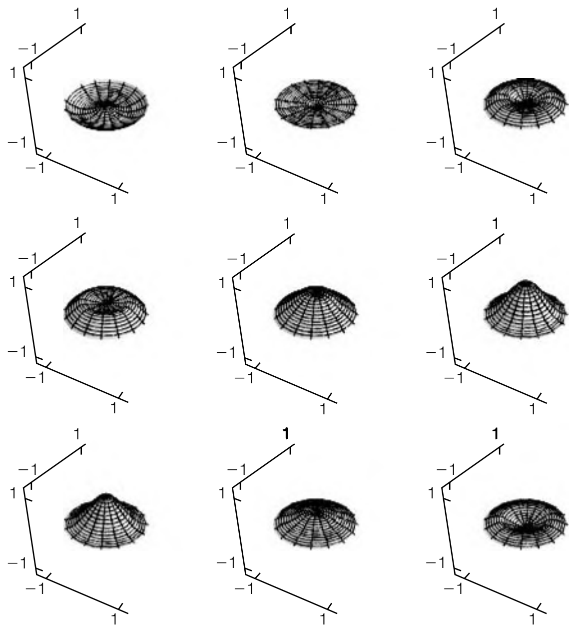


**FIGURE 6.51**

The drumhead for nine equally spaced values of $t$ between 0 and 1.5

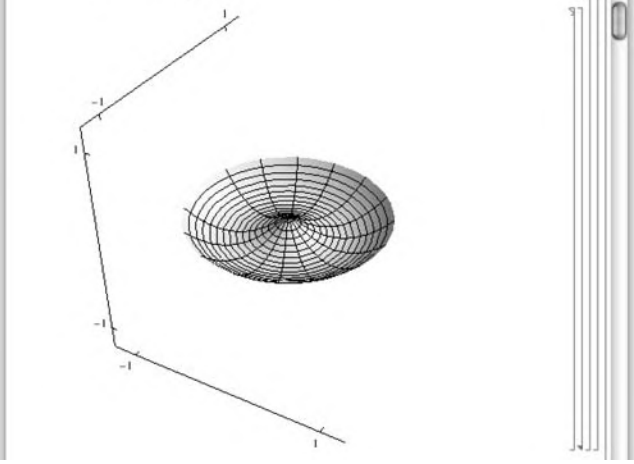
```
    {r, 0, 1}, {θ, −π, π}, Boxed → False,
        PlotRange → {−1.25, 1.25}, BoxRatios → {1, 1, 1},
        Ticks → {{−1, 1}, {−1, 1}, {−1, 1}}], {t, 0, 1.5, 1.5/8}] ;
toshow = Partition[somegraphs, 3];
Show[GraphicsGrid[toshow]]
```

In order to actually watch the drumhead move, we can use a Do loop to generate several graphs and animate the result. Be aware, however, that generating many three-dimensional graphics and then animating the results uses a great deal of memory and can take considerable time, even on a relatively powerful computer. We show one frame from the animation that results from the following Do loop.

```
Do[ParametricPlot3D[{rCos[θ], rSin[θ], uapprox[r, t]},
    {r, 0, 1}, {θ, −π, π}, Boxed → False,
        PlotRange → {−1.25, 1.25}, BoxRatios → {1, 1, 1},
        Ticks → {{−1, 1}, {−1, 1}, {−1, 1}}]//Print,
        {t, 0, 1.5, 1.5/15}] ;
```

If the displacement of the drumhead is not radially symmetric, the problem that describes the displacement of a circular membrane in its general

```
Do[ParametricPlot3D[{r Cos[θ], r Sin[θ], uapprox[r, t]},
    {r, 0, 1}, {θ, -π, π}, Boxed → False,
    PlotRange → {-1.25, 1.25}, BoxRatios → {1, 1, 1},
    Ticks → {{-1, 1}, {-1, 1}, {-1, 1}}] // Print,
  {t, 0, 1.5, 1.5/15}];
```




case is

$$\begin{cases} c^2\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}\right)=\dfrac{\partial^2 u}{\partial t^2},\ 0<r<\rho,\ -\pi<\theta<\pi,\ t>0 \\ u(\rho,\theta,t)=0,\ |u(0,\theta,t)|<\infty,\ -\pi\le\theta\le\pi,\ t>0 \\ u(r,\pi,t)=u(r,-\pi,t),\ \dfrac{\partial u}{\partial \theta}(r,\pi,t)=\dfrac{\partial u}{\partial \theta}(r,-\pi,t),\ 0<r<\rho,\ t>0 \\ u(r,\theta,0)=f(r,\theta),\ \dfrac{\partial u}{\partial t}(r,\pi,0)=g(r,\theta),\ 0<r<\rho,\ -\pi<\theta<\pi. \end{cases} \tag{6.43}$$

Using separation of variables and assuming that $u(r,\theta,t)=R(t)H(\theta)T(t)$, we obtain that a general solution is given by

$$\begin{aligned} u(r,\theta,t) = {} & \sum_n a_{0n}J_0\left(\lambda_{0n}r\right)\cos\left(\lambda_{0n}ct\right)+\sum_{m,n} a_{mn}J_m\left(\lambda_{mn}r\right)\cos\left(m\theta\right)\cos\left(\lambda_{mn}ct\right) \\ & +\sum_{m,n} b_{mn}J_m\left(\lambda_{mn}r\right)\sin\left(m\theta\right)\cos\left(\lambda_{mn}ct\right)+\sum_n A_{0n}J_0\left(\lambda_{0n}r\right)\sin\left(\lambda_{0n}ct\right) \\ & +\sum_{m,n} A_{mn}J_m\left(\lambda_{mn}r\right)\cos\left(m\theta\right)\sin\left(\lambda_{mn}ct\right) \\ & +\sum_{m,n} B_{mn}J_m\left(\lambda_{mn}r\right)\sin\left(m\theta\right)\sin\left(\lambda_{mn}ct\right), \end{aligned}$$

where $J_m$ represents the $m$th Bessel function of the first kind, $\alpha_{mn}$ denotes the $n$th zero of the Bessel function $y=J_m(x)$, and $\lambda_{mn}=\alpha_{mn}/\rho$. The coefficients are given by the following formulas:

$$a_{0n} = \frac{\int_0^{2\pi}\int_0^{\rho} f(r,\theta) J_0\left(\lambda_{0n} r\right) r\,dr\,d\theta}{2\pi\int_0^{\rho}\left[J_0\left(\lambda_{0n} r\right)\right]^2 r\,dr}$$

$$a_{mn} = \frac{\int_0^{2\pi}\int_0^{\rho} f(r,\theta) J_m\left(\lambda_{mn} r\right)\cos\left(m\theta\right) r\,dr\,d\theta}{\pi\int_0^{\rho}\left[J_m\left(\lambda_{mn} r\right)\right]^2 r\,dr}$$

$$b_{mn} = \frac{\int_0^{2\pi}\int_0^{\rho} f(r,\theta) J_m\left(\lambda_{mn} r\right)\sin\left(m\theta\right) r\,dr\,d\theta}{\pi\int_0^{\rho}\left[J_m\left(\lambda_{mn} r\right)\right]^2 r\,dr}$$

$$A_{0n} = \frac{\int_0^{2\pi}\int_0^{\rho} g(r,\theta) J_0\left(\lambda_{0n} r\right) r\,dr\,d\theta}{2\pi\lambda_{0n} c\pi\int_0^{\rho}\left[J_0\left(\lambda_{0n} r\right)\right]^2 r\,dr}$$

$$A_{mn} = \frac{\int_0^{2\pi}\int_0^{\rho} g(r,\theta) J_m\left(\lambda_{mn} r\right)\cos\left(m\theta\right) r\,dr\,d\theta}{\pi\lambda_{mn} c\int_0^{\rho}\left[J_m\left(\lambda_{mn} r\right)\right]^2 r\,dr}$$

$$B_{mn} = \frac{\int_0^{2\pi}\int_0^{\rho} g(r,\theta) J_m\left(\lambda_{mn} r\right)\sin\left(m\theta\right) r\,dr\,d\theta}{\pi\lambda_{mn} c\int_0^{\rho}\left[J_m\left(\lambda_{mn} r\right)\right]^2 r\,dr}$$

---

**Example 6.5.3**

Solve
$$\begin{cases} 10^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}\right) = \dfrac{\partial^2 u}{\partial t^2}, \\ 0 < r < 1,\ -\pi < \theta < \pi,\ t > 0 \\ u(1,\theta,t) = 0,\ \left|u(0,\theta,t)\right| < \infty,\ -\pi \le \theta \le \pi,\ t > 0 \\ u(r,\pi,t) = u(r,-\pi,t),\ \dfrac{\partial u}{\partial \theta}(r,\pi,t) = \dfrac{\partial u}{\partial \theta}(r,-\pi,t). \\ 0 < r < 1,\ t > 0 \\ u(r,\theta,0) = \cos\left(\pi r/2\right)\sin\theta, \\ \dfrac{\partial u}{\partial t}(r,\pi,0) = (r-1)\cos\left(\pi\theta/2\right),\ 0 < r < 1,\ -\pi < \theta < \pi \end{cases}$$

***Solution*** To calculate the coefficients, we will need to have approximations of the zeros of the Bessel functions, so we use BesselJZero together with N and Evaluate to define $\alpha_{mn}$ to be an approximation of the $n$th zero of $y = J_m(x)$. We illustrate the use of $\alpha_{mn}$ by using it to compute the first five zeros of $y = J_0(x)$.

```
α_{m_,n_}:=α_{m,n} = Evaluate[BesselJZero[m, n]//N]
Table[α_{0,n}, {n, 1, 5}]
```

{2.40483, 5.52008, 8.65373, 11.7915, 14.9309}

The appropriate parameter values as well as the initial condition functions are defined as follows. Notice that the functions describing the initial displacement and velocity are defined as the product of functions. This enables the subsequent calculations to be carried out using NIntegrate.

```
Clear[a, f, f1, f2, g1, g2, A, c, g, capa, capb, b]
c = 10;
ρ = 1;
f1[r_] = Cos[πr/2];
f2[θ_] = Sin[θ];
f[r_, θ_]:=f[r, θ] = f1[r]f2[θ];
g1[r_] = r − 1;
g2[θ_] − Cos[πθ/2];
g[r_, θ_]:=g[r, θ] = g1[r]g2[θ];
```

The coefficients $a_{0n}$ are determined with the function a.

```
Clear[a]
a[n_]:=a[n] =
    N[
        (NIntegrate[f1[r]BesselJ[0, α0,n r] r, {r, 0, ρ}]
        NIntegrate[f2[t], {t, 0, 2π}])/
            (2πNIntegrate[rBesselJ[0, α0,n r]^2, {r, 0, ρ}])];
```

Hence, as represents a table of the first five values of $a_{0n}$. Chop is used to round off very small numbers to zero.

```
as = Table[a[n]//Chop, {n, 1, 5}]
{0.0, 0, 0, 0}
```

Because the denominator of each integral formula used to find $a_{mn}$ and $b_{mn}$ is the same, the function bjmn which computes this value is defined next. A table of nine values of this coefficient is then determined.

```
bjmn[m_, n_]:=
bjmn[m, n] = N[NIntegrate[rBesselJ[m, αm,n r]^2, {r, 0, ρ}]]
Table[Chop[bjmn[m, n]], {m, 1, 3}, {n, 1, 3}]

{{0.0811076, 0.0450347, 0.0311763},
{0.0576874, 0.0368243, 0.0270149},
{0.0444835, 0.0311044, 0.0238229}}
```

We also note that in evaluating the numerators of $a_{mn}$ and $b_{mn}$ we must compute $\int_0^\rho r f_1(r) J_m\left(\alpha_{mn} r\right)\, dr$. This integral is defined in fbjmn and the corresponding values are found for $n = 1,\ 2,\ 3$ and $m = 1,\ 2,\ 3$.

```
Clear[fbjmn]
fbjmn[m_, n_]:=fbjmn[m, n] =
N[NIntegrate[f1[r]BesselJ[m, αm,n r] r, {r, 0, ρ}]]
```

```
Table[Chop[fbjmn[m, n]], {m, 1, 3}, {n, 1, 3}]
{{0.103574, 0.020514, 0.0103984},
{0.0790948, 0.0275564, 0.0150381},
{0.0628926, 0.0290764, 0.0171999}}
```

The formula to compute $a_{mn}$ is then defined and uses the information calculated in fbjmn and bjmn. As in the previous calculation, the coefficient values for $n = 1$, $2$, $3$ and $m = 1$, $2$, $3$ are determined.

```
a[m_, n_]:=
        a[m, n] =
        N[(fbjmn[m, n]NIntegrate[f2[t]Cos[mt], {t, 0, 2π}])/
            (πbjmn[m, n])];
Table[Chop[a[m, n]], {m, 1, 3}, {n, 1, 3}]

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

A similar formula is then defined for the computation of $b_{mn}$.

```
b[m_, n_]:=b[m, n] =
        N[(fbjmn[m, n]NIntegrate[f2[t]Sin[mt], {t, 0, 2π}])/
        (πbjmn[m, n])];
Table[Chop[b[m, n]], {m, 1, 3}, {n, 1, 3}]

{{1.277, 0.455514, 0.333537}, {0, 0, 0}, {0, 0, 0}}
```

Note that defining the coefficients in this manner a[m_,n_]:= a[m,n]=... and b[m_,n_]:=b[m,n]=... so that Mathematica "remembers" previously computed values, which reduces computation time. The values of $A_{0n}$ are found similarly to those of $a_{0n}$. After defining the function capa to calculate these coefficients, a table of values is then found.

```
capa[n_]:=capa[n] =
        N [(NIntegrate [g1[r]BesselJ [0, α0,nr] r, {r, 0, ρ}]
        NIntegrate[g2[t], {t, 0, 2π}])/
            (2πcα0,nNIntegrate [rBesselJ [0, α0,nr] ^2, {r, 0, ρ}])] ;
Table[Chop[capa[n]], {n, 1, 6}]

{0.00142231, 0.0000542518, 0.0000267596, 6.419764234815093`*^-6,
4.958428464118819`*^-6, 1.8858472721004333`*^-6}
```

The value of the integral of the component of $g$, g1, which depends on $r$ and the appropriate Bessel functions, is defined as gbjmn.

```
gbjmn[m_, n_]:=gbjmn[m, n] = NIntegrate[g1[r]*
        BesselJ [m, αm,nr] r, {r, 0, ρ}] //N
Table[gbjmn[m, n]//Chop, {m, 1, 3}, {n, 1, 3}]

{{-0.0743906, -0.019491, -0.00989293},
{-0.0554379, -0.0227976, -0.013039},
{-0.0433614, -0.0226777, -0.0141684}}
```

Then, $A_{mn}$ is found by taking the product of integrals, gbjmn depending on $r$ and one depending on $\theta$. A table of coefficient values is generated in this case as well.

```
capa[m_, n_]:=capa[m, n] =
        N[(gbjmn[m, n]NIntegrate[g2[t]Cos[mt], {t, 0, 2π}])/
            (πα_{m,n}cbjmn[m, n])] ;
Table[Chop[capa[m, n]], {m, 1, 3}, {n, 1, 3}]
```

{{0.0035096,0.000904517,0.000457326},
{−0.00262692,−0.00103252,−0.000583116},
{−0.000503187,−0.000246002,−0.000150499}}

Similarly, the $B_{mn}$ are determined.

```
capb[m_, n_]:=capb[m, n] =
        N[(gbjmn[m, n]NIntegrate[g2[t]Sin[mt], {t, 0, 2π}])/
            (πα_{m,n}cbjmn[m, n])] ;
Table[Chop[capb[m, n]], {m, 1, 3}, {n, 1, 3}]
```

{{0.00987945, 0.00254619, 0.00128736},
{−0.0147894, −0.00581305, −0.00328291},
{−0.00424938, −0.00207747, −0.00127095}}

Now that the necessary coefficients have been found, we construct an approximate solution to the wave equation by using our results. In the following, term1 represents those terms of the expansion involving $a_{0n}$, term2 those terms involving $a_{mn}$, term3 those involving $b_{mn}$, term4 those involving $A_{0n}$, term5 those involving $A_{mn}$, and term6 those involving $B_{mn}$.

```
Clear[term1, term2, term3, term4, term5, term6]
term1[r_, t_, n_]=a[n]BesselJ [0, α_{0,n}r] Cos [α_{0,n}ct];
term2[r_, t_, θ_, m_, n_]=
        a[m, n]BesselJ [m, α_{m,n}r] Cos[mθ]Cos [α_{m,n}ct];
term3[r_, t_, θ_, m_, n_]=
        b[m, n]BesselJ [m, α_{m,n}r] Sin[mθ]Cos [α_{m,n}ct];
term4[r_, t_, n_]=capa[n]BesselJ [0, α_{0,n}r] Sin [α_{0,n}ct];
term5[r_, t_, θ_, m_, n_]=
        capa[m, n]BesselJ [m, α_{m,n}r] Cos[mθ]Sin [α_{m,n}ct];
term6[r_, t_, θ_, m_, n_]=
        capb[m, n]BesselJ [m, α_{m,n}r] Sin[mθ]Sin [α_{m,n}ct];
```

Therefore, our approximate solution is given as the sum of these terms as computed in u.

```
Clear[u]
u[r_, t_, th_]=Σ_{n=1}^{5} term1[r, t, n] + Σ_{m=1}^{3} Σ_{n=1}^{3} term2[r, t, th, m, n]
```

```
        + Σ_{m=1}^{3} Σ_{n=1}^{3} term3[r, t, th, m, n] + Σ_{n=1}^{5} term4[r, t, n]
        + Σ_{m=1}^{3} Σ_{n=1}^{3} term5[r, t, th, m, n] + Σ_{m=1}^{3} Σ_{n=1}^{3} term6[r, t, th, m, n];
uc = Compile[{r, t, th}, u[r, t, th]]
```

CompiledFunction[{r, t, th}, u[r, t, th], -CompiledCode-]

The solution is *compiled* in uc. The command Compile is used to compile functions. Compile returns a CompiledFunction that represents the compiled code. Generally, compiled functions take less time to perform computations than uncompiled functions, although compiled functions can only be evaluated for numerical arguments.

Next, we define the function tplot, which uses ParametricPlot3D to produce the graph of the solution for a particular value of $t$. Note that the $x$ and $y$ coordinates are given in terms of polar coordinates.

```
Clear[tplot]
tplot[t_]:=ParametricPlot3D[{rCos[θ], rSin[θ], uc[r, t, θ]},
        {r, 0, 1}, {θ, −π, π}, PlotPoints → {20, 20},
        BoxRatios → {1, 1, 1}, Axes → False, Boxed → False]
```

A table of nine plots for nine equally spaced values of $t$ from $t = 0$ to $t = 1$ using increments of $1/8$ is then generated. This table of graphs is displayed as a graphics array in Figure 6.52.

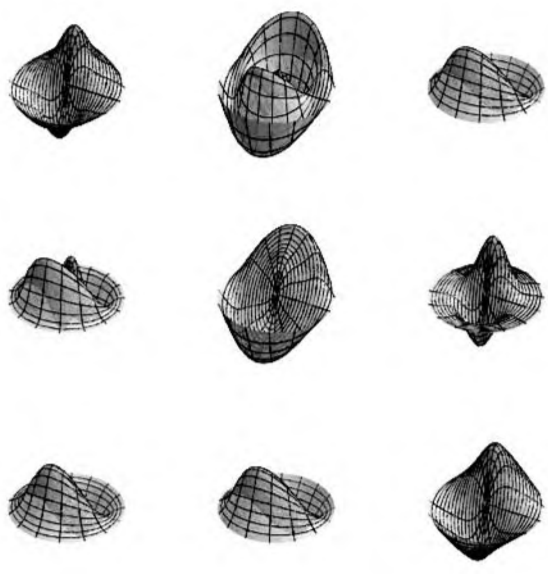


**FIGURE 6.52**

The drumhead for nine equally spaced values of $t$ from $t = 0$ to $t = 1$

```
somegraphs = Table[tplot[t], {t, 0, 1, 1/8}];
toshow = Partition[somegraphs, 3];
Show[GraphicsGrid[toshow]]
```

Of course, we can generate many graphs with a Do loop and animate the result as in the previous example. Be aware, however, that generating many three-dimensional graphics and then animating the results uses a great deal of memory and can take considerable time, even on a relatively powerful computer.

---

## 6.5.3 Other Partial Differential Equations

A partial differential equation of the form

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = 0c(x,y,u) \tag{6.44}$$

is called a **first-order, quasilinear partial differential equation**. In the case in which $c(x,y,u) = 0$, equation (6.44) is **homogeneous**; if $a$ and $b$ are independent of $u$, equation (6.44) is **almost linear**; and when $c(x,y,u)$ can be written in the form $c(x,y,u) = d(x,y)u + s(x,y)$, equation (6.44) is **linear**. Quasilinear partial differential equations can frequently be solved using the *method of characteristics*.

---

**Example 6.5.4** Use the *method of characteristics* to solve the initial-value problem $\begin{cases} -3xtu_x + u_t = xt \\ u(x,0) = x. \end{cases}$

***Solution*** For this problem, the *characteristic system* is

$$\begin{aligned} \partial x/\partial r &= -3xt, & x(0,s) &= s \\ \partial t/\partial r &= 1, & t(0,s) &= 0 \\ \partial u/\partial r &= xt, & u(0,s) &= s. \end{aligned}$$

We begin by using DSolve to solve $\partial t/\partial r = 1,\ t(0,s) = 0$

```
d1 = DSolve[{D[t[r], r]==1, t[0]==0}, t[r], r]
{{t[r] → r}}
```

and obtain $t = r$. Thus, $\partial x/\partial r = -3xr,\ x(0,s) = s$, which we solve next

```
d2 = DSolve[{D[x[r], r]==-3x[r]r, x[0]==s}, x[r], r]
```

$$\left\{\left\{x[r] \to e^{-\frac{3r^2}{2}} s\right\}\right\}$$

and obtain $x = se^{-3r^2/2}$. Substituting $r = t$ and $x = se^{-3r^2/2}$ into $\partial u/\partial r = xt,\ u(0,s) = s$ and using DSolve to solve the resulting equation yields the following result, named d3.

**d3 = DSolve$\left[\{D[u[r],r]==E^{-\frac{3r^2}{2}}sr, u[0]==s\}, u[r], r\right]$**

$\left\{\left\{u[r] \to \frac{1}{3}e^{-\frac{3r^2}{2}}\left(-1+4e^{\frac{3r^2}{2}}\right)s\right\}\right\}$

To find $u(x,t)$, we must solve the system of equations

$$\begin{cases} t = r \\ x = se^{-3r^2/2} \end{cases}$$

for $r$ and $s$. Substituting $r = t$ into $x = se^{-3r^2/2}$ and solving for $s$ yields $s = xe^{3t^2}/2$. Thus, the solution is given by replacing the values obtained previously in the solution obtained in d3. We do this below by using ReplaceAll (/.) to replace each occurrence of $r$ and $s$ in d3[[1,1,2]], the solution obtained in d3, by the values $r = t$ and $s = xe^{3t^2}/2$. The resulting output represents the solution to the initial-value problem.

**d3[[1, 1, 2]]/.{r− >t, s− >x Exp[3/2t^2]}//Simplify**

$\frac{1}{3}\left(-1+4e^{\frac{3t^2}{2}}\right)x$

In this example, DSolve can also solve this first-order partial differential equation.

Next, we use DSolve to find a general solution of $-3xtu_x + u_t = xt$ and name the resulting output gensol.

**gensol = DSolve[−3xtD[u[x, t], x] + D[u[x, t], t]==xt,
u[x, t], {x, t}]**

$\left\{\left\{u[x,t] \to \frac{1}{3}\left(-x+3C[1]\left[\frac{1}{6}\left(3t^2+2Log[x]\right)\right]\right)\right\}\right\}$

The output

$C[1]\left[-\frac{3t^2}{2}-Log[x]\right]$

represents an arbitrary function of $-\frac{3}{2}t^2 - \ln x$. The explicit solution is extracted from gensol with gensol[[1,1,2]], the same way that results are extracted from the output of DSolve commands involving ordinary differential equations.

**gensol[[1, 1, 2]]**

$\frac{1}{3}\left(-x+3C[1]\left[\frac{1}{6}\left(3t^2+2Log[x]\right)\right]\right)$

To find the solution that satisfies $u(x,0) = x$, we replace each occurrence of $t$ in the solution by $0$.

**gensol[[1, 1, 2]]/.t− >0**

$\frac{1}{3}\left(-x+3C[1]\left[\frac{Log[x]}{3}\right]\right)$

Thus, we must find a function $f(x)$ so that

$$-\frac{1}{2}x + f(\ln x) = x$$

$$f(\ln x) = \frac{3}{2}x.$$

Certainly $f(t) = \frac{4}{3}e^{-t}$ satisfies the previous criteria. We define $f(t) = \frac{4}{3}e^{-t}$ and then compute $f(\ln x)$ to verify that $f(\ln x) = \frac{3}{2}x$.

```
Clear[f]
f[t_] = 4Exp[-t]/3;
f[-Log[x]]
```

$\frac{4x}{3}$

Thus, the solution to the initial-value problem is given by $-\frac{1}{3}x + f\left(-\frac{3}{2}t^2 - \ln x\right)$, which is computed and named **sol**. Of course, the result returned is the same as that obtained previously.

**sol = Simplify** $\left[-\frac{x}{3} + f\left[-\frac{3t^2}{2} - \text{Log}[x]\right]\right]$

$\frac{1}{3}\left(-1 + 4e^{\frac{3t^2}{2}}\right)x$

Last, we use Plot3D to graph **sol** on the rectangle $[0, 20] \times [-2, 2]$ in Figure 6.53.

```
Plot3D[sol, {x, 0, 20}, {t, -2, 2}, PlotRange → {0, 30},
    PlotPoints → 30, ClippingStyle → None]
```

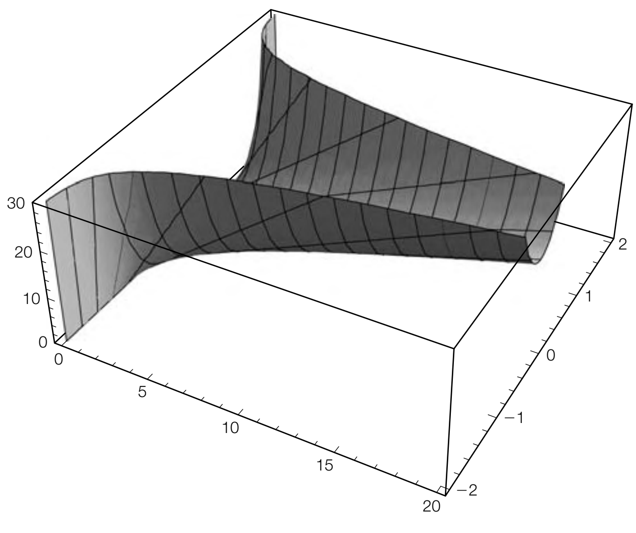



**FIGURE 6.53**

Plot of $u(x, t) = \frac{1}{3}x\left(4e^{3t^2/2} - 1\right)$

## 6.6 EXERCISES

1. (a) Solve $\left(1+y^2\right)y' = y\cos x$. (b) Explain the functionality of ProductLog. (c) Show that an implicit solution of the equation is $\frac{1}{2}y^2 + \ln|y| = \sin x + C$. (d) Use ContourPlot to graph various solutions on the rectangle $[0, 10] \times [0, 10]$.
2. Solve $xyy' = y^2 - x^2$ and graph several integral curves of the equation. (See Figure 6.54(a).)
3. Solve $(-1 + ye^{xy} + y\cos xy)\,dx + (1 + xe^{xy} + x\cos xy)\,dy = 0$ and graph several integral curves of the equation. (See Figure 6.54(b).)
4. Solve $y' = \sin(2x - y)$, $y(0) = 0.5$. What is the value of $y(1)$? Graph for $0 \le x \le 15$.
5. Graph the solution of $y' = \sin(ty)$, $y(0) = j$ on $[0, 7]$ for $j = 0.5, 1, \ldots, 2.5$.
6. Create a Manipulate object that lets you compare the solution of $x'' + ax' + \sin x = 0$ to $x'' + ax' + x = 0$.
7. Solve each of the following differential equations or initial-value problems by hand and then verify your results with Mathematica.

   **(a)** $2y'' + 5y' + 5y = 0$, $y(0) = 0$, $y'(0) = 1/2$
   **(b)** $y'' + 4y' + 13y = t\cos^2 3t$

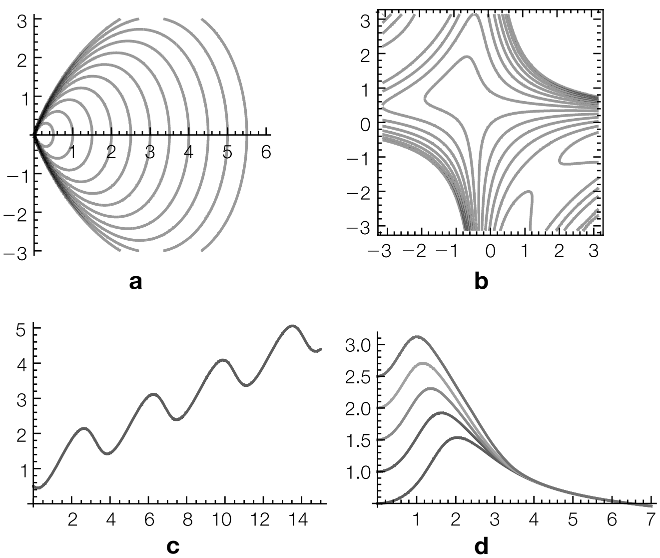


**FIGURE 6.54**

(a) Integral curves of $xyy' = y^2 - x^2$. (b) Integral curves of $(-1 + ye^{xy} + y\cos xy)\,dx + (1 + xe^{xy} + x\cos xy)\,dy = 0$. (c) The solution of an initial-value problem. (d) Solutions to several initial-value problems

**(c)** $y'' - 2y' + y = e^t \ln t$
**(d)** $t^3 y''' + 16t^2 y'' + 79ty' + 125y = 0$

**8.** Two lines, $l_1$ and $l_2$, with slopes $m_1$ and $m_2$, respectively, are **orthogonal** (or **perpendicular**) if their slopes satisfy the relationship $m_1 = -1/m_2$. Two curves, $C_1$ and $C_2$, are **orthogonal** (or **perpendicular**) at a point if their respective tangent lines to the curves at that point are perpendicular. Now we want to determine the set of orthogonal curves to a given family of curves. We refer to this set of orthogonal curves as the **family of orthogonal trajectories**. Suppose that a family of curves is defined as $F(x, y) = C$ and that the slope of the tangent line at any point on these curves is $dy/dx = f(x, y)$. Then, the slope of the tangent line on the orthogonal trajectory is $dy/dx = -1/f(x, y)$ so the family of orthogonal trajectories is found by solving the first-order equation $dy/dx = -1/f(x, y)$.

**(a)** Determine the family of orthogonal trajectories to the family of curves $y = cx^2$. Confirm your result graphically by graphing members of both families of curves on the same axes.
**(b)** Determine the orthogonal trajectories of the family of curves given by $y^2 - 2cx = c^2$. Graph several members of both families of curves on the same set of axes. Why are these two families of curves said to be **self-orthogonal**?

**9.** If we are given a family of curves that satisfies the differential equation $dy/dx = f(x, y)$ and we want to find a family of curves that intersects this family at a constant angle $\theta$, we must solve the differential equation

$$\frac{dy}{dx} = \frac{f(x, y) \pm \tan\theta}{1 \mp f(x, y)\tan\theta}.$$

Find a family of curves that intersects the family of curves $x^2 + y^2 = c^2$ at an angle of $\pi/6$. Confirm your result graphically by graphing members of both families of curves on the same axes.

**10.** Find a linear differential equation with general solution $y = c_1 \cos t + c_2 \sin t + e^{t/3}(c_3 \cos 2t + c_4 \sin 2t) + \frac{1}{2} t \sin t$.

**11.** Solve each system and graph various solutions together with the direction field: (a) $\mathbf{X}' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{X}$, (b) $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X}$, and (c) $x' = -5x + 3y$, $y' = -2x - 10y$

**12.** Solve $x' - y = e^{-t}$, $y' + 5x + 2y = \sin 3t$, $x(0) = x_0$, $y(0) = y_0$. Parametrically graph the solution for $(x_0, y_0) = (i, j)$, where $i, j$ take on four equally spaced values between $-1$ and $1$.

**13.** Solve $\mathbf{X}' = \begin{pmatrix} -\alpha & \beta \\ -\beta & 0 \end{pmatrix} \mathbf{X}$ if the eigenvalues of the coefficient matrix are (a) real and distinct, (b) real and equal, and (c) complex conjugates. *Hint:* Both DSolve and Assumptions might be helpful.

**14.** Under certain assumptions, the **FitzHugh–Nagumo equation** that arises in the study of the impulses in a nerve fiber can be written as the system of ordinary differential equations

$$\begin{cases} dV/d\xi = W \\ dW/d\xi = F(V) + R - uW \\ dR/d\xi = \dfrac{\epsilon}{u}(bR - V - a) \\ V(0) = v_0,\ W(0) = W_0,\ R(0) = R_0 \end{cases},$$

where $F(V) = \frac{1}{3}V^3 - V$. (a) Graph the solution to the FitzHugh–Nagumo equation that satisfies the initial conditions $V(0) = 1$, $W(0) = 0$, and $R(0) = 1$ if $\epsilon = 0.08$, $a = 0.7$, $b = 0$, and $u = 1$. (b) Graph the solution that satisfies the initial conditions $V(0) = 1$, $W(0) = 0.5$, and $R(0) = 0.5$ if $\epsilon = 0.08$, $a = 0.7$, $b = 0.8$, and $u = 0.6$.

**15. (Controlling the Spread of a Disease).**

**Sources:** Herbert W. Hethcote, "Three basic epidemiological models," *Applied Mathematical Ecology*, edited by Simon A. Levin, Thomas G. Hallan, and Louis J. Gross, Springer-Verlag (1989), pp. 119–143; Roy M. Anderson and Robert M. May, "Directly transmitted infectious diseases: Control by vaccination," *Science*, Volume 215, (February 26, 1982), pp. 1053–1060; and J. D. Murray, *Mathematical Biology*, Springer-Verlag (1990), pp. 611–618.

If a person becomes immune to a disease after recovering from it and births and deaths in the population are not taken into account, then the percentage (or proportion) of persons susceptible to becoming infected with the disease, $S(t)$, the percentage of people in the population infected with the disease, $I(t)$, and the percentage of the population recovered and immune to the disease, $R(t)$, can be modeled by the system

$$\begin{cases} S' = -\lambda SI \\ I' = \lambda SI - \gamma I \\ R' = \gamma I \\ S(0) = S_0,\ I(0) = I_0,\ R(0) = 0 \end{cases}. \tag{6.45}$$

Because $S(t) + I(t) + R(t) = 1$, once we know $S(t)$ and $I(t)$, we can compute $R(t)$ with $R(t) = 1 - S(t) - I(t)$. This model is called an **SIR model without vital dynamics** because once a person has had the disease, the person becomes immune to the disease, and because births and deaths are not taken into consideration. This model might be used to model diseases that are **epidemic** to a population—those diseases that persist in a population for short periods of time (less than 1 year). Such diseases typically include influenza, measles, rubella, and chickenpox.

If $S_0 < \gamma/\lambda$, $I'(0) = \lambda S_0 I_0 - \gamma I_0 < \lambda \frac{\gamma}{\lambda} I_0 - \gamma I_0 = 0$. Thus, the rate of infection immediately begins to decrease; the disease dies out. On the other hand, if $S_0 > \gamma/\lambda$, $I'(0) > \lambda S_0 I_0 - \gamma I_0 > 0$, so the rate of infection first increases; an epidemic results.

Although we cannot find explicit formulas for $S$, $I$, and $R$ as functions of $t$, we can, for example, solve for $I$ in terms of $S$.

**(a)** Solve the equation $\dfrac{dI}{dS} = -\dfrac{(\lambda S - \gamma)I}{\lambda SI} = -1 + \dfrac{\rho}{S}$, $\rho = \gamma/\lambda$.

When diseases persist in a population for long periods of time, births and deaths must be taken into consideration. If a person becomes immune to a disease after recovering from it and births and deaths in the population are taken into account, then the percentage of persons susceptible to becoming infected with the disease, $S(t)$, and the percentage of people in the population infected with the disease, $I(t)$, can be modeled by the system

$$\begin{cases} S' = -\lambda SI + \mu - \mu S \\ I' = \lambda SI - \gamma I - \mu I \\ S(0) = S_0,\ I(0) = I_0 \end{cases}.$$

This model is called an **SIR model with vital dynamics** because once a person has had the disease, the person becomes immune to the disease, and because births and deaths are taken into consideration. This model might be used to model diseases that are **endemic** to a population—those diseases that persist in a population for long periods of time (10 or 20 years). Smallpox is an example of a disease that was endemic until it was eliminated in 1977.

**(b)** Find and classify the equilibrium points of this system.

Because $S(t) + I(t) + R(t) = 1$, it follows that $S(t) + I(t) \leq 1$. The following table shows the average infectious period, $1/\gamma$, $\gamma$, and typical contact numbers, $\sigma$, for several diseases during certain epidemics.

| **Disease** | $1/\gamma$ | $\gamma$ | $\sigma$ |
|---|---|---|---|
| Measles | 6.5 | 0.153846 | 14.9667 |
| Chickenpox | 10.5 | 0.0952381 | 11.3 |
| Mumps | 19 | 0.0526316 | 8.1 |
| Scarlet fever | 17.5 | 0.0571429 | 8.5 |

Let us assume that the average lifetime, $1/\mu$, is 70 years so that $\mu = 0.0142857$.

For each of the diseases listed in the previous table, we use the formula $\sigma = \lambda/(\gamma + \mu)$ to calculate the daily contact rate $\lambda$.

| Disease | $\lambda$ |
|---|---|
| Measles | 2.51638 |
| Chickenpox | 1.23762 |
| Mumps | 0.54203 |
| Scarlet fever | 0.607143 |

Diseases such as those listed here can be controlled once an effective and inexpensive vaccine has been developed. Since it is virtually impossible to vaccinate everybody against a disease, we want to know what percentage of a population needs to be vaccinated to eliminate a disease. A population of people has **herd immunity** to a disease if enough people are immune to the disease so that if it is introduced into the population, it will not spread throughout the population. In order to have herd immunity, an infected person must infect less than one uninfected person during the time the person is infectious. Thus, we must have

$$\sigma S < 1.$$

Since $I + S + R = 1$, when $I = 0$ we have that $S = 1 - R$ and, consequently, herd immunity is achieved when

$$\begin{aligned} \sigma(1 - R) &< 1 \\ \sigma - \sigma R &< 1 \\ -\sigma R &< 1 - \sigma \\ R &> \frac{\sigma - 1}{\sigma} = 1 - \frac{1}{\sigma}. \end{aligned}$$

**(c)** For each of the diseases listed previously, create a table that estimates the minimum percentage of a population that needs to be vaccinated to achieve herd immunity.

**(d)** Using the values in the previous tables, for each disease graph the direction field and several solutions $\begin{cases} S = S(t) \\ I = I(t) \end{cases}$ parametrically.

See texts such as Jordan and Smith's *Nonlinear Ordinary Differential Equations* [23] for discussions of ways to analyze systems such as the Rössler attractor and the Lorenz equations.

**16.** The **Rössler attractor** is the system

$$\begin{cases} x' = -y - z \\ y' = x + ay \\ z' = bx - cz + xz \end{cases}.$$

Observe that this system is nonlinear because of the product of the $x$ and $z$ terms in the $z'$ equation.

If $a = 0.4$, $b = 0.3$, $x_0 = 1$, $y_0 = 0.4$, and $z(0) = 0.7$, how does the value of $c$ affect solutions to the initial-value problem

$$\begin{cases} x' = -y - z \\ y' = x + ay \\ z' = bx - cz + xz \\ x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0 \end{cases} ?$$

*Suggestion:* Use Manipulate.

**17.** *Challenge:* Using the linear approximation $\sin\theta = \theta$ for small displacements, derive the equations for a triple pendulum if *theta*$_1$ represents the displacement of the upper pendulum (with mass $m_1$ and length $l_1$), *theta*$_2$ represents the displacement of the upper pendulum (with mass $m_2$ and length $l_2$), and *theta*$_3$ represents the displacement of the upper pendulum (with mass $m_3$ and length $l_3$). Using $g = 32$, illustrate the solution graphically if $m_1 = 3$, $m_2 = 2$, and $m_3 = 1$, $l_1 = 16$, $l_2 = 8$, $l_3 = 16$, $\theta_1(0) = 0$, $\theta_1'(0) = 1$, $\theta_2(0) = 0$, $\theta_2'(0) = 0$, $\theta_3(0) = 0$, and $\theta_3'(0) = -1$.

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