Preface

The purpose of this handbook is to supply a collection of mathematical formulas and tables which will prove to be valuable to students and research workers in the fields of mathematics, physics, engineering and other sciences. To accomplish this, care has been taken to include those formulas and tables which are most likely to be needed in practice rather than highly specialized results which are rarely used. Every effort has been made to present results concisely as well as precisely so that they may be referred to with a maximum of ease as well as confidence.

Topics covered range from elementary to advanced. Elementary topics include those from algebra, geometry, trigonometry, analytic geometry and calculus. Advanced topics include those from differential equations, vector analysis, Fourier series, gamma and beta functions, Bessel and Legendre functions, Fourier and Laplace transforms, elliptic functions and various other special functions of importance. This wide coverage of topics has been adopted so as to provide within a single volume most of the important mathematical results needed by the student or research worker regardless of his particular field of interest or level of attainment.

The book is divided into two main parts. Part I presents mathematical formulas together with other material, such as definitions, theorems, graphs, diagrams, etc., essential for proper understanding and application of the formulas. Included in this first part are extensive tables of integrals and Laplace transforms which should be extremely useful to the student and research worker. Part II presents numerical tables such as the values of elementary functions (trigonometric, logarithmic, exponential, hyperbolic, etc.) as well as advanced functions (Bessel, Legendre, elliptic, etc.). In order to eliminate confusion, especially to the beginner in mathematics, the numerical tables for each function are separated. Thus, for example, the sine and cosine functions for angles in degrees and minutes are given in separate tables rather than in one table so that there is no need to be concerned about the possibility of error due to looking in the wrong column or row.

I wish to thank the various authors and publishers who gave me permission to adapt data from their books for use in several tables of this handbook. Appropriate references to such sources are given next to the corresponding tables. In particular I am indebted to the Literary Executor of the late Sir Ronald A. Fisher, F.R.S., to Dr. Frank Yates, F.R.S., and to Oliver and Boyd Ltd., Edinburgh, for permission to use data from Table III of their book *Statistical Tables for Biological, Agricultural and Medical Research.*

I also wish to express my gratitude to Nicola Monti, Henry Hayden and Jack Margolin for their excellent editorial cooperation.

M. R. SPIEGEL

Rensselaer Polytechnic Institute
September, 1968

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Part I

FORMULAS

THE GREEK ALPHABET

Greek name	Greek letter	
	Lower case	Capital
Alpha	α	A
Beta	β	B
Gamma	γ	Γ
Delta	δ	Δ
Epsilon	ϵ	E
Zeta	ζ	Z
Eta	η	H
Theta	θ	Θ
Iota	ι	I
Kappa	κ	K
Lambda	λ	Λ
Mu	μ	M

Greek name	Greek letter	
	Lower case	Capital
Nu	ν	N
Xi	ξ	Ξ
Omicron	o	O
Pi	π	Π
Rho	ρ	P
Sigma	σ	Σ
Tau	τ	T
Upsilon	υ	Υ
Phi	ϕ	Φ
Chi	χ	X
Psi	ψ	Ψ
Omega	ω	Ω

1 SPECIAL CONSTANTS

1.1 $\pi = 3.14159\ 26535\ 89793\ 23846\ 2643\ldots$

1.2 $e = 2.71828\ 18284\ 59045\ 23536\ 0287\ldots = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
$=$ natural base of logarithms

1.3 $\sqrt{2} = 1.41421\ 35623\ 73095\ 0488?\ldots$

1.4 $\sqrt{3} = 1.73205\ 08075\ 68877\ 2935\ldots$

1.5 $\sqrt{5} = 2.23606\ 79774\ 99789\ 6964\ldots$

1.6 $\sqrt[3]{2} = 1.25992\ 1050\ldots$

1.7 $\sqrt[3]{3} = 1.44224\ 9570\ldots$

1.8 $\sqrt[5]{2} = 1.14869\ 8355\ldots$

1.9 $\sqrt[5]{3} = 1.24573\ 0940\ldots$

1.10 $e^{\pi} = 23.14069\ 26327\ 79269\ 006\ldots$

1.11 $\pi^{e} = 22.45915\ 77183\ 61045\ 47342\ 715\ldots$

1.12 $e^{e} = 15.15426\ 22414\ 79264\ 190\ldots$

1.13 $\log_{10} 2 = 0.30102\ 99956\ 63981\ 19521\ 37389\ldots$

1.14 $\log_{10} 3 = 0.47712\ 12547\ 19662\ 43729\ 50279\ldots$

1.15 $\log_{10} e = 0.43429\ 44819\ 03251\ 82765\ldots$

1.16 $\log_{10} \pi = 0.49714\ 98726\ 94133\ 85435\ 12683\ldots$

1.17 $\log_{e} 10 = \ln 10 = 2.30258\ 50929\ 94045\ 68401\ 7991\ldots$

1.18 $\log_{e} 2 = \ln 2 = 0.69314\ 71805\ 59945\ 30941\ 7232\ldots$

1.19 $\log_{e} 3 = \ln 3 = 1.09861\ 22886\ 68109\ 69139\ 5245\ldots$

1.20 $\gamma = 0.57721\ 56649\ 01532\ 86060\ 6512\ldots =$ *Euler's constant*
$= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)$

1.21 $e^{\gamma} = 1.78107\ 24179\ 90197\ 9852\ldots$ [see 1.20]

1.22 $\sqrt{e} = 1.64872\ 12707\ 00128\ 1468\ldots$

1.23 $\sqrt{\pi} = \Gamma(\frac{1}{2}) = 1.77245\ 38509\ 05516\ 02729\ 8167\ldots$
where Γ is the *gamma function* [see pages 101-102].

1.24 $\Gamma(\frac{1}{3}) = 2.67893\ 85347\ 07748\ldots$

1.25 $\Gamma(\frac{1}{4}) = 3.62560\ 99082\ 21908\ldots$

1.26 1 radian $= 180°/\pi = 57.29577\ 95130\ 8232\ldots°$

1.27 $1° = \pi/180$ radians $= 0.01745\ 32925\ 19943\ 29576\ 92\ldots$ radians

2 SPECIAL PRODUCTS and FACTORS

2.1 $(x+y)^2 = x^2 + 2xy + y^2$

2.2 $(x-y)^2 = x^2 - 2xy + y^2$

2.3 $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2.4 $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

2.5 $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

2.6 $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

2.7 $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

2.8 $(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

2.9 $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

2.10 $(x-y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$

The results 2.1 to 2.10 above are special cases of the *binomial formula* [see page 3].

2.11 $x^2 - y^2 = (x-y)(x+y)$

2.12 $x^3 - y^3 = (x-y)(x^2+xy+y^2)$

2.13 $x^3 + y^3 = (x+y)(x^2-xy+y^2)$

2.14 $x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$

2.15 $x^5 - y^5 = (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$

2.16 $x^5 + y^5 = (x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$

2.17 $x^6 - y^6 = (x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$

2.18 $x^4 + x^2y^2 + y^4 = (x^2+xy+y^2)(x^2-xy+y^2)$

2.19 $x^4 + 4y^4 = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$

Some generalizations of the above are given by the following results where n is a positive integer.

2.20
$$\begin{aligned} x^{2n+1} - y^{2n+1} &= (x-y)(x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + \cdots + y^{2n}) \\ &= (x-y)\left(x^2 - 2xy\cos\frac{2\pi}{2n+1} + y^2\right)\left(x^2 - 2xy\cos\frac{4\pi}{2n+1} + y^2\right) \\ &\quad \cdots\left(x^2 - 2xy\cos\frac{2n\pi}{2n+1} + y^2\right) \end{aligned}$$

2.21
$$\begin{aligned} x^{2n+1} + y^{2n+1} &= (x+y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \cdots + y^{2n}) \\ &= (x+y)\left(x^2 + 2xy\cos\frac{2\pi}{2n+1} + y^2\right)\left(x^2 + 2xy\cos\frac{4\pi}{2n+1} + y^2\right) \\ &\quad \cdots\left(x^2 + 2xy\cos\frac{2n\pi}{2n+1} + y^2\right) \end{aligned}$$

2.22
$$\begin{aligned} x^{2n} - y^{2n} &= (x-y)(x+y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots) \\ &= (x-y)(x+y)\left(x^2 - 2xy\cos\frac{\pi}{n} + y^2\right)\left(x^2 - 2xy\cos\frac{2\pi}{n} + y^2\right) \\ &\quad \cdots\left(x^2 - 2xy\cos\frac{(n-1)\pi}{n} + y^2\right) \end{aligned}$$

2.23
$$\begin{aligned} x^{2n} + y^{2n} &= \left(x^2 + 2xy\cos\frac{\pi}{2n} + y^2\right)\left(x^2 + 2xy\cos\frac{3\pi}{2n} + y^2\right) \\ &\quad \cdots\left(x^2 + 2xy\cos\frac{(2n-1)\pi}{2n} + y^2\right) \end{aligned}$$

3 The BINOMIAL FORMULA and BINOMIAL COEFFICIENTS

FACTORIAL n

If $n = 1, 2, 3, \ldots$ *factorial* n or n *factorial* is defined as

3.1 $$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$

We also define *zero factorial* as

3.2 $$0! = 1$$

BINOMIAL FORMULA FOR POSITIVE INTEGRAL n

If $n = 1, 2, 3, \ldots$ then

3.3 $$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n$$

This is called the *binomial formula.* It can be extended to other values of n and then is an infinite series [see *Binomial Series*, page 110].

BINOMIAL COEFFICIENTS

The result 3.3 can also be written

3.4 $$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n}y^n$$

where the coefficients, called *binomial coefficients*, are given by

3.5 $$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$$

PROPERTIES OF BINOMIAL COEFFICIENTS

3.6 $$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

This leads to *Pascal's triangle* [see page 236].

3.7 $$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

3.8 $$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots (-1)^n \binom{n}{n} = 0$$

3.9 $$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+m}{n} = \binom{n+m+1}{n+1}$$

3.10 $$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$$

3.11 $$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$$

3.12 $$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

3.13 $$\binom{m}{0}\binom{n}{p} + \binom{m}{1}\binom{n}{p-1} + \cdots + \binom{m}{p}\binom{n}{0} = \binom{m+n}{p}$$

3.14 $$(1)\binom{n}{1} + (2)\binom{n}{2} + (3)\binom{n}{3} + \cdots + (n)\binom{n}{n} = n2^{n-1}$$

3.15 $$(1)\binom{n}{1} - (2)\binom{n}{2} + (3)\binom{n}{3} - \cdots (-1)^{n+1}(n)\binom{n}{n} = 0$$

MULTINOMIAL FORMULA

3.16 $$(x_1 + x_2 + \cdots + x_p)^n = \sum \frac{n!}{n_1!\, n_2! \cdots n_p!} x_1^{n_1} x_2^{n_2} \cdots x_p^{n_p}$$

where the sum, denoted by Σ, is taken over all nonnegative integers $n_1, n_2, \ldots, n_p$ for which $n_1 + n_2 + \cdots + n_p = n$.

4 GEOMETRIC FORMULAS

RECTANGLE OF LENGTH b AND WIDTH a

4.1 Area $= ab$

4.2 Perimeter $= 2a + 2b$

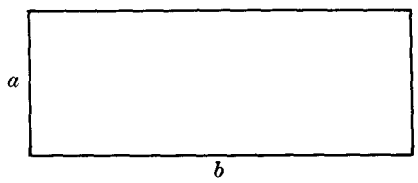


Fig. 4-1

PARALLELOGRAM OF ALTITUDE h AND BASE b

4.3 Area $= bh = ab \sin\theta$

4.4 Perimeter $= 2a + 2b$

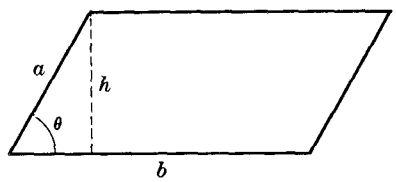


Fig. 4-2

TRIANGLE OF ALTITUDE h AND BASE b

4.5 Area $= \frac{1}{2}bh = \frac{1}{2}ab \sin\theta$

$= \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{1}{2}(a+b+c) =$ semiperimeter

4.6 Perimeter $= a + b + c$

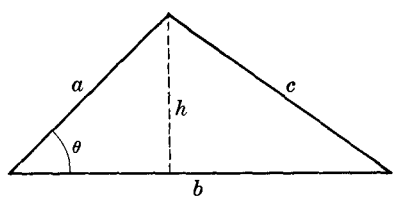


Fig. 4-3

TRAPEZOID OF ALTITUDE h AND PARALLEL SIDES a AND b

4.7 Area $= \frac{1}{2}h(a+b)$

4.8 Perimeter $= a + b + h\left(\frac{1}{\sin\theta} + \frac{1}{\sin\phi}\right)$

$= a + b + h(\csc\theta + \csc\phi)$

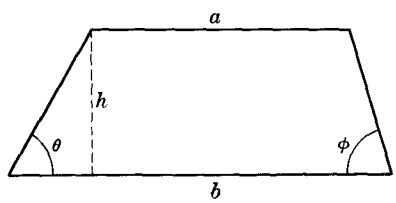


Fig. 4-4

REGULAR POLYGON OF n SIDES EACH OF LENGTH b

4.9 Area $= \frac{1}{4}nb^2 \cot\frac{\pi}{n} = \frac{1}{4}nb^2 \frac{\cos(\pi/n)}{\sin(\pi/n)}$

4.10 Perimeter $= nb$

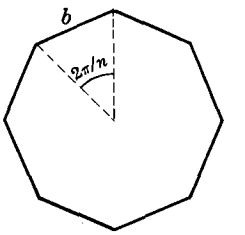


Fig. 4-5

CIRCLE OF RADIUS r

4.11 Area $= \pi r^2$

4.12 Perimeter $= 2\pi r$

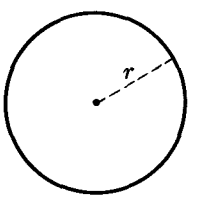


Fig. 4-6

SECTOR OF CIRCLE OF RADIUS r

4.13 Area $= \frac{1}{2}r^2\theta$ [θ in radians]

4.14 Arc length $s = r\theta$

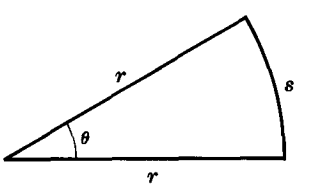


Fig. 4-7

RADIUS OF CIRCLE INSCRIBED IN A TRIANGLE OF SIDES a, b, c

4.15 $$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

where $s = \frac{1}{2}(a+b+c) =$ semiperimeter

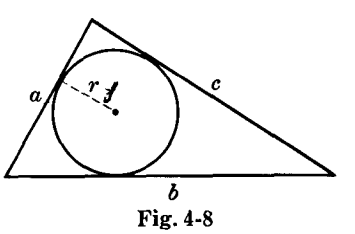


Fig. 4-8

RADIUS OF CIRCLE CIRCUMSCRIBING A TRIANGLE OF SIDES a, b, c

4.16 $$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

where $s = \frac{1}{2}(a+b+c) =$ semiperimeter

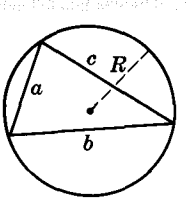


Fig. 4-9

REGULAR POLYGON OF n SIDES INSCRIBED IN CIRCLE OF RADIUS r

4.17 Area $= \frac{1}{2}nr^2 \sin\frac{2\pi}{n} = \frac{1}{2}nr^2 \sin\frac{360^\circ}{n}$

4.18 Perimeter $= 2nr \sin\frac{\pi}{n} = 2nr \sin\frac{180^\circ}{n}$

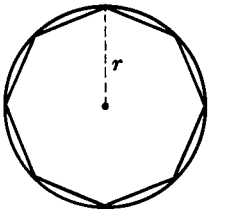


Fig. 4-10

REGULAR POLYGON OF n SIDES CIRCUMSCRIBING A CIRCLE OF RADIUS r

4.19 Area $= nr^2 \tan\frac{\pi}{n} = nr^2 \tan\frac{180^\circ}{n}$

4.20 Perimeter $= 2nr \tan\frac{\pi}{n} = 2nr \tan\frac{180^\circ}{n}$

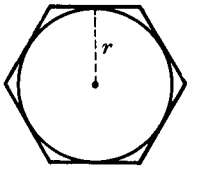


Fig. 4-11

SEGMENT OF CIRCLE OF RADIUS r

4.21 Area of shaded part $= \frac{1}{2}r^2(\theta - \sin\theta)$

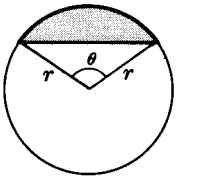


Fig. 4-12

ELLIPSE OF SEMI-MAJOR AXIS a AND SEMI-MINOR AXIS b

4.22 Area $= \pi ab$

4.23 Perimeter $= 4a\int_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\, d\theta$

$= 2\pi\sqrt{\frac{1}{2}(a^2 + b^2)}$ [approximately]

where $k = \sqrt{a^2 - b^2}/a$. See page 254 for numerical tables.

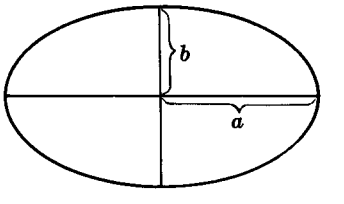


Fig. 4-13

SEGMENT OF A PARABOLA

4.24 Area $= \frac{2}{3}ab$

4.25 Arc length $ABC = \frac{1}{2}\sqrt{b^2 + 16a^2} + \frac{b^2}{8a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$

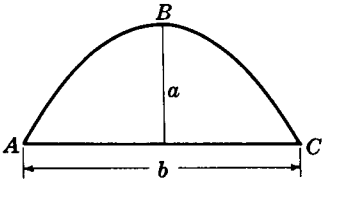


Fig. 4-14

RECTANGULAR PARALLELEPIPED OF LENGTH a, HEIGHT l, WIDTH c

4.26 Volume $= abc$

4.27 Surface area $= 2(ab + ac + bc)$

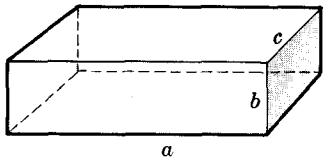


Fig. 4-15

PARALLELEPIPED OF CROSS-SECTIONAL AREA A AND HEIGHT h

4.28 Volume $= Ah = abc \sin\theta$

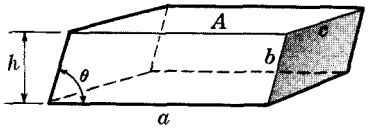


Fig. 4-16

SPHERE OF RADIUS r

4.29 Volume $= \frac{4}{3}\pi r^3$

4.30 Surface area $= 4\pi r^2$

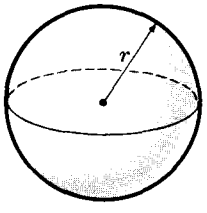


Fig. 4-17

RIGHT CIRCULAR CYLINDER OF RADIUS r AND HEIGHT h

4.31 Volume $= \pi r^2 h$

4.32 Lateral surface area $= 2\pi rh$

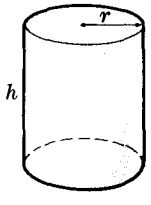


Fig. 4-18

CIRCULAR CYLINDER OF RADIUS r AND SLANT HEIGHT l

4.33 Volume $= \pi r^2 h = \pi r^2 l \sin\theta$

4.34 Lateral surface area $= 2\pi rl = \dfrac{2\pi rh}{\sin\theta} = 2\pi rh \csc\theta$

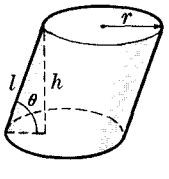


Fig. 4-19

CYLINDER OF CROSS-SECTIONAL AREA A AND SLANT HEIGHT l

4.35 Volume $= Ah = Al \sin\theta$

4.36 Lateral surface area $= pl = \dfrac{ph}{\sin\theta} = ph \csc\theta$

Note that formulas 4.31 to 4.34 are special cases.

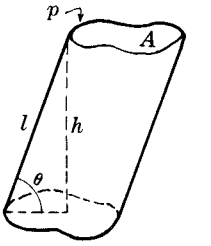


Fig. 4-20

RIGHT CIRCULAR CONE OF RADIUS r AND HEIGHT h

4.37 Volume $= \frac{1}{3}\pi r^2 h$

4.38 Lateral surface area $= \pi r\sqrt{r^2 + h^2} = \pi r l$

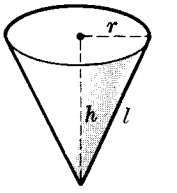


Fig. 4-21

PYRAMID OF BASE AREA A AND HEIGHT h

4.39 Volume $= \frac{1}{3}Ah$

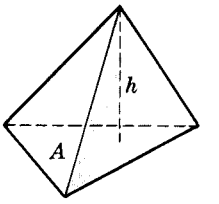


Fig. 4-22

SPHERICAL CAP OF RADIUS r AND HEIGHT h

4.40 Volume (shaded in figure) $= \frac{1}{3}\pi h^2(3r - h)$

4.41 Surface area $= 2\pi r h$

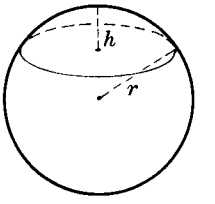


Fig. 4-23

FRUSTRUM OF RIGHT CIRCULAR CONE OF RADII a, b AND HEIGHT h

4.42 Volume $= \frac{1}{3}\pi h(a^2 + ab + b^2)$

4.43 Lateral surface area $= \pi(a + b)\sqrt{h^2 + (b - a)^2}$
$= \pi(a + b)l$

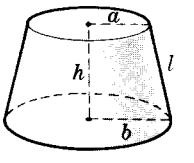


Fig. 4-24

SPHERICAL TRIANGLE OF ANGLES A, B, C ON SPHERE OF RADIUS r

4.44 Area of triangle $ABC = (A + B + C - \pi)r^2$

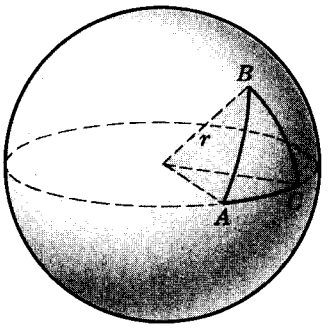


Fig. 4-25

TORUS OF INNER RADIUS a AND OUTER RADIUS b

4.45 Volume $= \frac{1}{4}\pi^2(a + b)(b - a)^2$

4.46 Surface area $= \pi^2(b^2 - a^2)$

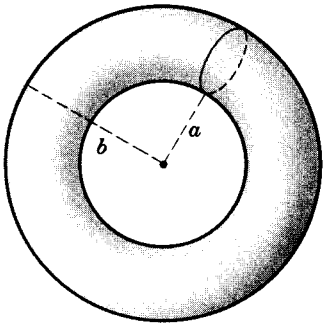


Fig. 4-26

ELLIPSOID OF SEMI-AXES a, b, c

4.47 Volume $= \frac{4}{3}\pi abc$

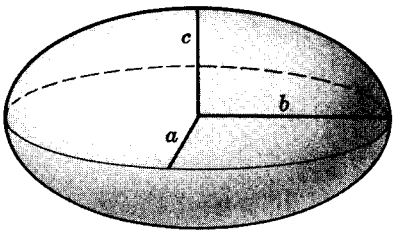


Fig. 4-27

PARABOLOID OF REVOLUTION

4.48 Volume $= \frac{1}{2}\pi b^2 a$

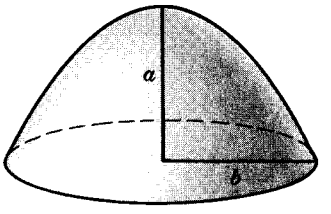


Fig. 4-28

5 TRIGONOMETRIC FUNCTIONS

DEFINITION OF TRIGONOMETRIC FUNCTIONS FOR A RIGHT TRIANGLE

Triangle ABC has a right angle (90°) at C and sides of length a, b, c. The trigonometric functions of angle A are defined as follows.

5.1 *sine* of $A = \sin A = \frac{a}{c} = \frac{\text{opposite}}{\text{hypotenuse}}$

5.2 *cosine* of $A = \cos A = \frac{b}{c} = \frac{\text{adjacent}}{\text{hypotenuse}}$

5.3 *tangent* of $A = \tan A = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent}}$

5.4 *cotangent* of $A = \cot A = \frac{b}{a} = \frac{\text{adjacent}}{\text{opposite}}$

5.5 *secant* of $A = \sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent}}$

5.6 *cosecant* of $A = \csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite}}$

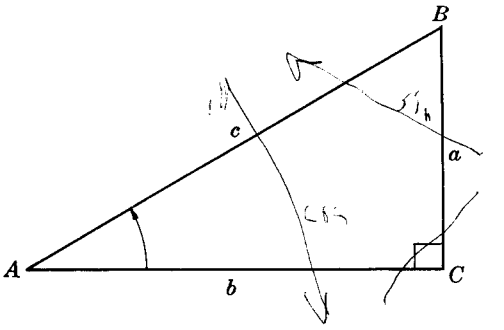


Fig. 5-1

EXTENSIONS TO ANGLES WHICH MAY BE GREATER THAN 90°

Consider an xy coordinate system [see Fig. 5-2 and 5-3 below]. A point P in the xy plane has coordinates (x, y) where x is considered as positive along OX and negative along OX' while y is positive along OY and negative along OY'. The distance from origin O to point P is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle A described *counterclockwise* from OX is considered *positive*. If it is described *clockwise* from OX it is considered *negative*. We call $X'OX$ and $Y'OY$ the x and y axis respectively.

The various quadrants are denoted by I, II, III and IV called the first, second, third and fourth quadrants respectively. In Fig. 5-2, for example, angle A is in the second quadrant while in Fig. 5-3 angle A is in the third quadrant.

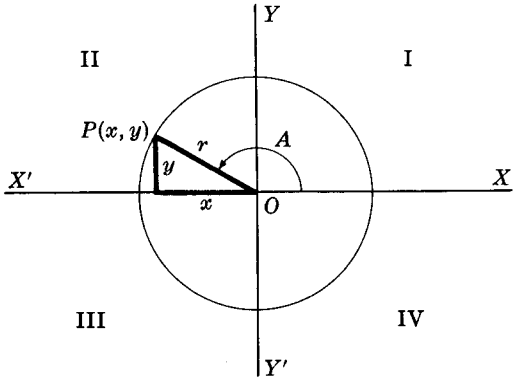


Fig. 5-2

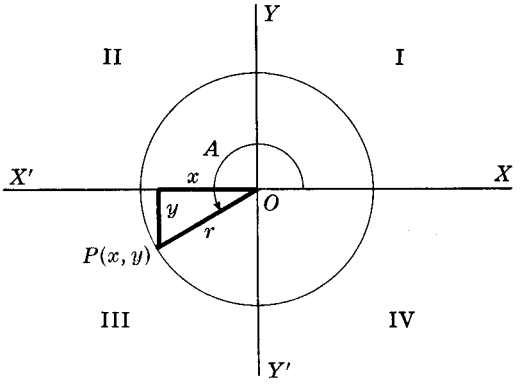


Fig. 5-3

For an angle A in any quadrant the trigonometric functions of A are defined as follows.

5.7 $\sin A = y/r$

5.8 $\cos A = x/r$

5.9 $\tan A = y/x$

5.10 $\cot A = x/y$

5.11 $\sec A = r/x$

5.12 $\csc A = r/y$

RELATIONSHIP BETWEEN DEGREES AND RADIANS

A *radian* is that angle θ subtended at center O of a circle by an arc MN equal to the radius r.

Since 2π radians $= 360°$ we have

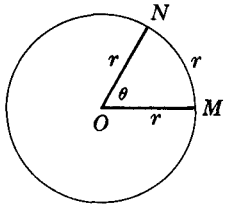


Fig. 5-4

5.13 1 radian $= 180°/\pi = 57.29577\ 95130\ 8232\ldots°$

5.14 $1° = \pi/180$ radians $= 0.01745\ 32925\ 19943\ 29576\ 92\ldots$radians

RELATIONSHIPS AMONG TRIGONOMETRIC FUNCTIONS

5.15 $\tan A = \dfrac{\sin A}{\cos A}$

5.16 $\cot A = \dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A}$

5.17 $\sec A = \dfrac{1}{\cos A}$

5.18 $\csc A = \dfrac{1}{\sin A}$

5.19 $\sin^2 A + \cos^2 A = 1$

5.20 $\sec^2 A - \tan^2 A = 1$

5.21 $\csc^2 A - \cot^2 A = 1$

SIGNS AND VARIATIONS OF TRIGONOMETRIC FUNCTIONS

Quadrant	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
I	+ 0 to 1	+ 1 to 0	+ 0 to ∞	+ ∞ to 0	+ 1 to ∞	+ ∞ to 1
II	+ 1 to 0	− 0 to −1	− $-\infty$ to 0	− 0 to $-\infty$	− $-\infty$ to −1	+ 1 to ∞
III	− 0 to −1	− −1 to 0	+ 0 to ∞	+ ∞ to 0	− −1 to $-\infty$	− $-\infty$ to −1
IV	− −1 to 0	+ 0 to 1	− $-\infty$ to 0	− 0 to $-\infty$	+ ∞ to 1	− −1 to $-\infty$

EXACT VALUES FOR TRIGONOMETRIC FUNCTIONS OF VARIOUS ANGLES

Angle A in degrees	Angle A in radians	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
0°	0	0	1	0	∞	1	∞
15°	$\pi/12$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
30°	$\pi/6$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
75°	$5\pi/12$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
90°	$\pi/2$	1	0	$\pm\infty$	0	$\pm\infty$	1
105°	$7\pi/12$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$-(2-\sqrt{3})$	$-(\sqrt{6}+\sqrt{2})$	$\sqrt{6}-\sqrt{2}$
120°	$2\pi/3$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$3\pi/4$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$5\pi/6$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
165°	$11\pi/12$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-(2-\sqrt{3})$	$-(2+\sqrt{3})$	$-(\sqrt{6}-\sqrt{2})$	$\sqrt{6}+\sqrt{2}$
180°	π	0	-1	0	$\mp\infty$	-1	$\pm\infty$
195°	$13\pi/12$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$	$-(\sqrt{6}-\sqrt{2})$	$-(\sqrt{6}+\sqrt{2})$
210°	$7\pi/6$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$5\pi/4$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$4\pi/3$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
255°	$17\pi/12$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$	$-(\sqrt{6}+\sqrt{2})$	$-(\sqrt{6}-\sqrt{2})$
270°	$3\pi/2$	-1	0	$\pm\infty$	0	$\mp\infty$	-1
285°	$19\pi/12$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$-(2-\sqrt{3})$	$\sqrt{6}+\sqrt{2}$	$-(\sqrt{6}-\sqrt{2})$
300°	$5\pi/3$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$7\pi/4$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$11\pi/6$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
345°	$23\pi/12$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-(2-\sqrt{3})$	$-(2+\sqrt{3})$	$\sqrt{6}-\sqrt{2}$	$-(\sqrt{6}+\sqrt{2})$
360°	2π	0	1	0	$\mp\infty$	1	$\mp\infty$

For tables involving other angles see pages 206-211 and 212-215.

GRAPHS OF TRIGONOMETRIC FUNCTIONS

In each graph x is in radians.

5.22 $y = \sin x$

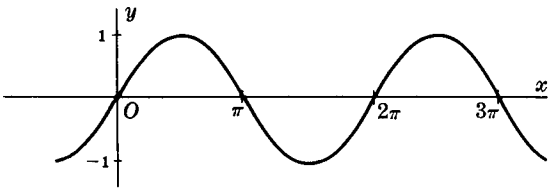


Fig. 5-5

5.23 $y = \cos x$

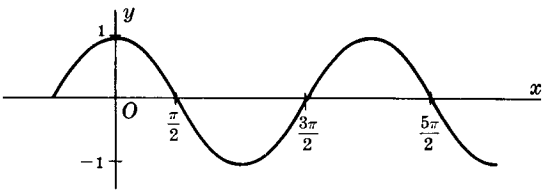


Fig. 5-6

5.24 $y = \tan x$

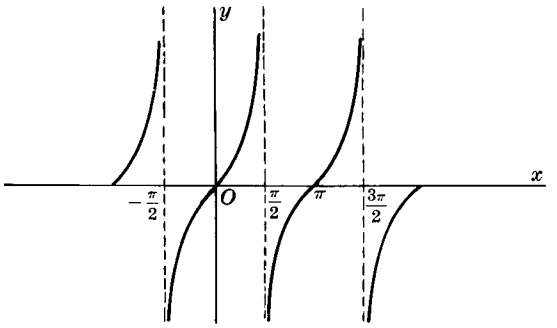


Fig. 5-7

5.25 $y = \cot x$

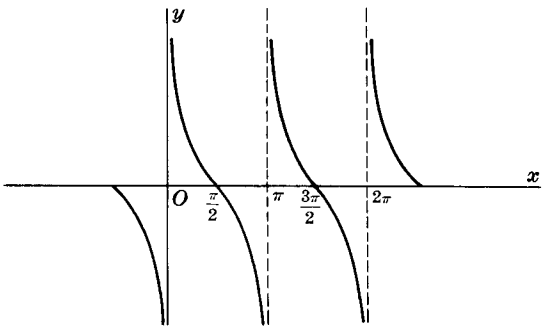


Fig. 5-8

5.26 $y = \sec x$

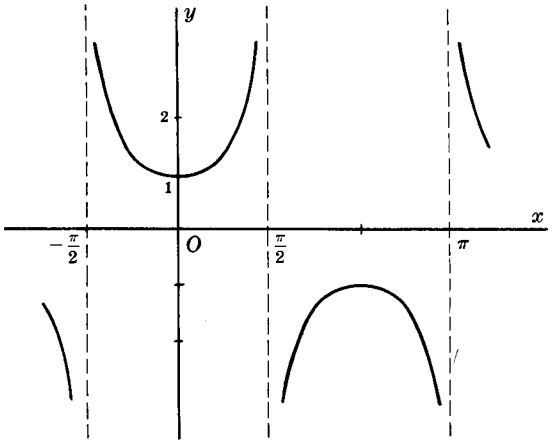


Fig. 5-9

5.27 $y = \csc x$

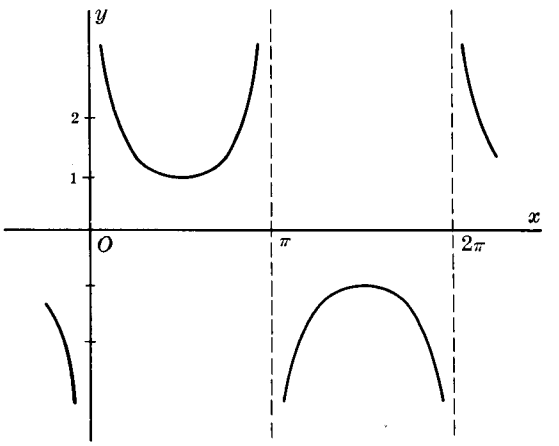


Fig. 5-10

FUNCTIONS OF NEGATIVE ANGLES

5.28 $\sin(-A) = -\sin A$

5.29 $\cos(-A) = \cos A$

5.30 $\tan(-A) = -\tan A$

5.31 $\csc(-A) = -\csc A$

5.32 $\sec(-A) = \sec A$

5.33 $\cot(-A) = -\cot A$

ADDITION FORMULAS

5.34 $$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

5.35 $$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

5.36 $$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

5.37 $$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

FUNCTIONS OF ANGLES IN ALL QUADRANTS IN TERMS OF THOSE IN QUADRANT I

	$-A$	$90^\circ \pm A$ $\frac{\pi}{2} \pm A$	$180^\circ \pm A$ $\pi \pm A$	$270^\circ \pm A$ $\frac{3\pi}{2} \pm A$	$k(360^\circ) \pm A$ $2k\pi \pm A$ k = integer
sin	$-\sin A$	$\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$
cos	$\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$	$\cos A$
tan	$-\tan A$	$\mp \cot A$	$\pm \tan A$	$\mp \cot A$	$\pm \tan A$
csc	$-\csc A$	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$
sec	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$	$\sec A$
cot	$-\cot A$	$\mp \tan A$	$\pm \cot A$	$\mp \tan A$	$\pm \cot A$

RELATIONSHIPS AMONG FUNCTIONS OF ANGLES IN QUADRANT I

	$\sin A = u$	$\cos A = u$	$\tan A = u$	$\cot A = u$	$\sec A = u$	$\csc A = u$
$\sin A$	u	$\sqrt{1-u^2}$	$u/\sqrt{1+u^2}$	$1/\sqrt{1+u^2}$	$\sqrt{u^2-1}/u$	$1/u$
$\cos A$	$\sqrt{1-u^2}$	u	$1/\sqrt{1+u^2}$	$u/\sqrt{1+u^2}$	$1/u$	$\sqrt{u^2-1}/u$
$\tan A$	$u/\sqrt{1-u^2}$	$\sqrt{1-u^2}/u$	u	$1/u$	$\sqrt{u^2-1}$	$1/\sqrt{u^2-1}$
$\cot A$	$\sqrt{1-u^2}/u$	$u/\sqrt{1-u^2}$	$1/u$	u	$1/\sqrt{u^2-1}$	$\sqrt{u^2-1}$
$\sec A$	$1/\sqrt{1-u^2}$	$1/u$	$\sqrt{1+u^2}$	$\sqrt{1+u^2}/u$	u	$u/\sqrt{u^2-1}$
$\csc A$	$1/u$	$1/\sqrt{1-u^2}$	$\sqrt{1+u^2}/u$	$\sqrt{1+u^2}$	$u/\sqrt{u^2-1}$	$\sqrt{1+u^2}$

For extensions to other quadrants use appropriate signs as given in the preceding table.

DOUBLE ANGLE FORMULAS

5.38 $\sin 2A = 2 \sin A \cos A$

5.39 $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$

5.40 $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

HALF ANGLE FORMULAS

5.41 $\sin \dfrac{A}{2} = \pm \sqrt{\dfrac{1 - \cos A}{2}}$ $\left[\begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or II} \\ - \text{ if } A/2 \text{ is in quadrant III or IV} \end{array}\right]$

5.42 $\cos \dfrac{A}{2} = \pm \sqrt{\dfrac{1 + \cos A}{2}}$ $\left[\begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or IV} \\ - \text{ if } A/2 \text{ is in quadrant II or III} \end{array}\right]$

5.43 $\tan \dfrac{A}{2} = \pm \sqrt{\dfrac{1 - \cos A}{1 + \cos A}}$ $\left[\begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or III} \\ - \text{ if } A/2 \text{ is in quadrant II or IV} \end{array}\right]$

$= \dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A} = \csc A - \cot A$

MULTIPLE ANGLE FORMULAS

5.44 $\sin 3A = 3 \sin A - 4 \sin^3 A$

5.45 $\cos 3A = 4 \cos^3 A - 3 \cos A$

5.46 $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

5.47 $\sin 4A = 4 \sin A \cos A - 8 \sin^3 A \cos A$

5.48 $\cos 4A = 8 \cos^4 A - 8 \cos^2 A + 1$

5.49 $\tan 4A = \dfrac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$

5.50 $\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A$

5.51 $\cos 5A = 16 \cos^5 A - 20 \cos^3 A + 5 \cos A$

5.52 $\tan 5A = \dfrac{\tan^5 A - 10 \tan^3 A + 5 \tan A}{1 - 10 \tan^2 A + 5 \tan^4 A}$

See also formulas 5.68 and 5.69.

POWERS OF TRIGONOMETRIC FUNCTIONS

5.53 $\sin^2 A = \frac{1}{2} - \frac{1}{2} \cos 2A$

5.54 $\cos^2 A = \frac{1}{2} + \frac{1}{2} \cos 2A$

5.55 $\sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A$

5.56 $\cos^3 A = \frac{3}{4} \cos A + \frac{1}{4} \cos 3A$

5.57 $\sin^4 A = \frac{3}{8} - \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A$

5.58 $\cos^4 A = \frac{3}{8} + \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A$

5.59 $\sin^5 A = \frac{5}{8} \sin A - \frac{5}{16} \sin 3A + \frac{1}{16} \sin 5A$

5.60 $\cos^5 A = \frac{5}{8} \cos A + \frac{5}{16} \cos 3A + \frac{1}{16} \cos 5A$

See also formulas 5.70 through 5.73.

SUM, DIFFERENCE AND PRODUCT OF TRIGONOMETRIC FUNCTIONS

5.61 $$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B)$$

5.62 $$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B)$$

5.63 $$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B)$$

5.64 $$\cos A - \cos B = 2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(B-A)$$

5.65 $$\sin A \sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

5.66 $$\cos A \cos B = \tfrac{1}{2}\{\cos(A-B) + \cos(A+B)\}$$

5.67 $$\sin A \cos B = \tfrac{1}{2}\{\sin(A-B) + \sin(A+B)\}$$

GENERAL FORMULAS

5.68 $$\sin nA = \sin A \left\{(2\cos A)^{n-1} - \binom{n-2}{1}(2\cos A)^{n-3} + \binom{n-3}{2}(2\cos A)^{n-5} - \cdots\right\}$$

5.69 $$\cos nA = \frac{1}{2}\left\{(2\cos A)^n - \frac{n}{1}(2\cos A)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos A)^{n-4} - \frac{n}{3}\binom{n-4}{2}(2\cos A)^{n-6} + \cdots\right\}$$

5.70 $$\sin^{2n-1} A = \frac{(-1)^{n-1}}{2^{2n-2}}\left\{\sin(2n-1)A - \binom{2n-1}{1}\sin(2n-3)A + \cdots (-1)^{n-1}\binom{2n-1}{n-1}\sin A\right\}$$

5.71 $$\cos^{2n-1} A = \frac{1}{2^{2n-2}}\left\{\cos(2n-1)A + \binom{2n-1}{1}\cos(2n-3)A + \cdots + \binom{2n-1}{n-1}\cos A\right\}$$

5.72 $$\sin^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\left\{\cos 2nA - \binom{2n}{1}\cos(2n-2)A + \cdots (-1)^{n-1}\binom{2n}{n-1}\cos 2A\right\}$$

5.73 $$\cos^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\left\{\cos 2nA + \binom{2n}{1}\cos(2n-2)A + \cdots + \binom{2n}{n-1}\cos 2A\right\}$$

INVERSE TRIGONOMETRIC FUNCTIONS

If $x = \sin y$ then $y = \sin^{-1} x$, i.e. *the angle whose sine is* x or *inverse sine of* x, is a many-valued function of x which is a collection of single-valued functions called *branches*. Similarly the other inverse trigonometric functions are multiple-valued.

For many purposes a particular branch is required. This is called the *principal branch* and the values for this branch are called *principal values*.

PRINCIPAL VALUES FOR INVERSE TRIGONOMETRIC FUNCTIONS

Principal values for $x \geqq 0$	Principal values for $x < 0$
$0 \leqq \sin^{-1} x \leqq \pi/2$	$-\pi/2 \leqq \sin^{-1} x < 0$
$0 \leqq \cos^{-1} x \leqq \pi/2$	$\pi/2 < \cos^{-1} x \leqq \pi$
$0 \leqq \tan^{-1} x < \pi/2$	$-\pi/2 < \tan^{-1} x < 0$
$0 < \cot^{-1} x \leqq \pi/2$	$\pi/2 < \cot^{-1} x < \pi$
$0 \leqq \sec^{-1} x < \pi/2$	$\pi/2 < \sec^{-1} x \leqq \pi$
$0 < \csc^{-1} x \leqq \pi/2$	$-\pi/2 \leqq \csc^{-1} x < 0$

RELATIONS BETWEEN INVERSE TRIGONOMETRIC FUNCTIONS

In all cases it is assumed that principal values are used.

5.74 $\sin^{-1} x + \cos^{-1} x = \pi/2$

5.75 $\tan^{-1} x + \cot^{-1} x = \pi/2$

5.76 $\sec^{-1} x + \csc^{-1} x = \pi/2$

5.77 $\csc^{-1} x = \sin^{-1}(1/x)$

5.78 $\sec^{-1} x = \cos^{-1}(1/x)$

5.79 $\cot^{-1} x = \tan^{-1}(1/x)$

5.80 $\sin^{-1}(-x) = -\sin^{-1} x$

5.81 $\cos^{-1}(-x) = \pi - \cos^{-1} x$

5.82 $\tan^{-1}(-x) = -\tan^{-1} x$

5.83 $\cot^{-1}(-x) = \pi - \cot^{-1} x$

5.84 $\sec^{-1}(-x) = \pi - \sec^{-1} x$

5.85 $\csc^{-1}(-x) = -\csc^{-1} x$

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

In each graph y is in radians. Solid portions of curves correspond to principal values.

5.86 $y = \sin^{-1} x$

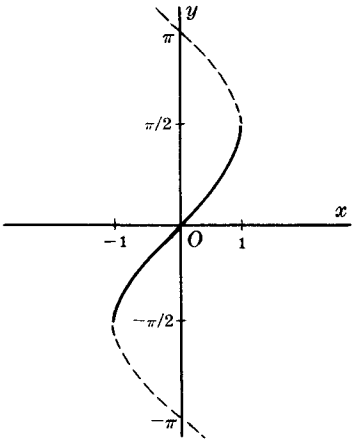


Fig. 5-11

5.87 $y = \cos^{-1} x$

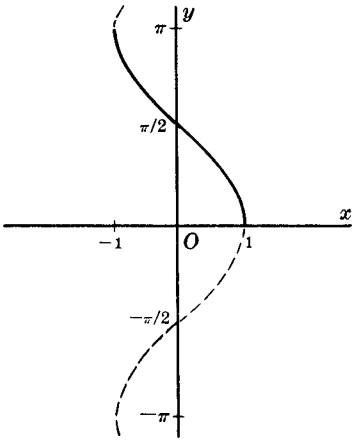


Fig. 5-12

5.88 $y = \tan^{-1} x$

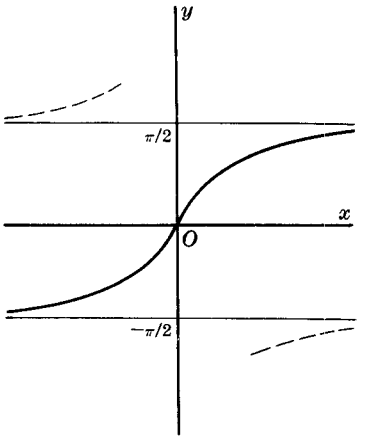


Fig. 5-13

5.89 $y = \cot^{-1} x$

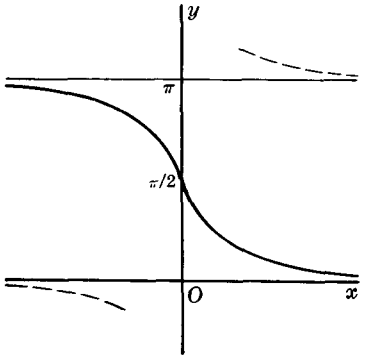


Fig. 5-14

5.90 $y = \sec^{-1} x$

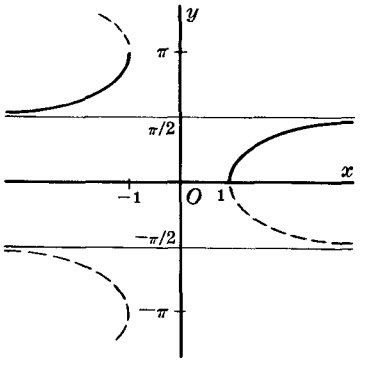


Fig. 5-15

5.91 $y = \csc^{-1} x$

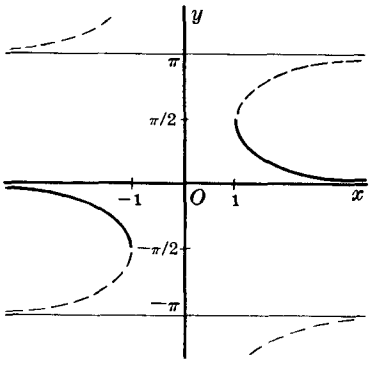


Fig. 5-16

RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A PLANE TRIANGLE

The following results hold for any plane triangle ABC with sides a, b, c and angles A, B, C.

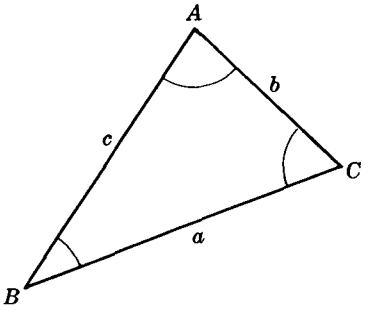


Fig. 5-17

5.92 Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

5.93 Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

with similar relations involving the other sides and angles.

5.94 Law of Tangents

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

with similar relations involving the other sides and angles.

5.95

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle. Similar relations involving angles B and C can be obtained.

See also formulas 4.5, page 5; 4.15 and 4.16, page 6.

RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A SPHERICAL TRIANGLE

Spherical triangle ABC is on the surface of a sphere as shown in Fig. 5-18. Sides a, b, c [which are arcs of great circles] are measured by their angles subtended at center O of the sphere. A, B, C are the angles opposite sides a, b, c respectively. Then the following results hold.

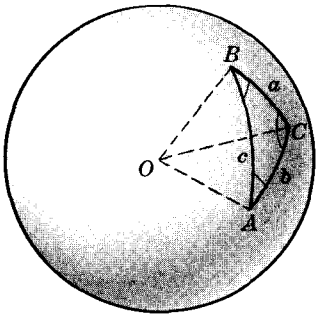


Fig. 5-18

5.96 Law of Sines

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

5.97 Law of Cosines

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

with similar results involving other sides and angles.

5.98 Law of Tangents

$$\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

with similar results involving other sides and angles.

5.99

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin (s-c)}{\sin b \sin c}}$$

where $s = \frac{1}{2}(a+b+c)$. Similar results hold for other sides and angles.

5.100

$$\cos \frac{a}{2} = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}$$

where $S = \frac{1}{2}(A+B+C)$. Similar results hold for other sides and angles.

See also formula 4.44, page 10.

NAPIER'S RULES FOR RIGHT ANGLED SPHERICAL TRIANGLES

Except for right angle C, there are five parts of spherical triangle ABC which if arranged in the order as given in Fig. 5-19 would be a, b, A, c, B.

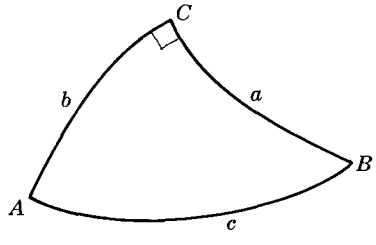


Fig. 5-19

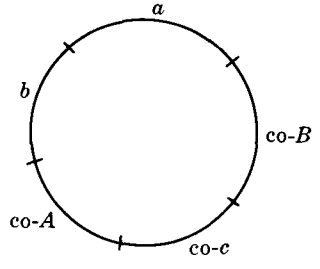


Fig. 5-20

Suppose these quantities are arranged in a circle as in Fig. 5-20 where we attach the prefix co [indicating *complement*] to hypotenuse c and angles A and B.

Any one of the parts of this circle is called a *middle part*, the two neighboring parts are called *adjacent parts* and the two remaining parts are called *opposite parts*. Then Napier's rules are

5.101 The sine of any middle part equals the product of the tangents of the adjacent parts.

5.102 The sine of any middle part equals the product of the cosines of the opposite parts.

Example: Since co-$A = 90° - A$, co-$B = 90° - B$, we have

$$\sin a = \tan b \tan (\text{co-}B) \quad \text{or} \quad \sin a = \tan b \cot B$$

$$\sin (\text{co-}A) = \cos a \cos (\text{co-}B) \quad \text{or} \quad \cos A = \cos a \sin B$$

These can of course be obtained also from the results 5.97 on page 19.

6 COMPLEX NUMBERS

DEFINITIONS INVOLVING COMPLEX NUMBERS

A *complex number* is generally written as $a + bi$ where a and b are real numbers and i, called the *imaginary unit*, has the property that $i^2 = -1$. The real numbers a and b are called the *real* and *imaginary parts* of $a + bi$ respectively.

The complex numbers $a + bi$ and $a - bi$ are called *complex conjugates* of each other.

EQUALITY OF COMPLEX NUMBERS

6.1 $$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

ADDITION OF COMPLEX NUMBERS

6.2 $$(a + bi) + (c + di) = (a + c) + (b + d)i$$

SUBTRACTION OF COMPLEX NUMBERS

6.3 $$(a + bi) - (c + di) = (a - c) + (b - d)i$$

MULTIPLICATION OF COMPLEX NUMBERS

6.4 $$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

DIVISION OF COMPLEX NUMBERS

6.5 $$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right) i$$

Note that the above operations are obtained by using the ordinary rules of algebra and replacing i^2 by -1 wherever it occurs.

GRAPH OF A COMPLEX NUMBER

A complex number $a + bi$ can be plotted as a point (a, b) on an xy plane called an *Argand diagram* or *Gaussian plane*. For example in Fig. 6-1 P represents the complex number $-3 + 4i$.

A complex number can also be interpreted as a *vector* from O to P.

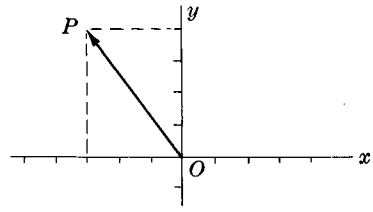


Fig. 6-1

POLAR FORM OF A COMPLEX NUMBER

In Fig. 6-2 point P with coordinates (x, y) represents the complex number $x + iy$. Point P can also be represented by *polar coordinates* (r, θ). Since $x = r\cos\theta$, $y = r\sin\theta$ we have

6.6 $$x + iy = r(\cos\theta + i\sin\theta)$$

called the *polar form* of the complex number. We often call $r = \sqrt{x^2 + y^2}$ the *modulus* and θ the *amplitude* of $x + iy$.

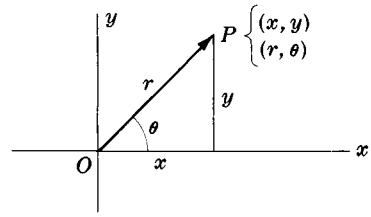


Fig. 6-2

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS IN POLAR FORM

6.7 $$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

6.8 $$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

DE MOIVRE'S THEOREM

If p is any real number, De Moivre's theorem states that

6.9 $$[r(\cos\theta + i\sin\theta)]^p = r^p(\cos p\theta + i\sin p\theta)$$

ROOTS OF COMPLEX NUMBERS

If $p = 1/n$ where n is any positive integer, 6.9 can be written

6.10 $$[r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right]$$

where k is any integer. From this the n nth roots of a complex number can be obtained by putting $k = 0, 1, 2, \ldots, n - 1$.

7 EXPONENTIAL and LOGARITHMIC FUNCTIONS

LAWS OF EXPONENTS

In the following p, q are real numbers, a, b are positive numbers and m, n are positive integers.

7.1 $a^p \cdot a^q = a^{p+q}$ **7.2** $a^p/a^q = a^{p-q}$ **7.3** $(a^p)^q = a^{pq}$

7.4 $a^0 = 1, \ a \neq 0$ **7.5** $a^{-p} = 1/a^p$ **7.6** $(ab)^p = a^p b^p$

7.7 $\sqrt[n]{a} = a^{1/n}$ **7.8** $\sqrt[n]{a^m} = a^{m/n}$ **7.9** $\sqrt[n]{a/b} = \sqrt[n]{a}/\sqrt[n]{b}$

In a^p, p is called the *exponent*, a is the *base* and a^p is called the pth *power of* a. The function $y = a^x$ is called an *exponential function.*

LOGARITHMS AND ANTILOGARITHMS

If $a^p = N$ where $a \neq 0$ or 1, then $p = \log_a N$ is called the *logarithm* of N to the base a. The number $N = a^p$ is called the *antilogarithm* of p to the base a, written $\text{antilog}_a p$.

Example: Since $3^2 = 9$ we have $\log_3 9 = 2$, $\text{antilog}_3 2 = 9$.

The function $y = \log_a x$ is called a *logarithmic function.*

LAWS OF LOGARITHMS

7.10 $\log_a MN = \log_a M + \log_a N$

7.11 $\log_a \frac{M}{N} = \log_a M - \log_a N$

7.12 $\log_a M^p = p \log_a M$

COMMON LOGARITHMS AND ANTILOGARITHMS

Common logarithms and antilogarithms [also called *Briggsian*] are those in which the base $a = 10$. The common logarithm of N is denoted by $\log_{10} N$ or briefly $\log N$. For tables of common logarithms and antilogarithms, see pages 202-205. For illustrations using these tables see pages 194-196.

NATURAL LOGARITHMS AND ANTILOGARITHMS

Natural logarithms and antilogarithms [also called Napierian] are those in which the base $a = e = 2.71828\,18\ldots$ [see page 1]. The natural logarithm of N is denoted by $\log_e N$ or $\ln N$. For tables of natural logarithms see pages 224-225. For tables of natural antilogarithms [i.e. tables giving e^x for values of x] see pages 226-227. For illustrations using these tables see pages 196 and 200.

CHANGE OF BASE OF LOGARITHMS

The relationship between logarithms of a number N to different bases a and b is given by

7.13 $$\log_a N = \frac{\log_b N}{\log_b a}$$

In particular,

7.14 $$\log_e N = \ln N = 2.30258\ 50929\ 94\ldots \log_{10} N$$

7.15 $$\log_{10} N = \log N = 0.43429\ 44819\ 03\ldots \log_e N$$

RELATIONSHIP BETWEEN EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

7.16 $$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

These are called *Euler's identities*. Here i is the imaginary unit [see page 21].

7.17 $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

7.18 $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

7.19 $$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

7.20 $$\cot\theta = i\left(\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}\right)$$

7.21 $$\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$$

7.22 $$\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$$

PERIODICITY OF EXPONENTIAL FUNCTIONS

7.23 $$e^{i(\theta + 2k\pi)} = e^{i\theta} \qquad k = \text{integer}$$

From this it is seen that e^x has period $2\pi i$.

POLAR FORM OF COMPLEX NUMBERS EXPRESSED AS AN EXPONENTIAL

The polar form of a complex number $x + iy$ can be written in terms of exponentials [see 6.6, page 22] as

7.24 $$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

OPERATIONS WITH COMPLEX NUMBERS IN POLAR FORM

Formulas 6.7 through 6.10 on page 22 are equivalent to the following.

7.25 $$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$$

7.26 $$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

7.27 $$(re^{i\theta})^p = r^pe^{ip\theta} \qquad \text{[De Moivre's theorem]}$$

7.28 $$(re^{i\theta})^{1/n} = [re^{i(\theta+2k\pi)}]^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}$$

LOGARITHM OF A COMPLEX NUMBER

7.29 $$\ln(re^{i\theta}) = \ln r + i\theta + 2k\pi i \qquad k = \text{integer}$$

8 HYPERBOLIC FUNCTIONS

DEFINITION OF HYPERBOLIC FUNCTIONS

8.1 *Hyperbolic sine* of x $= \sinh x = \dfrac{e^x - e^{-x}}{2}$

8.2 *Hyperbolic cosine* of x $= \cosh x = \dfrac{e^x + e^{-x}}{2}$

8.3 *Hyperbolic tangent* of x $= \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

8.4 *Hyperbolic cotangent* of x $= \coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

8.5 *Hyperbolic secant* of x $= \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

8.6 *Hyperbolic cosecant* of x $= \operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$

RELATIONSHIPS AMONG HYPERBOLIC FUNCTIONS

8.7 $\tanh x = \dfrac{\sinh x}{\cosh x}$

8.8 $\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$

8.9 $\operatorname{sech} x = \dfrac{1}{\cosh x}$

8.10 $\operatorname{csch} x = \dfrac{1}{\sinh x}$

8.11 $\cosh^2 x - \sinh^2 x = 1$

8.12 $\operatorname{sech}^2 x + \tanh^2 x = 1$

8.13 $\coth^2 x - \operatorname{csch}^2 x = 1$

FUNCTIONS OF NEGATIVE ARGUMENTS

8.14 $\sinh(-x) = -\sinh x$ **8.15** $\cosh(-x) = \cosh x$ **8.16** $\tanh(-x) = -\tanh x$

8.17 $\operatorname{csch}(-x) = -\operatorname{csch} x$ **8.18** $\operatorname{sech}(-x) = \operatorname{sech} x$ **8.19** $\coth(-x) = -\coth x$

ADDITION FORMULAS

8.20 $$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

8.21 $$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

8.22 $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

8.23 $$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$$

DOUBLE ANGLE FORMULAS

8.24 $$\sinh 2x = 2 \sinh x \cosh x$$

8.25 $$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

8.26 $$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$$

HALF ANGLE FORMULAS

8.27 $$\sinh\frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}} \quad [+ \text{ if } x > 0, \; - \text{ if } x < 0]$$

8.28 $$\cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

8.29 $$\tanh\frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \quad [+ \text{ if } x > 0, \; - \text{ if } x < 0]$$
$$= \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x}$$

MULTIPLE ANGLE FORMULAS

8.30 $$\sinh 3x = 3\sinh x + 4\sinh^3 x$$

8.31 $$\cosh 3x = 4\cosh^3 x - 3\cosh x$$

8.32 $$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

8.33 $$\sinh 4x = 8\sinh^3 x \cosh x + 4\sinh x \cosh x$$

8.34 $$\cosh 4x = 8\cosh^4 x - 8\cosh^2 x + 1$$

8.35 $$\tanh 4x = \frac{4\tanh x + 4\tanh^3 x}{1 + 6\tanh^2 x + \tanh^4 x}$$

POWERS OF HYPERBOLIC FUNCTIONS

8.36 $\sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$

8.37 $\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$

8.38 $\sinh^3 x = \frac{1}{4}\sinh 3x - \frac{3}{4}\sinh x$

8.39 $\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$

8.40 $\sinh^4 x = \frac{3}{8} - \frac{1}{2}\cosh 2x + \frac{1}{8}\cosh 4x$

8.41 $\cosh^4 x = \frac{3}{8} + \frac{1}{2}\cosh 2x + \frac{1}{8}\cosh 4x$

SUM, DIFFERENCE AND PRODUCT OF HYPERBOLIC FUNCTIONS

8.42 $\sinh x + \sinh y = 2\sinh \frac{1}{2}(x+y)\cosh \frac{1}{2}(x-y)$

8.43 $\sinh x - \sinh y = 2\cosh \frac{1}{2}(x+y)\sinh \frac{1}{2}(x-y)$

8.44 $\cosh x + \cosh y = 2\cosh \frac{1}{2}(x+y)\cosh \frac{1}{2}(x-y)$

8.45 $\cosh x - \cosh y = 2\sinh \frac{1}{2}(x+y)\sinh \frac{1}{2}(x-y)$

8.46 $\sinh x \sinh y = \frac{1}{2}\{\cosh(x+y) - \cosh(x-y)\}$

8.47 $\cosh x \cosh y = \frac{1}{2}\{\cosh(x+y) + \cosh(x-y)\}$

8.48 $\sinh x \cosh y = \frac{1}{2}\{\sinh(x+y) + \sinh(x-y)\}$

EXPRESSION OF HYPERBOLIC FUNCTIONS IN TERMS OF OTHERS

In the following we assume $x > 0$. If $x < 0$ use the appropriate sign as indicated by formulas 8.14 to 8.19.

	$\sinh x = u$	$\cosh x = u$	$\tanh x = u$	$\coth x = u$	$\operatorname{sech} x = u$	$\operatorname{csch} x = u$
$\sinh x$	u	$\sqrt{u^2-1}$	$u/\sqrt{1-u^2}$	$1/\sqrt{u^2-1}$	$\sqrt{1-u^2}/u$	$1/u$
$\cosh x$	$\sqrt{1+u^2}$	u	$1/\sqrt{1-u^2}$	$u/\sqrt{u^2-1}$	$1/u$	$\sqrt{1+u^2}/u$
$\tanh x$	$u/\sqrt{1+u^2}$	$\sqrt{u^2-1}/u$	u	$1/u$	$\sqrt{1-u^2}$	$1/\sqrt{1+u^2}$
$\coth x$	$\sqrt{u^2+1}/u$	$u/\sqrt{u^2-1}$	$1/u$	u	$1/\sqrt{1-u^2}$	$\sqrt{1+u^2}$
$\operatorname{sech} x$	$1/\sqrt{1+u^2}$	$1/u$	$\sqrt{1-u^2}$	$\sqrt{u^2-1}/u$	u	$u/\sqrt{1+u^2}$
$\operatorname{csch} x$	$1/u$	$1/\sqrt{u^2-1}$	$\sqrt{1-u^2}/u$	$\sqrt{u^2-1}$	$u/\sqrt{1-u^2}$	u

GRAPHS OF HYPERBOLIC FUNCTIONS

8.49 $y = \sinh x$

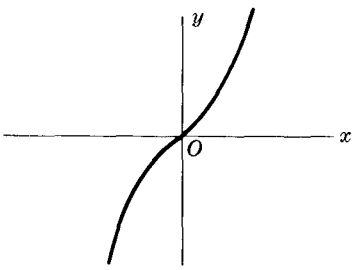


Fig. 8-1

8.50 $y = \cosh x$

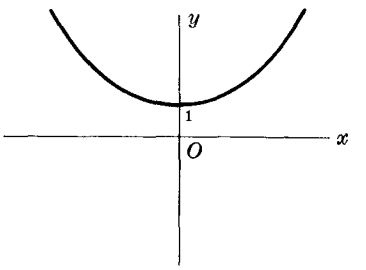


Fig. 8-2

8.51 $y = \tanh x$

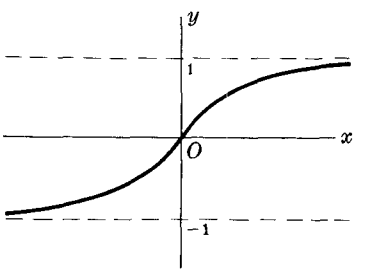


Fig. 8-3

8.52 $y = \coth x$

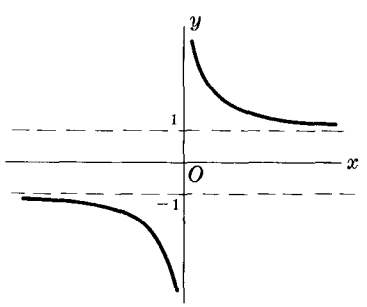


Fig. 8-4

8.53 $y = \operatorname{sech} x$

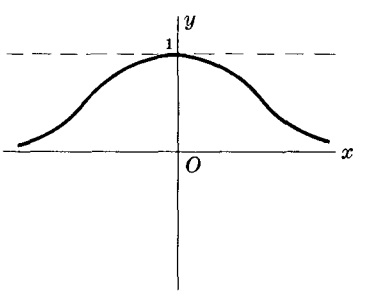


Fig. 8-5

8.54 $y = \operatorname{csch} x$

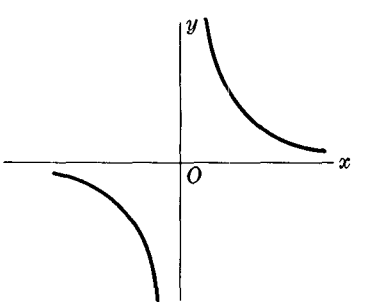


Fig. 8-6

INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh y$, then $y = \sinh^{-1} x$ is called the *inverse hyperbolic sine* of x. Similarly we define the other inverse hyperbolic functions. The inverse hyperbolic functions are multiple-valued and as in the case of inverse trigonometric functions [see page 17] we restrict ourselves to principal values for which they can be considered as single-valued.

The following list shows the principal values [unless otherwise indicated] of the inverse hyperbolic functions expressed in terms of logarithmic functions which are taken as real valued.

8.55 $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ $\quad -\infty < x < \infty$

8.56 $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$ $\quad x \geqq 1$ $\quad [\cosh^{-1} x > 0$ is principal value]

8.57 $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $\quad -1 < x < 1$

8.58 $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ $\quad x > 1$ or $x < -1$

8.59 $\operatorname{sech}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right)$ $\quad 0 < x \leqq 1$ $\quad [\operatorname{sech}^{-1} x > 0$ is principal value]

8.60 $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right)$ $\quad x \neq 0$

RELATIONS BETWEEN INVERSE HYPERBOLIC FUNCTIONS

8.61 $\operatorname{csch}^{-1} x = \sinh^{-1}(1/x)$

8.62 $\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$

8.63 $\coth^{-1} x = \tanh^{-1}(1/x)$

8.64 $\sinh^{-1}(-x) = -\sinh^{-1} x$

8.65 $\tanh^{-1}(-x) = -\tanh^{-1} x$

8.66 $\coth^{-1}(-x) = -\coth^{-1} x$

8.67 $\operatorname{csch}^{-1}(-x) = -\operatorname{csch}^{-1} x$

GRAPHS OF INVERSE HYPERBOLIC FUNCTIONS

8.68 $y = \sinh^{-1} x$

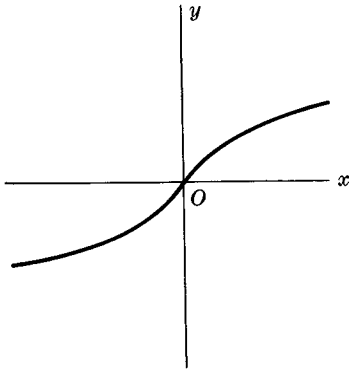


Fig. 8-7

8.69 $y = \cosh^{-1} x$

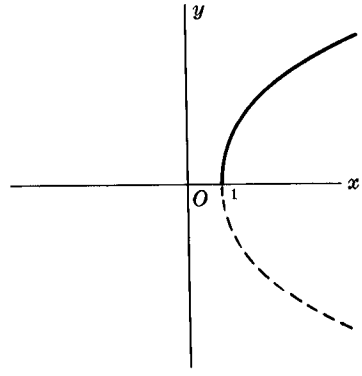


Fig. 8-8

8.70 $y = \tanh^{-1} x$

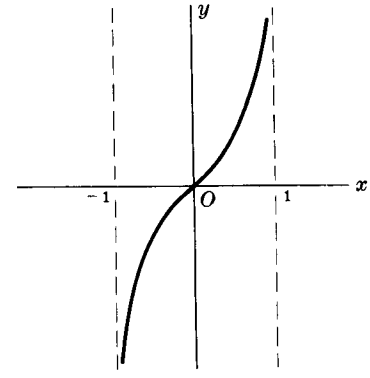


Fig. 8-9

8.71 $y = \coth^{-1} x$

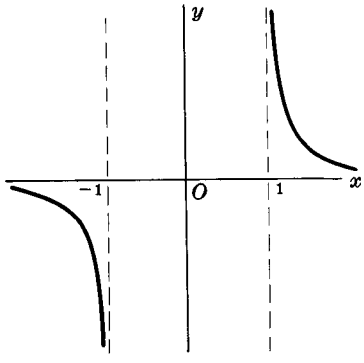


Fig. 8-10

8.72 $y = \operatorname{sech}^{-1} x$

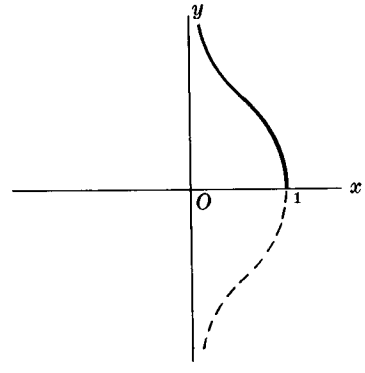


Fig. 8-11

8.73 $y = \operatorname{csch}^{-1} x$

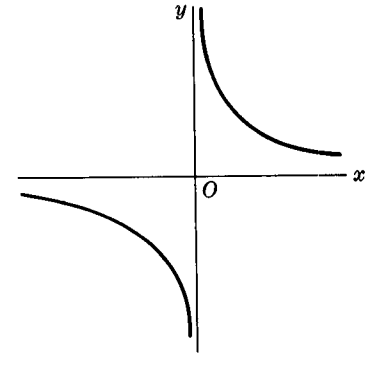


Fig. 8-12

RELATIONSHIP BETWEEN HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS

8.74 $\sin(ix) = i \sinh x$ **8.75** $\cos(ix) = \cosh x$ **8.76** $\tan(ix) = i \tanh x$

8.77 $\csc(ix) = -i \operatorname{csch} x$ **8.78** $\sec(ix) = \operatorname{sech} x$ **8.79** $\cot(ix) = -i \coth x$

8.80 $\sinh(ix) = i \sin x$ **8.81** $\cosh(ix) = \cos x$ **8.82** $\tanh(ix) = i \tan x$

8.83 $\operatorname{csch}(ix) = -i \csc x$ **8.84** $\operatorname{sech}(ix) = \sec x$ **8.85** $\coth(ix) = -i \cot x$

PERIODICITY OF HYPERBOLIC FUNCTIONS

In the following k is any integer.

8.86 $\sinh(x + 2k\pi i) = \sinh x$ **8.87** $\cosh(x + 2k\pi i) = \cosh x$ **8.88** $\tanh(x + k\pi i) = \tanh x$

8.89 $\operatorname{csch}(x + 2k\pi i) = \operatorname{csch} x$ **8.90** $\operatorname{sech}(x + 2k\pi i) = \operatorname{sech} x$ **8.91** $\coth(x + k\pi i) = \coth x$

RELATIONSHIP BETWEEN INVERSE HYPERBOLIC AND INVERSE TRIGONOMETRIC FUNCTIONS

8.92 $\sin^{-1}(ix) = i \sinh^{-1} x$ **8.93** $\sinh^{-1}(ix) = i \sin^{-1} x$

8.94 $\cos^{-1} x = \pm i \cosh^{-1} x$ **8.95** $\cosh^{-1} x = \pm i \cos^{-1} x$

8.96 $\tan^{-1}(ix) = i \tanh^{-1} x$ **8.97** $\tanh^{-1}(ix) = i \tan^{-1} x$

8.98 $\cot^{-1}(ix) = -i \coth^{-1} x$ **8.99** $\coth^{-1}(ix) = -i \cot^{-1} x$

8.100 $\sec^{-1} x = \pm i \operatorname{sech}^{-1} x$ **8.101** $\operatorname{sech}^{-1} x = \pm i \sec^{-1} x$

8.102 $\csc^{-1}(ix) = -i \operatorname{csch}^{-1} x$ **8.103** $\operatorname{csch}^{-1}(ix) = -i \csc^{-1} x$

9 SOLUTIONS of ALGEBRAIC EQUATIONS

QUADRATIC EQUATION: $ax^2 + bx + c = 0$

9.1 **Solutions:** $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If a, b, c are real and if $D = b^2 - 4ac$ is the *discriminant*, then the roots are

(i) real and unequal if $D > 0$

(ii) real and equal if $D = 0$

(iii) complex conjugate if $D < 0$

9.2 If x_1, x_2 are the roots, then $x_1 + x_2 = -b/a$ and $x_1 x_2 = c/a$.

CUBIC EQUATION: $x^3 + a_1 x^2 + a_2 x + a_3 = 0$

Let $$Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54},$$
$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

9.3 **Solutions:**
$$\begin{cases} x_1 = S + T - \frac{1}{3}a_1 \\ x_2 = -\frac{1}{2}(S+T) - \frac{1}{3}a_1 + \frac{1}{2}i\sqrt{3}\,(S - T) \\ x_3 = -\frac{1}{2}(S+T) - \frac{1}{3}a_1 - \frac{1}{2}i\sqrt{3}\,(S - T) \end{cases}$$

If a_1, a_2, a_3 are real and if $D = Q^3 + R^2$ is the *discriminant*, then

(i) one root is real and two complex conjugate if $D > 0$

(ii) all roots are real and at least two are equal if $D = 0$

(iii) all roots are real and unequal if $D < 0$.

If $D < 0$, computation is simplified by use of trigonometry.

9.4 **Solutions if $D < 0$:**
$$\begin{cases} x_1 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta) \\ x_2 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta + 120^\circ) \\ x_3 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta + 240^\circ) \end{cases} \quad \text{where } \cos\theta = -R/\sqrt{-Q^3}$$

9.5 $$x_1 + x_2 + x_3 = -a_1, \quad x_1x_2 + x_2x_3 + x_3x_1 = a_2, \quad x_1x_2x_3 = -a_3$$

where x_1, x_2, x_3 are the three roots.

QUARTIC EQUATION: $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$

Let y_1 be a real root of the cubic equation

9.6 $$y^3 - a_2y^2 + (a_1a_3 - 4a_4)y + (4a_2a_4 - a_3^2 - a_1^2a_4) = 0$$

9.7 **Solutions:** The 4 roots of $z^2 + \frac{1}{2}\{a_1 \pm \sqrt{a_1^2 - 4a_2 + 4y_1}\}z + \frac{1}{2}\{y_1 \pm \sqrt{y_1^2 - 4a_4}\} = 0$

If all roots of 9.6 are real, computation is simplified by using that particular real root which produces all real coefficients in the quadratic equation 9.7.

9.8
$$\begin{cases} x_1 + x_2 + x_3 + x_4 = -a_1 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 + x_1x_3 + x_2x_4 = a_2 \\ x_1x_2x_3 + x_2x_3x_4 + x_1x_2x_4 + x_1x_3x_4 = -a_3 \\ x_1x_2x_3x_4 = a_4 \end{cases}$$

where x_1, x_2, x_3, x_4 are the four roots.

10 FORMULAS from PLANE ANALYTIC GEOMETRY

DISTANCE d BETWEEN TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

10.1 $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

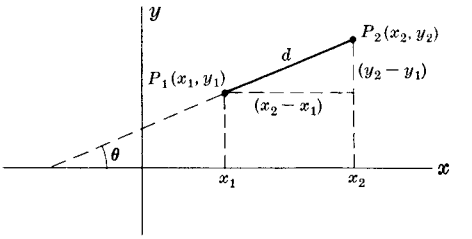


Fig. 10-1

SLOPE m OF LINE JOINING TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

10.2 $$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

EQUATION OF LINE JOINING TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

10.3 $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m \quad \text{or} \quad y - y_1 = m(x - x_1)$$

10.4 $$y = mx + b$$

where $b = y_1 - mx_1 = \dfrac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$ is the *intercept* on the y axis, i.e. the y *intercept*.

EQUATION OF LINE IN TERMS OF x INTERCEPT $a \neq 0$ AND y INTERCEPT $b \neq 0$

10.5 $$\frac{x}{a} + \frac{y}{b} = 1$$

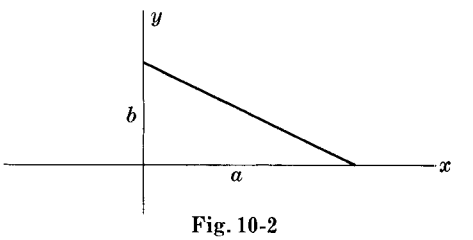


Fig. 10-2

NORMAL FORM FOR EQUATION OF LINE

10.6 $$x \cos \alpha + y \sin \alpha = p$$

where p = perpendicular distance from origin O to line

and α = angle of inclination of perpendicular with positive x axis.

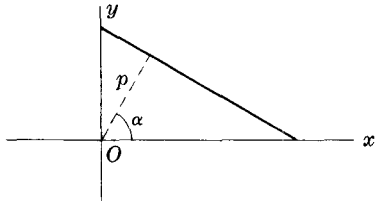


Fig. 10-3

INTEGRALS INVOLVING $\cos ax$

14.369 $$\int \cos ax\, dx = \frac{\sin ax}{a}$$

14.370 $$\int x \cos ax\, dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$$

14.371 $$\int x^2 \cos ax\, dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

14.372 $$\int x^3 \cos ax\, dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$$

14.373 $$\int \frac{\cos ax}{x}\, dx = \ln x - \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} - \frac{(ax)^6}{6 \cdot 6!} + \cdots$$

14.374 $$\int \frac{\cos ax}{x^2}\, dx = -\frac{\cos ax}{x} - a\int \frac{\sin ax}{x}\, dx$$ [See 14.343]

14.375 $$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

14.376 $$\int \frac{x\, dx}{\cos ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.377 $$\int \cos^2 ax\, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

14.378 $$\int x \cos^2 ax\, dx = \frac{x^2}{4} + \frac{x \sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

14.379 $$\int \cos^3 ax\, dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a}$$

14.380 $$\int \cos^4 ax\, dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

14.381 $$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a}$$

14.382 $$\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a \cos^2 ax} + \frac{1}{2a}\ln \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

14.383 $$\int \cos ax \cos px\, dx = \frac{\sin(a-p)x}{2(a-p)} + \frac{\sin(a+p)x}{2(a+p)}$$ [If $a = \pm p$, see 14.377.]

14.384 $$\int \frac{dx}{1 - \cos ax} = -\frac{1}{a}\cot \frac{ax}{2}$$

14.385 $$\int \frac{x\, dx}{1 - \cos ax} = -\frac{x}{a}\cot \frac{ax}{2} + \frac{2}{a^2}\ln \sin \frac{ax}{2}$$

14.386 $$\int \frac{dx}{1 + \cos ax} = \frac{1}{a}\tan \frac{ax}{2}$$

14.387 $$\int \frac{x\, dx}{1 + \cos ax} = \frac{x}{a}\tan \frac{ax}{2} + \frac{2}{a^2}\ln \cos \frac{ax}{2}$$

14.388 $$\int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a}\cot \frac{ax}{2} - \frac{1}{6a}\cot^3 \frac{ax}{2}$$

14.389 $$\int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a}\tan \frac{ax}{2} + \frac{1}{6a}\tan^3 \frac{ax}{2}$$

14.390 $$\int \frac{dx}{p+q\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}} \tan^{-1}\sqrt{(p-q)/(p+q)}\,\tan\tfrac{1}{2}ax \\ \dfrac{1}{a\sqrt{q^2-p^2}} \ln\left(\dfrac{\tan\frac{1}{2}ax+\sqrt{(q+p)/(q-p)}}{\tan\frac{1}{2}ax-\sqrt{(q+p)/(q-p)}}\right) \end{cases}$$ [If $p=\pm q$ see 14.384 and 14.386.]

14.391 $$\int \frac{dx}{(p+q\cos ax)^2} = \frac{q\sin ax}{a(q^2-p^2)(p+q\cos ax)} - \frac{p}{q^2-p^2}\int \frac{dx}{p+q\cos ax}$$ [If $p=\pm q$ see 14.388 and 14.389.]

14.392 $$\int \frac{dx}{p^2+q^2\cos^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}} \tan^{-1}\frac{p\tan ax}{\sqrt{p^2+q^2}}$$

14.393 $$\int \frac{dx}{p^2-q^2\cos^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}} \tan^{-1}\dfrac{p\tan ax}{\sqrt{p^2-q^2}} \\ \dfrac{1}{2ap\sqrt{q^2-p^2}} \ln\left(\dfrac{p\tan ax-\sqrt{q^2-p^2}}{p\tan ax+\sqrt{q^2-p^2}}\right) \end{cases}$$

14.394 $$\int x^m \cos ax\,dx = \frac{x^m\sin ax}{a} + \frac{mx^{m-1}}{a^2}\cos ax - \frac{m(m-1)}{a^2}\int x^{m-2}\cos ax\,dx$$

14.395 $$\int \frac{\cos ax}{x^n}\,dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax}{x^{n-1}}\,dx$$ [See 14.365]

14.396 $$\int \cos^n ax\,dx = \frac{\sin ax\cos^{n-1}ax}{an} + \frac{n-1}{n}\int \cos^{n-2}ax\,dx$$

14.397 $$\int \frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1}ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}ax}$$

14.477 $\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$

14.478 $\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{x\sqrt{a^2-x^2}}{4}$

14.479 $\int x^2\cos^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cos^{-1}\frac{x}{a} - \frac{(x^2+2a^2)\sqrt{a^2-x^2}}{9}$

14.480 $\int \frac{\cos^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int \frac{\sin^{-1}(x/a)}{x}\,dx$ [See 14.474]

14.481 $\int \frac{\cos^{-1}(x/a)}{x^2}\,dx = -\frac{\cos^{-1}(x/a)}{x} + \frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

14.482 $\int\left(\cos^{-1}\frac{x}{a}\right)^2 dx = x\left(\cos^{-1}\frac{x}{a}\right)^2 - 2x - 2\sqrt{a^2-x^2}\cos^{-1}\frac{x}{a}$

14.483 $\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2)$

14.484 $\int x\tan^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a} - \frac{ax}{2}$

14.485 $\int x^2\tan^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\tan^{-1}\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(x^2+a^2)$

14.486 $\int \frac{\tan^{-1}(x/a)}{x}\,dx = \frac{x}{a} - \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} - \frac{(x/a)^7}{7^2} + \cdots$

14.487 $\int \frac{\tan^{-1}(x/a)}{x^2}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a} - \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$

14.488 $\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln(x^2+a^2)$

14.489 $\int x\cot^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\cot^{-1}\frac{x}{a} + \frac{ax}{2}$

14.490 $\int x^2\cot^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cot^{-1}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(x^2+a^2)$

14.491 $\int \frac{\cot^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int \frac{\tan^{-1}(x/a)}{x}\,dx$ [See 14.486]

14.492 $\int \frac{\cot^{-1}(x/a)}{x^2}\,dx = -\frac{\cot^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$

14.493 $\int \sec^{-1}\frac{x}{a}\,dx = \begin{cases} x\sec^{-1}\frac{x}{a} - a\ln(x+\sqrt{x^2-a^2}) & 0 < \sec^{-1}\frac{x}{a} < \frac{\pi}{2} \\ x\sec^{-1}\frac{x}{a} + a\ln(x+\sqrt{x^2-a^2}) & \frac{\pi}{2} < \sec^{-1}\frac{x}{a} < \pi \end{cases}$

14.494 $\int x\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{x^2}{2}\sec^{-1}\frac{x}{a} - \frac{a\sqrt{x^2-a^2}}{2} & 0 < \sec^{-1}\frac{x}{a} < \frac{\pi}{2} \\ \frac{x^2}{2}\sec^{-1}\frac{x}{a} + \frac{a\sqrt{x^2-a^2}}{2} & \frac{\pi}{2} < \sec^{-1}\frac{x}{a} < \pi \end{cases}$

14.495 $\int x^2\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{x^3}{3}\sec^{-1}\frac{x}{a} - \frac{ax\sqrt{x^2-a^2}}{6} - \frac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & 0 < \sec^{-1}\frac{x}{a} < \frac{\pi}{2} \\ \frac{x^3}{3}\sec^{-1}\frac{x}{a} + \frac{ax\sqrt{x^2-a^2}}{6} + \frac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & \frac{\pi}{2} < \sec^{-1}\frac{x}{a} < \pi \end{cases}$

14.496 $$\int \frac{\sec^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x + \frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(a/x)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots$$

14.497 $$\int \frac{\sec^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\sec^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ -\dfrac{\sec^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.498 $$\int \csc^{-1}\frac{x}{a}\,dx = \begin{cases} x\csc^{-1}\dfrac{x}{a} + a\ln\left(x+\sqrt{x^2-a^2}\right) & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ x\csc^{-1}\dfrac{x}{a} - a\ln\left(x+\sqrt{x^2-a^2}\right) & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.499 $$\int x\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.500 $$\int x^2\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln\left(x+\sqrt{x^2-a^2}\right) & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln\left(x+\sqrt{x^2-a^2}\right) & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.501 $$\int \frac{\csc^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(a/x)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots\right)$$

14.502 $$\int \frac{\csc^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\csc^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ -\dfrac{\csc^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.503 $$\int x^m \sin^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sin^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

14.504 $$\int x^m \cos^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cos^{-1}\frac{x}{a} + \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

14.505 $$\int x^m \tan^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tan^{-1}\frac{x}{a} - \frac{a}{m+1}\int \frac{x^{m+1}}{x^2+a^2}\,dx$$

14.506 $$\int x^m \cot^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cot^{-1}\frac{x}{a} + \frac{a}{m+1}\int \frac{x^{m+1}}{x^2+a^2}\,dx$$

14.507 $$\int x^m \sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.508 $$\int x^m \csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

INTEGRALS INVOLVING $\ln x$

14.525 $\int \ln x \, dx = x \ln x - x$

14.526 $\int x \ln x \, dx = \frac{x^2}{2}(\ln x - \tfrac{1}{2})$

14.527 $\int x^m \ln x \, dx = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right)$ [If $m = -1$ see 14.528.]

14.528 $\int \frac{\ln x}{x} \, dx = \frac{1}{2}\ln^2 x$

14.529 $\int \frac{\ln x}{x^2} \, dx = -\frac{\ln x}{x} - \frac{1}{x}$

14.530 $\int \ln^2 x \, dx = x \ln^2 x - 2x \ln x + 2x$

14.531 $\int \frac{\ln^n x \, dx}{x} = \frac{\ln^{n+1} x}{n+1}$ [If $n = -1$ see 14.532.]

14.532 $\int \frac{dx}{x \ln x} = \ln(\ln x)$

14.533 $\int \frac{dx}{\ln x} = \ln(\ln x) + \ln x + \frac{\ln^2 x}{2 \cdot 2!} + \frac{\ln^3 x}{3 \cdot 3!} + \cdots$

14.534 $\int \frac{x^m \, dx}{\ln x} = \ln(\ln x) + (m+1)\ln x + \frac{(m+1)^2 \ln^2 x}{2 \cdot 2!} + \frac{(m+1)^3 \ln^3 x}{3 \cdot 3!} + \cdots$

14.535 $\int \ln^n x \, dx = x \ln^n x - n \int \ln^{n-1} x \, dx$

14.536 $\int x^m \ln^n x \, dx = \frac{x^{m+1} \ln^n x}{m+1} - \frac{n}{m+1} \int x^m \ln^{n-1} x \, dx$

If $m = -1$ see 14.531.

14.537 $\int \ln(x^2 + a^2) \, dx = x \ln(x^2 + a^2) - 2x + 2a \tan^{-1} \frac{x}{a}$

14.538 $\int \ln(x^2 - a^2) \, dx = x \ln(x^2 - a^2) - 2x + a \ln\left(\frac{x+a}{x-a}\right)$

14.539 $\int x^m \ln(x^2 \pm a^2) \, dx = \frac{x^{m+1} \ln(x^2 \pm a^2)}{m+1} - \frac{2}{m+1} \int \frac{x^{m+2}}{x^2 \pm a^2} \, dx$

INTEGRALS INVOLVING $\sinh ax$

14.540 $\int \sinh ax \, dx = \frac{\cosh ax}{a}$

14.541 $\int x \sinh ax \, dx = \frac{x \cosh ax}{a} - \frac{\sinh ax}{a^2}$

14.542 $\int x^2 \sinh ax \, dx = \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\cosh ax - \frac{2x}{a^2} \sinh ax$

INTEGRALS INVOLVING $\cosh ax$

14.562 $$\int \cosh ax\, dx = \frac{\sinh ax}{a}$$

14.563 $$\int x \cosh ax\, dx = \frac{x \sinh ax}{a} - \frac{\cosh ax}{a^2}$$

14.564 $$\int x^2 \cosh ax\, dx = -\frac{2x \cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3}\right) \sinh ax$$

14.565 $$\int \frac{\cosh ax}{x}\, dx = \ln x + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} + \frac{(ax)^6}{6 \cdot 6!} + \cdots$$

14.566 $$\int \frac{\cosh ax}{x^2}\, dx = -\frac{\cosh ax}{x} + a \int \frac{\sinh ax}{x}\, dx \qquad \text{[See 14.543]}$$

14.567 $$\int \frac{dx}{\cosh ax} = \frac{2}{a} \tan^{-1} e^{ax}$$

14.568 $$\int \frac{x\, dx}{\cosh ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{(-1)^n E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.569 $$\int \cosh^2 ax\, dx = \frac{x}{2} + \frac{\sinh ax \cosh ax}{2a}$$

14.570 $$\int x \cosh^2 ax\, dx = \frac{x^2}{4} + \frac{x \sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2}$$

14.571 $$\int \frac{dx}{\cosh^2 ax} = \frac{\tanh ax}{a}$$

14.572 $$\int \cosh ax \cosh px\, dx = \frac{\sinh (a-p)x}{2(a-p)} + \frac{\sinh (a+p)x}{2(a+p)}$$

14.573 $$\int \cosh ax \sin px\, dx = \frac{a \sinh ax \sin px - p \cosh ax \cos px}{a^2 + p^2}$$

14.574 $$\int \cosh ax \cos px\, dx = \frac{a \sinh ax \cos px + p \cosh ax \sin px}{a^2 + p^2}$$

14.575 $$\int \frac{dx}{\cosh ax + 1} = \frac{1}{a} \tanh \frac{ax}{2}$$

14.576 $$\int \frac{dx}{\cosh ax - 1} = -\frac{1}{a} \coth \frac{ax}{2}$$

14.577 $$\int \frac{x\, dx}{\cosh ax + 1} = \frac{x}{a} \tanh \frac{ax}{2} - \frac{2}{a^2} \ln \cosh \frac{ax}{2}$$

14.578 $$\int \frac{x\, dx}{\cosh ax - 1} = -\frac{x}{a} \coth \frac{ax}{2} + \frac{2}{a^2} \ln \sinh \frac{ax}{2}$$

14.579 $$\int \frac{dx}{(\cosh ax + 1)^2} = \frac{1}{2a} \tanh \frac{ax}{2} - \frac{1}{6a} \tanh^3 \frac{ax}{2}$$

14.580 $$\int \frac{dx}{(\cosh ax - 1)^2} = \frac{1}{2a} \coth \frac{ax}{2} - \frac{1}{6a} \coth^3 \frac{ax}{2}$$

14.581 $$\int \frac{dx}{p + q \cosh ax} = \begin{cases} \dfrac{2}{a\sqrt{q^2 - p^2}} \tan^{-1} \dfrac{qe^{ax} + p}{\sqrt{q^2 - p^2}} \\ \dfrac{1}{a\sqrt{p^2 - q^2}} \ln \left(\dfrac{qe^{ax} + p - \sqrt{p^2 - q^2}}{qe^{ax} + p + \sqrt{p^2 - q^2}}\right) \end{cases}$$

14.582 $$\int \frac{dx}{(p + q \cosh ax)^2} = \frac{q \sinh ax}{a(q^2 - p^2)(p + q \cosh ax)} - \frac{p}{q^2 - p^2} \int \frac{dx}{p + q \cosh ax}$$

14.602 $$\int \frac{dx}{\sinh ax\,(\cosh ax+1)} = \frac{1}{2a}\ln\tanh\frac{ax}{2}+\frac{1}{2a(\cosh ax+1)}$$

14.603 $$\int \frac{dx}{\sinh ax\,(\cosh ax-1)} = -\frac{1}{2a}\ln\tanh\frac{ax}{2}-\frac{1}{2a(\cosh ax-1)}$$

INTEGRALS INVOLVING tanh ax

14.604 $$\int \tanh ax\,dx = \frac{1}{a}\ln\cosh ax$$

14.605 $$\int \tanh^2 ax\,dx = x-\frac{\tanh ax}{a}$$

14.606 $$\int \tanh^3 ax\,dx = \frac{1}{a}\ln\cosh ax-\frac{\tanh^2 ax}{2a}$$

14.607 $$\int \tanh^n ax\,\operatorname{sech}^2 ax\,dx = \frac{\tanh^{n+1} ax}{(n+1)a}$$

14.608 $$\int \frac{\operatorname{sech}^2 ax}{\tanh ax}\,dx = \frac{1}{a}\ln\tanh ax$$

14.609 $$\int \frac{dx}{\tanh ax} = \frac{1}{a}\ln\sinh ax$$

14.610 $$\int x\tanh ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3}-\frac{(ax)^5}{15}+\frac{2(ax)^7}{105}-\cdots\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!}+\cdots\right\}$$

14.611 $$\int x\tanh^2 ax\,dx = \frac{x^2}{2}-\frac{x\tanh ax}{a}+\frac{1}{a^2}\ln\cosh ax$$

14.612 $$\int \frac{\tanh ax}{x}\,dx = ax-\frac{(ax)^3}{9}+\frac{2(ax)^5}{75}-\cdots\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!}+\cdots$$

14.613 $$\int \frac{dx}{p+q\tanh ax} = \frac{px}{p^2-q^2}-\frac{q}{a(p^2-q^2)}\ln(q\sinh ax+p\cosh ax)$$

14.614 $$\int \tanh^n ax\,dx = \frac{-\tanh^{n-1} ax}{a(n-1)}+\int \tanh^{n-2} ax\,dx$$

INTEGRALS INVOLVING coth ax

14.615 $$\int \coth ax\,dx = \frac{1}{a}\ln\sinh ax$$

14.616 $$\int \coth^2 ax\,dx = x-\frac{\coth ax}{a}$$

14.617 $$\int \coth^3 ax\,dx = \frac{1}{a}\ln\sinh ax-\frac{\coth^2 ax}{2a}$$

14.618 $$\int \coth^n ax\,\operatorname{csch}^2 ax\,dx = -\frac{\coth^{n+1} ax}{(n+1)a}$$

14.619 $$\int \frac{\operatorname{csch}^2 ax}{\coth ax}\,dx = -\frac{1}{a}\ln\coth ax$$

14.620 $$\int \frac{dx}{\coth ax} = \frac{1}{a}\ln\cosh ax$$

14.621 $$\int x \coth ax\, dx = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{9} - \frac{(ax)^5}{225} + \cdots \frac{(-1)^{n-1}2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.622 $$\int x \coth^2 ax\, dx = \frac{x^2}{2} - \frac{x \coth ax}{a} + \frac{1}{a^2}\ln \sinh ax$$

14.623 $$\int \frac{\coth ax}{x}\, dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{(ax)^3}{135} + \cdots \frac{(-1)^n 2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.624 $$\int \frac{dx}{p + q\coth ax} = \frac{px}{p^2 - q^2} - \frac{q}{a(p^2 - q^2)}\ln (p \sinh ax + q \cosh ax)$$

14.625 $$\int \coth^n ax\, dx = -\frac{\coth^{n-1} ax}{a(n-1)} + \int \coth^{n-2} ax\, dx$$

INTEGRALS INVOLVING sech ax

14.626 $$\int \operatorname{sech} ax\, dx = \frac{2}{a}\tan^{-1} e^{ax}$$

14.627 $$\int \operatorname{sech}^2 ax\, dx = \frac{\tanh ax}{a}$$

14.628 $$\int \operatorname{sech}^3 ax\, dx = \frac{\operatorname{sech} ax \tanh ax}{2a} + \frac{1}{2a}\tan^{-1}\sinh ax$$

14.629 $$\int \operatorname{sech}^n ax \tanh ax\, dx = -\frac{\operatorname{sech}^n ax}{na}$$

14.630 $$\int \frac{dx}{\operatorname{sech} ax} = \frac{\sinh ax}{a}$$

14.631 $$\int x \operatorname{sech} ax\, dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots \frac{(-1)^n E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.632 $$\int x \operatorname{sech}^2 ax\, dx = \frac{x \tanh ax}{a} - \frac{1}{a^2}\ln \cosh ax$$

14.633 $$\int \frac{\operatorname{sech} ax}{x}\, dx = \ln x - \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} - \frac{61(ax)^6}{4320} + \cdots \frac{(-1)^n E_n(ax)^{2n}}{2n(2n)!} + \cdots$$

14.634 $$\int \frac{dx}{q + p\operatorname{sech} ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q\cosh ax}$$ [See 14.581]

14.635 $$\int \operatorname{sech}^n ax\, dx = \frac{\operatorname{sech}^{n-2} ax \tanh ax}{a(n-1)} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2} ax\, dx$$

INTEGRALS INVOLVING csch ax

14.636 $$\int \operatorname{csch} ax\, dx = \frac{1}{a}\ln \tanh \frac{ax}{2}$$

14.637 $$\int \operatorname{csch}^2 ax\, dx = -\frac{\coth ax}{a}$$

14.638 $$\int \operatorname{csch}^3 ax\, dx = -\frac{\operatorname{csch} ax \coth ax}{2a} - \frac{1}{2a}\ln \tanh \frac{ax}{2}$$

14.639 $$\int \operatorname{csch}^n ax \coth ax\, dx = -\frac{\operatorname{csch}^n ax}{na}$$

15.13 $$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx \qquad \text{where } c \text{ is between } a \text{ and } b$$

This is a generalization of 15.12 and is valid if $f(x)$ and $g(x)$ are continuous in $a \leqq x \leqq b$ and $g(x) \geqq 0$.

LEIBNITZ'S RULE FOR DIFFERENTIATION OF INTEGRALS

15.14 $$\frac{d}{d\alpha}\int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha)\,dx = \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial F}{\partial \alpha}\,dx + F(\phi_2,\alpha)\frac{d\phi_1}{d\alpha} - F(\phi_1,\alpha)\frac{d\phi_2}{d\alpha}$$

APPROXIMATE FORMULAS FOR DEFINITE INTEGRALS

In the following the interval from $x = a$ to $x = b$ is subdivided into n equal parts by the points $a = x_0$, $x_1, x_2, \ldots, x_{n-1}, x_n = b$ and we let $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $\ldots$, $y_n = f(x_n)$, $h = (b-a)/n$.

Rectangular formula

15.15 $$\int_a^b f(x)\,dx \approx h(y_0 + y_1 + y_2 + \cdots + y_{n-1})$$

Trapezoidal formula

15.16 $$\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

Simpson's formula (or parabolic formula) for *n* even

15.17 $$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

DEFINITE INTEGRALS INVOLVING RATIONAL OR IRRATIONAL EXPRESSIONS

15.18 $$\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$$

15.19 $$\int_0^\infty \frac{x^{p-1}\,dx}{1+x} = \frac{\pi}{\sin p\pi}, \qquad 0 < p < 1$$

15.20 $$\int_0^\infty \frac{x^m\,dx}{x^n + a^n} = \frac{\pi a^{m+1-n}}{n \sin[(m+1)\pi/n]}, \qquad 0 < m+1 < n$$

15.21 $$\int_0^\infty \frac{x^m\,dx}{1 + 2x\cos\beta + x^2} = \frac{\pi}{\sin m\pi}\frac{\sin m\beta}{\sin\beta}$$

15.22 $$\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}$$

15.23 $$\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{4}$$

15.24 $$\int_0^a x^m(a^n - x^n)^p\,dx = \frac{a^{m+1+np}\,\Gamma[(m+1)/n]\,\Gamma(p+1)}{n\Gamma[(m+1)/n + p + 1]}$$

15.25 $$\int_0^\infty \frac{x^m\,dx}{(x^n + a^n)^r} = \frac{(-1)^{r-1}\pi a^{m+1-nr}\,\Gamma[(m+1)/n]}{n \sin[(m+1)\pi/n](r-1)!\,\Gamma[(m+1)/n - r + 1]}, \qquad 0 < m+1 < nr$$

15.46 $$\int_0^{2\pi} \frac{dx}{(a+b\sin x)^2} = \int_0^{2\pi} \frac{dx}{(a+b\cos x)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

15.47 $$\int_0^{2\pi} \frac{dx}{1-2a\cos x+a^2} = \frac{2\pi}{1-a^2}, \qquad 0<a<1$$

15.48 $$\int_0^{\pi} \frac{x\sin x\,dx}{1-2a\cos x+a^2} = \begin{cases} (\pi/a)\ln(1+a) & |a|<1 \\ \pi\ln(1+1/a) & |a|>1 \end{cases}$$

15.49 $$\int_0^{\pi} \frac{\cos mx\,dx}{1-2a\cos x+a^2} = \frac{\pi a^m}{1-a^2}, \qquad a^2<1, \quad m=0,1,2,\ldots$$

15.50 $$\int_0^{\infty} \sin ax^2\,dx = \int_0^{\infty} \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$

15.51 $$\int_0^{\infty} \sin ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\sin\frac{\pi}{2n}, \qquad n>1$$

15.52 $$\int_0^{\infty} \cos ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\cos\frac{\pi}{2n}, \qquad n>1$$

15.53 $$\int_0^{\infty} \frac{\sin x}{\sqrt{x}}\,dx = \int_0^{\infty} \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

15.54 $$\int_0^{\infty} \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \qquad 0<p<1$$

15.55 $$\int_0^{\infty} \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \qquad 0<p<1$$

15.56 $$\int_0^{\infty} \sin ax^2\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a}-\sin\frac{b^2}{a}\right)$$

15.57 $$\int_0^{\infty} \cos ax^2\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a}+\sin\frac{b^2}{a}\right)$$

15.58 $$\int_0^{\infty} \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

15.59 $$\int_0^{\infty} \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$$

15.60 $$\int_0^{\infty} \frac{\tan x}{x}\,dx = \frac{\pi}{2}$$

15.61 $$\int_0^{\pi/2} \frac{dx}{1+\tan^m x} = \frac{\pi}{4}$$

15.62 $$\int_0^{\pi/2} \frac{x}{\sin x}\,dx = 2\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right\}$$

15.63 $$\int_0^{1} \frac{\tan^{-1}x}{x}\,dx = \frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots$$ ✓ Catalan

15.64 $$\int_0^{1} \frac{\sin^{-1}x}{x}\,dx = \frac{\pi}{2}\ln 2$$

15.65 $$\int_0^{1} \frac{1-\cos x}{x}\,dx - \int_1^{\infty} \frac{\cos x}{x}\,dx = \gamma$$

15.66 $$\int_0^{\infty} \left(\frac{1}{1+x^2}-\cos x\right)\frac{dx}{x} = \gamma$$

15.67 $$\int_0^{\infty} \frac{\tan^{-1}px-\tan^{-1}qx}{x}\,dx = \frac{\pi}{2}\ln\frac{p}{q}$$

Extensions to sums of more than two vectors are immediate. Thus Fig. 22-3 shows how to obtain the sum **E** of the vectors **A**, **B**, **C** and **D**.

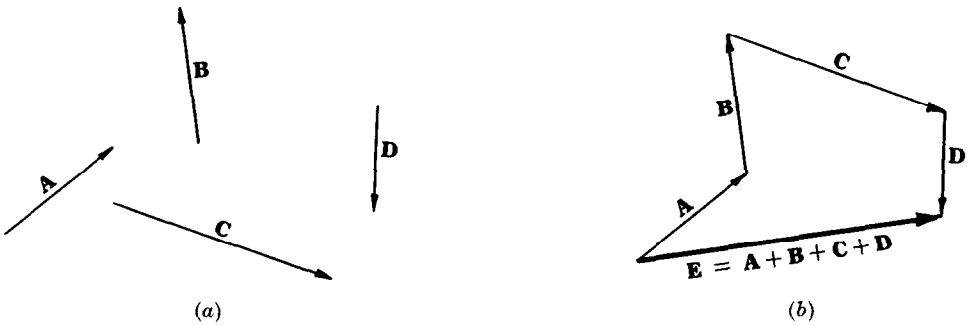


Fig. 22-3

4. **Unit vectors.** A *unit vector* is a vector with unit magnitude. If **A** is a vector, then a unit vector in the direction of **A** is $\mathbf{a} = \mathbf{A}/A$ where $A > 0$.

LAWS OF VECTOR ALGEBRA

If **A**, **B**, **C** are vectors and m, n are scalars, then

22.1 $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ Commutative law for addition

22.2 $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ Associative law for addition

22.3 $m(n\mathbf{A}) = (mn)\mathbf{A} = n(m\mathbf{A})$ Associative law for scalar multiplication

22.4 $(m + n)\mathbf{A} = m\mathbf{A} + n\mathbf{A}$ Distributive law

22.5 $m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}$ Distributive law

COMPONENTS OF A VECTOR

A vector **A** can be represented with initial point at the origin of a rectangular coordinate system. If **i**, **j**, **k** are unit vectors in the directions of the positive x, y, z axes, then

22.6 $$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

where $A_1\mathbf{i}, A_2\mathbf{j}, A_3\mathbf{k}$ are called *component vectors* of **A** in the **i**, **j**, **k** directions and A_1, A_2, A_3 are called the *components* of **A**.

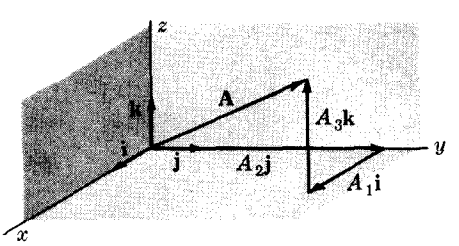


Fig. 22-4

DOT OR SCALAR PRODUCT

22.7 $$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \qquad 0 \leqq \theta \leqq \pi$$

where θ is the angle between **A** and **B**.

Fundamental results are

22.8 $$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$ Commutative law

22.9 $$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$ Distributive law

22.10 $$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

where $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$.

CROSS OR VECTOR PRODUCT

22.11 $$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{u} \qquad 0 \leqq \theta \leqq \pi$$

where θ is the angle between **A** and **B** and **u** is a unit vector perpendicular to the plane of **A** and **B** such that **A**, **B**, **u** form a *right-handed system* [i.e. a right-threaded screw rotated through an angle less than 180° from **A** to **B** will advance in the direction of **u** as in Fig. 22-5].

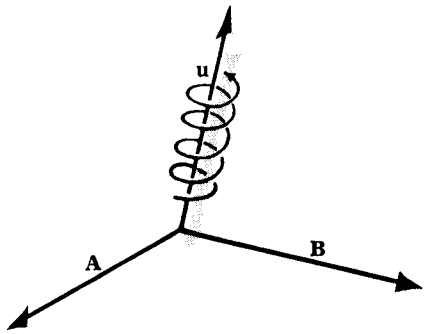


Fig. 22-5

Fundamental results are

22.12 $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}$$

22.13 $$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

22.14 $$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

22.15 $|\mathbf{A}\times\mathbf{B}|$ = area of parallelogram having sides **A** and **B**

MISCELLANEOUS FORMULAS INVOLVING DOT AND CROSS PRODUCTS

22.16 $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2$$

22.17 $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ = volume of parallelepiped with sides **A**, **B**, **C**

22.18 $$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

22.19 $$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$$

22.20 $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

where $\phi(P_1)$ and $\phi(P_2)$ denote the values of ϕ at P_1 and P_2 respectively. In particular if C is a closed curve,

22.53 $$\int_C \mathbf{A}\cdot d\mathbf{r} = \oint_C \mathbf{A}\cdot d\mathbf{r} = 0$$

where the circle on the integral sign is used to emphasize that C is closed.

MULTIPLE INTEGRALS

Let $F(x, y)$ be a function defined in a region $\mathcal{R}$ of the xy plane as in Fig. 22-7. Subdivide the region into n parts by lines parallel to the x and y axes as indicated. Let $\Delta A_p = \Delta x_p\,\Delta y_p$ denote an area of one of these parts. Then the integral of $F(x, y)$ over $\mathcal{R}$ is defined as

22.54 $$\int_{\mathcal{R}} F(x, y)\,dA = \lim_{n\to\infty} \sum_{p=1}^{n} F(x_p, y_p)\,\Delta A_p$$

provided this limit exists.

In such case the integral can also be written as

22.55 $$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} F(x, y)\,dy\,dx = \int_{x=a}^{b}\left\{\int_{y=f_1(x)}^{f_2(x)} F(x, y)\,dy\right\}dx$$

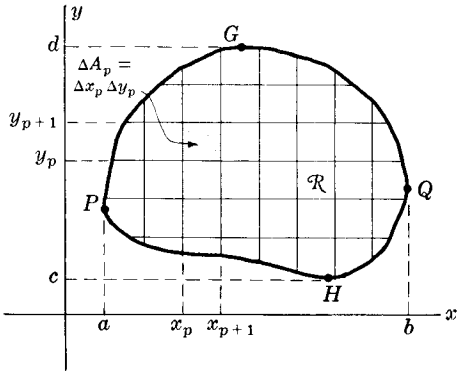


Fig. 22-7

where $y = f_1(x)$ and $y = f_2(x)$ are the equations of curves PHQ and PGQ respectively and a and b are the x coordinates of points P and Q. The result can also be written as

22.56 $$\int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} F(x, y)\,dx\,dy = \int_{y=c}^{d}\left\{\int_{x=g_1(y)}^{g_2(y)} F(x, y)\,dx\right\}dy$$

where $x = g_1(y)$, $x = g_2(y)$ are the equations of curves HPG and HQG respectively and c and d are the y coordinates of H and G.

These are called *double integrals* or *area integrals*. The ideas can be similarly extended to *triple* or *volume integrals* or to higher *multiple integrals*.

SURFACE INTEGRALS

Subdivide the surface S [see Fig. 22-8] into n elements of area ΔS_p, $p = 1, 2, \ldots, n$. Let $\mathbf{A}(x_p, y_p, z_p) = \mathbf{A}_p$ where (x_p, y_p, z_p) is a point P in ΔS_p. Let $\mathbf{N}_p$ be a unit normal to ΔS_p at P. Then the surface integral of the normal component of $\mathbf{A}$ over S is defined as

22.57 $$\int_S \mathbf{A}\cdot\mathbf{N}\,dS = \lim_{n\to\infty} \sum_{p=1}^{n} \mathbf{A}_p\cdot\mathbf{N}_p\,\Delta S_p$$

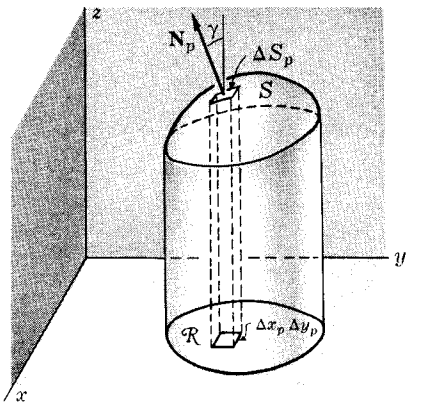


Fig. 22-8

RELATION BETWEEN SURFACE AND DOUBLE INTEGRALS

If $\mathcal{R}$ is the projection of S on the xy plane, then [see Fig. 22-8]

22.58
$$\int_S \mathbf{A}\cdot\mathbf{N}\,dS = \iint_{\mathcal{R}} \mathbf{A}\cdot\mathbf{N}\,\frac{dx\,dy}{|\mathbf{N}\cdot\mathbf{k}|}$$

THE DIVERGENCE THEOREM

Let S be a closed surface bounding a region of volume V; then if $\mathbf{N}$ is the positive (outward drawn) normal and $d\mathbf{S} = \mathbf{N}\,dS$, we have [see Fig. 22-9]

22.59
$$\int_V \nabla\cdot\mathbf{A}\,dV = \int_S \mathbf{A}\cdot d\mathbf{S}$$

The result is also called *Gauss' theorem* or *Green's theorem.*

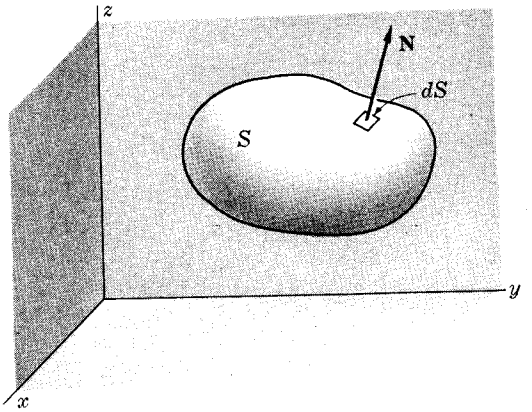


Fig. 22-9

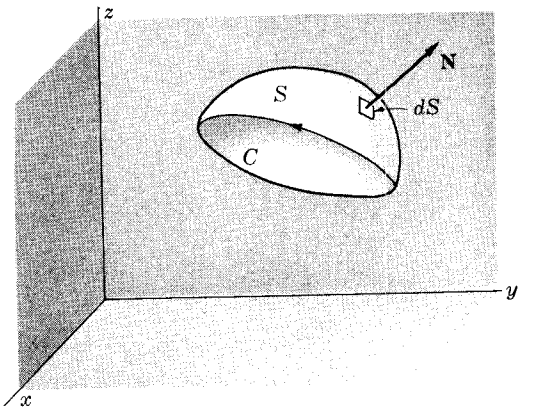


Fig. 22-10

STOKE'S THEOREM

Let S be an open two-sided surface bounded by a closed non-intersecting curve C [simple closed curve] as in Fig. 22-10. Then

22.60
$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where the circle on the integral is used to emphasize that C is closed.

GREEN'S THEOREM IN THE PLANE

22.61
$$\oint_C (P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where R is the area bounded by the closed curve C. This result is a special case of the divergence theorem or Stoke's theorem.

GREEN'S FIRST IDENTITY

22.62 $$\int_V \{\phi\,\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)\}\,dV = \int (\phi\,\nabla\psi)\cdot d\mathbf{S}$$

where ϕ and ψ are scalar functions.

GREEN'S SECOND IDENTITY

22.63 $$\int_V (\phi\,\nabla^2\psi - \psi\,\nabla^2\phi)\,dV = \int_S (\phi\,\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$$

MISCELLANEOUS INTEGRAL THEOREMS

22.64 $$\int_V \nabla\times\mathbf{A}\,dV = \int_S d\mathbf{S}\times\mathbf{A}$$ **22.65** $$\int_C \phi\,d\mathbf{r} = \int_S d\mathbf{S}\times\nabla\phi$$

CURVILINEAR COORDINATES

A point P in space [see Fig. 22-11] can be located by rectangular coordinates (x, y, z) or curvilinear coordinates (u_1, u_2, u_3) where the transformation equations from one set of coordinates to the other are given by

22.66
$$x = x(u_1, u_2, u_3)$$
$$y = y(u_1, u_2, u_3)$$
$$z = z(u_1, u_2, u_3)$$

If u_2 and u_3 are constant, then as u_1 varies, the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of P describes a curve called the u_1 coordinate curve. Similarly we define the u_2 and u_3 coordinate curves through P. The vectors $\partial\mathbf{r}/\partial u_1$, $\partial\mathbf{r}/\partial u_2$, $\partial\mathbf{r}/\partial u_3$ represent tangent vectors to the u_1, u_2, u_3 coordinate curves. Letting $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit tangent vectors to these curves, we have

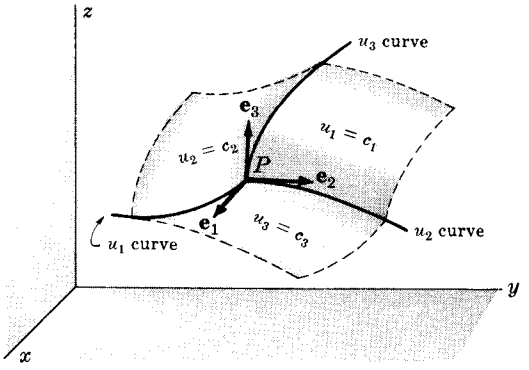


Fig. 22-11

22.67 $$\frac{\partial\mathbf{r}}{\partial u_1} = h_1\mathbf{e}_1, \quad \frac{\partial\mathbf{r}}{\partial u_2} = h_2\mathbf{e}_2, \quad \frac{\partial\mathbf{r}}{\partial u_3} = h_3\mathbf{e}_3$$

where

22.68 $$h_1 = \left|\frac{\partial\mathbf{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial\mathbf{r}}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial\mathbf{r}}{\partial u_3}\right|$$

are called *scale factors*. If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are mutually perpendicular, the curvilinear coordinate system is called *orthogonal*.

FORMULAS INVOLVING ORTHOGONAL CURVILINEAR COORDINATES

22.69 $$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3$$

22.70 $$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

where ds is the element of arc length.

If dV is the element of volume, then

22.71 $$dV = |(h_1\mathbf{e}_1\, du_1) \cdot (h_2\mathbf{e}_2\, du_2) \times (h_3\mathbf{e}_3\, du_3)| = h_1 h_2 h_3\, du_1\, du_2\, du_3$$

$$= \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| du_1\, du_2\, du_3 = \left| \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \right| du_1\, du_2\, du_3$$

where

22.72 $$\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix} \partial x/\partial u_1 & \partial x/\partial u_2 & \partial x/\partial u_3 \\ \partial y/\partial u_1 & \partial y/\partial u_2 & \partial y/\partial u_3 \\ \partial z/\partial u_1 & \partial z/\partial u_2 & \partial z/\partial u_3 \end{vmatrix}$$

is called the *Jacobian* of the transformation.

TRANSFORMATION OF MULTIPLE INTEGRALS

The result 22.72 can be used to transform multiple integrals from rectangular to curvilinear coordinates. For example, we have

22.73 $$\iiint_{\mathcal{R}} F(x, y, z)\, dx\, dy\, dz = \iiint_{\mathcal{R}'} G(u_1, u_2, u_3) \left| \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \right| du_1\, du_2\, du_3$$

where $\mathcal{R}'$ is the region into which $\mathcal{R}$ is mapped by the transformation and $G(u_1, u_2, u_3)$ is the value of $F(x, y, z)$ corresponding to the transformation.

GRADIENT, DIVERGENCE, CURL AND LAPLACIAN

In the following, Φ is a scalar function and $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ a vector function of orthogonal curvilinear coordinates u_1, u_2, u_3.

22.74 Gradient of Φ = grad Φ = $\nabla\Phi$ = $$\frac{1}{h_1}\frac{\partial \Phi}{\partial u_1} + \frac{1}{h_2}\frac{\partial \Phi}{\partial u_2} + \frac{1}{h_3}\frac{\partial \Phi}{\partial u_3}$$

22.75 Divergence of $\mathbf{A}$ = div $\mathbf{A}$ = $\nabla \cdot \mathbf{A}$ = $$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

22.76 Curl of $\mathbf{A}$ = curl $\mathbf{A}$ = $\nabla \times \mathbf{A}$ = $$\frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 A_3) - \frac{\partial}{\partial u_3}(h_2 A_2)\right]\mathbf{e}_1 + \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial u_3}(h_1 A_1) - \frac{\partial}{\partial u_1}(h_3 A_3)\right]\mathbf{e}_2$$

$$+ \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 A_2) - \frac{\partial}{\partial u_2}(h_1 A_1)\right]\mathbf{e}_3$$

22.77 Laplacian of Φ = $\nabla^2\Phi$ = $$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \Phi}{\partial u_3}\right)\right]$$

Note that the biharmonic operator $\nabla^4\Phi = \nabla^2(\nabla^2\Phi)$ can be obtained from 22.77.

SPECIAL ORTHOGONAL COORDINATE SYSTEMS

Cylindrical Coordinates (r, θ, z) [See Fig. 22-12]

22.78 $$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

22.79 $$h_1^2 = 1, \quad h_2^2 = r^2, \quad h_3^2 = 1$$

22.80 $$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

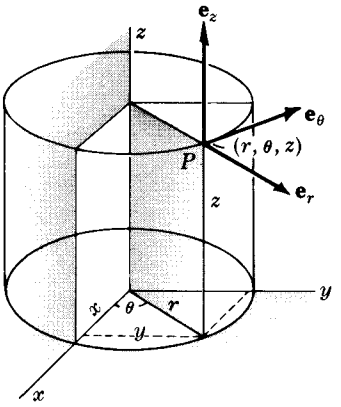


Fig. 22-12. Cylindrical coordinates.

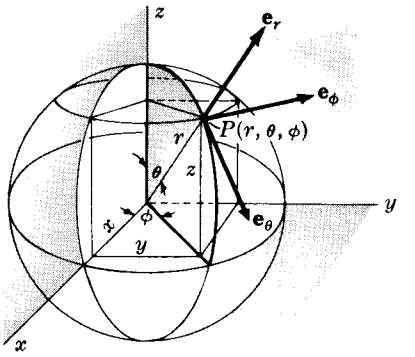


Fig. 22-13. Spherical coordinates.

Spherical Coordinates (r, θ, ϕ) [See Fig. 22-13]

22.81 $$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

22.82 $$h_1^2 = 1, \quad h_2^2 = r^2, \quad h_3^2 = r^2\sin^2\theta$$

22.83 $$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Parabolic Cylindrical Coordinates (u, v, z)

22.84 $$x = \tfrac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z$$

22.85 $$h_1^2 = h_2^2 = u^2 + v^2, \quad h_3^2 = 1$$

22.86 $$\nabla^2\Phi = \frac{1}{u^2+v^2}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 22-14. They are confocal parabolas with a common axis.

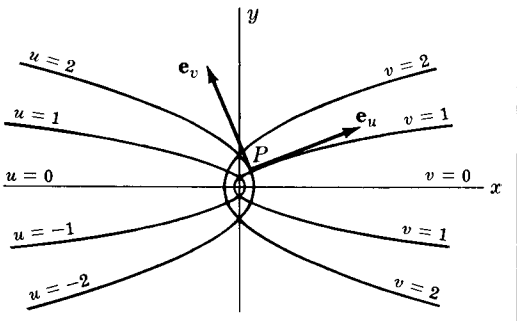


Fig. 22-14

Paraboloidal Coordinates (u, v, ϕ)

22.87 $$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \tfrac{1}{2}(u^2 - v^2)$$

where $$u \geqq 0, \quad v \geqq 0, \quad 0 \leqq \phi < 2\pi$$

22.88 $$h_1^2 = h_2^2 = u^2 + v^2, \quad h_3^2 = u^2v^2$$

22.89 $$\nabla^2\Phi = \frac{1}{u(u^2+v^2)}\frac{\partial}{\partial u}\left(u\frac{\partial\Phi}{\partial u}\right) + \frac{1}{v(u^2+v^2)}\frac{\partial}{\partial v}\left(v\frac{\partial\Phi}{\partial v}\right) + \frac{1}{u^2v^2}\frac{\partial^2\Phi}{\partial\phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the parabolas of Fig. 22-14 about the x axis which is then relabeled the z axis.

Elliptic Cylindrical Coordinates (u, v, z)

22.90 $$x = a\cosh u\cos v, \quad y = a\sinh u\sin v, \quad z = z$$

where $$u \geqq 0, \quad 0 \leqq v < 2\pi, \quad -\infty < z < \infty$$

22.91 $$h_1^2 = h_2^2 = a^2(\sinh^2 u + \sin^2 v), \quad h_3^2 = 1$$

22.92 $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 22-15. They are confocal ellipses and hyperbolas.

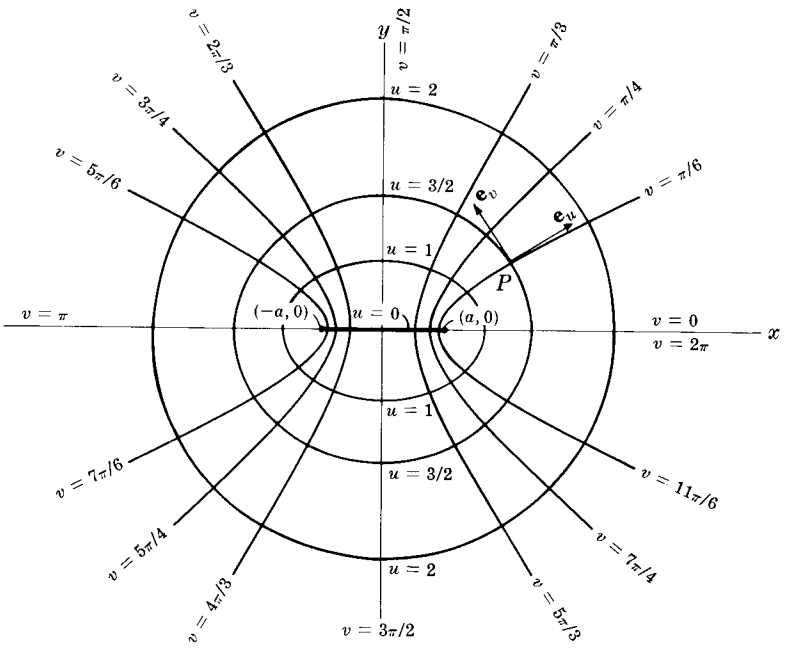


Fig. 22-15. Elliptic cylindrical coordinates.

Prolate Spheroidal Coordinates (ξ, η, ϕ)

22.93 $$x = a \sinh \xi \sin \eta \cos \phi, \quad y = a \sinh \xi \sin \eta \sin \phi, \quad z = a \cosh \xi \cos \eta$$

where $$\xi \geqq 0, \quad 0 \leqq \eta \leqq \pi, \quad 0 \leqq \phi < 2\pi$$

22.94 $$h_1^2 = h_2^2 = a^2(\sinh^2 \xi + \sin^2 \eta), \quad h_3^2 = a^2 \sinh^2 \xi \sin^2 \eta$$

22.95 $$\nabla^2 \Phi = \frac{1}{a^2(\sinh^2 \xi + \sin^2 \eta) \sinh \xi} \frac{\partial}{\partial \xi}\left(\sinh \xi \frac{\partial \Phi}{\partial \xi}\right) + \frac{1}{a^2(\sinh^2 \xi + \sin^2 \eta) \sin \eta} \frac{\partial}{\partial \eta}\left(\sin \eta \frac{\partial \Phi}{\partial \eta}\right) + \frac{1}{a^2 \sinh^2 \xi \sin^2 \eta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the x axis which is relabeled the z axis. The third set of coordinate surfaces consists of planes passing through this axis.

Oblate Spheroidal Coordinates (ξ, η, ϕ)

22.96 $$x = a \cosh \xi \cos \eta \cos \phi, \quad y = a \cosh \xi \cos \eta \sin \phi, \quad z = a \sinh \xi \sin \eta$$

where $$\xi \geqq 0, \quad -\pi/2 \leqq \eta \leqq \pi/2, \quad 0 \leqq \phi < 2\pi$$

22.97 $$h_1^2 = h_2^2 = a^2(\sinh^2 \xi + \sin^2 \eta), \quad h_3^2 = a^2 \cosh^2 \xi \cos^2 \eta$$

22.98 $$\nabla^2 \Phi = \frac{1}{a^2(\sinh^2 \xi + \sin^2 \eta) \cosh \xi} \frac{\partial}{\partial \xi}\left(\cosh \xi \frac{\partial \Phi}{\partial \xi}\right) + \frac{1}{a^2(\sinh^2 \xi + \sin^2 \eta) \cos \eta} \frac{\partial}{\partial \eta}\left(\cos \eta \frac{\partial \Phi}{\partial \eta}\right) + \frac{1}{a^2 \cosh^2 \xi \cos^2 \eta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the y axis which is relabeled the z axis. The third set of coordinate surfaces are planes passing through this axis.

Bipolar Coordinates (u, v, z)

22.99 $$x = \frac{a \sinh v}{\cosh v - \cos u}, \quad y = \frac{a \sin u}{\cosh v - \cos u}, \quad z = z$$

where $$0 \leqq u < 2\pi, \quad -\infty < v < \infty, \quad -\infty < z < \infty$$

or

22.100 $$x^2 + (y - a \cot u)^2 = a^2 \csc^2 u, \quad (x - a \coth v)^2 + y^2 = a^2 \operatorname{csch}^2 v, \quad z = z$$

22.101 $$h_1^2 = h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}, \quad h_3^2 = 1$$

22.102 $$\nabla^2 \Phi = \frac{(\cosh v - \cos u)^2}{a^2}\left(\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2}\right) + \frac{\partial^2 \Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 22-16 below.

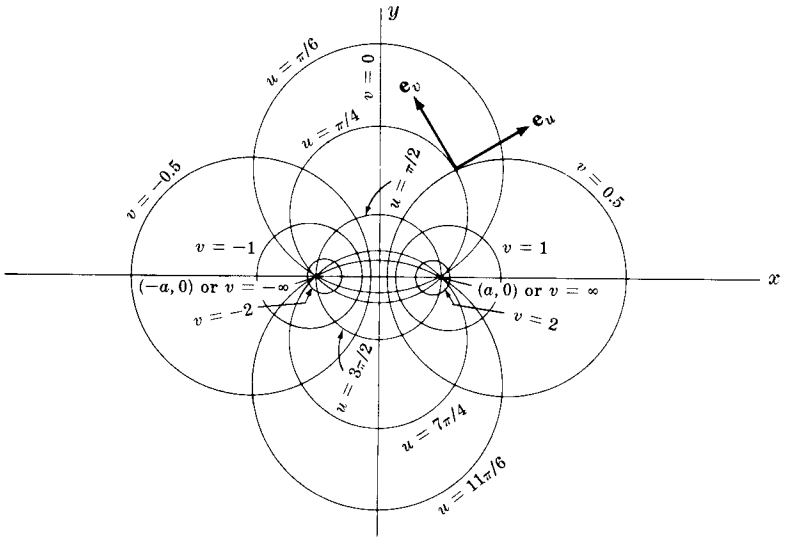


Fig. 22-16. Bipolar coordinates.

Toroidal Coordinates (u, v, ϕ)

22.103
$$x = \frac{a \sinh v \cos \phi}{\cosh v - \cos u}, \quad y = \frac{a \sinh v \sin \phi}{\cosh v - \cos u}, \quad z = \frac{a \sin u}{\cosh v - \cos u}$$

22.104
$$h_1^2 = h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}, \quad h_3^2 = \frac{a^2 \sinh^2 v}{(\cosh v - \cos u)^2}$$

22.105
$$\nabla^2 \Phi = \frac{(\cosh v - \cos u)^3}{a^2} \frac{\partial}{\partial u}\left(\frac{1}{\cosh v - \cos u} \frac{\partial \Phi}{\partial u}\right) + \frac{(\cosh v - \cos u)^3}{a^2 \sinh v} \frac{\partial}{\partial v}\left(\frac{\sinh v}{\cosh v - \cos u} \frac{\partial \Phi}{\partial v}\right) + \frac{(\cosh v - \cos u)^2}{a^2 \sinh^2 v} \frac{\partial^2 \Phi}{\partial \phi^2}$$

The coordinate surfaces are obtained by revolving the curves of Fig. 22-16 about the y axis which is relabeled the z axis.

Conical Coordinates (λ, μ, ν)

22.106
$$x = \frac{\lambda \mu \nu}{ab}, \quad y = \frac{\lambda}{a}\sqrt{\frac{(\mu^2 - a^2)(\nu^2 - a^2)}{a^2 - b^2}}, \quad z = \frac{\lambda}{b}\sqrt{\frac{(\mu^2 - b^2)(\nu^2 - b^2)}{b^2 - a^2}}$$

22.107
$$h_1^2 = 1, \quad h_2^2 = \frac{\lambda^2(\mu^2 - \nu^2)}{(\mu^2 - a^2)(b^2 - \mu^2)}, \quad h_3^2 = \frac{\lambda^2(\mu^2 - \nu^2)}{(\nu^2 - a^2)(\nu^2 - b^2)}$$

Confocal Ellipsoidal Coordinates (λ, μ, ν)

22.108

$$\begin{cases} \dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} + \dfrac{z^2}{c^2-\lambda} = 1 & \lambda < c^2 < b^2 < a^2 \\ \dfrac{x^2}{a^2-\mu} + \dfrac{y^2}{b^2-\mu} + \dfrac{z^2}{c^2-\mu} = 1 & c^2 < \mu < b^2 < a^2 \\ \dfrac{x^2}{a^2-\nu} + \dfrac{y^2}{b^2-\nu} + \dfrac{z^2}{c^2-\nu} = 1 & c^2 < b^2 < \nu < a^2 \end{cases}$$

or

22.109

$$\begin{cases} x^2 = \dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)} \\ y^2 = \dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-a^2)(b^2-c^2)} \\ z^2 = \dfrac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(c^2-a^2)(c^2-b^2)} \end{cases}$$

22.110

$$\begin{cases} h_1^2 = \dfrac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)} \\ h_2^2 = \dfrac{(\nu-\mu)(\lambda-\mu)}{4(a^2-\mu)(b^2-\mu)(c^2-\mu)} \\ h_3^2 = \dfrac{(\lambda-\nu)(\mu-\nu)}{4(a^2-\nu)(b^2-\nu)(c^2-\nu)} \end{cases}$$

Confocal Paraboloidal Coordinates (λ, μ, ν)

22.111

$$\begin{cases} \dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} = z - \lambda & -\infty < \lambda < b^2 \\ \dfrac{x^2}{a^2-\mu} + \dfrac{y^2}{b^2-\mu} = z - \mu & b^2 < \mu < a^2 \\ \dfrac{x^2}{a^2-\nu} + \dfrac{y^2}{b^2-\nu} = z - \nu & a^2 < \nu < \infty \end{cases}$$

or

22.112

$$\begin{cases} x^2 = \dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2} \\ y^2 = \dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2} \\ z = \lambda + \mu + \nu - a^2 - b^2 \end{cases}$$

22.113

$$\begin{cases} h_1^2 = \dfrac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)} \\ h_2^2 = \dfrac{(\nu-\mu)(\lambda-\mu)}{4(a^2-\mu)(b^2-\mu)} \\ h_3^2 = \dfrac{(\lambda-\nu)(\mu-\nu)}{16(a^2-\nu)(b^2-\nu)} \end{cases}$$

23 FOURIER SERIES

DEFINITION OF A FOURIER SERIES

The Fourier series corresponding to a function $f(x)$ defined in the interval $c \leqq x \leqq c+2L$ where c and $L > 0$ are constants, is defined as

23.1
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

23.2
$$\begin{cases} a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos \frac{n\pi x}{L} \, dx \\ b_n = \frac{1}{L} \int_c^{c+2L} f(x) \sin \frac{n\pi x}{L} \, dx \end{cases}$$

If $f(x)$ and $f'(x)$ are piecewise continuous and $f(x)$ is defined by periodic extension of period $2L$, i.e. $f(x+2L) = f(x)$, then the series converges to $f(x)$ if x is a point of continuity and to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ if x is a point of discontinuity.

COMPLEX FORM OF FOURIER SERIES

Assuming that the series 23.1 converges to $f(x)$, we have

23.3
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where

23.4
$$c_n = \frac{1}{L} \int_c^{c+2L} f(x) e^{-in\pi x/L} \, dx = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \\ \frac{1}{2}a_0 & n = 0 \end{cases}$$

PARSEVAL'S IDENTITY

23.5
$$\frac{1}{L} \int_c^{c+2L} \{f(x)\}^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

GENERALIZED PARSEVAL IDENTITY

23.6
$$\frac{1}{L} \int_c^{c+2L} f(x) \, g(x) \, dx = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n)$$

where a_n, b_n and c_n, d_n are the Fourier coefficients corresponding to $f(x)$ and $g(x)$ respectively.

SPECIAL FOURIER SERIES AND THEIR GRAPHS

23.7 $f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$

$$\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

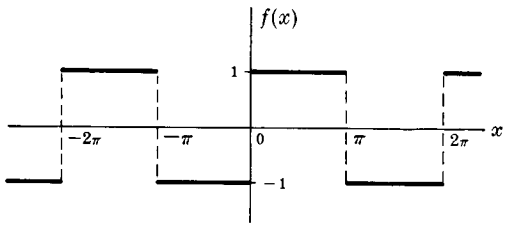


Fig. 23-1

23.8 $f(x) = |x| = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases}$

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

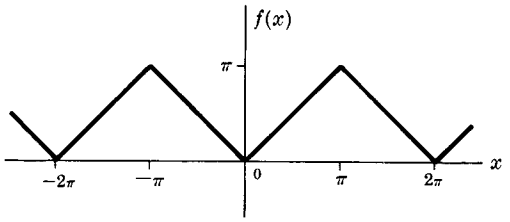


Fig. 23-2

23.9 $f(x) = x, \quad -\pi < x < \pi$

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

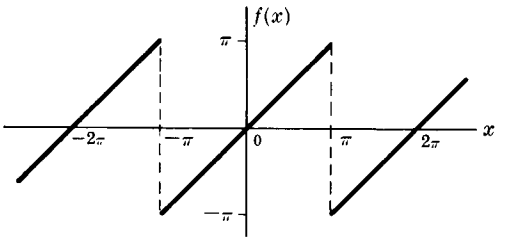


Fig. 23-3

23.10 $f(x) = x, \quad 0 < x < 2\pi$

$$\pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

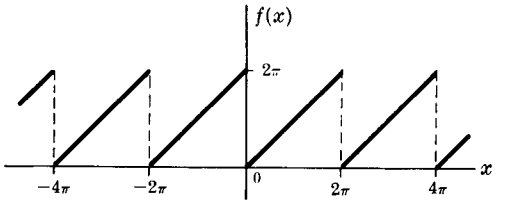


Fig. 23-4

23.11 $f(x) = |\sin x|, \quad -\pi < x < \pi$

$$\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1 \cdot 3} + \frac{\cos 4x}{3 \cdot 5} + \frac{\cos 6x}{5 \cdot 7} + \cdots\right)$$

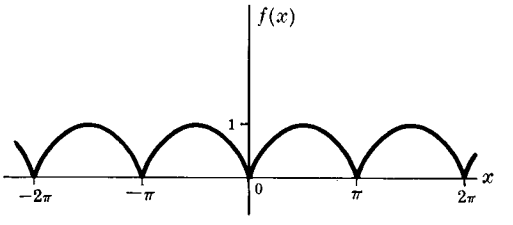


Fig. 23-5

23.12 $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$

$$\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \cdots\right)$$

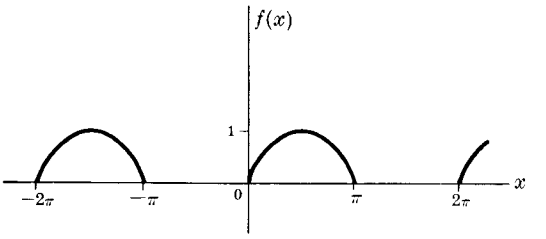


Fig. 23-6

23.13 $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ -\cos x & -\pi < x < 0 \end{cases}$

$$\frac{8}{\pi}\left(\frac{\sin 2x}{1\cdot 3} + \frac{2\sin 4x}{3\cdot 5} + \frac{3\sin 6x}{5\cdot 7} + \cdots\right)$$

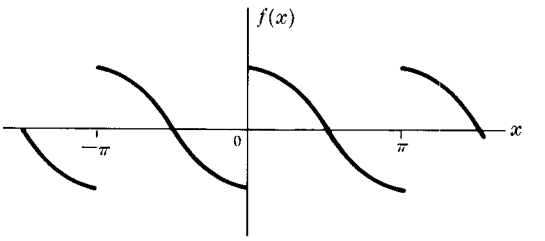


Fig. 23-7

23.14 $f(x) = x^2, \ -\pi < x < \pi$

$$\frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

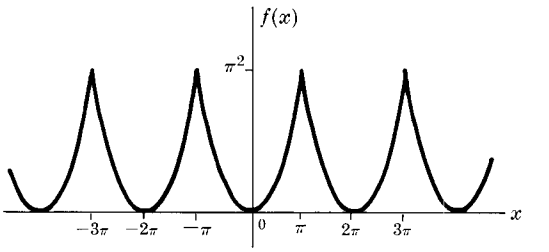


Fig. 23-8

23.15 $f(x) = x(\pi - x), \ 0 < x < \pi$

$$\frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right)$$

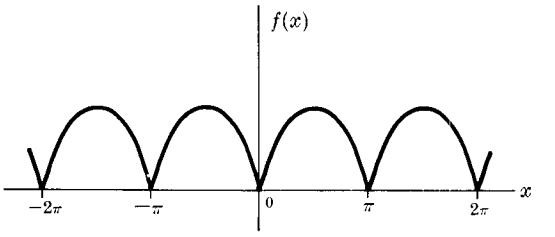


Fig. 23-9

23.16 $f(x) = x(\pi - x)(\pi + x), \ -\pi < x < \pi$

$$12\left(\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots\right)$$

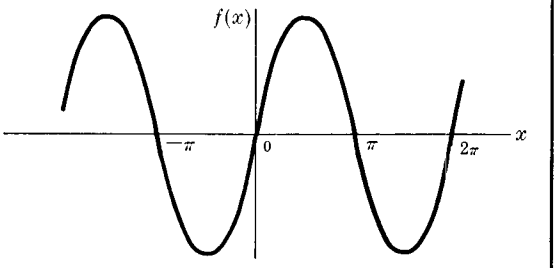


Fig. 23-10

23.17 $$f(x) = \begin{cases} 0 & 0 < x < \pi - \alpha \\ 1 & \pi - \alpha < x < \pi + \alpha \\ 0 & \pi + \alpha < x < 2\pi \end{cases}$$

$$\frac{\alpha}{\pi} - \frac{2}{\pi}\left(\frac{\sin\alpha\cos x}{1} - \frac{\sin 2\alpha\cos 2x}{2} + \frac{\sin 3\alpha\cos 3x}{3} - \cdots\right)$$

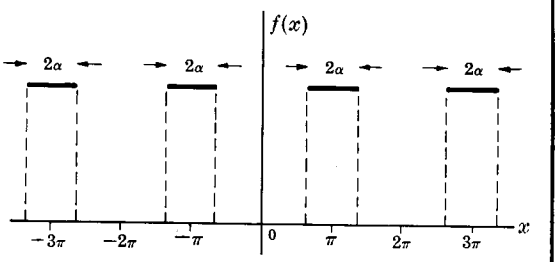


Fig. 23-11

23.18 $$f(x) = \begin{cases} x(\pi - x) & 0 < x < \pi \\ -x(\pi - x) & -\pi < x < 0 \end{cases}$$

$$\frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right)$$

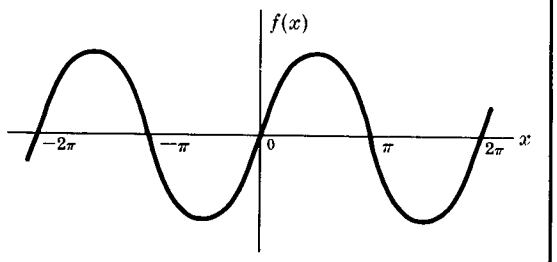


Fig. 23-12

MISCELLANEOUS FOURIER SERIES

23.19 $f(x) = \sin\mu x, \quad -\pi < x < \pi, \quad \mu \neq \text{integer}$

$$\frac{2\sin\mu\pi}{\pi}\left(\frac{\sin x}{1^2 - \mu^2} - \frac{2\sin 2x}{2^2 - \mu^2} + \frac{3\sin 3x}{3^2 - \mu^2} - \cdots\right)$$

23.20 $f(x) = \cos\mu x, \quad -\pi < x < \pi, \quad \mu \neq \text{integer}$

$$\frac{2\mu\sin\mu\pi}{\pi}\left(\frac{1}{2\mu^2} + \frac{\cos x}{1^2 - \mu^2} - \frac{\cos 2x}{2^2 - \mu^2} + \frac{\cos 3x}{3^2 - \mu^2} - \cdots\right)$$

23.21 $f(x) = \tan^{-1}[(a\sin x)/(1 - a\cos x)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a\sin x + \frac{a^2}{2}\sin 2x + \frac{a^3}{3}\sin 3x + \cdots$$

23.22 $f(x) = \ln(1 - 2a\cos x + a^2), \quad -\pi < x < \pi, \quad |a| < 1$

$$-2\left(a\cos x + \frac{a^2}{2}\cos 2x + \frac{a^3}{3}\cos 3x + \cdots\right)$$

23.23 $f(x) = \frac{1}{2}\tan^{-1}[(2a\sin x)/(1 - a^2)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a\sin x + \frac{a^3}{3}\sin 3x + \frac{a^5}{5}\sin 5x + \cdots$$

23.24 $f(x) = \frac{1}{2}\tan^{-1}[(2a\cos x)/(1-a^2)], \quad -\pi < x < \pi, \quad |a| < 1$

$$a\cos x - \frac{a^3}{3}\cos 3x + \frac{a^5}{5}\cos 5x - \cdots$$

23.25 $f(x) = e^{\mu x}, \quad -\pi < x < \pi$

$$\frac{2\sinh \mu\pi}{\pi}\left(\frac{1}{2\mu} + \sum_{n=1}^{\infty}\frac{(-1)^n(\mu\cos nx - n\sin nx)}{\mu^2 + n^2}\right)$$

23.26 $f(x) = \sinh \mu x, \quad -\pi < x < \pi$

$$\frac{2\sinh \mu\pi}{\pi}\left(\frac{\sin x}{1^2+\mu^2} - \frac{2\sin 2x}{2^2+\mu^2} + \frac{3\sin 3x}{3^2+\mu^2} - \cdots\right)$$

23.27 $f(x) = \cosh \mu x, \quad -\pi < x < \pi$

$$\frac{2\mu\sinh \mu\pi}{\pi}\left(\frac{1}{2\mu^2} - \frac{\cos x}{1^2+\mu^2} + \frac{\cos 2x}{2^2+\mu^2} - \frac{\cos 3x}{3^2+\mu^2} + \cdots\right)$$

23.28 $f(x) = \ln|\sin \tfrac{1}{2}x|, \quad 0 < x < \pi$

$$-\left(\ln 2 + \frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots\right)$$

23.29 $f(x) = \ln|\cos \tfrac{1}{2}x|, \quad -\pi < x < \pi$

$$-\left(\ln 2 - \frac{\cos x}{1} + \frac{\cos 2x}{2} - \frac{\cos 3x}{3} + \cdots\right)$$

23.30 $f(x) = \frac{1}{6}\pi^2 - \frac{1}{2}\pi x + \frac{1}{4}x^2, \quad 0 \leqq x \leqq 2\pi$

$$\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$$

23.31 $f(x) = \frac{1}{12}x(x-\pi)(x-2\pi), \quad 0 \leqq x \leqq 2\pi$

$$\frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \cdots$$

23.32 $f(x) = \frac{1}{90}\pi^4 - \frac{1}{12}\pi^2x^2 + \frac{1}{12}\pi x^3 - \frac{1}{48}x^4, \quad 0 \leqq x \leqq 2\pi$

$$\frac{\cos x}{1^4} + \frac{\cos 2x}{2^4} + \frac{\cos 3x}{3^4} + \cdots$$

24 BESSEL FUNCTIONS

BESSEL'S DIFFERENTIAL EQUATION

24.1
$$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad n \geq 0$$

Solutions of this equation are called *Bessel functions of order n.*

BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER n

24.2
$$J_n(x) = \frac{x^n}{2^n\,\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

24.3
$$J_{-n}(x) = \frac{x^{-n}}{2^{-n}\,\Gamma(1-n)}\left\{1 - \frac{x^2}{2(2-2n)} + \frac{x^4}{2\cdot 4(2-2n)(4-2n)} - \cdots\right\}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\,\Gamma(k+1-n)}$$

24.4
$$J_{-n}(x) = (-1)^n J_n(x) \qquad n = 0, 1, 2, \ldots$$

If $n \neq 0, 1, 2, \ldots$, $J_n(x)$ and $J_{-n}(x)$ are linearly independent.

If $n \neq 0, 1, 2, \ldots$, $J_n(x)$ is bounded at $x = 0$ while $J_{-n}(x)$ is unbounded.

For $n = 0, 1$ we have

24.5
$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

24.6
$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} - \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \cdots$$

24.7
$$J_0'(x) = -J_1(x)$$

BESSEL FUNCTIONS OF THE SECOND KIND OF ORDER n

24.8
$$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} & n \neq 0, 1, 2, \ldots \\ \lim\limits_{p\to n} \dfrac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} & n = 0, 1, 2, \ldots \end{cases}$$

This is also called *Weber's function* or *Neumann's function* [also denoted by $N_n(x)$].

For $n = 0, 1, 2, \ldots$, L'Hospital's rule yields

24.9
$$Y_n(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\}J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(x/2)^{2k-n} - \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\{\Phi(k) + \Phi(n+k)\}\frac{(x/2)^{2k+n}}{k!\,(n+k)!}$$

where $\gamma = .5772156\ldots$ is Euler's constant [page 1] and

24.10
$$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}, \qquad \Phi(0) = 0$$

For $n = 0$,

24.11
$$Y_0(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\}J_0(x) + \frac{2}{\pi}\left\{\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}(1 + \tfrac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) - \cdots\right\}$$

24.12
$$Y_{-n}(x) = (-1)^n Y_n(x) \qquad n = 0, 1, 2, \ldots$$

For any value $n \geqq 0$, $J_n(x)$ is bounded at $x = 0$ while $Y_n(x)$ is unbounded.

GENERAL SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION

24.13 $y = AJ_n(x) + BJ_{-n}(x)$ $\qquad n \neq 0, 1, 2, \ldots$

24.14 $y = AJ_n(x) + BY_n(x)$ $\qquad$ all n

24.15 $y = AJ_n(x) + BJ_n(x)\int \frac{dx}{xJ_n^2(x)}$ $\qquad$ all n

where A and B are arbitrary constants.

GENERATING FUNCTION FOR $J_n(x)$

24.16
$$e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

RECURRENCE FORMULAS FOR BESSEL FUNCTIONS

24.17
$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

24.18
$$J_n'(x) = \tfrac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$$

24.19
$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$$

24.20
$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

24.21
$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

24.22
$$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$$

The functions $Y_n(x)$ satisfy identical relations.

BESSEL FUNCTIONS OF ORDER EQUAL TO HALF AN ODD INTEGER

In this case the functions are expressible in terms of sines and cosines.

24.23 $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

24.24 $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

24.25 $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

24.26 $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$

24.27 $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \left(\frac{3}{x^2} - 1 \right) \sin x - \frac{3}{x} \cos x \right\}$

24.28 $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \frac{3}{x} \sin x + \left(\frac{3}{x^2} - 1 \right) \cos x \right\}$

For further results use the recurrence formula. Results for $Y_{1/2}(x), Y_{3/2}(x), \ldots$ are obtained from 24.8.

HANKEL FUNCTIONS OF FIRST AND SECOND KINDS OF ORDER n

24.29 $H_n^{(1)}(x) = J_n(x) + i Y_n(x)$

24.30 $H_n^{(2)}(x) = J_n(x) - i Y_n(x)$

BESSEL'S MODIFIED DIFFERENTIAL EQUATION

24.31 $$x^2y'' + xy' - (x^2 + n^2)y = 0 \qquad n \geqq 0$$

Solutions of this equation are called *modified Bessel functions of order n.*

MODIFIED BESSEL FUNCTIONS OF THE FIRST KIND OF ORDER n

24.32
$$I_n(x) = i^{-n} J_n(ix) = e^{-n\pi i/2} J_n(ix)$$
$$= \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} + \cdots \right\} = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\, \Gamma(n+k+1)}$$

24.33
$$I_{-n}(x) = i^n J_{-n}(ix) = e^{n\pi i/2} J_{-n}(ix)$$
$$= \frac{x^{-n}}{2^{-n} \Gamma(1-n)} \left\{ 1 + \frac{x^2}{2(2-2n)} + \frac{x^4}{2 \cdot 4(2-2n)(4-2n)} + \cdots \right\} = \sum_{k=0}^{\infty} \frac{(x/2)^{2k-n}}{k!\, \Gamma(k+1-n)}$$

24.34 $$I_{-n}(x) = I_n(x) \qquad n = 0, 1, 2, \ldots$$

If $n \neq 0, 1, 2, \ldots$, then $I_n(x)$ and $I_{-n}(x)$ are linearly independent.

For $n = 0, 1$, we have

24.35 $I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

24.36 $I_1(x) = \frac{x}{2} + \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots$

24.37 $I_0'(x) = I_1(x)$

MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND OF ORDER n

24.38
$$K_n(x) = \begin{cases} \dfrac{\pi}{2\sin n\pi}\{I_{-n}(x) - I_n(x)\} & n \neq 0, 1, 2, \ldots \\ \lim\limits_{p\to n} \dfrac{\pi}{2\sin p\pi}\{I_{-p}(x) - I_p(x)\} & n = 0, 1, 2, \ldots \end{cases}$$

For $n = 0, 1, 2, \ldots$, L'Hospital's rule yields

24.39
$$K_n(x) = (-1)^{n+1}\{\ln(x/2) + \gamma\}I_n(x) + \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k(n-k-1)!\,(x/2)^{2k-n} + \frac{(-1)^n}{2}\sum_{k=0}^{\infty}\frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k) + \Phi(n+k)\}$$

where $\Phi(p)$ is given by 24.10.

For $n = 0$,

24.40
$$K_0(x) = -\{\ln(x/2) + \gamma\}I_0(x) + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2}(1 + \tfrac{1}{2}) + \frac{x^6}{2^2\cdot 4^2\cdot 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \cdots$$

24.41
$$K_{-n}(x) = K_n(x) \qquad n = 0, 1, 2, \ldots$$

GENERAL SOLUTION OF BESSEL'S MODIFIED EQUATION

24.42
$$y = AI_n(x) + BI_{-n}(x) \qquad n \neq 0, 1, 2, \ldots$$

24.43
$$y = AI_n(x) + BK_n(x) \qquad \text{all } n$$

24.44
$$y = AI_n(x) + BI_n(x)\int \frac{dx}{xI_n^2(x)} \qquad \text{all } n$$

where A and B are arbitrary constants.

GENERATING FUNCTION FOR $I_n(x)$

24.45
$$e^{x(t+1/t)/2} = \sum_{n=-\infty}^{\infty} I_n(x)t^n$$

RECURRENCE FORMULAS FOR MODIFIED BESSEL FUNCTIONS

24.46 $I_{n+1}(x) = I_{n-1}(x) - \dfrac{2n}{x}I_n(x)$

24.47 $I_n'(x) = \frac{1}{2}\{I_{n-1}(x) + I_{n+1}(x)\}$

24.48 $xI_n'(x) = xI_{n-1}(x) - nI_n(x)$

24.49 $xI_n'(x) = xI_{n+1}(x) + nI_n(x)$

24.50 $\dfrac{d}{dx}\{x^nI_n(x)\} = x^nI_{n-1}(x)$

24.51 $\dfrac{d}{dx}\{x^{-n}I_n(x)\} = x^{-n}I_{n+1}(x)$

24.52 $K_{n+1}(x) = K_{n-1}(x) + \dfrac{2n}{x}K_n(x)$

24.53 $K_n'(x) = \frac{1}{2}\{K_{n-1}(x) + K_{n+1}(x)\}$

24.54 $xK_n'(x) = -xK_{n-1}(x) - nK_n(x)$

24.55 $xK_n'(x) = nK_n(x) - xK_{n+1}(x)$

24.56 $\dfrac{d}{dx}\{x^nK_n(x)\} = -x^nK_{n-1}(x)$

24.57 $\dfrac{d}{dx}\{x^{-n}K_n(x)\} = -x^{-n}K_{n+1}(x)$

MODIFIED BESSEL FUNCTIONS OF ORDER EQUAL TO HALF AN ODD INTEGER

In this case the functions are expressible in terms of hyperbolic sines and cosines.

24.58 $$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$$

24.59 $$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x$$

24.60 $$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\cosh x - \frac{\sinh x}{x}\right)$$

24.61 $$I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\sinh x - \frac{\cosh x}{x}\right)$$

24.62 $$I_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2}+1\right)\sinh x - \frac{3}{x}\cosh x\right\}$$

24.63 $$I_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2}+1\right)\cosh x - \frac{3}{x}\sinh x\right\}$$

For further results use the recurrence formula 24.46. Results for $K_{1/2}(x), K_{3/2}(x), \ldots$ are obtained from 24.38.

Ber AND Bei FUNCTIONS

The real and imaginary parts of $J_n(xe^{3\pi i/4})$ are denoted by $\mathrm{Ber}_n(x)$ and $\mathrm{Bei}_n(x)$ where

24.64 $$\mathrm{Ber}_n(x) = \sum_{k=0}^{\infty}\frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)}\cos\frac{(3n+2k)\pi}{4}$$

24.65 $$\mathrm{Bei}_n(x) = \sum_{k=0}^{\infty}\frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)}\sin\frac{(3n+2k)\pi}{4}$$

If $n = 0$,

24.66 $$\mathrm{Ber}(x) = 1 - \frac{(x/2)^4}{2!^2} + \frac{(x/2)^8}{4!^2} - \cdots$$

24.67 $$\mathrm{Bei}(x) = (x/2)^2 - \frac{(x/2)^6}{3!^2} + \frac{(x/2)^{10}}{5!^2} - \cdots$$

Ker AND Kei FUNCTIONS

The real and imaginary parts of $e^{-n\pi i/2}K_n(xe^{\pi i/4})$ are denoted by $\mathrm{Ker}_n(x)$ and $\mathrm{Kei}_n(x)$ where

24.68
$$\begin{aligned}\mathrm{Ker}_n(x) = {} & -\{\ln(x/2)+\gamma\}\,\mathrm{Ber}_n(x) + \tfrac{1}{4}\pi\,\mathrm{Bei}_n(x)\\ & + \frac{1}{2}\sum_{k=0}^{n-1}\frac{(n-k-1)!\,(x/2)^{2k-n}}{k!}\cos\frac{(3n+2k)\pi}{4}\\ & + \frac{1}{2}\sum_{k=0}^{\infty}\frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k)+\Phi(n+k)\}\cos\frac{(3n+2k)\pi}{4}\end{aligned}$$

24.69
$$\begin{aligned}\mathrm{Kei}_n(x) = {} & -\{\ln(x/2)+\gamma\}\,\mathrm{Bei}_n(x) - \tfrac{1}{4}\pi\,\mathrm{Ber}_n(x)\\ & - \frac{1}{2}\sum_{k=0}^{n-1}\frac{(n-k-1)!\,(x/2)^{2k-n}}{k!}\sin\frac{(3n+2k)\pi}{4}\\ & + \frac{1}{2}\sum_{k=0}^{\infty}\frac{(x/2)^{n+2k}}{k!\,(n+k)!}\{\Phi(k)+\Phi(n+k)\}\sin\frac{(3n+2k)\pi}{4}\end{aligned}$$

and Φ is given by 24.10, page 137.

If $n = 0$,

24.70 $$\mathrm{Ker}(x) = -\{\ln(x/2)+\gamma\}\,\mathrm{Ber}(x) + \frac{\pi}{4}\mathrm{Bei}(x) + 1 - \frac{(x/2)^4}{2!^2}(1+\tfrac{1}{2}) + \frac{(x/2)^8}{4!^2}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}) - \cdots$$

24.71 $$\mathrm{Kei}(x) = -\{\ln(x/2)+\gamma\}\,\mathrm{Bei}(x) - \frac{\pi}{4}\mathrm{Ber}(x) + (x/2)^2 - \frac{(x/2)^6}{3!^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \cdots$$

DIFFERENTIAL EQUATION FOR Ber, Bei, Ker, Kei FUNCTIONS

24.72
$$x^2y'' + xy' - (ix^2 + n^2)y = 0$$

The general solution of this equation is

24.73
$$y = A\{\mathrm{Ber}_n(x) + i\,\mathrm{Bei}_n(x)\} + B\{\mathrm{Ker}_n(x) + i\,\mathrm{Kei}_n(x)\}$$

GRAPHS OF BESSEL FUNCTIONS

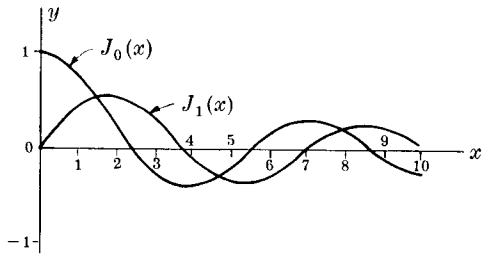


Fig. 24-1

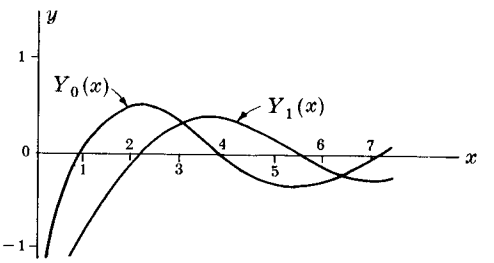


Fig. 24-2

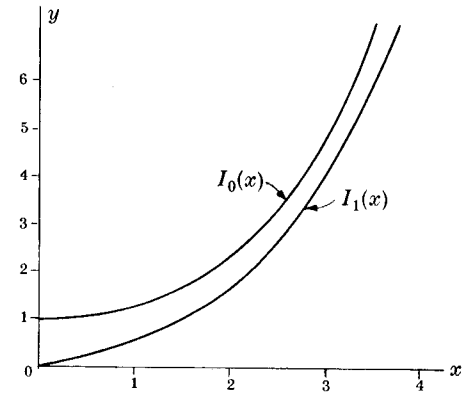


Fig. 24-3

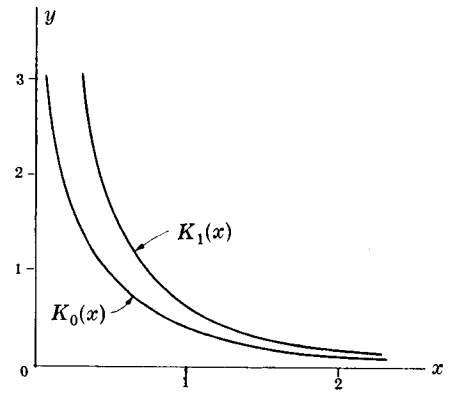


Fig. 24-4

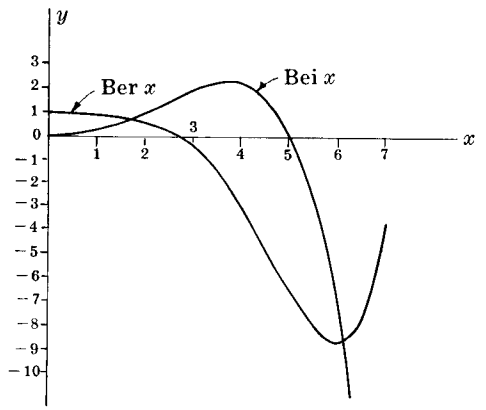


Fig. 24-5

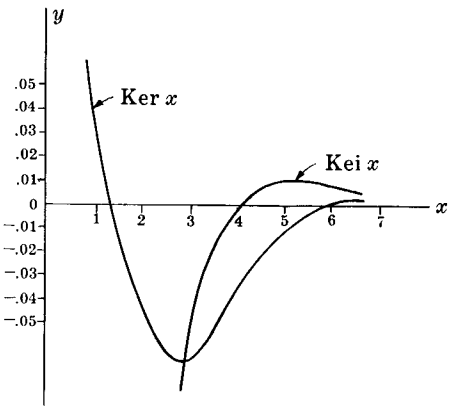


Fig. 24-6

INDEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

24.74 $$\int x J_0(x)\,dx = x J_1(x)$$

24.75 $$\int x^2 J_0(x)\,dx = x^2 J_1(x) + x J_0(x) - \int J_0(x)\,dx$$

24.76 $$\int x^m J_0(x)\,dx = x^m J_1(x) + (m-1)x^{m-1} J_0(x) - (m-1)^2 \int x^{m-2} J_0(x)\,dx$$

24.77 $$\int \frac{J_0(x)}{x^2}\,dx = J_1(x) - \frac{J_0(x)}{x} - \int J_0(x)\,dx$$

24.78 $$\int \frac{J_0(x)}{x^m}\,dx = \frac{J_1(x)}{(m-1)^2 x^{m-2}} - \frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2}\int \frac{J_0(x)}{x^{m-2}}\,dx$$

24.79 $$\int J_1(x)\,dx = -J_0(x)$$

24.80 $$\int x J_1(x)\,dx = -x J_0(x) + \int J_0(x)\,dx$$

24.81 $$\int x^m J_1(x)\,dx = -x^m J_0(x) + m\int x^{m-1} J_0(x)\,dx$$

24.82 $$\int \frac{J_1(x)}{x}\,dx = -J_1(x) + \int J_0(x)\,dx$$

24.83 $$\int \frac{J_1(x)}{x^m}\,dx = -\frac{J_1(x)}{mx^{m-1}} + \frac{1}{m}\int \frac{J_0(x)}{x^{m-1}}\,dx$$

24.84 $$\int x^n J_{n-1}(x)\,dx = x^n J_n(x)$$

24.85 $$\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x)$$

24.86 $$\int x^m J_n(x)\,dx = -x^m J_{n-1}(x) + (m+n-1)\int x^{m-1} J_{n-1}(x)\,dx$$

24.87 $$\int x J_n(\alpha x) J_n(\beta x)\,dx = \frac{x\{\alpha J_n(\beta x) J_n'(\alpha x) - \beta J_n(\alpha x) J_n'(\beta x)\}}{\beta^2 - \alpha^2}$$

24.88 $$\int x J_n^2(\alpha x)\,dx = \frac{x^2}{2}\{J_n'(\alpha x)\}^2 + \frac{x^2}{2}\left(1 - \frac{n^2}{\alpha^2 x^2}\right)\{J_n(\alpha x)\}^2$$

The above results also hold if we replace $J_n(x)$ by $Y_n(x)$ or, more generally, $A J_n(x) + B Y_n(x)$ where A and B are constants.

DEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

24.89 $$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$$

24.90 $$\int_0^\infty e^{-ax} J_n(bx)\,dx = \frac{(\sqrt{a^2+b^2}-a)^n}{b^n\sqrt{a^2+b^2}} \qquad n > -1$$

24.91 $$\int_0^\infty \cos ax\, J_0(bx)\,dx = \begin{cases} \dfrac{1}{\sqrt{a^2-b^2}} & a > b \\ 0 & a < b \end{cases}$$

24.92 $\displaystyle\int_0^\infty J_n(bx)\,dx = \frac{1}{b} \qquad n > -1$

24.93 $\displaystyle\int_0^\infty \frac{J_n(bx)}{x}\,dx = \frac{1}{n} \qquad n = 1, 2, 3, \ldots$

24.94 $\displaystyle\int_0^\infty e^{-ax} J_0(b\sqrt{x})\,dx = \frac{e^{-b^2/4a}}{a}$

24.95 $\displaystyle\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \frac{\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)}{\beta^2 - \alpha^2}$

24.96 $\displaystyle\int_0^1 x\,J_n^2(\alpha x)\,dx = \tfrac{1}{2}\{J_n'(\alpha)\}^2 + \tfrac{1}{2}(1 - n^2/\alpha^2)\{J_n(\alpha)\}^2$

24.97 $\displaystyle\int_0^1 x\,J_0(\alpha x)\,I_0(\beta x)\,dx = \frac{\beta J_0(\alpha) I_0'(\beta) - \alpha J_0'(\alpha) I_0(\beta)}{\alpha^2 + \beta^2}$

INTEGRAL REPRESENTATIONS FOR BESSEL FUNCTIONS

24.98 $\displaystyle J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin\theta)\,d\theta$

24.99 $\displaystyle J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta, \qquad n = \text{integer}$

24.100 $\displaystyle J_n(x) = \frac{x^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_0^\pi \cos(x\sin\theta)\cos^{2n}\theta\,d\theta, \qquad n > -\tfrac{1}{2}$

24.101 $\displaystyle Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh u)\,du$

24.102 $\displaystyle I_0(x) = \frac{1}{\pi}\int_0^\pi \cosh(x\sin\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{x\sin\theta}\,d\theta$

ASYMPTOTIC EXPANSIONS

24.103 $\displaystyle J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$ where x is large

24.104 $\displaystyle Y_n(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$ where x is large

24.105 $\displaystyle J_n(x) \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{ex}{2n}\right)^n$ where n is large

24.106 $\displaystyle Y_n(x) \sim -\sqrt{\frac{2}{\pi n}}\left(\frac{ex}{2n}\right)^{-n}$ where n is large

24.107 $\displaystyle I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}$ where x is large

24.108 $\displaystyle K_n(x) \sim \frac{e^{-x}}{\sqrt{2\pi x}}$ where x is large

ORTHOGONAL SERIES OF BESSEL FUNCTIONS

Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be the positive roots of $R J_n(x) + Sx J_n'(x) = 0$, $n > -1$. Then the following series expansions hold under the conditions indicated.

$S = 0$, $R \neq 0$, i.e. $\lambda_1, \lambda_2, \lambda_3, \ldots$ are positive roots of $J_n(x) = 0$

24.109
$$f(x) = A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + A_3 J_n(\lambda_3 x) + \cdots$$

where

24.110
$$A_k = \frac{2}{J_{n+1}^2(\lambda_k)} \int_0^1 x f(x) J_n(\lambda_k x)\, dx$$

In particular if $n = 0$,

24.111
$$f(x) = A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + A_3 J_0(\lambda_3 x) + \cdots$$

where

24.112
$$A_k = \frac{2}{J_1^2(\lambda_k)} \int_0^1 x f(x) J_0(\lambda_k x)\, dx$$

$R/S > -n$

24.113
$$f(x) = A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + A_3 J_n(\lambda_3 x) + \cdots$$

where

24.114
$$A_k = \frac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k) J_{n+1}(\lambda_k)} \int_0^1 x f(x) J_n(\lambda_k x)\, dx$$

In particular if $n = 0$,

24.115
$$f(x) = A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + A_3 J_0(\lambda_3 x) + \cdots$$

where

24.116
$$A_k = \frac{2}{J_0^2(\lambda_k) + J_1^2(\lambda_k)} \int_0^1 x f(x) J_0(\lambda_k x)\, dx$$

$R/S = -n$

24.117
$$f(x) = A_0 x^n + A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + \cdots$$

where

24.118
$$\begin{cases} A_0 = 2(n+1) \displaystyle\int_0^1 x^{n+1} f(x)\, dx \\ A_k = \dfrac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k) J_{n+1}(\lambda_k)} \displaystyle\int_0^1 x f(x) J_n(\lambda_k x)\, dx \end{cases}$$

In particular if $n = 0$ so that $R = 0$ [i.e. $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_1(x) = 0$],

24.119
$$f(x) = A_0 + A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + \cdots$$

where

24.120
$$\begin{cases} A_0 = 2 \displaystyle\int_0^1 x f(x)\, dx \\ A_k = \dfrac{2}{J_0^2(\lambda_k)} \displaystyle\int_0^1 x f(x) J_0(\lambda_k x)\, dx \end{cases}$$

$R/S < -n$

In this case there are two pure imaginary roots $\pm i\lambda_0$ as well as the positive roots $\lambda_1, \lambda_2, \lambda_3, \ldots$ and we have

24.121
$$f(x) = A_0 I_n(\lambda_0 x) + A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + \cdots$$

where

24.122
$$\begin{cases} A_0 = \dfrac{2}{I_n^2(\lambda_0) + I_{n-1}(\lambda_0) I_{n+1}(\lambda_0)} \displaystyle\int_0^1 x f(x) I_n(\lambda_0 x)\, dx \\ A_k = \dfrac{2}{J_n^2(\lambda_k) - J_{n-1}(\lambda_k) J_{n+1}(\lambda_k)} \displaystyle\int_0^1 x f(x) J_n(\lambda_k x)\, dx \end{cases}$$

MISCELLANEOUS RESULTS

24.123 $\cos(x \sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots$

24.124 $\sin(x \sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \cdots$

24.125 $J_n(x+y) = \displaystyle\sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y) \qquad n = 0, \pm 1, \pm 2, \ldots$

This is called the *addition formula* for Bessel functions.

24.126 $1 = J_0(x) + 2J_2(x) + \cdots + 2J_{2n}(x) + \cdots$

24.127 $x = 2\{J_1(x) + 3J_3(x) + 5J_5(x) + \cdots + (2n+1)J_{2n+1}(x) + \cdots\}$

24.128 $x^2 = 2\{4J_2(x) + 16J_4(x) + 36J_6(x) + \cdots + (2n)^2 J_{2n}(x) + \cdots\}$

24.129 $\dfrac{x J_1(x)}{4} = J_2(x) - 2J_4(x) + 3J_6(x) - \cdots$

24.130 $1 = J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + 2J_3^2(x) + \cdots$

24.131 $J_n''(x) = \frac{1}{4}\{J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\}$

24.132 $J_n'''(x) = \frac{1}{8}\{J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\}$

Formulas 24.131 and 24.132 can be generalized.

24.133 $J_n'(x) J_{-n}(x) - J_{-n}' J_n(x) = \dfrac{2\sin n\pi}{\pi x}$

24.134 $J_n(x) J_{-n+1}(x) + J_{-n}(x) J_{n-1}(x) = \dfrac{2\sin n\pi}{\pi x}$

24.135 $J_{n+1}(x) Y_n(x) - J_n(x) Y_{n+1}(x) = J_n(x) Y_n'(x) - J_n'(x) Y_n(x) = \dfrac{2}{\pi x}$

24.136 $\sin x = 2\{J_1(x) - J_3(x) + J_5(x) - \cdots\}$

24.137 $\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \cdots$

24.138 $\sinh x = 2\{I_1(x) + I_3(x) + I_5(x) + \cdots\}$

24.139 $\cosh x = I_0(x) + 2\{I_2(x) + I_4(x) + I_6(x) + \cdots\}$

25 LEGENDRE FUNCTIONS

LEGENDRE'S DIFFERENTIAL EQUATION

25.1 $$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Solutions of this equation are called *Legendre functions of order n.*

LEGENDRE POLYNOMIALS

If $n = 0, 1, 2, \ldots,$ solutions of 25.1 are Legendre polynomials $P_n(x)$ given by *Rodrigue's formula*

25.2 $$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

SPECIAL LEGENDRE POLYNOMIALS

25.3 $P_0(x) = 1$

25.4 $P_1(x) = x$

25.5 $P_2(x) = \frac{1}{2}(3x^2-1)$

25.6 $P_3(x) = \frac{1}{2}(5x^3-3x)$

25.7 $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

25.8 $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

25.9 $P_6(x) = \frac{1}{16}(231x^6-315x^4+105x^2-5)$

25.10 $P_7(x) = \frac{1}{16}(429x^7-693x^5+315x^3-35x)$

LEGENDRE POLYNOMIALS IN TERMS OF θ WHERE $x = \cos\theta$

25.11 $P_0(\cos\theta) = 1$

25.12 $P_1(\cos\theta) = \cos\theta$

25.13 $P_2(\cos\theta) = \frac{1}{4}(1+3\cos 2\theta)$

25.14 $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

25.15 $P_4(\cos\theta) = \frac{1}{64}(9+20\cos 2\theta + 35\cos 4\theta)$

25.16 $P_5(\cos\theta) = \frac{1}{128}(30\cos\theta + 35\cos 3\theta + 63\cos 5\theta)$

25.17 $P_6(\cos\theta) = \frac{1}{512}(50+105\cos 2\theta + 126\cos 4\theta + 231\cos 6\theta)$

25.18 $P_7(\cos\theta) = \frac{1}{1024}(175\cos\theta + 189\cos 3\theta + 231\cos 5\theta + 429\cos 7\theta)$

GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

25.19 $$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

RECURRENCE FORMULAS FOR LEGENDRE POLYNOMIALS

25.20 $$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

25.21 $$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)$$

25.22 $$xP'_n(x) - P'_{n-1}(x) = nP_n(x)$$

25.23 $$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

25.24 $$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$$

ORTHOGONALITY OF LEGENDRE POLYNOMIALS

25.25 $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad m \neq n$$

25.26 $$\int_{-1}^{1} \{P_n(x)\}^2\,dx = \frac{2}{2n+1}$$

Because of 25.25, $P_m(x)$ and $P_n(x)$ are called *orthogonal* in $-1 \leqq x \leqq 1$.

ORTHOGONAL SERIES OF LEGENDRE POLYNOMIALS

25.27 $$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \cdots$$

where

25.28 $$A_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\,dx$$

SPECIAL RESULTS INVOLVING LEGENDRE POLYNOMIALS

25.29 $P_n(1) = 1$ **25.30** $P_n(-1) = (-1)^n$ **25.31** $P_n(-x) = (-1)^nP_n(x)$

25.32 $$P_n(0) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2}\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} & n \text{ even} \end{cases}$$

25.33 $$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x + \sqrt{x^2-1}\cos\phi)^n\,d\phi$$

25.34 $$\int P_n(x)\,dx = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

25.35 $$|P_n(x)| \leqq 1$$

25.36 $$P_n(x) = \frac{1}{2^{n+1}\pi i}\oint_C \frac{(z^2-1)^n}{(z-x)^{n+1}}\,dz$$

where C is a simple closed curve having x as interior point.

GENERAL SOLUTION OF LEGENDRE'S EQUATION

The general solution of Legendre's equation is

25.37 $$y = A\,U_n(x) + B\,V_n(x)$$

where

25.38 $$U_n(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots$$

25.39 $$V_n(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots$$

These series converge for $-1 < x < 1$.

LEGENDRE FUNCTIONS OF THE SECOND KIND

If $n = 0, 1, 2, \ldots$ one of the series 25.38, 25.39 terminates. In such cases,

25.40 $$P_n(x) = \begin{cases} U_n(x)/U_n(1) & n = 0, 2, 4, \ldots \\ V_n(x)/V_n(1) & n = 1, 3, 5, \ldots \end{cases}$$

where

25.41 $$U_n(1) = (-1)^{n/2}\,2^n\left[\left(\frac{n}{2}\right)!\right]^2\Big/n! \qquad n = 0, 2, 4, \ldots$$

25.42 $$V_n(1) = (-1)^{(n-1)/2}\,2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2\Big/n! \qquad n = 1, 3, 5, \ldots$$

The nonterminating series in such case with a suitable multiplicative constant is denoted by $Q_n(x)$ and is called *Legendre's function of the second kind of order n.* We define

25.43 $$Q_n(x) = \begin{cases} U_n(1)\,V_n(x) & n = 0, 2, 4, \ldots \\ -V_n(1)\,U_n(x) & n = 1, 3, 5, \ldots \end{cases}$$

SPECIAL LEGENDRE FUNCTIONS OF THE SECOND KIND

25.44 $$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

25.45 $$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

25.46 $$Q_2(x) = \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

25.47 $$Q_3(x) = \frac{5x^3-3x}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}$$

The functions $Q_n(x)$ satisfy recurrence formulas exactly analogous to 25.20 through 25.24. Using these, the general solution of Legendre's equation can also be written

25.48 $$y = A\,P_n(x) + B\,Q_n(x)$$

26 ASSOCIATED LEGENDRE FUNCTIONS

LEGENDRE'S ASSOCIATED DIFFERENTIAL EQUATION

26.1
$$(1-x^2)y'' - 2xy' + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\} y = 0$$

Solutions of this equation are called *associated Legendre functions.* We restrict ourselves to the important case where m, n are nonnegative integers.

ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

26.2
$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) = \frac{(1-x^2)^{m/2}}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n$$

where $P_n(x)$ are Legendre polynomials [page 146]. We have

26.3
$$P_n^0(x) = P_n(x)$$

26.4
$$P_n^m(x) = 0 \quad \text{if } m > n$$

SPECIAL ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

26.5 $P_1^1(x) = (1-x^2)^{1/2}$

26.6 $P_2^1(x) = 3x(1-x^2)^{1/2}$

26.7 $P_2^2(x) = 3(1-x^2)$

26.8 $P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$

26.9 $P_3^2(x) = 15x(1-x^2)$

26.10 $P_3^3(x) = 15(1-x^2)^{3/2}$

GENERATING FUNCTION FOR $P_n^m(x)$

26.11
$$\frac{(2m)!\,(1-x^2)^{m/2}t^m}{2^m m!\,(1-2tx+t^2)^{m+1/2}} = \sum_{n=m}^{\infty} P_n^m(x)t^n$$

RECURRENCE FORMULAS

26.12
$$(n+1-m)\,P_{n+1}^m(x) - (2n+1)x\,P_n^m(x) + (n+m)\,P_{n-1}^m(x) = 0$$

26.13
$$P_n^{m+2}(x) - \frac{2(m+1)x}{(1-x^2)^{1/2}}P_n^{m+1}(x) + (n-m)(n+m+1)\,P_n^m(x) = 0$$

ORTHOGONALITY OF $P_n^m(x)$

26.14
$$\int_{-1}^{1} P_n^m(x)\, P_l^m(x)\, dx = 0 \qquad \text{if } n \neq l$$

26.15
$$\int_{-1}^{1} \{P_n^m(x)\}^2\, dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

ORTHOGONAL SERIES

26.16
$$f(x) = A_m P_m^m(x) + A_{m+1} P_{m+1}^m(x) + A_{m+2} P_{m+2}^m(x) + \cdots$$

where

26.17
$$A_k = \frac{2k+1}{2}\frac{(k-m)!}{(k+m)!}\int_{-1}^{1} f(x)\, P_k^m(x)\, dx$$

ASSOCIATED LEGENDRE FUNCTIONS OF THE SECOND KIND

26.18
$$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x)$$

where $Q_n(x)$ are Legendre functions of the second kind [page 148].

These functions are unbounded at $x = \pm 1$, whereas $P_n^m(x)$ are bounded at $x = \pm 1$.

The functions $Q_n^m(x)$ satisfy the same recurrence relations as $P_n^m(x)$ [see 26.12 and 26.13].

GENERAL SOLUTION OF LEGENDRE'S ASSOCIATED EQUATION

26.19
$$y = A\, P_n^m(x) + B\, Q_n^m(x)$$

27 HERMITE POLYNOMIALS

HERMITE'S DIFFERENTIAL EQUATION

27.1 $$y'' - 2xy' + 2ny = 0$$

HERMITE POLYNOMIALS

If $n = 0, 1, 2, \ldots$ then solutions of Hermite's equation are Hermite polynomials $H_n(x)$ given by *Rodrigue's formula*

27.2 $$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

SPECIAL HERMITE POLYNOMIALS

27.3 $H_0(x) = 1$

27.4 $H_1(x) = 2x$

27.5 $H_2(x) = 4x^2 - 2$

27.6 $H_3(x) = 8x^3 - 12x$

27.7 $H_4(x) = 16x^4 - 48x^2 + 12$

27.8 $H_5(x) = 32x^5 - 160x^3 + 120x$

27.9 $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

27.10 $H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$

GENERATING FUNCTION

27.11 $$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}$$

RECURRENCE FORMULAS

27.12 $$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$$

27.13 $$H_n'(x) = 2n\, H_{n-1}(x)$$

ORTHOGONALITY OF HERMITE POLYNOMIALS

27.14 $$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 0 \qquad m \neq n$$

27.15 $$\int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2\, dx = 2^n n! \sqrt{\pi}$$

ORTHOGONAL SERIES

27.16 $$f(x) = A_0 H_0(x) + A_1 H_1(x) + A_2 H_2(x) + \cdots$$

where

27.17 $$A_k = \frac{1}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_k(x)\, dx$$

SPECIAL RESULTS

27.18 $$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \cdots$$

27.19 $H_n(-x) = (-1)^n H_n(x)$ **27.20** $H_{2n-1}(0) = 0$

27.21 $$H_{2n}(0) = (-1)^n 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

27.22 $$\int_0^x H_n(t)\, dt = \frac{H_{n+1}(x)}{2(n+1)} - \frac{H_{n+1}(0)}{2(n+1)}$$

27.23 $$\frac{d}{dx}\{e^{-x^2} H_n(x)\} = -e^{-x^2} H_{n+1}(x)$$

27.24 $$\int_0^x e^{-t^2} H_n(t)\, dt = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$$

27.25 $$\int_{-\infty}^{\infty} t^n e^{-t^2} H_n(xt)\, dt = \sqrt{\pi}\, n!\, P_n(x)$$

27.26 $$H_n(x+y) = \sum_{k=0}^{n} \frac{1}{2^{n/2}} \binom{n}{k} H_k(x\sqrt{2})\, H_{n-k}(y\sqrt{2})$$

This is called the *addition formula* for Hermite polynomials.

27.27 $$\sum_{k=0}^{n} \frac{H_k(x) H_k(y)}{2^k k!} = \frac{H_{n+1}(x) H_n(y) - H_n(x) H_{n+1}(y)}{2^{n+1} n!\,(x-y)}$$

28 LAGUERRE POLYNOMIALS

LAGUERRE'S DIFFERENTIAL EQUATION

28.1 $$xy'' + (1-x)y' + ny = 0$$

LAGUERRE POLYNOMIALS

If $n = 0, 1, 2, \ldots$ then solutions of Laguerre's equation are Laguerre polynomials $L_n(x)$ and are given by *Rodrigue's formula.*

28.2 $$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

SPECIAL LAGUERRE POLYNOMIALS

28.3 $L_0(x) = 1$

28.4 $L_1(x) = -x + 1$

28.5 $L_2(x) = x^2 - 4x + 2$

28.6 $L_3(x) = -x^3 + 9x^2 - 18x + 6$

28.7 $L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$

28.8 $L_5(x) = -x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120$

28.9 $L_6(x) = x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$

28.10 $L_7(x) = -x^7 + 49x^6 - 882x^5 + 7350x^4 - 29{,}400x^3 + 52{,}920x^2 - 35{,}280x + 5040$

GENERATING FUNCTION

28.11 $$\frac{e^{-xt/1-t}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)\, t^n}{n!}$$

RECURRENCE FORMULAS

28.12 $$L_{n+1}(x) - (2n+1-x)\, L_n(x) + n^2 L_{n-1}(x) = 0$$

28.13 $$L_n'(x) - n\, L_{n-1}'(x) + n\, L_{n-1}(x) = 0$$

28.14 $$x\, L_n'(x) = n\, L_n(x) - n^2 L_{n-1}(x)$$

ORTHOGONALITY OF LAGUERRE POLYNOMIALS

28.15 $$\int_0^\infty e^{-x} L_m(x)\, L_n(x)\, dx = 0 \qquad m \neq n$$

28.16 $$\int_0^\infty e^{-x} \{L_n(x)\}^2\, dx = (n!)^2$$

ORTHOGONAL SERIES

28.17 $$f(x) = A_0 L_0(x) + A_1 L_1(x) + A_2 L_2(x) + \cdots$$

where

28.18 $$A_k = \frac{1}{(k!)^2} \int_0^\infty e^{-x} f(x)\, L_k(x)\, dx$$

SPECIAL RESULTS

28.19 $L_n(0) = n!$ **28.20** $$\int_0^x L_n(t)\, dt = L_n(x) - \frac{L_{n+1}(x)}{n+1}$$

28.21 $$L_n(x) = (-1)^n \left\{ x^n - \frac{n^2 x^{n-1}}{1!} + \frac{n^2(n-1)^2 x^{n-2}}{2!} - \cdots (-1)^n n! \right\}$$

28.22 $$\int_0^\infty x^p e^{-x} L_n(x)\, dx = \begin{cases} 0 & \text{if } p < n \\ (-1)^n (n!)^2 & \text{if } p = n \end{cases}$$

28.23 $$\sum_{k=0}^{n} \frac{L_k(x)\, L_k(y)}{(k!)^2} = \frac{L_n(x)\, L_{n+1}(y) - L_{n+1}(x)\, L_n(y)}{(n!)^2 (x-y)}$$

28.24 $$\sum_{k=0}^{\infty} \frac{t^k L_k(x)}{(k!)^2} = e^t J_0(2\sqrt{xt})$$

28.25 $$L_n(x) = \int_0^\infty u^n e^{x-u} J_0(2\sqrt{xu})\, du$$

29 ASSOCIATED LAGUERRE POLYNOMIALS

LAGUERRE'S ASSOCIATED DIFFERENTIAL EQUATION

29.1 $$xy'' + (m+1-x)y' + (n-m)y = 0$$

ASSOCIATED LAGUERRE POLYNOMIALS

Solutions of 29.1 for nonnegative integers m and n are given by the associated Laguerre polynomials

29.2 $$L_n^m(x) = \frac{d^m}{dx^m} L_n(x)$$

where $L_n(x)$ are Laguerre polynomials [see page 153].

29.3 $$L_n^0(x) = L_n(x)$$

29.4 $$L_n^m(x) = 0 \quad \text{if } m > n$$

SPECIAL ASSOCIATED LAGUERRE POLYNOMIALS

29.5 $L_1^1(x) = -1$

29.6 $L_2^1(x) = 2x - 4$

29.7 $L_2^2(x) = 2$

29.8 $L_3^1(x) = -3x^2 + 18x - 18$

29.9 $L_3^2(x) = -6x + 18$

29.10 $L_3^3(x) = -6$

29.11 $L_4^1(x) = 4x^3 - 48x^2 + 144x - 96$

29.12 $L_4^2(x) = 12x^2 - 96x + 144$

29.13 $L_4^3(x) = 24x - 96$

29.14 $L_4^4(x) = 24$

GENERATING FUNCTION FOR $L_n^m(x)$

29.15 $$\frac{(-1)^m t^m}{(1-t)^{m+1}} e^{-xt/(1-t)} = \sum_{n=m}^{\infty} \frac{L_n^m(x)}{n!} t^n$$

RECURRENCE FORMULAS

29.16 $$\frac{n-m+1}{n+1}L_{n+1}^{m}(x) + (x+m-2n-1)L_n^m(x) + n^2 L_{n-1}^{m}(x) = 0$$

29.17 $$\frac{d}{dx}\{L_n^m(x)\} = L_n^{m+1}(x)$$

29.18 $$\frac{d}{dx}\{x^m e^{-x} L_n^m(x)\} = (m-n-1)x^{m-1}e^{-x}L_n^{m-1}(x)$$

29.19 $$x\frac{d}{dx}\{L_n^m(x)\} = (x-m)L_n^m(x) + (m-n-1)L_n^{m-1}(x)$$

ORTHOGONALITY

29.20 $$\int_0^\infty x^m e^{-x} L_n^m(x) L_p^m(x)\,dx = 0 \qquad p \neq n$$

29.21 $$\int_0^\infty x^m e^{-x}\{L_n^m(x)\}^2\,dx = \frac{(n!)^3}{(n-m)!}$$

ORTHOGONAL SERIES

29.22 $$f(x) = A_m L_m^m(x) + A_{m+1}L_{m+1}^m(x) + A_{m+2}L_{m+2}^m(x) + \cdots$$

where

29.23 $$A_k = \frac{(k-m)!}{(k!)^3}\int_0^\infty x^m e^{-x} L_k^m(x) f(x)\,dx$$

SPECIAL RESULTS

29.24

$$L_n^m(x) = (-1)^n \frac{n!}{(n-m)!}\left\{x^{n-m} - \frac{n(n-m)}{1!}x^{n-m-1} + \frac{n(n-1)(n-m)(n-m-1)}{2!}x^{n-m-2} + \cdots\right\}$$

29.25 $$\int_0^\infty x^{m+1}e^{-x}\{L_n^m(x)\}^2\,dx = \frac{(2n-m+1)(n!)^3}{(n-m)!}$$

30 CHEBYSHEV POLYNOMIALS

CHEBYSHEV'S DIFFERENTIAL EQUATION

30.1 $$(1-x^2)y'' - xy' + n^2y = 0 \qquad n = 0, 1, 2, \ldots$$

CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

Solutions of 30.1 are given by

30.2 $$T_n(x) = \cos(n \cos^{-1} x) = x^n - \binom{n}{2} x^{n-2}(1-x^2) + \binom{n}{4} x^{n-4}(1-x^2)^2 - \cdots$$

SPECIAL CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

30.3 $T_0(x) = 1$

30.4 $T_1(x) = x$

30.5 $T_2(x) = 2x^2 - 1$

30.6 $T_3(x) = 4x^3 - 3x$

30.7 $T_4(x) = 8x^4 - 8x^2 + 1$

30.8 $T_5(x) = 16x^5 - 20x^3 + 5x$

30.9 $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

30.10 $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

GENERATING FUNCTION FOR $T_n(x)$

30.11 $$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{\infty} T_n(x)\, t^n$$

SPECIAL VALUES

30.12 $T_n(-x) = (-1)^n T_n(x)$

30.13 $T_n(1) = 1$

30.14 $T_n(-1) = (-1)^n$

30.15 $T_{2n}(0) = (-1)^n$

30.16 $T_{2n+1}(0) = 0$

RECURSION FORMULA FOR $T_n(x)$

30.17 $$T_{n+1}(x) - 2x\,T_n(x) + T_{n-1}(x) = 0$$

ORTHOGONALITY

30.18 $$\int_{-1}^{1} \frac{T_m(x)\,T_n(x)}{\sqrt{1-x^2}}\,dx = 0 \qquad m \neq n$$

30.19 $$\int_{-1}^{1} \frac{\{T_n(x)\}^2}{\sqrt{1-x^2}}\,dx = \begin{cases} \pi & \text{if } n = 0 \\ \pi/2 & \text{if } n = 1, 2, \ldots \end{cases}$$

ORTHOGONAL SERIES

30.20 $$f(x) = \tfrac{1}{2}A_0\,T_0(x) + A_1\,T_1(x) + A_2\,T_2(x) + \cdots$$

where

30.21 $$A_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\,T_k(x)}{\sqrt{1-x^2}}\,dx$$

CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

30.22 $$U_n(x) = \frac{\sin\{(n+1)\cos^{-1}x\}}{\sin(\cos^{-1}x)} = \binom{n+1}{1}x^n - \binom{n+1}{3}x^{n-2}(1-x^2) + \binom{n+1}{5}x^{n-4}(1-x^2)^2 - \cdots$$

SPECIAL CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

30.23 $U_0(x) = 1$

30.24 $U_1(x) = 2x$

30.25 $U_2(x) = 4x^2 - 1$

30.26 $U_3(x) = 8x^3 - 4x$

30.27 $U_4(x) = 16x^4 - 12x^2 + 1$

30.28 $U_5(x) = 32x^5 - 32x^3 + 6x$

30.29 $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$

30.30 $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$

GENERATING FUNCTION FOR $U_n(x)$

30.31 $$\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_n(x)\,t^n$$

SPECIAL VALUES

30.32 $U_n(-x) = (-1)^n U_n(x)$

30.33 $U_n(1) = n + 1$

30.34 $U_n(-1) = (-1)^n (n + 1)$

30.35 $U_{2n}(0) = (-1)^n$

30.36 $U_{2n+1}(0) = 0$

RECURSION FORMULA FOR $U_n(x)$

30.37 $$U_{n+1}(x) - 2x\,U_n(x) + U_{n-1}(x) = 0$$

ORTHOGONALITY

30.38 $$\int_{-1}^{1} \sqrt{1-x^2}\,U_m(x)\,U_n(x)\,dx = 0 \qquad m \neq n$$

30.39 $$\int_{-1}^{1} \sqrt{1-x^2}\,\{U_n(x)\}^2\,dx = \frac{\pi}{2}$$

ORTHOGONAL SERIES

30.40 $$f(x) = A_0\,U_0(x) + A_1\,U_1(x) + A_2\,U_2(x) + \cdots$$

where

30.41 $$A_k = \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-x^2}\,f(x)\,U_k(x)\,dx$$

RELATIONSHIPS BETWEEN $T_n(x)$ AND $U_n(x)$

30.42 $$T_n(x) = U_n(x) - x\,U_{n-1}(x)$$

30.43 $$(1-x^2)\,U_{n-1}(x) = x\,T_n(x) - T_{n+1}(x)$$

30.44 $$U_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{T_{n+1}(v)\,dv}{(v-x)\sqrt{1-v^2}}$$

30.45 $$T_n(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-v^2}\,U_{n-1}(v)}{x-v}\,dv$$

GENERAL SOLUTION OF CHEBYSHEV'S DIFFERENTIAL EQUATION

30.46 $$y = \begin{cases} A\,T_n(x) + B\sqrt{1-x^2}\,U_{n-1}(x) & \text{if } n = 1, 2, 3, \ldots \\ A + B\,\sin^{-1} x & \text{if } n = 0 \end{cases}$$

31 HYPERGEOMETRIC FUNCTIONS

HYPERGEOMETRIC DIFFERENTIAL EQUATION

31.1 $$x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0$$

HYPERGEOMETRIC FUNCTIONS

A solution of 31.1 is given by

31.2 $$F(a,b;c;x) = 1 + \frac{a\cdot b}{1\cdot c}x + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3 + \cdots$$

If a, b, c are real, then the series converges for $-1 < x < 1$ provided that $c-(a+b) > -1$.

SPECIAL CASES

31.3 $F(-p,1;1;-x) = (1+x)^p$

31.4 $F(1,1;2;-x) = [\ln(1+x)]/x$

31.5 $\lim_{n\to\infty} F(1,n;1;x/n) = e^x$

31.6 $F(\frac{1}{2},-\frac{1}{2};\frac{1}{2};\sin^2 x) = \cos x$

31.7 $F(\frac{1}{2},1;1;\sin^2 x) = \sec x$

31.8 $F(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2) = (\sin^{-1}x)/x$

31.9 $F(\frac{1}{2},1;\frac{3}{2};-x^2) = (\tan^{-1}x)/x$

31.10 $F(1,p;p;x) = 1/(1-x)$

31.11 $F(n+1,-n;1;(1-x)/2) = P_n(x)$

31.12 $F(n,-n;\frac{1}{2};(1-x)/2) = T_n(x)$

GENERAL SOLUTION OF THE HYPERGEOMETRIC EQUATION

If c, $a-b$ and $c-a-b$ are all nonintegers, the general solution valid for $|x| < 1$ is

31.13 $$y = A\,F(a,b;c;x) + Bx^{1-c}F(a-c+1,\,b-c+1;\,2-c;\,x)$$

MISCELLANEOUS PROPERTIES

31.14 $$F(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

31.15 $$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x)$$

31.16 $$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}\,du$$

31.17 $$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x)$$

32 LAPLACE TRANSFORMS

DEFINITION OF THE LAPLACE TRANSFORM OF $F(t)$

TABLE OF GENERAL PROPERTIES OF LAPLACE TRANSFORMS

	$f(s)$	$F(t)$
32.3	$a f_1(s) + b f_2(s)$	$a F_1(t) + b F_2(t)$
32.4	$f(s/a)$	$a F(at)$
32.5	$f(s-a)$	$e^{at} F(t)$
32.6	$e^{-as} f(s)$	$\mathcal{U}(t-a) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$
32.7	$s f(s) - F(0)$	$F'(t)$
32.8	$s^2 f(s) - s F(0) - F'(0)$	$F''(t)$
32.9	$s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \cdots - F^{(n-1)}(0)$	$F^{(n)}(t)$
32.10	$f'(s)$	$-t F(t)$
32.11	$f''(s)$	$t^2 F(t)$
32.12	$f^{(n)}(s)$	$(-1)^n t^n F(t)$
32.13	$\frac{f(s)}{s}$	$\int_0^t F(u)\, du$
32.14	$\frac{f(s)}{s^n}$	$\int_0^t \cdots \int_0^t F(u)\, du^n = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} F(u)\, du$
32.15	$f(s)\, g(s)$	$\int_0^t F(u)\, G(t-u)\, du$

	$f(s)$	$F(t)$
32.16	$\int_s^\infty f(u)\,du$	$\frac{F(t)}{t}$
32.17	$\frac{1}{1-e^{-sT}}\int_0^T e^{-su}F(u)\,du$	$F(t) = F(t+T)$
32.18	$\frac{f(\sqrt{s})}{s}$	$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-u^2/4t}F(u)\,du$
32.19	$\frac{1}{s}f(1/s)$	$\int_0^\infty J_0(2\sqrt{ut})\,F(u)\,du$
32.20	$\frac{1}{s^{n+1}}f(1/s)$	$t^{n/2}\int_0^\infty u^{-n/2}J_n(2\sqrt{ut})\,F(u)\,du$
32.21	$\frac{f(s+1/s)}{s^2+1}$	$\int_0^t J_0(2\sqrt{u(t-u)})\,F(u)\,du$
32.22	$\frac{1}{2\sqrt{\pi}}\int_0^\infty u^{-3/2}e^{-s^2/4u}f(u)\,du$	$F(t^2)$
32.23	$\frac{f(\ln s)}{s\ln s}$	$\int_0^\infty \frac{t^u F(u)}{\Gamma(u+1)}\,du$
32.24	$\frac{P(s)}{Q(s)}$ $P(s)$ = polynomial of degree less than n, $Q(s) = (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_n)$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all distinct.	$\sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$

TABLE OF SPECIAL LAPLACE TRANSFORMS

	$f(s)$	$F(t)$
32.25	$\frac{1}{s}$	1
32.26	$\frac{1}{s^2}$	t
32.27	$\frac{1}{s^n}$ $\quad n = 1, 2, 3, \ldots$	$\frac{t^{n-1}}{(n-1)!}$, $\quad 0! = 1$
32.28	$\frac{1}{s^n}$ $\quad n > 0$	$\frac{t^{n-1}}{\Gamma(n)}$
32.29	$\frac{1}{s-a}$	e^{at}
32.30	$\frac{1}{(s-a)^n}$ $\quad n = 1, 2, 3, \ldots$	$\frac{t^{n-1}e^{at}}{(n-1)!}$, $\quad 0! = 1$
32.31	$\frac{1}{(s-a)^n}$ $\quad n > 0$	$\frac{t^{n-1}e^{at}}{\Gamma(n)}$
32.32	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
32.33	$\frac{s}{s^2+a^2}$	$\cos at$
32.34	$\frac{1}{(s-b)^2+a^2}$	$\frac{e^{bt}\sin at}{a}$
32.35	$\frac{s-b}{(s-b)^2+a^2}$	$e^{bt}\cos at$
32.36	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
32.37	$\frac{s}{s^2-a^2}$	$\cosh at$
32.38	$\frac{1}{(s-b)^2-a^2}$	$\frac{e^{bt}\sinh at}{a}$

	$f(s)$	$F(t)$
32.39	$\dfrac{s-b}{(s-b)^2-a^2}$	$e^{bt}\cosh at$
32.40	$\dfrac{1}{(s-a)(s-b)}$ $a \neq b$	$\dfrac{e^{bt}-e^{at}}{b-a}$
32.41	$\dfrac{s}{(s-a)(s-b)}$ $a \neq b$	$\dfrac{be^{bt}-ae^{at}}{b-a}$
32.42	$\dfrac{1}{(s^2+a^2)^2}$	$\dfrac{\sin at - at\cos at}{2a^3}$
32.43	$\dfrac{s}{(s^2+a^2)^2}$	$\dfrac{t\sin at}{2a}$
32.44	$\dfrac{s^2}{(s^2+a^2)^2}$	$\dfrac{\sin at + at\cos at}{2a}$
32.45	$\dfrac{s^3}{(s^2+a^2)^2}$	$\cos at - \frac{1}{2}at\sin at$
32.46	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
32.47	$\dfrac{1}{(s^2-a^2)^2}$	$\dfrac{at\cosh at - \sinh at}{2a^3}$
32.48	$\dfrac{s}{(s^2-a^2)^2}$	$\dfrac{t\sinh at}{2a}$
32.49	$\dfrac{s^2}{(s^2-a^2)^2}$	$\dfrac{\sinh at + at\cosh at}{2a}$
32.50	$\dfrac{s^3}{(s^2-a^2)^2}$	$\cosh at + \frac{1}{2}at\sinh at$
32.51	$\dfrac{s^2+a^2}{(s^2-a^2)^2}$	$t\cosh at$
32.52	$\dfrac{1}{(s^2+a^2)^3}$	$\dfrac{(3-a^2t^2)\sin at - 3at\cos at}{8a^5}$
32.53	$\dfrac{s}{(s^2+a^2)^3}$	$\dfrac{t\sin at - at^2\cos at}{8a^3}$
32.54	$\dfrac{s^2}{(s^2+a^2)^3}$	$\dfrac{(1+a^2t^2)\sin at - at\cos at}{8a^3}$
32.55	$\dfrac{s^3}{(s^2+a^2)^3}$	$\dfrac{3t\sin at + at^2\cos at}{8a}$

	$f(s)$	$F(t)$
32.56	$\frac{s^4}{(s^2+a^2)^3}$	$\frac{(3-a^2t^2)\sin at + 5at\cos at}{8a}$
32.57	$\frac{s^5}{(s^2+a^2)^3}$	$\frac{(8-a^2t^2)\cos at - 7at\sin at}{8}$
32.58	$\frac{3s^2-a^2}{(s^2+a^2)^3}$	$\frac{t^2\sin at}{2a}$
32.59	$\frac{s^3-3a^2s}{(s^2+a^2)^3}$	$\frac{1}{2}t^2\cos at$
32.60	$\frac{s^4-6a^2s^2+a^4}{(s^2+a^2)^4}$	$\frac{1}{6}t^3\cos at$
32.61	$\frac{s^3-a^2s}{(s^2+a^2)^4}$	$\frac{t^3\sin at}{24a}$
32.62	$\frac{1}{(s^2-a^2)^3}$	$\frac{(3+a^2t^2)\sinh at - 3at\cosh at}{8a^5}$
32.63	$\frac{s}{(s^2-a^2)^3}$	$\frac{at^2\cosh at - t\sinh at}{8a^3}$
32.64	$\frac{s^2}{(s^2-a^2)^3}$	$\frac{at\cosh at + (a^2t^2-1)\sinh at}{8a^3}$
32.65	$\frac{s^3}{(s^2-a^2)^3}$	$\frac{3t\sinh at + at^2\cosh at}{8a}$
32.66	$\frac{s^4}{(s^2-a^2)^3}$	$\frac{(3+a^2t^2)\sinh at + 5at\cosh at}{8a}$
32.67	$\frac{s^5}{(s^2-a^2)^3}$	$\frac{(8+a^2t^2)\cosh at + 7at\sinh at}{8}$
32.68	$\frac{3s^2+a^2}{(s^2-a^2)^3}$	$\frac{t^2\sinh at}{2a}$
32.69	$\frac{s^3+3a^2s}{(s^2-a^2)^3}$	$\frac{1}{2}t^2\cosh at$
32.70	$\frac{s^4+6a^2s^2+a^4}{(s^2-a^2)^4}$	$\frac{1}{6}t^3\cosh at$
32.71	$\frac{s^3+a^2s}{(s^2-a^2)^4}$	$\frac{t^3\sinh at}{24a}$
32.72	$\frac{1}{s^3+a^3}$	$\frac{e^{at/2}}{3a^2}\left\{\sqrt{3}\sin\frac{\sqrt{3}\,at}{2} - \cos\frac{\sqrt{3}\,at}{2} + e^{-3at/2}\right\}$

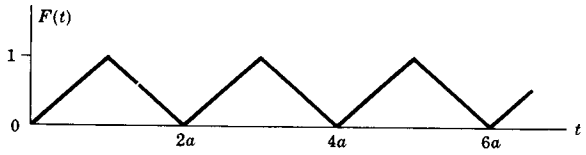
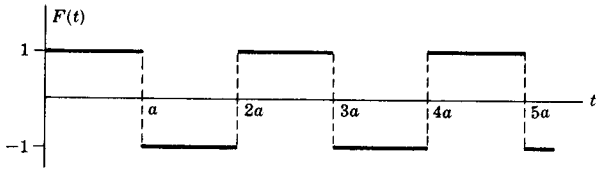
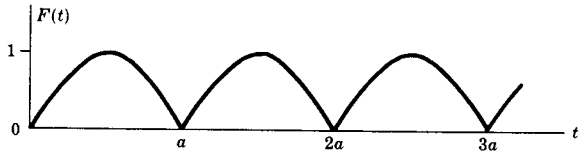
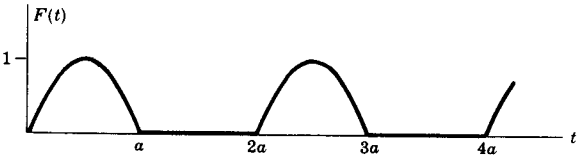
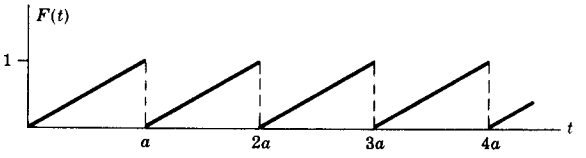
	$f(s)$	$F(t)$
32.73	$\frac{s}{s^3+a^3}$	$\frac{e^{at/2}}{3a}\left\{\cos\frac{\sqrt{3}\,at}{2}+\sqrt{3}\sin\frac{\sqrt{3}\,at}{2}-e^{-3at/2}\right\}$
32.74	$\frac{s^2}{s^3+a^3}$	$\frac{1}{3}\left(e^{-at}+2e^{at/2}\cos\frac{\sqrt{3}\,at}{2}\right)$
32.75	$\frac{1}{s^3-a^3}$	$\frac{e^{-at/2}}{3a^2}\left\{e^{3at/2}-\cos\frac{\sqrt{3}\,at}{2}-\sqrt{3}\sin\frac{\sqrt{3}\,at}{2}\right\}$
32.76	$\frac{s}{s^3-a^3}$	$\frac{e^{-at/2}}{3a}\left\{\sqrt{3}\sin\frac{\sqrt{3}\,at}{2}-\cos\frac{\sqrt{3}\,at}{2}+e^{3at/2}\right\}$
32.77	$\frac{s^2}{s^3-a^3}$	$\frac{1}{3}\left(e^{at}+2e^{-at/2}\cos\frac{\sqrt{3}\,at}{2}\right)$
32.78	$\frac{1}{s^4+4a^4}$	$\frac{1}{4a^3}(\sin at\cosh at-\cos at\sinh at)$
32.79	$\frac{s}{s^4+4a^4}$	$\frac{\sin at\sinh at}{2a^2}$
32.80	$\frac{s^2}{s^4+4a^4}$	$\frac{1}{2a}(\sin at\cosh at+\cos at\sinh at)$
32.81	$\frac{s^3}{s^4+4a^4}$	$\cos at\cosh at$
32.82	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at-\sin at)$
32.83	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2}(\cosh at-\cos at)$
32.84	$\frac{s^2}{s^4-a^4}$	$\frac{1}{2a}(\sinh at+\sin at)$
32.85	$\frac{s^3}{s^4-a^4}$	$\frac{1}{2}(\cosh at+\cos at)$
32.86	$\frac{1}{\sqrt{s+a}+\sqrt{s+b}}$	$\frac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$
32.87	$\frac{1}{s\sqrt{s+a}}$	$\frac{\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
32.88	$\frac{1}{\sqrt{s}\,(s-a)}$	$\frac{e^{at}\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
32.89	$\frac{1}{\sqrt{s-a}+b}$	$e^{at}\left\{\frac{1}{\sqrt{\pi t}}-b\,e^{b^2t}\operatorname{erfc}(b\sqrt{t})\right\}$

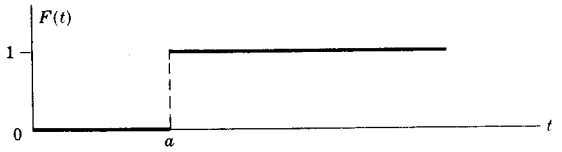
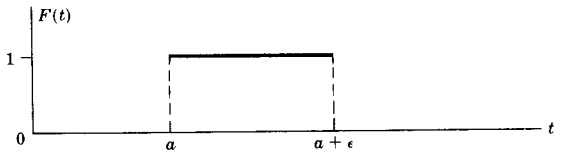
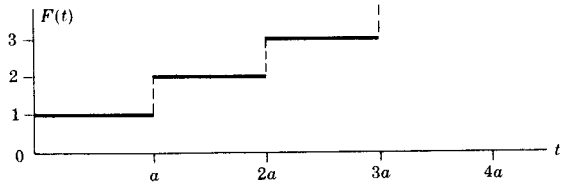
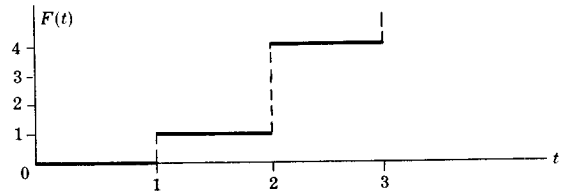
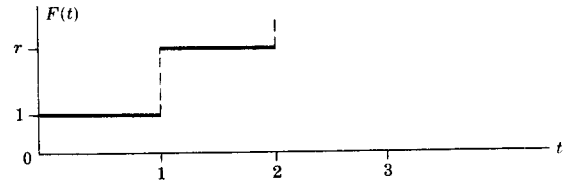
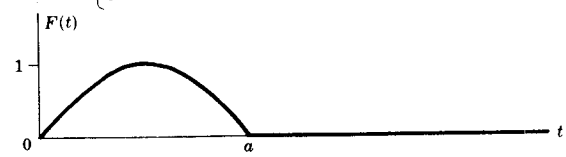
	$f(s)$	$F(t)$
32.90	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
32.91	$\dfrac{1}{\sqrt{s^2-a^2}}$	$I_0(at)$
32.92	$\dfrac{(\sqrt{s^2+a^2}-s)^n}{\sqrt{s^2+a^2}}$ $n>-1$	$a^n J_n(at)$
32.93	$\dfrac{(s-\sqrt{s^2-a^2})^n}{\sqrt{s^2-a^2}}$ $n>-1$	$a^n I_n(at)$
32.94	$\dfrac{e^{b(s-\sqrt{s^2+a^2})}}{\sqrt{s^2+a^2}}$	$J_0(a\sqrt{t(t+2b)})$
32.95	$\dfrac{e^{-b\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}$	$\begin{cases} J_0(a\sqrt{t^2-b^2}) & t>b \\ 0 & t<b \end{cases}$
32.96	$\dfrac{1}{(s^2+a^2)^{3/2}}$	$\dfrac{tJ_1(at)}{a}$
32.97	$\dfrac{s}{(s^2+a^2)^{3/2}}$	$tJ_0(at)$
32.98	$\dfrac{s^2}{(s^2+a^2)^{3/2}}$	$J_0(at) - atJ_1(at)$
32.99	$\dfrac{1}{(s^2-a^2)^{3/2}}$	$\dfrac{tI_1(at)}{a}$
32.100	$\dfrac{s}{(s^2-a^2)^{3/2}}$	$tI_0(at)$
32.101	$\dfrac{s^2}{(s^2-a^2)^{3/2}}$	$I_0(at) + atI_1(at)$
32.102	$\dfrac{1}{s(e^s-1)} = \dfrac{c^{-s}}{s(1-e^{-s})}$ See also entry 32.165.	$F(t)=n,\ n \leqq t < n+1,\ n=0,1,2,\ldots$
32.103	$\dfrac{1}{s(e^s-r)} = \dfrac{e^{-s}}{s(1-re^{-s})}$	$F(t) = \sum_{k=1}^{[t]} r^k$ where $[t]$ = greatest integer $\leqq t$
32.104	$\dfrac{e^s-1}{s(e^s-r)} = \dfrac{1-e^{-s}}{s(1-re^{-s})}$ See also entry 32.167.	$F(t)=r^n,\ n \leqq t < n+1,\ n=0,1,2,\ldots$
32.105	$\dfrac{e^{-a/s}}{\sqrt{s}}$	$\dfrac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$

	$f(s)$	$F(t)$
32.106	$\frac{e^{-a/s}}{s^{3/2}}$	$\frac{\sin 2\sqrt{at}}{\sqrt{\pi a}}$
32.107	$\frac{e^{-a/s}}{s^{n+1}} \quad n > -1$	$\left(\frac{t}{a}\right)^{n/2} J_n(2\sqrt{at})$
32.108	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$\frac{e^{-a^2/4t}}{\sqrt{\pi t}}$
32.109	$e^{-a\sqrt{s}}$	$\frac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$
32.110	$\frac{1 - e^{-a\sqrt{s}}}{s}$	$\operatorname{erf}(a/2\sqrt{t})$
32.111	$\frac{e^{-a\sqrt{s}}}{s}$	$\operatorname{erfc}(a/2\sqrt{t})$
32.112	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s}+b)}$	$e^{b(bt+a)} \operatorname{erfc}\left(b\sqrt{t} + \frac{a}{2\sqrt{t}}\right)$
32.113	$\frac{e^{-a/\sqrt{s}}}{s^{n+1}} \quad n > -1$	$\frac{1}{\sqrt{\pi t}\, a^{2n+1}} \int_0^\infty u^n e^{-u^2/4a^2t} J_{2n}(2\sqrt{u})\, du$
32.114	$\ln\left(\frac{s+a}{s+b}\right)$	$\frac{e^{-bt} - e^{-at}}{t}$
32.115	$\frac{\ln[(s^2+a^2)/a^2]}{2s}$	$Ci(at)$
32.116	$\frac{\ln[(s+a)/a]}{s}$	$Ei(at)$
32.117	$-\frac{(\gamma + \ln s)}{s}$ γ = Euler's constant = .5772156...	$\ln t$
32.118	$\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)$	$\frac{2(\cos at - \cos bt)}{t}$
32.119	$\frac{\pi^2}{6s} + \frac{(\gamma + \ln s)^2}{s}$ γ = Euler's constant = .5772156...	$\ln^2 t$
32.120	$\frac{\ln s}{s}$	$-(\ln t + \gamma)$ γ = Euler's constant = .5772156...
32.121	$\frac{\ln^2 s}{s}$	$(\ln t + \gamma)^2 - \frac{1}{6}\pi^2$ γ = Euler's constant = .5772156...

	$f(s)$	$F(t)$
32.122	$\dfrac{\Gamma'(n+1) - \Gamma(n+1)\ln s}{s^{n+1}} \qquad n > -1$	$t^n \ln t$
32.123	$\tan^{-1}(a/s)$	$\dfrac{\sin at}{t}$
32.124	$\dfrac{\tan^{-1}(a/s)}{s}$	$Si(at)$
32.125	$\dfrac{e^{a/s}}{\sqrt{s}}\operatorname{erfc}(\sqrt{a/s})$	$\dfrac{e^{-2\sqrt{at}}}{\sqrt{\pi t}}$
32.126	$e^{s^2/4a^2}\operatorname{erfc}(s/2a)$	$\dfrac{2a}{\sqrt{\pi}}e^{-a^2t^2}$
32.127	$\dfrac{e^{s^2/4a^2}\operatorname{erfc}(s/2a)}{s}$	$\operatorname{erf}(at)$
32.128	$\dfrac{e^{as}\operatorname{erfc}\sqrt{as}}{\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi(t+a)}}$
32.129	$e^{as}\,Ei(as)$	$\dfrac{1}{t+a}$
32.130	$\dfrac{1}{a}\left[\cos as\left\{\dfrac{\pi}{2} - Si(as)\right\} - \sin as\ Ci(as)\right]$	$\dfrac{1}{t^2+a^2}$
32.131	$\sin as\left\{\dfrac{\pi}{2} - Si(as)\right\} + \cos as\ Ci(as)$	$\dfrac{t}{t^2+a^2}$
32.132	$\dfrac{\cos as\left\{\dfrac{\pi}{2} - Si(as)\right\} - \sin as\ Ci(as)}{s}$	$\tan^{-1}(t/a)$
32.133	$\dfrac{\sin as\left\{\dfrac{\pi}{2} - Si(as)\right\} + \cos as\ Ci(as)}{s}$	$\dfrac{1}{2}\ln\left(\dfrac{t^2+a^2}{a^2}\right)$
32.134	$\left[\dfrac{\pi}{2} - Si(as)\right]^2 + Ci^2(as)$	$\dfrac{1}{t}\ln\left(\dfrac{t^2+a^2}{a^2}\right)$
32.135	0	$\mathcal{N}(t)$ = null function
32.136	1	$\delta(t)$ = delta function
32.137	e^{-as}	$\delta(t-a)$
32.138	$\dfrac{e^{-as}}{s}$ See also entry 32.163.	$\mathcal{U}(t-a)$

	$f(s)$	$F(t)$
32.139	$\frac{\sinh sx}{s \sinh sa}$	$\frac{x}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi x}{a} \cos\frac{n\pi t}{a}$
32.140	$\frac{\sinh sx}{s \cosh sa}$	$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sin\frac{(2n-1)\pi x}{2a} \sin\frac{(2n-1)\pi t}{2a}$
32.141	$\frac{\cosh sx}{s \sinh as}$	$\frac{t}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos\frac{n\pi x}{a} \sin\frac{n\pi t}{a}$
32.142	$\frac{\cosh sx}{s \cosh sa}$	$1 + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \cos\frac{(2n-1)\pi x}{2a} \cos\frac{(2n-1)\pi t}{2a}$
32.143	$\frac{\sinh sx}{s^2 \sinh sa}$	$\frac{xt}{a} + \frac{2a}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin\frac{n\pi x}{a} \sin\frac{n\pi t}{a}$
32.144	$\frac{\sinh sx}{s^2 \cosh sa}$	$x + \frac{8a}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{2a} \cos\frac{(2n-1)\pi t}{2a}$
32.145	$\frac{\cosh sx}{s^2 \sinh sa}$	$\frac{t^2}{2a} + \frac{2a}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{a}\left(1 - \cos\frac{n\pi t}{a}\right)$
32.146	$\frac{\cosh sx}{s^2 \cosh sa}$	$t + \frac{8a}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \cos\frac{(2n-1)\pi x}{2a} \sin\frac{(2n-1)\pi t}{2a}$
32.147	$\frac{\cosh sx}{s^3 \cosh sa}$	$\frac{1}{2}(t^2 + x^2 - a^2) - \frac{16a^2}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos\frac{(2n-1)\pi x}{2a} \cos\frac{(2n-1)\pi t}{2a}$
32.148	$\frac{\sinh x\sqrt{s}}{\sinh a\sqrt{s}}$	$\frac{2\pi}{a^2}\sum_{n=1}^{\infty} (-1)^n n\, e^{-n^2\pi^2 t/a^2} \sin\frac{n\pi x}{a}$
32.149	$\frac{\cosh x\sqrt{s}}{\cosh a\sqrt{s}}$	$\frac{\pi}{a^2}\sum_{n=1}^{\infty} (-1)^{n-1} (2n-1)\, e^{-(2n-1)^2\pi^2 t/4a^2} \cos\frac{(2n-1)\pi x}{2a}$
32.150	$\frac{\sinh x\sqrt{s}}{\sqrt{s} \cosh a\sqrt{s}}$	$\frac{2}{a}\sum_{n=1}^{\infty} (-1)^{n-1} e^{-(2n-1)^2\pi^2 t/4a^2} \sin\frac{(2n-1)\pi x}{2a}$
32.151	$\frac{\cosh x\sqrt{s}}{\sqrt{s} \sinh a\sqrt{s}}$	$\frac{1}{a} + \frac{2}{a}\sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2 t/a^2} \cos\frac{n\pi x}{a}$
32.152	$\frac{\sinh x\sqrt{s}}{s \sinh a\sqrt{s}}$	$\frac{x}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2\pi^2 t/a^2} \sin\frac{n\pi x}{a}$
32.153	$\frac{\cosh x\sqrt{s}}{s \cosh a\sqrt{s}}$	$1 + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2 t/4a^2} \cos\frac{(2n-1)\pi x}{2a}$
32.154	$\frac{\sinh x\sqrt{s}}{s^2 \sinh a\sqrt{s}}$	$\frac{xt}{a} + \frac{2a^2}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}(1 - e^{-n^2\pi^2 t/a^2}) \sin\frac{n\pi x}{a}$
32.155	$\frac{\cosh x\sqrt{s}}{s^2 \cosh a\sqrt{s}}$	$\frac{1}{2}(x^2 - a^2) + t - \frac{16a^2}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} e^{-(2n-1)^2\pi^2 t/4a^2} \cos\frac{(2n-1)\pi x}{2a}$

	$f(s)$	$F(t)$
32.156	$\dfrac{J_0(ix\sqrt{s})}{s\,J_0(ia\sqrt{s})}$	$1 - 2\sum_{n=1}^{\infty} \dfrac{e^{-\lambda_n^2 t/a^2}\,J_0(\lambda_n x/a)}{\lambda_n J_1(\lambda_n)}$ where $\lambda_1, \lambda_2, \ldots$ are the positive roots of $J_0(\lambda) = 0$
32.157	$\dfrac{J_0(ix\sqrt{s})}{s^2\,J_0(ia\sqrt{s})}$	$\frac{1}{4}(x^2 - a^2) + t + 2a^2\sum_{n=1}^{\infty} \dfrac{e^{-\lambda_n^2 t/a^2}\,J_0(\lambda_n x/a)}{\lambda_n^3 J_1(\lambda_n)}$ where $\lambda_1, \lambda_2, \ldots$ are the positive roots of $J_0(\lambda) = 0$
32.158	$\dfrac{1}{as^2}\tanh\left(\dfrac{as}{2}\right)$	**Triangular wave function**  **Fig. 32-1**
32.159	$\dfrac{1}{s}\tanh\left(\dfrac{as}{2}\right)$	**Square wave function**  **Fig. 32-2**
32.160	$\dfrac{\pi a}{a^2s^2 + \pi^2}\coth\left(\dfrac{as}{2}\right)$	**Rectified sine wave function**  **Fig. 32-3**
32.161	$\dfrac{\pi a}{(a^2s^2 + \pi^2)(1 - e^{-as})}$	**Half rectified sine wave function**  **Fig. 32-4**
32.162	$\dfrac{1}{as^2} - \dfrac{e^{-as}}{s(1 - e^{-as})}$	**Saw tooth wave function**  **Fig. 32-5**

	$f(s)$	$F(t)$
32.163	$\frac{e^{-as}}{s}$ See also entry 32.138.	**Heaviside's unit function** $\mathcal{U}(t-a)$  **Fig. 32-6**
32.164	$\frac{e^{-as}(1-e^{-\epsilon s})}{s}$	**Pulse function**  **Fig. 32-7**
32.165	$\frac{1}{s(1-e^{-as})}$ See also entry 32.102.	**Step function**  **Fig. 32-8**
32.166	$\frac{e^{-s}+e^{-2s}}{s(1-e^{-s})^2}$	$F(t) = n^2,\ n \leqq t < n+1,\ n = 0, 1, 2, \ldots$  **Fig. 32-9**
32.167	$\frac{1-e^{-s}}{s(1-re^{-s})}$ See also entry 32.104.	$F(t) = r^n,\ n \leqq t < n+1,\ n = 0, 1, 2, \ldots$  **Fig. 32-10**
32.168	$\frac{\pi a(1+e^{-as})}{a^2s^2+\pi^2}$	$F(t) = \begin{cases} \sin(\pi t/a) & 0 \leqq t \leqq a \\ 0 & t > a \end{cases}$  **Fig. 32-11**

33 FOURIER TRANSFORMS

FOURIER'S INTEGRAL THEOREM

33.1
$$f(x) = \int_0^\infty \{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\}\,d\alpha$$

where

33.2
$$\begin{cases} A(\alpha) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx \\ B(\alpha) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx \end{cases}$$

Sufficient conditions under which this theorem holds are:

(i) $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval $-L < x < L$;

(ii) $\displaystyle\int_{-\infty}^{\infty} |f(x)|\,dx$ converges;

(iii) $f(x)$ is replaced by $\frac{1}{2}\{f(x+0) + f(x-0)\}$ if x is a point of discontinuity.

EQUIVALENT FORMS OF FOURIER'S INTEGRAL THEOREM

33.3
$$f(x) = \frac{1}{2\pi}\int_{\alpha=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha\,(x-u)\,du\,d\alpha$$

33.4
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha\int_{-\infty}^{\infty} f(u)\,e^{-i\alpha u}\,du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\,e^{i\alpha(x-u)}\,du\,d\alpha$$

33.5
$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\alpha x\,d\alpha\int_0^\infty f(u)\sin\alpha u\,du$$

where $f(x)$ is an *odd function* $[f(-x) = -f(x)]$.

33.6
$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x\,d\alpha\int_0^\infty f(u)\cos\alpha u\,du$$

where $f(x)$ is an *even function* $[f(-x) = f(x)]$.

FOURIER TRANSFORMS

The Fourier transform of $f(x)$ is defined as

33.7 $$\mathcal{F}\{f(x)\} = F(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

Then from 33.7 the inverse Fourier transform of $F(\alpha)$ is

33.8 $$\mathcal{F}^{-1}\{F(\alpha)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha$$

We call $f(x)$ and $F(\alpha)$ *Fourier transform pairs.*

CONVOLUTION THEOREM FOR FOURIER TRANSFORMS

If $F(\alpha) = \mathcal{F}\{f(x)\}$ and $G(\alpha) = \mathcal{F}\{g(x)\}$, then

33.9 $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, G(\alpha)\, e^{i\alpha x}\, d\alpha = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du = f^*g$$

where f^*g is called the *convolution* of f and g. Thus

33.10 $$\mathcal{F}\{f^*g\} = \mathcal{F}\{f\}\,\mathcal{F}\{g\}$$

PARSEVAL'S IDENTITY

If $F(\alpha) = \mathcal{F}\{f(x)\}$, then

33.11 $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha$$

More generally if $F(\alpha) = \mathcal{F}\{f(x)\}$ and $G(\alpha) = \mathcal{F}\{g(x)\}$, then

33.12 $$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,\overline{G(\alpha)}\, d\alpha$$

where the bar denotes complex conjugate.

FOURIER SINE TRANSFORMS

The Fourier sine transform of $f(x)$ is defined as

33.13 $$F_S(\alpha) = \mathcal{F}_S\{f(x)\} = \int_0^{\infty} f(x)\, \sin \alpha x\, dx$$

Then from 33.13 the inverse Fourier sine transform of $F_S(\alpha)$ is

33.14 $$f(x) = \mathcal{F}_S^{-1}\{F_S(\alpha)\} = \frac{2}{\pi}\int_0^{\infty} F_S(\alpha)\, \sin \alpha x\, d\alpha$$

FOURIER COSINE TRANSFORMS

The Fourier cosine transform of $f(x)$ is defined as

33.15 $$F_C(\alpha) = \mathcal{F}_C\{f(x)\} = \int_0^\infty f(x) \cos \alpha x \, dx$$

Then from 33.15 the inverse Fourier cosine transform of $F_C(\alpha)$ is

33.16 $$f(x) = \mathcal{F}_C^{-1}\{F_C(\alpha)\} = \frac{2}{\pi} \int_0^\infty F_C(\alpha) \cos \alpha x \, d\alpha$$

SPECIAL FOURIER TRANSFORM PAIRS

	$f(x)$	$F(\alpha)$
33.17	$\begin{cases} 1 & \lvert x \rvert < b \\ 0 & \lvert x \rvert > b \end{cases}$	$\dfrac{2 \sin b\alpha}{\alpha}$
33.18	$\dfrac{1}{x^2 + b^2}$	$\dfrac{\pi e^{-b\alpha}}{b}$
33.19	$\dfrac{x}{x^2 + b^2}$	$-\dfrac{\pi i \alpha}{b} e^{-b\alpha}$
33.20	$f^{(n)}(x)$	$i^n \alpha^n F(\alpha)$
33.21	$x^n f(x)$	$i^n \dfrac{d^n F}{d\alpha^n}$
33.22	$f(bx)e^{itx}$	$\dfrac{1}{b} F\left(\dfrac{\alpha - t}{b}\right)$

SPECIAL FOURIER SINE TRANSFORMS

	$f(x)$	$F_C(\alpha)$
33.23	$\begin{cases} 1 & 0 < x < b \\ 0 & x > b \end{cases}$	$\dfrac{1 - \cos b\alpha}{\alpha}$
33.24	x^{-1}	$\dfrac{\pi}{2}$
33.25	$\dfrac{x}{x^2 + b^2}$	$\dfrac{\pi}{2} e^{-b\alpha}$
33.26	e^{-bx}	$\dfrac{\alpha}{\alpha^2 + b^2}$
33.27	$x^{n-1} e^{-bx}$	$\dfrac{\Gamma(n) \sin (n \tan^{-1} \alpha/b)}{(\alpha^2 + b^2)^{n/2}}$
33.28	xe^{-bx^2}	$\dfrac{\sqrt{\pi}}{4b^{3/2}} \alpha e^{-\alpha^2/4b}$
33.29	$x^{-1/2}$	$\sqrt{\dfrac{\pi}{2\alpha}}$
33.30	x^{-n}	$\dfrac{\pi \alpha^{n-1} \csc (n\pi/2)}{2\,\Gamma(n)}$ $\quad 0 < n < 2$
33.31	$\dfrac{\sin bx}{x}$	$\dfrac{1}{2} \ln \left(\dfrac{\alpha + b}{\alpha - b} \right)$
33.32	$\dfrac{\sin bx}{x^2}$	$\begin{cases} \pi\alpha/2 & \alpha < b \\ \pi b/2 & \alpha > b \end{cases}$
33.33	$\dfrac{\cos bx}{x}$	$\begin{cases} 0 & \alpha < b \\ \pi/4 & \alpha = b \\ \pi/2 & \alpha > b \end{cases}$
33.34	$\tan^{-1} (x/b)$	$\dfrac{\pi}{2\alpha} e^{-b\alpha}$
33.35	$\csc bx$	$\dfrac{\pi}{2b} \tanh \dfrac{\pi\alpha}{2b}$
33.36	$\dfrac{1}{e^{2x} - 1}$	$\dfrac{\pi}{4} \coth \left(\dfrac{\pi\alpha}{2} \right) - \dfrac{1}{2\alpha}$

SPECIAL FOURIER COSINE TRANSFORMS

	$f(x)$	$F_C(\alpha)$
33.37	$\begin{cases} 1 & 0 < x < b \\ 0 & x > b \end{cases}$	$\dfrac{\sin b\alpha}{\alpha}$
33.38	$\dfrac{1}{x^2 + b^2}$	$\dfrac{\pi e^{-b\alpha}}{2b}$
33.39	e^{-bx}	$\dfrac{b}{\alpha^2 + b^2}$
33.40	$x^{n-1} e^{-bx}$	$\dfrac{\Gamma(n) \cos (n \tan^{-1} \alpha/b)}{(\alpha^2 + b^2)^{n/2}}$
33.41	e^{-bx^2}	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{b}} e^{-\alpha^2/4b}$
33.42	$x^{-1/2}$	$\sqrt{\dfrac{\pi}{2\alpha}}$
33.43	x^{-n}	$\dfrac{\pi\alpha^{n-1} \sec (n\pi/2)}{2\,\Gamma(n)}, \quad 0 < n < 1$
33.44	$\ln\left(\dfrac{x^2 + b^2}{x^2 + c^2}\right)$	$\dfrac{e^{-c\alpha} - e^{-b\alpha}}{\pi\alpha}$
33.45	$\dfrac{\sin bx}{x}$	$\begin{cases} \pi/2 & \alpha < b \\ \pi/4 & \alpha = b \\ 0 & \alpha > b \end{cases}$
33.46	$\sin bx^2$	$\sqrt{\dfrac{\pi}{8b}}\left(\cos\dfrac{\alpha^2}{4b} - \sin\dfrac{\alpha^2}{4b}\right)$
33.47	$\cos bx^2$	$\sqrt{\dfrac{\pi}{8b}}\left(\cos\dfrac{\alpha^2}{4b} + \sin\dfrac{\alpha^2}{4b}\right)$
33.48	$\operatorname{sech} bx$	$\dfrac{\pi}{2b} \operatorname{sech} \dfrac{\pi\alpha}{2b}$
33.49	$\dfrac{\cosh (\sqrt{\pi}\, x/2)}{\cosh (\sqrt{\pi}\, x)}$	$\sqrt{\dfrac{\pi}{2}} \dfrac{\cosh (\sqrt{\pi}\, \alpha/2)}{\cosh (\sqrt{\pi}\, \alpha)}$
33.50	$\dfrac{e^{-b\sqrt{x}}}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{2\alpha}}\{\cos (2b\sqrt{\alpha}) - \sin (2b\sqrt{\alpha})\}$

34 ELLIPTIC FUNCTIONS

INCOMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

34.1
$$u = F(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{x} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$$

where $\phi = \operatorname{am} u$ is called the *amplitude* of u and $x = \sin\phi$, and where here and below $0 < k < 1$.

COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

34.2
$$K = F(k,\pi/2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{1} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$$
$$= \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right\}$$

INCOMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND

34.3
$$E(k,\phi) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{x} \frac{\sqrt{1-k^2v^2}}{\sqrt{1-v^2}}\, dv$$

COMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND

34.4
$$E = E(k,\pi/2) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{1} \frac{\sqrt{1-k^2v^2}}{\sqrt{1-v^2}}\, dv$$
$$= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right\}$$

INCOMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

34.5
$$\Pi(k,n,\phi) = \int_0^{\phi} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \int_0^{x} \frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}$$

COMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

34.6 $$\Pi(k, n, \pi/2) = \int_0^{\pi/2} \frac{d\theta}{(1 + n \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} = \int_0^1 \frac{dv}{(1 + nv^2)\sqrt{(1 - v^2)(1 - k^2v^2)}}$$

LANDEN'S TRANSFORMATION

34.7 $$\tan\phi = \frac{\sin 2\phi_1}{k + \cos 2\phi_1} \quad \text{or} \quad k \sin\phi = \sin(2\phi_1 - \phi)$$

This yields

34.8 $$F(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \frac{2}{1 + k} \int_0^{\phi_1} \frac{d\theta_1}{\sqrt{1 - k_1^2 \sin^2\theta_1}}$$

where $k_1 = 2\sqrt{k}/(1 + k)$. By successive applications, sequences $k_1, k_2, k_3, \ldots$ and $\phi_1, \phi_2, \phi_3, \ldots$ are obtained such that $k < k_1 < k_2 < k_3 < \cdots < 1$ where $\lim_{n \to \infty} k_n = 1$. It follows that

34.9 $$F(k, \Phi) = \sqrt{\frac{k_1 k_2 k_3 \ldots}{k}} \int_0^{\Phi} \frac{d\theta}{\sqrt{1 - \sin^2\theta}} = \sqrt{\frac{k_1 k_2 k_3 \ldots}{k}} \ln \tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)$$

where

34.10 $$k_1 = \frac{2\sqrt{k}}{1 + k}, \quad k_2 = \frac{2\sqrt{k_1}}{1 + k_1}, \quad \ldots \quad \text{and} \quad \Phi = \lim_{n \to \infty} \phi_n$$

The result is used in the approximate evaluation of $F(k, \phi)$.

JACOBI'S ELLIPTIC FUNCTIONS

From 34.1 we define the following elliptic functions.

34.11 $$x = \sin(\text{am}\, u) = \text{sn}\, u$$

34.12 $$\sqrt{1 - x^2} = \cos(\text{am}\, u) = \text{cn}\, u$$

34.13 $$\sqrt{1 - k^2x^2} = \sqrt{1 - k^2\, \text{sn}^2 u} = \text{dn}\, u$$

We can also define the inverse functions $\text{sn}^{-1} x$, $\text{cn}^{-1} x$, $\text{dn}^{-1} x$ and the following

34.14 $$\text{ns}\, u = \frac{1}{\text{sn}\, u}$$

34.15 $$\text{nc}\, u = \frac{1}{\text{cn}\, u}$$

34.16 $$\text{nd}\, u = \frac{1}{\text{dn}\, u}$$

34.17 $$\text{sc}\, u = \frac{\text{sn}\, u}{\text{cn}\, u}$$

34.18 $$\text{sd}\, u = \frac{\text{sn}\, u}{\text{dn}\, u}$$

34.19 $$\text{cd}\, u = \frac{\text{cn}\, u}{\text{dn}\, u}$$

34.20 $$\text{cs}\, u = \frac{\text{cn}\, u}{\text{sn}\, u}$$

34.21 $$\text{dc}\, u = \frac{\text{dn}\, u}{\text{cn}\, u}$$

34.22 $$\text{ds}\, u = \frac{\text{dn}\, u}{\text{sn}\, u}$$

ADDITION FORMULAS

34.23 $$\text{sn}(u + v) = \frac{\text{sn}\, u\, \text{cn}\, v\, \text{dn}\, v + \text{cn}\, u\, \text{sn}\, v\, \text{dn}\, u}{1 - k^2\, \text{sn}^2 u\, \text{sn}^2 v}$$

34.24 $$\text{cn}(u + v) = \frac{\text{cn}\, u\, \text{cn}\, v - \text{sn}\, u\, \text{sn}\, v\, \text{dn}\, u\, \text{dn}\, v}{1 - k^2\, \text{sn}^2 u\, \text{sn}^2 v}$$

34.25 $$\text{dn}(u + v) = \frac{\text{dn}\, u\, \text{dn}\, v - k^2\, \text{sn}\, u\, \text{sn}\, v\, \text{cn}\, u\, \text{cn}\, v}{1 - k^2\, \text{sn}^2 u\, \text{sn}^2 v}$$

DERIVATIVES

34.26 $\frac{d}{du}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u$

34.27 $\frac{d}{du}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u$

34.28 $\frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u$

34.29 $\frac{d}{du}\operatorname{sc} u = \operatorname{dc} u \operatorname{nc} u$

SERIES EXPANSIONS

34.30 $$\operatorname{sn} u = u - (1+k^2)\frac{u^3}{3!} + (1+14k^2+k^4)\frac{u^5}{5!} - (1+135k^2+135k^4+k^6)\frac{u^7}{7!} + \cdots$$

34.31 $$\operatorname{cn} u = 1 - \frac{u^2}{2!} + (1+4k^2)\frac{u^4}{4!} - (1+44k^2+16k^4)\frac{u^6}{6!} + \cdots$$

34.32 $$\operatorname{dn} u = 1 - k^2\frac{u^2}{2!} + k^2(4+k^2)\frac{u^4}{4!} - k^2(16+44k^2+k^4)\frac{u^6}{6!} + \cdots$$

CATALAN'S CONSTANT

34.33 $$\frac{1}{2}\int_0^1 K\,dk = \frac{1}{2}\int_{k=0}^{1}\int_{\theta=0}^{\pi/2}\frac{d\theta\,dk}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots = .915965594\ldots$$

PERIODS OF ELLIPTIC FUNCTIONS

Let

34.34 $$K = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad K' = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}} \quad \text{where } k' = \sqrt{1-k^2}$$

Then

34.35 $\operatorname{sn} u$ has periods $4K$ and $2iK'$

34.36 $\operatorname{cn} u$ has periods $4K$ and $2K + 2iK'$

34.37 $\operatorname{dn} u$ has periods $2K$ and $4iK'$

IDENTITIES INVOLVING ELLIPTIC FUNCTIONS

34.38 $\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$

34.39 $\operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1$

34.40 $\operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u = k'^2$ where $k' = \sqrt{1-k^2}$

34.41 $\operatorname{sn}^2 u = \frac{1 - \operatorname{cn} 2u}{1 + \operatorname{dn} 2u}$

34.42 $\operatorname{cn}^2 u = \frac{\operatorname{dn} 2u + \operatorname{cn} 2u}{1 + \operatorname{dn} 2u}$

34.43 $\operatorname{dn}^2 u = \frac{1 - k^2 + \operatorname{dn} 2u + k^2 \operatorname{cn} u}{1 + \operatorname{dn} 2u}$

34.44 $\sqrt{\frac{1 - \operatorname{cn} 2u}{1 + \operatorname{cn} 2u}} = \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}$

34.45 $\sqrt{\frac{1 - \operatorname{dn} 2u}{1 + \operatorname{dn} 2u}} = \frac{k \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}$

SPECIAL VALUES

34.46 $\operatorname{sn} 0 = 0$ **34.47** $\operatorname{cn} 0 = 1$ **34.48** $\operatorname{dn} 0 = 1$ **34.49** $\operatorname{sc} 0 = 0$ **34.50** $\operatorname{am} 0 = 0$

INTEGRALS

34.51 $$\int \operatorname{sn} u\, du = \frac{1}{k} \ln (\operatorname{dn} u - k \operatorname{cn} u)$$

34.52 $$\int \operatorname{cn} u\, du = \frac{1}{k} \cos^{-1} (\operatorname{dn} u)$$

34.53 $$\int \operatorname{dn} u\, du = \sin^{-1} (\operatorname{sn} u)$$

34.54 $$\int \operatorname{sc} u\, du = \frac{1}{\sqrt{1-k^2}} \ln (\operatorname{dc} u + \sqrt{1-k^2} \operatorname{nc} u)$$

34.55 $$\int \operatorname{cs} u\, du = \ln (\operatorname{ns} u - \operatorname{ds} u)$$

34.56 $$\int \operatorname{cd} u\, du = \frac{1}{k} \ln (\operatorname{nd} u + k \operatorname{sd} u)$$

34.57 $$\int \operatorname{dc} u\, du = \ln (\operatorname{nc} u + \operatorname{sc} u)$$

34.58 $$\int \operatorname{sd} u\, du = \frac{-1}{k\sqrt{1-k^2}} \sin^{-1} (k \operatorname{cd} u)$$

34.59 $$\int \operatorname{ds} u\, du = \ln (\operatorname{ns} u - \operatorname{cs} u)$$

34.60 $$\int \operatorname{ns} u\, du = \ln (\operatorname{ds} u - \operatorname{cs} u)$$

34.61 $$\int \operatorname{nc} u\, du = \frac{1}{\sqrt{1-k^2}} \ln \left(\operatorname{dc} u + \frac{\operatorname{sc} u}{\sqrt{1-k^2}} \right)$$

34.62 $$\int \operatorname{nd} u\, du = \frac{1}{\sqrt{1-k^2}} \cos^{-1} (\operatorname{cd} u)$$

LEGENDRE'S RELATION

34.63 $$EK' + E'K - KK' = \pi/2$$

where

34.64 $$E = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \qquad K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

34.65 $$E' = \int_0^{\pi/2} \sqrt{1 - k'^2 \sin^2 \theta} \qquad K' = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}}$$

35 MISCELLANEOUS SPECIAL FUNCTIONS

ERROR FUNCTION $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$

35.1 $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots\right)$$

35.2 $$\operatorname{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}\, x}\left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + \cdots\right)$$

35.3 $\operatorname{erf}(-x) = -\operatorname{erf}(x)$, $\operatorname{erf}(0) = 0$, $\operatorname{erf}(\infty) = 1$

COMPLEMENTARY ERROR FUNCTION $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du$

35.4 $$\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots\right)$$

35.5 $$\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}\, x}\left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + \cdots\right)$$

35.6 $\operatorname{erfc}(0) = 1$, $\operatorname{erfc}(\infty) = 0$

EXPONENTIAL INTEGRAL $Ei(x) = \int_x^\infty \frac{e^{-u}}{u}\, du$

35.7 $$Ei(x) = -\gamma - \ln x + \int_0^x \frac{1 - e^{-u}}{u}\, du$$

35.8 $$Ei(x) = -\gamma - \ln x + \left(\frac{x}{1 \cdot 1!} - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} - \cdots\right)$$

35.9 $$Ei(x) \sim \frac{e^{-x}}{x}\left(1 - \frac{1!}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots\right)$$

35.10 $Ei(\infty) = 0$

SINE INTEGRAL $Si(x) = \int_0^x \frac{\sin u}{u}\, du$

35.11 $$Si(x) = \frac{x}{1 \cdot 1!} - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

35.12 $$Si(x) \sim \frac{\pi}{2} - \frac{\sin x}{x}\left(\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots\right) - \frac{\cos x}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right)$$

35.13 $Si(-x) = -Si(x)$, $Si(0) = 0$, $Si(\infty) = \pi/2$

COSINE INTEGRAL $Ci(x) = \int_x^\infty \frac{\cos u}{u}\,du$

35.14 $Ci(x) = -\gamma - \ln x + \int_0^x \frac{1-\cos u}{u}\,du$

35.15 $Ci(x) = -\gamma - \ln x + \frac{x^2}{2\cdot 2!} - \frac{x^4}{4\cdot 4!} + \frac{x^6}{6\cdot 6!} - \frac{x^8}{8\cdot 8!} + \cdots$

35.16 $Ci(x) \sim \frac{\cos x}{x}\left(\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots\right) - \frac{\sin x}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right)$

35.17 $Ci(\infty) = 0$

FRESNEL SINE INTEGRAL $S(x) = \sqrt{\frac{2}{\pi}}\int_0^x \sin u^2\,du$

35.18 $S(x) = \sqrt{\frac{2}{\pi}}\left(\frac{x^3}{3\cdot 1!} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\right)$

35.19 $S(x) \sim \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\left\{(\cos x^2)\left(\frac{1}{x} - \frac{1\cdot 3}{2^2x^5} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4x^9} - \cdots\right) + (\sin x^2)\left(\frac{1}{2x^3} - \frac{1\cdot 3\cdot 5}{2^3x^7} + \cdots\right)\right\}$

35.20 $S(-x) = -S(x), \quad S(0) = 0, \quad S(\infty) = \frac{1}{2}$

FRESNEL COSINE INTEGRAL $C(x) = \sqrt{\frac{2}{\pi}}\int_0^x \cos u^2\,du$

35.21 $C(x) = \sqrt{\frac{2}{\pi}}\left(\frac{x}{1!} - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \cdots\right)$

35.22 $C(x) \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left\{(\sin x^2)\left(\frac{1}{x} - \frac{1\cdot 3}{2^2x^5} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4x^9} - \cdots\right) - (\cos x^2)\left(\frac{1}{2x^3} - \frac{1\cdot 3\cdot 5}{2^3x^7} + \cdots\right)\right\}$

35.23 $C(-x) = -C(x), \quad C(0) = 0, \quad C(\infty) = \frac{1}{2}$

RIEMANN ZETA FUNCTION $\zeta(x) = \frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \cdots$

35.24 $\zeta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{u^{x-1}}{e^u - 1}\,du, \quad x > 1$

35.25 $\zeta(1-x) = 2^{1-x}\pi^{-x}\,\Gamma(x)\cos(\pi x/2)\,\zeta(x)$ [extension to other values]

35.26 $\zeta(2k) = \frac{2^{2k-1}\pi^{2k}B_k}{(2k)!} \quad k = 1, 2, 3, \ldots$

36 INEQUALITIES

TRIANGLE INEQUALITY

36.1 $$|a_1| - |a_2| \leqq |a_1 + a_2| \leqq |a_1| + |a_2|$$

36.2 $$|a_1 + a_2 + \cdots + a_n| \leqq |a_1| + |a_2| + \cdots + |a_n|$$

CAUCHY-SCHWARZ INEQUALITY

36.3 $$|a_1b_1 + a_2b_2 + \cdots + a_nb_n|^2 \leqq (|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2)(|b_1|^2 + |b_2|^2 + \cdots + |b_n|^2)$$

The equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

INEQUALITIES INVOLVING ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

If A, G and H are the arithmetic, geometric and harmonic means of the positive numbers $a_1, a_2, \ldots, a_n$, then

36.4 $$H \leqq G \leqq A$$

where

36.5 $$A = \frac{a_1 + a_2 + \cdots + a_n}{n}$$ **36.6** $$G = \sqrt[n]{a_1a_2 \ldots a_n}$$ **36.7** $$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

The equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

HOLDER'S INEQUALITY

36.8 $$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leqq (|a_1|^p + |a_2|^p + \cdots + |a_n|^p)^{1/p}(|b_1|^q + |b_2|^q + \cdots + |b_n|^q)^{1/q}$$

where

36.9 $$\frac{1}{p} + \frac{1}{q} = 1 \qquad p > 1,\ q > 1$$

The equality holds if and only if $|a_1|^{p-1}/|b_1| = |a_2|^{p-1}/|b_2| = \cdots = |a_n|^{p-1}/|b_n|$. For $p = q = 2$ it reduces to 36.3.

CHEBYSHEV'S INEQUALITY

If $a_1 \geqq a_2 \geqq \cdots \geqq a_n$ and $b_1 \geqq b_2 \geqq \cdots \geqq b_n$, then

36.10 $$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \leqq \frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n}$$

or

36.11 $$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \leqq n(a_1b_1 + a_2b_2 + \cdots + a_nb_n)$$

MINKOWSKI'S INEQUALITY

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_n$ are all positive and $p > 1$, then

36.12 $$\{(a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p\}^{1/p} \leqq (a_1^p + a_2^p + \cdots + a_n^p)^{1/p} + (b_1^p + b_2^p + \cdots + b_n^p)^{1/p}$$

The equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

CAUCHY-SCHWARZ INEQUALITY FOR INTEGRALS

36.13 $$\left|\int_a^b f(x)\, g(x)\, dx\right|^2 \leqq \left\{\int_a^b |f(x)|^2\, dx\right\}\left\{\int_a^b |g(x)|^2\, dx\right\}$$

The equality holds if and only if $f(x)/g(x)$ is a constant.

HOLDER'S INEQUALITY FOR INTEGRALS

36.14 $$\int_a^b |f(x)\, g(x)|\, dx \leqq \left\{\int_a^b |f(x)|^p\, dx\right\}^{1/p}\left\{\int_a^b |g(x)|^q\, dx\right\}^{1/q}$$

where $1/p + 1/q = 1$, $p > 1$, $q > 1$. If $p = q = 2$, this reduces to 36.13.

The equality holds if and only if $|f(x)|^{p-1}/|g(x)|$ is a constant.

MINKOWSKI'S INEQUALITY FOR INTEGRALS

If $p > 1$,

36.15 $$\left\{\int_a^b |f(x) + g(x)|^p\, dx\right\}^{1/p} \leqq \left\{\int_a^b |f(x)|^p\, dx\right\}^{1/p} + \left\{\int_a^b |g(x)|^p\, dx\right\}^{1/p}$$

The equality holds if and only if $f(x)/g(x)$ is a constant.

37 PARTIAL FRACTION EXPANSIONS

37.1 $$\cot x = \frac{1}{x} + 2x\left\{\frac{1}{x^2-\pi^2} + \frac{1}{x^2-4\pi^2} + \frac{1}{x^2-9\pi^2} + \cdots\right\}$$

37.2 $$\csc x = \frac{1}{x} - 2x\left\{\frac{1}{x^2-\pi^2} - \frac{1}{x^2-4\pi^2} + \frac{1}{x^2-9\pi^2} - \cdots\right\}$$

37.3 $$\sec x = 4\pi\left\{\frac{1}{\pi^2-4x^2} - \frac{3}{9\pi^2-4x^2} + \frac{5}{25\pi^2-4x^2} - \cdots\right\}$$

37.4 $$\tan x = 8x\left\{\frac{1}{\pi^2-4x^2} + \frac{1}{9\pi^2-4x^2} + \frac{1}{25\pi^2-4x^2} + \cdots\right\}$$

37.5 $$\sec^2 x = 4\left\{\frac{1}{(\pi-2x)^2} + \frac{1}{(\pi+2x)^2} + \frac{1}{(3\pi-2x)^2} + \frac{1}{(3\pi+2x)^2} + \cdots\right\}$$

37.6 $$\csc^2 x = \frac{1}{x^2} + \frac{1}{(x-\pi)^2} + \frac{1}{(x+\pi)^2} + \frac{1}{(x-2\pi)^2} + \frac{1}{(x+2\pi)^2} + \cdots$$

37.7 $$\coth x = \frac{1}{x} + 2x\left\{\frac{1}{x^2+\pi^2} + \frac{1}{x^2+4\pi^2} + \frac{1}{x^2+9\pi^2} + \cdots\right\}$$

37.8 $$\operatorname{csch} x = \frac{1}{x} - 2x\left\{\frac{1}{x^2+\pi^2} - \frac{1}{x^2+4\pi^2} + \frac{1}{x^2+9\pi^2} - \cdots\right\}$$

37.9 $$\operatorname{sech} x = 4\pi\left\{\frac{1}{\pi^2+4x^2} - \frac{3}{9\pi^2+4x^2} + \frac{5}{25\pi^2+4x^2} - \cdots\right\}$$

37.10 $$\tanh x = 8x\left\{\frac{1}{\pi^2+4x^2} + \frac{1}{9\pi^2+4x^2} + \frac{1}{25\pi^2+4x^2} + \cdots\right\}$$

38 INFINITE PRODUCTS

38.1 $$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

38.2 $$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{9\pi^2}\right)\left(1-\frac{4x^2}{25\pi^2}\right)\cdots$$

38.3 $$\sinh x = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\cdots$$

38.4 $$\cosh x = \left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{9\pi^2}\right)\left(1+\frac{4x^2}{25\pi^2}\right)\cdots$$

38.5 $$\frac{1}{\Gamma(x)} = xe^{\gamma x}\left\{\left(1+\frac{x}{1}\right)e^{-x}\right\}\left\{\left(1+\frac{x}{2}\right)e^{-x/2}\right\}\left\{\left(1+\frac{x}{3}\right)e^{-x/3}\right\}\cdots$$

See also 16.12, page 102.

38.6 $$J_0(x) = \left(1-\frac{x^2}{\lambda_1^2}\right)\left(1-\frac{x^2}{\lambda_2^2}\right)\left(1-\frac{x^2}{\lambda_3^2}\right)\cdots$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_0(x) = 0$.

38.7 $$J_1(x) = x\left(1-\frac{x^2}{\lambda_1^2}\right)\left(1-\frac{x^2}{\lambda_2^2}\right)\left(1-\frac{x^2}{\lambda_3^2}\right)\cdots$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_1(x) = 0$.

38.8 $$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cos\frac{x}{16}\cdots$$

38.9 $$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

This is called *Wallis' product.*

39 PROBABILITY DISTRIBUTIONS

BINOMIAL DISTRIBUTION

39.1

$$\Phi(x) = \sum_{t \leqq x} \binom{n}{t} p^t q^{n-t} \qquad p > 0,\ q > 0,\ p + q = 1$$

POISSON DISTRIBUTION

39.2

$$\Phi(x) = \sum_{t \leqq x} \frac{\lambda^t e^{-\lambda}}{t!} \qquad \lambda > 0$$

HYPERGEOMETRIC DISTRIBUTION

39.3

$$\Phi(x) = \sum_{t \leqq x} \frac{\binom{r}{t}\binom{s}{n-t}}{\binom{r+s}{n}}$$

NORMAL DISTRIBUTION

39.4

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2}\, dt$$

STUDENT'S t DISTRIBUTION

39.5

$$\Phi(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)} \int_{-\infty}^{x} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} dt$$

CHI SQUARE DISTRIBUTION

39.6

$$\Phi(x) = \frac{1}{2^{n/2}\,\Gamma(n/2)} \int_{0}^{x} t^{(n-2)/2}\, e^{-t/2}\, dt$$

F DISTRIBUTION

39.7

$$\Phi(x) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right) n_1^{n_1/2} n_2^{n_2/2}}{\Gamma(n_1/2)\,\Gamma(n_2/2)} \int_{0}^{x} t^{n_1/2} (n_2 + n_1 t)^{-(n_1+n_2)/2}\, dt$$

40 SPECIAL MOMENTS OF INERTIA

The table below shows the moments of inertia of various rigid bodies of mass M. In all cases it is assumed the body has uniform [i.e. constant] density.

TYPE OF RIGID BODY	MOMENT OF INERTIA
40.1 Thin rod of length a	
(a) about axis perpendicular to the rod through the center of mass,	$\frac{1}{12}Ma^2$
(b) about axis perpendicular to the rod through one end.	$\frac{1}{3}Ma^2$
40.2 Rectangular parallelepiped with sides a, b, c	
(a) about axis parallel to c and through center of face ab,	$\frac{1}{12}M(a^2+b^2)$
(b) about axis through center of face bc and parallel to c.	$\frac{1}{12}M(4a^2+b^2)$
40.3 Thin rectangular plate with sides a, b	
(a) about axis perpendicular to the plate through center,	$\frac{1}{12}M(a^2+b^2)$
(b) about axis parallel to side b through center.	$\frac{1}{12}Ma^2$
40.4 Circular cylinder of radius a and height h	
(a) about axis of cylinder,	$\frac{1}{2}Ma^2$
(b) about axis through center of mass and perpendicular to cylindrical axis,	$\frac{1}{12}M(h^2+3a^2)$
(c) about axis coinciding with diameter at one end.	$\frac{1}{12}M(4h^2+3a^2)$
40.5 Hollow circular cylinder of outer radius a, inner radius b and height h	
(a) about axis of cylinder,	$\frac{1}{2}M(a^2+b^2)$
(b) about axis through center of mass and perpendicular to cylindrical axis,	$\frac{1}{12}M(3a^2+3b^2+h^2)$
(c) about axis coinciding with diameter at one end.	$\frac{1}{12}M(3a^2+3b^2+4h^2)$

40.6 Circular plate of radius a	
(a) about axis perpendicular to plate through center,	$\frac{1}{2}Ma^2$
(b) about axis coinciding with a diameter.	$\frac{1}{4}Ma^2$
40.7 Hollow circular plate or ring with outer radius a and inner radius b	
(a) about axis perpendicular to plane of plate through center,	$\frac{1}{2}M(a^2+b^2)$
(b) about axis coinciding with a diameter.	$\frac{1}{4}M(a^2+b^2)$
40.8 Thin circular ring of radius a	
(a) about axis perpendicular to plane of ring through center,	Ma^2
(b) about axis coinciding with diameter.	$\frac{1}{2}Ma^2$
40.9 Sphere of radius a	
(a) about axis coinciding with a diameter,	$\frac{2}{5}Ma^2$
(b) about axis tangent to the surface.	$\frac{7}{5}Ma^2$
40.10 Hollow sphere of outer radius a and inner radius b	
(a) about axis coinciding with a diameter,	$\frac{2}{5}M(a^5-b^5)/(a^3-b^3)$
(b) about axis tangent to the surface.	$\frac{2}{5}M(a^5-b^5)/(a^3-b^3)+Ma^2$
40.11 Hollow spherical shell of radius a	
(a) about axis coinciding with a diameter,	Ma^2
(b) about axis tangent to the surface.	$2Ma^2$
40.12 Ellipsoid with semi-axes a, b, c	
(a) about axis coinciding with semi-axis c,	$\frac{1}{5}M(a^2+b^2)$
(b) about axis tangent to surface, parallel to semi-axis c and at distance a from center.	$\frac{1}{5}M(6a^2+b^2)$
40.13 Circular cone of radius a and height h	
(a) about axis of cone,	$\frac{3}{10}Ma^2$
(b) about axis through vertex and perpendicular to axis,	$\frac{3}{20}M(a^2+4h^2)$
(c) about axis through center of mass and perpendicular to axis.	$\frac{3}{80}M(4a^2+h^2)$
40.14 Torus with outer radius a and inner radius b	
(a) about axis through center of mass and perpendicular to plane of torus,	$\frac{1}{4}M(7a^2-6ab+3b^2)$
(b) about axis through center of mass and in the plane of the torus.	$\frac{1}{4}M(9a^2-10ab+5b^2)$

41 CONVERSION FACTORS

Length

1 kilometer (km)	= 1000 meters (m)	1 inch (in.)	= 2.540 cm
1 meter (m)	= 100 centimeters (cm)	1 foot (ft)	= 30.48 cm
1 centimeter (cm)	= 10^{-2} m	1 mile (mi)	= 1.609 km
1 millimeter (mm)	= 10^{-3} m	1 mil	= 10^{-3} in.
1 micron (μ)	= 10^{-6} m	1 centimeter	= 0.3937 in.
1 millimicron (mμ)	= 10^{-9} m	1 meter	= 39.37 in.
1 angstrom (A)	= 10^{-10} m	1 kilometer	= 0.6214 mile

Area

1 square meter (m^2) = 10.76 ft^2 1 square mile (mi^2) = 640 acres
1 square foot (ft^2) = 929 cm^2 1 acre = 43,560 ft^2

Volume

1 liter (l) = 1000 cm^3 = 1.057 quart (qt) = 61.02 in^3 = 0.03532 ft^3
1 cubic meter (m^3) = 1000 l = 35.32 ft^3
1 cubic foot (ft^3) = 7.481 U.S. gal = 0.02832 m^3 = 28.32 l
1 U.S. gallon (gal) = 231 in^3 = 3.785 l; 1 British gallon = 1.201 U.S. gallon = 277.4 in^3

Mass

1 kilogram (kg) = 2.2046 pounds (lb) = 0.06852 slug; 1 lb = 453.6 gm = 0.03108 slug
1 slug = 32.174 lb = 14.59 kg

Speed

1 km/hr = 0.2778 m/sec = 0.6214 mi/hr = 0.9113 ft/sec
1 mi/hr = 1.467 ft/sec = 1.609 km/hr = 0.4470 m/sec

Density

1 gm/cm^3 = 10^3 kg/m^3 = 62.43 lb/ft^3 = 1.940 $slug/ft^3$
1 lb/ft^3 = 0.01602 gm/cm^3; 1 $slug/ft^3$ = 0.5154 gm/cm^3

Force

1 newton (nt) = 10^5 dynes = 0.1020 kgwt = 0.2248 lbwt
1 pound weight (lbwt) = 4.448 nt = 0.4536 kgwt = 32.17 poundals
1 kilogram weight (kgwt) = 2.205 lbwt = 9.807 nt
1 U.S. short ton = 2000 lbwt; 1 long ton = 2240 lbwt; 1 metric ton = 2205 lbwt

Energy

1 joule = 1 nt m = 10^7 ergs = 0.7376 ft lbwt = 0.2389 cal = 9.481×10^{-4} Btu
1 ft lbwt = 1.356 joules = 0.3239 cal = 1.285×10^{-3} Btu
1 calorie (cal) = 4.186 joules = 3.087 ft lbwt = 3.968×10^{-3} Btu
1 Btu (British thermal unit) = 778 ft lbwt = 1055 joules = 0.293 watt hr
1 kilowatt hour (kw hr) = 3.60×10^6 joules = 860.0 kcal = 3413 Btu
1 electron volt (ev) = 1.602×10^{-19} joule

Power

1 watt = 1 joule/sec = 10^7 ergs/sec = 0.2389 cal/sec
1 horsepower (hp) = 550 ft lbwt/sec = 33,000 ft lbwt/min = 745.7 watts
1 kilowatt (kw) = 1.341 hp = 737.6 ft lbwt/sec = 0.9483 Btu/sec

Pressure

1 nt/m^2 = 10 dynes/cm^2 = 9.869×10^{-6} atmosphere = 2.089×10^{-2} $lbwt/ft^2$
1 $lbwt/in^2$ = 6895 nt/m^2 = 5.171 cm mercury = 27.68 in. water
1 atmosphere (atm) = 1.013×10^5 nt/m^2 = 1.013×10^6 dynes/cm^2 = 14.70 $lbwt/in^2$
= 76 cm mercury = 406.8 in. water

Part II

TABLES

SAMPLE PROBLEMS

ILLUSTRATING USE OF THE TABLES

COMMON LOGARITHMS

1. Find log 2.36.

We must find the number p such that $10^p = 2.36 = N$. Since $10^0 = 1$ and $10^1 = 10$, p lies between 0 and 1 and can be found from the tables of common logarithms on page 202.

Thus to find log 2.36 we glance down the *left* column headed N until we come to the first two digits, 23. Then we proceed *right* to the column headed 6. We find the entry 3729. Thus log 2.36 = 0.3729, i.e. $2.36 = 10^{0.3729}$.

2. Find (*a*) log 23.6, (*b*) log 236, (*c*) log 2360.

From Problem 1, $2.36 = 10^{0.3729}$. Then multiplying successively by 10 we have

$$23.6 = 10^{1.3729}, \quad 236 = 10^{2.3729}, \quad 2360 = 10^{3.3729}$$

Thus

(*a*) log 23.6 = 1.3729

(*b*) log 236 = 2.3729

(*c*) log 2360 = 3.3729.

The number .3729 obtained from the table is called the *mantissa* of the logarithm. The number before the decimal point is called the *characteristic*. Thus in (*b*) the characteristic is 2.

The following rule is easily demonstrated.

Rule 1. For a number greater than 1, the characteristic is one less than the number of digits before the decimal point. For example since 2360 has four digits before the decimal point, the characteristic is $4 - 1 = 3$.

3. Find (*a*) log .236, (*b*) log .0236, (*c*) log .00236.

From Problem 1, $2.36 = 10^{0.3729}$. Then dividing successively by 10 we have

$$.236 = 10^{0.3729-1} = 10^{9.3729-10} = 10^{-.6271}$$

$$.0236 = 10^{0.3729-2} = 10^{8.3729-10} = 10^{-1.6271}$$

$$.00236 = 10^{0.3729-3} = 10^{7.3729-10} = 10^{-2.6271}$$

Then

(*a*) log .236 = 9.3729 − 10 = −.6271

(*b*) log .0236 = 8.3729 − 10 = −1.6271

(*c*) log .00236 = 7.3729 − 10 = −2.6271.

The number .3729 is the mantissa of the logarithm. The number apart from the mantissa [for example 9 − 10, 8 − 10 or 7 − 10] is the characteristic.

The following rule is easily demonstrated.

Rule 2. For a positive number less than 1, the characteristic is negative and numerically one more than the number of zeros immediately following the decimal point. For example since .00236 has two zeros immediately following the decimal point, the characteristic is −3 or 7 − 10.

4. Verify each of the following logarithms.

   (*a*) log 87.2. Mantissa = .9405, characteristic = 1; then log 87.2 = 1.9405.

   (*b*) log 395,000 = 5.5966.

   (*c*) log .0482. Mantissa = .6830, characteristic = 8 − 10; then $\log_{10}$ .0482 = 8.6830 − 10.

   (*d*) log .000827 = 6.9175 − 10.

5. Find log 4.638.

   Since the number has four digits, we must use interpolation to find the mantissa. The mantissa of log 4638 is .8 of the way between the mantissas of log 4630 and log 4640.

Mantissa of log 4640	= .6665	Mantissa of log 4.638	= .6656 + (.8)(.0009)
Mantissa of log 4630	= .6656		= .6663 to four digits
Tabular difference	= .0009	Then log 4.638	= 0.6663

   If desired the proportional parts table on page 202 can be used to give the mantissa directly (6656 + 7).

6. Verify each of the following logarithms.

   (*a*) log 183.2 = 2.2630 (2625 + 5)

   (*b*) log 87,640 = 4.9427 (9425 + 2)

   (*c*) log .2548 = 9.4062 − 10 (4048 + 14)

   (*d*) log .009848 = 7.9933 − 10 (9930 + 3)

COMMON ANTILOGARITHMS

7. Find (*a*) antilog 1.7530, (*b*) antilog (7.7530 − 10).

   (*a*) We must find the value of $10^{1.7530}$. Since the mantissa is .7530 we glance down the *left* column headed *p* in the table on page 205 until we come to the first two digits 75. Then we proceed *right* to the column headed 3. We find the entry 5662. Since the characteristic is 1, there are two digits before the decimal point. Then the required number is 56.62.

   (*b*) As in part (*a*) we find the entry 5662 corresponding to the mantissa .7530. Then since the characteristic is 7 − 10, the number must have two zeros immediately following the decimal point. Thus the required number is .005662.

8. Find antilog (9.3842 − 10).

   The mantissa .3842 lies between .3840 and .3850 and we must use interpolation. From the table on page 204 we have

Number corresponding to .3850	= 2427	Given mantissa	= .3842
Number corresponding to .3840	= 2421	Next smaller mantissa	= .3840
Tabular difference	= 6	Difference	= .0002

   Then $2421 + \frac{2}{10}(2427 - 2421) = 2422$ to four digits, and the required number is 0.2422.

   The proportional parts table on page 204 can also be used.

9. Verify each of the following antilogarithm.

   (*a*) antilog 2.6715 = 469.3

   (*b*) antilog 9.6089 − 10 = .4063

   (*c*) antilog 4.2023 = 15,930

COMPUTATIONS USING LOGARITHMS

10. $P = \dfrac{(784.6)(.0431)}{28.23}$. $\log P = \log 784.6 + \log .0431 - \log 28.23$.

$$\begin{aligned} \log 784.6 &= 2.8947 \\ (+)\ \log .0431 &= 8.6345 - 10 \\ &\quad\ 11.5292 - 10 \\ (-)\ \log 28.23 &= 1.4507 \\ \log P &= 10.0785 - 10 = .0785. \end{aligned}$$ Then $P = 1.198$.

Note the exponential significance of the computation, i.e.

$$\frac{(784.6)(.0431)}{28.23} = \frac{(10^{2.8947})(10^{8.6345-10})}{10^{1.4507}} = 10^{2.8947+8.6345-10-1.4507} = 10^{.0785} = 1.198$$

11. $P = (5.395)^8$. $\log P = 8 \log 5.395 = 8(0.7320) = 5.8560$, and $P = 717{,}800$.

12. $P = \sqrt{387.2} = (387.2)^{1/2}$. $\log P = \frac{1}{2} \log 387.2 = \frac{1}{2}(2.5879) = 1.2940$ and $P = 19.68$.

13. $P = \sqrt[5]{.08317} = (.08317)^{1/5}$. $\log P = \frac{1}{5} \log .08317 = \frac{1}{5}(8.9200 - 10) = \frac{1}{5}(48.9200 - 50) = 9.7840 - 10$ and $P = .6081$.

14. $P = \dfrac{\sqrt{.003654}\,(18.37)^3}{(8.724)^4 \sqrt[4]{743.8}}$. $\log P = \frac{1}{2} \log .003654 + 3 \log 18.37 - (4 \log 8.724 + \frac{1}{4} \log 743.8)$

Numerator N

$$\begin{aligned} \tfrac{1}{2} \log .003654 &= \tfrac{1}{2}(7.5628 - 10) \\ &= \tfrac{1}{2}(17.5628 - 20) = 8.7814 - 10 \\ 3 \log 18.37 &= 3(1.2641) = 3.7923 \\ \text{Add:}\quad \log N &= 12.5737 - 10 \end{aligned}$$

Denominator D

$$\begin{aligned} 4 \log 8.724 &= 4(0.9407) = 3.7628 \\ \tfrac{1}{4} \log 743.6 &= \tfrac{1}{4}(2.8714) = 0.7178 \\ \text{Add:}\quad \log D &= 4.4806 \end{aligned}$$

$$\begin{aligned} \log N &= 12.5737 - 10 \\ (-)\ \log D &= 4.4806 \\ \log P &= 8.0931 - 10. \end{aligned}$$ Then $P = .01239$

NATURAL OR NAPIERIAN LOGARITHMS

15. Find (*a*) ln 7.236, (*b*) ln 836.2, (*c*) ln .002548.

(*a*) Use the table on page 225.

$$\begin{aligned} \ln 7.240 &= 1.97962 \\ \ln 7.230 &= 1.97824 \\ \text{Tabular difference} &= .00138 \end{aligned}$$

Then $\ln 7.236 = 1.97824 + \frac{6}{10}(.00138) = 1.97907$

In terms of exponentials this means that $e^{1.97907} = 7.236$.

(*b*) As in part (*a*) we find

$$\ln 8.362 = 2.12346 + \tfrac{2}{10}(2.12465 - 2.12346) = 2.12370$$

Then

$$\ln 836.2 = \ln (8.362 \times 10^2) = \log 8.362 + 2 \ln 10 = 2.12370 + 4.60517 = 6.72887$$

In terms of exponentials this means that $e^{6.72887} = 836.2$.

(*c*) As in part (*a*) we find

$$\ln 2.548 = 0.93216 + \tfrac{8}{10}(0.93609 - 0.93216) = 0.93530$$

Then

$$\ln .002548 = \ln (2.548 \times 10^{-3}) = \ln 2.548 - 3 \ln 10 = 0.93530 - 6.90776 = -5.97246$$

In terms of exponentials this means that $e^{-5.97246} = .002548$.

TRIGONOMETRIC FUNCTIONS (DEGREES AND MINUTES)

16. Find (*a*) sin 74°23′, (*b*) cos 35°42′, (*c*) tan 82°56′.

(*a*) Refer to the table on page 206.

$$\begin{aligned} \sin 74°30' &= .9636 \\ \sin 74°20' &= .9628 \\ \text{Tabular difference} &= .0008 \end{aligned}$$

Then $\sin 74°23' = .9628 + \frac{3}{10}(.0008) = .9630$

(*b*) Refer to the table on page 207.

$$\begin{aligned} \cos 35°40' &= .8124 \\ \cos 35°50' &= .8107 \\ \text{Tabular difference} &= .0017 \end{aligned}$$

Then $\cos 35°42' = .8124 - \frac{2}{10}(.0017) = .8121$

or $\cos 35°42' = .8107 + \frac{8}{10}(.0017) = .8121$

(*c*) Refer to the table on page 208.

$$\begin{aligned} \tan 82°60' = \tan 83°0' &= 8.1443 \\ \tan 82°50' &= 7.9530 \\ \text{Tabular difference} &= .1913 \end{aligned}$$

Then $\tan 82°56' = 7.9530 + \frac{6}{10}(.1913) = 8.0678$

17. Find (*a*) cot 45°16′, (*b*) sec 73°48′, (*c*) csc 28°33′.

(*a*) Refer to the table on page 209.

$$\begin{aligned} \cot 45°10' &= .9942 \\ \cot 45°20' &= .9884 \\ \text{Tabular difference} &= .0058 \end{aligned}$$

Then $\cot 45°16' = .9942 - \frac{6}{10}(.0058) = .9907$

or $\cot 45°16' = .9884 + \frac{4}{10}(.0058) = .9907$

(*b*) Refer to the table on page 210.

$$\begin{aligned} \sec 73°50' &= 3.592 \\ \sec 73°40' &= 3.556 \\ \text{Tabular difference} &= .036 \end{aligned}$$

Then $\sec 73°48' = 3.556 + \frac{8}{10}(.036) = 3.585$

(*c*) Refer to the table on page 211.

$$\begin{aligned} \csc 28°30' &= 2.096 \\ \csc 28°40' &= 2.085 \\ \text{Tabular difference} &= .011 \end{aligned}$$

Then $\csc 28°33' = 2.096 - \frac{3}{10}(.011) = 2.093$

or $\csc 28°33' = 2.085 + \frac{7}{10}(.011) = 2.093$

INVERSE TRIGONOMETRIC FUNCTIONS (DEGREES AND MINUTES)

18. Find (*a*) $\sin^{-1}(.2143)$, (*b*) $\cos^{-1}(.5412)$, (*c*) $\tan^{-1}(1.1536)$.

(*a*) Refer to the table on page 206.

$$\begin{aligned} \sin 12°30' &= .2164 \\ \sin 12°20' &= .2136 \\ \text{Tabular difference} &= .0028 \end{aligned}$$

Since .2143 is $\dfrac{.2143 - .2136}{.0028} = \frac{1}{4}$ of the way between .2136 and .2164, the required angle is $12°20' + \frac{1}{4}(10') = 12°22.5'$.

(*b*) Refer to the table on page 207.

$$\begin{aligned} \cos 57°10' &= .5422 \\ \cos 57°20' &= .5398 \\ \text{Tabular difference} &= .0024 \end{aligned}$$

Then $$\cos^{-1}(.5412) = 57°20' - \frac{.5412 - .5398}{.0024}(10') = 57°14.2'$$

or $$\cos^{-1}(.5412) = 57°10' + \frac{.5422 - .5412}{.0024}(10') = 57°14.2'$$

(*c*) Refer to the table on page 208.

$$\begin{aligned} \tan 49°10' &= 1.1571 \\ \tan 49°0' &= 1.1504 \\ \text{Tabular difference} &= .0067 \end{aligned}$$

Then $$\tan^{-1}(1.1536) = 49°0' + \frac{1.1536 - 1.1504}{.0067}(10') = 49°4.8'$$

Other inverse trigonometric functions can be obtained similarly.

TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS (RADIANS)

19. Find (*a*) $\sin(.627)$, (*b*) $\cos(1.056)$, (*c*) $\tan(.153)$.

(*a*) Refer to the table on page 213.

$$\begin{aligned} \sin(.630) &= .58914 \\ \sin(.620) &= .58104 \\ \text{Tabular difference} &= .00810 \end{aligned}$$

Then $$\sin(.627) = .58104 + \tfrac{7}{10}(.00810) = .58671$$

(*b*) Refer to the table on page 214.

$$\begin{aligned} \cos(1.050) &= .49757 \\ \cos(1.060) &= .48887 \\ \text{Tabular difference} &= .00870 \end{aligned}$$

Then $$\cos(1.056) = .49757 - \tfrac{6}{10}(.00870) = .49235$$

or $$\cos(1.056) = .48887 + \tfrac{4}{10}(.00870) = .49235$$

(*c*) Refer to the table on page 212.

$$\begin{aligned} \tan(.160) &= .16138 \\ \tan(.150) &= .15114 \\ \text{Tabular difference} &= .01024 \end{aligned}$$

Then $$\tan(.153) = .15114 + \tfrac{3}{10}(.01024) = .15421$$

Similarly other trigonometric functions are obtained.

20. Find $\sin^{-1}(.512)$ in radians.

Refer to the table on page 213.

$$\begin{aligned} \sin(.540) &= .51414 \\ \sin(.530) &= .50553 \\ \text{Tabular difference} &= .00861 \end{aligned}$$

Then $$\sin^{-1}(.512) = .530 + \frac{.512 - .50553}{.00861}(.01) = .5375 \text{ radians}$$

Similarly the other inverse trigonometric functions are obtained.

COMMON LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

21. Find (*a*) $\log \sin 63°17'$, (*b*) $\log \cos 48°44'$.

(*a*) Refer to the table on page 217.

$$\begin{aligned} \log \sin 63°20' &= 9.9512 - 10 \\ \log \sin 63°10' &= 9.9505 - 10 \\ \text{Tabular difference} &= .0007 \end{aligned}$$

Then $$\log \sin 63°17' = 9.9505 - 10 + \tfrac{7}{10}(.0007) = 9.9510 - 10$$

(*b*) Refer to the table on page 219.

$$\begin{aligned} \log \cos 48°40' &= 9.8198 - 10 \\ \log \cos 48°50' &= 9.8184 - 10 \\ \text{Tabular difference} &= .0014 \end{aligned}$$

Then $$\log \cos 48°44' = 9.8198 - 10 - \tfrac{4}{10}(.0014) = 9.8192 - 10$$

or $$\log \cos 48°44' = 9.8184 - 10 + \tfrac{6}{10}(.0014) = 9.8192 - 10$$

Similarly we can find logarithms of other trigonometric functions. Note that $\log \sec x = -\log \cos x$, $\log \cot x = -\log \tan x$, $\log \csc x = -\log \sin x$.

22. If $\log \tan x = 9.6845 - 10$, find x.

Refer to the table on page 220.

$$\begin{aligned} \log \tan 25°50' &= 9.6850 - 10 \\ \log \tan 25°40' &= 9.6817 - 10 \\ \text{Tabular difference} &= .0033 \end{aligned}$$

Then $$x = 25°40' + \frac{9.6845 - 9.6817}{.0033}(10') = 25°48.5'$$

CONVERSION OF DEGREES, MINUTES AND SECONDS TO RADIANS

23. Find $75° 28' 47''$ in radians.

Refer to the table on page 223.

$$\begin{aligned} 70° &= 1.221730 \text{ radians} \\ 5° &= .087267 \\ 20' &= .005818 \\ 8' &= .002327 \\ 40'' &= .000194 \\ 7'' &= .000034 \end{aligned}$$

Adding, $$75° 28' 47'' = 1.317370 \text{ radians}$$

CONVERSION OF RADIANS TO DEGREES, MINUTES AND SECONDS

24. Find 2.547 radians in degrees, minutes and seconds.

Refer to the table on page 222.

	2	radians	=	114°	35′	29.6″	
	.5		=	28°	38′	52.4″	
	.04		=	2°	17′	30.6″	
	.007		=	0°	24′	3.9″	
Adding,	2.547	radians	=	144°	114′	116.5″	= 145° 55′ 56.5″

CONVERSION OF RADIANS TO FRACTIONS OF A DEGREE

25. Find 1.382 radians in terms of degrees.

Refer to the table on page 222.

	1	radian	= 57.2958°
	.3		= 17.1887°
	.08		= 4.5837°
	.002		= .1146°
Adding,	1.382	radians	= 79.1828°

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

26. Find (*a*) $e^{5.24}$, (*b*) $e^{-.158}$.

(*a*) Refer to the table on page 226.

$$e^{5.30} = 200.34$$
$$e^{5.20} = 181.27$$
$$\text{Tabular difference} = 19.07$$

Then $e^{5.24} = 181.27 + \frac{4}{10}(19.07) = 188.90$

(*b*) Refer to the table on page 227.

$$e^{-.150} = .86071$$
$$e^{-.160} = .85214$$
$$\text{Tabular difference} = .00857$$

Then $e^{-.158} = .86071 - \frac{8}{10}(.00857) = .85385$

or $e^{-.158} = .85214 + \frac{2}{10}(.00857) = .85385$

27. Find (*a*) sinh (4.846), (*b*) sech (.163).

(*a*) Refer to the table on page 229.

$$\sinh(4.850) = 63.866$$
$$\sinh(4.840) = 63.231$$
$$\text{Tabular difference} = .635$$

Then $\sinh(4.846) = 63.231 + \frac{6}{10}(.635) = 63.612$

(*b*) Refer to the table on page 230.

$$\cosh(.170) = 1.0145$$
$$\cosh(.160) = 1.0128$$
$$\text{Tabular difference} = .0017$$

Then $\cosh(.163) = 1.0128 + \frac{3}{10}(.0017) = 1.0133$

and so $\operatorname{sech}(.163) = \dfrac{1}{\cosh(.163)} = \dfrac{1}{1.0133} = .98687$

28. Find $\tanh^{-1}(.71423)$.

Refer to the table on page 232.

$$\begin{aligned} \tanh(.900) &= .71630 \\ \tanh(.890) &= .71139 \\ \text{Tabular difference} &= .00491 \end{aligned}$$

Then $$\tanh^{-1}(.71423) = .890 + \frac{.71423 - .71139}{.00491}(10) = .8958$$

INTEREST AND ANNUITIES

29. A man deposits **$2800** in a bank which pays 5% compounded quarterly. What will the deposit amount to in 8 years?

There are $n = 8 \cdot 4 = 32$ payment periods at interest rate $r = .05/4 = .0125$ per period. Then the amount is

$$A = \$2800(1 + .0125)^{32} = \$2800(1.4881) = \$4166.68$$

using the table on page 240.

30. A man expects to receive $12,000 in 10 years. How much is that money worth now, considering interest at 6% compounded semi-annually?

We are asked for the present value P which will amount to $A = \$12{,}000$ in 10 years. Since there are $n = 10 \cdot 2 = 20$ payment periods at interest rate $r = .06/2 = .03$ per period, the present value is

$$P = \$12{,}000(1 + .03)^{-20} = \$12{,}000(.55368) = \$6644.16$$

using the table on page 241.

31. An investor has an annuity in which a payment of $500 is made at the end of each year. If interest is 4% compounded annually, what is the amount of the annuity after 20 years?

Here $r = .04$, $n = 20$ and the amount is [see table on page 242],

$$\$500\left[\frac{(1 + .04)^{20} - 1}{.04}\right] = \$500(29.7781) = \$14{,}889.05$$

32. What is the present value of an annuity of $120 at the end of each 3 months for 12 years at 6% compounded quarterly?

Here $n = 4 \cdot 12 = 48$ payment periods, $r = .06/4 = .015$ and the present value is

$$\$120\left[\frac{1 - (1.015)^{-48}}{.015}\right] = \$120(34.0426) = \$4085.11$$

using the table on page 243.

TABLE 1

FOUR PLACE COMMON LOGARITHMS

$\log_{10} N$ or $\log N$

N	0	1	2	3	4	5	6	7	8	9	1	2
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2

12	17	21	25	29	33	37						
11	15	19	23	26	30	34						
10	14	17	21	24	28	31						
10	13	16	19	23	26	29						
9	12	15	18	21	24	27						
8	11	14	17	20	22	25						
8	11	13	16	18	21	24						
7	10	12	15	17	20	22						
7	9	12	14	16	19	21						
7	9	11	13	16	18	20						
6	8	11	13	15	17	19						
6	8	10	12	14	16	18						
6	8	10	12	14	15	17						
6	7	9	11	13	15	17						
5	7	9	11	12	14	16						
5	7	9	10	12	14	15						
5	7	8	10	11	13	15						
5	6	8	9	11	13	14						
5	6	8	9	11	12	14						
4	6	7	9	10	12	13						
4	6	7	9	10	11	13						
4	6	7	8	10	11	12						
4	5	7	8	9	11	12						
4	5	6	8	9	10	12						
4	5	6	8	9	10	11						
4	5	6	7	9	10	11						
4	5	6	7	8	10	11						
3	5	6	7	8	9	10						
3	5	6	7	8	9	10						
3	4	5	7	8	9	10						
3	4	5	6	8	9	10						
3	4	5	6	7	8	9						
3	4	5	6	7	8	9						
3	4	5	6	7	8	9						
3	4	5	6	7	8	9						
3	4	5	6	7	7	8						
3	4	5	5	6	7	8						
3	4	4	5	6	7	8						
3	4	4	5	6	7	8						
3	3	4	5	6	7	8						
3	3	4	5	6	7	8						
2	3	4	5	6	7	7						
2	3	4	5	6	6	7						
2	3	4	5	6	6	7						
3	4	5	6	7	8	9						

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TABLE 24

PRESENT VALUE OF AN AMOUNT: $(1+r)^{-n}$

The present value P which will amount to A in n years at an interest rate of r (in decimals) compounded annually is $P = A(1+r)^n$.

n \ r	1%	1¼%	1½%	2%	2½%	3%	4%	5%	6%
1	.99010	.98765	.98522	.98039	.97561	.97087	.96154	.95238	.94340
2	.98030	.97546	.97066	.96117	.95181	.94260	.92456	.90703	.89000
3	.97059	.96342	.95632	.94232	.92860	.91514	.88900	.86384	.83962
4	.96098	.95152	.94218	.92385	.90595	.88849	.85480	.82270	.79209
5	.95147	.93978	.92826	.90573	.88385	.86261	.82193	.78353	.74726
6	.94205	.92817	.91454	.88797	.86230	.83748	.79031	.74622	.70496
7	.93272	.91672	.90103	.87056	.84127	.81309	.75992	.71068	.66506
8	.92348	.90540	.88771	.85349	.82075	.78941	.73069	.67684	.62741
9	.91434	.89422	.87459	.83676	.80073	.76642	.70259	.64461	.59190
10	.90529	.88318	.86167	.82035	.78120	.74409	.67556	.61391	.55839
11	.89632	.87228	.84893	.80426	.76214	.72242	.64958	.58468	.52679
12	.88745	.86151	.83639	.78849	.74356	.70138	.62460	.55684	.49697
13	.87866	.85087	.82403	.77303	.72542	.68095	.60057	.53032	.46884
14	.86996	.84037	.81185	.75788	.70773	.66112	.57748	.50507	.44230
15	.86135	.82999	.79985	.74301	.69047	.64186	.55526	.48102	.41727
16	.85282	.81975	.78803	.72845	.67362	.62317	.53391	.45811	.39365
17	.84438	.80963	.77639	.71416	.65720	.60502	.51337	.43630	.37136
18	.83602	.79963	.76491	.70016	.64117	.58739	.49363	.41552	.35034
19	.82774	.78976	.75361	.68643	.62553	.57029	.47464	.39573	.33051
20	.81954	.78001	.74247	.67297	.61027	.55368	.45639	.37689	.31180
21	.81143	.77038	.73150	.65978	.59539	.53755	.43883	.35894	.29416
22	.80340	.76087	.72069	.64684	.58086	.52189	.42196	.34185	.27751
23	.79544	.75147	.71004	.63416	.56670	.50669	.40573	.32557	.26180
24	.78757	.74220	.69954	.62172	.55288	.49193	.39012	.31007	.24698
25	.77977	.73303	.68921	.60953	.53939	.47761	.37512	.29530	.23300
26	.77205	.72398	.67902	.59758	.52623	.46369	.36069	.28124	.21981
27	.76440	.71505	.66899	.58586	.51340	.45019	.34682	.26785	.20737
28	.75684	.70622	.65910	.57437	.50088	.43708	.33348	.25509	.19563
29	.74934	.69750	.64936	.56311	.48866	.42435	.32065	.24295	.18456
30	.74192	.68889	.63976	.55207	.47674	.41199	.30832	.23138	.17411
31	.73458	.68038	.63031	.54125	.46511	.39999	.29646	.22036	.16425
32	.72730	.67198	.62099	.53063	.45377	.38834	.28506	.20987	.15496
33	.72010	.66369	.61182	.52023	.44270	.37703	.27409	.19987	.14619
34	.71297	.65549	.60277	.51003	.43191	.36604	.26355	.19035	.13791
35	.70591	.64740	.59387	.50003	.42137	.35538	.25342	.18129	.13011
36	.69892	.63941	.58509	.49022	.41109	.34503	.24367	.17266	.12274
37	.69200	.63152	.57644	.48061	.40107	.33498	.23430	.16444	.11579
38	.68515	.62372	.56792	.47119	.39128	.32523	.22529	.15661	.10924
39	.67837	.61602	.55953	.46195	.38174	.31575	.21662	.14915	.10306
40	.67165	.60841	.55126	.45289	.37243	.30656	.20829	.14205	.09722
41	.66500	.60090	.54312	.44401	.36335	.29763	.20028	.13528	.09172
42	.65842	.59348	.53509	.43530	.35448	.28896	.19257	.12884	.08653
43	.65190	.58616	.52718	.42677	.34584	.28054	.18517	.12270	.08163
44	.64545	.57892	.51939	.41840	.33740	.27237	.17805	.11686	.07701
45	.63905	.57177	.51171	.41020	.32917	.26444	.17120	.11130	.07265
46	.63273	.56471	.50415	.40215	.32115	.25674	.16461	.10600	.06854
47	.62646	.55774	.49670	.39427	.31331	.24926	.15828	.10095	.06466
48	.62026	.55086	.48936	.38654	.30567	.24200	.15219	.09614	.06100
49	.61412	.54406	.48213	.37896	.29822	.23495	.14634	.09156	.05755
50	.60804	.53734	.47500	.37153	.29094	.22811	.14071	.08720	.05429

TABLE 25

AMOUNT OF AN ANNUITY: $\dfrac{(1+r)^n - 1}{r}$

If a principal P is deposited at the end of each year at interest rate r (in decimals) compounded annually, then at the end of n years the accumulated amount is $P\left[\dfrac{(1+r)^n - 1}{r}\right]$. The process is often called an *annuity*.

n \ r	1%	1¼%	1½%	2%	2½%	3%	4%	5%	6%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0125	2.0150	2.0200	2.0250	2.0300	2.0400	2.0500	2.0600
3	3.0301	3.0377	3.0452	3.0604	3.0756	3.0909	3.1216	3.1525	3.1836
4	4.0604	4.0756	4.0909	4.1216	4.1525	4.1836	4.2465	4.3101	4.3746
5	5.1010	5.1266	5.1523	5.2040	5.2563	5.3091	5.4163	5.5256	5.6371
6	6.1520	6.1907	6.2296	6.3081	6.3877	6.4684	6.6330	6.8019	6.9753
7	7.2135	7.2680	7.3230	7.4343	7.5474	7.6625	7.8983	8.1420	8.3938
8	8.2857	8.3589	8.4328	8.5830	8.7361	8.8923	9.2142	9.5491	9.8975
9	9.3685	9.4634	9.5593	9.7546	9.9545	10.1591	10.5828	11.0266	11.4913
10	10.4622	10.5817	10.7027	10.9497	11.2034	11.4639	12.0061	12.5779	13.1808
11	11.5668	11.7139	11.8633	12.1687	12.4835	12.8078	13.4864	14.2068	14.9716
12	12.6825	12.8604	13.0412	13.4121	13.7956	14.1920	15.0258	15.9171	16.8699
13	13.8093	14.0211	14.2368	14.6803	15.1404	15.6178	16.6268	17.7130	18.8821
14	14.9474	15.1964	15.4504	15.9739	16.5190	17.0863	18.2919	19.5986	21.0151
15	16.0969	16.3863	16.6821	17.2934	17.9319	18.5989	20.0236	21.5786	23.2760
16	17.2579	17.5912	17.9324	18.6393	19.3802	20.1569	21.8245	23.6575	25.6725
17	18.4304	18.8111	19.2014	20.0121	20.8647	21.7616	23.6975	25.8404	28.2129
18	19.6147	20.0462	20.4894	21.4123	22.3863	23.4144	25.6454	28.1324	30.9057
19	20.8109	21.2968	21.7967	22.8406	23.9460	25.1169	27.6712	30.5390	33.7600
20	22.0190	22.5630	23.1237	24.2974	25.5447	26.8704	29.7781	33.0660	36.7856
21	23.2392	23.8450	24.4705	25.7833	27.1833	28.6765	31.9692	35.7193	39.9927
22	24.4716	25.1431	25.8376	27.2990	28.8629	30.5368	34.2480	38.5052	43.3923
23	25.7163	26.4574	27.2251	28.8450	30.5844	32.4529	36.6179	41.4305	46.9958
24	26.9735	27.7881	28.6335	30.4219	32.3490	34.4265	39.0826	44.5020	50.8156
25	28.2432	29.1354	30.0630	32.0303	34.1578	36.4593	41.6459	47.7271	54.8645
26	29.5256	30.4996	31.5140	33.6709	36.0117	38.5530	44.3117	51.1135	59.1564
27	30.8209	31.8809	32.9867	35.3443	37.9120	40.7096	47.0842	54.6691	63.7058
28	32.1291	33.2794	34.4815	37.0512	39.8598	42.9309	49.9676	58.4026	68.5281
29	33.4504	34.6954	35.9987	38.7922	41.8563	45.2189	52.9663	62.3227	73.6398
30	34.7849	36.1291	37.5387	40.5681	43.9027	47.5754	56.0849	66.4388	79.0582
31	36.1327	37.5807	39.1018	42.3794	46.0003	50.0027	59.3283	70.7608	84.8017
32	37.4941	39.0504	40.6883	44.2270	48.1503	52.5028	62.7015	75.2988	90.8898
33	38.8690	40.5386	42.2986	46.1116	50.3540	55.0778	66.2095	80.0638	97.3432
34	40.2577	42.0453	43.9331	48.0338	52.6129	57.7302	69.8579	85.0670	104.1838
35	41.6603	43.5709	45.5921	49.9945	54.9282	60.4621	73.6522	90.3203	111.4348
36	43.0769	45.1155	47.2760	51.9944	57.3014	63.2759	77.5983	95.8363	119.1209
37	44.5076	46.6794	48.9851	54.0343	59.7339	66.1742	81.7022	101.6281	127.2681
38	45.9527	48.2629	50.7199	56.1149	62.2273	69.1594	85.9703	107.7095	135.9042
39	47.4123	49.8662	52.4807	58.2372	64.7830	72.2342	90.4091	114.0950	145.0585
40	48.8864	51.4896	54.2679	60.4020	67.4026	75.4013	95.0255	120.7998	154.7620
41	50.3752	53.1332	56.0819	62.6100	70.0876	78.6633	99.8265	127.8398	165.0477
42	51.8790	54.7973	57.9231	64.8622	72.8398	82.0232	104.8196	135.2318	175.9505
43	53.3978	56.4823	59.7920	67.1595	75.6608	85.4839	110.0124	142.9933	187.5076
44	54.9318	58.1883	61.6889	69.5027	78.5523	89.0484	115.4129	151.1430	199.7580
45	56.4811	59.9157	63.6142	71.8927	81.5161	92.7199	121.0294	159.7002	212.7435
46	58.0459	61.6646	65.5684	74.3306	84.5540	96.5015	126.8706	168.6852	226.5081
47	59.6263	63.4354	67.5519	76.8172	87.6679	100.3965	132.9454	178.1194	241.0986
48	61.2226	65.2284	69.5652	79.3535	90.8596	104.4084	139.2632	188.0254	256.5645
49	62.8348	67.0437	71.6087	81.9406	94.1311	108.5406	145.8337	198.4267	272.9584
50	64.4632	68.8818	73.6828	84.5794	97.4843	112.7969	152.6671	209.3480	290.3359

TABLE 26

PRESENT VALUE OF AN ANNUITY: $\frac{1-(1+r)^{-n}}{r}$

An annuity in which the yearly payment at the end of each of n years is A at an interest rate r (in decimals) compounded annually has present value $A\left[\frac{1-(1+r)^{-n}}{r}\right]$.

n \ r	1%	1¼%	1½%	2%	2½%	3%	4%	5%	6%
1	0.9901	0.9877	0.9852	0.9804	0.9756	0.9709	0.9615	0.9524	0.9434
2	1.9704	1.9631	1.9559	1.9416	1.9274	1.9135	1.8861	1.8594	1.8334
3	2.9410	2.9265	2.9122	2.8839	2.8560	2.8286	2.7751	2.7232	2.6730
4	3.9020	3.8781	3.8544	3.8077	3.7620	3.7171	3.6299	3.5460	3.4651
5	4.8534	4.8178	4.7826	4.7135	4.6458	4.5797	4.4518	4.3295	4.2124
6	5.7955	5.7460	5.6972	5.6014	5.5081	5.4172	5.2421	5.0757	4.9173
7	6.7282	6.6627	6.5982	6.4720	6.3494	6.2303	6.0021	5.7864	5.5824
8	7.6517	7.5681	7.4859	7.3255	7.1701	7.0197	6.7327	6.4632	6.2098
9	8.5660	8.4623	8.3605	8.1622	7.9709	7.7861	7.4353	7.1078	6.8017
10	9.4713	9.3455	9.2222	8.9826	8.7521	8.5302	8.1109	7.7217	7.3601
11	10.3676	10.2178	10.0711	9.7868	9.5142	9.2526	8.7605	8.3064	7.8869
12	11.2551	11.0793	10.9075	10.5753	10.2578	9.9540	9.3851	8.8633	8.3838
13	12.1337	11.9302	11.7315	11.3484	10.9832	10.6350	9.9856	9.3936	8.8527
14	13.0037	12.7706	12.5434	12.1062	11.6909	11.2961	10.5631	9.8986	9.2950
15	13.8651	13.6005	13.3432	12.8493	12.3814	11.9379	11.1184	10.3797	9.7122
16	14.7179	14.4203	14.1313	13.5777	13.0550	12.5611	11.6523	10.8378	10.1059
17	15.5623	15.2299	14.9076	14.2919	13.7122	13.1661	12.1657	11.2741	10.4773
18	16.3983	16.0295	15.6726	14.9920	14.3534	13.7535	12.6593	11.6896	10.8276
19	17.2260	16.8193	16.4262	15.6785	14.9789	14.3238	13.1339	12.0853	11.1581
20	18.0456	17.5993	17.1686	16.3514	15.5892	14.8775	13.5903	12.4622	11.4699
21	18.8570	18.3697	17.9001	17.0112	16.1845	15.4150	14.0292	12.8212	11.7641
22	19.6604	19.1306	18.6208	17.6580	16.7654	15.9369	14.4511	13.1630	12.0416
23	20.4558	19.8820	19.3309	18.2922	17.3321	16.4436	14.8568	13.4886	12.3034
24	21.2434	20.6242	20.0304	18.9139	17.8850	16.9355	15.2470	13.7986	12.5504
25	22.0232	21.3573	20.7196	19.5235	18.4244	17.4131	15.6221	14.0939	12.7834
26	22.7952	22.0813	21.3986	20.1210	18.9506	17.8768	15.9828	14.3752	13.0032
27	23.5596	22.7963	22.0676	20.7069	19.4640	18.3270	16.3296	14.6430	13.2105
28	24.3164	23.5025	22.7267	21.2813	19.9649	18.7641	16.6631	14.8981	13.4062
29	25.0658	24.2000	23.3761	21.8444	20.4535	19.1885	16.9837	15.1411	13.5907
30	25.8077	24.8889	24.0158	22.3965	20.9303	19.6004	17.2920	15.3725	13.7648
31	26.5423	25.5693	24.6461	22.9377	21.3954	20.0004	17.5885	15.5928	13.9291
32	27.2696	26.2413	25.2671	23.4683	21.8492	20.3888	17.8736	15.8027	14.0840
33	27.9897	26.9050	25.8790	23.9886	22.2919	20.7658	18.1476	16.0025	14.2302
34	28.7027	27.5605	26.4817	24.4986	22.7238	21.1318	18.4112	16.1929	14.3681
35	29.4086	28.2079	27.0756	24.9986	23.1452	21.4872	18.6646	16.3742	14.4982
36	30.1075	28.8473	27.6607	25.4888	23.5563	21.8323	18.9083	16.5469	14.6210
37	30.7995	29.4788	28.2371	25.9695	23.9573	22.1672	19.1426	16.7113	14.7368
38	31.4847	30.1025	28.8051	26.4406	24.3486	22.4925	19.3679	16.8679	14.8460
39	32.1630	30.7185	29.3646	26.9026	24.7303	22.8082	19.5845	17.0170	14.9491
40	32.8347	31.3269	29.9158	27.3555	25.1028	23.1148	19.7928	17.1591	15.0463
41	33.4997	31.9278	30.4590	27.7995	25.4661	23.4124	19.9931	17.2944	15.1380
42	34.1581	32.5213	30.9941	28.2348	25.8206	23.7014	20.1856	17.4232	15.2245
43	34.8100	33.1075	31.5212	28.6616	26.1664	23.9819	20.3708	17.5459	15.3062
44	35.4555	33.6864	32.0406	29.0800	26.5038	24.2543	20.5488	17.6628	15.3832
45	36.0945	34.2582	32.5523	29.4902	26.8330	24.5187	20.7200	17.7741	15.4558
46	36.7272	34.8229	33.0565	29.8923	27.1542	24.7754	20.8847	17.8801	15.5244
47	37.3537	35.3806	33.5532	30.2866	27.4675	25.0247	21.0429	17.9810	15.5890
48	37.9740	35.9315	34.0426	30.6731	27.7732	25.2667	21.1951	18.0772	15.6500
49	38.5881	36.4755	34.5247	31.0521	28.0714	25.5017	21.3415	18.1687	15.7076
50	39.1961	37.0129	34.9997	31.4236	28.3623	25.7298	21.4822	18.2559	15.7619

TABLE 27

BESSEL FUNCTIONS $J_0(x)$

x	0	1	2	3	4	5	6	7	8	9
0.	1.0000	.9975	.9900	.9776	.9604	.9385	.9120	.8812	.8463	.8075
1.	.7652	.7196	.6711	.6201	.5669	.5118	.4554	.3980	.3400	.2818
2.	.2239	.1666	.1104	.0555	.0025	−.0484	−.0968	−.1424	−.1850	−.2243
3.	−.2601	−.2921	−.3202	−.3443	−.3643	−.3801	−.3918	−.3992	−.4026	−.4018
4.	−.3971	−.3887	−.3766	−.3610	−.3423	−.3205	−.2961	−.2693	−.2404	−.2097
5.	−.1776	−.1443	−.1103	−.0758	−.0412	−.0068	.0270	.0599	.0917	.1220
6.	.1506	.1773	.2017	.2238	.2433	.2601	.2740	.2851	.2931	.2981
7.	.3001	.2991	.2951	.2882	.2786	.2663	.2516	.2346	.2154	.1944
8.	.1717	.1475	.1222	.0960	.0692	.0419	.0146	−.0125	−.0392	−.0653
9.	−.0903	−.1142	−.1367	−.1577	−.1768	−.1939	−.2090	−.2218	−.2323	−.2403

TABLE 28

BESSEL FUNCTIONS $J_1(x)$

x	0	1	2	3	4	5	6	7	8	9
0.	.0000	.0499	.0995	.1483	.1960	.2423	.2867	.3290	.3688	.4059
1.	.4401	.4709	.4983	.5220	.5419	.5579	.5699	.5778	.5815	.5812
2.	.5767	.5683	.5560	.5399	.5202	.4971	.4708	.4416	.4097	.3754
3.	.3391	.3009	.2613	.2207	.1792	.1374	.0955	.0538	.0128	−.0272
4.	−.0660	−.1033	−.1386	−.1719	−.2028	−.2311	−.2566	−.2791	−.2985	−.3147
5.	−.3276	−.3371	−.3432	−.3460	−.3453	−.3414	−.3343	−.3241	−.3110	−.2951
6.	−.2767	−.2559	−.2329	−.2081	−.1816	−.1538	−.1250	−.0953	−.0652	−.0349
7.	−.0047	.0252	.0543	.0826	.1096	.1352	.1592	.1813	.2014	.2192
8.	.2346	.2476	.2580	.2657	.2708	.2731	.2728	.2697	.2641	.2559
9.	.2453	.2324	.2174	.2004	.1816	.1613	.1395	.1166	.0928	.0684

TABLE 29

BESSEL FUNCTIONS $Y_0(x)$

x	0	1	2	3	4	5	6	7	8	9
0.	$-\infty$	−1.5342	−1.0811	−.8073	−.6060	−.4445	−.3085	−.1907	−.0868	.0056
1.	.0883	.1622	.2281	.2865	.3379	.3824	.4204	.4520	.4774	.4968
2.	.5104	.5183	.5208	.5181	.5104	.4981	.4813	.4605	.4359	.4079
3.	.3769	.3431	.3071	.2691	.2296	.1890	.1477	.1061	.0645	.0234
4.	−.0169	−.0561	−.0938	−.1296	−.1633	−.1947	−.2235	−.2494	−.2723	−.2921
5.	−.3085	−.3216	−.3313	−.3374	−.3402	−.3395	−.3354	−.3282	−.3177	−.3044
6.	−.2882	−.2694	−.2483	−.2251	−.1999	−.1732	−.1452	−.1162	−.0864	−.0563
7.	−.0259	.0042	.0339	.0628	.0907	.1173	.1424	.1658	.1872	.2065
8.	.2235	.2381	.2501	.2595	.2662	.2702	.2715	.2700	.2659	.2592
9.	.2499	.2383	.2245	.2086	.1907	.1712	.1502	.1279	.1045	.0804

TABLE 30

BESSEL FUNCTIONS $Y_1(x)$

x	0	1	2	3	4	5	6	7	8	9
0.	$-\infty$	−6.4590	−3.3238	−2.2931	−1.7809	−1.4715	−1.2604	−1.1032	−.9781	−.8731
1.	−.7812	−.6981	−.6211	−.5485	−.4791	−.4123	−.3476	−.2847	−.2237	−.1644
2.	−.1070	−.0517	.0015	.0523	.1005	.1459	.1884	.2276	.2635	.2959
3.	.3247	.3496	.3707	.3879	.4010	.4102	.4154	.4167	.4141	.4078
4.	.3979	.3846	.3680	.3484	.3260	.3010	.2737	.2445	.2136	.1812
5.	.1479	.1137	.0792	.0445	.0101	−.0238	−.0568	−.0887	−.1192	−.1481
6.	−.1750	−.1998	−.2223	−.2422	−.2596	−.2741	−.2857	−.2945	−.3002	−.3029
7.	−.3027	−.2995	−.2934	−.2846	−.2731	−.2591	−.2428	−.2243	−.2039	−.1817
8.	−.1581	−.1331	−.1072	−.0806	−.0535	−.0262	.0011	.0280	.0544	.0799
9.	.1043	.1275	.1491	.1691	.1871	.2032	.2171	.2287	.2379	.2447

TABLE 31

BESSEL FUNCTIONS $I_0(x)$

x	0	1	2	3	4	5	6	7	8	9
0.	1.000	1.003	1.010	1.023	1.040	1.063	1.092	1.126	1.167	1.213
1.	1.266	1.326	1.394	1.469	1.553	1.647	1.750	1.864	1.990	2.128
2.	2.280	2.446	2.629	2.830	3.049	3.290	3.553	3.842	4.157	4.503
3.	4.881	5.294	5.747	6.243	6.785	7.378	8.028	8.739	9.517	10.37
4.	11.30	12.32	13.44	14.67	16.01	17.48	19.09	20.86	22.79	24.91
5.	27.24	29.79	32.58	35.65	39.01	42.69	46.74	51.17	56.04	61.38
6.	67.23	73.66	80.72	88.46	96.96	106.3	116.5	127.8	140.1	153.7
7.	168.6	185.0	202.9	222.7	244.3	268.2	294.3	323.1	354.7	389.4
8.	427.6	469.5	515.6	566.3	621.9	683.2	750.5	824.4	905.8	995.2
9.	1094	1202	1321	1451	1595	1753	1927	2119	2329	2561

TABLE 32

BESSEL FUNCTIONS $I_1(x)$

x	0	1	2	3	4	5	6	7	8	9
0.	.0000	.0501	.1005	.1517	.2040	.2579	.3137	.3719	.4329	.4971
1.	.5652	.6375	.7147	.7973	.8861	.9817	1.085	1.196	1.317	1.448
2.	1.591	1.745	1.914	2.098	2.298	2.517	2.755	3.016	3.301	3.613
3.	3.953	4.326	4.734	5.181	5.670	6.206	6.793	7.436	8.140	8.913
4.	9.759	10.69	11.71	12.82	14.05	15.39	16.86	18.48	20.25	22.20
5.	24.34	26.68	29.25	32.08	35.18	38.59	42.33	46.44	50.95	55.90
6.	61.34	67.32	73.89	81.10	89.03	97.74	107.3	117.8	129.4	142.1
7.	156.0	171.4	188.3	206.8	227.2	249.6	274.2	301.3	331.1	363.9
8.	399.9	439.5	483.0	531.0	583.7	641.6	705.4	775.5	852.7	937.5
9.	1031	1134	1247	1371	1508	1658	1824	2006	2207	2428

TABLE 33

BESSEL FUNCTIONS $K_0(x)$

x	0	1	2	3	4	5	6	7	8	9
0.	∞	2.4271	1.7527	1.3725	1.1145	.9244	.7775	.6605	.5653	.4867
1.	.4210	.3656	.3185	.2782	.2437	.2138	.1880	.1655	.1459	.1288
2.	.1139	.1008	.08927	.07914	.07022	.06235	.05540	.04926	.04382	.03901
3.	.03474	.03095	.02759	.02461	.02196	.01960	.01750	.01563	.01397	.01248
4.	.01116	$.0^{2}9980$	$.0^{2}8927$	$.0^{2}7988$	$.0^{2}7149$	$.0^{2}6400$	$.0^{2}5730$	$.0^{2}5132$	$.0^{2}4597$	$.0^{2}4119$
5.	$.0^{2}3691$	$.0^{2}3308$	$.0^{2}2966$	$.0^{2}2659$	$.0^{2}2385$	$.0^{2}2139$	$.0^{2}1918$	$.0^{2}1721$	$.0^{2}1544$	$.0^{2}1386$
6.	$.0^{2}1244$	$.0^{2}1117$	$.0^{2}1003$	$.0^{3}9001$	$.0^{3}8083$	$.0^{3}7259$	$.0^{3}6520$	$.0^{3}5857$	$.0^{3}5262$	$.0^{3}4728$
7.	$.0^{3}4248$	$.0^{3}3817$	$.0^{3}3431$	$.0^{3}3084$	$.0^{3}2772$	$.0^{3}2492$	$.0^{3}2240$	$.0^{3}2014$	$.0^{3}1811$	$.0^{3}1629$
8.	$.0^{3}1465$	$.0^{3}1317$	$.0^{3}1185$	$.0^{3}1066$	$.0^{4}9588$	$.0^{4}8626$	$.0^{4}7761$	$.0^{4}6983$	$.0^{4}6283$	$.0^{4}5654$
9.	$.0^{4}5088$	$.0^{4}4579$	$.0^{4}4121$	$.0^{4}3710$	$.0^{4}3339$	$.0^{4}3006$	$.0^{4}2706$	$.0^{4}2436$	$.0^{4}2193$	$.0^{4}1975$

TABLE 34

BESSEL FUNCTIONS $K_1(x)$

x	0	1	2	3	4	5	6	7	8	9
0.	∞	9.8538	4.7760	3.0560	2.1844	1.6564	1.3028	1.0503	.8618	.7165
1.	.6019	.5098	.4346	.3725	.3208	.2774	.2406	.2094	.1826	.1597
2.	.1399	.1227	.1079	.09498	.08372	.07389	.06528	.05774	.05111	.04529
3.	.04016	.03563	.03164	.02812	.02500	.02224	.01979	.01763	.01571	.01400
4.	.01248	.01114	$.0^{2}9938$	$.0^{2}8872$	$.0^{2}7923$	$.0^{2}7078$	$.0^{2}6325$	$.0^{2}5654$	$.0^{2}5055$	$.0^{2}4521$
5.	$.0^{2}4045$	$.0^{2}3619$	$.0^{2}3239$	$.0^{2}2900$	$.0^{2}2597$	$.0^{2}2326$	$.0^{2}2083$	$.0^{2}1866$	$.0^{2}1673$	$.0^{2}1499$
6.	$.0^{2}1344$	$.0^{2}1205$	$.0^{2}1081$	$.0^{3}9691$	$.0^{3}8693$	$.0^{3}7799$	$.0^{3}6998$	$.0^{3}6280$	$.0^{3}5636$	$.0^{3}5059$
7.	$.0^{3}4542$	$.0^{3}4078$	$.0^{3}3662$	$.0^{3}3288$	$.0^{3}2953$	$.0^{3}2653$	$.0^{3}2383$	$.0^{3}2141$	$.0^{3}1924$	$.0^{3}1729$
8.	$.0^{3}1554$	$.0^{3}1396$	$.0^{3}1255$	$.0^{3}1128$	$.0^{3}1014$	$.0^{4}9120$	$.0^{4}8200$	$.0^{4}7374$	$.0^{4}6631$	$.0^{4}5964$
9.	$.0^{4}5364$	$.0^{4}4825$	$.0^{4}4340$	$.0^{4}3904$	$.0^{4}3512$	$.0^{4}3160$	$.0^{4}2843$	$.0^{4}2559$	$.0^{4}2302$	$.0^{4}2072$

TABLE 35

BESSEL FUNCTIONS
Ber (x)

x	0	1	2	3	4	5	6	7	8	9
0.	1.0000	1.0000	1.0000	.9999	.9996	.9990	.9980	.9962	.9936	.9898
1.	.9844	.9771	.9676	.9554	.9401	.9211	.8979	.8700	.8367	.7975
2.	.7517	.6987	.6377	.5680	.4890	.4000	.3001	.1887	.06511	−.07137
3.	−.2214	−.3855	−.5644	−.7584	−.9680	−1.1936	−1.4353	−1.6933	−1.9674	−2.2576
4.	−2.5634	−2.8843	−3.2195	−3.5679	−3.9283	−4.2991	−4.6784	−5.0639	−5.4531	−5.8429
5.	−6.2301	−6.6107	−6.9803	−7.3344	−7.6674	−7.9736	−8.2466	−8.4794	−8.6644	−8.7937
6.	−8.8583	−8.8491	−8.7561	−8.5688	−8.2762	−7.8669	−7.3287	−6.6492	−5.8155	−4.8146
7.	−3.6329	−2.2571	−.6737	1.1308	3.1695	5.4550	7.9994	10.814	13.909	17.293
8.	20.974	24.957	29.245	33.840	38.738	43.936	49.423	55.187	61.210	67.469
9.	73.936	80.576	87.350	94.208	101.10	107.95	114.70	121.26	127.54	133.43

TABLE 36

BESSEL FUNCTIONS
Bei (x)

x	0	1	2	3	4	5	6	7	8	9
0.	.0000	$.0^{2}2500$	.01000	.02250	.04000	.06249	.08998	.1224	.1599	.2023
1.	.2496	.3017	.3587	.4204	.4867	.5576	.6327	.7120	.7953	.8821
2.	.9723	1.0654	1.1610	1.2585	1.3575	1.4572	1.5569	1.6557	1.7529	1.8472
3.	1.9376	2.0228	2.1016	2.1723	2.2334	2.2832	2.3199	2.3413	2.3454	2.3300
4.	2.2927	2.2309	2.1422	2.0236	1.8726	1.6860	1.4610	1.1946	.8837	.5251
5.	.1160	−.3467	−.8658	−1.4443	−2.0845	−2.7890	−3.5597	−4.3986	−5.3068	−6.2854
6.	−7.3347	−8.4545	−9.6437	−10.901	−12.223	−13.607	−15.047	−16.538	−18.074	−19.644
7.	−21.239	−22.848	−24.456	−26.049	−27.609	−29.116	−30.548	−31.882	−33.092	−34.147
8.	−35.017	−35.667	−36.061	−36.159	−35.920	−35.298	−34.246	−32.714	−30.651	−28.003
9.	−24.713	−20.724	−15.976	−10.412	−3.9693	3.4106	11.787	21.218	31.758	43.459

TABLE 37

BESSEL FUNCTIONS
Ker (x)

x	0	1	2	3	4	5	6	7	8	9
0.	∞	2.4205	1.7331	1.3372	1.0626	.8559	.6931	.5614	.4529	.3625
1.	.2867	.2228	.1689	.1235	.08513	.05293	.02603	$.0^{2}3691$	−.01470	−.02966
2.	−.04166	−.05111	−.05834	−.06367	−.06737	−.06969	−.07083	−.07097	−.07030	−.06894
3.	−.06703	−.06468	−.06198	−.05903	−.05590	−.05264	−.04932	−.04597	−.04265	−.03937
4.	−.03618	−.03308	−.03011	−.02726	−.02456	−.02200	−.01960	−.01734	−.01525	−.01330
5.	−.01151	$-.0^{2}9865$	$-.0^{2}8359$	$-.0^{2}6989$	$-.0^{2}5749$	$-.0^{2}4632$	$-.0^{2}3632$	$-.0^{2}2740$	$-.0^{2}1952$	$-.0^{2}1258$
6.	$-.0^{3}6530$	$-.0^{3}1295$	$.0^{3}3191$	$.0^{3}6991$	$.0^{2}1017$	$.0^{2}1278$	$.0^{2}1488$	$.0^{2}1653$	$.0^{2}1777$	$.0^{2}1866$
7.	$.0^{2}1922$	$.0^{2}1951$	$.0^{2}1956$	$.0^{2}1940$	$.0^{2}1907$	$.0^{2}1860$	$.0^{2}1800$	$.0^{2}1731$	$.0^{2}1655$	$.0^{2}1572$
8.	$.0^{2}1486$	$.0^{2}1397$	$.0^{2}1306$	$.0^{2}1216$	$.0^{2}1126$	$.0^{2}1037$	$.0^{3}9511$	$.0^{3}8675$	$.0^{3}7871$	$.0^{3}7102$
9.	$.0^{3}6372$	$.0^{3}5681$	$.0^{3}5030$	$.0^{3}4422$	$.0^{3}3855$	$.0^{3}3330$	$.0^{3}2846$	$.0^{3}2402$	$.0^{3}1996$	$.0^{3}1628$

TABLE 38

BESSEL FUNCTIONS
Kei (x)

x	0	1	2	3	4	5	6	7	8	9
0.	−.7854	−.7769	−.7581	−.7331	−.7038	−.6716	−.6374	−.6022	−.5664	−.5305
1.	−.4950	−.4601	−.4262	−.3933	−.3617	−.3314	−.3026	−.2752	−.2494	−.2251
2.	−.2024	−.1812	−.1614	−.1431	−.1262	−.1107	−.09644	−.08342	−.07157	−.06083
3.	−.05112	−.04240	−.03458	−.02762	−.02145	−.01600	−.01123	$-.0^{2}7077$	$-.0^{2}3487$	$-.0^{3}4108$
4.	$.0^{2}2198$	$.0^{2}4386$	$.0^{2}6194$	$.0^{2}7661$	$.0^{2}8826$	$.0^{2}9721$	.01038	.01083	.01110	.01121
5.	.01119	.01105	.01082	.01051	.01014	$.0^{2}9716$	$.0^{2}9255$	$.0^{2}8766$	$.0^{2}8258$	$.0^{2}7739$
6.	$.0^{2}7216$	$.0^{2}6696$	$.0^{2}6183$	$.0^{2}5681$	$.0^{2}5194$	$.0^{2}4724$	$.0^{2}4274$	$.0^{2}3846$	$.0^{2}3440$	$.0^{2}3058$
7.	$.0^{2}2700$	$.0^{2}2366$	$.0^{2}2057$	$.0^{2}1770$	$.0^{2}1507$	$.0^{2}1267$	$.0^{2}1048$	$.0^{3}8498$	$.0^{3}6714$	$.0^{3}5117$
8.	$.0^{3}3696$	$.0^{3}2440$	$.0^{3}1339$	$.0^{4}3809$	$-.0^{4}4449$	$-.0^{3}1149$	$-.0^{3}1742$	$-.0^{3}2233$	$-.0^{3}2632$	$-.0^{3}2949$
9.	$-.0^{3}3192$	$-.0^{3}3368$	$-.0^{3}3486$	$-.0^{3}3552$	$-.0^{3}3574$	$-.0^{3}3557$	$-.0^{3}3508$	$-.0^{3}3430$	$-.0^{3}3329$	$-.0^{3}3210$

TABLE 39

VALUES FOR APPROXIMATE ZEROS OF BESSEL FUNCTIONS

The following table lists the first few positive roots of various equations. Note that for all cases listed the successive large roots differ approximately by $\pi = 3.14159\ldots$.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$J_n(x) = 0$	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715	9.9361
	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386	13.5893
	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002	17.0038
	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801	20.3208
	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178	23.5861
	18.0711	19.6159	21.1170	22.5827	24.0190	25.4303	26.8202
$Y_n(x) = 0$	0.8936	2.1971	3.3842	4.5270	5.6452	6.7472	7.8377
	3.9577	5.4297	6.7938	8.0976	9.3616	10.5972	11.8110
	7.0861	8.5960	10.0235	11.3965	12.7301	14.0338	15.3136
	10.2223	11.7492	13.2100	14.6231	15.9996	17.3471	18.6707
	13.3611	14.8974	16.3790	17.8185	19.2244	20.6029	21.9583
	16.5009	18.0434	19.5390	20.9973	22.4248	23.8265	25.2062
$J_n'(x) = 0$	0.0000	1.8412	3.0542	4.2012	5.3176	6.4156	7.5013
	3.8317	5.3314	6.7061	8.0152	9.2824	10.5199	11.7349
	7.0156	8.5363	9.9695	11.3459	12.6819	13.9872	15.2682
	10.1735	11.7060	13.1704	14.5859	15.9641	17.3128	18.6374
	13.3237	14.8636	16.3475	17.7888	19.1960	20.5755	21.9317
	16.4706	18.0155	19.5129	20.9725	22.4010	23.8036	25.1839
$Y_n'(x) = 0$	2.1971	3.6830	5.0026	6.2536	7.4649	8.6496	9.8148
	5.4297	6.9415	8.3507	9.6988	11.0052	12.2809	13.5328
	8.5960	10.1234	11.5742	12.9724	14.3317	15.6608	16.9655
	11.7492	13.2858	14.7609	16.1905	17.5844	18.9497	20.2913
	14.8974	16.4401	17.9313	19.3824	20.8011	22.1928	23.5619
	18.0434	19.5902	21.0929	22.5598	23.9970	25.4091	26.7995

TABLE 40

EXPONENTIAL, SINE AND COSINE INTEGRALS

$$Ei(x) = \int_{\infty}^{x} \frac{e^{-u}}{u}\,du \qquad Si(x) = \int_{0}^{x} \frac{\sin u}{u}\,du \qquad Ci(x) = \int_{x}^{\infty} \frac{\cos u}{u}\,du$$