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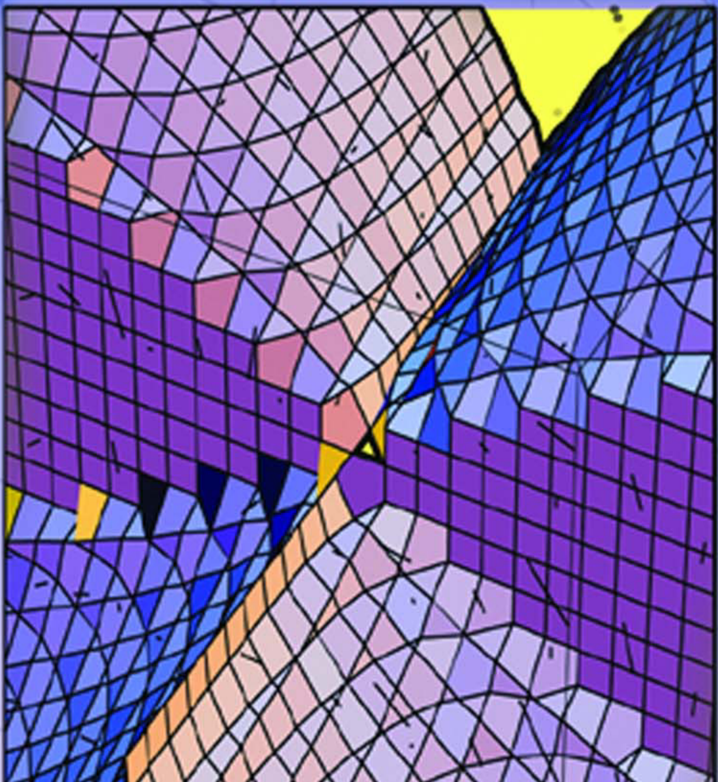
DIFFERENTIAL EQUATIONS WITH *MATHEMATICA*®

THIRD EDITION



Martha L. Abell & James P. Braselton

Differential Equations with *Mathematica*

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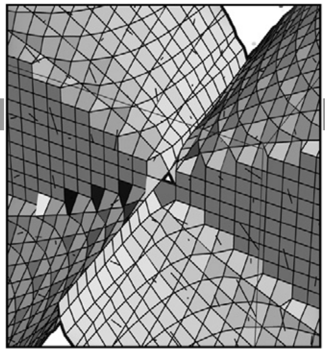

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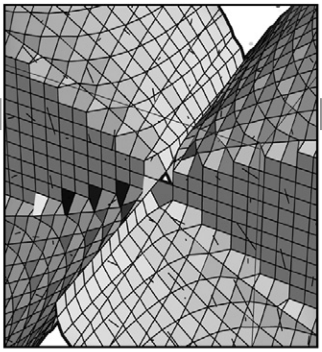
Preface



Mathematica's diversity makes it particularly well suited to performing many calculations encountered when solving many ordinary and partial differential equations. In some cases, Mathematica's built-in functions can immediately solve a differential equation by providing an explicit, implicit, or numerical solution; in other cases, Mathematica can be used to perform the calculations encountered when solving a differential equation. Because one goal of elementary differential equations courses is to introduce students to basic methods and algorithms and have the student gain proficiency in them, nearly every topic covered in *Differential Equations with Mathematica*, Third Edition, includes typical examples solved by traditional methods and examples solved using Mathematica. *Differential Equations with Mathematica* introduces basic commands and includes typical examples of applications of them. A study of differential equations relies on concepts from calculus and linear algebra so the text also includes discussions of relevant commands useful in those areas. In many cases, seeing a solution graphically is most meaningful so *Differential Equations with Mathematica* relies heavily on Mathematica's outstanding graphics capabilities.

Differential Equations with Mathematica is an appropriate reference for all users of Mathematica who encounter differential equations in their profession, in particular, for beginning users like students, instructors, engineers, business people, and other professionals using Mathematica to solve and visualize solutions to differential equations. *Differential Equations with Mathematica* is a valuable supplement for students and instructors at engineering schools that use Mathematica.

Taking advantage of Version 5 of Mathematica, *Differential Equations with Mathematica*, Third Edition, introduces the fundamental concepts of Mathematica to solve (analytically, numerically, and/or graphically) differential equations of interest to students, instructors, and scientists. Other features to help make *Differential Equations with Mathematica*, Third Edition, as easy to use and as useful as possible include the following.

1. **Version 5 Compatibility.** All examples illustrated in *Differential Equations with Mathematica*, Third Edition, were completed using Version 5 of Mathematica. Although most computations can continue to be carried out with earlier versions of Mathematica, like Versions 2, 3, and 4, we have taken advantage of the new features in Version 5 as much as possible.
2. **Applications.** New applications, many of which are documented by references, from a variety of fields, especially biology, physics, and engineering, are included throughout the text.
3. **Detailed Table of Contents.** The table of contents includes all chapter, section, and subsection headings. Along with the comprehensive index, we hope that users will be able to locate information quickly and easily.
4. **Additional Examples.** We have considerably expanded the topics in Chapters 1 through 6. The results should be more useful to instructors, students, business people, engineers, and other professionals using Mathematica on a variety of platforms. In addition, several sections have been added to help make locating information easier for the user.
5. **Comprehensive Index.** In the index, mathematical examples and applications are listed by topic, or name, as well as commands along with frequently used options: particular mathematical examples as well as examples illustrating how to use frequently used commands are easy to locate. In addition, commands in the index are cross-referenced with frequently used options. Functions available in the various packages are cross-referenced both by package and alphabetically.
6. **Included CD.** All Mathematica code that appears in *Differential Equations with Mathematica*, Third Edition, is included on the CD packaged with the text.
7. **Getting Started.** The Appendix provides a brief introduction to Mathematica, including discussion about entering and evaluating commands, loading packages, and taking advantage of Mathematica's extensive help facilities. Appropriate references to *The Mathematica Book* are included as well.

We began *Differential Equations with Mathematica* in 1990 and the first edition was published in 1991. Back then, we were on top of the world using Macintosh IIcx's with 8 megs of RAM and 40 meg hard drives. We tried to choose examples that we thought would be relevant to typical users — typically in the context of differential equations encountered in the undergraduate curriculum. Those examples could also be carried out by Mathematica in a timely manner on a computer as powerful as a Macintosh IIcx.

Now, we are on top of the world with Power Macintosh G4's with 768 megs of RAM and 50 gig hard drives, which will almost certainly be obsolete by the time you are reading this. The examples presented in *Differential Equations with Mathematica* continue to be the ones that we think are most similar to the problems encountered by beginning users and are presented in the context of someone familiar with mathematics typically encountered by undergraduates. However, for this third edition of *Differential Equations with Mathematica* we have taken the opportunity to expand on several of our favorite examples because the machines now have the speed and power to explore them in greater detail.

Other improvements to the third edition include:

1. Throughout the text, we have attempted to eliminate redundant examples and added several interesting ones. The following changes are especially worth noting.
   (a) In Chapter 2, First-Order Ordinary Differential Equations, we present the integrating factor approach, variation of parameters, and method of undetermined coefficients when solving first-order linear equations.
   (b) In Chapter 3, we discuss the Logistic difference equation and give some surprisingly simple ways to generate the classic "Pitchfork diagram" with Mathematica.
   (c) Chapter 4, Higher-Order Equations, has been completely reorganized; a new section on nonlinear equations has been added.
   (d) Chapter 5, Applications of Higher-Order Equations, has also been completely reorganized. The catenary is now included in the Other Applications section.
   (e) Chapter 6, Systems of Ordinary Differential Equations, includes several new examples. See especially Example 6.2.5.
   (f) Chapter 7, Applications of Systems, includes several new examples. See especially Examples 7.3.3, 7.3.4, and 7.3.6.
   (g) We have included references that we find particularly interesting in the **Bibliography**, even if they are not specific Mathematica-related texts. A comprehensive list of Mathematica-related publications can be found at the Wolfram website.

`http://store.wolfram.com/catalog/books/.`

Finally, we must express our appreciation to those who assisted in this project. We would like to express appreciation to our editors, Tom Singer and Barbara Holland, and our production editor, Brandy Palacios, at Academic Press for providing a pleasant environment in which to work. In addition, Wolfram Research, especially Misty Mosely, have been most helpful in providing us up-to-date information about Mathematica. Finally, we thank those close to us, especially Imogene Abell, Lori Braselton, Ada Braselton, and Mattie Braselton for enduring with us the pressures of meeting a deadline and for graciously accepting our demanding work schedules. We certainly could not have completed this task without their care and understanding.

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Introduction to Differential Equations

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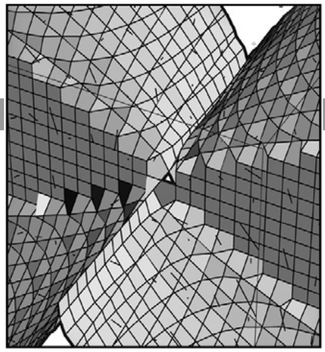


The purpose of *Differential Equations with Mathematica*, Third Edition, is twofold. First, we introduce and discuss the topics covered in typical undergraduate and beginning graduate courses in ordinary and partial differential equations including topics such as Laplace transforms, Fourier series, eigenvalue problems, and boundary-value problems. Second, we illustrate how Mathematica is used to enhance the study of differential equations not only by eliminating the computational difficulties, but also by overcoming the visual limitations associated with the explicit solutions to differential equations, which are often quite complicated. In each chapter, we first briefly present the material in a manner similar to most differential equations texts and then illustrate how Mathematica can be used to solve some typical problems. For example, in Chapter 2, we introduce the topic of first-order equations. First, we show how to solve certain types of problems by hand and then show how Mathematica can be used to assist in the same solution procedures. Finally, we illustrate how Mathematica commands like `DSolve` and `NDSolve` can be used to solve some frequently encountered equations exactly and/or numerically. In Chapter 3 we discuss some applications of first-order equations. Since we are experienced and understand the methods of solution covered in Chapter 2, we make use of `DSolve` and similar commands to obtain solutions. In doing so, we are able to emphasize the applications themselves as opposed to becoming bogged down in calculations.

The advantages of using Mathematica in the study of differential equations are numerous, but perhaps the most useful is that of being able to produce the graphics associated with solutions of differential equations. This is particularly beneficial in the discussion of applications because many physical situations are modeled with differential equations. For example, we will see that the motion of a pendulum can be modeled by a differential equation. When we solve the problem of the motion of a pendulum, we use technology to actually watch the pendulum move. The same is true for the motion of a mass attached to the end of a spring as well as many other problems. In having this ability, the study of differential equations becomes much more meaningful as well as interesting.

Numerous references like Abell and Braselton's *Mathematica By Example* [1] are also available to beginning users of Mathematica.

If you are a beginning Mathematica user and, especially, new to Version 5.0, the **Appendix** contains an introduction to Mathematica, including discussions about entering and evaluating commands, loading packages, and taking advantage of Mathematica's extensive help facility.

Although Chapter 1 is short in length, Chapter 1 introduces examples that will be investigated in subsequent chapters. Also, the vocabulary introduced in Chapter 1 will be used throughout the text. Consequently, even though, to a large extent, it may be read quickly, subsequent chapters will take advantage of the terminology and techniques discussed here.

1.1 Definitions and Concepts

We begin our study of differential equations by explaining what a differential equation is.

Definition 1 (Differential Equation). *A* ***differential equation*** *is an equation that contains the derivative or differentials of one or more dependent variables with respect to one or more independent variables. If the equation contains only ordinary derivatives (of one or more dependent variables) with respect to a single independent variable, the equation is called an* ***ordinary differential equation****.*

EXAMPLE 1.1.1: Thus, $dy/dx = x^2/\left(y^2 \cos y\right)$ and $dy/dx + du/dx = u + x^2 y$ are examples of *ordinary differential equations*.

The equation $(y-1)dx + x \cos y \, dy = 1$ is an *ordinary differential equation* written in *differential form*.

Using *prime notation*, a *solution* of the *ordinary differential equation* $xy'' + xy' + \left(x^2 - n^2\right) y = 0$, which is called **Bessel's equation**, is a function $y = y(x)$ with the property that $x\, d^2y/dx^2 + x\, dy/dx + \left(x^2 - n^2\right) y$ is identically the 0 function.

On the other hand,

$$\begin{cases} \dfrac{dx}{dt} = (a - by)x \\ \dfrac{dy}{dt} = (-m + nx)y \end{cases} \tag{1.1}$$

where a, b, m, and n are positive constants, is a *system* of two ordinary differential equations, called the **predator–prey equations**. A *solution* consists of two functions $x = x(t)$ and $y = y(t)$ that satisfy **both** equations. Predator–prey models can exhibit *very* interesting behavior as we will see when we study systems in more detail.

See texts like Giordano, Weir, and Fox's *A First Course in Mathematical Modeling* [12] and similar texts for detailed descriptions of predator–prey models.

Note that a system of differential equations can consist of more than two equations. For example, the basic equations that describe the competition between two organisms, with population densities x_1 and x_2, respectively, in a chemostat are

See Smith and Waltman's *The Theory of the Chemostat* [24] for a detailed discussion of chemostat models.

$$\begin{cases} S' = 1 - S - \dfrac{m_1 S}{a_1 + S} x_1 - \dfrac{m_2 S}{a_2 + S} x_2 \\ {x_1}' = x_1 \left(\dfrac{m_1 S}{a_1 + S} - 1 \right) \\ {x_2}' = x_2 \left(\dfrac{m_2 S}{a_2 + S} - 1 \right) \end{cases} \tag{1.2}$$

where $'$ denotes differentiation with respect to t; $S = S(t)$, $x_1 = x_1(t)$, and $x_2 = x_2(t)$. For equations (1.2), we remark that S denotes the concentration of the nutrient available to the competitors with population densities x_1 and x_2. We investigate chemostat models in more detail in Chapter 9.

If the equation contains partial derivatives of one or more dependent variables, then the equation is called a **partial differential equation**.

EXAMPLE 1.1.2: Because the equations involve partial derivatives of an unknown function, equations like $u\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}$ and $uu_x + u = u_{yy}$ are partial differential equations. For **Laplace's equation**, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ a *solution* would be a function $u = u(x, y)$ such that $u_{xx} + u_{yy}$ is identically the 0 function. A *solution* $u = u(x, t)$ of the **wave equation** is a function satisfying $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$.

The partial differential equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ is known as the **heat equation**.

As with systems of ordinary differential equations, systems of partial differential equations can be considered. With exceptions, their study is beyond the scope of this text.

Generally, given a differential equation, our goal in this course will most often be to construct a *solution* (or a numerical approximation of the *solution*). The approach to solving an equation depends on various features of the equation. The first level of classification, distinguishing between *ordinary* and *partial* differential equations, was discussed above. Generally, equations with higher *order* are more difficult to solve than those with lower *order*.

Definition 2 (Order). *The* ***order*** *of a differential equation is the order of the highest-order derivative appearing in the equation.*

EXAMPLE 1.1.3: Determine the order of each of the following differential equations: (a) $dy/dx = x^2/\left(y^2 \cos y\right)$; (b) $u_{xx}+u_{yy} = 0$; (c) $(dy/dx)^4 = y+x$; and (d) $y^3 + dy/dx = 1$.

SOLUTION: (a) The order of this equation is one because the only derivative it includes is a first-order derivative, dy/dx. (b) This equation is classified as second-order because the highest-order derivatives, both u_{xx}, representing $\partial^2 u/\partial x^2$, and u_{yy}, representing $\partial^2 u/\partial y^2$, are of order two. Hence, **Laplace's equation**, $u_{xx} + u_{yy} = 0$, is a second-order partial differential equation. (c) This is a first-order equation because the highest-order derivative is the first derivative. Raising that derivative to the fourth power does not affect the order of the equation. The expressions

$$\left(\frac{dy}{dx}\right)^4 \quad \text{and} \quad \frac{d^4 y}{dx^4}$$

do not represent the same quantities: $(dy/dx)^4$ represents the derivative of y with respect to x raised to the fourth power; d^4y/dx^4 represents the fourth derivative of y with respect to x. (d) Again, we have a first-order equation, because the highest-order derivative is the first derivative.

■

Linear differential equations are defined in a manner similar to algebraic linear equations that are introduced in algebra and pre-calculus courses.

Definition 3 (Linear Differential Equation). *An ordinary differential equation (of order n) is **linear** if it is of the form*

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \qquad (1.3)$$

where the functions $a_i(x)$, $i = 0, 1, \ldots, n$, *and* $f(x)$ *are given and* $a_n(x)$ *is not the zero function.*

For the linear differential equation (1.3), $f(x)$ is called the **forcing function**.

If the equation does not meet the requirements of this definition, then the equation is said to be **nonlinear**. If $f(x)$ is identically equal to the zero function, the linear equation (1.3) is said to be **homogeneous**.

A similar classification is followed for partial differential equations. In this case, the coefficients in a linear partial differential equation are functions of the independent variables.

EXAMPLE 1.1.4: Determine which of the following differential equations are linear: (a) $dy/dx = x^3$; (b) $d^2u/dx^2+u = x^x$; (c) $(y-1)dx+x\cos y\,dy = 0$; (d) $y^{(3)} + yy' = x$; (e) $y' + x^2y = x$; (f) $x'' + \sin x = 0$; (g) $u_{xx} + yu_y = 0$; and (h) $u_{xx} + u\,u_y = 0$.

SOLUTION: (a) This equation is linear, because the nonlinear term x^3 is the function $f(x)$ of the independent variable in equation (1.3). (b) This equation is also linear. Using u as the dependent variable name does not affect the linearity. (c) Solving for dy/dx we have $dy/dx = (1 - y)/(x\cos y)$. Because the right-hand side of this equation includes a nonlinear function of y, the equation is *nonlinear* in y. However, solving for dx/dy, we see that

$$\frac{dx}{dy} = \frac{\cos y}{1-y}x \qquad \text{or} \qquad \frac{dx}{dy} - \frac{\cos y}{1-y}x = 0.$$

This equation is linear in the variable x, if we take the dependent variable to be x and the independent variable to be y in this equation. (d) The coefficient of the y' term is y and, thus, depends on y. Hence, this equation is nonlinear. (e) This equation is linear. The term x^2 is the coefficient function $a_0(x) = x^2$ of y. (f) This equation, known as the **pendulum equation** because it models the motion of a pendulum, is nonlinear because it involves a nonlinear function of x, the dependent variable in this case. (t is assumed to be the independent variable.) For this equation, the nonlinear function of x is $\sin x$. (g) This partial differential equation is linear because the coefficient of u_y is a function of one of the independent variables. (h) In this case, there is a product of u and one of its derivatives. Therefore, the equation is nonlinear.

■

In the same manner that we consider systems of equations in algebra, we can also consider systems of differential equations. For example, if x and y represent functions of t, we will learn to solve the **system of linear equations**

$$\begin{cases} dx/dt = ax + by \\ dy/dt = cx + dy \end{cases}$$

where a, b, c, and d represent constants and differentiation is with respect to t in Chapter 8. On the other hand, systems (1.1) and (1.2) involve products of the dependent variables (x and y; S, x_1, and x_2, respectively) so are **nonlinear systems of ordinary differential equations**.

We will see that linear and nonlinear systems of differential equations arise naturally in many physical situations that are modeled with more than one equation and involve more than one dependent variable.

1.2 Solutions of Differential Equations

When faced with a differential equation, our goal is frequently, but not always, to determine explicit and/or numerical *solutions* to the equation.

Definition 4 (Solution). *A **solution** to the nth-order ordinary differential equation*

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0 \tag{1.4}$$

on the interval $a < x < b$ is a function $\phi(x)$ that is continuous on the interval $a < x < b$ and has all the derivatives present in the differential equation such that

$$F\left(x, \phi, \phi', \phi'', \ldots, \phi^{(n)}\right) = 0$$

on $a < x < b$.

In subsequent chapters, we will discuss methods for solving differential equations. Here, in order to understand what is meant to be a solution, we either give both the equation and a solution and then verify the solution or use Mathematica to solve equations directly.

EXAMPLE 1.2.1: Verify that the given function is a solution to the corresponding differential equation: (a) $dy/dx = 3y$, $y(x) = e^{3x}$; (b) $u'' + 16u = 0$, $u(x) = \cos 4x$; and (c) $y'' + 2y' + y = 0$, $y(x) = xe^{-x}$.

SOLUTION: (a) Differentiating y we have $dy/dx = 3e^{3x}$ so that substitution yields

$$\frac{dy}{dx} = 3y \qquad \text{or} \qquad 3e^{3x} = 3e^{3x}.$$

(b) Two derivatives are required in this case: $u' = -4\sin 4x$ and $u'' = -16\cos 4x$. Therefore,

$$u'' + 16u = -16\cos 4x + 16\cos 4x = 0.$$

(c) In this case, we illustrate how to use Mathematica. After defining y,

If you are a beginning Mathematica user, see the **Appendix** for help getting started with Mathematica.

```
In[1]:= y[x_] = x Exp[-x]
Out[1]= e^-x x
```

we use `'` to compute $y' = e^{-x} - xe^{-x}$, naming the resulting output `dy`.

```
In[2]:= dy = y'[x]
Out[2]= e^-x - e^-x x
```

Similarly, we use `''` to compute $y'' = -2e^{-x} + xe^{-x}$, naming the resulting output `d2y`.

```
In[3]:= d2y = y''[x]
Out[3]= -2 e^-x + e^-x x
```

Finally, we compute $y'' + 2y' + y = -2e^{-x} + 2\left(e^{-x} - xe^{-x}\right) + xe^{-x} = 0$. The result is not automatically simplified so we use `Simplify` to simplify the output.

```
In[4]:= d2y + 2dy + y[x]
Out[4]= -2 e^-x + 2 e^-x x + 2 (e^-x - e^-x x)

In[5]:= Simplify[d2y + 2dy + y[x]]
Out[5]= 0
```

We obtain the same result by entering

```
In[6]:= Simplify[y''[x] + 2 y'[x] + y[x]]
Out[6]= 0
```

which first computes $y'' + 2y' + y$ and then applies the `Simplify` command to the result. We graph this solution with `Plot`. Entering

```
In[7]:= Plot[y[x], {x, -1, 1}]
```

graphs $y(x) = xe^{-x}$ on the interval $[-1, 1]$. See Figure 1-1.

■

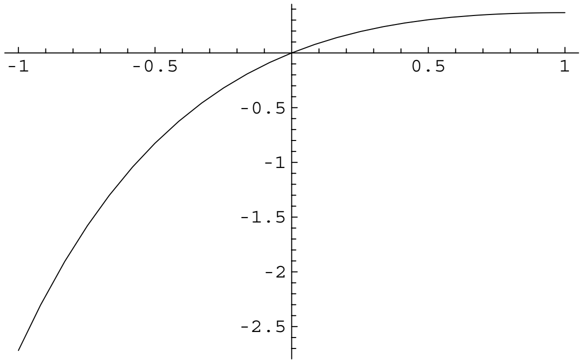


Figure 1-1 Plot of $y(x) = xe^{-x}$ on the interval $[-1, 1]$

In the previous example, the solution is given as a function of the independent variable. In these cases, the solution is said to be explicit. In solving some differential equations, however, we can only find an equation involving x and y that the solution satisfies. In this case, the solution is said to be **implicit**.

EXAMPLE 1.2.2: Verify that the given implicit function satisfies the differential equation.

$$\text{Function: } 2x^2 + y^2 - 2xy + 5x = 0$$
$$\text{Differential Equation: } \frac{dy}{dx} = \frac{2y - 4x - 5}{2y - 2x}$$

Assuming that $y = y(x)$, $\frac{dy}{dx} = y'$.

SOLUTION: We use implicit differentiation to compute the derivative of the equation $2x^2 + y^2 - 2xy + 5y = 0$:

$$4x + 2y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y + 5 = 0$$
$$(2y - 2x)\frac{dy}{dx} = 2y - 4x - 5$$
$$\frac{dy}{dx} = \frac{2y - 4x - 5}{2y - 2x}.$$

Hence, the given implicit solution satisfies the differential equation.

We also illustrate how to use Mathematica to differentiate the equation $2x^2 + y^2 - 2xy + 5x = 0$ with respect to x. After clearing all prior definitions of `x`, `y`, and `eq`, if any, with `Clear` we define `eq` to be the equation $2x^2 + y^2 - 2xy + 5x = 0$. Note how we use a double equals sign (==) to separate the left and right-hand sides of the equation.

```
In[8]:= Clear[x, y]

        eq = 2x^2 + y^2 - 2xy + 5x == 0
Out[8]= 5 x + 2 x^2 - 2 x y + y^2 == 0
```

Next, we use `Dt` to differentiate `eq` with respect to x, naming the resulting output `step1`. The symbol `Dt[y,x]` appearing in the result represents dy/dx; `step1` represents the equation $4x + 2yy' - 2xy' - 2y + 5 = 0$.

```
In[9]:= step1 = Dt[eq, x]
Out[9]= 5 + 4 x - 2 y - 2 x Dt[y, x] + 2 y Dt[y, x] == 0
```

Finally, we obtain $y' = dy/dx$ by solving the equation `step1` for `Dt[y,x]` with `Solve`.

```
In[10]:= step2 = Solve[step1, Dt[y, x]]
Out[10]= {{Dt[y, x] → (5 + 4 x - 2 y)/(2 (x - y))}}
```

Generally, to graph an equation of the form $f(x, y) = C$, where C is a constant, we use the `ContourPlot` command which is used to graph level curves of surfaces: the graph of $f(x, y) = C$ is the same as the graph of the level curve of $z = f(x, y)$ corresponding to $z = C$. Thus, the graph of the equation $2x^2 + y^2 - 2xy + 5x = 0$ is the same as the graph of the level curve of $z = f(x, y) = 2x^2 + y^2 - 2xy + 5x$ corresponding to 0. Note how $2x^2 + y^2 - 2xy + 5x$, the left-hand side of the equation `eq`, is extracted from `eq` with `Part` (`[[...]]`): $2x^2 + y^2 - 2xy + 5x$ is the first part of `eq`.

```
In[11]:= eq[[1]]
Out[11]= 5 x + 2 x^2 - 2 x y + y^2
```

Thus, entering

```
In[12]:= ContourPlot[Evaluate[eq[[1]]], {x, -7, 2},
           {y, -7, 2}, Frame → False, Axes → Automatic,
           AxesOrigin → {0, 0},
           AxesStyle → GrayLevel[0.5],
           PlotPoints → 100, Contours → {0},
           ContourShading → False]
```

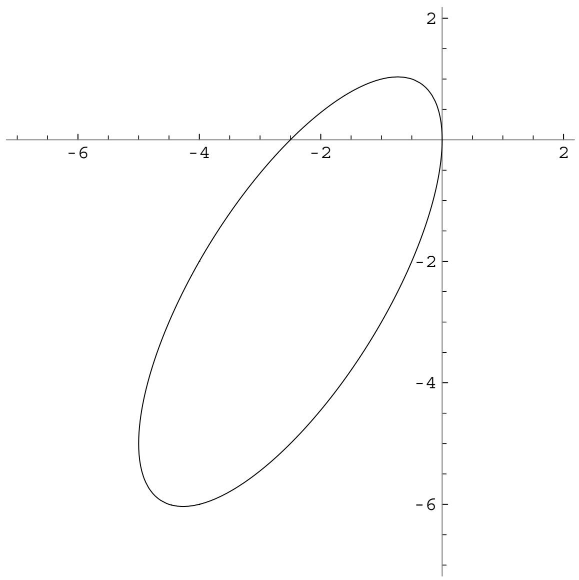


Figure 1-2 Graph of $2x^2 + y^2 - 2xy + 5x = 0$

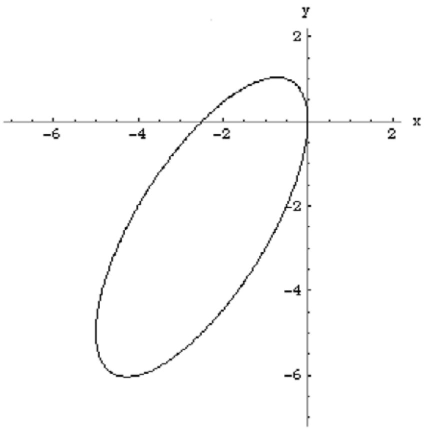
graphs the equation $2x^2 + y^2 - 2xy + 5x = 0$ as shown in Figure 1-2 for $-7 \leq x \leq 2$ and $-7 \leq y \leq 2$ (the option `Contours->{0}` instructs Mathematica to graph only the level curve corresponding to 0). The option `ContourShading->False` specifies to not shade the regions between contours, `Frame->False` specifies that a frame is not to be placed around the resulting graphics object, `Axes->Automatic` specifies that axes are to be placed on the resulting graphics object while the option `AxesOrigin->{0,0}` specifies that they intersect at the point (0, 0) and the option `AxesStyle->GrayLevel[.5]` specifies that they be drawn in a medium shade of gray. The option `PlotPoints->100` instructs Mathematica to increase the number of sample points to 100 (the default is 15), helping assure that the resulting graphics object appears smooth. Be sure to enclose `eq[[1]]` in `Evaluate` as shown: this ensures that Mathematica evaluates `eq[[1]]` before sampling points; if `Evaluate` is not included error messages result. (Note that an alternative way to graph equations is to use the `ImplicitPlot` command which is contained in the **ImplicitPlot** package located in the **Graphics** folder (or directory).) If you are using Version 5.0 (or later) and wish to avoid using `Part`, you can select the left-hand side of the equation, copy it,

```
Clear[x, y]
eq = 2 x^2 + y^2 - 2 x y + 5 x == 0

5 x + 2 x^2 - 2 x y + y^2 == 0

step1 = Dt[eq, x]

5 + 4 x - 2 y - 2 x Dt[y, x] + 2 y Dt[y, x] == 0
```

and then paste it into the `ContourPlot` command.

See **Getting Started** in the **Appendix**.

```
step1 = Dt[eq, x]

5 + 4 x - 2 y - 2 x Dt[y, x] + 2 y Dt[y, x] == 0

ContourPlot[5 x + 2 x^2 - 2 x y + y^2, {x, -7, 2}, {y, -7, 2},
  Frame → False, Axes → Automatic, AxesOrigin → {0, 0},
  AxesStyle → GrayLevel[.5], PlotPoints → 200,
  Contours → {0}, ContourShading → False,
  AxesLabel → {"x", "y"}]
```




```
- ContourGraphics -
```

■

EXAMPLE 1.2.3: On a rectangular membrane, the solution of the **wave equation**,

For details, see Graff's *Wave Motion in Elastic Solids* [13].

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c_0^2} \frac{\partial^2 w}{\partial t^2} \tag{1.5}$$

takes the form

$$w = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t)\, W_{mn},$$

where A_{mn} and B_{mn} are constants, $\omega_{mn}{}^2 = c_0{}^2\left(\xi_n{}^2 + \zeta_m{}^2\right)$, $\xi_n = n\pi/a$, $\zeta_m = m\pi/b$, and the **normal modes**, W_{nm}, are given by

$$W_{nm}(x, y) = \sin\xi_n x\, \sin\zeta_m y.$$

(a) Verify that $f_c(x, y, t) = \cos\omega_{mn}t\, W_{mn}(x, y)$ and $f_s(x, y, t) = \sin\omega_{mn}t\, W_{mn}(x, y)$ satisfy the wave equation (1.5) on a rectangular membrane. (b) Plot the first few normal modes of the membrane.

SOLUTION: After defining $\omega_{mn}{}^2 = c_0{}^2\left(\xi_n{}^2 + \zeta_m{}^2\right)$, $\xi_n = n\pi/a$, $\zeta_m = m\pi/b$, we define $f_c(x, y, t) = \cos\omega_{mn}tW_{mn}(x, y)$ and $f_s(x, y, t) = \sin\omega_{mn}tW_{mn}(x, y)$.

```
In[13]:= ω[m_, n_] = π c0 Sqrt[n^2/a^2 + m^2/b^2];

In[14]:= ξ[n_] = n π/a;

         ζ[m_] = m π/b;

In[15]:= fc[x_, y_, t_] =
           Cos[ω[m, n] t] Sin[ξ[n] x] Sin[ζ[m] y];

In[16]:= fs[x_, y_, t_] =
           Sin[ω[m, n] t] Sin[ξ[n] x] Sin[ζ[m] y];
```

To verify that $f_c(x, y, t)$ satisfies equation (1.5), we compute $\dfrac{\partial^2 f_c}{\partial x^2}$, $\dfrac{\partial^2 f_c}{\partial y^2}$, and $\dfrac{\partial^2 f_c}{\partial t^2}$ in `fcxx`, `fcyy`, and `fczz`, respectively.

```
In[17]:= fcxx = D[fc[x, y, t], {x, 2}]
```

$$\text{Out[17]= } -\frac{n^2\pi^2 \,\text{Cos}\left[c0\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}\,\pi\, t\right]\text{Sin}\left[\frac{n\pi x}{a}\right]\text{Sin}\left[\frac{m\pi y}{b}\right]}{a^2}$$

```
In[18]:= fcyy = D[fc[x, y, t], {y, 2}]
```

$$\text{Out[18]= } -\frac{m^2\pi^2 \,\text{Cos}\left[c0\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}\,\pi\, t\right]\text{Sin}\left[\frac{n\pi x}{a}\right]\text{Sin}\left[\frac{m\pi y}{b}\right]}{b^2}$$

```
In[19]:= fctt = D[fc[x, y, t], {t, 2}]
```

$$\text{Out[19]= } -c0^2\left(\frac{m^2}{b^2}+\frac{n^2}{a^2}\right)\pi^2\,\text{Cos}\left[c0\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}\,\pi\, t\right]\text{Sin}\left[\frac{n\pi x}{a}\right]\text{Sin}\left[\frac{m\pi y}{b}\right]$$

`fcxx` + `fcyy` − $1/c_0^2$`fctt` is simplified with `Simplify`; the result is 0 so $f_c(x, y, t)$ satisfies equation (1.5).

```
In[20]:= Simplify[fcxx + fcyy - 1/c0^2 fctt]//Simplify
Out[20]= 0
```

On the other hand, to verify that $f_s(x, y, t)$ satisfies equation (1.5), we compute and simplify

$$\frac{\partial^2 f_s}{\partial x^2} + \frac{\partial^2 f_s}{\partial y^2} - \frac{1}{c_0^2}\frac{\partial^2 f_s}{\partial t^2}$$

in a single step. The result is identically equal to 0 so $f_s(x, y, t)$ also satisfies equation (1.5).

```
In[21]:= Simplify[D[fs[x, y, t], {x, 2}] + D[fs[x, y, t],
           {y, 2}] - 1/c0^2 D[fs[x, y, t],
           {t, 2}]]//Simplify
Out[21]= 0
```

(b) To graph the normal modes, we choose $a = b = 1$. We then use `Table` and `Plot3D` to plot $W_{nm}(x, y)$, $0 \le x \le 1$ and $0 \le y \le 1$, for $n = 1$, 2, 3, and 4, $m = 1$, 2, and 3. The resulting array of graphics is displayed as a graphics array using `Show` and `GraphicsArray`. In Figure 1-3, the first row corresponds to $n = 1$, the second to $n = 2$, and so on; the first column corresponds to $m = 1$, the second to $m = 3$, and so on.

```
In[22]:= tp = Table[Plot3D[Sin[n π x] Sin[m π y], {x, 0, 1},
              {y, 0, 1}, DisplayFunction → Identity,
              BoxRatios → {1, 1, 1}, PlotPoints → 45],
              {n, 1, 4}, {m, 1, 3}];

         Show[GraphicsArray[tp]]
```

■

As indicated in the previous example, without added initial or boundary conditions many differential equations have more than one solution. We further illustrate this property in the following examples where we use the Mathematica command `DSolve` to solve the indicated equations. Generally, the command

```
DSolve[F[x,y[x],y'[x],...,D[y[x],{x,n}]]==0,y[x],x]
```

attempts to solve the differential equation (1.4) for y. Detailed help regarding `DSolve` is obtained by entering `?DSolve` or by going to **Help** under the Mathematica menu

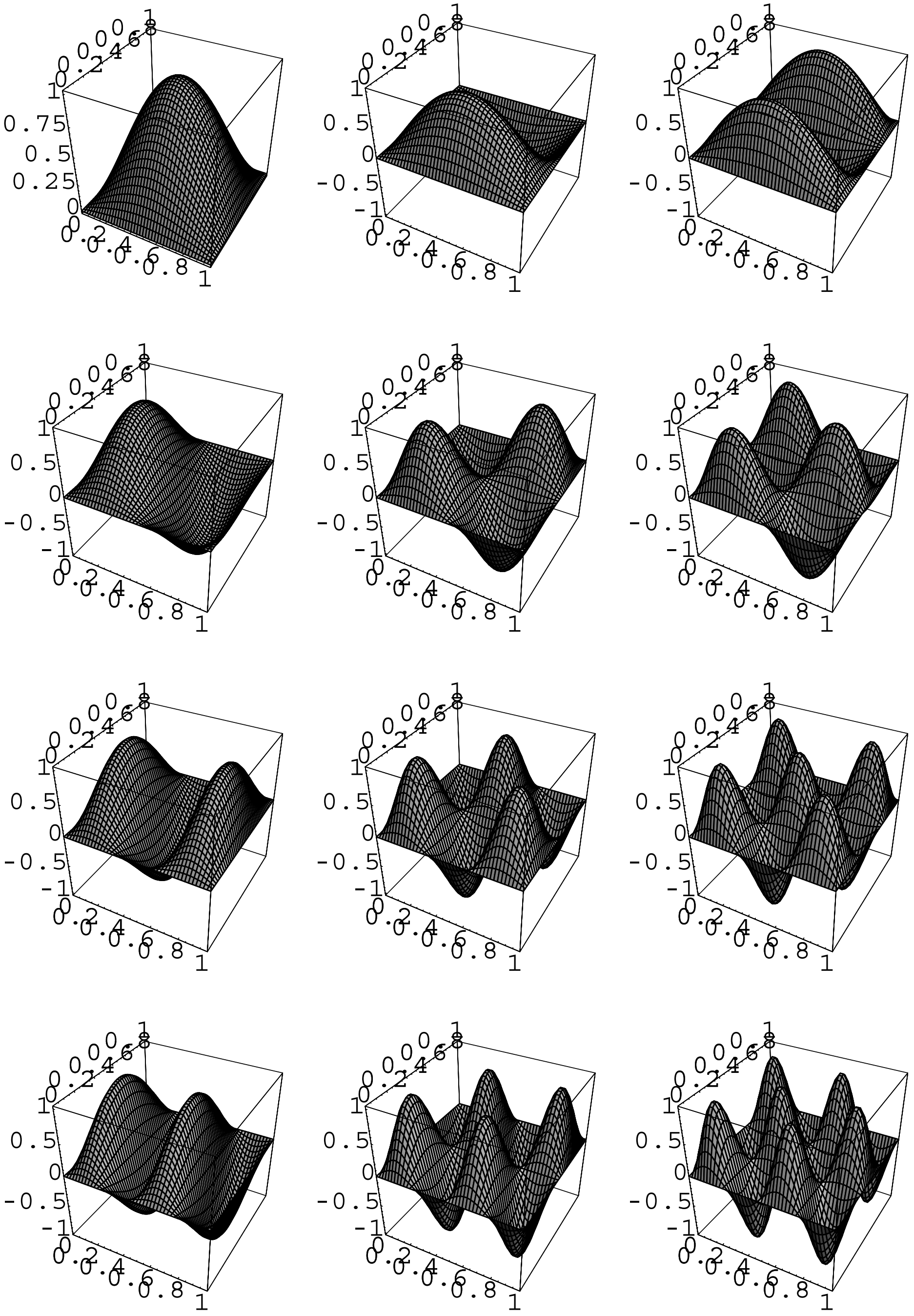
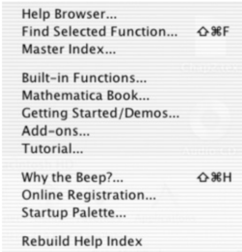
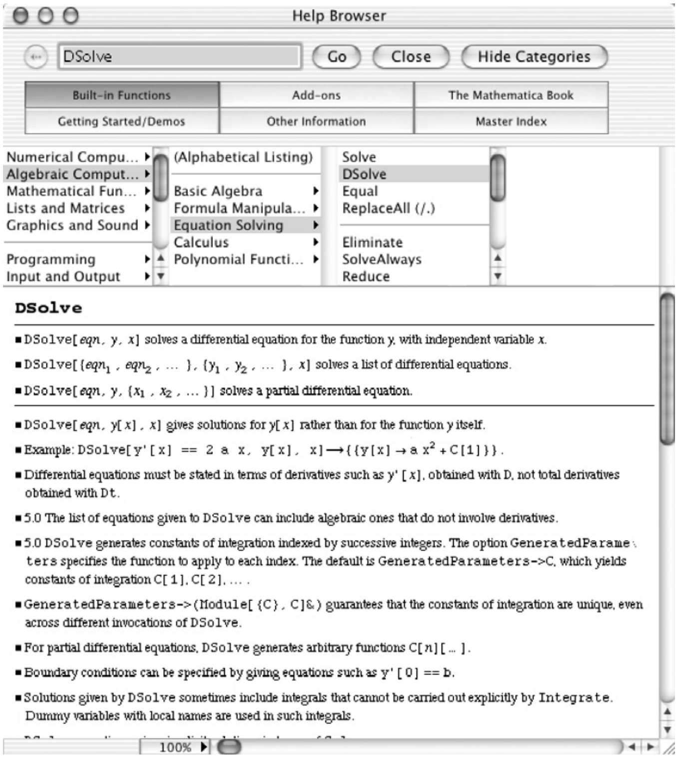


Figure 1-3 W_{nm} for $n = 1, 2, 3$, and 4, $m = 1, 2$, and 3



and then selecting **Help...** Once Mathematica opens the **Help Browser**, you can either type `DSolve` and select **Go To** or select **Algebraic Computation** followed by **Equation Solving** and `DSolve` to obtain a description of the `DSolve` command, a discussion of its various options, and several examples, as illustrated in the following screen shot.



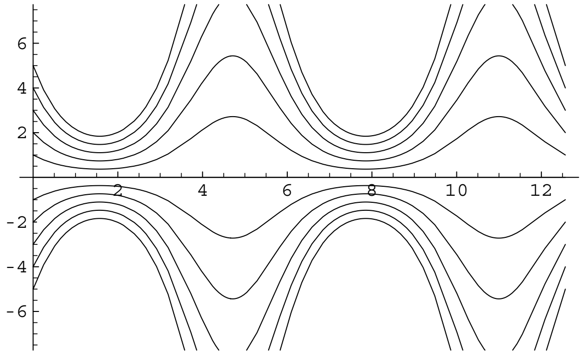


Figure 1-4 $y = Ce^{-\sin x}$ for various values of C

EXAMPLE 1.2.4: Verify that the differential equation $dy/dx = -y\cos x$ has infinitely many solutions.

SOLUTION: We use `DSolve` to solve this first-order linear equation and name the resulting list `sol`. We see that the solution is given in terms of a *replacement rule* and interpret the result to mean that if `C[1]` is any number, a solution to the equation is y = C[1]E$^{-\text{Sin}[x]}$. In traditional mathematical notation, we could write that $y = Ce^{-\sin x}$ is a solution of $dy/dx = -y\cos x$ for any value of C.

The formula for the solution is extracted from `sol` with `sol[[1,1,1]]` or, if you are using Version 5, by selecting and copying.

```
In[23]:= sol = DSolve[y′[x] == -Cos[x] * y[x], y[x], x]
Out[23]= {{y[x] → e^(-Sin[x]) C[1]}}
```

Thus, the equation has infinitely many solutions. We graph several solutions with `Plot` in Figure 1-4.

```
In[24]:= toplot = Table[e^(-Sin[x]) C[1]/.C[1]- > i, {i, -5, 5}];

In[25]:= Plot[Evaluate[toplot], {x, 0, 4π}]
```

■

EXAMPLE 1.2.5: Verify that $y'' + y = 0$ has infinitely many solutions.

SOLUTION: We use `DSolve` to solve this second-order linear equation and name the resulting list `sol`. We interpret the result to mean that if C[1] and C[2] are any numbers, a solution to the equation is y = C[1]Cos[x] + C[2]Sin[x]. In traditional mathematical notation, we write that $y = C_1 \cos x + C_2 \sin x$ is a solution of $y'' + y = 0$ for any constant values of C_1 and C_2.

```
In[26]:= sol = DSolve[y''[x] + y[x] == 0, y[x], x]

Out[26]= {{y[x] → C[1] Cos[x] + C[2] Sin[x]}}
```

In particular, this result indicates that $y = C \cos x$ is a solution of $y'' + y = 0$ for any value of C (set $C_2 = 0$) and that $y = C \sin x$ is a solution of $y'' + y = 0$ for any value of C (set $C_1 = 0$). Some of the members of the family of solutions are graphed with Plot. First, we use Table to generate a set of eleven functions obtained by replacing C in $y = C \cos x$ by $-2.5, -2, -1.5, \ldots, 1.5, 2,$ and 2.5, naming the resulting set toplot1 and then a set of eleven functions obtained by replacing C in $y = C \sin x$ by $-2.5, -2, -1.5, \ldots, 1.5, 2,$ and 2.5, naming the resulting set toplot2.

```
In[27]:= toplot1 =
           Table[C[2] Cos[x] - C[1] Sin[x] /.{C[2]- > i,
           C[1]- > 0}, {i, -2.5, 2.5, 0.5}];

         toplot2 =
           Table[C[2] Cos[x] - C[1] Sin[x] /.{C[2]- > 0,
           C[1]- > i}, {i, -2.5, 2.5, 0.5}];
```

Then, the set of functions toplot1 and toplot2 are graphed with Plot for $0 \le x \le 4\pi$. Be sure to include toplot1 and toplot2 within the Evaluate command because Mathematica must evaluate each set of functions before sampling points. Neither graph is displayed as it is generated because we include the option DisplayFunction-> Identity in each Plot command. Instead, we show the graphs side-by-side using GraphicsArray in Figure 1-5.

```
In[28]:= plot1 = Plot[Evaluate[toplot1], {x, 0, 4π},
                DisplayFunction- > Identity];

         plot2 = Plot[Evaluate[toplot2], {x, 0, 4π},
                DisplayFunction- > Identity];

         Show[GraphicsArray[{plot1, plot2}]];
```

■

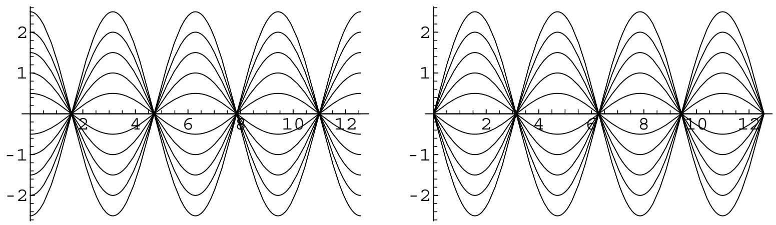


Figure 1-5 Plots of $y = C\cos x$ and $y = C\sin x$ for various values of C

1.3 Initial and Boundary-Value Problems

In many applications, we are not only given a differential equation to solve but we are given one or more conditions that must be satisfied by the solution(s) as well. For example, suppose that we want to find an antiderivative of the function $f(x) = 3x^2 - 4x$. Then, we solve the differential equation $dy/dx = 3x^2 - 4x$ by integrating:

$$\frac{dy}{dx} = 3x^2 - 4x \Longrightarrow y = \int \left(3x^2 - 4x\right) dx \Longrightarrow y = x^3 - 2x^2 + C.$$

```
In[29]:= ∫ (3x^2 - 4x) dx
Out[29]= -2 x^2 + x^3
```

Because the solution involves an arbitrary constant and all solutions to the equation can be obtained from it, we call this a **general solution**. On the other hand, if we want to find a solution that passes through the point (1, 4), we must find a solution that satisfies the *auxiliary condition* $y(1) = 4$. Substitution into $y = x^3 - 2x^2 + C$ yields $y(1) = 1^3 - 2 \cdot 1^2 + C = 4 \Longrightarrow C = 5$. Therefore, *the* member of the family of solutions $y = x^3 - 2x^2 + C$ that satisfies $y(1) = 4$ is $y = x^3 - 2x^2 + 5$. The following commands illustrate how to graph some members of the family of solutions by substituting various values of C into the general solution. We also graph the solution to the problem

$$\begin{cases} y' = 3x^2 - 4x \\ y(1) = 4. \end{cases}$$

First, we use `Table` to generate a table of functions $x^3 - 2x^2 + C$ for $C = -10, -8, \ldots, 8, 10$, naming the resulting set of functions `toplot`. Note that we use `c` to represent C to avoid conflict with the built-in symbol `C`. The set of functions `toplot` is not displayed (for length reasons) because a semi-colon (`;`) is included at the end of the command. However, we are able to view an abbreviation of `toplot` with `Short`. Using `Length`, we see that `toplot` contains eleven functions.

```
In[30]:= toplot = Table[-2 x^2 + x^3 + c, {c, -10, 10, 2}];
         Short[toplot]
         Length[toplot]

Out[30]= {-10 - 2 x^2 + x^3, <<9>>, 10 - 2 x^2 + x^3}

Out[30]= 11
```

To graph the eleven functions contained in `toplot` and distinguish between the graphs, we use `Table` and `GrayLevel` to generate a list of eleven different gray levels. Then, we graph `toplot` with `Plot` in Figure 1-6(a). The option `PlotStyle->grays` instructs Mathematica to display each graph using the corresponding shade of gray.

```
In[31]:= grays = Table[GrayLevel[i], {i, 0, 0.5, 0.5/10}];

         Plot[Evaluate[toplot], {x, -2, 3}, PlotStyle → grays,
           AxesStyle → GrayLevel[0.5], PlotRange → {-15, 15}]

In[32]:= Plot[x^3 - 2x^2 + 5, {x, -2, 3}, PlotRange → {-15, 15}]
```

Notice that this first-order equation requires one auxiliary condition to eliminate the unknown coefficient in the general solution. Frequently, the independent variable in a problem is t, which usually represents time. Therefore, we call the *auxiliary condition* of a first-order equation an **initial condition**, because it indicates the initial-value (at $t = t_0$) of the dependent variable. Problems that involve an initial condition are called **initial-value problems**.

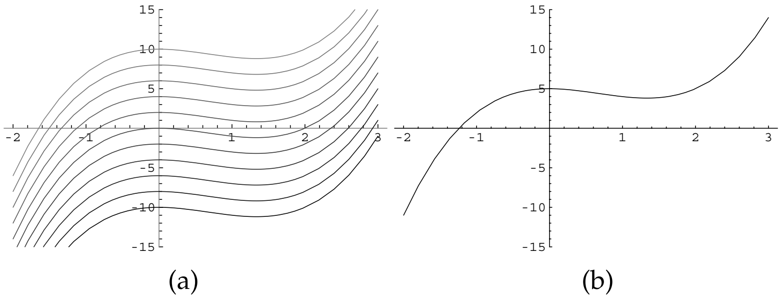


Figure 1-6 (a) Plot of $y = x^3 - 2x^2 + C$ for various values of C. (b) Plot of *the* solution that satisfies $y(1) = 4$

EXAMPLE 1.3.1: Consider the first-order equation

$$\frac{dv}{dt} = 32 - v,$$

which is solved to determine the velocity at time t, $v(t)$, of an object of mass $m = 1$ subjected to air resistance equivalent to the instantaneous velocity of the object. If the initial velocity of the object is $v(0) = 0$, determine the solution that satisfies this initial condition.

SOLUTION: A general solution to this equation is found to be $v(t) = 32 + Ce^{-t}$, where C is a constant, with `DSolve`.

```
In[33]:= gensol = DSolve[v'[t] == 32 - v[t], v[t], t]

Out[33]= {{v[t] → 32 + e^-t C[1]}}
```

Substituting into the general solution, we have $v(0) = 32 + C = 0$. Hence, $C = -32$, and the solution to the initial-value problem is $v(t) = 32 - 32e^{-t}$. `DSolve` can be used to solve this initial-value problem as well.

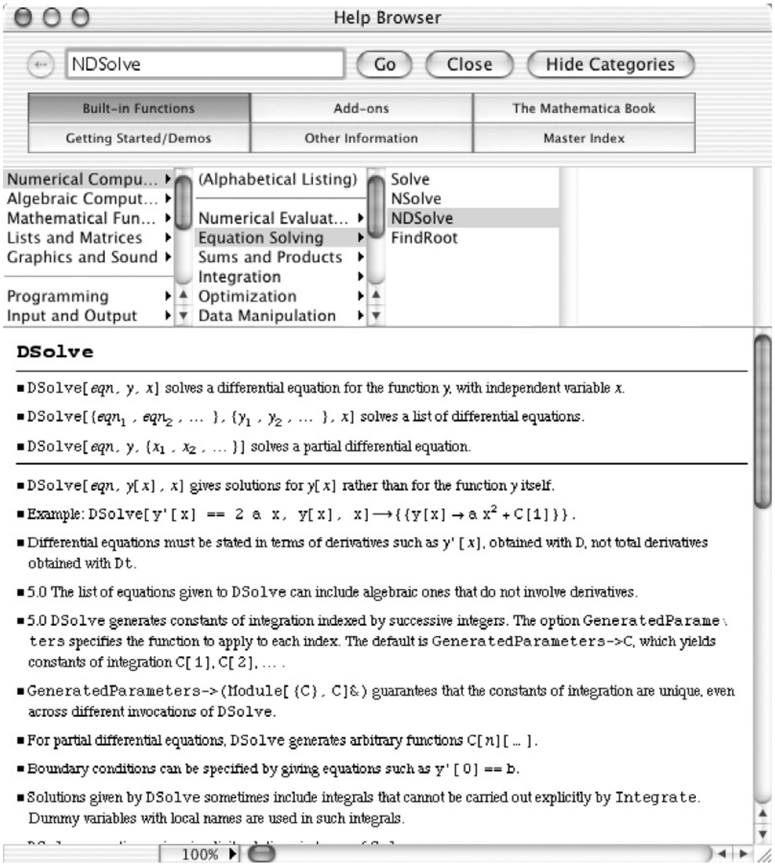
```
In[34]:= gensol = DSolve[{v'[t] == 32 - v[t],
           v[0] == 0}, v[t], t]

Out[34]= {{v[t] → 32 e^-t (-1 + e^t)}}
```

■

If `DSolve` cannot find an exact solution to an initial-value problem or if numerical results are desired, the command

```
NDSolve[{y'[x]==f[x,y[x]],y[x0]==y0},y[x],{x,a,b}]
```

attempts to find a numerical solution to the initial-value problem $\{y' = f(x, y), y(x_0) = y_0\}$ valid for $a \le x \le b$. We use the **Help Browser** to obtain information about `NDSolve` and its options as well as several examples illustrating its use.



Notice that the syntax of the `NDSolve` command is almost identical to that of the `DSolve` command, except that we must specify an interval $[a, b]$ on which we want the numerical solution to be valid.

EXAMPLE 1.3.2: Graph the solution to the initial-value problem $\begin{cases} y' = \sin x^2 \\ y(0) = 0 \end{cases}$ on the interval $[0, 10]$. Evaluate $y(5)$.

SOLUTION: In this case, we see that `DSolve` is able to solve the initial-value problem although the result is given in terms of the `FresnelS` function.

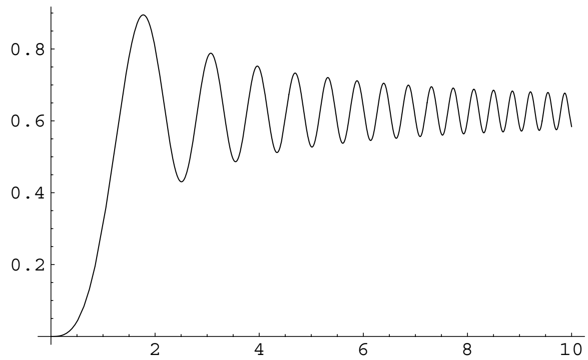


Figure 1-7 Plot of the solution of $\begin{cases} y' = \sin x^2 \\ y(0) = 0 \end{cases}$ for $0 \le x \le 10$

```
In[35]:= exactsol = DSolve[{y'[x] == Sin[x^2],
          y[0] == 0}, y[x], x]
```

Out[35]= $\left\{\left\{y[x] \to \sqrt{\frac{\pi}{2}}\,\text{FresnelS}\left[\sqrt{\frac{2}{\pi}}\,x\right]\right\}\right\}$

Here is Mathematica's description of the `FresnelS` function.

```
In[36]:= ?FresnelS
         FresnelS[z] gives the Fresnel integral S(z).
```

Using `NDSolve`, we obtain a numerical solution to the initial-value problem valid on the interval [0, 10]:

```
In[37]:= numsol = NDSolve[{y'[x] == Sin[x^2], y[0] == 0},
          y[x], {x, 0, 10}]
Out[37]= {{y[x] → InterpolatingFunction[{{0., 10.}},
             <>][x]}}
```

which we graph with `Plot` in Figure 1-7.

```
In[38]:= Plot[y[x]/.numsol, {x, 0, 10}, PlotRange → All]
```

The value of $y(5)$ is found with `ReplaceAll` (`/.`)

```
In[39]:= numsol/.x → 5
Out[39]= {{y[5] → 0.527917}}
```

and indicates that $y(5) \approx 0.5279$.

■

Because first-order equations involve a single auxiliary condition, which is usually referred to as an initial condition, we use the following examples to distinguish between **initial-value** and **boundary-value** problems which involve higher-order equations.

EXAMPLE 1.3.3: Consider the second-order differential equation $x'' + x = 0$, which models the motion of a mass with $m = 1$ attached to the end of a spring with spring constant $k = 1$, where $x(t)$ represents the displacement of the mass from the equilibrium position $x = 0$ at time t. A general solution of this differential equation is found to be $x(t) = A\cos t + B\sin t$, where A and B are arbitrary constants, with `DSolve`.

```
In[40]:= sol = DSolve[x''[t] + x[t] == 0, x[t], t]
Out[40]= {{x[t] → C[1] Cos[t] + C[2] Sin[t]}}
```

Because this is a second-order equation, we need two auxiliary conditions to determine the two unknown constants. Suppose that the initial displacement of the mass is $x(0) = 0$ and the initial velocity is $x'(0) = 1$. This is an **initial-value problem** because we have two auxiliary conditions given at the same value of t, namely $t = 0$. Use these initial conditions to determine the solution of this problem.

SOLUTION: Because we need the first derivative of the general solution, we calculate $x'(t) = B\cos t - A\sin t$. Substitution yields $x(0) = A = 0$ and $x'(0) = B = 1$. Hence, the solution is $x(t) = \sin t$. `DSolve` can solve this initial-value problem as well.

```
In[41]:= DSolve[{x''[t] + x[t] == 0, x[0] == 0, x'[0] == 1},
           x[t], t]
Out[41]= {{x[t] → Sin[t]}}
```

■

EXAMPLE 1.3.4: The shape of a bendable beam of length 1 unit that is subjected to a compressive force at one end is described by the graph of the solution $y(x)$ of the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{\pi^2}{4}y = 0$, $0 < x < 1$. If the height of the beam above the x-axis is known at the endpoints $x = 0$ and $x = 1$, then we have a **boundary-value problem**. Use the boundary conditions $y(0) = 0$ and $y(1) = 2$ to find the shape of the beam.

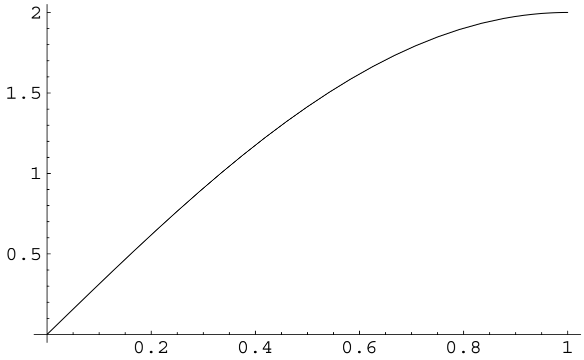


Figure 1-8 A plot of a solution to the beam equation

Later, we will see that under reasonable conditions, initial-value problems have unique solutions. On the other hand, boundary-value problems may have no solutions, infinitely many solutions, or a unique solution.

SOLUTION: First, we use `DSolve` to find a general solution to the equation. The result indicates that a general solution is $y(x) = A\cos\left(\frac{\pi}{2}x\right) + B\sin\left(\frac{\pi}{2}x\right)$.

```
In[42]:= DSolve[y''[x] + π²/4 y[x] == 0, y[x], x]
Out[42]= {{y[x] → C[1] Cos[πx/2] + C[2] Sin[πx/2]}}
```

Applying the condition $y(0) = 0$ to the general solution yields

$$y(0) = A\cos 0 + B\sin 0 = A = 0.$$

Similarly, $y(1) = 2$ indicates that

$$y(1) = B\sin\frac{\pi}{2} = B = 2,$$

so the solution to the boundary value problem is $y(x) = 2\sin\left(\frac{\pi}{2}x\right)$, $0 < x < 1$. `DSolve` is also able to solve this boundary value problem.

```
In[43]:= DSolve[{y''[x] + π²/4 y[x] == 0, y[0] == 0, y[1] == 2},
           y[x], x]
Out[43]= {{y[x] → 2 Sin[πx/2]}}
```

This function that describes the shape of the beam is graphed with `Plot` in Figure 1-8.

```
In[44]:= Plot[2 Sin[πx/2], {x, 0, 1}]
```

■

We will see that it is usually impossible to find exact solutions of higher-order nonlinear initial-value problems. In those cases, we can often use NDSolve to generate an accurate approximation of the solution.

Also see Example 1.4.4.

EXAMPLE 1.3.5 (Rayleigh's Equation): Rayleigh's equation is the nonlinear equation

$$\frac{d^2x}{dt^2} + \left[\frac{1}{3}\left(\frac{dx}{dt}\right)^2 - 1\right]\frac{dx}{dt} + x = 0 \tag{1.6}$$

and arises in the study of the motion of a violin string. Graph the solution to Rayleigh's equation on the interval [0, 15] if (a) $x(0) = 1, x'(0) = 0$; (b) $x(0) = 0.1, x'(0) = 0$; and (c) $x(0) = 0, x'(0) = 1.9$.

SOLUTION: In each case, we use NDSolve to approximate the solution to the initial-value problem, naming the results numsol1, numsol2, and numsol3, respectively.

```
In[45]:= numsol1 =
           NDSolve[{x''[t] + (1/3 x'[t]^2 - 1) x'[t] + x[t] == 0,
           x[0] == 1, x'[0] == 0}, x[t], {t, 0, 15}]
Out[45]= {{x[t] → InterpolatingFunction[{{0., 15.}},
           <>][t]}}
```

```
In[46]:= numsol2 =
           NDSolve[{x''[t] + (1/3 x'[t]^2 - 1) x'[t] + x[t] == 0,
           x[0] == 0.1, x'[0] == 0}, x[t], {t, 0, 15}]
Out[46]= {{x[t] → InterpolatingFunction[{{0., 15.}},
           <>][t]}}
```

```
In[47]:= numsol3 =
           NDSolve[{x''[t] + (1/3 x'[t]^2 - 1) x'[t] + x[t] == 0,
           x[0] == 0, x'[0] == 1.9}, x[t], {t, 0, 15}]
Out[47]= {{x[t] → InterpolatingFunction[{{0., 15.}},
           <>][t]}}
```

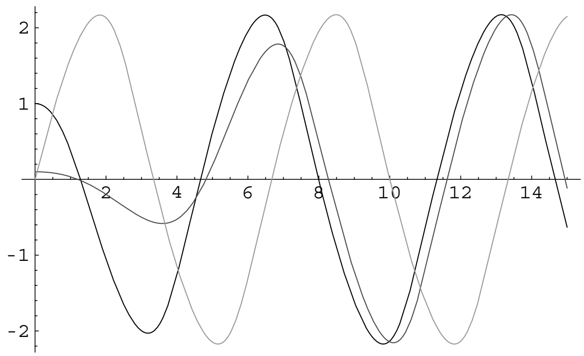


Figure 1-9 Plot of three solutions of Rayleigh's equation, (1.6)

All three solutions are graphed together on the interval [0, 15] with `Plot` in Figure 1-9. Notice that the solution to (c) appears to be periodic.

```
In[48]:= Plot[Evaluate[x[t]/
           .{numsol1, numsol2, numsol3}],
           {t, 0, 15}, PlotStyle → {GrayLevel[0],
            GrayLevel[0.3], GrayLevel[0.6]}]
```

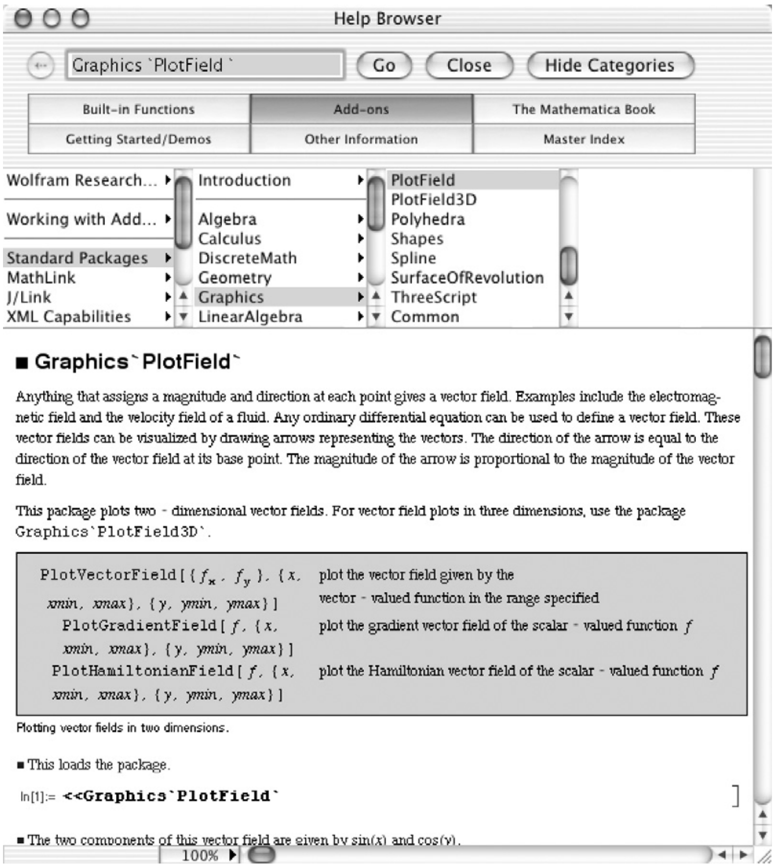
■

1.4 Direction Fields

The geometrical interpretation of solutions to first-order differential equations of the form $dy/dx = f(x, y)$ is important to the basic understanding of problems of this type. Suppose that a solution to this equation is a function $y = \psi(x)$, so a solution is the graph of the function ψ. Therefore, if (x, y) is a point on this graph, the slope of the tangent line is given by $f(x, y)$. A set of short line segments representing the tangent lines can be constructed for a large number of points. This collection of line segments is known as the **direction field** of the differential equation and provides a great deal of information concerning the behavior of the family of solutions. This is due to the fact that by determining the slope of the tangent line for a large number of points in the plane, the shape of the graphs of the solutions can be seen without actually having a formula for them. The direction field for a differential equation provides a geometric interpretation about the behavior of the solutions of the equation. Throughout this text, we will frequently display graphs of various solutions to a differential equation along with a graph of the direction field. Direction fields are generated with the `PlotVectorField` command, which is contained in the **PlotField** package. After loading the **PlotField** package, which is contained in the **Graphics** folder (or directory) the command

```
PlotVectorField[{1,f[x,y]},{x,x0,x1},{y,y0,y1}]
```

graphs the direction field associated with $dy/dx = f(x, y)$ for $x_0 \le x \le x_1$ and $y_0 \le y \le y_1$. We use the **Help Browser** to obtain information abut the commands contained in the **PlotField** package in the same way that we use the **Help Browser** to obtain information about built-in Mathematica functions.



EXAMPLE 1.4.1: Graph the direction field associated with the differential equation $dy/dx = e^{-x} - 2y$.

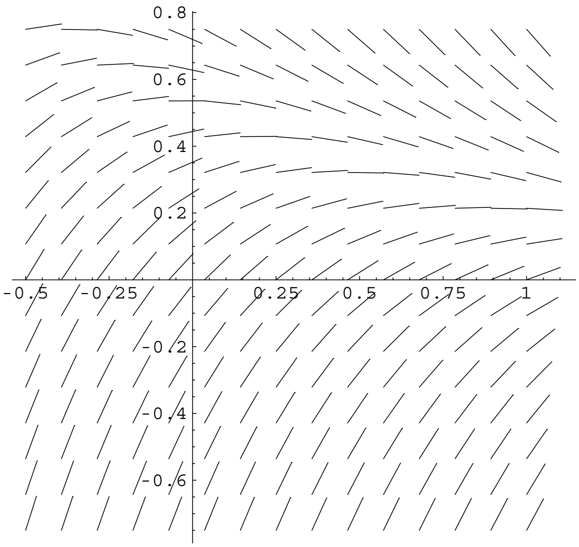


Figure 1-10 Direction field associated with $dy/dx = e^{-x} - 2y$

SOLUTION: Entering

```
In[49]:= << Graphics`PlotField`
```

```
In[50]:= p1 = PlotVectorField[{1, Exp[-x] - 2y},
             {x, -1/2, 1}, {y, -3/4, 3/4}, Axes → Automatic,
             ScaleFunction → (1&), AxesOrigin → {0, 0},
             HeadLength- > 0]
```

first loads the **PlotField** package and then graphs the direction field associated with the equation $dy/dx = e^{-x} - 2y$ in Figure 1-10 for $-1/2 \le x \le 1$ and $-3/4 \le y \le 3/4$, naming the resulting graphics object `p1`. The option `Axes->Automatic` specifies that axes are to be placed on the resulting graph, `AxesOrigin->{0,0}` specifies that the axes intersect at the point (0, 0), the option `ScaleFunction->(1&)` specifies that the magnitude of each line segment be 1 (this makes them easier to see in the resulting graph), and the option `HeadLength->0` instructs Mathematica to not place arrows on the line segments, which would result in vectors. For a complete list of the options and their default values associated with this command, enter `Options[PlotVectorField]`.

A general solution of this first-order linear equation is found to be $y = e^{-x} + Ce^{-2x}$ with `DSolve`.

Generally, Mathematica is able to find a general solution of first-order linear equations like this with `DSolve`.

```
In[51]:= Clear[x, y, diffeq]

         gensol = DSolve[y'[x] == Exp[-x] - 2 y[x], y[x], x]
Out[51]= {{y[x] → e^-x + e^-2 x C[1]}}
```

first clears all prior definitions of `x`, `y`, and `gensol`, if any, and then solves the equation $dy/dx = e^{-x} - 2y$ for $y = y(x)$, naming the resulting output `gensol`. In the `DSolve` command, the first argument (y′[x] == E^-x – 2y[x]) represents the equation $dy/dx = e^{-x} - 2y$, the second argument (`y[x]`) instructs Mathematica that we are solving for $y = y(x)$, and the third argument (x) instructs Mathematica that the independent variable is x. Note that `gensol` is a nested list. The first part of `gensol`, extracted with `gensol[[1]]`, is the list {y(x) → e^-x + e^-2x C[1]}; the first part of this list, extracted with `gensol[[1,1]]`, is the list y(x) → e^-x + e^-2x C[1]; and the first part of this list, extracted with `gensol[[1,1,1]]`, is $y(x)$ while the second part of this list (which represents the formula for the solution), extracted with `gensol[[1,1,2]]`, is $y = e^{-x} + Ce^{-2x}$. Of course, if you are using Version 5.0 (or later), you can extract these results by selecting, copying, and pasting the results to the desired location in your Mathematica notebook.

```
In[52]:= gensol[[1, 1, 2]]
Out[52]= e^-x + e^-2 x C[1]
```

Note that in the formula for the solution the built-in symbol `C` is used to denote arbitrary constants. Here `C[1]` represents C in the solution $y = e^{-x} + Ce^{-2x}$.

To graph the solution for various values of the arbitrary constant, we use `Table` and `ReplaceAll` (`/.`) to replace `C[1]` in the formula for the general solution obtained in `gensol` by i for $i = -2, -1.75, -1.50, \ldots, 1.50, 1.75$, and 2, naming the resulting set of functions `toplot`. The list `toplot` is not displayed because a semi-colon (`;`) is included at the end of the `Table` command.

```
In[53]:= toplot = Table[(gensol[[1, 1, 2]])/.C[1] → i,
           {i, -2, 2, 0.25}];
```

We then use `Plot` to graph the set of functions `toplot` for $-1/2 \le x \le 1$ in Figure 1-11, naming the resulting graphics object `p2`. The option `PlotRange->{-3/4,3/4}` instructs Mathematica that the range of

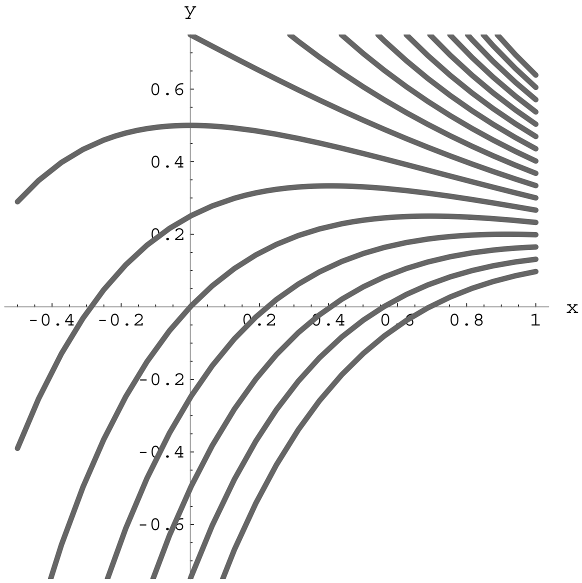


Figure 1-11 Various solutions of $dy/dx = e^{-x} - 2y$

y-values displayed corresponds to the interval [−3/4, 3/4], `AspectRatio->1` specifies that the ratio of the lengths of *x* and *y*-axes in the resulting graph is to be 1, `AxesStyle->GrayLevel[.5]` specifies that the axes are to be shaded a light gray,

`PlotStyle->{{GrayLevel[.4],Thickness[0.01]}}`

specifies that the graphs are to be displayed in a slightly darker shade of gray and that the thickness of each graph be 1/100 of the width of the total graph, and `AxesLabel->{x,y}` specifies that the *x* and *y*-axes are to be labeled by *x* and *y*, respectively.

```
In[54]:= p2 = Plot[Evaluate[toplot], {x, -1/2, 1},
           PlotRange → {-3/4, 3/4}, AspectRatio → 1,
           AxesStyle → GrayLevel[0.5],
           PlotStyle → {{GrayLevel[0.4],
           Thickness[0.01]}},
           AxesLabel → {x, y}]
```

Notice that we can predict the behavior of the solutions of this equation by observing the direction field, as we confirm with the following

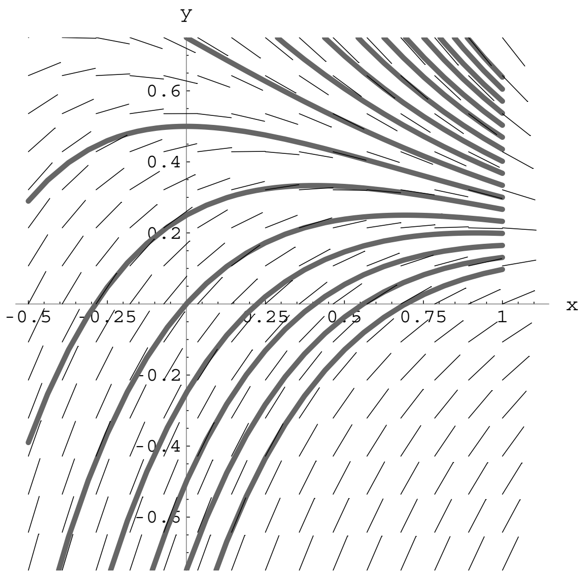


Figure 1-12 Direction field associated with $dy/dx = e^{-x} - 2y$ together with plots of several solutions

Show command in Figure 1-12. This is the purpose of direction fields: most differential equations cannot be solved by the elementary methods covered in an introductory text on differential equations.

```
In[55]:= Show[p2, p1]
```

■

Mathematica allows us to graph solutions to equations and associated direction fields that would be nearly impossible by traditional methods.

EXAMPLE 1.4.2: Graph the direction field associated with the differential equation

$$\frac{dy}{dx} = \frac{\cos y - y \cos x}{x \sin y + \sin x - 1}.$$

SOLUTION: As in the previous example, we use PlotVectorField to graph the direction field associated with the equation, in this case for

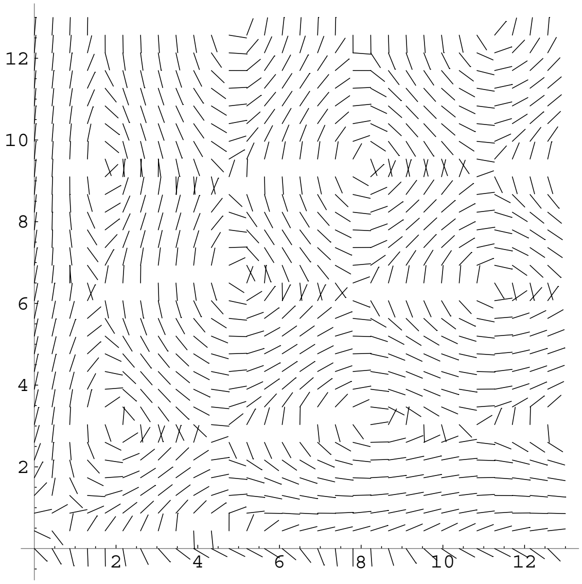


Figure 1-13 Direction field associated with $dy/dx = (\cos y - y\cos x)/(x\sin y + \sin x - 1)$

$0 \le x \le 4\pi$ and $0 \le y \le 4\pi$ in Figure 1-13, naming the resulting graphics object `p1`.

If you have already loaded the **PlotField** package during your *current* Mathematica session and have not quit Mathematica, you need not reload the **PlotField** package.

```
In[56]:= << Graphics`PlotField`

In[57]:= p1 = PlotVectorField[{1, (Cos[y] - y Cos[x])/(x Sin[y] + Sin[x] - 1)},
            {x, 0, 4π}, {y, 0, 4π}, Frame → False,
            Axes → Automatic, AxesOrigin → {0, 0},
            AxesStyle → GrayLevel[0.5],
            ScaleFunction → (0.5&),
            PlotPoints- > 30, HeadLength- > 0]
```

In this case, we see that we are able to find a general solution of the equation with `DSolve` if we rewrite the equation as $(x\sin y + \sin x - 1)\,y' = \cos y - y\cos x$.

```
In[58]:= Clear[x, y]

         gensol =
           DSolve[(x Sin[y[x]] + Sin[x] - 1)y'[x] ==
             Cos[y[x]] - y[x] Cos[x], y[x], x]
```

```
Solve :: tdep : The equations appear
to involve the variables to be solved
for in an essentially non-algebraic way.
     Out[58]= Solve[-x Cos[y[x]] - y[x]
              + Sin[x] y[x] == C[1], y[x]]
```

The result indicates that a general solution is $y \sin x - x \cos y - y = C$.

To graph solutions for various values of C, we note that the graph of the equation $y \sin x - x \cos y - y = C$ for various values of C is the same as the graph of the level curves of $z = f(x, y) = y \sin x - x \cos y - y$ for various values of z.

```
In[59]:= toplot = -x Cos[y[x]] - y[x]
         + Sin[x] y[x] /. y[x]- > y
Out[59]= -y - x Cos[y] + y Sin[x]
```

We now generate several level curves of `toplot` for $0 \leq x \leq 4\pi$ and $0 \leq y \leq 4\pi$ with `ContourPlot` in Figure 1-14, naming the resulting graphics object `cp1`.

```
In[60]:= cp1 = ContourPlot[toplot, {x, 0, 4π}, {y, 0, 4π},
                ContourShading → False, Frame → False,
                PlotPoints → 150, Axes → Automatic,
                AxesOrigin → {0, 0}, AxesStyle →
                GrayLevel[0.5], Contours → 20,
                ContourStyle → {{GrayLevel[0.4],
                Thickness[0.01]}}]
```

Finally, we use `Show` to display `cp1` and `p1` together in Figure 1-15. From these graphs we see that the behavior of the solution depends on the initial condition. Some solutions follow a closed path while others do not.

```
In[61]:= Show[cp1, p1]
```

■

Mathematica is particularly useful in graphing the direction field associated with a system of equations. After the **PlotField** package has been loaded by entering `<<Graphics'PlotField'`, the command

```
PlotVectorField[{f[x,y],g[x,y]},{x,x0,x1},{y,y0,y1}]
```

graphs the direction field associated with the system $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$ for $x_0 \leq x \leq x_1$ and $y_0 \leq y \leq y_1$.

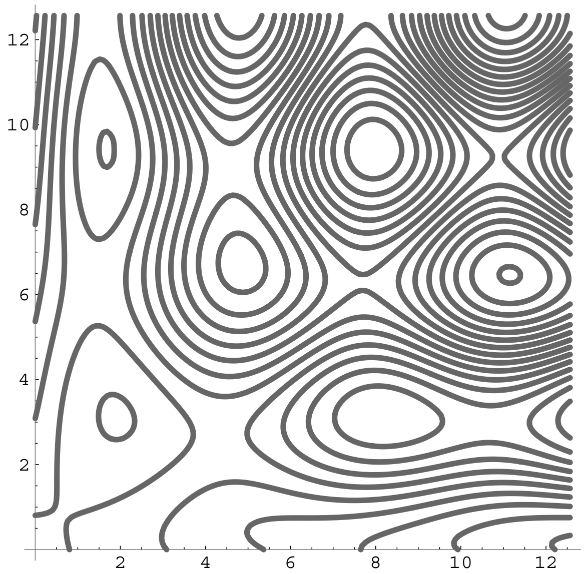


Figure 1-14 Various solutions of $dy/dx = (\cos y - y\cos x)/(x\sin y + \sin x - 1)$

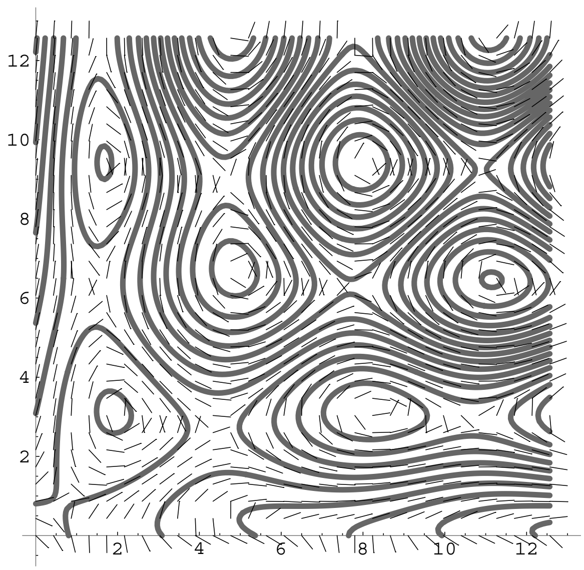


Figure 1-15 Direction field and various solutions of $dy/dx = (\cos y - y\cos x)/(x\sin y + \sin x - 1)$

EXAMPLE 1.4.3 (Competing Species): Under certain assumptions the system of equations

$$\begin{cases} \dfrac{dx}{dt} = x\,(a - b_1 x - b_2 y) \\ \dfrac{dy}{dt} = y\,(c - d_1 x - d_2 y) \end{cases} \tag{1.7}$$

where a, b_1, b_2, c, d_1, and d_2 represent positive constants, can be used to model the population of two species, represented by $x(t)$ and $y(t)$, competing for a common food supply. Graph the direction field associated with the system if (a) $a = 1$, $b_1 = 2$, $b_2 = 1$, $c = 1$, $d_1 = 0.75$, and $d_2 = 2$; and (b) $a = 1$, $b_1 = 1$, $b_2 = 1$, $c = 0.67$, $d_1 = 0.75$, and $d_2 = 1$.

SOLUTION: After identifying $f(x, y) = x\,(a - b_1 x - b_2 y)$ and $g(x, y) = y\,(c - d_1 x - d_2 y)$, we define f and g.

```
In[62]:= Clear[g, f, x, y]

         f[x_, y_] = x (a - b1x - b2y);

         g[x_, y_] = y (c - d1x - d2y);
```

Then, for (a) we define $a = 1$, $b_1 = 2$, $b_2 = 1$, $c = 1$, $d_1 = 0.75$, and $d_2 = 2$, load the **PlotField** package, and graph the direction field associated with the system for $0 \le x \le 1$ and $0 \le y \le 1$ with `PlotVectorField` in Figure 1-16.

Remember that if you have previously loaded the **PlotField** package during your current Mathematica session, you do not need to reload the **PlotField** package.

```
In[63]:= a = 1; b1 = 2; b2 = 1; c = 1; d1 = 0.75; d2 = 2;

In[64]:= << Graphics`PlotField`

         PlotVectorField[{f[x, y], g[x, y]}, {x, 0, 1},
           {y, 0, 1}, Frame → False, Axes → Automatic,
           AxesLabel → {x, y}, AxesOrigin → {0, 0},
           AxesStyle → GrayLevel[0.5],
           ScaleFunction → (1&)]
```

In this case, we see that both the species appear to approach some equilibrium population. In fact, later we will see that this equilibrium

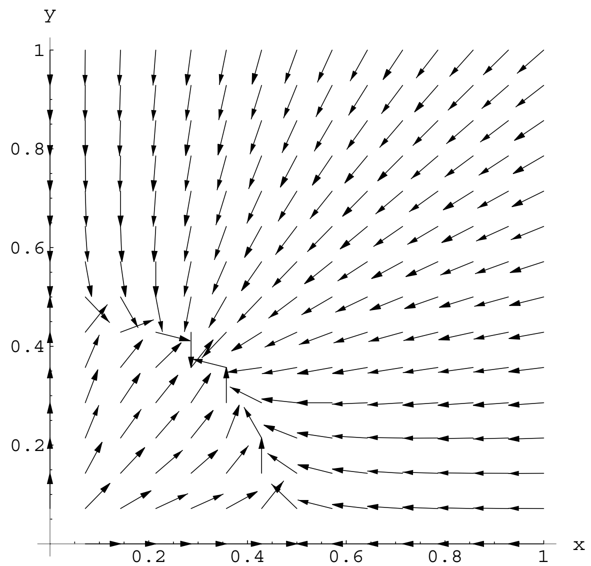


Figure 1-16 Direction field associated with the competing species system using the parameter values in (a)

population is obtained by solving the system of equations $\begin{cases} a - b_1 x - b_2 y = 0 \\ c - d_1 x - d_2 y = 0 \end{cases}$ for x and y. For (b), we redefine $a = 1, b_1 = 1, b_2 = 1,$ $c = 0.67, d_1 = 0.75,$ and $d_2 = 1$ and then re-enter the `PlotVectorField` command. See Figure 1-17.

```
In[65]:= a = 1; b1 = 1; b2 = 1; c = 0.67; d1 = 0.75; d2 = 1;

In[66]:= PlotVectorField[{f[x, y], g[x, y]}, {x, 0, 1},
           {y, 0, 1}, Frame → False, Axes → Automatic,
           AxesLabel → {x, y}, AxesOrigin → {0, 0},
           AxesStyle → GrayLevel[0.5],
           ScaleFunction → (1&)]
```

In this case, we see that it appears as though the species with population given by $y(t)$ eventually dies out while the species with population given by $x(t)$ eventually dominates and approaches some equilibrium population. Later, we will see that this is true and the equilibrium population of the species with population given by $x(t)$ will be found by

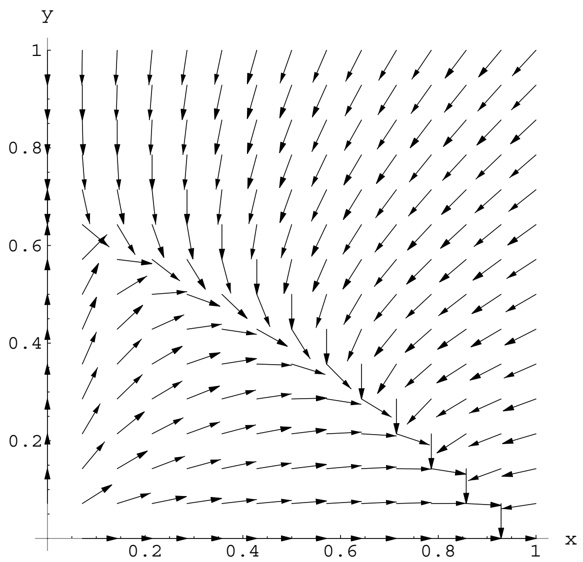


Figure 1-17 Direction field associated with the competing species system using the parameter values in (b)

computing the limit as $t \to \infty$ of the solution to the differential equation $dx/dt = ax - b_1 x^2$.

■

Often, we can generate the direction field of a higher-order equation by rewriting it as a system of first-order equations.

Also see Example 1.3.5.

EXAMPLE 1.4.4 (Rayleigh's Equation): Write **Rayleigh's equation**, (1.6), as a system of two first-order equations. Graph the direction field associated with the resulting system on the rectangle $[-4, 4] \times [-4, 4]$.

SOLUTION: We write Rayleigh's equation as a system by letting $y = x'$. Then Rayleigh's equation, (1.6), becomes

$$y' = x'' = -\left[\frac{1}{3}\left(x'\right)^2 - 1\right]x' - x = -\left(\frac{1}{3}y^2 - 1\right)y - x$$

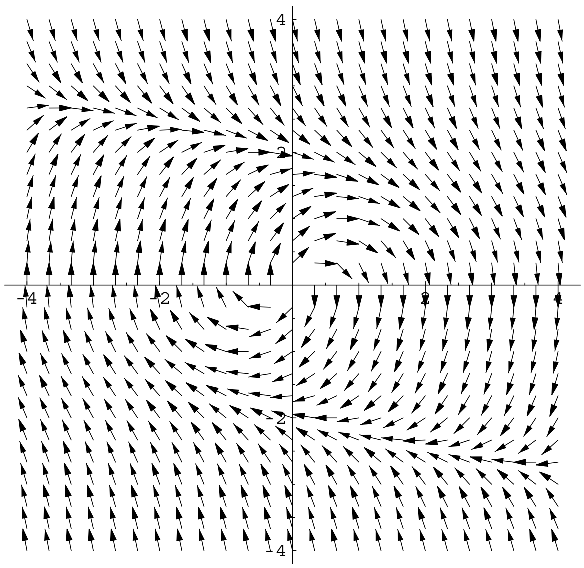


Figure 1-18 Direction field associated with the Rayleigh system

so Rayleigh's equation is equivalent to the system

$$\begin{cases} x' = y \\ y' = -\left(\frac{1}{3}y^2 - 1\right)y - x. \end{cases}$$

The direction field associated with this system is then graphed with `PlotVectorField` in Figure 1-18.

```
In[67]:= << Graphics`PlotField`

         pvf = PlotVectorField[{y, -(1/3y^2 - 1)y - x},
                {x, -4, 4}, {y, -4, 4}, Frame- > False,
                Axes- > Automatic, AxesOrigin- > {0, 0},
                ScaleFunction- > (1&), PlotPoints- > 25]
```

In the direction field, we see that solutions appear to tend to a closed curve, C. We can accurately approximate C. First, we use `NDSolve` to approximate the solution to the equation if (a) $x(0) = 1$, $y(0) = 0$; (b) $x(0) = 0.1$, $y(0) = 0$; (c) $x(0) = 0$, $y(0) = 1.9$; and (d) $x(0) = -4$, $y(0) = 4$.

```
In[68]:= numsol1 =
          NDSolve[{x'[t] == y[t],
            y'[t] == -(1/3y[t]^2 - 1)y[t] - x[t],
            x[0] == 1, y[0] == 0}, {x[t], y[t]},
            {t, 0, 15}]

Out[68]= {{x[t] → InterpolatingFunction[{{0., 15.}},
            <>][t], y[t] → InterpolatingFunction
              [{{0., 15.}}, <>][t]}}

In[69]:= numsol2 =
          NDSolve[{x'[t] == y[t],
            y'[t] == -(1/3y[t]^2 - 1)y[t] - x[t],
            x[0] == 0.1, y[0] == 0}, {x[t], y[t]},
            {t, 0, 15}]

Out[69]= {{x[t] → InterpolatingFunction[{{0., 15.}},
            <>][t], y[t] → InterpolatingFunction
              [{{0., 15.}}, <>][t]}}

In[70]:= numsol3 =
          NDSolve[{x'[t] == y[t],
            y'[t] == -(1/3y[t]^2 - 1)y[t] - x[t],
            x[0] == 0, y[0] == 1.9},
            {x[t], y[t]}, {t, 0, 15}]

Out[70]= {{x[t] → InterpolatingFunction[{{0., 15.}},
            <>][t], y[t] → InterpolatingFunction
              [{{0., 15.}}, <>][t]}}

In[71]:= numsol4 =
          NDSolve[{x'[t] == y[t],
              y'[t] == -(1/3y[t]^2 - 1)y[t] - x[t],
              x[0] == -4, y[0] == 4},
              {x[t], y[t]}, {t, 0, 15}]

Out[71]= {{x[t] → InterpolatingFunction[{{0., 15.}},
            <>][t], y[t] → InterpolatingFunction
              [{{0., 15.}}, <>][t]}}
```

We then graph all four solutions for $0 \leq t \leq 15$ with `ParametricPlot` and show these graphs together with the direction field in Figure 1-19.

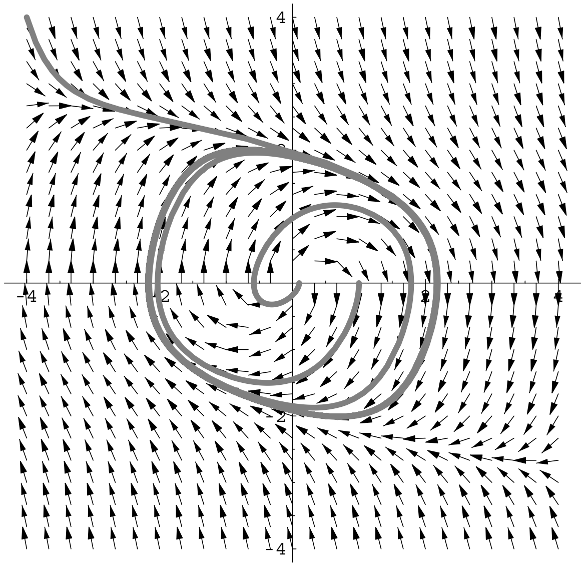


Figure 1-19 Solutions of Rayleigh's equation tend to a limit cycle

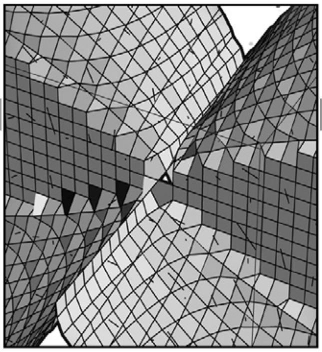
In the graph, we see that the graph of solution (c) corresponds to C; the graphs of the other solutions all tend to C, which is called a *limit cycle*.

```
In[72]:= parplot = ParametricPlot[
            Evaluate[{x[t],y[t]}/.
               {numsol1,numsol2,numsol3,numsol4}],
            {t,0,15},Compiled- > False,
            PlotStyle- > {{GrayLevel[0.5],
            Thickness[0.01]}},
            DisplayFunction- > Identity];

In[73]:= Show[pvf,parplot]
```

■

First-Order Ordinary Differential Equations

2



We will devote a considerable amount of time in this text to developing explicit, implicit, numerical, and graphical solutions of differential equations. In this chapter we introduce frequently encountered forms of first-order ordinary differential equations and methods to construct explicit, numerical, and graphical solutions of them. Several of the equations along with the methods of solution discussed here will be used in subsequent chapters of the text.

2.1 Theory of First-Order Equations: A Brief Discussion

In order to understand the types of first-order initial-value problems that have a unique solution, the following theorem is stated.

Theorem 1 (Existence and Uniqueness). *Consider the initial-value problem*

$$\begin{cases} dy/dx = f(x, y) \\ y(x_0) = y_0. \end{cases} \tag{2.1}$$

If f and $\partial f/\partial y$ are continuous functions on the rectangular region R,

See texts like [6], [7], or [3].

$$R = \{(x, y) | a < x < b,\ c < y < d\},$$

containing the point (x_0, y_0), *there exists an interval* $|x - x_0| < h$ *centered at* x_0 *on which there exists one and only one solution to the differential equation that satisfies the initial condition.*

Often, we can use the command

```
DSolve[{y'[x]==f[x,y[x]],y[x0]==y0},y[x],x]
```

to solve the initial-value problem (2.1);

```
DSolve[y'[x]==f[x,y[x]],y[x],x]
```

attempts to find a general solution of $y' = f(x, y)$.

EXAMPLE 2.1.1: Solve the initial-value problem

$$\begin{cases} dy/dx = x/y \\ y(0) = 0. \end{cases}$$

Does this result contradict the Existence and Uniqueness Theorem?

SOLUTION: This equation is solved with `DSolve` to determine the family of solutions $y^2 - x^2 = C$.

```
In[74]:= Clear[x, y]
         DSolve[y'[x] == x/y[x], y[x], x]

Out[74]= {{y[x] → -Sqrt[x^2 + 2 C[1]]}, {y[x] → Sqrt[x^2 + 2 C[1]]}}
```

We note that the graph of $y^2 - x^2 = C$ for various values of C is the same as the graph of the level curves of $f(x, y) = y^2 - x^2$. Members of this family corresponding to $C = -40, -38, \ldots, 38, 40$ are graphed with `ContourPlot` in Figure 2-1.

```
In[75]:= cvals = Table[i, {i, -40, 40, 2}];

In[76]:= ContourPlot[y^2 - x^2, {x, -6, 6}, {y, -6, 6},
           PlotPoints → 120, Contours- > cvals,
           ContourShading → False]
```

Application of the initial condition yields $0^2 - 0^2 = C$, so $C = 0$. Therefore, solutions that pass through $(0, 0)$, satisfy $y^2 - x^2 = 0$, so there

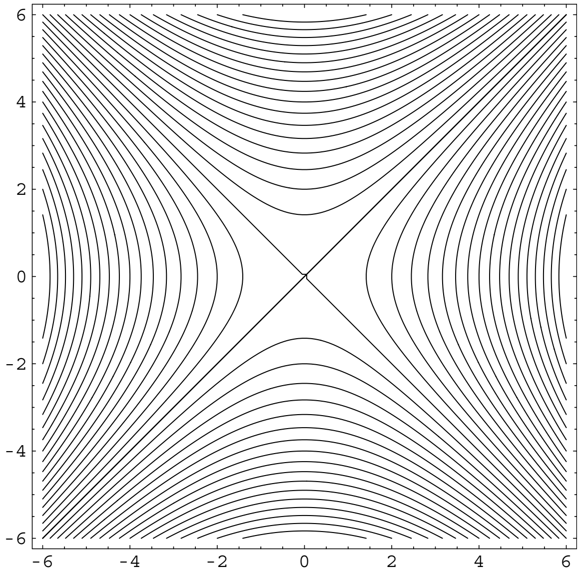


Figure 2-1 Plot of $f(x, y) = C$ for various values of C

are two solutions, $y = x$ and $y = -x$, that satisfy the differential equation and the initial condition. Although more than one solution satisfies this initial-value problem, the Existence and Uniqueness Theorem is *not* contradicted because the function $f(x, y) = x/y$ is not continuous at the point (0, 0); the requirements of the theorem are not met.

■

EXAMPLE 2.1.2: Verify that the initial-value problem $\{dy/dx = y, y(0) = 1\}$ has a unique solution.

SOLUTION: In this case, $f(x, y) = y$, $x_0 = 0$, and $y_0 = 1$. Hence, both f and $\partial f/\partial y = 1$ are continuous on all rectangular regions containing the point $(x_0, y_0) = (0, 1)$. Therefore by the Existence and Uniqueness Theorem, there exists a unique solution to the differential equation that satisfies the initial condition $y(0) = 1$.

We can verify this by solving the initial-value problem. The unique solution is $y = e^x$, which is computed with `DSolve` and then graphed

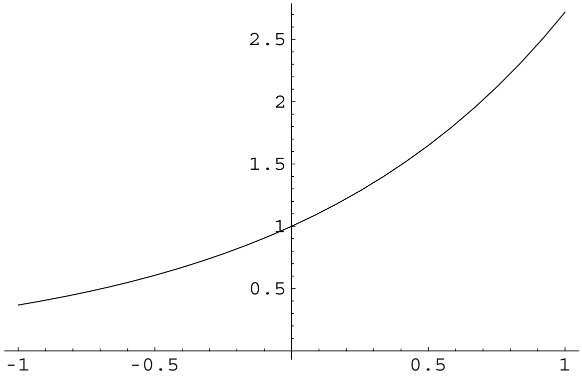


Figure 2-2 Plot of $y = e^x$

with `Plot` in Figure 2-2. Notice that the graph passes through the point (0, 1), as required by the initial condition.

```
In[77]:= Clear[x, y, sol]

         sol = DSolve[{y'[x] == y[x], y[0] == 1}, y[x], x]
Out[77]= {{y[x] → e^x}}

In[78]:= Plot[y[x] /. sol, {x, -1, 1}]
```

■

EXAMPLE 2.1.3: Show that the initial-value problem

$$\begin{cases} x\dfrac{dy}{dx} - y = x^2 \cos x \\ y(0) = 0 \end{cases}$$

has infinitely many solutions.

SOLUTION: Writing $xy' - y = x^2 \cos x$ in the form $y' = f(x, y)$ results in

$$\frac{dy}{dx} = \frac{x^2 \cos x + y}{x}$$

and because $f(x, y) = \left(x^2 \cos x + y\right)/x$ is not continuous on an interval containing $x = 0$, the Existence and Uniqueness Theorem does not guarantee the existence or uniqueness of a solution. In fact, using `DSolve` we see that a general solution of the equation is $y = x \sin x + Cx$ and for every value of C, $y(0) = 0$.

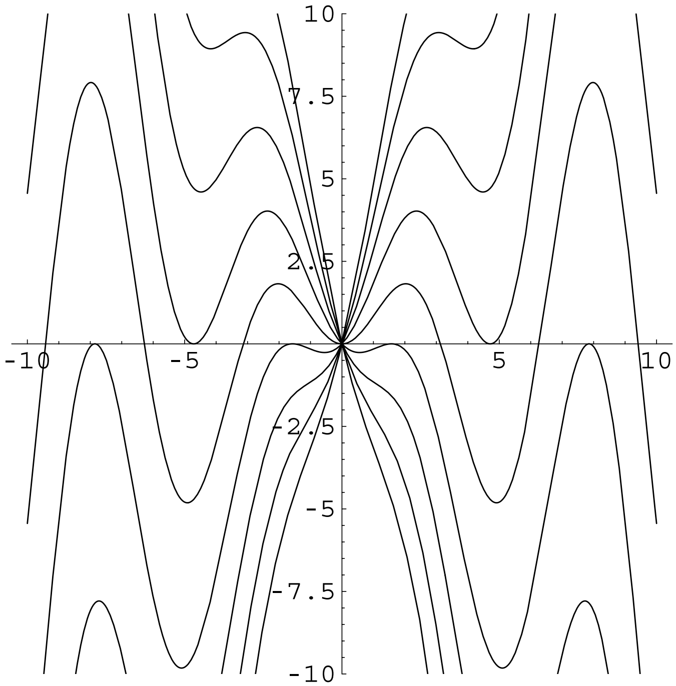


Figure 2-3 Every solution satisfies $y(0) = 0$

```
In[79]:= Clear[y]

         sol = DSolve[x y'[x] - y[x] == x^2 Cos[x], y[x], x]
Out[79]= {{y[x] → x C[1] + x Sin[x]}}
```

We confirm this graphically by graphing several solutions. First, we use `Table` to define `toplot` to be a set of functions obtained by replacing the arbitrary constant in $y(x)$ by $-4, -3, \ldots, 3, 4$.

```
In[80]:= toplot = Table[sol[[1, 1, 2]]/.C[1]- > i,
           {i, -4, 4}]
Out[80]= {-4 x + x Sin[x], -3 x + x Sin[x], -2 x + x Sin[x],
           -x + x Sin[x], x Sin[x], x + x Sin[x],
           2 x + x Sin[x], 3 x + x Sin[x], 4 x + x Sin[x]}
```

These functions are then graphed with `Plot` in Figure 2-3.

```
In[81]:= Plot[Evaluate[toplot], {x, -10, 10},
           PlotRange- > {-10, 10}, AspectRatio- > 1]
```

■

2.2 Separation of Variables

Definition 5 (Separable Differential Equation). *A differential equation that can be written in the form $g(y)y' = f(x)$ or $g(y)\,dy = f(x)\,dx$ is called a* ***separable differential equation****.*

Separable differential equations are solved by collecting all the terms involving y on one side of the equation, all the terms involving x on the other side of the equation, and integrating:

$$g(y)\,dy = f(x)\,dx \Longrightarrow \int g(y)\,dy = \int f(x)\,dx + C,$$

where C is a constant.

EXAMPLE 2.2.1: Show that the equation

$$\frac{dy}{dx} = \frac{2\sqrt{y} - 2y}{x}$$

is separable, and solve by separation of variables.

SOLUTION: The equation $y' = \left(2\sqrt{y} - 2y\right)/x$ is separable because it can be written in the form

$$\frac{1}{2\sqrt{y} - 2y}dy = \frac{1}{x}dx.$$

To solve the equation, we integrate both sides and simplify. Observe that we can write this equation as

$$\int \frac{1}{2\sqrt{y}}\frac{1}{1 - \sqrt{y}}dy = \int \frac{1}{x}dx + C.$$

To evaluate the integral on the left-hand side, let $u = 1 - \sqrt{y}$ so $-du = \dfrac{1}{2\sqrt{y}}dy$. We then obtain

$$-\int \frac{1}{u}du = \int \frac{1}{x}dx + C_1$$

so that $-\ln|u| = \ln|x| + C_1$. Recall that $-\ln|u| = \ln|u|^{-1}$, so we have

$$\ln\frac{1}{|u|} = \ln|x| + C_1.$$

Using Mathematica, we have

```
In[82]:= ∫ 1/(2√y - 2y) dy
Out[82]= 2 (-1 + √y) √y Log[-1 + √y] / (2 √y - 2 y)
```

which we then simplify with `Simplify`.

Note that `Log[x]` represents the **natural logarithm function**, $y = \ln x$.

```
In[83]:= Simplify[2 (-1 + √y) √y Log[-1 + √y] / (2 √y - 2y)]
Out[83]= -Log[-1 + √y]
```

The integral on the right-hand side of the equation is computed in the same way.

```
In[84]:= ∫ 1/x dx
Out[84]= Log[x]
```

Simplification yields

$$\frac{1}{|u|} = e^{\ln|x|+C_1} = C_2|x|$$

where $C_2 = e^{C_1}$. Resubstituting we find that

$$\frac{1}{\left|1-\sqrt{y}\right|} = C_2|x| \qquad \text{or} \qquad x = \frac{C_3}{1-\sqrt{y}}.$$

Solving for y shows us that

$$\sqrt{y} - 1 = \frac{C}{x}$$

$$\sqrt{y} = \frac{x+C}{x}$$

$$y = \left(\frac{x+C}{x}\right)^2$$

is a general solution of the equation $y' = \left(2\sqrt{y} - 2y\right)/x$. We obtain the same results with Mathematica,

We use `cons` to represent the arbitrary constant C to avoid ambiguity with the built-in symbol `C`.

```
In[85]:= Solve[-Log[-1 + √y] == Log[x]
         + cons, y]//Simplify
Out[85]= {{y → e^(-2 cons) (1 + e^cons x)^2 / x^2}}
```

where `E`$^{-\mathrm{cons}}$ represents the arbitrary constant in the solution. We obtain an equivalent result with `DSolve`. Entering

```
In[86]:= Clear[x, y]

         gensol = DSolve[y'[x] == (2 Sqrt[y[x]] - 2 y[x])/x,
           y[x], x]
Solve::ifun :
  Inverse functions are being used by Solve, so some solutions may not be
found.
```

$$\text{Out[86]=}\ \left\{\left\{y[x]\rightarrow \frac{\left(e^{\frac{C[1]}{2}}+x\right)^2}{x^2}\right\}\right\}$$

finds a general solution of the equation which is equivalent to the one we obtained by hand and names the result `gensol`. The formula for the solution, which is the second part of the first part of the first part of `gensol`, is extracted from `gensol` with `gensol[[1,1,2]]`. Alternatively, if you are using Version 5, you can select, copy, and paste the result to any location in the notebook.

```
In[87]:= gensol[[1, 1, 2]]//Simplify
```

$$\text{Out[87]=}\ \frac{\left(e^{\frac{C[1]}{2}}+x\right)^2}{x^2}$$

To graph the solution for various values of `C[1]`, which represents the arbitrary constant in the formula for the solution, we use `Table` together with `ReplaceAll` (`/.`) to generate a set of functions obtained by replacing `C[1]` in the formula for the solution by i for $i = -3, -2.50, \ldots, 2.50$, and 3, naming the resulting set of functions `toplot`. We view an abbreviation of `toplot` with `Short`.

```
In[88]:= toplot = Table[gensol[[1, 1, 2]]/.C[1] →
           i, {i, -3, 3, 1/2}];

         Short[toplot, 4]
```

$$\text{Out[88]=}\ \left\{\frac{\left(\frac{1}{e^{3/2}}+x\right)^2}{x^2}, \frac{\left(\frac{1}{e^{5/4}}+x\right)^2}{x^2}, \frac{\left(\frac{1}{e}+x\right)^2}{x^2}, \frac{\left(\frac{1}{e^{3/4}}+x\right)^2}{x^2},\right.$$
$$\frac{\left(\frac{1}{\sqrt{e}}+x\right)^2}{x^2}, \frac{\left(\frac{1}{e^{1/4}}+x\right)^2}{x^2}, \frac{(1+x)^2}{x^2}, \frac{\left(e^{1/4}+x\right)^2}{x^2},$$
$$\frac{\left(\sqrt{e}+x\right)^2}{x^2}, \frac{\left(e^{3/4}+x\right)^2}{x^2}, \frac{(e+x)^2}{x^2}, \frac{\left(e^{5/4}+x\right)^2}{x^2},$$
$$\left.\frac{\left(e^{3/2}+x\right)^2}{x^2}\right\}$$

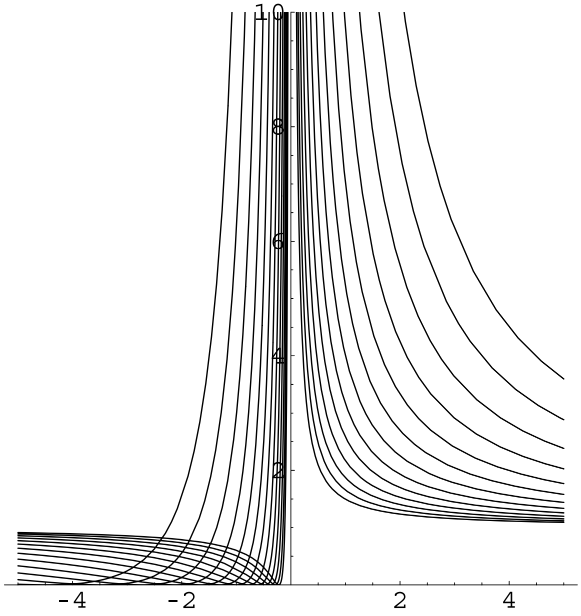


Figure 2-4 Various solutions of $y' = \left(2\sqrt{y} - 2y\right)/x$

We then graph the set of functions `toplot` with `Plot` in Figure 2-4.

```
In[89]:= Plot[Evaluate[toplot], {x, -5, 5},
           PlotRange- > {0, 10}, AspectRatio → 1]
```

■

An initial-value problem involving a separable equation is solved through the following steps.

1. Find a general solution of the differential equation using separation of variables.
2. Use the initial condition to determine the unknown constant in the general solution.

EXAMPLE 2.2.2: Solve (a) $y \cos x \, dx - \left(1 + y^2\right) dy = 0$ and (b) the initial-value problem $\left\{y \cos x \, dx - \left(1 + y^2\right) dy = 0, \ y(0) = 1\right\}$.

SOLUTION: (a) Note that this equation can be rewritten as $dy/dx = (y \cos x)/\left(1 + y^2\right)$. We first use `DSolve` to solve the equation.

```
In[90]:= sol1 = DSolve[y'[x] == y[x] Cos[x]/(1 + y[x]^2),
           y[x], x]
```

```
InverseFunction::ifun :
Inverse functions are being used. Values may be lost for multivalued
inverses.

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be
found.
```

```
Out[90]= {{y[x] → -√(ProductLog[e^(2 C[1]+2 Sin[x])])},
          {y[x] → √(ProductLog[e^(2 C[1]+2 Sin[x])])}}
```

In this case, we see that DSolve is able to solve the nonlinear equation, although the result contains the ProductLog function. Given z, the **Product Log function** returns the principal value of w that satisfies $z = we^w$. A more familiar form of the solution is found using traditional techniques. Separating and integrating gives us

$$\frac{1+y^2}{y}\,dy = \cos x\,dx$$
$$\left(\frac{1}{y}+y\right)dy = \cos x\,dx$$
$$\ln|y| + \frac{1}{2}y^2 = \sin x + C.$$

We can also use Mathematica to implement the steps necessary to solve the equation by hand. To solve the equation, we must integrate both the left and right-hand sides which we do with Integrate, naming the resulting output lhs and rhs, respectively.

```
In[91]:= lhs = Integrate[(1 + y^2)/y, y]

         rhs = Integrate[Cos[x], x]

Out[91]= y^2/2 + Log[y]

Out[91]= Sin[x]
```

Therefore, a general solution to the equation is $\ln|y| + \frac{1}{2}y^2 = \sin x + C$. We now use ContourPlot to graph $\ln|y| + \frac{1}{2}y^2 = \sin x + C$ in Figure 2-5 for various values of C by observing that the level curves of $f(x,y) = \ln|y| + \frac{1}{2}y^2 - \sin x$ correspond to the graph of $\ln|y| + \frac{1}{2}y^2 = \sin x + C$ for various values of C.

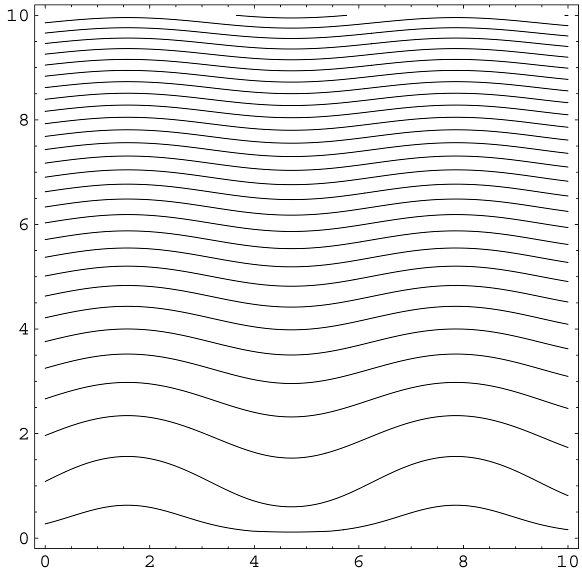


Figure 2-5 Plot of $\ln|y| + \frac{1}{2}y^2 = \sin x + C$ for various values of C

```
In[92]:= ContourPlot[lhs - rhs, {x, 0, 10}, {y, 0, 10},
           Contours- > 30, PlotPoints- > 150,
           ContourShading- > False]
```

By substituting $y(0) = 1$ into this equation, we find that $C = 1/2$, so the implicit solution is given by $\ln|y| + \frac{1}{2}y^2 = \sin x + 1/2$.

```
In[93]:= Clear[x, y, c]

In[94]:= Solve[Evaluate[lhs == rhs + c]/.
           {x- > 0, y- > 1}, c]
```

Out[94]= $\{\{c \to \frac{1}{2}\}\}$

We can also use `DSolve` to solve the initial value problem as well. The solution is then graphed in Figure 2-6 with `Plot`.

```
In[95]:= sol2 = DSolve[{y'[x] == y[x] Cos[x]/(1 + y[x]^2),
                y[0] == 1}, y[x], x]
```

Out[95]= $\{\{y[x] \to \sqrt{\text{ProductLog}[e^{1+2\,\text{Sin}[x]}]}\}\}$

```
In[96]:= Plot[y[x]/.sol2, {x, 0, 10}]
```

■

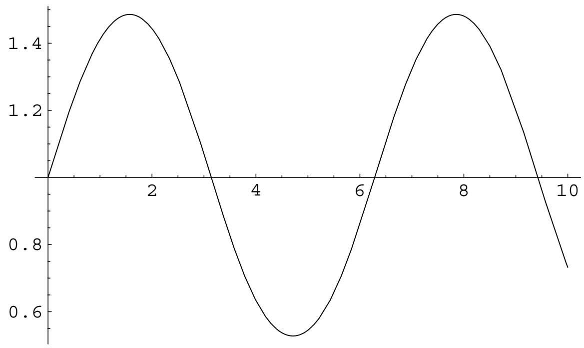


Figure 2-6 Plot of the solution that satisfies $y(0) = 1$

EXAMPLE 2.2.3: Solve each of the following equations: (a) $y' - y^2 \sin t = 0$; (b) $y' = \alpha y\left(1 - \frac{1}{K}y\right)$, K, $\alpha > 0$ constant.

SOLUTION: (a) The equation is separable:

$$\frac{1}{y^2}dy = \sin t\, dt$$
$$\int \frac{1}{y^2}dy = \int \sin t\, dt$$
$$-\frac{1}{y} = -\cos t + C$$
$$y = \frac{1}{\cos t + C}.$$

We check our result with `DSolve`.

```
In[97]:= sola = DSolve[y'[t] - y[t]^2 Sin[t] == 0, y[t], t]
Out[97]= {{y[t] → 1/(-C[1] + Cos[t])}}
```

Observe that the result is given as a list. The formula for the solution is the second part of the first part of the first part of `sola`.

```
In[98]:= sola[[1, 1, 2]]
Out[98]= 1/(-C[1] + Cos[t])
```

We then graph the solution for various values of C with `Plot` in Figure 2-7.

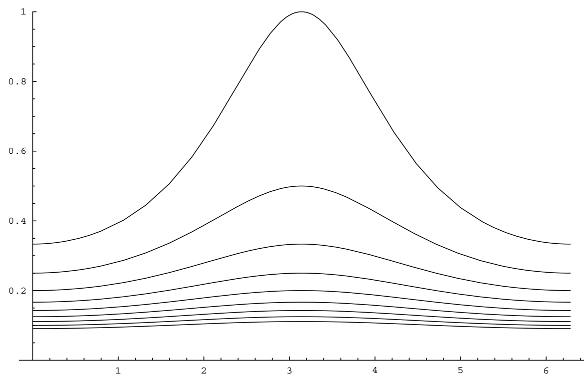


Figure 2-7 Several solutions of $y' - y^2 \sin t = 0$

```
In[99]:= toplota = Table[sola[[1,1,2]]/.
           C[1]->-i,{i,2,10}]
```

Out[99]= $\left\{\frac{1}{2+\text{Cos}[t]}, \frac{1}{3+\text{Cos}[t]}, \frac{1}{4+\text{Cos}[t]}, \frac{1}{5+\text{Cos}[t]}, \frac{1}{6+\text{Cos}[t]}, \frac{1}{7+\text{Cos}[t]}, \frac{1}{8+\text{Cos}[t]}, \frac{1}{9+\text{Cos}[t]}, \frac{1}{10+\text{Cos}[t]}\right\}$

```
In[100]:= Plot[Evaluate[toplota],{t,0,2π},
            PlotRange->{0,1},AxesOrigin->{0,0}]
```

(b) After separating variables, we use partial fractions to integrate.

$$y' = \alpha y\left(1 - \frac{1}{K}y\right)$$
$$\frac{1}{\alpha y\left(1 - \frac{1}{K}y\right)}dy = dt$$
$$\frac{1}{\alpha}\left(\frac{1}{y} + \frac{1}{K-y}\right) = dt$$
$$\frac{1}{\alpha}\left(\ln|y| - \ln|K-y|\right) = C_1 + t$$
$$\frac{y}{K-y} = Ce^{\alpha t}$$
$$y = \frac{CKe^{\alpha t}}{Ce^{\alpha t}+1}$$

`expression /. x->y` replaces all occurrences of x in `expression` by y.
`Table[a[k],{k,n,m}]` generates the list $a_n, a_{n+1}, \ldots, a_{m-1}, a_m$.
To graph the list of functions `list` for $a \le x \le b$, enter `Plot[Evaluate[list],{x,a,b}]`.

We check the calculations with Mathematica. First, we use `Apart` to find the partial fraction decomposition of $\frac{1}{\alpha y\left(1 - \frac{1}{K}y\right)}$.

```
In[101]:= s1 = Apart[1/(α y (1 - 1/k y)), y]
```

Out[101]= $\frac{1}{y\ \alpha} - \frac{1}{(-k + y)\ \alpha}$

Then, we use `Integrate` to check the integration.

```
In[102]:= s2 = Integrate[s1, y]
```

Out[102]= $\frac{\text{Log}[y]}{\alpha} - \frac{\text{Log}[-k + y]}{\alpha}$

Last, we use `Solve` to solve $\frac{1}{\alpha}(\ln|y| - \ln|K - y|) = ct$ for y.

```
In[103]:= Solve[s2 == c t, y]
```

Out[103]= $\{\{y \to \frac{e^{c\ t\ \alpha}\ k}{-1 + e^{c\ t\ \alpha}}\}\}$

We can use `DSolve` to find a general solution of the equation

```
In[104]:= solb = DSolve[y'[t] == α y[t] (1 - 1/k y[t]),
               y[t], t]
```

Out[104]= $\{\{y[t] \to \frac{e^{t\ \alpha}\ k}{e^{t\ \alpha} - e^{C[1]}}\}\}$

as well as find the solution that satisfies the initial condition $y(0) = y_0$.

```
In[105]:= solc = DSolve[{y'[t] ==  y[t] (1 -  y[t]),
               y[0] == y0}, y[t], t]
```

Out[105]= $\{\{y[t] \to \frac{e^{t}\ y0}{1 - y0 + e^{t}\ y0}\}\}$

The equation $y' = \alpha y\left(1 - \frac{1}{K}y\right)$ is called the **Logistic equation** (or **Verhulst equation**) and is used to model the size of a population that is not allowed to grow in an unbounded manner. Assuming that $y(0) > 0$, then all solutions of the equation have the property that $\lim_{t\to\infty} y(t) = K$.

Logistic growth is discussed in more detail in Section 3.2.2.

To see this, we set $\alpha = K = 1$ and use `PlotVectorField`, which is contained in the **PlotField** package that is located in the **Graphics** directory to graph the direction field associated with the equation in Figure 2-8.

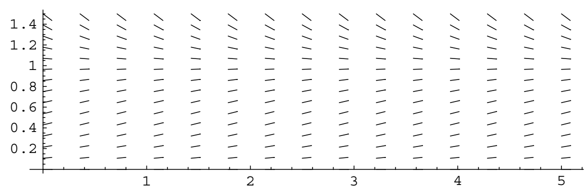


Figure 2-8 A typical direction field for the Logistic equation

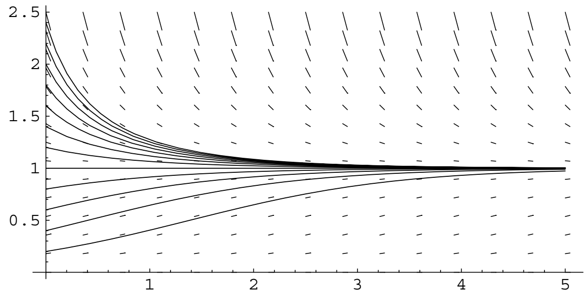


Figure 2-9 A typical direction field for the Logistic equation along with several solutions

```
In[106]:= << Graphics`PlotField`;
          pvf = PlotVectorField[{1, y(1 - y)}, {t, 0, 5},
                 {y, 0, 5/2}, HeadLength- > 0,
                 Axes- > Automatic]
```

The property is more easily seen when we graph various solutions along with the direction field as done next in Figure 2-9.

```
In[107]:= toplot = Table[solc[[1, 1, 2]]/.y0- > i/5,
                   {i, 1, 12}];
          sols = Plot[Evaluate[toplot],
                 {t, 0, 5}, DisplayFunction- > Identity];
          Show[pvf, sols]
```

■

Application: Kidney Dialysis

The primary purpose of the kidney is to remove waste products, like urea, creatinine, and excess fluid, from blood. When kidneys are not working properly, wastes accumulate in the blood; when toxic levels are reached, death is certain. The leading causes of chronic kidney failure in the United States are hypertension (high blood pressure) and diabetes mellitus. In fact, one-quarter of all patients requiring **kidney dialysis** have diabetes. Fortunately, **kidney dialysis** removes waste products from the blood of patients with improperly working kidneys. During the hemodialysis process, the patient's blood is pumped through a **dialyser**, usually at a rate of 1 to 3 deciliters per minute. The patient's blood is separated from the "cleaning fluid" by a semi-permeable membrane, which permits wastes (but not blood cells) to diffuse to the cleaning fluid; the cleaning fluid contains some substances beneficial to the body which diffuse to the blood. The "cleaning fluid," called the **dialysate**, is flowing in the *opposite* direction as the blood, usually at a

Sources: D. N. Burghes and M. S. Borrie, *Modeling with Differential Equations*, Ellis Horwood Limited, pp. 41–45. Joyce M. Black and Esther Matassarin-Jacobs, *Luckman and Sorensen's Medical–Surgical Nursing: A Psychophysiologic Approach*, Fourth Edition, W. B. Saunders Company (1993), pp. 1509–1519, 1775–1808.

rate of 2 to 6 deciliters per minute. Waste products from the blood diffuse to the dialysate through the membrane at a rate proportional to the difference in concentration of the waste products in the blood and dialysate. If we let $u(x)$ represent the concentration of wastes in blood, $v(x)$ represent the concentration of wastes in the dialysate, where x is the distance along the dialyser, Q_D represent the flow rate of the dialysate through the machine, and Q_B represent the flow rate of the blood through the machine, then

$$\begin{cases} Q_B u' = -k(u-v) \\ -Q_D v' = k(u-v) \end{cases}$$

where k is the proportionality constant.

If we let L denote the length of the dialyser and the initial concentration of wastes in the blood is $u(0) = u_0$ while the initial concentration of wastes in the dialysate is $v(L) = 0$, then we must solve the initial-value problem

$$\begin{cases} Q_B u' = -k(u-v) \\ -Q_D v' = k(u-v) \\ u(0) = u_0, \; v(L) = 0. \end{cases}$$

Solving the first equation for u' and the second equation for $-v'$, we obtain the equivalent system

$$\begin{cases} u' = -\dfrac{k}{Q_B}(u-v) \\ -v' = \dfrac{k}{Q_D}(u-v) \\ u(0) = u_0, \; v(L) = 0. \end{cases}$$

Adding these two equations results in a separable (and linear) equation in $u - v$,

$$\begin{aligned} u' - v' &= -\frac{k}{Q_B}(u-v) + \frac{k}{Q_D}(u-v) \\ (u-v)' &= -\left(\frac{k}{Q_B} - \frac{k}{Q_D}\right)(u-v). \end{aligned}$$

Let $\alpha = k/Q_B - k/Q_D$ and $y = u - v$. Then we must solve the separable equation $y' = -\alpha y$, which is done with `DSolve`, naming the resulting output `step1`. We then name `y` the result obtained in `step1` by extracting the formula for `y[x]` from `step1` with `Part` (`[[...]]`) and replacing `C[1]` by `c` with `ReplaceAll` (`/.`).

```
In[108]:= Clear[x, y]

          step1 = DSolve[y'[x] == -α y[x], y[x], x]
Out[108]= {{y[x] → e^(-x α) C[1]}}
```

```
In[109]:= y = step1[[1, 1, 2]]/.C[1] → c
Out[109]= c e^(-x α)
```

Using the facts that $u' = -\frac{k}{Q_B}(u - v)$ and $v = u - y$, we are able to use `DSolve` to find $u(x)$.

```
In[110]:= step2 = DSolve[{u'[x] == -(k/Q_B) c e^(-x α), u[0] == u0},
            u[x], x]
Out[110]= {{u[x] → (e^(-x α) (c k - c e^(x α) k + e^(x α) u0 α Q_B))/(α Q_B)}}
```

Because $y = u - v$, $v = u - y$. Consequently, because $v(L) = 0$ we are able to compute `c`

Note that we use `capl` to represent L.

```
In[111]:= leftside = (e^(-x α) (c k + e^(x α) α (u0 - (c k)/(α Q_B)) Q_B))/(α Q_B) - y/.
            x → capl
Out[111]= -c e^(-capl α) + (e^(-capl α) (c k + e^(capl α) α (u0 - (c k)/(α Q_B)) Q_B))/(α Q_B)

In[112]:= cval = Solve[leftside == 0, c]
Out[112]= {{c → (e^(capl α) u0 α Q_B)/(-k + e^(capl α) k + α Q_B)}}
```

and determine u and v. Next, we substitute the value of C into the formula for u and v.

```
In[113]:= u = Simplify[(e^(-x α) (c k + e^(x α) α (u0 - (c k)/(α Q_B)) Q_B))/(α Q_B)/.
            cval[[1]]]
Out[113]= (u0 ((-1 + e^((capl-x) α)) k + α Q_B))/((-1 + e^(capl α)) k + α Q_B)

In[114]:= v = Simplify[u - y/.cval[[1]]]
Out[114]= (e^(-x α) (e^(capl α) - e^(x α)) u0 (k - α Q_B))/((-1 + e^(capl α)) k + α Q_B)
```

For example, in healthy adults, typical urea nitrogen levels are 11 to 23 milligrams per deciliter, while serum creatinine levels range from 0.6 to 1.2 milligrams per deciliter and the total volume of blood is 4 to 5 liters.

Suppose that hemodialysis is performed on a patient with urea nitrogen level of 34 mg/dl and serum creatinine level of 1.8 using a dialyser with $k = 2.25$ and $L = 1$. If the flow rate of blood, Q_B, is 2 dl/minute while the flow rate of the dialysate, Q_D, is 4 dl/minute, will the level of wastes in the patient's blood reach normal levels after dialysis is performed?

After defining the appropriate constants, we evaluate u and v

```
In[115]:= α = k/Q_B - k/Q_D;

           k = 2.25;

        capl = 1;

          Q_B = 2;

          Q_D = 4;

          u0 = 34 + 1.8;

           u

           v
Out[115]= 12.6776 (1.125 + 2.25 (-1 + e^(0.5625 (1-x))))
Out[115]= 14.2623 e^(-0.5625 x) (1.75505 - e^(0.5625 x))
```

and then graph u and v on the interval [0,1] with `Plot` in Figure 2-10. Remember that the dialysate is moving in the direction opposite the blood. Thus, we see from the graphs that as levels of waste in the blood decrease, levels of waste in the dialysate increase and at the end of the dialysis procedure, levels of waste in the blood are within normal ranges.

```
In[116]:= p1 = Plot[u, {x, 0, 1}, DisplayFunction → Identity];

          p2 = Plot[v, {x, 0, 1}, DisplayFunction → Identity];
               Show[GraphicsArray[{p1, p2}]]
```

Typically, hemodialysis is performed 3 to 4 hours at a time 3 or 4 times per week. In some cases, a kidney transplant can free patients from the restrictions of dialysis. Of course, transplants have other risks not necessarily faced by those on dialysis;

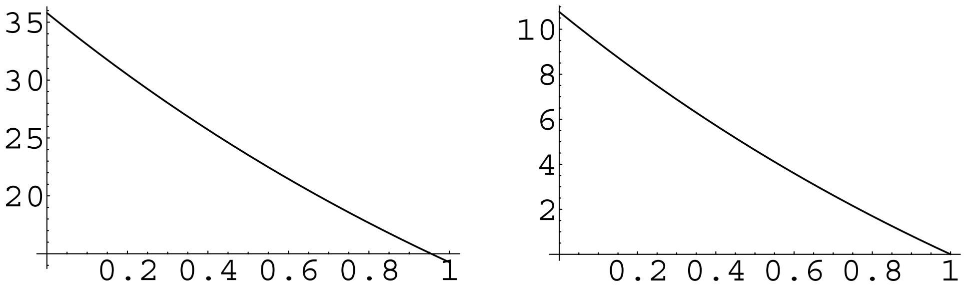


Figure 2-10 Remember that the dialysate moves in the opposite direction to the blood

the number of available kidneys also affects the number of transplants performed. For example, in 1991 over 130,000 patients were on dialysis while only 7000 kidney transplants had been performed.

2.3 Homogeneous Equations

Definition 6 (Homogeneous Differential Equation). *A differential equation that can be written in the form*

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

where

$$M(tx, ty) = t^n M(x, y) \qquad \text{and} \qquad N(tx, ty) = t^n N(x, y)$$

is called a ***homogeneous differential equation of degree*** *n.*

It is a good exercise to show that an equation is homogeneous if we can write it in either of the forms $dy/dx = F(y/x)$ or $dy/dx = G(x/y)$.

EXAMPLE 2.3.1: Show that the equation $\left(x^2 + xy\right) dx - y^2\,dy = 0$ is homogeneous.

SOLUTION: Let $M(x, y) = x^2 + xy$ and $N(x, y) = -y^2$. Because $M(tx, ty) = (tx)^2 + (tx)(ty) = t^2\left(x^2 + xy\right) = t^2 M(x, y)$ and $N(tx, ty) = -t^2 y^2 = t^2 N(x, y)$, the equation $\left(x^2 + xy\right) dx - y^2\,dy = 0$ is homogeneous of degree two.

■

Homogeneous equations can be reduced to separable equations by either of the substitutions

$$y = ux \qquad \text{or} \qquad x = vy.$$

Generally, use the substitution $y = ux$ if $N(x, y)$ is less complicated than $M(x, y)$ and use $x = vy$ if $M(x, y)$ is less complicated than $N(x, y)$. If a difficult integration problem is encountered after a substitution is made, try the other substitution to see if it yields an easier problem.

EXAMPLE 2.3.2: Solve the equation $\left(x^2 - y^2\right) dx + xy\, dy = 0$.

SOLUTION: In this case, $M(x, y) = x^2 - y^2$ and $N(x, y) = xy$. Then, $M(tx, ty) = t^2 M(x, y)$ and $N(tx, ty) = t^2 N(x, y)$ which means that $\left(x^2 - y^2\right) dx + xy\, dy = 0$ is a homogeneous equation of degree two. Assume $x = vy$. Then, $dx = v\, dy + y\, dv$ and substituting into the equation and simplifying yields

$$\begin{aligned} 0 &= \left(x^2 - y^2\right) dx + xy\, dy \\ &= \left(v^2 y^2 - y^2\right)(v\, dy + y\, dv) + vy \cdot y\, dy \\ &= \left(v^2 - 1\right)(v\, dy + y\, dv) + v\, dy \\ &= v^3\, dy + y\left(v^2 - 1\right) dv. \end{aligned}$$

We solve this equation by rewriting it in the form

$$\frac{1}{y} dy = \frac{1 - v^2}{v^3} dv = \left(\frac{1}{v^3} - \frac{1}{v}\right) dv$$

and integrating. This yields

$$\ln|y| = -\frac{1}{2v^2} - \ln|v| + C_1,$$

which can be simplified as

$$\ln|vy| = -\frac{1}{2v^2} + C_1, \qquad \text{so} \qquad vy = Ce^{-1/(2v^2)}, \text{ where } C = \pm e^{C_1}.$$

Because $x = vy$, $v = x/y$, and resubstituting into the above equation yields

$$x = Ce^{-y^2/(2x^2)}$$

as a general solution of the equation $\left(x^2 - y^2\right) dx + xy\, dy = 0$. We see that `DSolve` is able to solve the equation as well.

```
In[117]:= Clear[x, y]

In[118]:= gensol = DSolve[x^2 - y[x]^2 + x y[x] y'[x] == 0,
             y[x], x]
Out[118]= {{y[x] → -Sqrt[x^2 (C[1] - 2 Log[x])]},
            {y[x] → Sqrt[x^2 (C[1] - 2 Log[x])]}}
```

The result means that a general solution of the equation is $y^2 = x^2(C - 2\ln|x|)$. We can graph this implicit solution for various values of C by solving this equation for C

```
In[119]:= f = Solve[y^2 == gensol[[1, 1, 2]]^2, C[1]]
Out[119]= {{C[1] → (y^2 + 2 x^2 Log[x])/x^2}}
In[120]:= f[[1, 1, 2]]
Out[120]= (y^2 + 2 x^2 Log[x])/x^2
In[121]:= f[[1, 1, 2]]
Out[121]= (y^2 + 2 x^2 Log[x])/x^2
```

and then noting that graphs of the equation $y^2 = x^2(C - 2\ln|x|)$ for various values of C are the same as the graphs of the level curves of the function $f(x, y) = \left(y^2 + 2x^2 \ln|x|\right)/\left(2x^2\right)$.

The `ContourPlot` command graphs several level curves $C = f(x, y)$, C a constant, of the function $z = f(x, y)$. We may instruct Mathematica to graph the level curves of $z = f(x, y)$ for particular values of C by including the `Contours` option. For example, the level curves of $f(x, y) = \left(y^2 + 2x^2 \ln|x|\right)/\left(2x^2\right)$ that intersect the x-axis at $x = 1, 3/2, 2, \ldots, 19/2$, and 10 are the contours with values obtained by replacing each occurrence of y in $f(x, y)$ by 0 and x by $1, 3/2, 2, \ldots, 19/2$, and 10 which we do now with `Table` and `ReplaceAll` (`/.`), naming the resulting set of ten numbers `contourvals`.

```
In[122]:= contourvals = Table[(f[[1, 1, 2]])/.
            {x → i, y → 0}, {i, 1, 10, 1/2}];
```

Then, entering

```
In[123]:= cp1 = ContourPlot[f[[1, 1, 2]], {x, 0.01, 10},
                {y, -5, 5}, PlotPoints → 150, Frame →
                False, Contours → contourvals,
                Axes → Automatic, AxesOrigin → {0, 0},
                ContourStyle- > {{GrayLevel[0.4],
                Thickness[0.01]}}, ContourShading →
                False, DisplayFunction- > Identity];
```

graphs several level curves of $z = f(x, y)$ for $0.01 \le x \le 10$ and $-5 \le y \le 5$ and names the resulting graphics object `cp1`. `cp1` is not displayed because we include the `DisplayFunction->Identity` option in the `ContourPlot` command. The option `Contours->contourvals` instructs Mathematica to draw contours with values given in the list

We avoid $x = 0$ because $f(x, y)$ is undefined if $x = 0$.

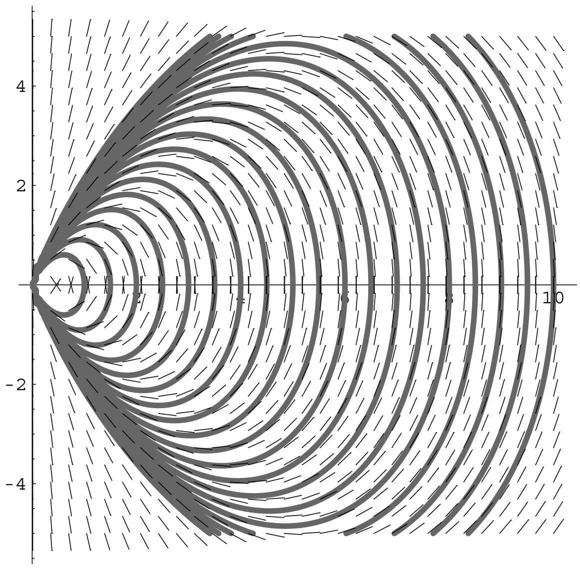


Figure 2-11 Various solutions and direction field for the homogeneous equation $\left(x^2 - y^2\right) dx + xy\, dy = 0$

of numbers `contourvals`. We use `PlotVectorField` to graph the direction field associated with the equation on the rectangle

```
In[124]:= << Graphics`PlotField`

In[125]:= pvf = PlotVectorField[{1, (y^2 - x^2)/(x y)},
              {x, 0.01, 10}, {y, -5, 5}, Frame →
              False, Axes → Automatic, AxesOrigin →
              {0, 0}, AxesStyle → GrayLevel[0.5],
              ScaleFunction → (0.5&),
              PlotPoints- > 30, HeadLength- > 0,
              DisplayFunction- > Identity];
```

and then display `cp1` and the direction field together with `Show` in Figure 2-11.

```
In[126]:= Show[cp1,
            pvf, DisplayFunction- > $DisplayFunction]
```

■

EXAMPLE 2.3.3: Solve $\left(y^2+2xy\right)dx - x^2 dy = 0$.

SOLUTION: In this case, letting $F(t) = t^2 + 2t$, we note that $dy/dx = F(y/x) = (y/x)^2 + 2(y/x)$ so the equation is homogeneous.

Let $y = ux$. Then, $dy = u\,dx + x\,du$. Substituting into $\left(y^2+2xy\right)dx - x^2 dy = 0$ and separating gives us

$$\begin{aligned}
\left(y^2+2xy\right)dx - x^2 dy &= 0\\
\left(u^2x^2+2ux^2\right)dx - x^2(u\,dx + x\,du) &= 0\\
\left(u^2+2u\right)dx - (u\,dx + x\,du) &= 0\\
\left(u^2+u\right)dx &= x\,du\\
\frac{1}{u\,(u+1)}du &= \frac{1}{x}dx.
\end{aligned}$$

Integrating the left and right-hand sides of this equation with `Integrate`,

```
In[127]:= Integrate[1/(u(u+1)),u]
Out[127]= Log[u] - Log[1+u]
```

```
In[128]:= Integrate[1/x,x]
Out[128]= Log[x]
```

exponentiating, resubstituting $u = y/x$, and solving for y gives us

$$\begin{aligned}
\ln|u| - \ln|u+1| &= -\ln|x| + C\\
\frac{u}{u+1} &= \frac{Cx}{x}\\
\frac{\frac{y}{x}}{\frac{y}{x}+1} &= \frac{Cx}{x}\\
y &= \frac{Cx^2}{1-Cx}.
\end{aligned}$$

```
In[129]:= Solve[(y/x)/(y/x+1) == cx, y]
```

Out[129]= $\left\{\left\{y \to -\frac{c\ x^2}{-1+xc}\right\}\right\}$

We confirm this result with `DSolve` and then graph several solutions with `Plot` in Figure 2-12.

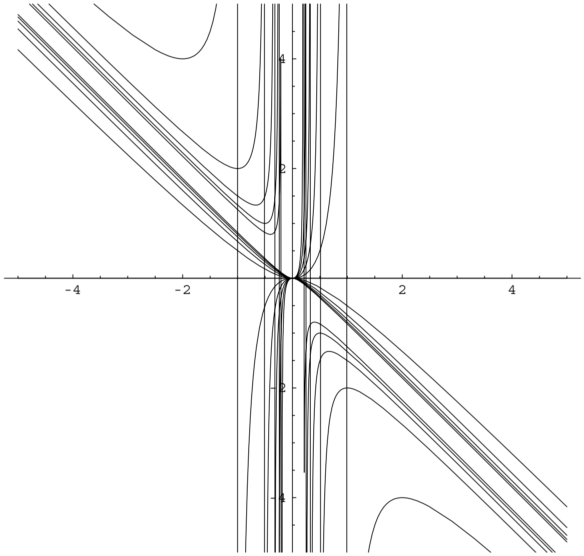


Figure 2-12 Graphs of several solutions of $(y^2 + 2xy)\,dx - x^2 dy = 0$

```
In[130]:= sol = DSolve[y[x]^2 + 2x y[x] - x^2y'[x] == 0,
              y[x], x]
```

Out[130]= $\left\{\left\{y[x] \to -\frac{x^2\ C[1]}{-1 + x\ C[1]}\right\}\right\}$

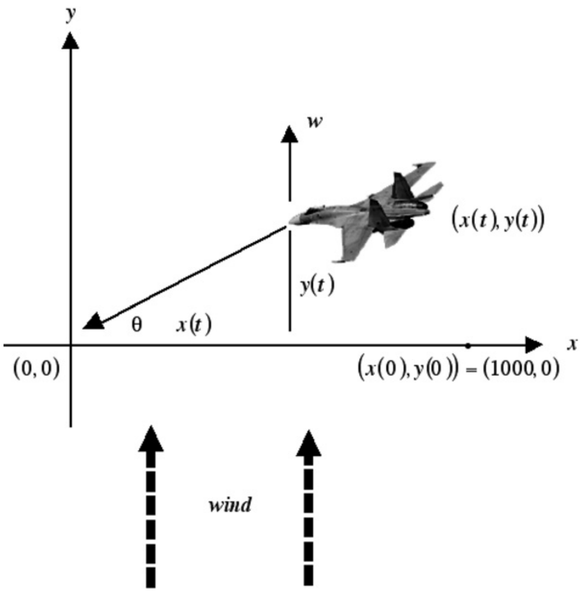
```
In[131]:= toplot = Table[sol[[1, 1, 2]]/.C[1]- > i,
               {i, -5, 5}]; Plot[Evaluate[toplot],
               {x, -5, 5}, PlotRange- > {-5, 5},
               AspectRatio- > Automatic]
```

■

Sources: A particularly interesting and fun-to-read discussion of flight paths and models of pursuit can be found in *Differential Equations: A Modeling Perspective* by Robert L. Borrelli and Courtney S. Coleman and published by John Wiley & Sons.

Application: Models of Pursuit

Suppose that one object pursues another whose motion is known by a predetermined strategy. For example, suppose that an airplane is positioned at $B(1000, 0)$ to fly to another airport $A(0, 0)$ that is 1000 miles directly west of its position B, as illustrated in the following figure. Assume that the airplane aims towards A at all times. If the wind goes from south to north at a constant speed, w, and the airplane's speed in still air is b, determine conditions on b so that the airplane eventually arrives at A and describe its path.



As described, the speed of the airplane, b, must be greater than the speed of the wind, w: $b > w$, in order for the plane to arrive at A. Observe that dx/dt describes the airplane's velocity in the x direction:

$$\frac{dx}{dt} = -b\cos\theta = \frac{-bx}{\sqrt{x^2+y^2}},$$

because from right-triangle trigonometry we know that $\cos\theta$ = adjacent/hypotenuse $= x/\sqrt{x^2+y^2}$. Similarly,

$$\frac{dy}{dt} = -b\sin\theta + w = \frac{-by}{\sqrt{x^2+y^2}} + w,$$

so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-by}{\sqrt{x^2+y^2}} + w}{\frac{-bx}{\sqrt{x^2+y^2}}} = \frac{by - w\sqrt{x^2+y^2}}{bx}.$$

This is a homogeneous equation (of degree one) because it can be written in the form $dy/dx = F(y/x)$:

$$\frac{dy}{dx} = \frac{by - w\sqrt{x^2+y^2}}{bx} = \frac{y}{x} - \frac{w}{b}\sqrt{1+\left(\frac{y}{x}\right)^2}.$$

Therefore, we must solve the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{by - w\sqrt{x^2 + y^2}}{bx} \\ y(1000) = 0. \end{cases}$$

In this case, we see `DSolve` is both able to find a general solution of the equation

```
In[132]:= Clear[x, y, w, b]

          DSolve[y'[x] == (b y[x] - w Sqrt[x^2 + y[x]^2])/(b x),
            y[x], x]
```

Out[132]= $\{\{y[x] \to x \,\text{Sinh}[C[1] - \frac{w\,\text{Log}[x]}{b}]\}\}$

as well as solve the initial-value problem.

```
In[133]:= Clear[x, y, b, w]

          DSolve[{y'[x] == (b y[x] - w Sqrt[x^2 + y[x]^2])/(b x),
            y[1000] == 0}, y[x], x]
```

Out[133]= $\{\{y[x] \to x \,\text{Sinh}[\frac{w\,\text{Log}[1000]}{b} - \frac{w\,\text{Log}[x]}{b}]\}\}$

Alternatively, letting $y = ux$, differentiating to obtain $dy = u\,dx + x\,du$, and substituting into the equation results in the separable equation

$$\frac{dy}{dx} = \frac{du}{dx}x + u = \frac{bux - w\sqrt{x^2 + u^2x^2}}{bx}$$
$$\frac{du}{dx}x + u = u - \frac{w}{b}\sqrt{1 + u^2}$$
$$\frac{1}{\sqrt{1 + u^2}}du = -\frac{w}{b}\frac{1}{x}dx.$$

```
In[134]:= Clear[x, y, u]

          y = u x;

          eqn = Dt[y] == (b y - w Sqrt[x^2 + y^2])/(b x);

          step1 = PowerExpand[Simplify[eqn]]
```

Out[134]= $\frac{\sqrt{1 + u^2}\,w}{b} + x\,\text{Dt}[u] + u\,\text{Dt}[x] == u$

Integrating the left-hand side of this equation yields $\int \frac{1}{\sqrt{1 + u^2}}du = \ln\left|u + \sqrt{1 + u^2}\right| + C_1$

```
In[135]:= leftint = Integrate[1/Sqrt[1 + u^2], u]
Out[135]= ArcSinh[u]

In[136]:= leftint = TrigToExp[leftint]
Out[136]= Log[u + Sqrt[1 + u^2]]
```

and integrating the right results in $-\frac{w}{b}\int\frac{1}{x}dx = -\frac{w}{b}\ln|x| + C_2$. Note that absolute value bars are not necessary because x and y and, hence, u are nonnegative. Thus, $\ln\left(u+\sqrt{1+u^2}\right) = -\frac{w}{b}\ln x + C$.

```
In[137]:= rightint = Integrate[-w/(b x), x] + c
Out[137]= c - (w Log[x])/b
```

Because $y(1000) = 0$, $C = \frac{w}{b}\ln 1000$

```
In[138]:= cval = Solve[leftint == rightint/.{x- > 1000, u- > 0}, c]
Out[138]= {{c → (w Log[1000])/b}}
```

and $\ln\left(u+\sqrt{1+u^2}\right) = -\frac{w}{b}\ln x + \frac{w}{b}\ln 1000$.

```
In[139]:= step2 = leftint == rightint/.cval[[1]]
Out[139]= Log[u + Sqrt[1 + u^2]] == (w Log[1000])/b - (w Log[x])/b
```

Solving for u gives us

$$
\begin{aligned}
\ln\left(u+\sqrt{1+u^2}\right) &= \ln\left(\frac{x}{1000}\right)^{-w/b}\\
u+\sqrt{1+u^2} &= \left(\frac{x}{1000}\right)^{-w/b}\\
\sqrt{1+u^2} &= \left(\frac{x}{1000}\right)^{-w/b} - u\\
1+u^2 &= \left(\frac{x}{1000}\right)^{-2w/b} - 2u\left(\frac{x}{1000}\right)^{-w/b} + u^2\\
2u\left(\frac{x}{1000}\right)^{-w/b} &= \left(\frac{x}{1000}\right)^{-2w/b} - 1\\
u &= \frac{1}{2}\left[\left(\frac{x}{1000}\right)^{-w/b} - \left(\frac{x}{1000}\right)^{w/b}\right].
\end{aligned}
$$

```
In[140]:= step3 = Solve[step2, u]
Out[140]= {{u → -2^(-1-3w/b) 125^(-w/b) x^(-w/b) (-1000^(2w/b) + x^(2w/b))}}
```

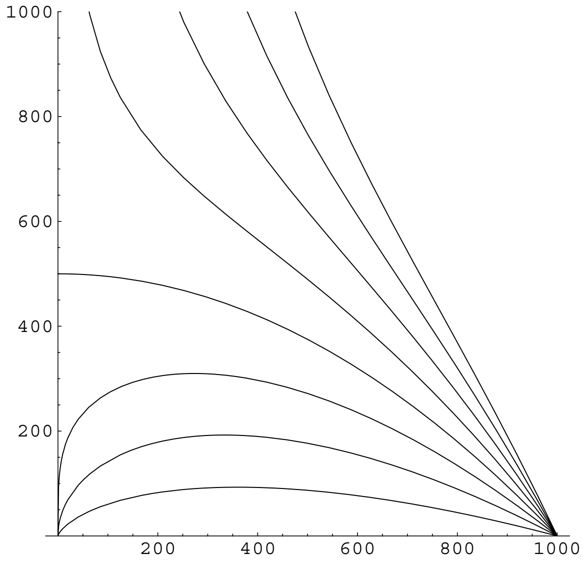


Figure 2-13 If $w/b \geq 1$, the airplane never reaches its destination

We solve for y by resubstituting $u = y/x$ and multiplying by x:

$$\frac{y}{x} = \frac{1}{2}\left[\left(\frac{x}{1000}\right)^{-w/b} - \left(\frac{x}{1000}\right)^{w/b}\right]$$
$$y = \frac{1}{2}x\left[\left(\frac{x}{1000}\right)^{-w/b} - \left(\frac{x}{1000}\right)^{w/b}\right].$$

```
In[141]:= Clear[y]

          y[x_] = x step3[[1, 1, 2]]
```

Out[141]= $-2^{-1-\frac{3w}{b}}\, 125^{-\frac{w}{b}}\, x^{1-\frac{w}{b}}\left(-1000^{\frac{2w}{b}} + x^{\frac{2w}{b}}\right)$

We graph y for various values of w/b by setting $b = 1$ and then using `Table` to generate the value of y for $w = 0.25, 0.50, \ldots, 2.0$. These functions are then graphed with `Plot` in Figure 2-13. Notice that the airplane never arrives at A if $w/b \geq 1$.

```
In[142]:= b = 1;

          toplot = Table[y[x], {w, 0.25, 2, 0.25}];

          Plot[Evaluate[toplot], {x, 0, 1000},
            PlotRange- > {0, 1000}, AspectRatio- > 1]
```

2.4 Exact Equations

Definition 7 (Exact Differential Equation). *A differential equation that can be written in the form*

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

where

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}(x, y)\,dx + \frac{\partial f}{\partial y}(x, y)\,dy$$

for some function $z = f(x, y)$ is called an ***exact differential equation****.*

We can show that the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact if and only if $\partial M/\partial y = \partial N/\partial x$.

EXAMPLE 2.4.1: Show that the equation $2xy^3\,dx + \left(1 + 3x^2y^2\right)dy = 0$ is exact and that the equation $x^2y\,dx + 5xy^2\,dy = 0$ is not exact.

SOLUTION: Because

$$\frac{\partial}{\partial y}\left(2xy^3\right) = 6xy^2 = \frac{\partial}{\partial x}\left(1 + 3x^2y^2\right),$$

the equation $2xy^3\,dx + \left(1 + 3x^2y^2\right)dy = 0$ is an exact equation. On the other hand, the equation $x^2y\,dx + 5xy^2\,dy = 0$ is not exact because

$$\frac{\partial}{\partial y}\left(x^2y\right) = x^2 \neq 5y^2 = \frac{\partial}{\partial x}\left(5xy^2\right).$$

(However, the equation $x^2y\,dx + 5xy^2\,dy = 0$ is separable.)

■

If an equation is exact, we can find a function $z = f(x, y)$ such that $M(x, y) = \frac{\partial f}{\partial x}(x, y)$ and $N(x, y) = \frac{\partial f}{\partial y}(x, y)$.

1. Assume that $M(x, y) = \frac{\partial f}{\partial x}(x, y)$ and $N(x, y) = \frac{\partial f}{\partial y}(x, y)$.
2. Integrate $M(x, y)$ with respect to x. (Add an arbitrary function of y, $g(y)$.)
3. Differentiate the result in Step 2 with respect to y and set the result equal to $N(x, y)$. Solve for $g'(y)$.
4. Integrate $g'(y)$ with respect to y to obtain an expression for $g(y)$. (There is no need to include an arbitrary constant.)

5. Substitute $g(y)$ into the result obtained in Step 2 for $f(x, y)$.
6. A general solution is $f(x, y) = C$ where C is a constant.
7. If given an initial-value problem, apply the initial condition to determine C.

Remark. A similar algorithm can be stated so that in Step 2 $N(x, y)$ is integrated with respect to y.

EXAMPLE 2.4.2: Solve $2x \sin y\, dx + \left(x^2 \cos y - 1\right) dy = 0$ subject to $y(0) = 1/2$.

SOLUTION: The equation $2x \sin y\, dx + \left(x^2 \cos y - 1\right) dy = 0$ is exact because

$$\frac{\partial}{\partial y}(2x \sin y) = 2x \cos y = \frac{\partial}{\partial x}\left(x^2 \cos y - 1\right).$$

Let $z = f(x, y)$ be a function with $\partial f/\partial x = 2x \sin y$ and $\partial f/\partial y = x^2 \cos y - 1$. Then, integrating $\partial f/\partial x$ with respect to x yields

$$f(x, y) = \int 2x \sin y\, dx = x^2 \sin y + g(y).$$

Notice that the arbitrary function $g = g(y)$ of y serves as a "constant" of integration with respect to x. Because we have $\partial f/\partial y = x^2 \cos y - 1$ from the differential equation, and

$$\frac{\partial f}{\partial y}(x, y) = x^2 \cos y + g'(y)$$

from differentiation of $f(x, y)$ with respect to y, $g'(y) = -1$. Integrating $g'(y)$ with respect to y gives us $g(y) = -y$. Therefore, $f(x, y) = x^2 \sin y - y$, so a general solution of the exact equation is $x^2 \sin y - y = C$, where C is a constant. Because our solution requires that $y(0) = 1/2$, we must find the solution in the family of solutions that passes through the point $(0, 1/2)$. Substituting these values of x and y into the general solution, we obtain $0^2 \cdot \sin(1/2) - 1/2 = C$ so that $C = -1/2$. Therefore, the desired solution is $x^2 \sin y - y = -1/2$. We are able to use DSolve to solve the initial-value problem implicitly as well.

Notice that we do not have to include the constant in calculating $g(y)$ because we combine it with the constant in the general solution.

```
In[143]:= Clear[x, y]

partsol = DSolve[{2x Sin[y[x]]
        + (x^2 Cos[y[x]] - 1) y'[x] == 0, y[0] == 1/2}, y[x], x]
```

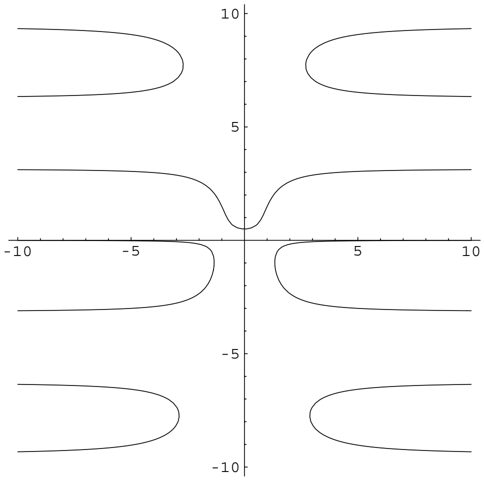


Figure 2-14 Plot of $x^2 \sin y - y = -1/2$

```
Solve::tdep : The equations appear to involve the
      variables to be solved for in an essentially
      non-algebraic way.
Solve::tdep : The equations appear to involve the
      variables to be solved for in an essentially
      non-algebraic way.
```

Out[143]= $\text{Solve}\left[x^2\ \text{Sin}[y[x]] - y[x] == -\frac{1}{2}, y[x]\right]$

To graph the equation `implicitsol`, we first load the **ImplicitPlot** package, which is located in the **Graphics** folder (or directory) and then we use `ImplicitPlot` to graph the equation for $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$ in Figure 2-14. We include the option `AxesOrigin->{0,0}` to specify that the axes intersect at the point (0, 0) and the option `PlotPoints->100` to help guarantee that the resulting curves appear smooth.

In[144]:= **toplot = x^2 Sin[y[x]] - y[x] == $-\frac{1}{2}$ /. y[x]->y**

Out[144]= $-y + x^2\ \text{Sin}[y] == -\frac{1}{2}$

In[145]:= **<< Graphics\`ImplicitPlot\`**

In[146]:= **ImplicitPlot[toplot, {x, -10, 10}, {y, -10, 10}, AxesOrigin → {0, 0}, PlotPoints → 100]**

■

The following example illustrates how we can use Mathematica to assist us in carrying out the necessary steps encountered when solving an exact equation.

EXAMPLE 2.4.3: Solve $\left(2x - y^2 \sin(xy)\right) dx + (\cos(xy) - xy \sin(xy))\, dy = 0$.

SOLUTION: We begin by identifying $M(x, y) = 2x - y^2 \sin(xy)$ and $N(x, y) = \cos(xy) - xy \sin(xy)$. We then define `capm`, corresponding to M, and `capn`, corresponding to N. We then see that the equation is exact because $\partial M/\partial y = \partial N/\partial x$.

```
In[147]:= capm[x_, y_] = 2x - y^2 Sin[xy];
          capn[x_, y_] = Cos[xy] - xy Sin[xy];
In[148]:= ∂_y capm[x, y] == ∂_x capn[x, y]
Out[148]= True
```

Next, we compute $\int M(x, y)\, dx$ and add an arbitrary function of y, `g[y]`, to the result.

```
In[149]:= f = ∫ capm[x, y] dx + g[y] //Simplify
Out[149]= x^2 + y Cos[x y] + g[y]
```

Differentiating f with respect to y gives us

```
In[150]:= ∂_y f
Out[150]= Cos[x y] - x y Sin[x y] + g′[y]
```

and because we must have that $\partial f/\partial y = N(x, y)$, we obtain the equation

```
In[151]:= ∂_y f == capn[x, y]
Out[151]= Cos[xy]-xy Sin[xy]+g′[y] == Cos[xy]-xy Sin[xy]
```

which we solve for $g'(y)$ with `Solve`.

```
In[152]:= Solve[∂_y f == capn[x, y], g′[y]]
Out[152]= {{g′[y] → 0}}
```

Thus, $g(y)$ is a (real-valued) constant and a general solution of the equation is $x^2 + y\cos(xy) = C$. We can graph this general solution for various values of C by observing that the level curves of the function $z = x^2 + y\cos(xy)$ correspond to the graphs of the equation $x^2 + y\cos(xy) = C$ for various values of C.

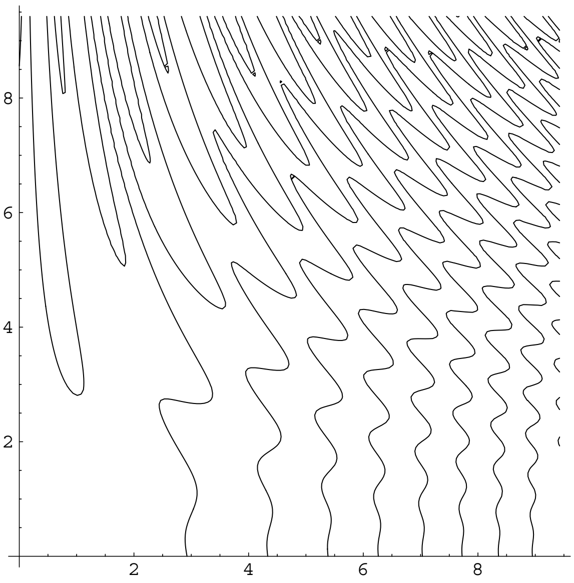


Figure 2-15 Level curves of $z = x^2 + y\cos(xy)$

```
In[153]:= f = f/.g[y] → 0
Out[153]= x² + y Cos[x y]
```

We now use `ContourPlot` to graph several level curves of $z = x^2 + y\cos(xy)$ on the rectangle $[0, 3\pi] \times [0, 3\pi]$ in Figure 2-15. In this case, the option `Frame->False` instructs Mathematica to not place a frame around the resulting graphics object, the option `Axes->Automatic` specifies that axes are to be placed on the graph, `AxesOrigin->{0,0}` specifies that the axes are to intersect at the point (0, 0), `AxesStyle->GrayLevel[.5]` specifies that the axes are to be drawn in a light gray, `ContourShading->False` specifies that the region between contours is to not be shaded and the option `PlotPoints->150` helps assure that the resulting contours appear smooth.

```
In[154]:= ContourPlot[f, {x, 0, 3π}, {y, 0, 3π},
            Frame → False, Axes → Automatic,
            AxesOrigin → {0, 0}, ContourShading → False,
          PlotPoints → 150]
```

We see that `DSolve` is able to find an implicit solution of the equation after we rewrite it in the form $\left(2x - y^2\sin(xy)\right) + (\cos(xy) - xy\sin(xy))\,y' = 0$.

```
In[155]:= gensol = DSolve[(2x - y[x]^2 Sin[x y[x]])
            + (Cos[x y[x]] - x y[x] Sin[x y[x]]) y'[x] == 0,
             y[x], x]
Solve :: tdep : The equations appear to involve the
      variables to be solved for in an essentially non-algebraic way.
Solve :: tdep : The equations appear to involve the
      variables to be solved for in an essentially non-algebraic way.
Out[155]= Solve[x^2 + Cos[x y[x]] y[x] == C[1], y[x]]
```

The implicit solution is the first part of `gensol`.

```
In[156]:= step1 = gensol[[1]]
Out[156]= x^2 + Cos[x y[x]] y[x] == C[1]
```

To graph the level curves of $z = x^2 + y\cos(xy)$, we extract the left-hand side of the equation `gensol[[1]]` by noting that it is the first part of the first part of `gensol`

```
In[157]:= step2 = gensol[[1, 1]]
Out[157]= x^2 + Cos[x y[x]] y[x]
```

and then replacing each occurrence of `y[x]` by `y` using `ReplaceAll` (`/.`).

```
In[158]:= implicitsol = step2/.y[x] → y
Out[158]= x^2 + y Cos[x y]
```

Entering

```
In[159]:= ContourPlot[implicitsol, {x, 0, 3π}, {y, 0, 3π},
             ContourShading → False, PlotPoints → 150]
```

produces the same graph as that obtained in Figure 2-15.

■

2.5 Linear Equations

Definition 8 (First-Order Linear Equation). *A differential equation of the form*

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{2.2}$$

where $a_1(x)$ is not identically the zero function, is a first-order ***linear differential equation****.*

Assuming that $a_1(x)$ is not identically the zero function, dividing equation (2.2) by $a_1(x)$ gives us the **standard form** of the first-order linear equation:

$$\frac{dy}{dx} + p(x)y = q(x). \tag{2.3}$$

If $q(x)$ is identically the zero function, we say that the equation is **homogeneous**. The **corresponding homogeneous equation** of equation (2.3) is

$$\frac{dy}{dx} + p(x)y = 0. \tag{2.4}$$

2.5.1 Integrating Factor Approach

Multiplying equation (2.3) by $e^{\int p(x)\,dx}$ yields

$$e^{\int p(x)\,dx}\frac{dy}{dx} + e^{\int p(x)\,dx}p(x)y = e^{\int p(x)\,dx}q(x).$$

By the product rule and the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}y\right) = e^{\int p(x)\,dx}\frac{dy}{dx} + e^{\int p(x)\,dx}p(x)y$$

so equation (2.3) becomes

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}y\right) = e^{\int p(x)\,dx}q(x).$$

Integrating and dividing by $e^{\int p(x)\,dx}$ yields a general solution of $y' + p(x)y = q(x)$:

$$\begin{aligned} e^{\int p(x)\,dx}y &= \int e^{\int p(x)\,dx}q(x)\,dx \\ y &= \frac{1}{e^{\int p(x)\,dx}}\int e^{\int p(x)\,dx}q(x)\,dx = e^{-\int p(x)\,dx}\int e^{\int p(x)\,dx}q(x)\,dx. \end{aligned}$$

The term $\mu(x) = e^{\int p(x)\,dx}$ is called an **integrating factor** for the linear equation (2.3). Thus, first-order linear equations can always be solved, although the resulting integrals may be difficult or impossible to evaluate exactly.

As we see with the following command, `DSolve` is always able to solve first-order linear differential equations, although the result might contain unevaluated integrals.

```
In[160]:= Clear[x, y, p, q]

          DSolve[y′[x] + p[x] y[x] == q[x], y[x], x]
Out[160]= {{y[x] → e^(∫_K$14523^x -p[K$14522] dK$14522) C[1]
          + e^(∫_K$14523^x -p[K$14522] dK$14522)
          ∫_K$14559^x e^(-∫_K$14523^K$14558 -p[K$14522] dK$14522)
          q[K$14558] dK$14558}}
```

EXAMPLE 2.5.1: Solve $x\,dy/dx + y = x\cos x$, $x > 0$.

SOLUTION: First, we place the equation in the form used in the derivation above. Dividing the equation by x yields

$$\frac{dy}{dx} + \frac{1}{x}y = \cos x, \tag{2.5}$$

where $p(x) = 1/x$ and $q(x) = \cos x$. Then, an integrating factor is

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x, \quad \text{for } x > 0,$$

and multiplying equation (2.5) by the integrating factor gives us

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = x\cos x.$$

Integrating once we have

$$xy = \int x\cos x\,dx.$$

Using the integration by parts formula, $\int u\,dv = uv - \int v\,du$, with $u = x$ and $dv = \cos x\,dx$, we obtain $du = dx$ and $v = \sin x$ so

$$xy = \int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Therefore, a general solution of the equation $x\,dy/dx + y = x\cos x$ for $x > 0$ is $y = (x\sin x + \cos x + C)/x$. We see that DSolve is also successful in finding a general solution of the equation.

If we want to solve the equation for $x < 0$, then we would have $e^{\int \frac{1}{x}dx} = e^{\ln|x|} = -x$ for $x < 0$.

```
In[161]:= gensol = DSolve[xy′[x]+y[x] == x Cos[x], y[x], x]
Out[161]= {{y[x] → C[1]/x + (Cos[x] + x Sin[x])/x}}
```

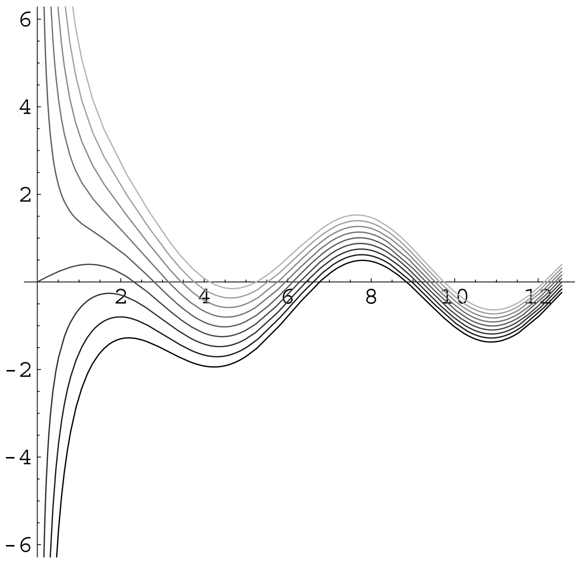


Figure 2-16 Various solutions of $x\,dy/dx + y = x\cos x$, $x > 0$

As we have seen in previous examples, we can graph the solution for various values of the arbitrary constant by generating a set of functions obtained by replacing the arbitrary constant with numbers using `Table` and `ReplaceAll` (`/.`).

```
In[162]:= toplot = Table[gensol[[1, 1, 2]]/.C[1] → i,
             {i, -4, 4}];
```

In this case, Mathematica generates several error messages, which are not displayed here because the solution is undefined if $x = 0$. Nevertheless, the resulting graph shown in Figure 2-16 is displayed correctly.

```
In[163]:= grays = Table[GrayLevel[i], {i, 0, 0.7, 0.7/8}];

          Plot[Evaluate[toplot], {x, 0, 4π},
            PlotRange- > {-2π, 2π}, AspectRatio- > 1,
            PlotStyle- > grays]
```

■

As with other types of equations, we solve initial-value problems by first finding a general solution of the equation and then applying the initial condition to determine the value of the constant.

EXAMPLE 2.5.2: Solve the initial-value problem $\begin{cases} dy/dx + 5x^4 y = x^4 \\ y(0) = -7. \end{cases}$

SOLUTION: As we have seen in many previous examples, `DSolve` can be used to find a general solution of the equation and the solution to the initial-value problem, as done in `gensol` and `partsol`, respectively.

```
In[164]:= Clear[x, y]

          gensol = DSolve[y'[x] + 5x^4 y[x] == x^4, y[x], x]
```

$$\text{Out[164]= } \left\{\left\{y[x] \to \frac{1}{5} + e^{-x^5} C[1]\right\}\right\}$$

```
In[165]:= partsol = DSolve[{y'[x] + 5x^4 y[x] == x^4,
               y[0] == -7}, y[x], x]
```

$$\text{Out[165]= } \left\{\left\{y[x] \to \frac{1}{5} e^{-x^5} \left(-36 + e^{x^5}\right)\right\}\right\}$$

We now graph the solution to the initial-value problem obtained in `partsol` with `Plot` in Figure 2-17.

```
In[166]:= Plot[y[x]/.partsol, {x, -1, 2}]
```

We can also use Mathematica to carry out the steps necessary to solve first-order linear equations. We begin by identifying the integrating factor $e^{\int 5x^4\,dx} = e^{x^5}$, computed as follows with `Integrate`.

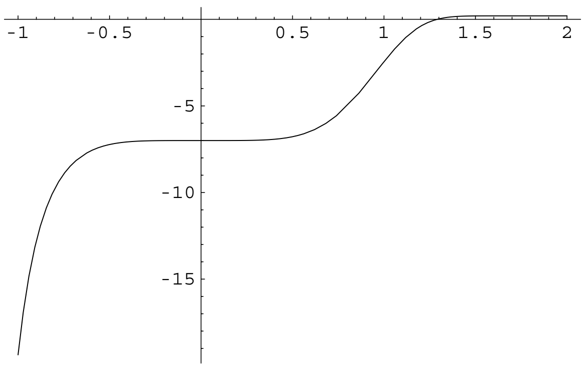


Figure 2-17 The solution of $dy/dx + 5x^4 y = x^4$ that satisfies $y(0) = -7$

```
In[167]:= intfac = Exp[∫ 5x^4 dx]
Out[167]= e^(x^5)
```

Therefore, the equation can be written as

$$\frac{d}{dx}\left(e^{x^5}y\right) = x^4 e^{x^5}$$

so that integration of both sides of the equation yields

$$e^{x^5}y = \frac{1}{5}e^{x^5} + C.$$

```
In[168]:= rightside = ∫ intfacx^4 dx
Out[168]= e^(x^5)/5
```

Hence, a general solution is $y = \frac{1}{5} + Ce^{-x^5}$. Note that we compute y by using `Solve` to solve the equation $e^{x^5}y = \frac{1}{5}e^{x^5} + C$ for y.

```
In[169]:= step1 = Solve[Exp[x^5] y == rightside + c, y]
Out[169]= {{y → 1/5 e^(-x^5) (5 c + e^(x^5))}}
```

We find the unknown constant C by substituting the initial condition $y(0) = -7$ into the general solution and solving for C.

```
In[170]:= findc = Solve[-7 == step1[[1, 1, 2]]/.x → 0]
Out[170]= {{c → -36/5}}
```

Therefore, the solution to the initial-value problem is $y = \frac{1}{5} - \frac{36}{5}e^{-x^5}$.

```
In[171]:= step1[[1, 1, 2]]/.findc[[1]]
Out[171]= 1/5 e^(-x^5) (-36 + e^(x^5))
```

■

We can use `DSolve` to solve a first-order linear equation even if the coefficient functions are discontinuous or piecewise-defined. In such situations, it is often useful to take advantage of the *unit step function*. The **unit step function**, $\mathcal{U}(t)$, is defined by

$$\mathcal{U}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

The Mathematica command `UnitStep[t]` returns $\mathcal{U}(t)$.

EXAMPLE 2.5.3 (Drug Concentration): If a drug is introduced into the bloodstream in dosages $D(t)$ and is removed at a rate proportional to the concentration, the concentration $C(t)$ at time t is given by

$$\begin{cases} dC/dt = D(t) - kC \\ C(0) = 0 \end{cases}$$

where $k > 0$ is the constant of proportionality.

See J. D. Murray's *Mathematical Biology*, Springer-Verlag, 1990, pp. 645–649.

Suppose that over a 24-hour period, a drug is introduced into the bloodstream at a rate of $24/t_0$ for exactly t_0 hours and then stopped so that $D_{t_0}(t) = \begin{cases} 24/t_0, & 0 \le t \le t_0 \\ 0, & t > t_0 \end{cases}$. Calculate and then graph $C(t)$ on the interval $[0, 30]$ if $k = 0.05, 0.10, 0.15, 0.20$, and 0.25 for $t_0 = 4, 8, 12, 16$, and 25. How does increasing t_0 affect the concentration of the drug in the bloodstream? Then consider the effect of increasing k.

SOLUTION: To compute $C(t)$, we must keep in mind that $D_{t_0}(t)$ is a piecewise-defined function. In terms of the unit step function, $\mathcal{U}(t)$, $D_{t_0}(t) = \dfrac{24}{t_0}\mathcal{U}(t_0 - t)$.

Note that we use lower-case letters to avoid any ambiguity with built-in objects like `C` and `D`.

```
In[172]:= d[t_, t0_] = 24/t0 UnitStep[t0 - t]

Out[172]= 24 UnitStep[-t + t0]
          --------------------
                  t0
```

For example, entering `d(t,4)` returns $D_4(t) = \begin{cases} 6, & 0 \le t \le 4 \\ 0, & t > 4 \end{cases}$.

```
In[173]:= d[t, 4]

Out[173]= 6 UnitStep[4 - t]
```

Given k and t_0, the function `sol` returns the solution to the initial-value problem $\begin{cases} dC/dt = D_{t_0}(t) - kC \\ C(0) = 0 \end{cases}$.

```
In[174]:= Clear[sol, k, c, t, t0]

          sol[k_, t0_] := DSolve[{c'[t] == d[t, t0] - k c[t],
            c[0] == 0}, c[t], t][[1, 1, 2]]
```

Then, for $k = 0.05$ we solve the initial-value probem $\begin{cases} dC/dt = D_{t_0}(t) - kC \\ C(0) = 0 \end{cases}$ for $t = 4, 8, 12, 16$, and 20 by applying `sol` to the list `{4,8,12,16,20}`.

```
In[175]:= toplot05 = Map[sol[0.05, #]&,
            {4, 8, 12, 16, 20}]
```

$$\text{Out[175]}= \{e^{-0.05t}(-120.+120.\,e^{0.05t} +146.568\,\text{UnitStep}[-4.+1.t] -120.\,2.71828^{0.05t}\,\text{UnitStep}[-4.+1.t]),$$
$$e^{-0.05t}(-60.+60.\,e^{0.05t} +89.5095\,\text{UnitStep}[-8.+1.t] -60.\,2.71828^{0.05t}\,\text{UnitStep}[-8.+1.t]),$$
$$e^{-0.05t}(-40.+40.\,e^{0.05t} +72.8848\,\text{UnitStep}[-12.+1.t] -40.\,2.71828^{0.05t}\,\text{UnitStep}[-12.+1.t]),$$
$$e^{-0.05t}(-30.+30.\,e^{0.05t} +66.7662\,\text{UnitStep}[-16.+1.t] -30.\,2.71828^{0.05t}\,\text{UnitStep}[-16.+1.t]),$$
$$e^{-0.05t}(-24.+24.\,e^{0.05t} +65.2388\,\text{UnitStep}[-20.+1.t] -24.\,2.71828^{0.05t}\,\text{UnitStep}[-20.+1.t])\}$$

These solutions are graphed with `Plot` in Figure 2-18.

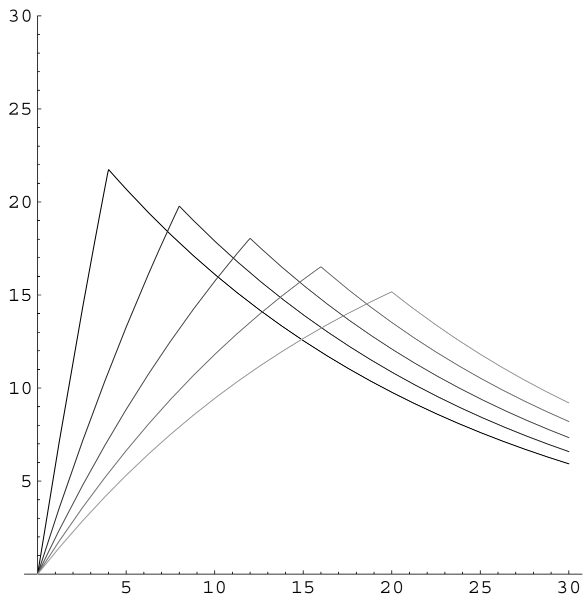


Figure 2-18 As t_0 increases, the maximum concentration of the drug decreases

```
In[176]:= grays = Table[GrayLevel[i],
            {i, 0, 0.6, 0.15}];

In[177]:= Plot[Evaluate[toplot05], {t, 0, 30},
            PlotRange- > {0, 30}, AspectRatio- > 1,
            PlotStyle- > grays]
```

Similar steps are repeated for $k = 0.10$, 0.15, 0.20, and 0.25 by defining the function `toplot`. Given k, `toplot[k]` solves the initial-value problem $\begin{cases} dC/dt = D_{t_0}(t) - kC \\ C(0) = 0 \end{cases}$ for $t_0 = 4, 8, 12, 16$, and 20.

```
In[178]:= Clear[toplot, sols]

          toplot[k_] := Map[sol[k, #]&, {4, 8, 12, 16, 20}];
```

We then apply `toplot` to the list `{0.1, 0.15, 0.20, 0.25}` naming the resulting lists of functions `sols`.

```
In[179]:= sols = Map[toplot, {0.1, 0.15, 0.2, 0.25}];
```

Each list of functions in `sols` is then graphed with `Plot` by applying the pure function

```
Plot[Evaluate[#], {t, 0, 30}, PlotRange ->
{0, 30}, AspectRatio -> 1, PlotStyle -> grays,
DisplayFunction -> Identity] &
```

to each element of `sols` with `Map`.

```
In[180]:= toshow = Map[Plot[Evaluate[#], {t, 0, 30},
                  PlotRange- > {0, 30}, AspectRatio- > 1,
                  PlotStyle- > grays,
                  DisplayFunction- > Identity]&, sols]
Out[180]= {-Graphics-, -Graphics-, -Graphics-,
            -Graphics-}
```

Finally, all four graphs are shown together as a graphics array using `Show` and `GraphicsArray` in Figure 2-19.

```
In[181]:= Show[GraphicsArray[Partition[toshow, 2]]]
```

From the graphs, we see that as t_0 is increased, the maximum concentration level decreases and occurs at later times, while increasing k increases the rate at which the drug is removed from the bloodstream.

■

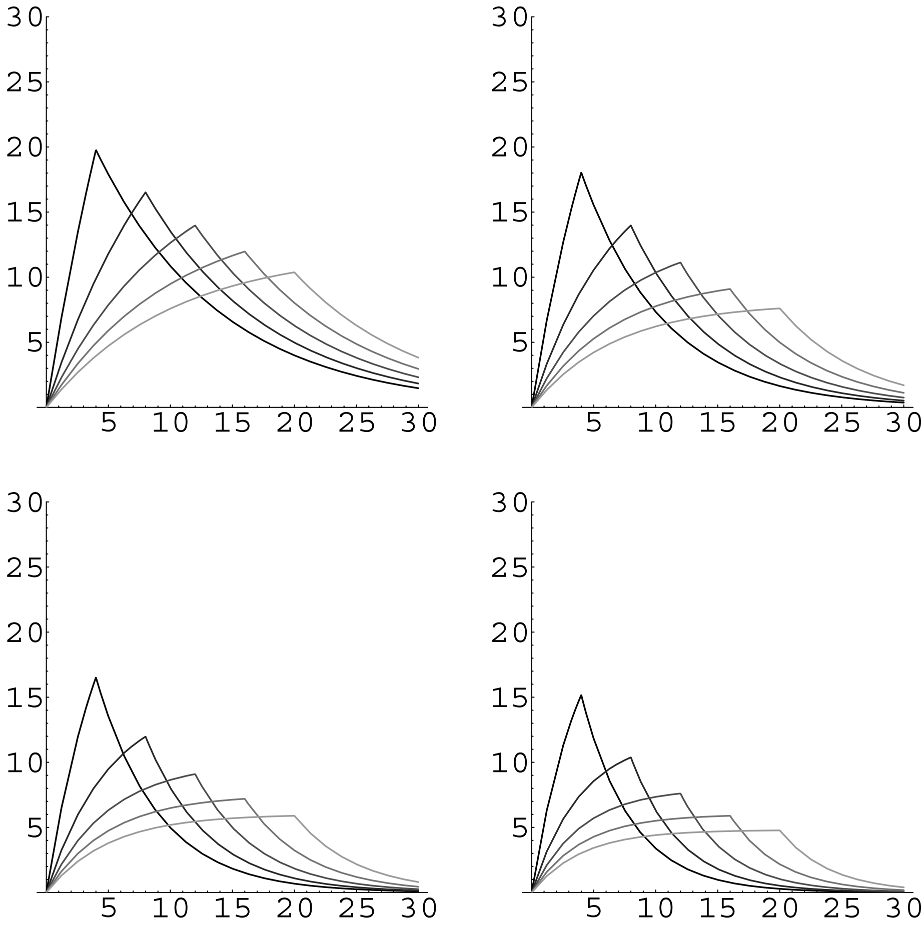


Figure 2-19 $C(t)$ for various values of t_0 and k

If the integration cannot be carried out, the solution can often be approximated numerically by taking advantage of numerical integration techniques.

EXAMPLE 2.5.4: Graph the solution to the initial-value problem $y' - y\sin(2\pi x) = 1$, $y(0) = 1$ on the interval $[0, 2\pi]$.

SOLUTION: Note that `DSolve` is successful in finding the solution to the initial-value problem even though the result contains unevaluated integrals.

```
In[182]:= Clear[x, y, partsol]
          partsol = DSolve[{y′[x] - Sin[2πx] y[x] == 1,
                    y[0] == 1}, y[x], x]
```

Out[182]= $\left\{\left\{y[x] \to e^{-\frac{\mathrm{Cos}[2\pi x]}{2\pi}}\left(e^{\frac{1}{2\pi}} - \int_{K\$16674}^{0} e^{\frac{\mathrm{Cos}[2\,K\$16673\,\pi]}{2\pi}}\, dK\$16673 + \int_{K\$16674}^{x} e^{\frac{\mathrm{Cos}[2\,K\$16673\,\pi]}{2\pi}}\, dK\$16673\right)\right\}\right\}$

We can evaluate the result for particular numbers. For example, entering

```
In[183]:= partsol[[1, 1, 2]]/.x → 1
```

Out[183]= $e^{-\frac{1}{2\pi}}\left(e^{\frac{1}{2\pi}} - \int_{K\$16674}^{0} e^{\frac{\mathrm{Cos}[2\,K\$16673\,\pi]}{2\pi}}\, dK\$16673 + \int_{K\$16674}^{1} e^{\frac{\mathrm{Cos}[2\,K\$16673\,\pi]}{2\pi}}\, dK\$16673\right)$

returns the value of the solution to the initial-value problem if $x = 1$. This result is a bit complicated to understand so we use `N` to obtain a numerical approximation.

In[184]:= $\mathbf{N}\left[e^{-\frac{1}{2\pi}}\left(e^{\frac{1}{2\pi}} + \mathbf{BesselI}\left[0, \frac{1}{2\pi}\right]\right)\right]$

```
Out[184]= 1.85827
```

To graph the solution on the interval $[0, 2\pi]$, we use `NDSolve` to generate a numerical solution to the initial-value problem valid for $0 \leq x \leq 2\pi$. Generally, the command

```
NDSolve[{deq, ics}, fun, {var, varmin, varmax}]
```

returns a numerical solution `fun` (which is a function of the variable `var`) of the differential equation `deq` that satisfies the initial conditions `ics` valid on the interval `[varmin, varmax]`. In some cases, the interval on which the solution is returned by `NDSolve` is smaller than the interval requested.

Note that the number of initial conditions in `ics` must equal the order of the differential equation `deq`.

We see that the syntax for the `NDSolve` command is nearly the same as the syntax of the `DSolve` command although we must specify an interval on which we want the approximation to be valid. In this case, including $\{x, 0, 2\pi\}$ in the `NDSolve` command instructs Mathematica to (try to) make the resulting numerical solution valid for $0 \leq x \leq 2\pi$.

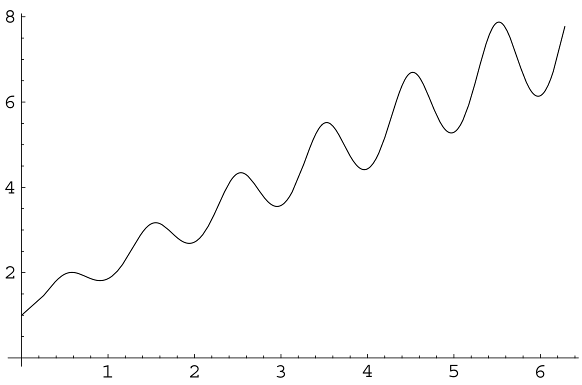


Figure 2-20 Plot of a numerical solution to a differential equation

```
In[185]:= numsol = NDSolve[{y′[x] - Sin[2πx] y[x] == 1,
                   y[0] == 1}, y[x], {x, 0, 2π}]

Out[185]= {{y[x] → InterpolatingFunction[{{0., 6.28319}},
            <>][x]}}
```

The resulting output is an `InterpolatingFunction` which represents an approximate function obtained through interpolation. We can evaluate the result for particular values of x as long as $0 \le x \le 2\pi$. For example, entering

```
In[186]:= numsol/.x → 1

Out[186]= {{y[1] → 1.85827}}
```

approximates the value of the solution to the initial-value problem if $x = 1$. Thus, the result means that $y(1) \approx 1.85828$. We can graph the result returned by `NDSolve` in the same way as we graph results returned by `DSolve`: entering

```
In[187]:= Plot[numsol[[1, 1, 2]], {x, 0, 2π}]
```

graphs the solution to the initial-value problem on the interval $[0, 2\pi]$ as shown in Figure 2-20. Note that we obtain the same graph by entering `Plot[y[x]/.numsol,{x,0,2Pi}]`.

■

2.5.2 Variation of Parameters and the Method of Undetermined Coefficients

Observe that equation (2.4) is separable:

$$\begin{aligned} \frac{dy}{dx} + p(x)y &= 0 \\ \frac{1}{y}dy &= -p(x)\,dx \\ \ln|y| &= -\int p(x)\,dx + C \\ y &= Ce^{-\int p(x)\,dx}. \end{aligned}$$

Notice that any constant multiple of a solution to a linear homogeneous equation is also a solution. Now suppose that y is any solution of equation (2.3) and y_p is a particular solution of equation (2.3). Then,

A **particular solution** is a specific solution to the equation that does not contain any arbitrary constants.

$$\begin{aligned} \left(y - y_p\right)' + p(x)\left(y - y_p\right) &= y' + p(x)y - \left({y_p}' + p(x)y_p\right) \\ &= q(x) - q(x) = 0. \end{aligned}$$

Thus, $y - y_p$ is a solution to the corresponding homogeneous equation of equation (2.3). Hence,

$$\begin{aligned} y - y_p &= Ce^{-\int p(x)\,dx} \\ y &= Ce^{-\int p(x)\,dx} + y_p \\ y &= y_h + y_p, \end{aligned}$$

where $y_h = Ce^{-\int p(x)\,dx}$. That is, a general solution of equation (2.3) is

$$y = y_h + y_p,$$

where y_p is a particular solution to the nonhomogeneous equation and y_h is a general solution to the corresponding homogeneous equation. Thus, to solve equation (2.3), we need to first find a general solution to the corresponding homogeneous equation, y_h, which we can accomplish through separation of variables, and then find a particular solution, y_p, to the nonhomogeneous equation.

If y_h is a solution to the corresponding homogeneous equation of equation (2.3) then for any constant C, Cy_h is also a solution to the corresponding homogeneous equation. Hence, it is impossible to find a particular solution to equation (2.3) of this form. Instead, we search for a particular solution of the form $y_p = u(x)y_h$, where $u(x)$ is *not* a constant function. Assuming that a particular solution, y_p, to equation (2.3) has the form $y_p = u(x)y_h$, differentiating gives us

$${y_p}' = u'y_h + u{y_h}'$$

and substituting into equation (2.3) results in

$$y_p' + p(x)y_p = u'y_h + uy_h' + p(x)uy_h = q(x).$$

Because $uy_h' + p(x)uy_h = u\,[y_h' + p(x)y_h] = u \cdot 0 = 0$, we obtain

y_h is a solution to the corresponding homogeneous equation so $y_h' + p(x)y_h = 0$.

$$\begin{aligned} u'y_h &= q(x) \\ u' &= \frac{1}{y_h}q(x) \\ u' &= e^{\int p(x)\,dx}q(x) \\ u &= \int e^{\int p(x)\,dx}q(x)\,dt \end{aligned}$$

so

$$y_p = u(x)\,y_h = Ce^{-\int p(x)\,dx}\int e^{\int p(x)\,dx}q(x)\,dx.$$

Because we can include an arbitrary constant of integration when evaluating $\int e^{\int p(x)\,dx}q(x)\,dx$, it follows that we can write a general solution of equation (2.3) as

$$y = e^{-\int p(x)\,dx}\int e^{\int p(x)\,dx}q(t)\,dt. \tag{2.6}$$

Exponential growth is discussed in more detail in Section 3.2.1.

EXAMPLE 2.5.5 (Exponential Growth): Let $y = y(t)$ denote the size of a population at time t. If y grows at a rate proportional to the amount present, y satisfies

$$\frac{dy}{dt} = \alpha y, \tag{2.7}$$

where α is the **growth constant**. If $y(0) = y_0$, using equation (2.6) results in $y = y_0e^{\alpha t}$. We use DSolve to confirm this result.

```
In[188]:= DSolve[{y'[t] == α y[t], y[0] == y0}, y[t], t]
Out[188]= {{y[t] → e^(t α) y0}}
```

$dy/dt = k\,(y - y_s)$ models *Newton's Law of Cooling*: the rate at which the temperature, $y(t)$, changes in a heating/cooling body is proportional to the difference between the temperature of the body and the constant temperature, y_s, of the surroundings. Newton's Law of Cooling is discussed in more detail in Section 3.3.

EXAMPLE 2.5.6: Solve each of the following equations: (a) $dy/dt = k\,(y - y_s)$, $y(0) = y_0$, k and y_s constant and (b) $y' - 2ty = t$.

SOLUTION: By hand, we rewrite the equation and obtain

$$\frac{dy}{dt} - ky = -ky_s.$$

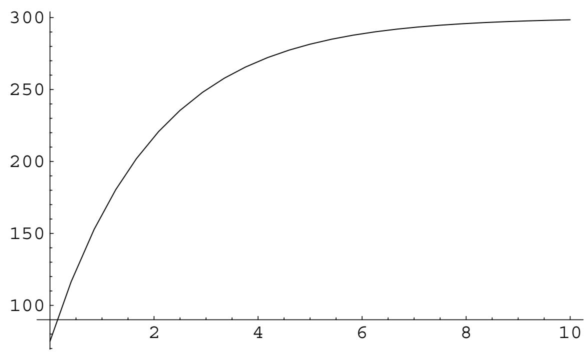


Figure 2-21 The temperature of the body approaches the temperature of its surroundings

A general solution of the corresponding homogeneous equation

$$\frac{dy}{dt} - ky = 0$$

is $y_h = Ce^{kt}$. Because k and $-ky_s$ are constants, we suppose that a particular solution of the nonhomogeneous equation, y_p, has the form $y_p = A$, where A is a constant.

This will turn out to be a lucky guess. If there is not a solution of this form, we would not find one of this form.

Assuming that $y_p = A$, we have $y_p' = 0$ and substitution into the nonhomogeneous equation gives us

$$\frac{dy_p}{dt} - ky_p = -kA = -ky_s \qquad \text{so} \qquad A = y_s.$$

Thus, a general solution is $y = y_h + y_p = Ce^{kt} + y_s$. Applying the initial condition $y(0) = y_0$ results in $y = y_s + (y_0 - y_s)e^{kt}$.

We obtain the same result with `DSolve`. We graph the solution satisfying $y(0) = 75$ assuming that $k = -1/2$ and $y_s = 300$ in Figure 2-21. Notice that $y(t) \to y_s$ as $t \to \infty$.

```
In[189]:= sola = DSolve[{y'[t] == k(y[t] - ys),
            y[0] == y0}, y[t], t]
Out[189]= {{y[t] → e^(k t) (y0 - ys) + ys}}

In[190]:= tp = sola[[1, 1, 2]]/.{k- > -1/2,
            ys- > 300, y0- > 75}; Plot[tp, {t, 0, 10}]
```

(b) The equation is in standard form and we identify $p(t) = -2t$. Then, the integrating factor is $\mu(t) = e^{\int p(t)\,dt} = e^{-t^2}$. Multiplying the equation by the integrating factor, $\mu(t)$, results in

$$e^{-t^2}(y' - 2ty) = te^{-t^2} \qquad \text{or} \qquad \frac{d}{dt}\left(ye^{-t^2}\right) = te^{-t^2}.$$

Integrating gives us

$$ye^{-t^2} = -\frac{1}{2}e^{-t^2} + C \qquad \text{or} \qquad y = -\frac{1}{2} + Ce^{t^2}.$$

We confirm the result with DSolve.

```
In[191]:= DSolve[y'[t] - 2t y[t] == t, y[t], t]
Out[191]= {{y[t] → -1/2 + e^(t^2) C[1]}}
```

■

Application: Antibiotic Production

When you are injured or sick, your doctor may prescribe antibiotics to prevent or cure infections. In the journal article "Changes in the Protein Profile of *Streptomyces Griseus* during a Cycloheximide Fermentation" we see that production of the antibiotic cycloheximide by *Streptomyces* is typical of antibiotic production. During the production of cycloheximide, the mass of *Streptomyces* grows relatively quickly and produces little cycloheximide. After approximately 24 hours, the mass of *Streptomyces* remains relatively constant and cycloheximide accumulates. However, once the level of cycloheximide reaches a certain level, extracellular cycloheximide is degraded (**feedback inhibited**). One approach to alleviating this problem to maximize cycloheximide production is to continuously remove extracellular cycloheximide. The rate of growth of *Streptomyces* can be described by the separable equation

Source: Kevin H. Dykstra and Henry Y. Wang, "Changes in the Protein Profile of *Streptomyces Griseus* during a Cycloheximide Fermentation," *Biochemical Engineering V*, Annals of the New York Academy of Sciences, Volume 56, New York Academy of Sciences (1987), pp. 511–522.

$$\frac{dX}{dt} = \mu_{\max}\left(1 - \frac{1}{X_{\max}}X\right)X,$$

where X represents the mass concentration in g/L, $\mu_{\max}$ is the maximum specific growth rate, and $X_{\max}$ represents the maximum mass concentration. We now solve the initial-value problem $\begin{cases} dX/dt = \mu_{\max}\left(1 - \frac{1}{X_{\max}}X\right)X \\ X(0) = 1 \end{cases}$ with DSolve, naming the result sol1.

Note that this equation can be converted to a linear equation with the substitution $y = X^{-1}$.

```
In[192]:= Clear[x]
          sol1 = DSolve[{x'[t] == μ (1 - x[t]/xmax) x[t],
            x[0] == 1}, x[t], t]
Out[192]= {{x[t] → (e^(μ t) xmax)/(-1 + e^(μ t) + xmax)}}
```

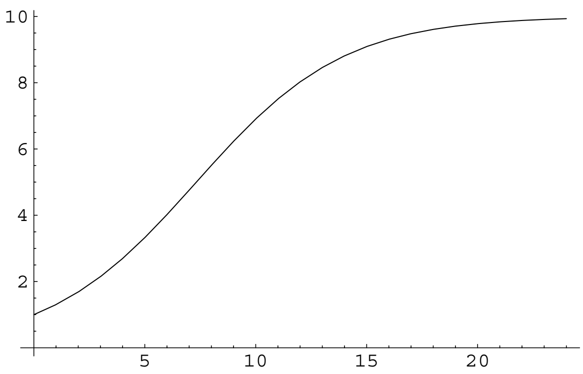


Figure 2-22 Plot of the mass concentration, $x(t)$

Experimental results have shown that $\mu_{\max} = 0.3\,\mathrm{hr}^{-1}$ and $X_{\max} = 10\,\mathrm{g/L}$. For these values, we use `Plot` to graph $X(t)$ on the interval $[0, 24]$ in Figure 2-22. Then, we use `Table` and `TableForm` to determine the mass concentration at the end of 4, 8, 12, 16, 20, and 24 hours.

```
In[193] := μ = 0.3; xmax = 10;

In[194] := Plot[Evaluate[x[t]/.sol1], {t, 0, 24}]

In[195] := TableForm[Table[{t, sol1[[1, 1, 2]]}, {t, 4, 24, 4}]]
           4  2.69487
           8  5.50521
           12 8.02624
Out[195] = 16 9.3104
           20 9.78178
           24 9.93326
```

The rate of accumulation of cycloheximide is the difference between the rate of synthesis and the rate of degradation:

$$\frac{dP}{dt} = R_s - R_d.$$

It is known that $R_d = K_d P$, where $K_d = 5 \times 10^{-3}\,\mathrm{h}^{-1}$, so $dP/dt = R_s - R_d$ is equivalent to $dP/dt = R_s - K_d P$. Furthermore,

$$R_s = Q_{po} E X \left(1 + \frac{P}{K_l}\right)^{-1},$$

where Q_{po} represents the specific enzyme activity with value $Q_{po} \approx 0.6\,\mathrm{g\,CH/g}$ protein $\cdot$ h and K_l represents the inhibition constant. E represents the intracellular

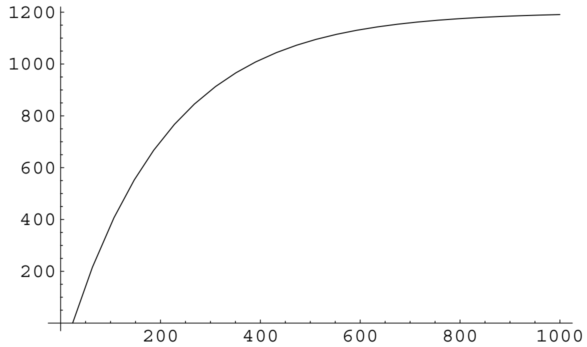


Figure 2-23 Accumulation of the antibiotic

concentration of an enzyme which we will assume is constant. For large values of K_l and t, $X(t) \approx 10$ and $(1 + P/K_l)^{-1} \approx 1$. Thus, $R_s \approx 10Q_{po}E$ so

$$\frac{dP}{dt} = 10Q_{po}E - K_d P.$$

After defining K_d and Q_{po}, we solve the initial-value problem $\begin{cases} dP/dt = 10Q_{po}E - K_d P \\ p(24) = 0 \end{cases}$

and then graph $\frac{1}{E}P(t)$ on the interval [0, 24] in Figure 2-23.

```
In[196]:= Clear[p]

          k_d = 5/1000;

          Q_po = 0.6;

          sol2 = DSolve[{p'[t] == 10Q_po cape - k_d p[t],
                 p[24] == 0}, p[t], t]//Chop
Out[196]= {{p[t] → (-1353. cape + 1200. 2.71828^(0.005 t) cape) ⅇ^(-0.005 t)}}

In[197]:= toplot = Expand[sol2[[1, 1, 2]]/cape]
Out[197]= -1353. ⅇ^(-0.005 t) + 1200. 2.71828^(0.005 t) ⅇ^(-0.005 t)

In[198]:= Plot[toplot, {t, 24, 1000}]
```

From the graph, we see that the total accumulation of the antibiotic approaches a limiting value, which in this case is 1200.

2.6 Numerical Approximations of Solutions to First-Order Equations

2.6.1 Built-In Methods

Numerical approximations of solutions to differential equations can be obtained with `NDSolve`, which is particularly useful when working with nonlinear equations for which `DSolve` alone is unable to find an explicit or implicit solution. The command

```
NDSolve[{y'[t]==f[t,y[t]],y[t0]==y0},y[t],{t,a,b}]
```

attempts to generate a numerical solution of

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

valid for $a \le t \le b$. In some cases, the interval on which the solution is returned by `NDSolve` is smaller than the interval requested. You can obtain basic information regarding `NDSolve` by entering `?NDSolve` or detailed information by accessing Mathematica's on-line help facility by selecting **Help** from the Mathematica menu.

EXAMPLE 2.6.1: Consider

$$\frac{dy}{dt} = \left(t^2 - y^2\right)\sin y,\ y(0) = -1.$$

(a) Determine $y(1)$. (b) Graph $y(t)$, $-1 \le t \le 10$.

SOLUTION: We first remark that `DSolve` can neither exactly solve the differential equation $y' = \left(t^2 - y^2\right)\sin y$ nor find the solution that satisfies $y(0) = -1$.

```
In[199]:= sol = DSolve[y'[t] == (t^2 - y[t]^2) Sin[t],
                    y[t], t]

Out[199]= BoxData(DSolve[y'[t] == Sin[t]  (t^2 - y[t]^2),
            y[t], t])
```

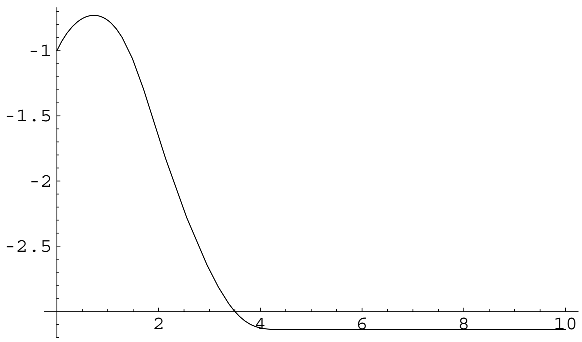


Figure 2-24 Graph of the solution to $y' = (t^2 - y^2)\sin y$, $y(0) = -1$

```
In[200]:= sol =
            DSolve[{y′[t] == (t^2 - y[t]^2) Sin[t],
            y[0] == y0}, y[t], t]

Out[200]= BoxData(DSolve[{y′[t] == Sin[t] (t² - y[t]²),
            y[0] == y0}, y[t], t])
```

However, we obtain a numerical solution valid for $0 \le t \le 1000$ using the `NDSolve` function.

```
In[201]:= sol = NDSolve[{y′[t] == (t^2 - y[t]^2) Sin[y[t]],
                y[0] == -1}, y[t], {t, 0, 1000}]

Out[201]= BoxData({{y[t] →
            InterpolatingFunction[{{0., 1000.}},
            " <> "][t]}})
```

Entering `sol /.t->1` evaluates the numerical solution if $t = 1$.

```
In[202]:= sol /.t- >1

Out[202]= {{y[1] → -0.766014}}
```

The result means that $y(1) \approx -.766$. We use the `Plot` command to graph the solution for $0 \le t \le 10$ in Figure 2-24.

```
In[203]:= Plot[Evaluate[y[t]/.sol], {t, 0, 10}]
```

■

EXAMPLE 2.6.2: Graph the solution to the initial-value problem

$$\begin{cases} dy/dx = \sin(2x - y) \\ y(0) = 0.5 \end{cases}$$

on the interval [0, 15]. What is the value of $y(1)$?

SOLUTION: We use `NDSolve` to approximate the solution to the initial-value problem, naming the resulting output `numsol`. The resulting `InterpolatingFunction` is a procedure that represents an approximate function obtained through interpolation.

```
In[204]:= Clear[x, y]

          numsol = NDSolve[{y'[x] == Sin[2x - y[x]],
            y[0] == 0.5}, y[x], {x, 0, 15}]
Out[204]= {{y[x] → InterpolatingFunction[{{0., 15.}},
            <>][x]}}
```

We can evaluate `numsol` for particular values of x. For example, entering

```
In[205]:= numsol/.x → 1

Out[205]= {{y[1] → 0.875895}}
```

returns a list corresponding to the value of $y(x)$ if $x = 1$. We interpret the result to mean that $y(1) \approx 0.875895$. We then graph the solution returned by `NDSolve` using `Plot` in the same way that we graph solutions returned by `DSolve`. As you probably expect, entering `Plot[numsol[[1,1,2]],{x,0,15}]` produces the same graph as the one shown in Figure 2-25 generated by the following `Plot` command.

```
In[206]:= p1 = Plot[y[x]/.numsol, {x, 0, 15}]
```

One way to graph solutions that satisfy different initial conditions is to define a function as we do here. Given i, `sol[i]` returns a numerical solution to the initial-value problem $y' = \sin(2x - y)$, $y(0) = i$.

```
In[207]:= Clear[x, y, i, sol]

          sol[i_] := NDSolve[{y'[x] == Sin[2x - y[x]],
            y[0] == i}, y[x], {x, 0, 7}];
```

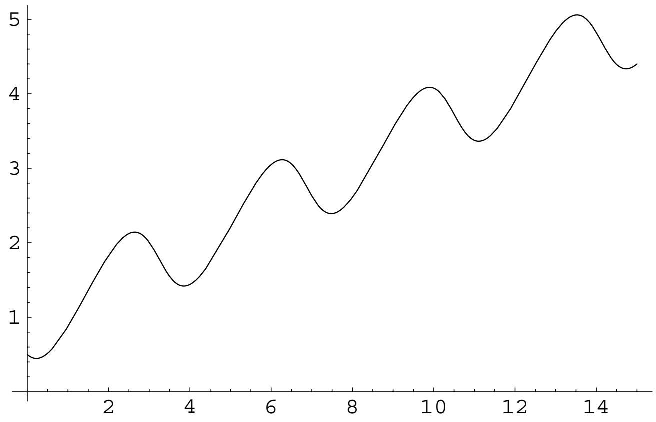


Figure 2-25 Graph of the solution to $y' = \sin(2x - y)$, $y(0) = 0.5$

For example, to use `sol`, we first use `Table` to define `inits` to be the list of numbers $i/2$ for $i = 1, 2, \ldots, 5$ and then use `Map` to apply `sol` to the list of numbers `inits`. The command

```
interpfunctions=Map[sol,inits]
```

computes `sol[i]` for each value of i in `inits`. The result is a nested list consisting of `InterpolatingFunction`'s.

```
In[208]:= inits = Table[i/2, {i, 1, 10}];
```

```
In[209]:= interpfunctions = Map[sol, inits];
```

We graph the set of `InterpolatingFunction`'s with `Plot` in the same way as we graph other sets of functions. See Figure 2-26.

```
In[210]:= plot1 = Plot[Evaluate[y[x] /. interpfunctions],
              {x, 0, 7}, PlotRange- > {0, 7},
              AspectRatio- > 1,
              PlotStyle- > GrayLevel[0]]
```

Last, we show these graphs together with the direction field associated with the equation in Figure 2-27.

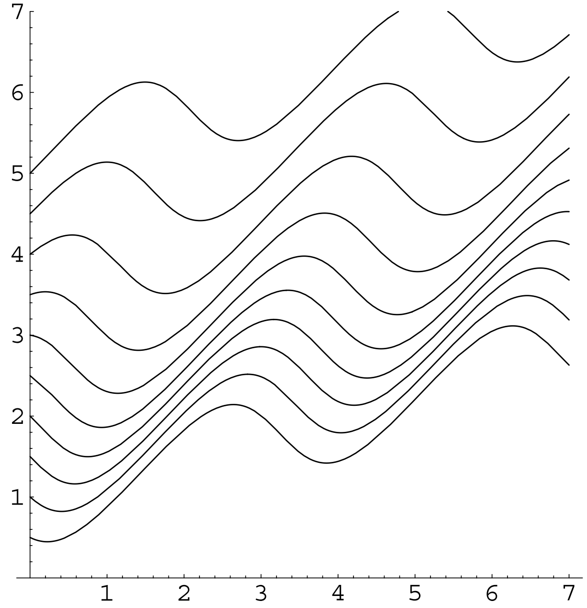


Figure 2-26 Various solutions of $y' = \sin(2x - y)$

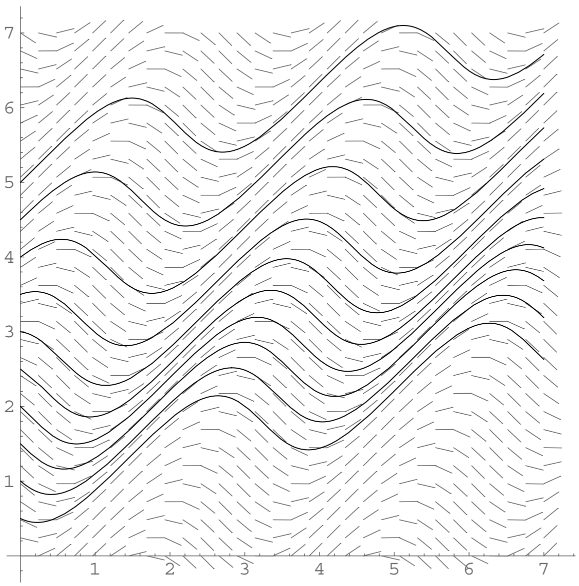


Figure 2-27 Direction field together with various solutions of $y' = \sin(2x - y)$

```
In[211]:= << Graphics`PlotField`

          pvf = PlotVectorField[{1, Sin[2x - y]},
                  {x, 0, 7}, {y, 0, 7}, Frame- > False,
                  Axes- > Automatic,
                  AxesOrigin- > {0, 0}, PlotPoints- > 30,
                  HeadLength- > 0,
                  ScaleFunction- > (1&),
                  DisplayFunction- > Identity,
                  DefaultColor- > GrayLevel[0.4]];

In[212]:= Show[pvf, plot1,
            DisplayFunction- > $DisplayFunction]
```

■

Application: Modeling the Spread of a Disease

Suppose that a disease is spreading among a population of size N. In some diseases, like chickenpox, once an individual has had the disease, the individual becomes immune to the disease. In other diseases, like most venereal diseases, once an individual has had the disease and recovers from the disease, the individual does not become immune to the disease; subsequent encounters can lead to recurrences of the infection.

Source: Herbert W. Hethcote, "Three Basic Epidemiological Models," in *Applied Mathematical Ecology*, edited by Simon A. Levin, Thomas G. Hallan, and Louis J. Gross, New York, Springer-Verlag (1989), pp. 119–143.

Let $S(t)$ denote the percent of the population susceptible to a disease at time t, $I(t)$ the percent of the population infected with the disease, and $R(t)$ the percent of the population unable to contract the disease. For example, $R(t)$ could represent the percent of persons who have had a particular disease, recovered, and have subsequently become immune to the disease. In order to model the spread of various diseases, we begin by making several assumptions and introducing some notation.

1. Susceptible and infected individuals die at a rate proportional to the number of susceptible and infected individuals with proportionality constant μ called the **daily death removal rate**; the number $1/\mu$ is the **average lifetime** or **life expectancy**.
2. The constant λ represents the **daily contact rate**: on average, an infected person will spread the disease to λ people per day.
3. Individuals recover from the disease at a rate proportional to the number infected with the disease with proportionality constant γ. The constant γ is called the **daily recovery removal rate**; the **average period of infectivity** is $1/\gamma$.
4. The **contact number** $\sigma = \lambda/(\gamma + \mu)$ represents the average number of contacts an infected person has with both susceptible and infected persons.

If a person becomes susceptible to a disease after recovering from it (like gonorrhea, meningitis, and streptococcal sore throat), then the percent of persons susceptible to becoming infected with the disease, $S(t)$, and the percent of people in the population infected with the disease, $I(t)$, can be modeled by the system of differential equations

$$\begin{cases} \dfrac{dS}{dt} = -\lambda IS + \gamma I + \mu - \mu S \\ \dfrac{dI}{dt} = \lambda IS - \gamma I - \mu I \\ S(0) = S_0,\ I(0) = I_0,\ S(t) + I(t) = 1. \end{cases} \tag{2.8}$$

This model is called an **SIS model** (susceptible–infected–susceptible model) because once an individual has recovered from the disease, the individual again becomes susceptible to the disease.

We can write $dI/dt = \lambda IS - \gamma I - \mu I$ as $dI/dt = \lambda I(1-I) - \gamma I - \mu I$ because $S(t) = 1 - I(t)$ and thus we need to solve the initial-value problem

$$\begin{cases} \dfrac{dI}{dt} = [\lambda - (\gamma + \mu)]\, I - \lambda I^2 \\ I(0) = I_0 . \end{cases} \tag{2.9}$$

In the following, we use `i` to represent I, thus avoiding conflict with the built-in constant `I` $= \sqrt{-1}$. After defining `eq`, we use `DSolve` to find the solution to the initial-value problem.

```
In[213]:= eq = i'[t] + (γ + μ - λ) i[t] == -λ i[t]^2;

In[214]:= sol = DSolve[{eq, i[0] == i0}, i[t], t]
```

$$\text{Out[214]= } \left\{\left\{\text{i[t]} \to -\frac{45.\left(\frac{\text{i0}}{9.+5.\,\text{i0}}\right)^{1.}}{-5.\,e^{0.9\,t} + 25.\left(\frac{\text{i0}}{9.+5.\,\text{i0}}\right)^{1.}}\right\}\right\}$$

We can use this result to see how a disease might spread through a population. For example, we compute the solution to the initial-value problem, which is extracted from `sol` with `sol[[1,1,2]]`, if $\lambda = 0.5$, $\gamma = 0.75$, and $\mu = 0.65$. In this case, we see that the contact number is $\sigma = \lambda/(\gamma + \mu) \approx 0.357143$.

```
In[215]:= λ = 0.5;

          γ = 0.75;

          μ = 0.65;

          σ = λ/(γ + μ)

          sol[[1, 1, 2]]
```

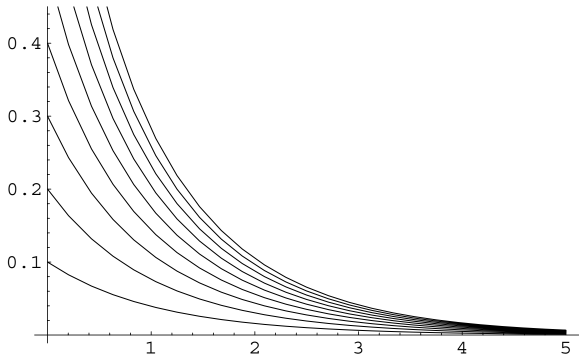


Figure 2-28 The disease is removed from the population

```
Out[215]= 0.357143
```

$$\text{Out[215]} = -\frac{45.\left(\frac{\text{i0}}{9.+5.\,\text{i0}}\right)^{1.}}{-5.\,e^{0.9t}+25.\left(\frac{\text{i0}}{9.+5.\,\text{i0}}\right)^{1.}}$$

Next, we use `Table` to substitute various initial conditions into `sol[[1,1,2]]`, naming the resulting set of nine functions `toplot1`. We then graph the functions in `toplot1` for $0 \leq t \leq 5$ in Figure 2-28. Apparently, regardless of the initial percent of the population infected, under these conditions, the disease is eventually removed from the population. This makes sense because the contact number is less than one.

```
In[216]:= toplot1 = Table[sol[[1, 1, 2]], {i0, 0.1, 0.9, 0.1}];

In[217]:= Plot[Evaluate[toplot1], {t, 0, 5}]
```

On the other hand, if $\lambda = 1.5$, $\gamma = 0.75$, and $\mu = 0.65$, we see that the contact number is $\sigma = \lambda/(\gamma + \mu)$.

```
In[218]:= Clear[λ, γ, μ, σ]

          eq = i′[t] + (γ + μ - λ) i[t] == -λ i[t]^2;
          sol = DSolve[{eq, i[0] == i0}, i[t], t];
          λ = 1.5;
          γ = 0.75;
          μ = 0.65;
          σ = λ/(γ + μ)

          sol[[1, 1, 2]]
Out[218]= 1.07143
```

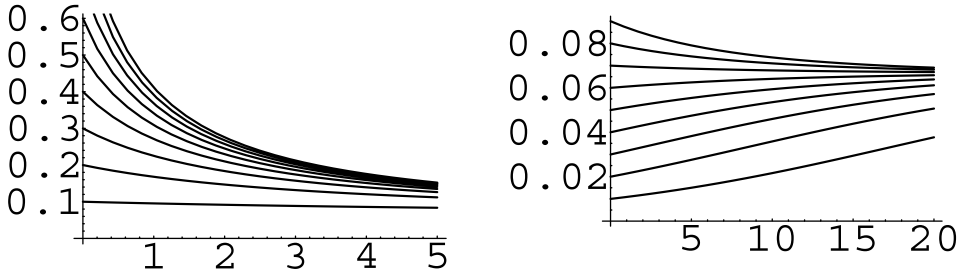


Figure 2-29 The disease persists

```
Out[218]= (0.1 e^(1.5 t) i0)/
           (0.1 e^(1.4 t) -
              1.5 e^(1.4 t) i0 + 1.5 e^(1.5 t) i0)
```

Proceeding as before, we graph the solution using different initial conditions in Figure 2-29. In this case, we see that no matter what percent of the population is initially infected, a certain percent of the population is always infected. This makes sense because the contact number is greater than one. In fact, it is a theorem that

$$\lim_{t\to\infty} I(t) = \begin{cases} 1 - 1/\sigma, & \text{if } \sigma > 1 \\ 0, & \text{if } \sigma \le 1. \end{cases}$$

```
In[219]:= toplot2 = Table[sol[[1, 1, 2]], {i0, 0.1, 0.9, 0.1}];

          p1 = Plot[Evaluate[toplot2], {t, 0, 5},
               DisplayFunction- > Identity];

In[220]:= toplot3 = Table[sol[[1, 1, 2]], {i0, 0.01, 0.09, 0.01}];

          p2 = Plot[Evaluate[toplot3], {t, 0, 20},
               DisplayFunction- > Identity];

In[221]:= Show[GraphicsArray[{p1, p2}]]
```

The incidence of some diseases, such as measles, rubella, and gonorrhea, oscillates seasonally. To model these diseases, we may wish to replace the constant contact rate λ, by a periodic function $\lambda(t)$. For example, to graph the solution to the SIS model for various initial conditions if (a) $\lambda(t) = 3 - 2.5\sin 6t$, $\gamma = 2$, and $\mu = 1$ and (b) $\lambda(t) = 3 - 2.5\sin 6t$, $\gamma = 1$, and $\mu = 1$ we proceed as follows. For (a), we begin by defining λ, γ, and μ, and eq.

```
In[222]:= Clear[λ, i, t, γ, μ]

          λ[t_] = 3 - 2.5 Sin[6t];

             γ = 2;

             μ = 1;

             eq = i'[t] == (λ[t] - (γ + μ)) i[t] - λ[t] i[t]^2
Out[222]= i'[t] == -i[t]^2 (3 - 2.5 Sin[6 t]) - 2.5 i[t] Sin[6 t]
```

We will graph the solutions satisfying the initial conditions $I(0) = I_0$ for $I_0 = 0.1, 0.2, \ldots, 0.9$. We begin by defining `graph`. Given `i0`, `graph[i0]` graphs the solution to the initial-value problem

$$\begin{cases} \frac{dI}{dt} = [\lambda(t) - (\gamma + \mu)]\, I - \lambda(t) I^2 \\ I(0) = I_0 \end{cases}$$

on the interval [0, 10]. The resulting graphics object is not displayed because we include the option `DisplayFunction->Identity` in the `Plot` command. Next, we use `Table` to define the list of numbers `inits`, corresponding to the initial conditions, and then use `Map` to apply the function `graph` to the list of numbers `inits`. We see that the result is a list of nine graphics objects that we name `toshow`.

```
In[223]:= graph[i0_] := Module[{numsol},
                numsol = NDSolve[{eq, i[0] == i0}, i[t], {t, 0, 10}];
              Plot[i[t]/.numsol, {t, 0, 10},
              DisplayFunction → Identity]]

In[224]:= inits = Table[i, {i, 0.1, 0.9, 0.1}];

          toshow = Map[graph, inits];
```

Finally, we use `Show` together with the option `DisplayFunction->` `$DisplayFunction` to view the list of nine graphs `toshow` in Figure 2-30.

```
In[225]:= Show[toshow, DisplayFunction → $DisplayFunction,
            PlotRange → {0, 1}]
```

For (b), we proceed in the same manner as in (a). See Figure 2-31.

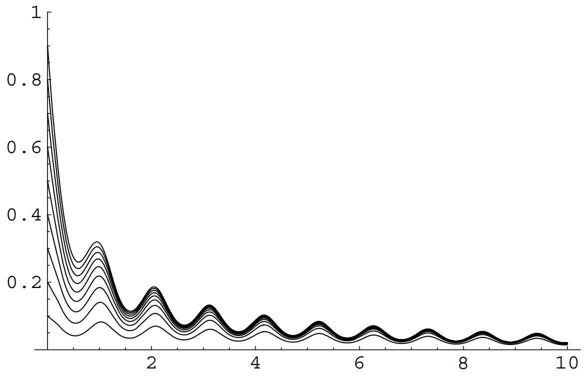


Figure 2-30 The disease is slowly eliminated from the population

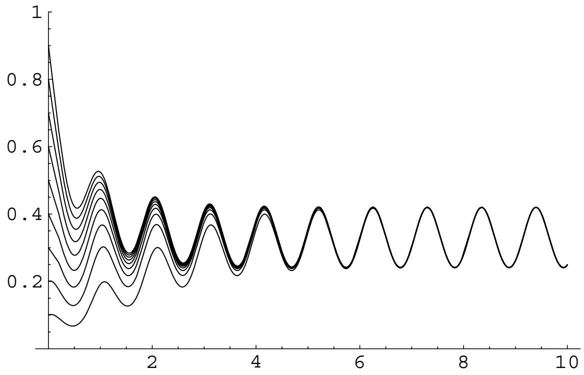


Figure 2-31 The disease persists periodically in the population

```
In[226]:= Clear[λ, i, t, γ, μ]

          λ[t_] = 3 - 2.5 Sin[6t];

            γ = 1;

            μ = 1;

In[227]:= eq = i′[t] == (λ[t] - (γ + μ)) i[t] - λ[t] i[t]^2
Out[227]= i′[t] == i[t] (1 - 2.5 Sin[6 t]) - i[t]^2 (3 - 2.5 Sin[6 t])

In[228]:= graph[i0_] := Module[{numsol},
             numsol = NDSolve[{eq, i[0] == i0}, i[t], {t, 0, 10}];
          Plot[i[t]/.numsol, {t, 0, 10},
          DisplayFunction → Identity]]
```

```
In[229]:= inits = Table[i, {i, 0.1, 0.9, 0.1}];
          toshow = graph/@inits;

In[230]:= Show[toshow, DisplayFunction → $DisplayFunction,
          PlotRange → {0, 1}]
```

2.6.2 Other Numerical Methods

In other cases, you may wish to implement your own numerical algorithms to approximate solutions of differential equations. We briefly discuss three familiar methods (Euler's method, the improved Euler's method, and the Runge–Kutta method) and illustrate how to implement these algorithms using Mathematica. Details regarding these and other algorithms, including discussions of the error involved in implementing them, can be found in most numerical analysis texts or other references like Zwillinger's *Handbook of Differential Equations* [28].

Euler's Method

In many cases, we cannot obtain an explicit formula for the solution to an initial-value problem of the form

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

but we can approximate the solution using a numerical method like **Euler's method**, which is based on tangent line approximations. Let h represent a small change, or **step size**, in the independent variable x. Then, we approximate the value of y at the sequence of x-values, $x_1, x_2, \ldots, x_n$, where

$$\begin{aligned} x_1 &= x_0 + h \\ x_2 &= x_1 + h = x_0 + 2h \\ x_3 &= x_2 + h = x_0 + 3h \\ &\vdots \\ x_n &= x_{n-1} + h = x_0 + nh. \end{aligned}$$

The slope of the tangent line to the graph of y at each value of x is found with the differential equation $y' = dy/dx = f(x, y)$. For example, at $x = x_0$, the slope of the tangent line is $f(x_0, y(x_0)) = f(x_0, y_0)$. Therefore, the tangent line to the graph of y is

$$y - y_0 = f(x_0, y_0)(x - x_0) \qquad \text{or} \qquad y = f(x_0, y_0)(x - x_0) + y_0.$$

Using this line to find the value of y, which we call y_1, at x_1 then yields

$$y_1 = f(x_0, y_0)(x_1 - x_0) + y_0 = hf(x_0, y_0) + y_0.$$

Therefore, we obtain the approximate value of y at x_1. Next, we use the point (x_1, y_1) to estimate the value of y when $x = x_2$. Using a similar procedure, we approximate the tangent line at $x = x_1$ with

$$y - y_1 = f(x_1, y_1)(x - x_1) \qquad \text{or} \qquad y = f(x_1, y_1)(x - x_1) + y_1.$$

Then, at $x = x_2$,

$$y_2 = f(x_1, y_1)(x_2 - x_1) + y_1 = hf(x_1, y_1) + y_1.$$

Continuing with this procedure, we see that at $x = x_n$,

$$y_n = hf(x_{n-1}, y_{n-1}) + y_{n-1}. \tag{2.10}$$

Using this formula, we obtain a sequence of points of the form (x_n, y_n), $n = 1, 2, \ldots$ where y_n is the approximate value of $y(x_n)$.

```
In[231]:= xe[n_] = x0 + nh;

          ye[n_] := ye[n] = h f[xe[n - 1], ye[n - 1]] + ye[n - 1];

          ye[0] = y0;
```

EXAMPLE 2.6.3: Use Euler's method with (a) $h = 0.1$ and (b) $h = 0.05$ to approximate the solution of $y' = xy$, $y(0) = 1$ on $0 \le x \le 1$. Also, determine the exact solution and compare the results.

SOLUTION: Because we will be considering this initial-value problem in subsequent examples, we first determine the exact solution with `DSolve` and graph the result with `Plot`, naming the graph `p1`.

```
In[232]:= Clear[x, y]

          exactsol = DSolve[{y'[x] == x y[x],
            y[0] == 1}, y[x], x]
Out[232]= {{y[x] → e^(x^2/2)}}

In[233]:= p1 = Plot[e^(x^2/2), {x, 0, 1},
                PlotStyle → GrayLevel[0.4],
                DisplayFunction- > Identity];
```

To implement Euler's method (2.10), we note that $f(x, y) = xy$, $x_0 = 0$, and $y_0 = 1$. (a) With $h = 0.1$, we have the formula

$$y_n = hf(x_{n-1}, y_{n-1}) + y_{n-1} = 0.1x_{n-1}y_{n-1} + y_{n-1}.$$

For $x_1 = x_0 + h = 0.1$, we have

$$y_1 = 0.1x_0y_0 + y_0 = 0.1 \cdot 0 \cdot 1 + 1 = 1.$$

Similarly, for $x_2 = x_0 + 2h = 0.2$,

$$y_2 = 0.1x_1y_1 + y_1 = 0.1 \cdot 0.1 \cdot 1 + 1 = 1.01.$$

In the following, we define f, h, x, and y to calculate y_n given by equation (2.10). We define ye using the form

```
ye[n_]:=ye[n]=...
```

so that Mathematica "remembers" the values of ye computed, and thus, when computing ye[n], Mathematica need not recompute ye[n-1] if ye[n-1] has previously been computed.

```
In[234]:= f[x_, y_] = xy;

          h = 0.1;

          x0 = 0;

          y0 = 1;

In[235]:= xe[n_] = x0 + nh;

          ye[n_] := ye[n] = h * f[xe[n - 1],
            ye[n - 1]] + ye[n - 1];

          ye[0] = y0;
```

Next, we use Table to calculate the set of ordered pairs (x_n, y_n) for $n = 0, 1, 2, \ldots, 9, 10$, naming the result first, and then TableForm to view first in traditional row-and-column form.

```
In[236]:= first = Table[{xe[n], ye[n]}, {n, 0, 10}];

          TableForm[first]
          0   1
          0.1 1
          0.2 1.01
          0.3 1.0302
          0.4 1.06111
Out[236]= 0.5 1.10355
          0.6 1.15873
          0.7 1.22825
          0.8 1.31423
          0.9 1.41937
          1.  1.54711
```

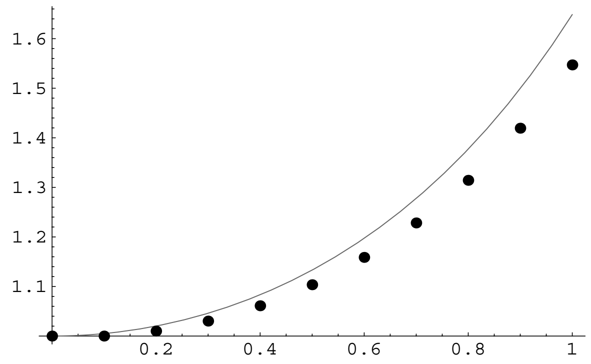


Figure 2-32 Comparison of Euler's method to the exact solution using $h = 0.1$

To compare these results to the exact solution, we use `ListPlot` to graph the list of ordered pairs `first` in `t2` and display `t2` together with `p1` with `Show` in Figure 2-32.

```
In[237]:= lp = Map[Point, first]
Out[237]= {Point[{0, 1}], Point[{0.1, 1}],
           Point[{0.2, 1.01}], Point[{0.3, 1.0302}],
           Point[{0.4, 1.06111}], Point[{0.5, 1.10355}],
           Point[{0.6, 1.15873}], Point[{0.7, 1.22825}],
           Point[{0.8, 1.31423}], Point[{0.9, 1.41937}],
           Point[{1., 1.54711}]}

In[238]:= t2 = Graphics[{PointSize[0.02], lp}];

In[239]:= Show[p1, t2,
           DisplayFunction- > $DisplayFunction]
```

Alternatively, we can produce Figure 2-32 using `ListPlot` together with the option `PlotStyle->PointSize[.02]` with the following command.

```
In[240]:= t2 = ListPlot[first,
           PlotStyle → PointSize[0.02],
           DisplayFunction → Identity];
           Show[p1,
           t2, DisplayFunction → $DisplayFunction]
```

(b) For $h = 0.05$, we use

$$y_n = hf(x_{n-1}, y_{n-1}) + y_{n-1} = 0.05x_{n-1}y_{n-1} + y_{n-1}$$

to obtain an approximation. In the same manner as in (a), we define f, h, x, and y to calculate y_n given by equation (2.10). Then, we use `Table` to calculate the set of ordered pairs for $n = 0, 1, 2, \ldots, 19, 20$, naming the result `second`, followed by `TableForm` to view `second` in traditional row-and-column form.

```
In[241]:= Remove[x, y, f]

          f[x_, y_] = x * y;

                h = 0.05;

               x0 = 0;

               y0 = 1;

In[242]:= xe[n_] = x0 + n * h;

          ye[n_] := ye[n] = h * f[xe[n - 1],
                   ye[n - 1]] + ye[n - 1];

          ye[0] = y0;

In[243]:= second = Table[{xe[n], ye[n]}, {n, 0, 20}];

          TableForm[second]
```

Out[243]=

0	1
0.05	1
0.1	1.0025
0.15	1.00751
0.2	1.01507
0.25	1.02522
0.3	1.03803
0.35	1.05361
0.4	1.07204
0.45	1.09348
0.5	1.11809
0.55	1.14604
0.6	1.17756
0.65	1.21288
0.7	1.2523
0.75	1.29613
0.8	1.34474
0.85	1.39853
0.9	1.45796
0.95	1.52357
1.	1.59594

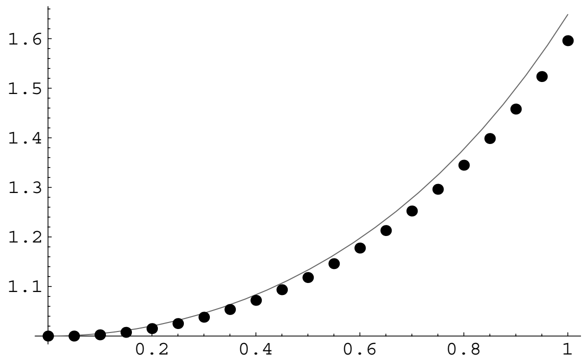


Figure 2-33 Comparison of Euler's method to the exact solution using $h = 0.05$

We graph the approximation obtained with $h = 0.05$ together with the graph of $y = e^{x^2/2}$ in Figure 2-33. Notice that the approximation is more accurate when h is decreased.

```
In[244]:= t3 = ListPlot[second,
               PlotStyle- > PointSize[0.02],
               DisplayFunction- > Identity];

In[245]:= Show[p1, t3,
            DisplayFunction- > $DisplayFunction]
```

■

Improved Euler's Method

Euler's method can be improved by using an average slope over each interval. Using the tangent line approximation of the curve through (x_0, y_0), $y = f(x_0, y_0)(x - x_0) + y_0$, we find the approximate value of y at $x = x_1$ which we now call $y_1{}^*$:

$$y_1{}^* = hf(x_0, y_0) + y_0.$$

With the differential equation $y' = f(x, y)$, we find that the approximate slope of the tangent line at $x = x_1$ is $f\left(x, y_1^1\right)$. Then, the average of the two slopes, $f(x_0, y_0)$ and $f(x_1, y_1{}^*)$, is $\frac{1}{2}(f(x_0, y_0) + f(x_1, y_1{}^*))$, and an equation of the line through (x_0, y_0) with slope $\frac{1}{2}(f(x_0, y_0) + f(x_1, y_1{}^*))$ is

$$y = \frac{1}{2}(f(x_0, y_0) + f(x_1, y_1{}^*))(x - x_0) + y_0.$$

Therefore, at $x = x_1$, we find the approximate value of f with

$$y_1 = \frac{1}{2}(f(x_0, y_0) + f(x_1, y_1{}^*))(x_1 - x_0) + y_0 = \frac{1}{2}h(f(x_0, y_0) + f(x_1, y_1{}^*)) + y_0.$$

Continuing in this manner, the approximation at each step of the **improved Euler's method** depends on the following two calculations:

$$y_n^* = hf(x_{n-1}, y_{n-1}) + y_{n-1}$$
$$y_n = \frac{1}{2}h(f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)) + y_{n-1}. \tag{2.11}$$

```
In[246]:= xi[n_] = x0 + nh;

          yi[n_] :=
            yi[n] = 1/2 h (f[xi[n - 1], yi[n - 1]]
                    +f[xi[n], h f[xi[n - 1], yi[n - 1]]
                    +yi[n - 1]]) + yi[n - 1];

          yi[0] = y0;
```

EXAMPLE 2.6.4: Use the improved Euler's method to approximate the solution of $y' = xy$, $y(0) = 1$ on $0 \le x \le 1$ for $h = 0.1$. Also, compare the results to the exact solution.

SOLUTION: In this case, $f(x, y) = xy$, $x_0 = 0$, and $y_0 = 1$ so equations (2.11) become

$$y_n^* = hx_{n-1}y_{n-1} + y_{n-1}$$
$$y_n = \frac{1}{2}h(x_{n-1}y_{n-1} + x_n y_n^*) + y_{n-1}$$

for $n = 1, 2, \ldots, 10$. For example, if $n = 1$, we have

$$y_1^* = hx_0y_0 + y_0 = 0.1 \cdot 0 \cdot 1 + 1 = 1$$

and

$$y_1 = \frac{1}{2}h(x_0y_0 + x_1y_1^*) + y_0 = \frac{1}{2} \cdot 0.1 \cdot (0 \cdot 1 + 0.1 \cdot 1) + 1 = 1.005.$$

Similarly,

$$y_2^* = hx_1y_1 + y_1 = 0.1 \cdot 0.1 \cdot 1.005 + 1.005 = 1.01505$$

and

$$y_2 = \frac{1}{2}h(x_1y_1 + x_2y_2^*) + y_1 =$$
$$\frac{1}{2} \cdot 0.1 \cdot (0.1 \cdot 1.005 + 0.2 \cdot 1.01505) + 1.005 = 1.0201755.$$

In the same way as in the previous example, we define f, x, h, and y. We define `yi` using the form

`yi[n_]:=yi[n]=...,`

so that Mathematica "remembers" the values of `ystar` and `y` computed. Thus, to compute `yi[n]`, Mathematica need not recompute `yi [n-1]` if `yi[n-1]` has previously been computed.

```
In[247]:= Remove[f,x,y]

          f[x_,y_] = x y;

             h = 0.1;

             x0 = 0;

             y0 = 1;

In[248]:= xi[n_] = x0 + n h;

          yi[n_] := yi[n] =
             N[1/2 h (f[xi[n - 1], yi[n - 1]]
             +f[xi[n], h f[xi[n - 1],
             yi[n - 1]] + yi[n - 1]]) + yi[n - 1]];


          yi[0] = y0;
```

We then compute (x_n, y_n) for $n = 0, 1, \ldots, 10$ and name the resulting list of ordered pairs `third`.

```
In[249]:= third = Table[{xi[n], yi[n]}, {n, 0, 10}];

          TableForm[third]
          0   1
          0.1 1.005
          0.2 1.02018
          0.3 1.04599
          0.4 1.08322
Out[249]= 0.5 1.13305
          0.6 1.19707
          0.7 1.27739
          0.8 1.37677
          0.9 1.49876
          1.  1.64788
```

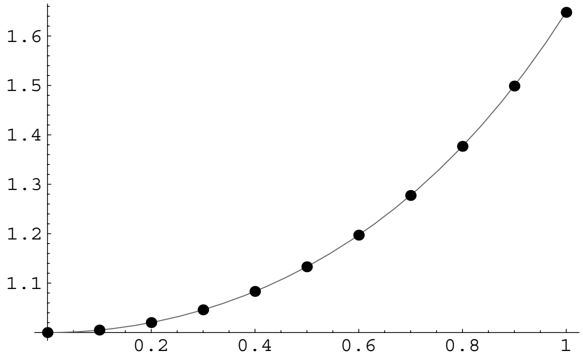


Figure 2-34 Comparison of the improved Euler's method to the exact solution using $h = 0.1$

We graph the approximation obtained using the improved Euler's method together with the graph of the exact solution in Figure 2-34. From the results, we see that the approximation using the improved Euler's method results in a slight improvement from that obtained in the previous example.

```
In[250]:= t4 = ListPlot[third,
                PlotStyle- > PointSize[0.02],
                DisplayFunction- > Identity];

          Show[p1, t4,
            DisplayFunction- > $DisplayFunction]
```

■

The Runge–Kutta Method

In an attempt to improve on the approximation obtained with Euler's method as well as avoid the analytic differentiation of the function $f(x, y)$ to obtain $y'', y''', \ldots,$ the *Runge–Kutta method* is introduced. Let us begin with the *Runge–Kutta method of order two*. Suppose that we know the value of y at x_n. We now use the point (x_n, y_n) to approximate the value of y at a nearby value $x = x_n + h$ by assuming that

$$y_{n+1} = y_n + Ak_1 + Bk_2,$$

where

$$k_1 = hf(x_n, y_n) \qquad \text{and} \qquad k_2 = hf(x_n + ah, y_n + bk_1).$$

We can use the Taylor series expansion of y to obtain another representation of $y_{n+1} = y(x_n + h)$ as follows:

$$y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \cdots = y_n + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \cdots$$

Now, because

$$y_{n+1} = y_n + Ak_1 + Bk_2 = y_n + Ahf(x_n, y_n) + Bhf(x_n + ah, y_n + bhf(x_n, y_m)),$$

we wish to determine values of A, B, a, and b such that these two representations of y_{n+1} agree. Notice that if we let $A = 1$ and $B = 0$, then the relationships match up to order h. However, we can choose these parameters more wisely so that agreement occurs up through terms of order h^2. This is accomplished by considering the Taylor series expansion of a function $z = F(x, y)$ of two variables about (x_0, y_0) which is given by

$$F(x_0, y_0) + \frac{\partial F}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0) + \cdots$$

In our case, we have

$$\begin{aligned} f(x_n + ah, y_n + bhf(x_n, y_m)) &= f(x_n, y_n) + ah\frac{\partial f}{\partial x}(x_n, y_n) \\ &\quad + bhf(x_n, y_n)\frac{\partial f}{\partial y}(x_n, y_n) + O\left(h^2\right). \end{aligned}$$

The power series is then substituted into the following expression and simplified to yield:

$$\begin{aligned} y_{n+1} &= y_n + Ahf(x_n, y_n) + Bhf(x_n + ah, y_n + bhf(x_n, y_m)) \\ &= y_n + (A + B)hf(x_n, y_n) + aBh^2\frac{\partial f}{\partial x}(x_n, y_n) + bBh^2 f(x_n, y_n)\frac{\partial f}{\partial y}(x_n, y_n) + O\left(h^3\right). \end{aligned}$$

Comparing this expression to the following power series obtained directly from the Taylor series of y,

$$y(x_n + h) = y(x_n) + hf(x_n, y_n) + \frac{1}{2}h^2\frac{\partial f}{\partial x}(x_n, y_n) + \frac{1}{2}h^2\frac{\partial f}{\partial y}(x_n, y_n) + O\left(h^3\right)$$

or

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{1}{2}h^2\frac{\partial f}{\partial x}(x_n, y_n) + \frac{1}{2}h^2\frac{\partial f}{\partial y}(x_n, y_n) + O\left(h^3\right),$$

we see that A, B, a, and b must satisfy the following system of nonlinear equations:

$$A + B = 1, \qquad aB = \frac{1}{2}, \qquad \text{and} \qquad bB = \frac{1}{2}.$$

Therefore, choosing $a = b = 1$, the Runge–Kutta method of order two uses the equation:

$$\begin{aligned} y_{n+1} &= y(x_n + h) = y_n + \frac{1}{2}hf(x_n, y_n) + \frac{1}{2}hf(x_n + h, y_n + hf(x_n, y_n)) \\ &= y_n + \frac{1}{2}(k_1 + k_2) \end{aligned} \tag{2.12}$$

where $k_1 = hf(x_n, y_n)$ and $k_2 = hf(x_n + h, y_n + k_1)$.

```
In[251]:= xr[n_] = x0 + nh;

          yr[n_] := yr[n] =
              yr[n - 1] + 1/2 h f[xr[n - 1], yr[n - 1]]
              + 1/2 h f[xr[n - 1] + h, yr[n - 1]
              +h f[xr[n - 1], yr[n - 1]]]

          yr[0] = y0;
```

EXAMPLE 2.6.5: Use the Runge–Kutta method of order two with $h = 0.1$ to approximate the solution of the initial-value problem $y' = xy$, $y(0) = 1$ on $0 \le x \le 1$.

SOLUTION: As with the previous examples, $f(x, y) = xy$, $x_0 = 0$, and $y_0 = 1$. Therefore, on each step we use the three equations

$$k_1 = hf(x_n, y_n) = 0.1 x_n y_n,$$

$$k_2 = hf(x_n + h, y_n + k_1) = 0.1(x_n + 0.1)(y_n + k_1),$$

and

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2).$$

For example, if $n = 0$, then

$$k_1 = 0.1 x_0 y_0 = 0.1 \cdot 0 \cdot 1 = 0,$$

$$k_2 = 0.1(x_0 + 0.1)(y_0 + k_1) = 0.1 \cdot 0.1 \cdot 1 = 0.01,$$

and

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2} \cdot 0.01 = 1.005.$$

Therefore, the Runge–Kutta method of order two approximates that the value of y at $x = 0.1$ is 1.005.

In the same manner as in the previous two examples, we define a function `yr` to implement the Runge–Kutta method of order two and use `Table` to generate a set of approximations for $n = 0, 1, \ldots, 10$.

```
In[252]:= Remove[f,x,y]

          f[x_,y_] = xy;

                h = 0.1;

               x0 = 0;

               y0 = 1;

In[253]:= xr[n_] = x0 + nh;

          yr[n_] := yr[n] =
              yr[n - 1] + 1/2 h f[xr[n - 1],
              yr[n - 1]] + 1/2 h f[xr[n - 1] + h, yr[n - 1]
              +h f[xr[n - 1], yr[n - 1]]]

          yr[0] = y0;

In[254]:= rktable1 = Table[{xr[i], yr[i]}, {i, 0, 10}];

          TableForm[rktable1]

          0   1
          0.1 1.005
          0.2 1.02018
          0.3 1.04599
          0.4 1.08322
Out[254]= 0.5 1.13305
          0.6 1.19707
          0.7 1.27739
          0.8 1.37677
          0.9 1.49876
          1.  1.64788
```

We then use `ListPlot` to graph the set of points determined in `rktable1`. The resulting graphics object, named `p2`, is not displayed because the option `DisplayFunction->Identity` is included in the `ListPlot` command. The graphs in `p1` and `p2` are shown together with `Show` in Figure 2-35.

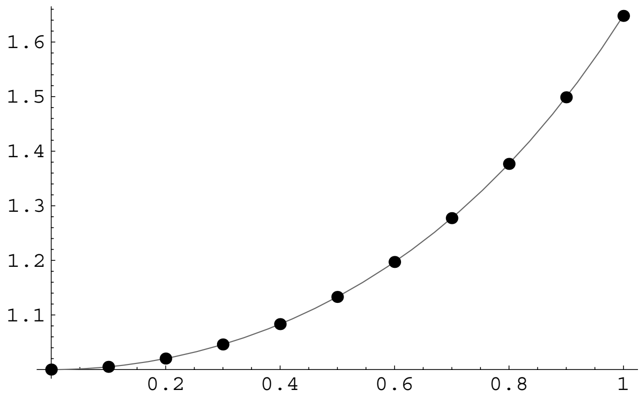


Figure 2-35 Comparison of the Runge-Kutta method of order two to the exact solution using $h = 0.1$

```
In[255]:= p2 = ListPlot[rktable1,
                PlotStyle- > PointSize[0.02],
                DisplayFunction → Identity];

          Show[p1,
            p2, DisplayFunction → $DisplayFunction]
```

■

The terms of the power series expansions used in the derivation of the Runge–Kutta method of order two can be made to match up to order four. These computations are rather complicated, so they will not be discussed here. However, after much work, the **fourth-order Runge–Kutta method** approximation at each step is found to be made with

$$y_{n+1} = y_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right),\ n = 0,\ 1,\ 2,\ \ldots$$

where

$$\begin{aligned} k_1 &= f\left(x_n, y_n\right) \\ k_2 &= f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \\ k_3 &= f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right) \end{aligned} \tag{2.13}$$

and

$$k_4 = f\left(x_{n+1}, y_n + hk_3\right).$$

EXAMPLE 2.6.6: Use the fourth-order Runge–Kutta method with $h = 0.1$ to approximate the solution of the problem $y' = xy$, $y(0) = 1$ on $0 \le x \le 1$.

SOLUTION: With $f(x, y) = xy$, $x_0 = 0$, and $y_0 = 1$, using equations (2.13), the formulas are

$$y_{n+1} = y_n + \frac{0.1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),\ n = 0,\ 1,\ 2,\ \ldots$$

where

$$k_1 = f\left(x_n, y_n\right) = x_n y_n$$

$$k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) = \left(x_n + \frac{1}{2}\cdot 0.1\right)\left(y_n + \frac{1}{2}\cdot 0.1k_1\right)$$

$$k_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right) = \left(x_n + \frac{1}{2}\cdot 0.1\right)\left(y_n + \frac{1}{2}\cdot 0.1k_2\right)$$

and

$$k_4 = f\left(x_{n+1}, y_n + hk_3\right) = x_{n+1}\left(y_n + 0.1k_3\right).$$

For $n = 0$, we have

$$k_1 = x_0 y_0 = 0\cdot 1 = 0$$

$$k_2 = \left(x_0 + \frac{1}{2}\cdot 0.1\right)\left(y_0 + \frac{1}{2}\cdot 0.1k_1\right) = 0.05\cdot 1 = 0.05$$

$$k_3 = \left(x_0 + \frac{1}{2}\cdot 0.1\right)\left(y_0 + \frac{1}{2}\cdot 0.1k_2\right) = 0.05\cdot(1 + 0.0025) = 0.050125$$

and

$$k_4 = x_1\left(y_0 + 0.1k_3\right) = 0.1\cdot(1 + 0.0050125) = 0.10050125.$$

Therefore,

$$y_1 = y_0 + \frac{0.1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 1.005012521.$$

We list the results for the Runge–Kutta method of order four and compare these results to the exact solution in Figure 2-36. Notice that this

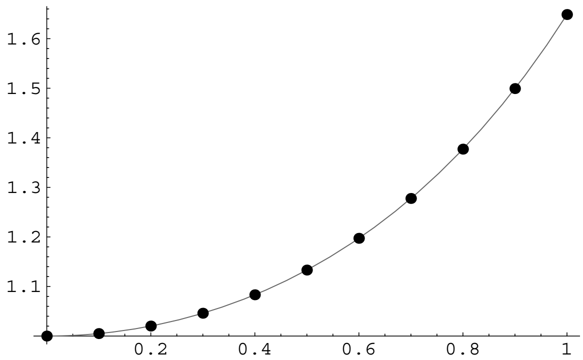


Figure 2-36 Comparison of the fourth-order Runge–Kutta method to the exact solution using $h = 0.1$

method yields the most accurate approximation of the methods used to this point.

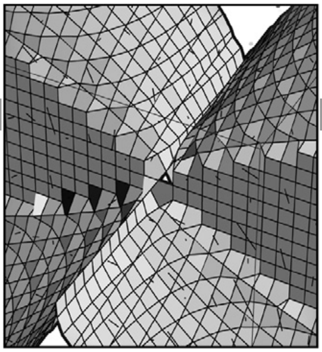
```
In[256]:= Remove[f, x, y]

          f[x_, y_] = xy;

                h = 0.1;

              x0 = 0;

              y0 = 1;

In[257]:= xr[n_] = x0 + nh;

          yr[n_] := yr[n] = yr[n - 1] + 1/6 h (k1[n - 1]
                +2 k2[n - 1] + 2 k3[n - 1] + k4[n - 1]);

          yr[0] = y0;

          k1[n_] := k1[n] = f[xr[n], yr[n]];

          k2[n_] := k2[n] = f[xr[n] + h/2, yr[n] + 1/2 h k1[n]];

          k3[n_] := k3[n] = f[xr[n] + h/2, yr[n] + 1/2 h k2[n]];

          k4[n_] := k4[n] = f[xr[n + 1], yr[n] + h k3[n]]
```

```
In[258]:= rktable2 = Table[{xr[i], yr[i]}, {i, 0, 10}];

          TableForm[rktable2]
          0   1
          0.1 1.00501
          0.2 1.0202
          0.3 1.04603
          0.4 1.08329
Out[258]= 0.5 1.13315
          0.6 1.19722
          0.7 1.27762
          0.8 1.37713
          0.9 1.4993
          1.  1.64872

In[259]:= p3 = ListPlot[rktable2,
               PlotStyle- > PointSize[0.02],
               DisplayFunction → Identity];

          Show[p1, p3,
            DisplayFunction → $DisplayFunction]
```

■

Applications of First-Order Ordinary Differential Equations

3



When the space shuttle is launched from the Kennedy Space Center, its escape velocity can be determined by solving a first-order ordinary differential equation. The same can be said for finding the flow of electromagnetic forces, the temperature of a cup of coffee, the population of a species, as well as numerous other applications. In this chapter, we show how these problems can be expressed as first-order equations. We will focus our attention on setting up the problems and explaining the meaning of the subsequent solutions because the techniques for solving these problems were discussed in Chapter 2.

3.1 Orthogonal Trajectories

We begin our discussion with *orthogonal trajectories*, a topic that is encountered in the study of electromagnetic fields and heat flow.

Definition 9 (Orthogonal Curves). *Two lines, l_1 and l_2, with slopes m_1 and m_2, respectively, are* ***orthogonal*** *(or* ***perpendicular****) if their slopes satisfy the relationship $m_1 = -1/m_2$. Two curves, C_1 and C_2, are* ***orthogonal*** *(or* ***perpendicular****) at a point if their respective tangent lines to the curves at that point are perpendicular.*

EXAMPLE 3.1.1: Use the definition of orthogonality to verify that the curves given by $y = x$ and $y = \sqrt{1-x^2}$ are orthogonal at the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$.

SOLUTION: First note that the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ lies on both the graph of $y = x$ and $y = \sqrt{1-x^2}$. The derivatives of the functions are given by $y' = 1$ and $y' = -x/\sqrt{1-x^2}$, respectively.

```
In[260]:= Clear[x, y]
  y1[x_] = x;
  y2[x_] = √(1 - x²);
In[261]:= y1'[x]
Out[261]= 1
In[262]:= y2'[x]
Out[262]= -x/√(1 - x²)
```

Hence, the slope of the tangent line to $y = x$ at $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ is 1. Substitution of $x = \sqrt{2}/2$ into $y' = -x/\sqrt{1-x^2}$ yields -1 as the slope of the tangent line at $\left(\sqrt{2}/2, \sqrt{2}/2\right)$.

```
In[263]:= y2'[√2/2]
Out[263]= -1
```

Thus, the curves are orthogonal at the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ because the slopes of the lines tangent to the graphs of $y = x$ and $y = \sqrt{1-x^2}$ at the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ are negative reciprocals. We graph these two curves along with the tangent line to $y = \sqrt{1-x^2}$ at $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ in Figure 3-1 to illustrate that the two are orthogonal. Note that the graphs are displayed correctly even though several error messages, which are not all displayed here, are generated because $y = \sqrt{1-x^2}$ is undefined

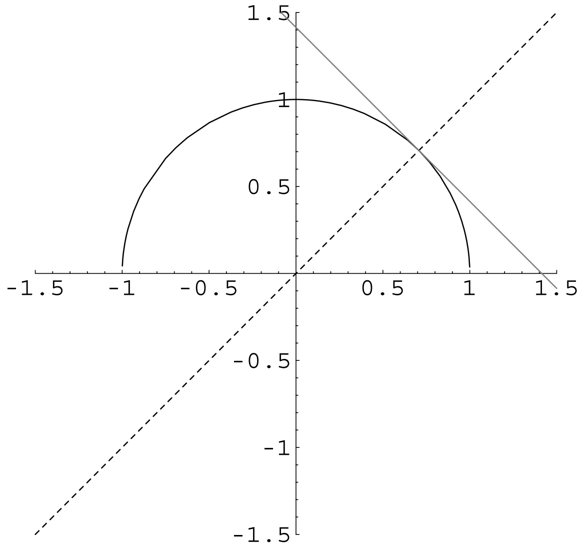


Figure 3-1 The curves are orthogonal at the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$

if $x < -1$ or $x > 1$. The option `AspectRatio->1` specifies that the ratio of lengths of the x-axis to the y-axis in the resulting graphics object be 1.

```
In[264]:= Plot[{x, √(1 - x²), -x + √2}, {x, -3/2, 3/2},
            PlotRange → {{-3/2, 3/2}, {-3/2, 3/2}},
            AspectRatio → 1, PlotStyle → {Dashing[{0.01}],
              GrayLevel[0], GrayLevel[0.5]}]

Plot::plnr : √(1 - x²)
    is not a machine - size real number at x = -1.5.
```

■

The next step in our discussion of orthogonal curves is to determine the set of orthogonal curves to a given family of curves. We refer to this set of orthogonal curves as the **family of orthogonal trajectories**. Suppose that a family of curves is defined as $F(x, y) = C$ and that the slope of the tangent line at any point on these curves is $dy/dx = f(x, y)$. Then, the slope of the tangent line on the orthogonal trajectory is $dy/dx = -1/f(x, y)$ so the family of orthogonal trajectories is found by solving the first-order equation $dy/dx = -1/f(x, y)$.

EXAMPLE 3.1.2: Determine the family of orthogonal trajectories to the family of curves $y = cx^2$.

SOLUTION: First, we must find the slope of the tangent line at any point on the parabola $y = cx^2$. Differentiating with respect to x results in $dy/dx = 2cx$. However from $y = cx^2$, we have that $c = y/x^2$. Substitution into $dy/dx = 2cx$ then yields $dy/dx = 2 \cdot y/x^2 \cdot x = 2y/x$ on the parabolas. Hence, we must solve $dy/dx = -x/(2y)$ to determine the orthogonal trajectories. This equation is separable, so we write it as $2y\,dy = -x\,dx$, and then integrating both sides gives us $2y^2 + x^2 = k$, where k is a constant, which we recognize as a family of ellipses. Note that an equivalent result is obtained with `DSolve`.

```
In[265]:= sol = DSolve[y'[x] == -x/(2 y[x]), y[x], x]

Out[265]= {{y[x] → -Sqrt[-x^2 + 4 C[1]]/Sqrt[2]}, {y[x] → Sqrt[-x^2 + 4 C[1]]/Sqrt[2]}}
```

We graph several members of the family of parabolas $y = cx^2$, the family of ellipses $2y^2 + x^2 = k$, and the two families of curves together. First, we define `parabs` to be the list of functions obtained by replacing c in $y = cx^2$ by nine equally spaced values of c between $-3/2$ and $3/2$.

```
In[266]:= parabs = Table[c x^2, {c, -3/2, 3/2, 1/8}];
```

Next, we graph the list of functions `parabs` for $-3 \le x \le 3$ with `Plot` and name the result `p1`. The graphs in `p1` are not displayed because the option `DisplayFunction->Identity` is included in the `Plot` command. We graph several ellipses $2y^2 + x^2 = k$ by using `ContourPlot` to graph several level curves of $f(x, y) = y^2 + \frac{1}{2}x^2$ and name the result `p2`. Including the option `PlotPoints->120` helps assure that the ellipses appear smooth in the result. Including the option `ContourStyle->GrayLevel[.4]` specifies that the contours be drawn in a light gray. (This will help us distinguish between the ellipses and the parabolas when we show the graphs together.) As with `p1`, `p2` is not displayed. Finally, `p1` and `p2` are displayed together with `Show` in Figure 3-2. Notice that these two families appear orthogonal, confirming the results we obtained.

```
In[267]:= p1 = Plot[Evaluate[parabs], {x, -3, 3},
              DisplayFunction → Identity];
```

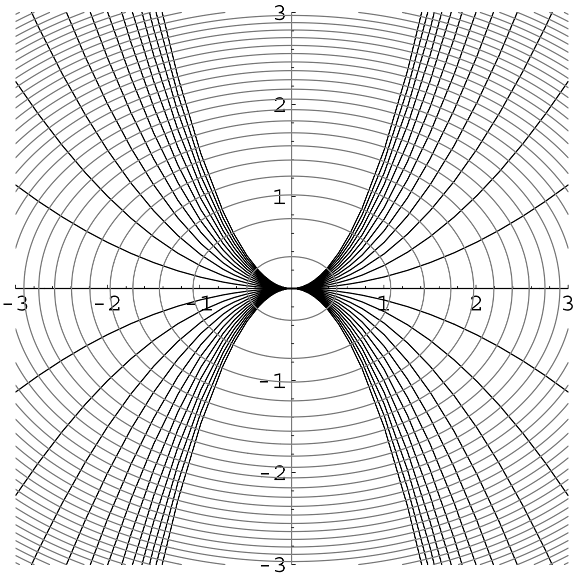


Figure 3-2 The two sets of curves are orthogonal to each other

```
In[268]:= p2 = ContourPlot[y^2 + x^2/2, {x, -3, 3}, {y, -3, 3},
              Contours → 30, ContourStyle → GrayLevel[0.5],
              ContourShading → False, PlotPoints → 120,
              DisplayFunction → Identity];

In[269]:= Show[p1, p2, PlotRange → {{-3, 3}, {-3, 3}},
            AspectRatio → 1,
            DisplayFunction → $DisplayFunction]
```

■

EXAMPLE 3.1.3 (Temperature): Let $T(x, y)$ represent the temperature at the point (x, y). The curves given by $T(x, y) = c$ (where c is constant) are called **isotherms**. The orthogonal trajectories are curves along which heat will flow. Determine the isotherms if the curves of heat flow are given by $y^2 + 2xy - x^2 = c$.

SOLUTION: We begin by finding the slope of the tangent line at each point on the heat flow curves $y^2 + 2xy - x^2 = c$ using implicit differentiation.

```
In[270]:= eq1 = y[x]^2 + 2x y[x] - x^2 == c
Out[270]= -x^2 + 2 x y[x] + y[x]^2 == c

In[271]:= step1 = Dt[eq1, x]
Out[271]= -2 x + 2 y[x] + 2 x y'[x] + 2 y[x] y'[x] == Dt[c, x]
```

Because c represents a constant, $d/dx\,(c) = 0$. We interpret `step2` to be equivalent to the equation $2yy' + 2y + 2xy' - 2x = 0$, where $y' = dy/dx$.

```
In[272]:= step2 = step1/.Dt[c, x] → 0
Out[272]= -2 x + 2 y[x] + 2 x y'[x] + 2 y[x] y'[x] == 0
```

We calculate $y' = dy/dx$ by solving `step2` for `y'[x]` with `Solve` and name the result `imderiv`.

```
In[273]:= imderiv = Solve[step2, y'[x]]
Out[273]= {{y'[x] → (x - y[x])/(x + y[x])}}
```

Thus, $dy/dx = (x - y)/(x + y)$ so the orthogonal trajectories satisfy the differential equation $dy/dx = -(x + y)/(x - y)$.

This equation is also homogeneous of degree one.

Writing this equation in differential form as $(x + y)dx + (x - y)dy = 0$, we see that this equation is exact because $\partial/\partial y\,(x + y) = 1 = \partial/\partial x\,(x - y)$. Thus, we solve the equation by integrating $x+y$ with respect to x to yield $f(x, y) = \frac{1}{2}x^2 + xy + g(y)$. Differentiating f with respect to y then gives us $f_y(x, y) = x + g'(y)$. Then, because the equation is exact, $x + g'(y) = x - y$. Therefore, $g'(y) = -y$ which implies that $g(y) = -\frac{1}{2}y^2$. This means that the family of orthogonal trajectories (isotherms) is given by $\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 = k$.

Note that `DSolve` is able to solve this differential equation.

```
In[274]:= DSolve[y'[x] == (x + y[x])/(-x + y[x]), y[x], x]//Simplify
Out[274]= {{y[x] → x - Sqrt[e^(2 C[1]) + 2 x^2]},
           {y[x] → x + Sqrt[e^(2 C[1]) + 2 x^2]}}
```

To graph $y^2 + 2xy - x^2 = c$ and $\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 = k$ for various values of c and k to see that the curves are orthogonal, we use `ContourPlot`. First, we graph several level curves of $y^2 + 2xy - x^2 = c$ on the rectangle $[-4, 4]\times[-4, 4]$ and name the result `cp1`. The option `Contours->40` instructs Mathematica to graph 40 contours instead of the default of ten.

```
In[275]:= cp1 = ContourPlot[y^2 + 2xy - x^2, {x, -4, 4},
             {y, -4, 4}, ContourShading → False,
             Axes → Automatic, Contours → 40,
             PlotPoints → 120, AxesOrigin → {0, 0},
             Frame → False,
             DisplayFunction → Identity];
```

Next we graph several level curves of $\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 = k$ on the same rectangle and name the result `cp2`. In this case, the option

```
ContourStyle->{{GrayLevel[0.4],Dashing[{0.01}]}}
```

specifies that the contours are to be dashed in a medium gray.

```
In[276]:= cp2 = ContourPlot[xy + x^2/2 - y^2/2, {x, -4, 4},
             {y, -4, 4}, ContourShading → False,
             Axes → Automatic, Contours → 40,
             PlotPoints → 120,
             ContourStyle → {{GrayLevel[0.4],
             Dashing[{0.01}]}}, AxesOrigin → {0, 0},
             Frame → False,
             DisplayFunction → Identity];
```

The graphs are then displayed side-by-side using `Show` and `GraphicsArray` in Figure 3-3 and together using `Show` in Figure 3-4.

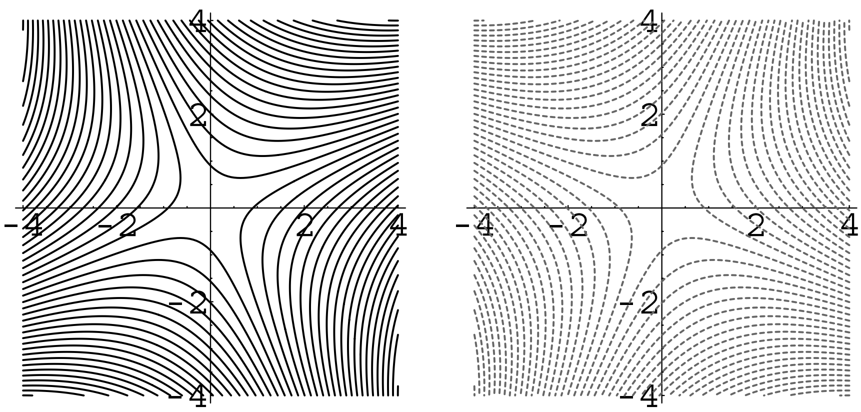


Figure 3-3 Several members of each family of curves

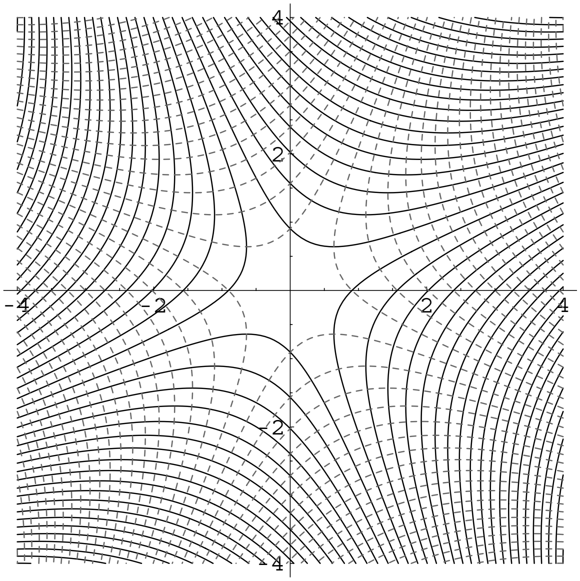


Figure 3-4 The two sets of curves are orthogonal to each other

```
In[277]:= Show[GraphicsArray[{cp1, cp2}]]

In[278]:= Show[cp1, cp2,
            DisplayFunction → $DisplayFunction]
```

■

EXAMPLE 3.1.4: Determine the orthogonal trajectories of the family of curves given by $y^2 - 2cx = c^2$. Graph several members of both families of curves on the same set of axes.

SOLUTION: After defining eq to be the equation $y^2 - 2cx = c^2$, we implicitly differentiate.

```
In[279]:= eq = y^2 - 2c x == c^2;

          step1 = Dt[eq]
Out[279]= -2 x Dt[c] - 2 c Dt[x] + 2 y Dt[y] == 2 c Dt[c]
```

As in the previous examples, we interpret Dt[x] to be 1, Dt[c] to be 0, and Dt[y] to represent dy/dx.

```
In[280]:= step2 = step1 /.{Dt[c] → 0, Dt[x] → 1}
Out[280]= -2 c + 2 y Dt[y] == 0
```

The equation $y^2 - 2cx = c^2$ is a quadratic in c. Solving for c,

```
In[281]:= cval = Solve[eq, c]
```

Out[281]= $\{\{c \to -x - \sqrt{x^2 + y^2}\}, \{c \to -x + \sqrt{x^2 + y^2}\}\}$

we choose to substitute the first value into the equation $y\,dy/dx = c$.

```
In[282]:= imderiv = Solve[step2, Dt[y]]/.cval[[1]]
```

Out[282]= $\{\{Dt[y] \to \frac{-x - \sqrt{x^2 + y^2}}{y}\}\}$

Then, we must solve $\frac{dy}{dx} = \frac{y}{x + \sqrt{x^2 + y^2}}$.

```
In[283]:= de =
            y'[x] == Evaluate[-1/imderiv[[1, 1, 2]]/.
            y → y[x]]
```

Out[283]= $y'[x] == -\frac{y[x]}{-x - \sqrt{x^2 + y[x]^2}}$

Note that Mathematica is able to solve this equation.

```
In[284]:= DSolve[de, y[x], x]//Simplify
```

Out[284]= $\{\{y[x] \to -e^{\frac{C[1]}{2}} \sqrt{e^{C[1]} + 2x}\},$
$\{y[x] \to e^{\frac{C[1]}{2}} \sqrt{e^{C[1]} + 2x}\}\}$

Thus, $y^2 = 4C^2 + 4Cx$ and replacing $2C$ with C yields $y^2 = C^2 + 2Cx$ or $y^2 - 2Cx = c^2$, which means that this family of curves is self-orthogonal. We confirm that the family is self-orthogonal with `ContourPlot` in Figures 3-5 and 3-6.

```
In[285]:= cp1 = ContourPlot[cval[[1, 1, 2]], {x, -10, 10},
                {y, -10, 10}, ContourShading → False,
                Frame → False, Axes → Automatic,
                AxesOrigin → {0, 0}, Contours → 30,
                PlotPoints → 120,
                DisplayFunction → Identity];

In[286]:= cp2 = ContourPlot[cval[[2, 1, 2]], {x, -10, 10},
                {y, -10, 10}, ContourShading → False,
                Frame → False, Axes → Automatic,
                AxesOrigin → {0, 0}, Contours → 30,
                PlotPoints → 120,
                DisplayFunction → Identity];
```

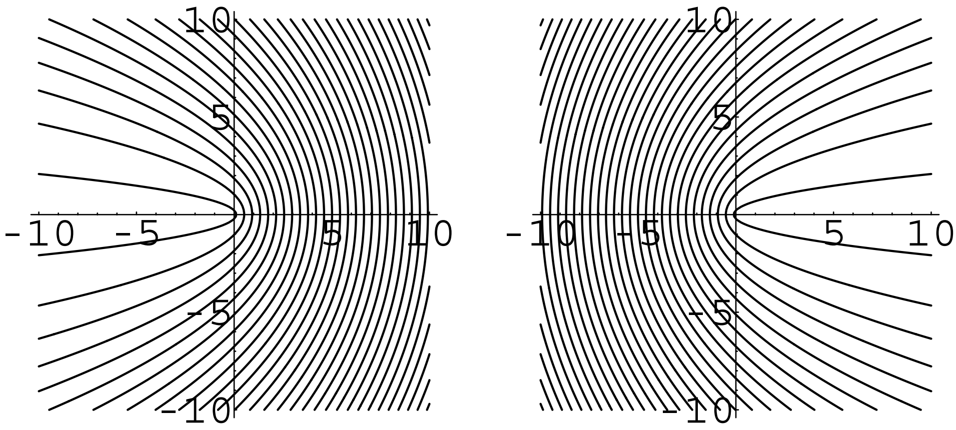


Figure 3-5 The plots are symmetric about the y-axis

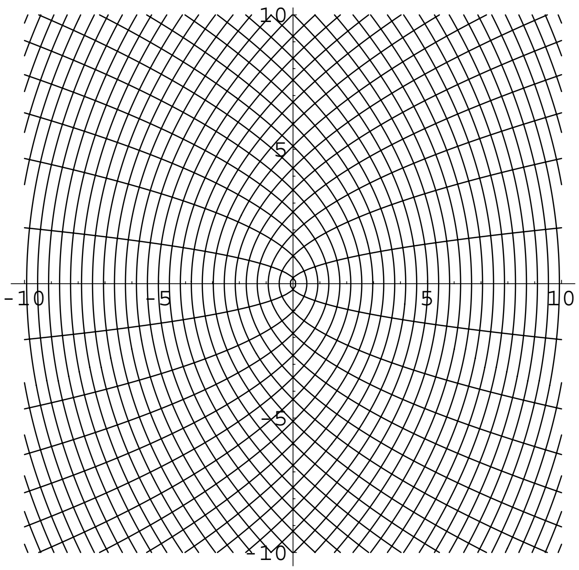


Figure 3-6 The family of curves is self-orthogonal

```
In[287]:= Show[GraphicsArray[{cp1, cp2}]]
```

```
In[288]:= Show[cp1, cp2,
            DisplayFunction → $DisplayFunction]
```

■

Application: Oblique Trajectories

If we are given a family of curves that satisfies the differential equation $dy/dx = f(x, y)$ and we want to find a family of curves that intersects this family at a constant angle θ, we must solve the differential equation

$$\frac{dy}{dx} = \frac{f(x, y) \pm \tan\theta}{1 \mp f(x, y)\tan\theta}.$$

For example, to find a family of curves that intersects the family of curves $x^2 + y^2 = c^2$ at an angle of $\pi/6$, we first implicitly differentiate the equation to obtain

$$2x + 2y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y} = f(x, y).$$

Because $\tan\theta = \tan\pi/6 = 1/\sqrt{3}$, we solve

$$\frac{dy}{dx} = \frac{-x/y + 1/\sqrt{3}}{1 - (-x/y)(1/\sqrt{3})} = \frac{-x\sqrt{3} + y}{y\sqrt{3} + x},$$

which is a first-order homogeneous equation. With the substitution $x = vy$, we obtain the separable equation

$$\frac{1 - v\sqrt{3}}{1 + v^2}dv = \frac{\sqrt{3}}{y}dy.$$

Integrating yields

$$-\frac{\sqrt{3}}{2}\ln\left(1 + v^2\right) + \tan^{-1} v = \sqrt{3}\ln|y| + k_1$$

so

$$-\frac{\sqrt{3}}{2}\ln\left(1 + \frac{x^2}{y^2}\right) + \tan^{-1}\frac{x}{y} = \sqrt{3}\ln|y| + k_1.$$

Mathematica finds an equivalent implicit solution.

```
In[289]:= sol1 = DSolve[y'[x] == (-√3x + y[x])/(√3y[x] + x), y[x], x]//
            FullSimplify
Out[289]= Solve[2 ArcTan[y[x]/x]
          +√3 (2 Log[x] + Log[1 + y[x]^2/x^2]) ==
           2 C[1], y[x]]
```

Similarly, for

$$\frac{dy}{dx} = \frac{-x/y - 1/\sqrt{3}}{1 + (-x/y)(1/\sqrt{3})} = \frac{-x\sqrt{3} - y}{y\sqrt{3} - x},$$

we obtain

$$\frac{1+v\sqrt{3}}{1+v^2}\,dv = -\frac{\sqrt{3}}{y}\,dy$$

so that the trajectories are

$$\frac{\sqrt{3}}{2}\ln\left(1+\frac{x^2}{y^2}\right)+\tan^{-1}\frac{x}{y} = -\sqrt{3}\ln|y|+k_1.$$

```
In[290]:= sol2 = DSolve[y'[x] == (-√3x - y[x])/(√3y[x] - x), y[x], x]
Out[290]= Solve[-ArcTan[y[x]/x] + 1/2 √3 Log[1 + y[x]^2/x^2] ==
            C[1] - √3 Log[x], y[x]]
```

To confirm the result graphically, we graph several members of each family of curves in Figure 3-7

```
In[291]:= toplot2 = Solve[sol2[[1]], C[1]]/.y[x] → y
Out[291]= {{C[1] → 1/2 (-2 ArcTan[y/x]
            +2 √3 Log[x] + √3 Log[1 + y^2/x^2])}}
```

```
In[292]:= cp1 = ContourPlot[x^2 + y^2, {x, -10, 10},
               {y, -10, 10}, Frame → False, Contours → 20,
               ContourStyle → GrayLevel[0.5],
               Axes → Automatic, ContourShading → False,
               PlotPoints → 120, AxesOrigin → {0, 0},
               DisplayFunction → Identity];
```

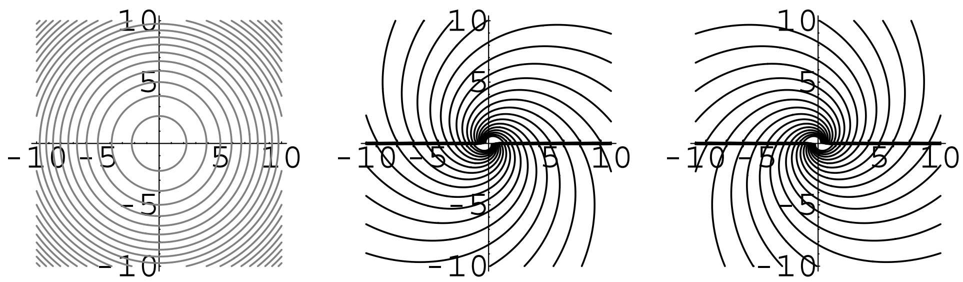


Figure 3-7 Several members of each family of curves

```
In[293]:= cp2 = ContourPlot[
              1/2 (-√3) Log[1 + x^2/y^2] + ArcTan[x/y]
              -√3 Log[Abs[y]], {x, -10, 10}, {y, -10, 10},
              Frame → False, Contours → 20,
              ContourStyle → GrayLevel[0], Axes → Automatic,
              ContourShading → False, PlotPoints → 120,
              AxesOrigin → {0, 0},
              DisplayFunction → Identity];
```

```
In[294]:= cp3 = ContourPlot[
              1/2 √3 Log[1 + x^2/y^2] + ArcTan[x/y] + √3 Log[Abs[y]],
              {x, -10, 10}, {y, -10, 10}, Frame → False,
              Contours → 20, ContourStyle → GrayLevel[0],
              Axes → Automatic, ContourShading → False,
              PlotPoints → 120, AxesOrigin → {0, 0},
              DisplayFunction → Identity];
```

```
In[295]:= Show[GraphicsArray[{cp1, cp2, cp3}]]
```

and then show the curves together in Figure 3-8 to see that they intersect at an angle of $\pi/6$.

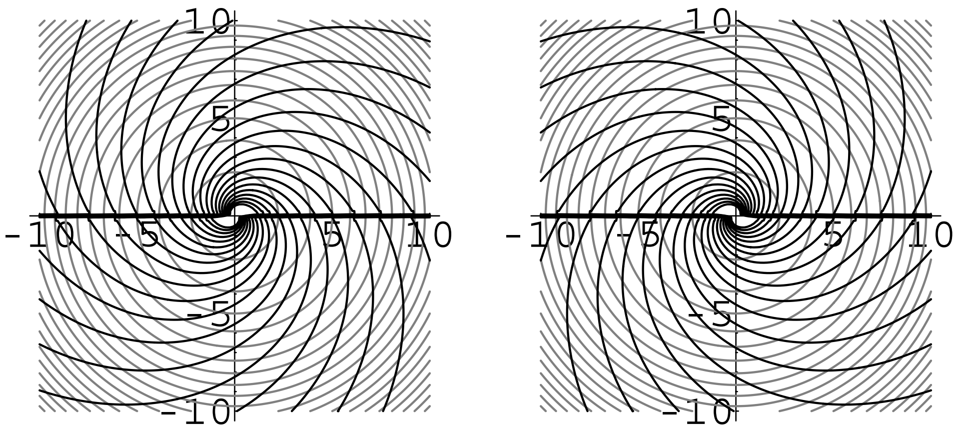


Figure 3-8 The curves intersect at an angle of $\pi/6$

```
In[296]:= cp4 = Show[cp1, cp2];
          cp5 = Show[cp1, cp3];
          Show[GraphicsArray[{cp4, cp5}]]
```

3.2 Population Growth and Decay

Many interesting problems involving population can be solved through the use of first-order differential equations. These include the determination of the number of cells in a bacteria culture, the number of citizens in a country, and the amount of radioactive substance remaining in a fossil. We begin our discussion by solving a population problem.

3.2.1 The Malthus Model

Suppose that the rate at which a population of size $y(t)$ at time t changes is proportional to the population, $y(t)$, at time t. Mathematically, this statement is represented as the first-order initial-value problem

$$\begin{cases} dy/dt = ky \\ y(0) = y_0 \end{cases} \tag{3.1}$$

where y_0 is the initial population. If $k > 0$, then the population increases (growth) while the population decreases (decay) if $k < 0$. Problems of this nature arise in such fields as cell population growth in biology as well as radioactive decay in physics. Equation (3.1) is known as the Malthus model due to the work of the English clergyman and economist Thomas R. Malthus. We solve the Malthus model for all values of k and y_0 which enables us to refer to the solution in other problems without solving the differential equation again. Rewriting $dy/dt = ky$ in the form $dy/y = k\,dt$, we see that this is a separable differential equation. Integrating and simplifying results in:

$$\int \frac{1}{y}\,dy = \int k\,dt$$
$$\ln|y| = kt + C_1$$
$$y = Ce^{kt}, \quad \text{where } C = e^{C_1}.$$

Notice that because y represents population, $y \geq 0$ and, therefore, $|y| = y$. To find C, we apply the initial condition obtaining $y_0 = y(0) = Ce^{k \cdot 0} = C$. Thus, the solution to the initial-value problem (3.1) is

$$y = y_0 e^{kt}. \tag{3.2}$$

We obtain the same result with `DSolve`:

```
In[297]:= Clear[x, y, t]

          DSolve[{y'[t] == k y[t], y[0] == y0}, y[t], t]
Out[297]= {{y[t] → e^(k t) y0}}
```

EXAMPLE 3.2.1 (Radioactive Decay): Forms of a given element with different numbers of neutrons are called **nuclides**. Some nuclides are not stable. For example, potassium-40 (^{40}K) naturally decays to reach argon-40 (^{40}K). This decay which occurs in some nuclides was first observed, but not understood, by Henri Becquerel (1852–1908) in 1896. Marie Curie, however, began studying this decay in 1898, named it **radioactivity**, and discovered the radioactive substances polonium and radium. Marie Curie (1867–1934), along with her husband, Pierre Curie (1859–1906), and Henri Becquerel, received the Nobel Prize in Physics in 1903 for their work on radioactivity. Marie Curie subsequently received the Nobel Prize in Chemistry in 1910 for discovering polonium and radium.

Given a sample of ^{40}Ar of sufficient size, after 1.2×10^9 years approximately half of the sample will have decayed to ^{40}Ar. The **half-life** of a nuclide is the time for half the nuclei in a given sample to decay. We see that the rate of decay of a nuclide is proportional to the amount present because the half-life of a given nuclide is constant and independent of the sample size.

If the half-life of polonium ^{209}Po is 100 years, determine the percentage of the original amount of ^{209}Po that remains after 50 years.

SOLUTION: Let y_0 represent the original amount of ^{209}Po that is present. Then the amount present after t years is $y(t) = y_0 e^{kt}$. Because $y(100) = y_0/2$ and $y(100) = y_0 e^{100k}$, we solve $y_0 e^{100k} = y_0/2$ for e^k:

$$e^{100k} = \frac{1}{2} \qquad \text{or} \qquad e^k = \left(\frac{1}{2}\right)^{1/100}$$

so

$$y(t) = y_0 e^{kt} = y_0 \left(\frac{1}{2}\right)^{t/100}.$$

```
In[298]:= k = -Log[2]/100;

In[299]:= y[t_] = y0 Exp[kt];

          Simplify[y[t]]
Out[299]= 2^(-t/100) y0
```

In order to determine the percentage of y_0 that remains, we evaluate $y(50) = y_0(1/2)^{50/100} = y_0/\sqrt{2} \approx 0.7071y_0$.

```
In[300]:= y[50]
Out[300]= y0/√2

In[301]:= N[y[50]]
Out[301]= 0.707107 y0.
```

Therefore, 70.71% of the original amount of ^{209}Po remains after 50 years.

■

In the previous example, we see that we can determine the percentage of y_0 that remains even though we do not know the value of y_0. Hence, instead of letting $y(t)$ represent the amount of the substance present after time t, we can let it represent the fraction (or percent) of y_0 that remains after time t. In doing this, we use the initial condition $y(0) = 1$ to indicate that 100% of y_0 is present at $t = 0$.

EXAMPLE 3.2.2: The wood of an Egyptian sarcophagus (burial case) is found to contain 63% of the carbon-14 found in a present day sample. What is the age of the sarcophagus?

SOLUTION: The half-life of carbon-14 is 5730 years. Let $y(t)$ be the percent of carbon-14 in the sample after t years. Then, $y(0) = 1$. Because $y(t) = y_0e^{kt}$, $y(5730) = e^{5730k}$. Solving for k yields:

$$\ln\left(e^{5730k}\right) = \ln\left(\frac{1}{2}\right)$$

$$5730k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5730} = -\frac{\ln 2}{5730}.$$

Thus, $y(t) = e^{kt} = e^{-\frac{\ln 2}{5730}t} = 2^{-t/5730}$.

```
In[302]:= Clear[k, y]

          k = -Log[2]/5730;

In[303]:= y[t_] = Exp[kt]
Out[303]= 2^(-t/5730)
```

In this problem, we must find the value of t for which $y(t) = 0.63$. Solving the equation $2^{-t/5730} = 0.63 = 63/100$ results in:

$$
\begin{aligned}
2^{-t/5730} &= \frac{63}{100} \\
\ln\left(2^{-t/5730}\right) &= \ln\frac{63}{100} \\
-\frac{t}{5730}\ln 2 &= \ln\frac{63}{100} \\
t &= -\frac{5730 \ln \frac{63}{100}}{\ln 2} \approx 3819.48.
\end{aligned}
$$

We conclude that the sarcophagus is approximately 3819 years old.

An alternative way to approximate the age of the sarcophagus is to first graph $y(t)$ and the line $y = 0.63$ with `Plot` as shown in Figure 3-9. The age of the sarcophagus is the t-coordinate of the point of intersection of $y(t)$ and $y = 0.63$.

```
In[304]:= Plot[{y[t], 0.63}, {t, 0, 6000},
            PlotStyle → {GrayLevel[0], GrayLevel[0.5]},
            PlotRange → {0, 1}]
```

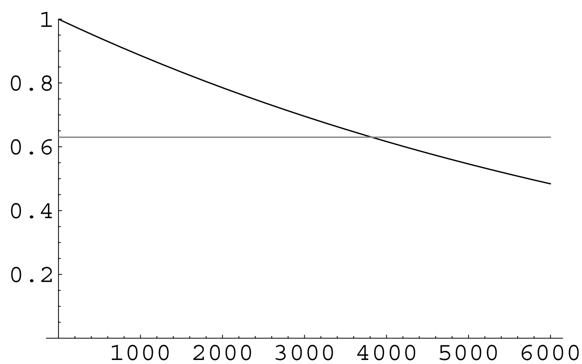


Figure 3-9 The age of the sarcophagus is the t-coordinate of the point of intersection of $y(t)$ and $y = 0.63$

We see that the t-coordinate of the point of intersection is approximately $t \approx 3770$. A more accurate approximation is obtained with `FindRoot`. The command

```
FindRoot[equation,{variable,firstguess}]
```

attempts to find a solution to the equation `equation` in the variable `variable` "near" the number `firstguess`. Thus, entering

```
In[305]:= FindRoot[y[t] == 0.63, {t, 3770}]
Out[305]= {t → 3819.48}
```

returns an approximation of the solution to the equation $y(t) = 0.63$ using an initial approximation of 3770.

■

To observe some of the limitations of the Malthus model, we next consider a population problem in which the rate of growth of the population does not exclusively depend on the population present.

EXAMPLE 3.2.3: The population of the United States was recorded as 5.3 million in 1800. Use the Malthus model to approximate the population for years after 1800 if k was experimentally determined to be 0.03. Compare these results to the actual population. Is this a good approximation for years after 1800?

SOLUTION: In this example, $k = 0.03$ and $y_0 = 5.3$ and our model for the population of the United States at time t (where t is the number of years from 1800) is $y(t) = 5.3e^{0.03t}$.

```
In[306]:= Clear[k, y, t]

          peq = DSolve[{y'[t] == k y[t], y[0] == y0},
          y[t], t]
Out[306]= {{y[t] → e^(k t) y0}}

In[307]:= pop[t_, k_, y0_] = e^(kt) y0
Out[307]= e^(k t) y0

In[308]:= pop[t, 0.03, 5.3]
Out[308]= 5.3 e^(0.03 t)
```

Table 3-1 Population of the United States for various years

Year (t)	Actual Population (in millions)	Value of $y(t) = 5.3e^{0.03t}$	Year (t)	Actual Population (in millions)	Value of $y(t) = 5.3e^{0.03t}$
1800 (0)	5.30	5.30	1870 (70)	38.56	43.28
1810 (10)	7.24	7.15	1880 (80)	50.19	58.42
1820 (20)	9.64	9.66	1890 (90)	62.98	78.86
1830 (30)	12.68	13.04	1900 (100)	76.21	106.45
1840 (40)	17.06	17.60	1910 (110)	92.23	143.70
1850 (50)	23.19	23.75	1920 (120)	106.02	193.97
1860 (60)	31.44	32.06	1930 (130)	123.20	261.83

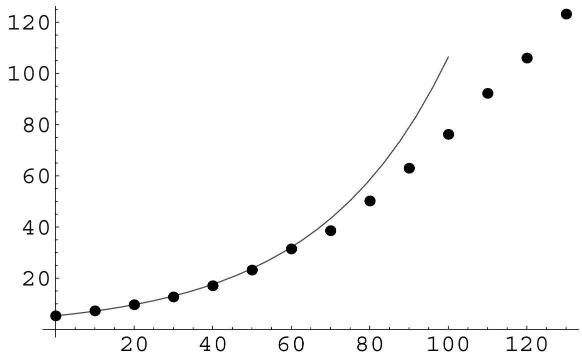


Figure 3-10 Over time, the Malthus model does not provide a good model of the growth of the population of the United States

In order to compare this model with the actual population of the United States, census figures for the population of the United States for various years are listed in Table 3-1 along with the corresponding value of $y(t)$.

Although the model appears to closely approximate the data for several years after 1800, the accuracy of the approximation diminishes over time. This is because the population of the United States does not exclusively increase at a rate proportional to the population. Hence, another model which better approximates the population taking other factors into account is needed. The graph of $y(t) = 5.3e^{0.03t}$ is shown along with the data points in Figure 3-10 to show how the approximation becomes less accurate as t increases.

```
In[309]:= realpop = {{0, 5.3}, {10, 7.24}, {20, 9.64},
            {30, 12.68}, {40, 17.06}, {50, 23.19},
            {60, 31.44}, {70, 38.56}, {80, 50.19},
            {90, 62.98}, {100, 76.21}, {110, 92.23},
            {120, 106.02}, {130, 123.2}};

In[310]:= toshow = ListPlot[realpop,
                  PlotStyle- > PointSize[0.02],
                  DisplayFunction → Identity];

In[311]:= popplot = Plot[pop[t, 0.03, 5.3], {t, 0, 100},
                  PlotStyle- > GrayLevel[0.3],
                  DisplayFunction- > Identity];

          Show[popplot, toshow,
            DisplayFunction- > $DisplayFunction]
```

■

3.2.2 The Logistic Equation

The **logistic equation** (or **Verhulst equation**) is the equation

$$\frac{dy}{dt} = (r - ay(t))\,y(t), \tag{3.3}$$

where r and a are constants. Equation (3.3) was first introduced by the Belgian mathematician Pierre Verhulst to study population growth. The logistic equation differs from the Malthus model in that the term $r - ay(t)$ is not constant. This equation can be written as $dy/dt = (r - ay)y = ry - ay^2$ where the term $-y^2$ represents an inhibitive factor. Under these assumptions the population is neither allowed to grow out of control nor grow or decay constantly as it was with the Malthus model.

The logistic equation is separable, and, thus, can be solved by separation of variables. We solve equation (3.3) subject to the condition $y(0) = y_0$.

Separating variables and using partial fractions to integrate with respect to y, we have

$$\begin{aligned}
\frac{1}{(r - ay)y}\,dy &= dt \\
\left(\frac{a}{r}\frac{1}{r - ay} + \frac{1}{r}\frac{1}{y}\right) dy &= dt \\
\left(a\frac{1}{r - ay} + \frac{1}{y}\right) dy &= r\,dt \\
-\ln|r - ay| + \ln|y| &= rt + C.
\end{aligned}$$

Using the properties of logarithms to solve this equation for y yields

$$\ln\left|\frac{y}{r-ay}\right| = rt + C$$

$$\frac{y}{r-ay} = \pm e^{rt+C} = Ke^{rt}, \text{ where } K = \pm e^{C}$$

$$y = r\left(\frac{1}{K}e^{-rt} + a\right)^{-1}.$$

```
In[312]:= DSolve[y'[t] == (r - a y[t]) y[t], y[t], t]
Out[312]= {{y[t] → (e^(r t) r)/(a e^(r t) + e^(r C[1]))}}
```

Applying the initial condition $y(0) = y_0$ and solving for K, we find that

$$K = \frac{y_0}{r - ay_0}.$$

After substituting this value into the general solution and simplifying, the solution of equation (3.3) that satisfies the initial condition $y(0) = y_0$ can be written as

$$y = \frac{ry_0}{ay_0 + (r - ay_0)\,e^{-rt}}. \tag{3.4}$$

Notice that if $r > 0$, $\lim_{t\to\infty} y(t) = r/a$ because $\lim_{t\to\infty} e^{-rt} = 0$. This makes the solution to the logistic equation different from that of the Malthus model in that the solution to the logistic equation approaches a finite nonzero limit as $t \to \infty$ while that of the Malthus model approaches either infinity or zero as $t \to \infty$.

```
In[313]:= DSolve[{y'[t] == (r - a y[t]) y[t],
            y[0] == y0}, y[t], t]
Out[313]= {{y[t] → (e^(r t) r y0)/(r - a y0 + a e^(r t) y0)}}
```

EXAMPLE 3.2.4: Use the logistic equation to approximate the population of the United States using $r = 0.03$, $a = 0.0001$, and $y_0 = 5.3$. Compare this result with the actual census values shown in Table 3-1. Use the model obtained to predict the population of the United States in the year 2000.

SOLUTION: We substitute the indicated values of r, a, and y_0 into equation (3.4) to obtain the approximation of the population of the United States at time t, where t represents the number of years since 1800,

$$y(t) = \frac{0.159}{0.00053 + 0.02947e^{-0.03t}}.$$

```
In[314]:= pop[t_] = 0.159 / (0.00053 + 0.0294 Exp[-0.03 t])
```

$$Out[314]= \frac{0.159}{0.00053 + 0.0294\, e^{-0.03\, t}}$$

```
In[315]:= realpop = Union[realpop, {{140, 132.16},
            {150, 151.33}, {160, 179.32},
            {170, 203.3}, {180, 226.54},
            {190, 248.71}}];
```

We compare the approximation of the population of the United States given by the approximation with the actual population obtained from census figures. Note that this model appears to more closely approximate the population over a longer period of time than the Malthus model which was considered in the previous examples as we can see in the graph shown in Figure 3-11.

Be sure that you have defined `realpop` as in Example 3.2.3 before entering the following commands.

```
In[316]:= toshow = ListPlot[realpop,
                     PlotStyle- > PointSize[0.02],
                     DisplayFunction → Identity];

In[317]:= popplot = Plot[pop[t], {t, 0, 200},
                     PlotStyle → GrayLevel[0.3],
                     DisplayFunction → Identity];

          Show[popplot, toshow,
            DisplayFunction → $DisplayFunction]
```

To predict the population of the United States in the year 2000 with this model, we evaluate $y(200)$. Thus, we predict that the population will be approximately 263.74 million in the year 2000. Note that projections of

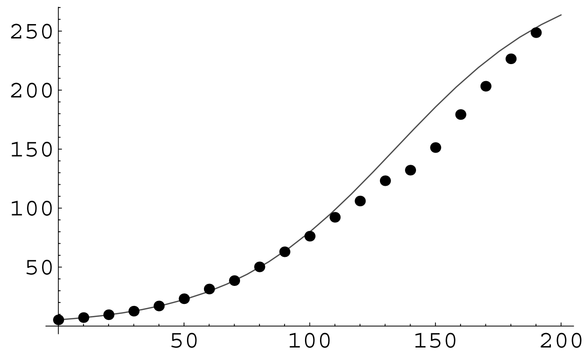


Figure 3-11 Over time, the logistic equation appears to provide a better model of the population of the United States than the Malthus model

the population of the United States in the year 2000 made by the Bureau of the Census range from 259.57 million to 278.23 million.

```
In[318]:= pop[200]
Out[318]= 263.736

In[319]:= pop[2010]
Out[319]= 300.
```

■

EXAMPLE 3.2.5 (Logistic Equation with Predation): Incorporating predation into the **logistic equation**, $y' = \alpha y\left(1 - \frac{1}{K}y\right)$, results in

$$\frac{dy}{dt} = \alpha y\left(1 - \frac{1}{K}y\right) - P(y),$$

where $P(y)$ is a function of y describing the rate of predation. A typical choice for P is $P(y) = ay^2/(b^2 + y^2)$ because $P(0) = 0$ and P is bounded above: $\lim_{t\to\infty} P(y) < \infty$.

Remark. Of course, if $\lim_{t\to\infty} y(t) = Y$, then $\lim_{t\to\infty} P(y) = aY^2/(b^2 + Y^2)$. Generally, however, $\lim_{t\to\infty} P(y) \neq a$ because $\lim_{t\to\infty} y(t) \leq K \neq \infty$, for some $K \geq 0$, in the predation situation.

If $\alpha = 1$, $a = 5$, and $b = 2$, graph the direction field associated with the equation as well as various solutions if (a) $K = 19$ and (b) $K = 20$.

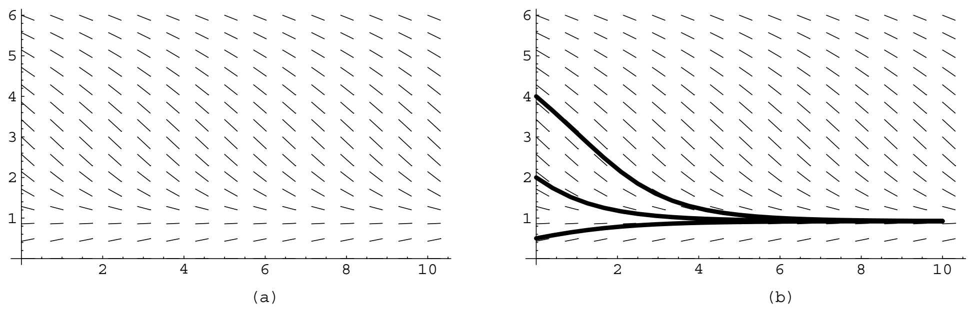
SOLUTION: (a) We define `eqn[k]` to be

$$\frac{dy}{dt} = y\left(1 - \frac{1}{K}y\right) - \frac{5y^2}{4 + y^2}.$$

```
In[320]:= << Graphics`PlotField`

In[321]:= eqn[k_] = y'[t] == y[t](1 - 1/k y[t])
          -5y[t]^2/(4 + y[t]^2);
```

We use `PlotVectorField` to graph the direction field in Figure 3-12 (a) and then the direction field along with the solutions that satisfy $y(0) = .5$, $y(0) = .2$, and $y(0) = 4$ in Figure 3-12 (b).

```
In[322]:= pvf19 = PlotVectorField[{1, y(1 - 1/19 y)
                  -5y^2/(4 + y^2)}, {t, 0, 10}, {y, 0, 6},
                  Axes- > Automatic, HeadLength- > 0,
                  DisplayFunction- > Identity];
```



(a) (b)

Figure 3-12 (a) Direction field and (b) direction field with three solutions

```
In[323]:= n1 = NDSolve[{eqn[19], y[0] == 0.5}, y[t],
                {t, 0, 10}];
          n2 = NDSolve[{eqn[19], y[0] == 2}, y[t],
                {t, 0, 10}];
          n3 = NDSolve[{eqn[19], y[0] == 4}, y[t],
                {t, 0, 10}];

In[324]:= solplot = Plot[Evaluate[y[t]/.{n1, n2, n3}],
                      {t, 0, 10}, PlotStyle- >
                      Thickness[0.01],
                      DisplayFunction- > Identity];
```

The same results can be obtained using `Map`.

```
In[325]:= numsols = Map[NDSolve[{eqn[19], y[0] == #},
                      y[t], {t, 0, 10}]&, {0.5, 2, 4}];
          solplot = Plot[Evaluate[y[t]/.numsols],
                      {t, 0, 10}, PlotStyle- >
                      Thickness[0.01],
                      DisplayFunction- > Identity];

In[326]:= Show[GraphicsArray[{pvf19, Show[pvf19,
            solplot]}]]
```

In the plot, notice that all nontrivial solutions appear to approach an equilibrium solution. We determine the equilibrium solution by solving $y' = 0$

```
In[327]:= eqn[19][[2]]
```

Out[327]= $\left(1 - \frac{y[t]}{19}\right) y[t] - \frac{5\, y[t]^2}{4 + y[t]^2}$

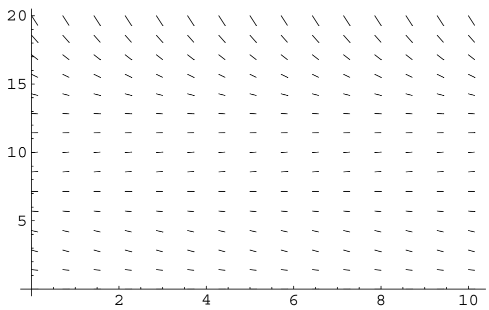


Figure 3-13 Direction field

```
In[328]:= Solve[eqn[19.][[2]] == 0, y[t]]
Out[328]= {{y[t] → 0.}, {y[t] → 0.923351},
           {y[t] → 9.03832 - 0.785875 i},
           {y[t] → 9.03832 + 0.785875 i}}
```

to see that it is $y \approx 0.923$.

(b) We carry out similar steps for (b). First, we graph the direction field with PlotVectorField in Figure 3-13.

```
In[329]:= pvf20 = PlotVectorField[{1, y(1 - 1/20 y)
                -5y^2/(4 + y^2)}, {t, 0, 10}, {y, 0, 20},
                Axes- > Automatic, HeadLength- > 0,
                AspectRatio- > 1/GoldenRatio];
```

We then use Map together with NDSolve to numerically find the solution satisfying $y(0) = .5i$, for $i = 1, 2, \ldots, 40$ and name the resulting list numsols. The functions contained in numsols are graphed with Plot in solplot.

```
In[330]:= numsols = Map[NDSolve[{eqn[20], y[0] == #},
              y[t], {t, 0, 10}]&, Table[0.5i, {i, 1, 40}]];
          solplot = Plot[Evaluate[y[t]/.numsols],
              {t, 0, 10}, PlotStyle- > Thickness[0.005],
              DisplayFunction- > Identity];
```

Last, we display the direction field along with the solution graphs in solplot using Show in Figure 3-14.

```
In[331]:= Show[pvf20, solplot]
```

Notice that there are three nontrivial equilibrium solutions that are found by solving $y' = 0$.

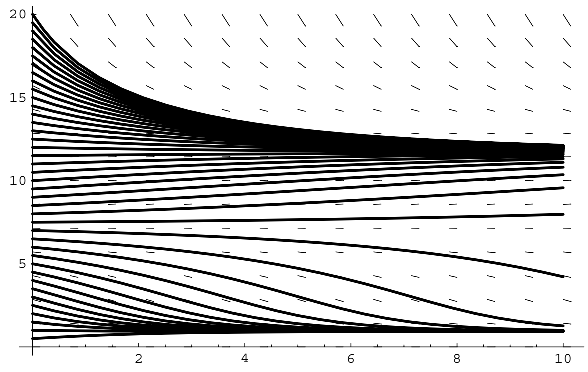


Figure 3-14 Direction field with several solutions

```
In[332]:= Solve[eqn[20.][[2]] == 0, y[t]]
Out[332]= {{y[t] → 0.}, {y[t] → 0.926741},
           {y[t] → 7.38645},
           {y[t] → 11.6868}}
```

In this case, $y \approx .926$ and $y \approx 11.687$ are stable while $y \approx 7.386$ is unstable.

■

See Smith and Waltman's, *The Theory of the Chemostat: Dynamics of Microbial Competition* [24], for a detailed discussion of various chemostat models.

EXAMPLE 3.2.6 (Growth in the Chemostat): The *scaled* equations for the growth of a population in a chemostat are

$$\begin{cases} \dfrac{dS}{dt} = 1 - S - \dfrac{mS}{a+S}x \\ \dfrac{dx}{dt} = x\left(\dfrac{mS}{a+S} - 1\right) \\ S(0) \geq 0, x(0) > 0 \end{cases} \tag{3.5}$$

where $S(t)$ denotes the concentration of the nutrient at time t for the organism with concentration $x(t)$ at time t. Letting $\Sigma = 1 - S - x$, we see that $\Sigma' = -\Sigma$

```
In[333]:= seq = 1 - s - m s x/(a + s);

          xeq = x (m s/(a + s) - 1);

In[334]:= σ = 1 - s - x;

In[335]:= seq + xeq - σ//Simplify
Out[335]= 0
```

and system (3.5) can be written as

$$\begin{cases} \dfrac{d\Sigma}{dt} = -\Sigma \\ \dfrac{dx}{dt} = x\left(\dfrac{m\,(1-\Sigma-x)}{a+(1-\Sigma-x)} - 1\right) \\ \Sigma(0) > 0,\, x(0) > 0. \end{cases} \tag{3.6}$$

Because $\Sigma(t) = \Sigma(0)e^{-t}$, $\lim_{t\to\infty} \Sigma(t) = 0$ so system (3.6) can be rewritten as the single first-order equation

$$\frac{dx}{dt} = x\left(\frac{m\,(1-x)}{a+(1-x)} - 1\right) \quad \text{or} \quad \frac{dx}{dt} = x\left[\frac{m(1-x)}{1+a-x} - 1\right],\ 0 \le x \le 1, \tag{3.7}$$

where $x(0) > 0$. The rest points (or equilibrium points) of equation (3.7) are found by solving

$$x\left[\frac{m(1-x)}{1+a-x} - 1\right] = 0$$

for x.

```
In[336]:= Solve[(-1 + m (1 - x)/(1 + a - x)) x == 0, x]
Out[336]= {{x → 0}, {x → (-1 - a + m)/(-1 + m)}}
```

For $m \neq 1$, observe that the nonzero rest point can be written as $x = 1-\lambda$, where $\lambda = a/(m-1)$ is called the **break-even** concentration.

```
In[337]:= Apart[(-1 - a + m)/(-1 + m)]
Out[337]= 1 - a/(-1 + m)
```

We use `Plot` to graph $y = 1 - \lambda$ in Figure 3-15.

```
In[338]:= Plot[1 - λ, {λ, 0, 1}, AspectRatio → Automatic,
            AxesLabel → {"λ", ""}]
```

On the other hand, in Figure 3-16 we use `ContourPlot` to generate a plot of the level curve of $f(a, m) = (m - a - 1)/(m - 1)$ corresponding to 0. Points (a, m) in the white region are points where $1 - \lambda$ is positive.

```
In[339]:= (-1 - a + m)/(-1 + m) /.{a → 2, m → 6}
Out[339]= 3/5

In[340]:= (-1 - a + m)/(-1 + m) /.{a → 8, m → 2}
Out[340]= -7
```

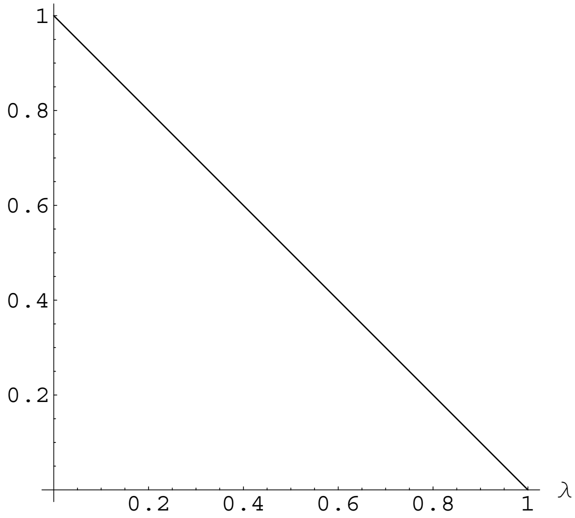


Figure 3-15 Plot of $y = 1 - \lambda$

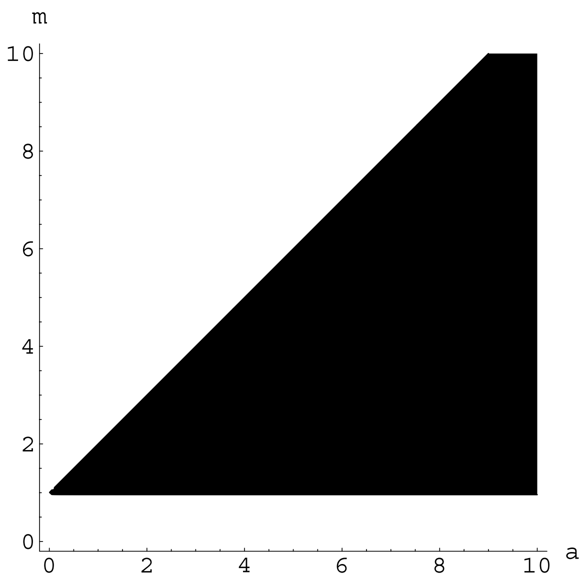


Figure 3-16 The black region corresponds to points (a, m) where $1 - \lambda$ is negative

```
In[341]:= ContourPlot[(-1 - a + m)/(-1 + m), {a, 0, 10}, {m, 0, 10},
            Contours → {0}, PlotPoints → 200,
            AxesLabel → {"a", "m"}, Frame → False,
            Axes → Automatic]
```

To see how a, m, and $x(0) = x_0$ affect the solutions of equation (3.7), we define the function x. The command `x[m,a][x0]` plots the solution of equation (3.7) satisfying $x(0) = x_0$ for $0 \le t \le 10$. You can include `Plot` options, `opts`, with `x[m,a][x0,opts]`.

```
In[342]:= Clear[x]

          x[m_, a_][x0_, opts___] := Module[{λ},
              λ = a/(m - 1);
              numsol = NDSolve[{x'[t] ==
                x[t] (m - 1)/(1 + a - x[t]) (1 - λ - x[t]),
                    x[0] == x0}, x[t], {t, 0, 100}];
              Plot[Evaluate[x[t]/.numsol],
                {t, 0, 10}, opts,
                DisplayFunction → Identity,
                PlotRange → {0, 1}]]
```

For example,

```
In[343]:= g1 = Table[x[4, 2][x0, PlotStyle → GrayLevel
                x[0 + x0/2]], {x0, 0.05, 0.95, 0.9/14}];

In[344]:= (-1 - a + m)/(-1 + m) /. {m → 4, a → 2}

In[345]:= g1b = Show[g1, DisplayFunction →
                  $DisplayFunction]
```

plots solutions to equation (3.7) if $m = 4$ and $a = 2$ for various initial conditions. The results are shown in Figure 3-17. In this case, we see that all solutions approach the equilibrium solution $x = 1/3$.

```
Out[345]= 1/3
```

On the other hand, entering

```
In[346]:= g2 = Table[x[m, 2][0.2,
                PlotStyle → GrayLevel[(9 - m)/8]],
                {m, 1.01, 9, 8.99/24}];

In[347]:= g2b = Show[g2,
                  DisplayFunction → $DisplayFunction];
```

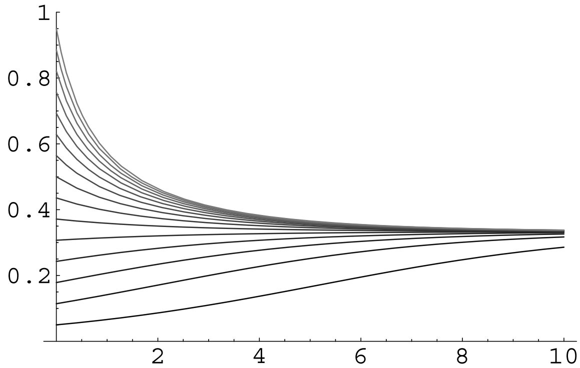


Figure 3-17 If $m = 4$ and $a = 2$, all solutions of equation (3.7) approach the equilibrium solution $x = 1/3$

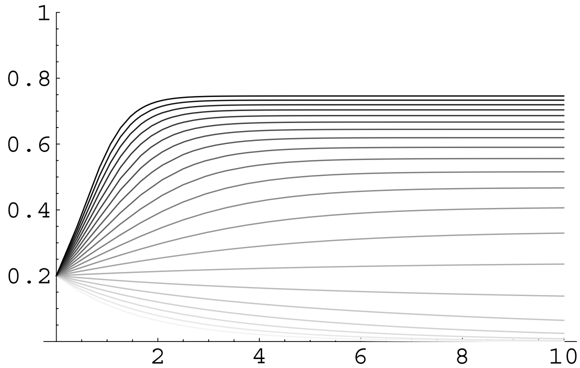


Figure 3-18 If m is sufficiently small, the organism becomes extinct

solves equation (3.7) if $a = 2$ and $x_0 = 0.2$ for various values of m. The results are shown in Figure 3-18.

Source: David A. Sanchez, "Populations and Harvesting," *Mathematical Modeling: Classroom Notes in Applied Mathematics*, Murray S. Klamkin, Editor, SIAM (1987), pp. 311–313.

Application: Harvesting

If we wish to take a constant harvest rate, H (like hunting, fishing, or disease) into consideration, then we might instead modify the logistic equation (3.3) and use the equation

$$\frac{dP}{dt} = rP - aP^2 - H$$

to model the population under consideration. Notice that Mathematica can find a solution to this equation if $r^2 - 4aH < 0$.

```
In[348]:= gensol = DSolve[p'[t] == r p[t] - ap[t]^2
            -h, p[t], t]
```

$$\text{Out[348]= } \left\{\left\{p[t] \to \frac{1}{2a}\left(r + \sqrt{4ah - r^2}\,\text{Tan}\left[\frac{1}{2}\left(-\sqrt{4ah - r^2}\,t + \sqrt{4ah - r^2}\,C[1]\right)\right]\right)\right\}\right\}$$

If H does not depend on P, the equilibrium solutions are found by solving $rP - aP^2 - H = 0$ for P.

```
In[349]:= eqsols = Solve[r p - a p^2 - h == 0, p]
```

$$\text{Out[349]= } \left\{\left\{p \to -\frac{-r - \sqrt{-4ah + r^2}}{2a}\right\}, \left\{p \to -\frac{-r + \sqrt{-4ah + r^2}}{2a}\right\}\right\}$$

Observe that there are two distinct equilibrium solutions if $r^2 - 4aH > 0$. There is one equilibrium solution if $r^2 - 4aH = 0$, and none if $r^2 - 4aH < 0$. Suppose that for a certain species it is found that $r = 7/10$, $a = 1/10$, and $H = 1$. Then, the model becomes $dP/dt = \frac{7}{10}P - \frac{1}{10}P^2 - 1$ with equilibrium solutions $P = 2$ and $P = 5$.

For these parameter values, the solution obtained in `gensol` is not valid because $r^2 - 4aH = 9/100 > 0$.

```
In[350]:= eqsols/.{r → 7/10, a → 1/10, h → 1}

Out[350]= {{p → 5}, {p → 2}}
```

Mathematica can solve the initial-value problem $dP/dt = \frac{7}{10}P - \frac{1}{10}P^2 - 1$, $P(0) = P_0$ although the result is very lengthy so we display an abbreviated portion of the solution with `Short`.

```
In[351]:= exactsol =
            DSolve[{p'[t] == 7/10 p[t] - 1/10p[t]^2 - 1,
              p[0] == p0}, p[t], t];

In[352]:= Short[exactsol]
```

$$\text{Out[352]= } \left\{\left\{p[t] \to \frac{\ll 5 \gg + 5 \ll 1 \gg p0}{\ll 5 \gg + e^{\ll 1 \gg}\,p0}\right\}\right\}$$

Proceeding numerically, we define the function `numgraph`. Given i, `numgraph[i]` attempts to graph the numerical solution to $dP/dt = \frac{7}{10}P - \frac{1}{10}P^2 - 1$, $P(0) = P_0$ on the rectangle $[-10, 10] \times [-10, 10]$. Any options specified are passed through to the `Plot` command.

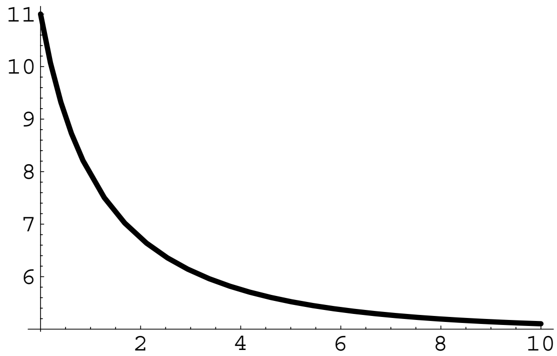


Figure 3-19 A solution to the logistic equation with harvesting

```
In[353]:= Clear[p, numgraph];
          numgraph[i_, opts___] :=
            Module[{numsol}, numsol =
                NDSolve[{p′[t] == 7/10p[t] - 1/10p[t]^2 - 1,
                    p[0] == i}, p[t], {t, 0, 10}];
              Plot[p[t]/.numsol, {t, 0, 10}, opts,
                DisplayFunction → Identity,
                PlotStyle → {{GrayLevel[0],
                Thickness[0.01]}},
                PlotRange → {{0, 10}, {0, 10}}]
            ]
```

For example, entering

```
In[354]:= numgraph[11, PlotRange → All,
            DisplayFunction → $DisplayFunction]
```

displays the graph of the solution to $dP/dt = \frac{7}{10}P - \frac{1}{10}P^2 - 1$, $P(0) = 11$ for $0 \le t \le 10$, which is shown in Figure 3-19. We then use `Map` to graph the solution of $dP/dt = \frac{7}{10}P - \frac{1}{10}P^2 - 1$, $P(0) = i$, $i = 1, 1/2, \ldots, 10$, naming the resulting set of graphs `toshow`. Notice that Mathematica generates several error messages, not all of which are shown here, because solutions satisfying $P(0) = P_0$ for $P_0 < 2$ become unbounded very quickly.

```
In[355]:= toshow = Map[numgraph, Table[i, {i, 1, 10, 1/2}]]
 NDSolve :: ndsz :
   At t == 4.62098, step size is effectively zero;
     singularity or stiff system suspected.
```

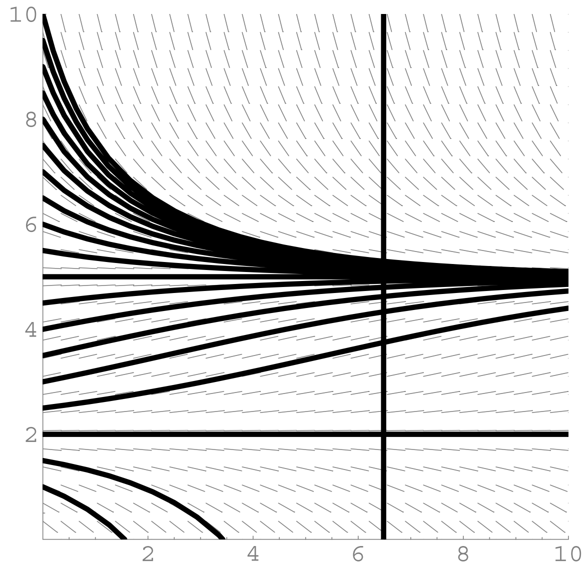


Figure 3-20 Solutions to the logistic equation with harvesting with the associated direction field

```
In[356]:= p1 = Show[toshow,
               DisplayFunction → $DisplayFunction];
```

The unbounded behavior of the solutions is particularly evident when we display the graphs in `toshow` together with the direction field associated with the equation in Figure 3-20.

```
In[357]:= << Graphics`PlotField`

          p2 = PlotVectorField[{1, 7/10y - 1/10y^2 - 1},
               {x, 0, 10}, {y, 0, 10}, Frame → False,
               Axes → Automatic, AxesOrigin → {0, 0},
               DefaultColor → GrayLevel[0.5],
               ScaleFunction → (0.5&), PlotPoints- > 30,
               HeadLength- > 0, DisplayFunction → Identity];

In[358]:= Show[p2, p1, DisplayFunction → $DisplayFunction,
            PlotRange → {{0, 10}, {0, 10}}, AspectRatio → 1]
```

Thus, if $P_0 < 2$ and harvesting is allowed to continue, the species becomes extinct. If $P_0 > 2$, the population of the species has an equilibrium population of $P = 5$.

Sources: See texts like Boyce and DiPrima's *Elementary Differential Equations and Boundary-Value Problems*, [5] and Edwards and Penney's *Differential Equations and Boundary Value Problems: Computing and Modeling*, [10] for elementary but precise discussions.

Application: The Logistic Difference Equation

Given x_0, the **Logistic difference equation** is

$$x_{n+1} = rx_n\,(1 - x_n)\,. \tag{3.8}$$

Assume that $x_0 = 0.5$.

Given r, we use Mathematica to define the function $x_r(n)$ using the form

```
x[r_][n_]: = x[r][n]=...
```

so that Mathematica "remembers" the values of $x_r(n)$ computed. In doing so, Mathematica need not recompute $x_r(n-1)$ to compute $x_r(n)$ if $x_r(n-1)$ has already been computed.

```
In[359]:= Clear[x]

          x[r_][0] = 0.5;

          x[r_][n_] := x[r][n] = r x[r][n - 1] (1 - x[r][n - 1]);
```

For example,

```
In[360]:= t4 = Table[x[3.83][n], {n, 1, 50}]
Out[360]= {0.9575, 0.155857, 0.503896, 0.957442, 0.156061,
           0.504433, 0.957425, 0.156121, 0.504592, 0.957419,
           0.15614, 0.504642, 0.957417, 0.156146, 0.504658,
           0.957417, 0.156148, 0.504664, 0.957417, 0.156149,
           0.504666, 0.957417, 0.156149, 0.504666, 0.957417,
           0.156149, 0.504666, 0.957417, 0.156149, 0.504666,
           0.957417, 0.156149, 0.504666, 0.957417, 0.156149,
           0.504666, 0.957417, 0.156149, 0.504666, 0.957417,
           0.156149, 0.504666, 0.957417, 0.156149, 0.504666,
           0.957417, 0.156149, 0.504666, 0.957417, 0.156149}
```

computes x_n for $n = 1, 2, \ldots, 50$ if $r = 3.83$ in equation (3.8) and

```
In[361]:= ListPlot[t4, PlotJoined → True]
```

plots the resulting list and connects successive points with line segments as shown in Figure 3-21.

To investigate how the behavior of equation (3.8) changes as r changes, we enter

```
In[362]:= t1 = Table[{r, x[r][n]}, {r, 2.8, 4, 1.2/249},
              {n, 101, 300}];
```

which computes a nested list: for 250 equally spaced values of r between 2.8 and 4.0, the list consisting of $(r, x_r(n))$ for $n = 101, \ldots, 300$ is returned. The nested list is converted to a list of ordered pairs with `Flatten`

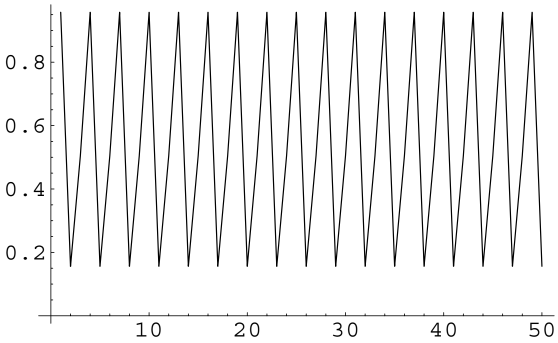


Figure 3-21 A 3-cycle

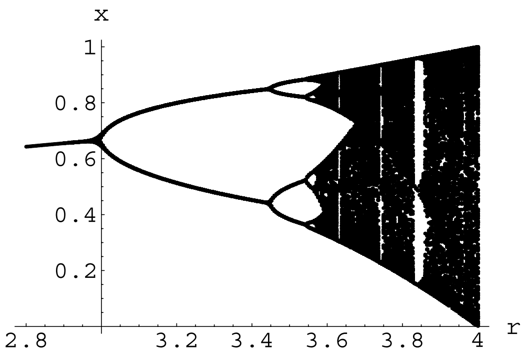


Figure 3-22 The "Pitchfork diagram" for $2.8 \le r \le 4.0$

```
In[363]:= toshow = Flatten[t1, 1];
```

and then plotted with `ListPlot` in Figure 3-22.

```
In[364]:= ListPlot[toshow, AxesLabel → {"r", "x"}]
```

However, if you immediately request `x[r][M]` for a large value of `M` without computing `x[r][n]` for `n` less than `M`, Mathematica has a difficult time working backwards. We finally abort the calculation.

```
In[365]:= x[3.9][1000]

$RecursionLimit :: reclim : Recursion depth of 256 exceeded.
$RecursionLimit :: reclim : Recursion depth of 256 exceeded.
$RecursionLimit :: reclim : Recursion depth of 256 exceeded.
General :: stop : Further output of
    $RecursionLimit :: reclim will be suppressed during this calculation.

Out[365]= $Aborted
```

When using a recursive definition like the one illustrated and to be on the safe side, a calculation like this should be carried out after the first 999 terms are computed. For situations like this, we prefer using `Nest`. For repeated compositions of a function with itself, `Nest[f,x,n]` computes the composition

$$\underbrace{(f \circ f \circ f \circ \cdots f)}_{n \text{ times}}(x) = \underbrace{(f(f(f \cdots)))}_{n \text{ times}}(x) = f^n(x).$$

In terms of a composition, computing x_n in equation (3.8) is equivalent to composing $f(x) = rx(1-x)$ with itself n times.

```
In[366]:= Clear[f, p]

          f[r_][x_] := r x(1 - x)//N;
```

Thus, entering

```
In[367]:= Table[Nest[f[2.5], 0.5, n], {n, 1, 50}]
Out[367]= {0.625, 0.585938, 0.606537, 0.596625, 0.601659,
            0.599164, 0.600416, 0.599791, 0.600104, 0.599948,
            0.600026, 0.599987, 0.600007, 0.599997, 0.600002,
            0.599999, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6,
            0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6,
            0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6,
            0.6, 0.6, 0.6, 0.6}
```

computes x_n for $n = 1, 2, \ldots, 50$ if $r = 2.5$ while entering

```
In[368]:= Table[Nest[f[3.5], 0.5, n], {n, 1, 50}]
Out[368]= {0.875, 0.382813, 0.826935, 0.500898, 0.874997,
            0.38282, 0.826941, 0.500884, 0.874997, 0.38282,
            0.826941, 0.500884, 0.874997, 0.38282, 0.826941,
            0.500884, 0.874997, 0.38282, 0.826941, 0.500884,
            0.874997, 0.38282, 0.826941, 0.500884, 0.874997,
            0.38282, 0.826941, 0.500884, 0.874997, 0.38282,
            0.826941, 0.500884, 0.874997, 0.38282, 0.826941,
            0.500884, 0.874997, 0.38282, 0.826941, 0.500884,
            0.874997, 0.38282, 0.826941, 0.500884, 0.874997,
            0.38282, 0.826941, 0.500884, 0.874997, 0.38282}
```

computes x_n for $n = 1, 2, \ldots, 50$ if $r = 3.5$.

You can see how solutions depend on r in a variety of ways. For example, entering the following command computes x_n for $n = 1000, \ldots, 1050$ for nine equally spaced values of r between 3.2 and 3.9.

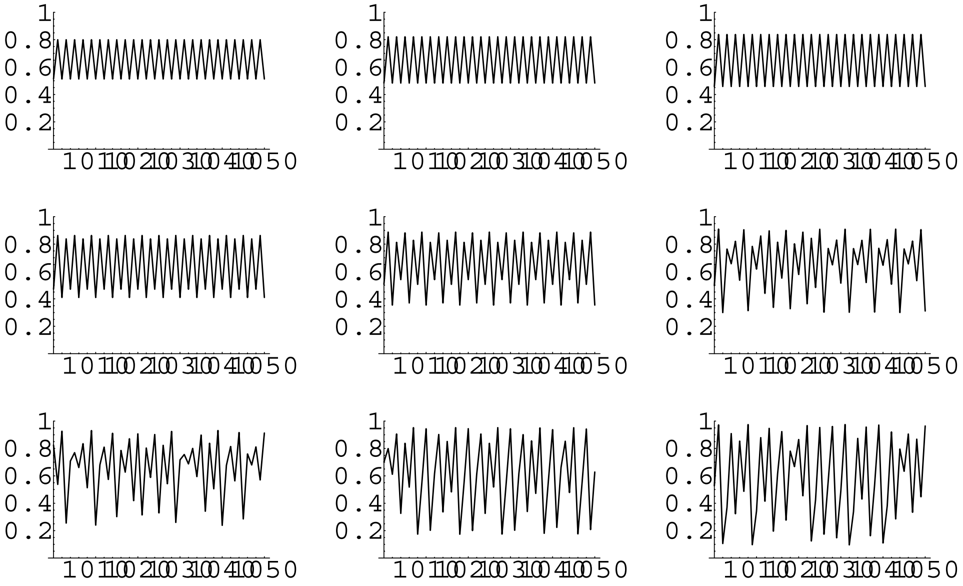


Figure 3-23 As *r* increases, we see *chaos*

```
In[369]:= t1 = Table[{n, Nest[f[r], 0.5, n]},
            {r, 3.2, 3.9, 0.7/8}, {n, 1000, 1050}];
```

Each list is plotted with `ListPlot`; successive points are connected with line segments because the option `PlotJoined->True` is included in the `ListPlot` command.

```
In[370]:= t1b = Map[ListPlot[#,
              PlotJoined → True,
              DisplayFunction → Identity,
              PlotRange → {0, 1}]&, t1];
```

All nine graphs are shown together as an array using `Show` and `GraphicsArray` in Figure 3-23.

```
In[371]:= Show[GraphicsArray[Partition[t1b, 3]]]
```

We regenerate the "Pitchfork diagram" with the following commands in Figure 3-24.

```
In[372]:= t1 = Table[{r, Nest[f[r], 0.5, n]},
            {r, 2.8, 4, 1.2/249}, {n, 101, 300}];
```

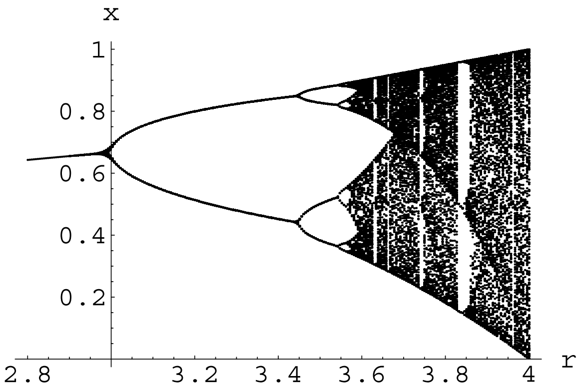


Figure 3-24 A copy of the "Pitchfork diagram"

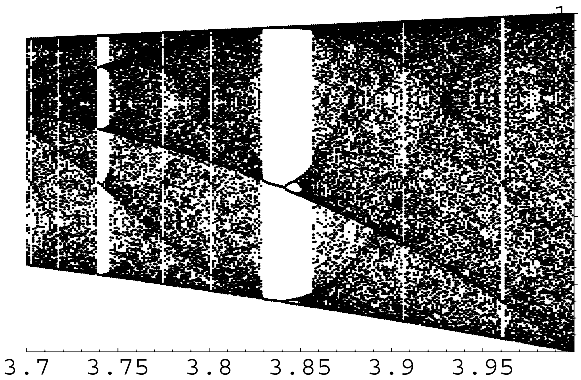


Figure 3-25 The "Pitchfork diagram" for $3.7 \le r \le 4.0$

```
In[373]:= toshow = Flatten[t1, 1];

In[374]:= ListPlot[toshow, AxesLabel → {"r", "x"},
            PlotStyle → PointSize[0.004]]
```

You can zoom in on areas of interest, as well. In Figure 3-25, we restrict r to $3.7 \le r \le 4.0$.

```
In[375]:= t3 = Table[{r, Nest[f[r], 0.5, n]},
               {r, 3.7, 4., 0.4/349}, {n, 101, 300}];

In[376]:= toshow = Flatten[t3, 1];

In[377]:= ListPlot[toshow, PlotRange → {{3.7, 4}, {0, 1}},
            PlotStyle → PointSize[0.004]]
```

3.3 Newton's Law of Cooling

First-order linear differential equations can be used to solve a variety of problems that involve temperature. For example, a medical examiner can find the time of death in a homicide case, a chemist can determine the time required for a plastic mixture to cool to a hardening temperature, and an engineer can design the cooling and heating system of a manufacturing facility. Although distinct, each of these problems depend on a basic principle, *Newton's Law of Cooling*, that is used to develop the associated differential equation.

> **Newton's Law of Cooling:** The rate at which the temperature $T(t)$ changes in a cooling body is proportional to the difference between the temperature of the body and the constant temperature T_s of the surrounding medium.

Newton's Law of Cooling is modeled with the first-order initial-value problem

$$\begin{cases} \dfrac{dT}{dt} = k\left(T - T_s\right) \\ T(0) = T_0 \end{cases} \tag{3.9}$$

where T_0 is the initial temperature of the body and k is the constant of proportionality. If T_s is constant, equation (3.9) is separable and separating variables gives us

Equation (3.9) is also linear. See Example 2.5.2 where we solved equation (3.9) by viewing it as a linear equation.

$$\frac{1}{T - T_s} dT = k\,dt \qquad \text{so} \qquad \ln|T - T_s| = kt + C_1.$$

Using the properties of the natural logarithm and simplifying yields $T(t) = Ce^{kt} + T_s$, where $C = \pm e^{C_1}$. Applying the initial condition implies that $T_0 = C + T_s$, so $C = T_0 - T_s$. Therefore, the solution of equation (3.9) is

$$T(t) = \left(T_0 - T_s\right) e^{kt} + T_s. \tag{3.10}$$

Recall that if $k < 0$, $\lim_{t\to\infty} e^{kt} = 0$. Therefore, $\lim_{t\to\infty} T(t) = T_s$, so the temperature of the body approaches that of its surroundings.

EXAMPLE 3.3.1: A pie is removed from a 350° F oven and placed to cool in a room with temperature 75° F. In 15 minutes, the pie has a temperature of 150° F. Determine the time required to cool the pie to a temperature of 80° F so that it may be eaten.

SOLUTION: In this example, $T_0 = 350$ and $T_s = 75$. Substituting these values into equation (3.11), we obtain $T(t) = (350 - 75)e^{kt} + 75 = 275e^{kt} + 75$.

```
In[378]:= step1 = capt_s + e^(kt) (-capt_s + t_0)/.
            {t_0 → 350, capt_s → 75}
Out[378]= 75 + 275 e^(k t)
```

To solve the problem we must find k or e^k. Because we also know that $T(15) = 150$, $T(15) = 275e^{15k} = 75$. Solving this equation for k or e^k gives us:

$$\begin{aligned} 275e^{15k} &= 75 \\ e^{15k} &= \frac{3}{11} \\ \ln\left(e^{15k}\right) &= \ln\left(\frac{3}{11}\right) \\ 15k &= \ln\left(\frac{3}{11}\right) \\ k &= \frac{\ln\left(\frac{3}{11}\right)}{15} \\ k &= -\frac{\ln\left(\frac{11}{3}\right)}{15} \end{aligned} \quad \text{or} \quad \begin{aligned} 275e^{15k} &= 75 \\ e^{15k} &= \frac{3}{11} \\ e^k &= \left(\frac{3}{11}\right)^{1/15} \\ e^k &= \left(\frac{11}{3}\right)^{-1/15}. \end{aligned}$$

Thus, $T(t) = 275e^{-t\ln(11/3)/15} + 75 = 275\left(\frac{11}{3}\right)^{-t/15} + 75$.

```
In[379]:= step2 = step1/.k → 1/15 Log[3/11]
Out[379]= 75 + 25 3^(t/15) 11^(1-t/15)
```

To find the value of t for which $T(t) = 80$, we solve the equation $275\left(\frac{11}{3}\right)^{-t/15} + 75 = 80$ for t:

$$\begin{aligned} 275\left(\frac{11}{3}\right)^{-t/15} &= 5 \\ \left(\frac{11}{3}\right)^{-t/15} &= \frac{1}{55} \\ \ln\left(\frac{11}{3}\right)^{-t/15} &= \ln\left(\frac{1}{55}\right) = -\ln 55 \\ -\frac{t}{15}\ln\left(\frac{11}{3}\right) &= -\ln 55 \\ t &= \frac{15\ \ln 55}{\ln\left(\frac{11}{3}\right)} \approx 46.264. \end{aligned}$$

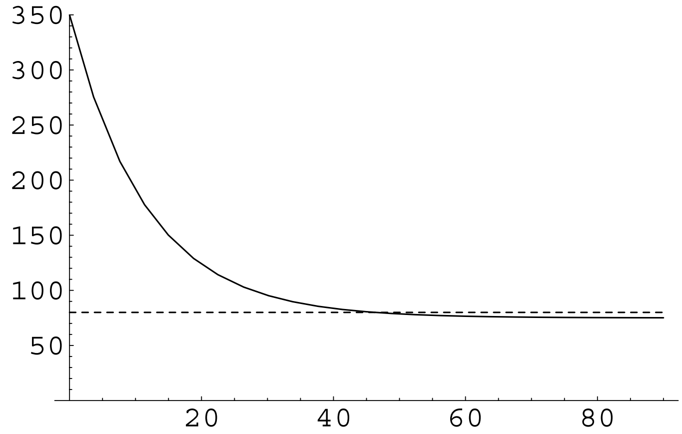


Figure 3-26 The value of t where $y = T(t)$ and $y = 80$ is the solution to the problem

Alternatively, we can graph the solution together with the line $y = 80$ as shown in Figure 3-26

```
In[380]:= Plot[{step2, 80}, {t, 0, 90},
            PlotStyle → {GrayLevel[0], Dashing[{0.01}]},
            PlotRange → {0, 350}]
```

and then use `FindRoot` to approximate the time at which the temperature of the pie reaches 80° F.

```
In[381]:= FindRoot[step2 == 80, {t, 45}]

Out[381]= {t → 46.264}
```

Thus, the pie will be ready to eat after approximately 46 minutes.

An interesting question associated with cooling problems is to determine if the pie reaches room temperature. From the formula, $T(t) = 275\left(\frac{11}{3}\right)^{-t/15} + 75$, we see that the component $275\left(\frac{11}{3}\right)^{-t/15} > 0$, so $T(t) = 275\left(\frac{11}{3}\right)^{-t/15} + 75 > 75$. Therefore, the pie never actually reaches room temperature according to our model. However, we see from the graph and from the values in the following table that its temperature approaches 75° F as t increases.

```
In[382]:= Table[{t,step2//N},{t,60,100,10}]//
            TableForm
           60  76.5214
           70  75.6398
Out[382]=  80  75.2691
           90  75.1132
           100 75.0476
```

■

If the temperature of the surroundings, T_s, varies the situation is more complicated. For example, consider the problem of heating and cooling a building. Over the span of a 24-hour day, the outside temperature, T_s, varies so the problem of determining the temperature inside the building becomes more complicated. Assuming that the building has no heating or air conditioning system, the differential equation that needs to be solved to find the temperature $u(t)$ at time t inside the building is

$$\frac{du}{dt} = k\,(C(t) - u(t)), \tag{3.11}$$

where $C(t)$ is a function that describes the outside temperature and $k > 0$ is a constant that depends on the insulation of the building. According to this equation, if $C(t) > u(t)$, then $du/dt > 0$, which implies that u increases. On the other hand, if $C(t) < u(t)$, then $du/dt < 0$ which means that u decreases.

The first choice of $C(t)$ has average value of 70° F; the second choice has an average value of 80° F.

EXAMPLE 3.3.2: (a) Suppose that during the month of April in Atlanta, Georgia, the outside temperature in degrees F is given by $C(t) = 70 - 10\cos(\pi t/12)$, $0 \le t \le 24$. Determine the temperature in a building that has an initial temperature of 60° F if $k = 1/4$. (b) Compare this to the temperature in June when the outside temperature is $C(t) = 80 - 10\cos(\pi t/12)$ and the initial temperature is 70° F.

SOLUTION: (a) The initial-value problem that we must solve is

$$\begin{cases} \dfrac{du}{dt} = k\left[70 - 10\cos\left(\dfrac{\pi}{12}t\right) - u\right] \\ u(0) = 60. \end{cases}$$

The differential equation can be solved if we write it as $du/dt + ku = k\,[70 - 10\cos(\pi t/12)]$ and then use an integrating factor. This gives us

$$\frac{d}{dt}\left(e^{kt}u\right) = ke^{kt}\,[70 - 10\cos(\pi t/12)],$$

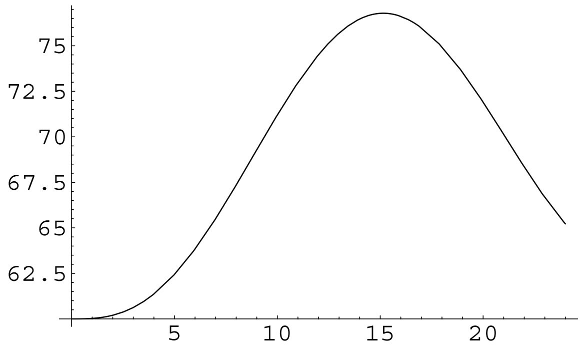


Figure 3-27 The temperature in a hypothetical building over a period of 24 hours

so we must integrate both sides of the equation. Of course, solving the equation is most easily carried out through the use of `DSolve`.

```
In[383]:= sol1 =
    DSolve[{u'[t] == 1/4 (70 - 10 Cos[πt/12] - u[t]),
     u[0] == 60}, u[t], t]//Simplify
```

```
Out[383]= {{u[t] → 1/(9 + π^2) (10 (63 + 7 π^2
    -e^(-t/4) π^2 - 9 Cos[π t/12]
    -3 π Sin[π t/12])))}}
```

We then use `Plot` to graph the solution for $0 \leq t \leq 24$ in Figure 3-27.

```
In[384]:= Plot[u[t]/.sol1, {t, 0, 24}]
```

Note that the temperature reaches its maximum (approximately 77°F) near $t \approx 15.5$ hours which corresponds to 3:30 p.m. A more accurate estimate is obtained with `FindRoot` by setting the first derivative of the solution equal to zero and solving for t.

```
In[385]:= FindRoot[Evaluate[∂t(sol1[[1, 1, 2]]) == 0],
    {t, 15}]

Out[385]= {t → 15.1506}
```

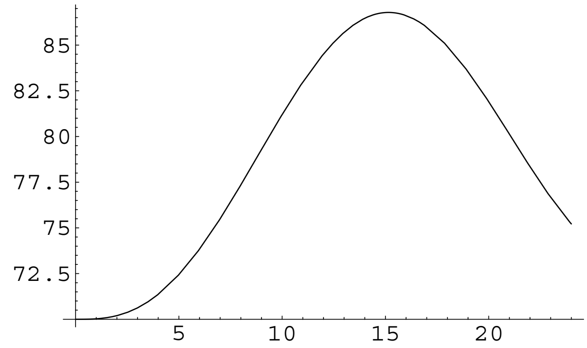


Figure 3-28 The plot is almost identical to the plot obtained in (a)

(b) This problem is solved in the same manner as the previous case.

```
In[386]:= sol2 = DSolve[{u'[t]
            == 1/4 (80 - 10 Cos[πt/12] - u[t]), u[0]
              == 70}, u[t], t]//Simplify

Out[386]= {{u[t] → 1/(9 + π^2) (10 (72 + 8 π^2
          -e^(-t/4) π^2 - 9 Cos[π t/12]
          -3 π Sin[π t/12])))}}
```

The solution is graphed with `Plot` in Figure 3-28. From the graph, we see that the maximum temperature appears to occur near $t \approx 15$ hours.

```
In[387]:= Plot[u[t]/.sol2, {t, 0, 24}]
```

Again, a more accurate value is obtained with `FindRoot` by setting the first derivative of the solution equal to zero and solving for t. This calculation yields 15.15 hours, the same as that in (a).

```
In[388]:= FindRoot[Evaluate[∂t (sol2[[1, 1, 2]]) == 0],
            {t, 15}]

Out[388]= {t → 15.1506}
```

■

3.4 Free-Falling Bodies

The motion of objects can be determined through the solution of first-order initial-value problems. We begin by explaining some of the theory that is needed to set up the differential equation that models the situation.

> **Newton's Second Law of Motion:** The rate at which the momentum of a body changes with respect to time is equal to the resultant force acting on the body.

Because the body's momentum is defined as the product of its mass and velocity, this statement is modeled as

$$\frac{d}{dt}(mv) = F,$$

where m and v represent the body's mass and velocity, respectively, and F is the sum of the forces (the resultant force) acting on the body. Because m is constant, differentiation leads to the well-known equation

$$m\frac{dv}{dt} = F.$$

If the body is subjected only to the force due to gravity, then its velocity is determined by solving the differential equation

$$m\frac{dv}{dt} = mg \qquad \text{or} \qquad \frac{dv}{dt} = g,$$

where $g = 32$ ft/s^2 (English system) and $g = 9.8$ m/s^2 (international system). This differential equation is applicable only when the resistive force due to the medium (such as air resistance) is ignored. If this offsetting resistance is considered, we must discuss all of the forces acting on the object. Mathematically, we write the equation as

$$m\frac{dv}{dt} = \sum(\text{forces acting on the object})$$

where the direction of motion is taken to be the positive direction. Because air resistance acts against the object as it falls and g acts in the same direction of the motion, we state the differential equation in the form

$$m\frac{dv}{dt} = mg + (-F_R) \qquad \text{or} \qquad m\frac{dv}{dt} = mg - F_R,$$

where F_R represents this resistive force. Note that down is assumed to be the positive direction. The resistive force is typically proportional to the body's velocity, v, or the square of its velocity, v^2. Hence, the differential equation is linear or nonlinear based on the resistance of the medium taken into account.

EXAMPLE 3.4.1: Determine the velocity and displacement functions of an object with $m = 1$ slug where 1 slug = lb s^2/ft, that is thrown downward with an initial velocity of 2 ft/s from a height of 1000 feet. Assume that the object is subjected to air resistance that is equivalent to the instantaneous velocity of the object. Also, determine the time at which the object strikes the ground and its velocity when it strikes the ground.

SOLUTION: First, we set up the initial-value problem to determine the velocity of the object. Because the air resistance is equivalent to the instantaneous velocity, we have $F_R = v$. The formula $m\,dv/dt = mg - F_R$ then gives us $dv/dt = 32 - v$. Of course, we must impose the initial velocity $v(0) = 2$. Therefore, the initial-value problem is

$$\begin{cases} dv/dt = 32 - v \\ v(0) = 2 \end{cases}$$

which is both separable and first-order linear. We solve it as a linear first-order equation and so we multiply both sides of the equation by the integrating factor e^t, which results in $d/dt\left(e^t v\right) = 32t$. Integrating both sides gives us $e^t v = 32e^t + C$, so $v = 32 + Ce^{-t}$. Applying the initial velocity, we have $v(0) = 32 + C = 2$. Therefore, the velocity of the object is $v = 32 - 30e^{-t}$. We obtain the same result with `DSolve`, naming the resulting output `step1`.

```
In[389]:= Clear[v, t]

          step1 = DSolve[{v'[t] == 32 - v[t], v[0] == 2},
            v[t], t]
Out[389]= {{v[t] → 2 e^-t (-15 + 16 e^t)}}
```

To determine the position, or distance traveled at time t, $s(t)$, we solve the first-order equation $ds/dt = 32 - 30e^{-t}$ with initial displacement $s(0) = 0$. Notice that we use the initial displacement as a reference and let $s = s(t)$ represent the distance traveled from this reference point.

```
In[390]:= step2 = DSolve[{s'[t] == 32 - 30e^-t, s[0] == 0},
            s[t], t]
Out[390]= {{s[t] → 2 e^-t (15 - 15 e^t + 16 e^t t)}}
```

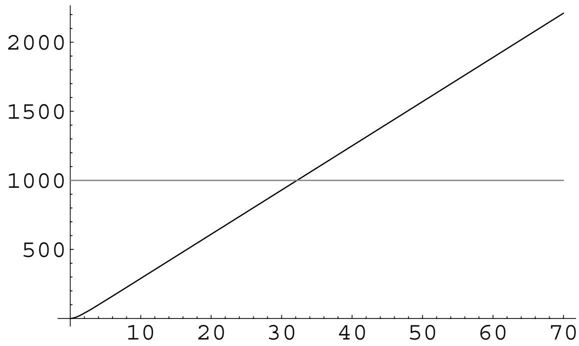


Figure 3-29 Plots of $s = s(t)$ and the line $s = 1000$

Thus, the displacement of the object at time t is given by $s = 32t + 30e^{-t} - 30$.

Because we are taking $s(0) = 0$ as our starting point, the object strikes the ground when $s(t) = 1000$. Therefore, we must solve $s = 32t + 30e^{-t} - 30 = 1000$. The roots of this equation can be approximated with `FindRoot`. We begin by graphing the function $s = s(t)$ and the line $s = 1000$ with `Plot` in Figure 3-29.

```
In[391]:= Plot[{-30 + 30e^-t + 32t, 1000}, {t, 0, 70},
             PlotStyle → {GrayLevel[0], GrayLevel[0.5]}]
```

From the graph of this function, we see that $s(t) = 1000$ near $t \approx 35$. To obtain a better approximation, we use `FindRoot`

```
In[392]:= t00 = FindRoot[-30 + 30e^-t + 32t == 1000,
             {t, 35}]

Out[392]= {t → 32.1875}
```

so the object strikes the ground after approximately 32.1875 seconds.

The velocity at the point of impact is found to be 32.0 ft/s by evaluating the derivative, $s'(t) = v(t) = 32 - 30e^{-t}$, at the time at which the object strikes the ground, $t \approx 32.1875$.

```
In[393]:= 32 - 30e^-t /. (t00[[1]])

Out[393]= 32.
```

■

EXAMPLE 3.4.2: Determine a solution (for the velocity and the displacement) of the differential equation that models the motion of an object of mass m when directed upward with an initial velocity of v_0 from an initial displacement y_0 assuming that the air resistance equals cv, where c is constant.

SOLUTION: Because the motion of the object is upward, mg and F_R act against the upward motion of the object; mg and F_R are in the negative direction. Therefore, the initial-value problem that must be solved in this case is the linear problem,

$$\begin{cases} \dfrac{dv}{dv} = -g - \dfrac{c}{m}v \\ v(0) = v_0 \end{cases}$$

which we solve with `DSolve`, naming the resulting output `sol`.

```
In[394]:= Clear[v, t, s]

          sol = DSolve[{v'[t] == -g - (c v[t])/m, v[0] == v0},
              v[t], t]

Out[394]= {{v[t] -> -(e^(-(c t)/m) (-g m + e^((c t)/m) g m - c v0))/c}}
```

Next, we use `sol` to define `velocity`. This function can be used to investigate numerous situations without re-solving the differential equation each time.

```
In[395]:= velocity[m_, c_, g_, v0_, t_] =
              (-g m + c e^(-(c t)/m) ((g m)/c + v0))/c;
```

For example, the velocity function for the case with $m = 128$ slugs, $c = 1/160$, $g = 32$ ft/s^2, and $v_0 = 48$ ft/s is $v(t) = 88e^{-4t/5} - 40$.

```
In[396]:= velocity[1/128, 1/160, 32, 48, t]//Expand
Out[396]= -40 + 88 e^(-4 t/5)
```

The displacement function $s(t)$ that represents the distance above the ground at time t is determined by integrating the velocity function. This is accomplished here with `DSolve` using the initial displacement y_0. As with the previous case, the output is named `pos` so that the displacement formula may be used to define the function `position`.

```
In[397]:= pos = DSolve[{y'[t]
                == velocity[m, c, g, v0, t],
                y[0] == y0}, y[t], t]
```

Out[397]= $\left\{\left\{y[t] \to \frac{1}{c^2}\left(e^{-\frac{ct}{m}}\left(-g m^2 + e^{\frac{ct}{m}} g m^2 - c e^{\frac{ct}{m}} g m t - c m v0 + c e^{\frac{ct}{m}} m v0 + c^2 e^{\frac{ct}{m}} y0\right)\right)\right\}\right\}$

In[398]:= position[m_, c_, g_, v0_, y0_, t_] =

$$-\frac{1}{c^2}\left(e^{-\frac{ct}{m}} gm^2 + cgmt + ce^{-\frac{ct}{m}} mv0 - c^2\left(\frac{gm^2}{c^2} + \frac{mv0}{c} + y0\right)\right);$$

The displacement and velocity functions are plotted in the following using the parameters $m = 128$ slugs, $c = 1/160$, $g = 32$ ft/s^2, and $v_0 = 48$ ft/s as well as $y_0 = 0$.

The time at which the object reaches its maximum height occurs when the derivative of the displacement is equal to zero. From the graph in Figure 3-30 we see that $s'(t) = v(t) = 0$ when $t \approx 1$.

In[399]:= Plot[{velocity[$\frac{1}{128}$, $\frac{1}{160}$, 32, 48, t], position[$\frac{1}{128}$, $\frac{1}{160}$, 32, 48, 0, t]}, {t, 0, 2}, PlotStyle → {GrayLevel[0], Dashing[{0.01}]}}]

A more accurate approximation, $t \approx 0.985572$, is obtained using `Solve` together with `N`

In[400]:= root = N[Solve[∂_tposition[$\frac{1}{128}$, $\frac{1}{160}$, 32, 48, 0, t] == 0, t]]

```
Out[400]= {{t → 0.985572}}
```

or with `FindRoot`.

In[401]:= FindRoot[velocity[$\frac{1}{128}$, $\frac{1}{160}$, 32, 48, t] == 0, {t, 1}]

```
Out[401]= {t → 0.985572}
```

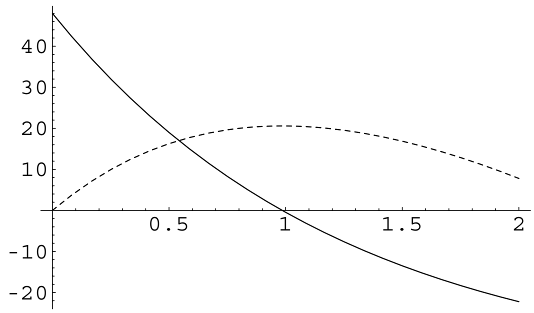


Figure 3-30 The maximum height of the object occurs when its velocity is 0

We now compare the effect that varying the initial velocity and displacement has on the displacement function. Suppose that we use the same values used earlier for *m*, *c*, and *g*. However, we let $v_0 = 48$ in one function and $v_0 = 36$ in the other. We also let $y_0 = 0$ and $y_0 = 6$ in these two functions, respectively. See Figure 3-31.

```
In[402]:= Plot[{position[1/128, 1/160, 32, 48, 0, t],
             position[1/128, 1/160, 32, 36, 6, t]},
             {t, 0, 2}, PlotStyle → {GrayLevel[0],
             Dashing[{0.01}]}]
```

Figure 3-32 demonstrates the effect that varying the initial velocity only has on the displacement function. The values of v_0 used are 48, 64, and 80. The darkest curve corresponds to $v_0 = 48$. Notice that as the

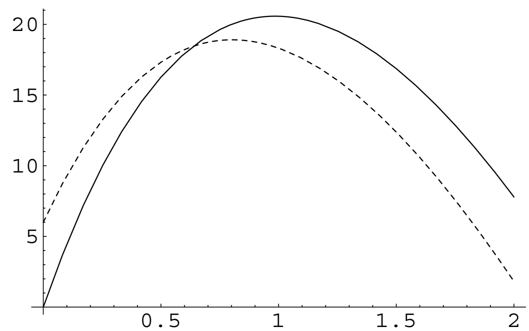


Figure 3-31 Varying v_0 and y_0

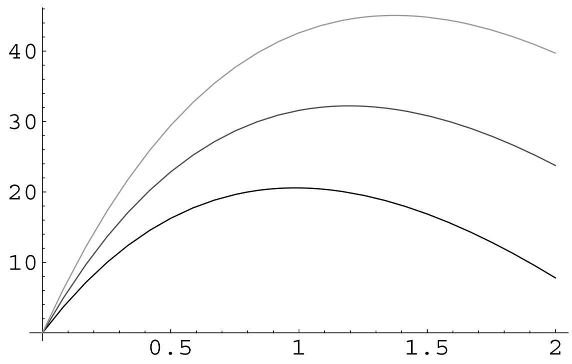


Figure 3-32 Varying v_0

initial velocity is increased the maximum height attained by the object is increased as well.

```
In[403]:= Plot[{position[1/128, 1/160, 32, 48, 0, t],
            position[1/128, 1/160, 32, 64, 0, t],
            position[1/128, 1/160, 32, 80, 0, t]},
            {t, 0, 2}, PlotStyle → {GrayLevel[0],
            GrayLevel[0.3], GrayLevel[0.6]}]
```

Figure 3-33 indicates the effect that varying the initial displacement and holding all other values constant has on the displacement function.

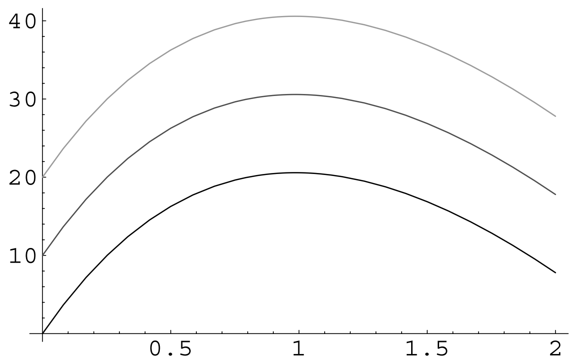


Figure 3-33 Varying y_0

We use values of 0, 10, and 20 for y_0. Notice that the value of the initial displacement vertically translates the displacement function.

```
In[404]:= Plot[{position[1/128, 1/160, 32, 48, 0, t],
              position[1/128, 1/160, 32, 48, 10, t],
              position[1/128, 1/160, 32, 48, 20, t]},
              {t, 0, 2}, PlotStyle → {GrayLevel[0],
              GrayLevel[0.3], GrayLevel[0.6]}]
```

■

We now combine several of the topics discussed in this section to solve the following problem.

EXAMPLE 3.4.3: An object of mass $m = 1$ slug is dropped from a height of 50 feet above the surface of a small pond. While the object is in the air, the force due to air resistance is v. However, when the object is in the pond, it is subjected to a buoyancy force equivalent to $6v$. Determine how much time is required for the object to reach a depth of 25 feet in the pond.

SOLUTION: This problem must be broken into two parts: an initial-value problem for the object above the pond, and an initial-value problem for the object below the surface of the pond. The initial-value problem above the pond's surface is found to be

$$\begin{cases} dv/dt = 32 - v \\ v(0) = 0. \end{cases}$$

However, to define the initial-value problem to find the velocity of the object beneath the pond's surface, the velocity of the object when it reaches the surface must be known. Hence, the velocity of the object above the surface must be determined by solving the initial-value problem above. The equation $dv/dt = 32 - v$ is separable and solved with `DSolve` in `d1`.

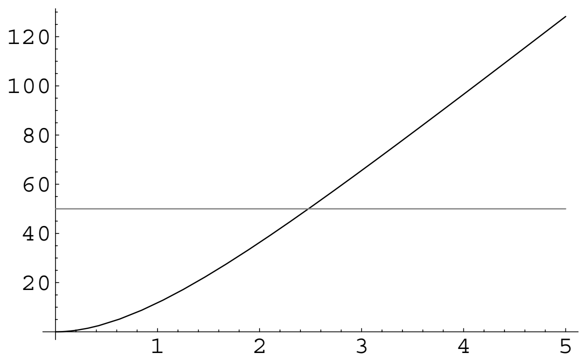


Figure 3-34 The object has traveled 50 feet when $t \approx 2.5$

```
In[405]:= Clear[v,y]

          d1 = DSolve[{v'[t] == 32 - v[t],
               v[0] == 0},v[t],t]
Out[405]= {{v[t] → 32 e^-t (-1 + e^t)}}
```

In order to find the velocity when the object hits the pond's surface we must know the time at which the distance traveled by the object (or the displacement of the object) is 50. Thus, we must find the displacement function which is done by integrating the velocity function obtaining $s(t) = 32e^{-t} + 32t - 32$.

```
In[406]:= p1 = DSolve[{y'[t] == e^-t (-32 + 32 e^t),
               y[0] == 0},y[t],t]
Out[406]= {{y[t] → 32 e^-t (1 - e^t + e^t t)}}
```

The displacement function is graphed with `Plot` in Figure 3-34. The value of t at which the object has traveled 50 feet is needed. This time appears to be approximately 2.5 seconds.

```
In[407]:= Plot[{e^-t (32 - 32 e^t + 32 e^t t),50},{t,0,5},
            PlotStyle → {GrayLevel[0],
            GrayLevel[0.5]}]
```

A more accurate value of the time at which the object hits the surface is found using `FindRoot`. In this case, we obtain $t \approx 2.47864$. The velocity at this time is then determined by substitution into the velocity function resulting in $v(2.47864) \approx 29.3166$. Note that this value is the initial velocity of the object when it hits the surface of the pond.

```
In[408]:= t1 = FindRoot[p1[[1, 1, 2]] == 50, {t, 2.5}]
Out[408]= {t → 2.47864}

In[409]:= v1 = d1[[1, 1, 2]] /. t1
Out[409]= 29.3166
```

Thus, the initial-value problem that determines the velocity of the object beneath the surface of the pond is given by

$$\begin{cases} dv/dt = 32 - 6v \\ v(0) = 29.3166. \end{cases}$$

The solution of this initial-value problem is $v(t) = \frac{16}{3} + 23.9833e^{-6t}$ and integrating to obtain the displacement function (the initial displacement is 0) we obtain $s(t) = 3.99722 - 3.99722e^{-6t} + \frac{16}{3}t$. These steps are carried out in `d2` and `p2`.

```
In[410]:= d2 = DSolve[{v'[t] == 32 - 6 v[t],
               v[0] == v1}, v[t], t]
Out[410]= {{v[t] → e^(-6 t) (23.9832 + 5.33333 e^(6 t))}}

In[411]:= p2 = DSolve[{y'[t] == d2[[1, 1, 2]],
               y[0] == 0}, y[t], t]
Out[411]= {{y[t] → 2.71828^(-6. t) (-3.99721
                +3.99721 2.71828^(6. t)
                +5.33333 2.71828^(6. t) t)}}
```

This displacement function is then plotted in Figure 3-35 to determine when the object is 25 feet beneath the surface of the pond. This time appears to be near 4 seconds.

```
In[412]:= Plot[{p2[[1, 1, 2]], 25}, {t, 0, 5},
            PlotStyle → {GrayLevel[0], GrayLevel[0.5]}]
```

A more accurate approximation of the time at which the object is 25 feet beneath the pond's surface is obtained with `FindRoot`. In this case, we obtain $t \approx 3.93802$. Finally, the time required for the object to reach the pond's surface is added to the time needed for it to travel 25 feet beneath the surface to see that approximately 6.41667 seconds are required for the object to travel from a height of 50 feet above the pond to a depth of 25 feet below the surface.

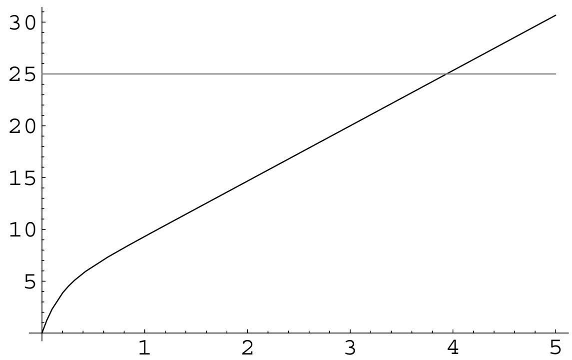


Figure 3-35 After approximately 4 seconds, the object is 25 feet below the surface of the pond

```
In[413]:= t2 = FindRoot[p2[[1, 1, 2]] == 25, {t, 4}]
Out[413]= {t → 3.93802}

In[414]:= t1[[1, 2]] + t2[[1, 2]]
Out[414]= 6.41667
```

■

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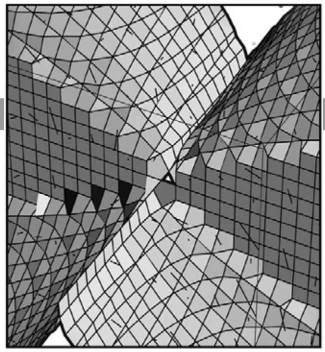
Higher-Order Differential Equations

4



In Chapters 2 and 3 we saw that first-order differential equations can be used to model a variety of physical situations. However, many physical situations need to be modeled by higher-order differential equations. In this chapter, we discuss several methods for solving higher-order differential equations.

4.1 Preliminary Definitions and Notation

4.1.1 Introduction

In the same way as in previous chapters, we can frequently use `DSolve` to generate exact solutions of higher-order equations and `NDSolve` to generate numerical solutions to higher-order initial-value problems.

EXAMPLE 4.1.1 (Van-der-Pol Equation): The **Van-der-Pol equation**, which arises in the study of nonlinear damping, is the nonlinear second-order equation

$$\frac{d^2x}{dt^2} + \mu\left(x^2 - 1\right)\frac{dx}{dt} + x = 0. \tag{4.1}$$

(a) If $x(0) = 1$ and $x'(0) = 0$, graph the solution to the Van-der-Pol equation (4.1) on the interval $[0, 15]$ for $\mu = 1/32, 1/16, 1/8, 1/4, 1/2, 1, 3/2, 2, 3, 5, 7$, and 9. (b) Compare the graphs of these solutions to the graph of the solution to the initial-value problem

$$\begin{cases} x'' + x = 0 \\ x(0) = 1, \; x'(0) = 0. \end{cases}$$

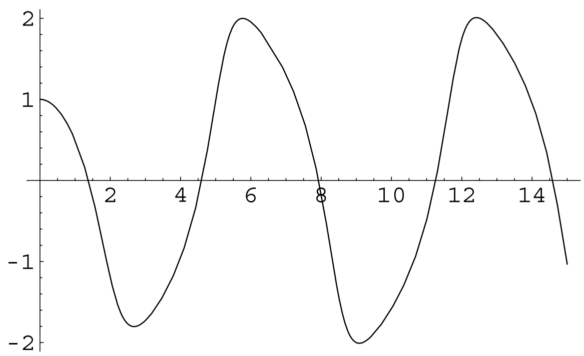


Figure 4-1 Plot of $x(t)$ if $\mu = 1$

SOLUTION: We begin by defining the function `vanderpol`. Given μ, `vanderpol`[μ] solves the initial-value problem

$$\begin{cases} x'' + \mu\left(x^2 - 1\right)x' + x = 0 \\ x(0) = 1,\ x'(0) = 0. \end{cases} \tag{4.2}$$

```
In[415]:= vanderpol[μ_] :=
            NDSolve[{x''[t] + μ (x[t]^2 - 1) x'[t] + x[t] == 0,
            x[0] == 1, x'[0] == 0}, x[t], {t, 0, 15}]
```

For example, entering

```
In[416]:= numsol1 = vanderpol[1]
Out[416]= {{x[t] →
               InterpolatingFunction[{{0., 15.}},
               <>][t]}}
In[417]:= Plot[x[t] /. numsol1, {t, 0, 15}]
```

returns a numerical solution to the initial-value problem (4.2) if $\mu = 1$ and then graphs the result on [0, 15], as shown in Figure 4-1. Entering

```
In[418]:= numsol1/.t → 1

Out[418]= {{x[1] → 0.497615}}
```

shows us that if $\mu = 1$, $x(1) \approx 0.497615$ and entering

```
In[419]:= Plot[Evaluate[D[numsol1[[1, 1, 2]], t]],
            {t, 0, 15}]
```

graphs the derivative of the numerical solution on the interval [0, 15] as shown in Figure 4-2.

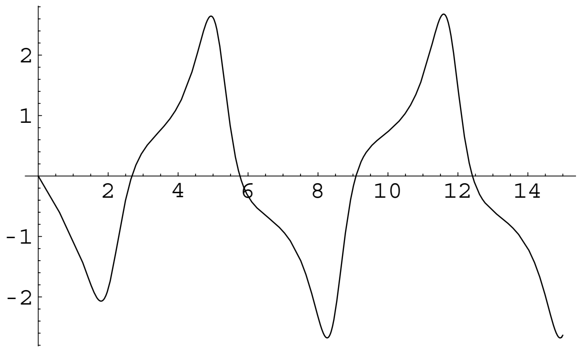


Figure 4-2 Plot of $x'(t)$ if $\mu = 1$

Because we will be graphing the solution for many values of μ, we now define the function `solgraph`. Given μ, `solgraph[`μ`]` graphs the solution to the initial-value problem (4.2) on [0, 15]. Note that the resulting graph is not displayed because the option `DisplayFunction->Identity` is included in the `Plot` command. Any options included in the `solgraph` command are passed to the `Plot` command.

```
In[420]:= Remove[solgraph]

          solgraph[μ_, opts___] := Module[{numsol},
            numsol = vanderpol[μ];
              Plot[x[t]/.numsol, {t, 0, 15}, opts,
                PlotRange → {-3, 3}, Ticks → {{0, 15},
                {-3, 3}}, DisplayFunction → Identity]]
```

For example, entering

```
In[421]:= solgraph[11,
           DisplayFunction → $DisplayFunction]
```

displays the graph of the solution to equation (4.2) on the interval [0, 15] if $\mu = 11$ shown in Figure 4-3. Thus, entering

```
In[422]:= muvals = {1/32, 1/16, 1/8, 1/4, 1/2, 1,
              3/2, 2, 3, 5, 7, 9};

In[423]:= graphs = Map[solgraph, muvals];
```

graphs the solution to the initial-value problem on the interval for $\mu =$ 1/32, 1/16, 1/8, 1/4, 1/2, 1, 3/2, 2, 3, 5, 7, and 9.

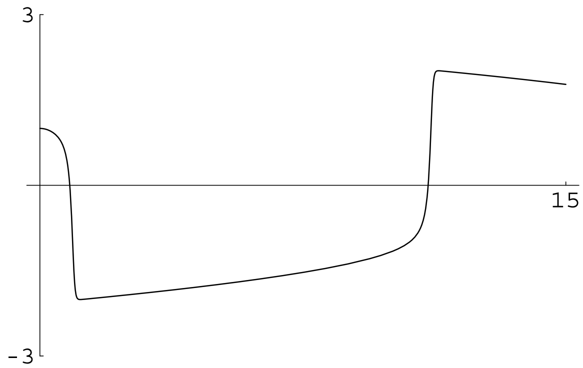


Figure 4-3 Plot of $x(t)$ if $\mu = 11$

After partitioning this list of graphics into three-element subsets in `toshow` with `Partition`, the resulting array of graphics is displayed with `Show` and `GraphicsArray` in Figure 4-4.

```
In[424]:= toshow = Partition[graphs, 3];

          Show[GraphicsArray[toshow]]
```

We find the solution to $\begin{cases} x'' + x = 0 \\ x(0) = 1, \ x'(0) = 0 \end{cases}$ with `DSolve`. The graph of $y = cost$ looks most like the first graph in `toshow`, corresponding to $\mu = 1/32$.

```
In[425]:= exactsol =
            DSolve[{x''[t] + x[t] == 0, x[0] == 1,
            x'[0] == 0}, x[t], t]
Out[425]= {{x[t] → Cos[t]}}
```

Last, we show the two graphs together to see how similar they are in Figure 4-5.

```
In[426]:= sol2 = vanderpol[1/32]
Out[426]= {{x[t] →
              InterpolatingFunction[{{0., 15.}},
              <>][t]}}

In[427]:= Plot[Evaluate[x[t]/.{exactsol, sol2}],
            {t, 0, 15},
            PlotStyle → {GrayLevel[0], GrayLevel[0.5]},
            PlotRange → {-2, 2}]
```

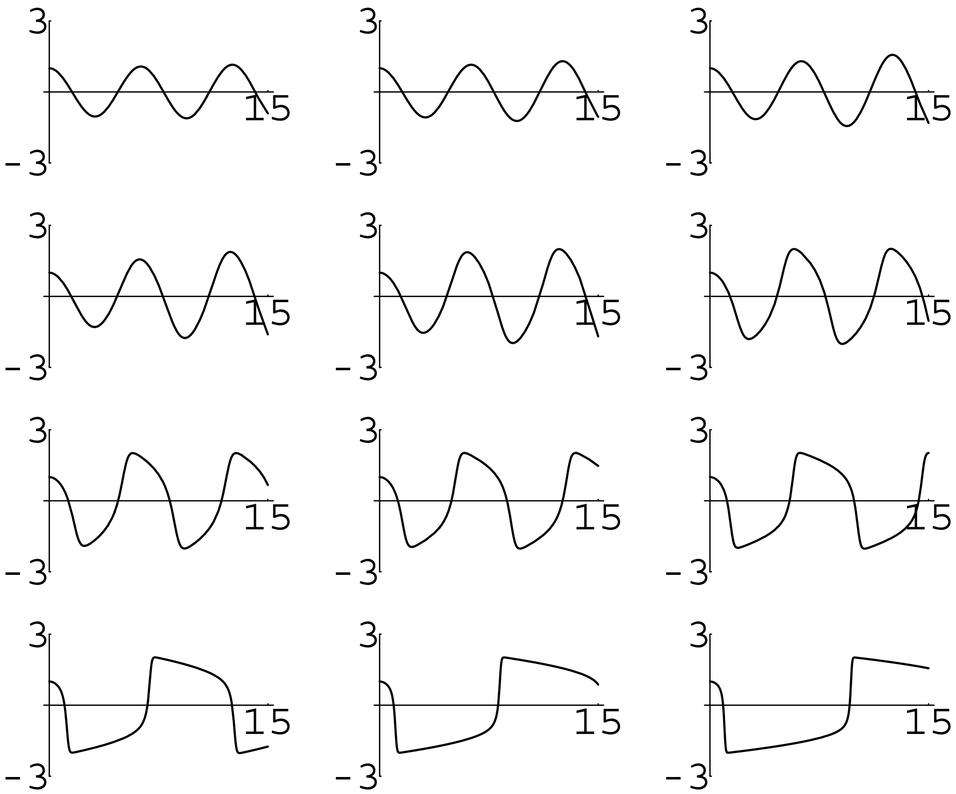


Figure 4-4 Plot of $x(t)$ if μ = 1/32, 1/16, 1/8, 1/4, 1/2, 1, 3/2, 2, 3, 5, 7, and 9

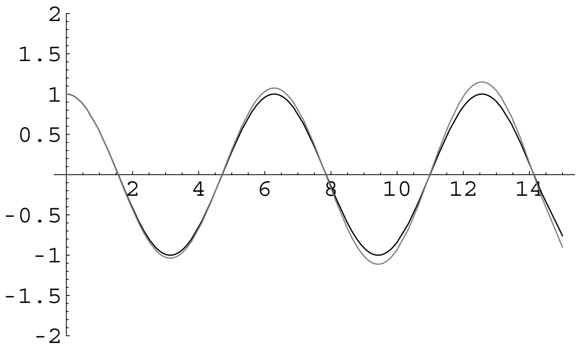


Figure 4-5 Plots of $x(t)$ if μ = 1/32 (in gray) and $y = \cos t$ (in black)

■

The example illustrates an important difference between linear and nonlinear equations. Exact solutions of linear equations with constant coefficents can often be found. Nonlinear equations can often be approximated by linear equations. Thus, we concentrate our study on linear differential equations.

4.1.2 The *n*th-Order Ordinary Linear Differential Equation

In order to develop the methods needed to solve higher-order differential equations, we must state several important definitions and theorems. We begin by introducing the types of higher-order equations that we will be solving in this chapter by restating the following definition that was given in Chapter 1.

Definition 10 (Linear Differential Equation). *An ordinary differential equation (of order n) is* ***linear*** *if it is of the form*

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{4.3}$$

For the linear differential equation (4.3), $f(x)$ is called the **forcing function**.

where the functions $a_i(x)$, $i = 0, 1, \ldots, n$, and $f(x)$ are given and $a_n(x)$ is not the zero function.

If $f(x)$ is identically the zero function, equation (4.3) is said to be **homogeneous**; if $f(x)$ is not the zero function, equation (4.3) is said to be **nonhomogeneous**; and if the functions $a_i(x)$, $i = 1, 2, \ldots, n$ are constants, equation (4.3) is said to have **constant coefficients**. An *n*th-order equation accompanied by the conditions

$$y(x_0) = y_0,\ y'(x_0) = y_0',\ \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)}$$

where $y_0, y_0', \ldots, y_0^{(n-1)}$ are constants is called an ***n*****th-order initial-value problem**. For equation (4.3), the **corresponding homogeneous equation** is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0. \tag{4.4}$$

The following theorem gives sufficient conditions for the existence of a unique solution of the *n*th-order initial-value problem.

Theorem 2 (Existence and Uniqueness). *If $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $f(x)$ are continuous throughout an interval I and $a_n(x) \neq 0$ for all x in the interval I, then for every x_0 in I there is a unique solution to the initial-value problem*

$$\begin{cases} a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = f(x) \\ y(x_0) = y_0,\ y'(x_0) = y_0',\ \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)} \end{cases} \tag{4.5}$$

on I where $y_0, y_0', \ldots, y_0^{(n-1)}$ represent arbitrary constants.

Now that we have conditions that indicate the existence of solutions, we become familiar with the properties of the functions that form the solution. We will see that solutions to nth-order ordinary linear differential equations require n solutions with the following property.

Definition 11 (Linearly Dependent and Linearly Independent Functions). *Let*

$$S = \{f_1(x),\ f_2(x),\ \ldots,\ f_n(x)\}$$

*be a set of n functions. S is **linearly dependent** on an interval I if there are constants c_1, $c_2, \ldots, c_n$, not all zero, so that*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

*for every value of x in the interval I. S is **linearly independent** if S is not linearly dependent.*

It is a good exercise to use the definition of linear dependence to show that a set of two functions is linearly dependent if and only if the two functions are constant multiples of each other.

Definition 12 (Wronskian). *Let $S = \{f_1(x),\ f_2(x),\ \ldots,\ f_n(x)\}$ be a set of n functions for which each is differentiable at least $n-1$ times. The **Wronskian** of S, $W(S)$, denoted by*

$$W(S) = W\left(\{f_1(x),\ f_2(x),\ \ldots,\ f_n(x)\}\right),$$

is the determinant

$$W(S) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f'_1(x) & f'_2(x) & \cdots & f'_n(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}. \tag{4.6}$$

EXAMPLE 4.1.2: Compute the Wronskian for each of the following sets of functions: (a) $S = \{\sin x,\ \cos x\}$ and (b) $S = \{\cos 2x,\ \sin 2x,\ \sin x \cos x\}$.

SOLUTION: The 2×2 determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is computed with $a_{11}a_{22} - a_{12}a_{21}$. Thus, for (a) we have

$$W(S) = \begin{vmatrix} \sin x & \cos x \\ \frac{d}{dx}(\sin x) & \frac{d}{dx}(\cos x) \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1.$$

For (b), we need to compute the determinant

$$\begin{vmatrix} \cos 2x & \sin 2x & \sin x \cos x \\ \frac{d}{dx}(\cos 2x) & \frac{d}{dx}(\sin 2x) & \frac{d}{dx}(\sin x \cos x) \\ \frac{d^2}{dx^2}(\cos 2x) & \frac{d^2}{dx^2}(\sin 2x) & \frac{d^2}{dx^2}(\sin x \cos x) \end{vmatrix}.$$

The 3×3 determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ can be computed in several equivalent ways. For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Here, we take advantage of the `Det` command, which computes the determinant of a square matrix.

First, we define `caps` to be the set of functions $S = \{\cos 2x, \ \sin 2x, \ \sin x \cos x\}$.

```
In[428]:= caps = {Cos[2x], Sin[2x], Sin[x] Cos[x]}
Out[428]= {Cos[2 x], Sin[2 x], Cos[x] Sin[x]}
```

Next, we use `D` to compute the list

$$\left\{ \frac{d}{dx}(\cos 2x), \ \frac{d}{dx}(\sin 2x), \ \frac{d}{dx}(\sin x \cos x) \right\}.$$

Note that `D` automatically computes the derivative (with respect to x) of each function in `caps`.

```
In[429]:= row2 = D[caps, x]
Out[429]= {-2 Sin[2 x], 2 Cos[2 x], Cos[x]^2 - Sin[x]^2}
```

Similarly, we use `D` to compute the list

$$\left\{ \frac{d^2}{dx^2}(\cos 2x), \ \frac{d^2}{dx^2}(\sin 2x), \ \frac{d^2}{dx^2}(\sin x \cos x) \right\}.$$

```
In[430]:= row3 = D[row2, x]
Out[430]= {-4 Cos[2 x], -4 Sin[2 x], -4 Cos[x] Sin[x]}
```

(Note that entering `row3=D[caps,{x,2}]` yields the same result.) Finally, we use `Det` to see that the determinant

```
In[431]:= {caps, row2, row3}//MatrixForm
```

$$\text{Out[431]=}\begin{pmatrix} \texttt{Cos[2 x]} & \texttt{Sin[2 x]} & \texttt{Cos[x] Sin[x]} \\ \texttt{-2 Sin[2 x]} & \texttt{2 Cos[2 x]} & \texttt{Cos[x]}^2 - \texttt{Sin[x]}^2 \\ \texttt{-4 Cos[2 x]} & \texttt{-4 Sin[2 x]} & \texttt{-4 Cos[x] Sin[x]} \end{pmatrix}$$

```
In[432]:= Det[{caps, row2, row3}]
Out[432]= 0
```

is zero.

■

In Example 4.1.2, we see that in (a) the Wronskian is not 0 while in (b) the Wronskian is 0. Moreover, the set of functions in (a) is linearly independent because $y = \sin x$ and $y = \cos x$ are not multiples of each other while the set of functions in (b) is linearly dependent: $\sin 2x = 2 \sin x \cos x$. In fact, we will see that we can often use the Wronskian to determine if a set of functions is linearly dependent or linearly independent.

Before doing so, we define a function `wronskian` that quickly computes the Wronskian of a set of functions. The command `wronskian` is defined to compute the Wronskian of a list of n functions `list` in the variable x by:

1. Defining the variables `n`, `r`, and `matrix` local to the procedure `wronskian`;
2. Defining `n` to be the number of functions in `list`;
3. Defining `r[1]` to be the $1 \times n$ matrix `list`. Note that `r[1]` corresponds to the row vector $\left(f_1(x)\ f_2(x)\ \cdots\ f_n(x)\right)$, which corresponds to the top row of the matrix

$$\begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}; \tag{4.7}$$

4. Defining `r[k]` to be the derivative (with respect to x) of `r[k-1]`. `r[k]` corresponds to the kth row of the matrix (4.7);
5. Defining `matrix` to be the matrix (4.7); and
6. Computing and returning the determinant of `matrix`, corresponding to the Wronskian of `list`.

```
In[433]:= Clear[wronskian]

          wronskian[list_] := Module[{n, r, matrix},
            n = Length[list];
            r[1] = list;
            r[k_] := r[k] = ∂x r[k - 1];
            matrix = Table[r[i], {i, 1, n}];
            Expand[Det[matrix], Trig → True]]
```

We illustrate the use of `wronskian` for the set of functions

$$\text{caps} = S = \left\{ \frac{1}{\sqrt{x}} \sin 4x,\ \frac{1}{\sqrt{x}} \cos 4x \right\}.$$

```
In[434]:= wronskian[{Sin[4x]/Sqrt[x], Cos[4x]/Sqrt[x]}]
Out[434]= -4/x
```

Because the Wronskian for these two functions is not 0 and they are both solutions of $4x^2y'' + 4xy' + \left(64x^2 - 1\right)y = 0$, as verified with the following commands,

```
In[435]:= y1[x_] = Sin[4x]/Sqrt[x];

In[436]:= y2[x_] = Cos[4x]/Sqrt[x];

In[437]:= -y[x] + 64 x^2 y[x] + 4 x y'[x] + 4 x^2 y''[x]//.
            {{y[x] → y1[x], y'[x] → y1'[x], y''[x] → y1''[x]},
             {y[x] → y2[x], y'[x] → y2'[x],
              y''[x] → y2''[x]}}//Simplify
Out[437]= {0, 0}
```

we can conclude they are linearly independent by the following theorem.

Theorem 3. *Let $S = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a set of n solutions of equation (4.4) on an interval I. S is linearly independent if and only if $W(S) \neq 0$ for at least one value of x in the interval I.*

EXAMPLE 4.1.3: Use the Wronskian to classify each of the following sets of functions as linearly independent or linearly dependent: (a) $S = \left\{1 - 2\sin^2 x,\ \cos 2x\right\}$ and (b) $S = \left\{e^x,\ xe^x,\ x^2e^x\right\}$.

SOLUTION: (a) Note that both functions in S are solutions of $y'' + 4y = 0$. Here, we must compute the determinant of the 2×2 matrix

$$\begin{vmatrix} 1 - 2\sin^2 x & \cos 2x \\ \dfrac{d}{dx}\left(1 - 2\sin^2 x\right) & \dfrac{d}{dx}(\cos 2x) \end{vmatrix}.$$

We use `wronskian` to compute the determinant

```
In[438]:= wronskian[{1 - 2Sin[x]^2, Cos[2x]}]
Out[438]= 0
```

and see that the result is 0. Therefore, the set of functions $S = \{1 - 2\sin^2 x, \cos 2x\}$ is linearly dependent. This makes sense because these functions are multiples of each other: $\cos 2x = 1 - 2\sin^2 x$.

(b) Note that all three functions in S are solutions of $y''' - 3y'' + 3y' - y = 0$. Here, we must compute the determinant

$$\begin{vmatrix} e^x & xe^x & x^2e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(xe^x) & \frac{d}{dx}(x^2e^x) \\ \frac{d^2}{dx^2}(e^x) & \frac{d^2}{dx^2}(xe^x) & \frac{d^2}{dx^2}(x^2e^x) \end{vmatrix}.$$

```
In[439]:= wronskian[{Exp[x], x Exp[x], x^2 Exp[x]}]
Out[439]= 2 e^(3 x)
```

We conclude that S is linearly independent because the Wronskian of S is not identically zero.

■

4.1.3 Fundamental Set of Solutions

Obtaining a collection of n linearly independent solutions to the nth-order linear homogeneous differential equation (4.4) is of great importance in solving it.

A **nontrivial solution** is one that is not identically the zero function.

Definition 13 (Fundamental Set of Solutions). *A set S of n linearly independent nontrivial solutions of the nth-order linear homogeneous equation* (4.4) *is called a* ***fundamental set of solutions*** *of the equation.*

EXAMPLE 4.1.4: Show that $S = \{e^{-5x}, e^{-x}\}$ is a fundamental set of solutions of the equation $y'' + 6y' + 5y = 0$.

SOLUTION: Because

$$\frac{d^2}{dx^2}\left(e^{-5x}\right) + 6\frac{d}{dx}\left(e^{-5x}\right) + 5e^{-5x} = 25e^{-5x} - 30e^{-5x} + 5e^{-5x} = 0$$

and

$$\frac{d^2}{dx^2}\left(e^{-x}\right) + 6\frac{d}{dx}\left(e^{-x}\right) + 5e^{-x} = e^{-x} - 6e^{-x} + 5e^{-x} = 0$$

each function is a solution of the differential equation. It follows that S is linearly independent because

$$W(S) = \begin{vmatrix} e^{-5x} & e^{-x} \\ -5e^{-5x} & -e^{-x} \end{vmatrix} = -e^{-6x} + 5e^{-6x} = 4e^{-x} \neq 0$$

so we conclude that S is a fundamental set of solutions of the equation.

Of course, we can perform the same steps with Mathematica. First, we define `caps` to be the set of functions S.

```
In[440]:= Clear[x, y, caps]

          caps = {Exp[-5 x], Exp[-x]};
```

To verify that each function in S is a solution of $y'' + 6y' + 5y = 0$, we define a function `f`. `f[y]` computes and returns $y'' + 6y' + 5y$. We then use `Map` to apply `f` to each function in `caps` to see that each function in `caps` is a solution of $y'' + 6y' + 5y = 0$, confirming the result we obtained previously.

```
In[441]:= Clear[f]

          f[y_] := D[y, {x, 2}] + 6D[y, x] + 5y;

In[442]:= f/@caps

Out[442]= {0, 0}
```

Next, we define `wmat` to be the matrix $\begin{pmatrix} e^{-5x} & e^{-x} \\ -5e^{-5x} & -e^{-x} \end{pmatrix}$ and display `wmat` in traditional row-and-column form with `MatrixForm`.

```
In[443]:= wmat = {caps, ∂x caps};

          MatrixForm[wmat]
```

Out[443]= $\begin{pmatrix} e^{-5x} & e^{-x} \\ -5\,e^{-5x} & -e^{-x} \end{pmatrix}$

`Det` is then used to compute $W(S)$.

```
In[444]:= Det[wmat]
```

Out[444]= $4\,e^{-6x}$

■

We use a fundamental set of solutions to create what is known as a *general solution* of an nth-order linear homogeneous differential equation.

Theorem 4 (Principle of Superposition). *If* $S = \{f_1(x), f_2(x), \ldots, f_k(x)\}$ *is a set of solutions of the nth-order linear homogeneous equation* (4.4) *and* $\{c_1, c_2, \ldots, c_k\}$ *is a set of k constants, then*

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)$$

is also a solution of equation (4.4).

$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)$ is called a **linear combination of functions** in the set $S = \{f_1(x), f_2(x), \ldots, f_k(x)\}$. A consequence of this fact is that the linear combination of the functions in a fundamental set of solutions of the nth-order linear homogeneous differential equation (4.4) is also a solution of the differential equation, and we call this linear combination a **general solution** of the differential equation.

Definition 14 (General Solution). *If* $S = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ *is a fundamental set of solutions of the nth-order linear homogeneous equation*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

then a ***general solution*** *of the equation is*

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$$

where $\{c_1, c_2, \ldots, c_n\}$ *is a set of n arbitrary constants.*

In other words, if we have a fundamental set of solutions S, then a general solution of the differential equation is formed by taking the linear combination of the functions in S.

EXAMPLE 4.1.5: Show that $S = \{\cos 2x, \sin 2x\}$ is a fundamental set of solutions of the second-order ordinary linear differential equation with constant coefficients $y'' + 4y = 0$.

SOLUTION: First, we verify that both functions are solutions of $y'' + 4y = 0$. Note that we have defined `caps` to be the set of functions $S = \{\cos 2x, \sin 2x\}$. Now, we use `Map` to apply the function $y'' + 4y$ to the functions in `caps`: the command `Map[D[#,{x,2}]+4#&,caps]` computes $y'' + 4y$ for each function y in `caps`. Thus, we see that given an argument `#`, the command `D[#,{x,2}]+4#&` computes the sum of the second derivative (with respect to x) of the argument and four times the argument. We conclude that both functions are solutions of $y'' + 4y$ because the result is a list of two zeros.

```
In[445]:= caps = {Cos[2x], Sin[2x]};

In[446]:= Map[D[#, {x, 2}] + 4#&, caps]
Out[446]= {0, 0}
```

Next, we compute the Wronskian

```
In[447]:= step1 = Det[{caps, ∂xcaps}]
Out[447]= 2 Cos[2 x]^2 + 2 Sin[2 x]^2

In[448]:= Expand[step1, Trig → True]
Out[448]= 2
```

to show that the functions in S are linearly independent.

By the Principle of Superposition, $y(x) = c_1 \cos 2x + c_2 \sin 2x$, where c_1 and c_2 are arbitrary constants, is also a solution of the equation. We now graph $y(x)$ for various values of c_1 and c_2. After defining y, we use `Table` to create a list obtained by replacing `c[1]` in `y[x]` by -1, 0, and 1 and `c[2]` by -1, 0, and 1. We name the resulting list `toplot`. Note that `toplot` is a list of lists: `toplot` consists of three elements each of which is a list consisting of three functions.

```
In[449]:= Clear[y]

          y[x_] = c[1] Cos[2x] + c[2] Sin[2x];

In[450]:= toplot = Table[y[x], {c[1], -1, 1},
                       {c[2], -1, 1}]
Out[450]= {{- Cos[2 x] - Sin[2 x], - Cos[2 x],
             - Cos[2 x] + Sin[2 x]},
           {- Sin[2 x], 0, Sin[2 x]}, {Cos[2 x] - Sin[2 x],
              Cos[2 x], Cos[2 x] + Sin[2 x]}}
```

Next, we use `Table` and `GrayLevel` to create a list of nine different levels of gray.

```
In[451]:= grays = Table[GrayLevel[i],
                     {i, 0, 0.45, 0.45/8}];
```

Finally, we use `Plot` to graph the nine functions in `toplot` for $0 \leq x \leq 2\pi$ in Figure 4-6. The option `PlotStyle->grays` specifies that the functions are to be graphed using the `GrayLevels` in the list `grays`, which helps us distinguish the graphs in the resulting plot.

```
In[452]:= Plot[Evaluate[toplot], {x, 0, 2π},
            PlotStyle → grays]
```

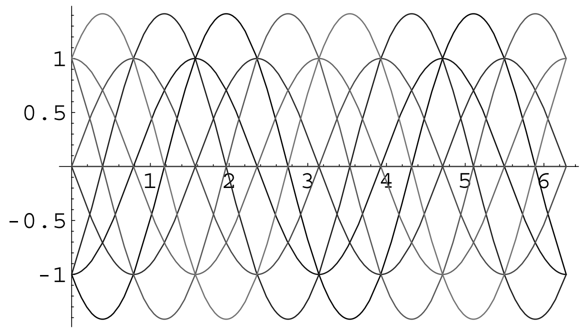


Figure 4-6 Graphs of various *linear combinations* of cos 2*x* and sin 2*x*

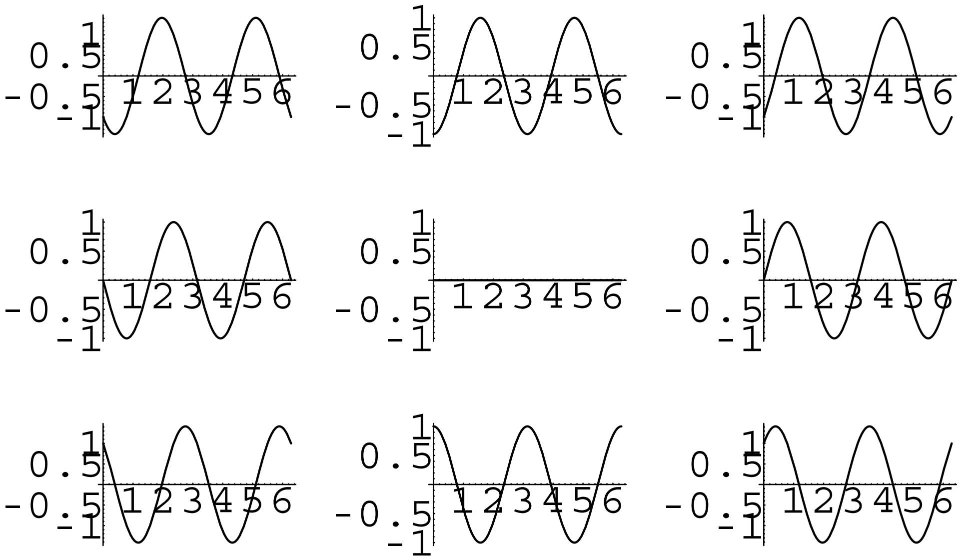


Figure 4-7 Graphs of various *linear combinations* of cos 2*x* and sin 2*x*

Alternatively, we can show the graphs individually in a graphics array as shown in Figure 4-7.

```
In[453]:= toshow = Map[Plot[#, {x, 0, 2π},
                    DisplayFunction → Identity]&,
                    Flatten[toplot]]
```

```
Out[453]= {-Graphics-, -Graphics-, -Graphics-,
           -Graphics-, -Graphics-, -Graphics-,
           -Graphics-, -Graphics-, -Graphics-}

In[454]:= Show[GraphicsArray[Partition[toshow, 3]]]
```

■

The Principle of Superposition is a *very* important property of linear homogeneous equations and is generally not valid for nonlinear equations and *never* valid for nonhomogeneous equations.

EXAMPLE 4.1.6: Is the Principle of Superposition valid for the nonlinear equation $tx'' - 2xx' = 0$?

SOLUTION: We see that `DSolve` is able to find a general solution of this nonlinear equation.

```
In[455]:= gensol = DSolve[t x''[t] - 2 x[t] x'[t] == 0,
                   x[t], t]
```

$$\text{Out[455]= } \left\{\left\{x[t] \to \frac{1}{2}\left(-1 + \sqrt{-1-8\,C[1]}\ \mathrm{Tan}\left[\frac{1}{2}\left(\sqrt{-1-8\,C[1]}\ C[2] + \sqrt{-1-8\,C[1]}\ \mathrm{Log}[t]\right)\right]\right)\right\}\right\}$$

$x(t) = -1/2$ is the solution that satisfies $x(1) = -1/2$ and $x'(1) = 0$.

```
In[456]:= gensol[[1, 1, 2]]/.{C[1] → 0, C[2] → 1/4}
```

$$\text{Out[456]= } \frac{1}{2}\left(-1 + i\ \mathrm{Tan}\left[\frac{1}{2}\left(\frac{i}{4} + i\ \mathrm{Log}[t]\right)\right]\right)$$

$$\text{In[457]:= } \frac{1}{2}\left(-1 + \sqrt{-1+4\,C[2]} \times \mathrm{Tan}\left[\frac{1}{2}\sqrt{-1+4\,C[2]}\ (C[1] + \mathrm{Log}[t])\right]\right) /. \{C[1]->0, C[2]->1/4\}$$

$$\text{Out[457]= } -\frac{1}{2}$$

$x(t) = \frac{1}{2}\left(-1 + \tan\left(\frac{1}{2}\ln t\right)\right)$ is the solution that satisfies $x(1) = -1/2$ and $x'(1) = 1/4$.

```
In[458]:= 1/2 (-1 + Sqrt[-1 + 4 C[2]]
          × Tan[1/2 Sqrt[-1 + 4 C[2]] (C[1] + Log[t])]) /.
          {C[1] -> 0, C[2] -> 1/2}
Out[458]= 1/2 (-1 + Tan[Log[t]/2])
```

However, the sum of these two solutions is not a solution to the nonlinear equation because $tf'' - 2ff' \neq 0$; the Principle of Superposition is not valid for this nonlinear equation.

```
In[459]:= f[t_] = -1/2 + 1/2 (-1 + Tan[Log[t]/2]);

In[460]:= Simplify[t f''[t] - 2 f[t] f'[t]]
Out[460]= Sec[Log[t]/2]^2 / (4 t)
```

■

4.1.4 Existence of a Fundamental Set of Solutions

The following two theorems tell us that under reasonable conditions, the nth-order linear homogeneous equation (4.4) has a fundamental set of n solutions.

Theorem 5. *If $a_i(x)$ is continuous on an open interval I for $i = 0, 1, \ldots, n$, and $a_n(x) \neq 0$ for all x in the interval I then the nth-order linear homogeneous equation* (4.4),

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

has a fundamental set of n solutions.

Theorem 6. *Any set of $n+1$ solutions of the nth-order linear homogeneous equation* (4.4) *is linearly dependent.*

We can summarize the results of these theorems by saying that in order to solve the nth-order linear homogeneous differential equation (4.4), we must find a set S of n functions that satisfy the differential equation such that $W(S) \neq 0$.

EXAMPLE 4.1.7: Show that $y = e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$ is a general solution of $y'' + 2y' + 17y = 0$.

SOLUTION: After defining y, we use `D` to compute the first and second derivatives (with respect to x) of y.

```
In[461]:= Clear[x, y, c1, c2]

          y[{c1_, c2_}] =
            Exp[-x] (c1 Cos[4x] + c2 Sin[4x]);

In[462]:= D[y[{c1, c2}], {x, 2}]

          D[y[{c1, c2}], x]

Out[462]= -2 e^-x (4 c2 Cos[4 x] - 4 c1 Sin[4 x])
          +e^-x (-16 c1 Cos[4 x] - 16 c2 Sin[4 x])
          +e^-x (c1 Cos[4 x] + c2 Sin[4 x])

Out[462]= e^-x (4 c2 Cos[4 x] - 4 c1 Sin[4 x])
          -e^-x (c1 Cos[4 x] + c2 Sin[4 x])
```

We then compute and simplify $y'' + 2y' + 17y$. Because the result is zero and the set of functions S is linearly independent, $y = e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$ is a general solution of the equation.

You should verify that $S = \{e^{-x} \cos 4x,\ e^{-x} \sin 4x\}$ is a linearly independent set of functions.

```
In[463]:= Simplify[D[y[{c1, c2}], {x, 2}]
          +2D[y[{c1, c2}], x] + 17 y[{c1, c2}]]

Out[463]= 0
```

Next, we define `cvals` to be the set of ordered pairs consisting of (0, 1), (1, 0), (2, 1), and (1, −2) and use `Map` to compute the value of y for each ordered pair in `cvals`, naming the resulting set of functions `toplot` and finally graphing them on the interval $[-1, 2]$ with `Plot` as shown in Figure 4-8.

```
In[464]:= cvals = {{0, 1}, {1, 0}, {2, 1}, {1, -2}};

          grays = Table[GrayLevel[i],
                    {i, 0, 0.4, 0.4/3}];

          toplot = y/@cvals;

In[465]:= Plot[Evaluate[toplot], {x, -1, 2},
            PlotStyle → grays]
```

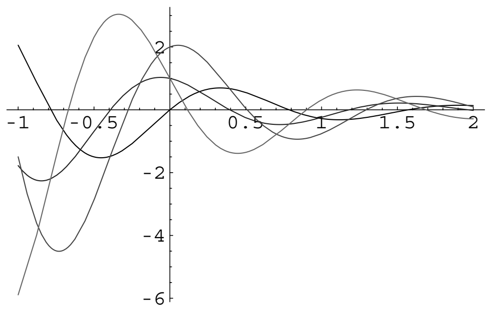


Figure 4-8 Various solutions of $y'' + 2y' + 17y = 0$

■

4.1.5 Reduction of Order

In the next section, we learn how to find solutions of homogeneous equations with constant coefficients. In doing so, we will find it necessary to determine a second solution from a known solution. We illustrate this procedure, called **reduction of order**, by considering a second-order equation. In certain situations, we can reduce a second-order equation by making an appropriate substitution to convert the second-order equation to a first-order equation (this reduction in order gives the name to the method). Consider the equation

$$y'' + p(x)y' + q(x)y = 0,$$

and suppose that $y_1 = f(x)$ is a solution to this equation. Of course we know from our previous discussion that in order to solve this second-order differential equation , we must have two linearly independent solutions. Hence, we must determine a second linearly independent solution. We accomplish this by attempting to find a solution of the form

$$y_2 = v(x)f(x),$$

where $v(x)$ is *not* a constant function. Differentiating $y_2 = v(x)f(x)$ twice we obtain

If $v(x)$ were constant, y_1 and y_2 would be linearly dependent.

$$y_2' = v'f + vf' \qquad \text{and} \qquad y_2'' = v''f + 2v'f' + vf''.$$

```
In[466]:= Clear[x, y, f, v]

          y[x_] = v[x] f[x];

In[467]:= y'[x]

          y''[x]
Out[467]= v[x] f'[x] + f[x] v'[x]
Out[467]= 2 f'[x] v'[x] + v[x] f''[x] + f[x] v''[x]
```

$f'' + p(x)f' + q(x)f = 0$ because f is a solution to $y'' + p(x)y' + q(x)y = 0$.

Notice that for convenience, we have omitted the argument of these functions. We now substitute y_2, y_2', and y_2'' into the equation $y'' + p(x)y' + q(x)y = 0$, which gives us

$$\begin{aligned} y'' + p(x)y' + q(x)y &= v''f + 2v'f' + vf'' + p(x)\left(v'f + vf'\right) + q(x)vf \\ &= fv'' + \left(2f' + p(x)f\right)v' + v\left(f'' + p(x)f' + q(x)f\right) \\ &= fv'' + \left(2f' + p(x)f\right)v' = 0. \end{aligned}$$

```
In[468]:= step1 = Collect[y''[x] + p[x] y'[x] + q[x] y[x],
              {v[x], v'[x], v''[x]}]
Out[468]= (f[x] p[x] + 2 f'[x]) v'[x]
          +v[x] (f[x] q[x] + p[x] f'[x] + f''[x]) + f[x] v''[x]

In[469]:= step2 =
            step1/.f''[x] + p[x] f'[x] + q[x] f[x] → 0
Out[469]= (f[x] p[x] + 2 f'[x]) v'[x] + f[x] v''[x]
```

Therefore, we have the equation $fv'' + (2f' + p(x)f)\,v' = 0$, which can be written as a first-order equation by letting $w = v'$. Making this substitution gives us the linear first-order equation

$$fw' + \left(2f' + p(x)f\right)w = 0 \qquad \text{or} \qquad f\frac{dw}{dx} + \left(2f' + p(x)f\right)w = 0,$$

which is separable, resulting in the separated equation

$$\frac{1}{w}dw = \left(-2\frac{f'}{f} - p(x)\right)dx.$$

```
In[470]:= step3 = step2/.{v''[x] → w'[x], v'[x] → w[x]}
Out[470]= w[x] (f[x] p[x] + 2 f'[x]) + f[x] w'[x]
```

We can solve this equation by integrating both sides of the equation to yield

$$\ln|w| = \ln\left(\frac{1}{f^2}\right) - \int p(x)\,dx \qquad \text{so} \qquad w = \frac{1}{f^2}e^{-\int p(x)\,dx}.$$

```
In[471]:= step4 = DSolve[step3 == 0, w[x], x]
```

$$\text{Out[471]= } \left\{\left\{w[x] \to e^{\int_{K\$439}^{x} \frac{-f[K\$438]\,p[K\$438]-2\,f'[K\$438]}{f[K\$438]}\,dK\$438}\,C[1]\right\}\right\}$$

```
In[472]:= step5 = Simplify[step4[[1, 1, 2]]]
```

$$\text{Out[472]= } e^{\int_{K\$439}^{x}\left(-p[K\$438]-\frac{2\,f'[K\$438]}{f[K\$438]}\right)dK\$438}\,C[1]$$

Thus, we have the formula

$$\frac{dv}{dx} = \frac{1}{f^2}e^{-\int p(x)\,dx} \qquad \text{or} \qquad v(x) = \int \frac{1}{[f(x)]^2}e^{-\int p(x)\,dx}.$$

```
In[473]:= step6 = ∫ step5 dx
```

$$\text{Out[473]= C[1]} \int e^{\int_{K\$439}^{x} \left(-p[K\$438] - \frac{2 f'[K\$438]}{f[K\$438]}\right) dK\$438} dx$$

Therefore, if we have the solution $y_1(x) = f(x)$ of the differential equation $y'' + p(x)y' + q(x)y = 0$, then we can obtain a second linearly independent solution of the form $y_2(x) = v(x)f(x) = v(x)y_1(x)$ where

It is a good exercise to show that y_1 and the solution $y_2 = vy_1$ obtained by reduction of order are linearly independent.

$$v(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int p(x)\,dx} dx \qquad \text{and} \qquad y_2(x) = y_1(x) \int \frac{1}{[y_1(x)]^2} e^{-\int p(x)\,dx} dx. \quad (4.8)$$

EXAMPLE 4.1.8: Determine a second linearly independent solution to the differential equation $4x^2y'' + 8xy' + y = 0$, $x > 0$, given that $y_1 = 1/\sqrt{x}$ is a solution.

SOLUTION: In this case, we must divide by $4x^2$ in order to obtain an equation of the form $y'' + p(x)y' + q(x)y = 0$. This gives us the equation $y'' + 2x^{-1}y' + \frac{1}{4}x^{-2}y = 0$. Therefore, $p(x) = 2x^{-1}$, and $y_1(x) = x^{-1/2}$. Using the formula for v, equation (4.8), we obtain

$$v(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int p(x)\,dx} dx = \int \frac{1}{\left[x^{-1/2}\right]^2} e^{-\int 2/x\,dx} dx$$
$$= \int \frac{1}{x^{-1}} e^{-2\ln x} dx = \int \frac{1}{x} dx = \ln x, \; x > 0.$$

Hence, a second linearly independent solution is $y_2 = x^{-1/2} \ln x$; a general solution is $y = x^{-1/2}(c_1 + c_2 \ln x)$. Of course, we can take advantage of commands like `Integrate` to carry out the steps encountered here.

```
In[474]:= p[x_] = 2/x;
          f[x_] = 1/√x;

In[475]:= v[x_] = ∫ (Exp[-∫ p[x] dx] / f[x]^2) dx
Out[475]= Log[x]

In[476]:= y[x_] = v[x] f[x];
```

Mathematica generates several error messages, which are not all displayed here, when we enter the following `Plot` command because both

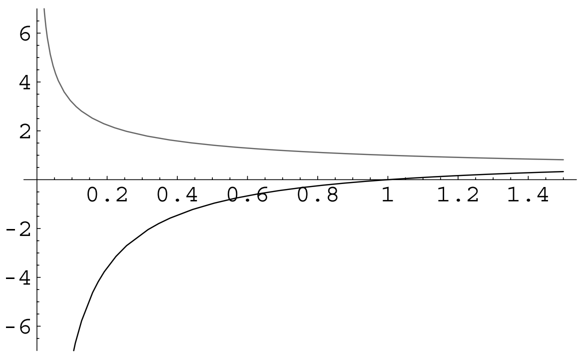


Figure 4-9 Plots of y_1 (in black) and y_2 (in gray)

$y_1(x)$ and $y_2(x)$ are undefined if $x = 0$. Nevertheless, the resulting graphs are displayed correctly in Figure 4-9.

```
In[477]:= Plot[{f[x], y[x]}, {x, 0, 3/2},
    PlotStyle → {GrayLevel[0.4],
    GrayLevel[0]}, PlotRange → {-7, 7}]
```

■

4.2 Solving Homogeneous Equations with Constant Coefficients

We now turn our attention to solving linear homogeneous equations with constant coefficients. Nonhomogeneous equations are considered in the following sections.

4.2.1 Second-Order Equations

Suppose that the coefficient functions of equation $a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$ are constants: $a_2(t) = a$, $a_1(t) = b$, and $a_0(t) = c$ and that $f(t)$ is identically the zero function. In this case, the equation $a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$ becomes

$$ay'' + by' + cy = 0. \tag{4.9}$$

Now suppose that $y = e^{kt}$, k constant, is a solution of equation (4.9). Then, $y' = ke^{kt}$ and $y'' = k^2e^{kt}$. Substitution into equation (4.9) then gives us

$$\begin{aligned} ay'' + by' + cy &= ak^2e^{kt} + bke^{kt} + ce^{kt} \\ &= e^{kt}\left(ak^2 + bk + c\right) = 0. \end{aligned}$$

Because $e^{kt} \neq 0$, the solutions of equation (4.9) are determined by the solutions of

$$ak^2 + bk + c = 0, \tag{4.10}$$

called the **characteristic equation** of equation (4.9).

Theorem 7. *Let k_1 and k_2 be the solutions of equation* (4.10).

1. *If $k_1 \neq k_2$ are real and distinct, two linearly independent solutions of equation* (4.9) *are $y_1 = e^{k_1t}$ and $y_2 = e^{k_2t}$; a general solution of equation* (4.9) *is*

$$y = c_1e^{k_1t} + c_2e^{k_2t}.$$

2. *If $k_1 = k_2$, two linearly independent solutions of equation* (4.9) *are $y_1 = e^{k_1t}$ and $y_2 = te^{k_1t}$; a general solution of equation* (4.9) *is*

$$y = c_1e^{k_1t} + c_2te^{k_1t}.$$

3. *If $k_{1,2} = \alpha \pm \beta i$, $\beta \neq 0$, two linearly independent solutions of equation* (4.9) *are $y_1 = e^{\alpha t}\cos\beta t$ and $y_2 = e^{\alpha t}\sin\beta t$; a general solution of equation* (4.9) *is*

$$y = e^{\alpha t}\left(c_1\cos\beta t + c_2\sin\beta t\right).$$

EXAMPLE 4.2.1: Solve each of the following equations: (a) $6y'' + y' - 2y = 0$; (b) $y'' + 2y' + y = 0$; (c) $16y'' + 8y' + 145y = 0$.

SOLUTION: (a) The characteristic equation is $6k^2 + k - 2 = (3k+2)(2k-1) = 0$ with solutions $k = -2/3$ and $k = 1/2$. We check with either `Factor` or `Solve`.

```
In[478]:= Factor[6k^2 + k - 2]

          Solve[6k^2 + k - 2 == 0]
Out[478]= (-1 + 2 k) (2 + 3 k)
Out[478]= {{k → -2/3}, {k → 1/2}}
```

Then, a fundamental set of solutions is $\left\{e^{-2t/3}, e^{t/2}\right\}$ and a general solution is

$$y = c_1e^{-2t/3} + c_2e^{t/2}.$$

Of course, we obtain the same result with `DSolve`.

```
In[479]:= DSolve[6y''[t] + y'[t] - 2y[t] == 0, y[t], t]
Out[479]= {{y[t] → e^(-2 t/3) C[1] + e^(t/2) C[2]}}
```

(b) The characteristic equation is $k^2 + 2k + 1 = (k + 1)^2 = 0$ with solution $k = -1$, which has multiplicity two, so a fundamental set of solutions is $\{e^{-t}, te^{-t}\}$ and a general solution is

$$y = c_1 e^{-t} + c_2 t e^{-t}.$$

We check the calculation in the same way as in (a).

```
In[480]:= Factor[k^2 + 2k + 1]

          Solve[k^2 + 2k + 1 == 0]

          DSolve[y''[t] + 2y'[t] + y[t] == 0, y[t], t]
Out[480]= (1 + k)^2
Out[480]= {{k → -1}, {k → -1}}
Out[480]= {{y[t] → e^-t C[1] + e^-t t C[2]}}
```

(c) The characteristic equation is $16k^2 + 8k + 145 = 0$ with solutions $k_{1,2} = -\frac{1}{4} \pm 3i$ so a fundamental set of solutions is $\{e^{-t/4}\cos 3t, e^{-t/4}\sin 3t\}$ and a general solution is

$$y = e^{-t/4}(c_1 \cos 3t + c_2 \sin 3t).$$

The calculation is verified in the same way as in (a) and (b).

```
In[481]:= Factor[16k^2 + 8k + 145,
            GaussianIntegers- > True]

          Solve[16k^2 + 8k + 145 == 0]

          DSolve[16y''[t] + 8y'[t] + 145y[t] == 0, y[t], t]
Out[481]= ((1 - 12 i) + 4 k) ((1 + 12 i) + 4 k)
Out[481]= {{k → -1/4 - 3 i}, {k → -1/4 + 3 i}}
Out[481]= {{y[t] → e^(-t/4) C[2] Cos[3 t] - e^(-t/4) C[1]
            Sin[3 t]}}
```

■

EXAMPLE 4.2.2: Solve

$$64\frac{d^2y}{dt^2} + 16\frac{dy}{dt} + 1025y = 0,\ y(0) = 1,\ \frac{dy}{dt}(0) = 2.$$

SOLUTION: A general solution of $64y'' + 16y' + 1025y = 0$ is $y = e^{-t/8}(c_1 \sin 4t + c_2 \cos 4t)$.

```
In[482]:= gensol = DSolve[64y''[t] + 16y'[t] + 1025y[t] == 0,
                  y[t], t]
Out[482]= {{y[t] → e^(-t/8) C[2] Cos[4 t] - e^(-t/8) C[1]
             Sin[4 t]}}
```

Applying $y(0) = 1$ shows us that $c_2 = 1$.

```
In[483]:= e1 = y[t]/.gensol[[1]]/.t- > 0
Out[483]= C[2]
```

Computing y'

```
In[484]:= D[y[t]/.gensol[[1]], t]
Out[484]= -4 e^(-t/8) C[1] Cos[4 t] - 1/8 e^(-t/8) C[2] Cos[4 t]
          + 1/8 e^(-t/8) C[1] Sin[4 t] - 4 e^(-t/8) C[2] Sin[4 t]
```

and then $y'(0)$, shows us that $-4c_1 - \frac{1}{8}c_2 = 2$.

```
In[485]:= e2 = D[y[t]/.gensol[[1]], t]/.t- > 0
Out[485]= -4 C[1] - C[2]/8
```

Solving for c_1 and c_2 with `Solve` shows us that $c_1 = -25/32$ and $c_1 = 1$.

```
In[486]:= cvals = Solve[{e1 == 1, e2 == 3}]
Out[486]= {{C[1] → -25/32, C[2] → 1}}
```

Thus, $y = e^{-t/8}\left(\frac{17}{32}\sin 4t + \cos 4t\right)$, which we graph with `Plot` in Figure 4-10.

```
In[487]:= sol = y[t]/.gensol[[1]]/.cvals[[1]]
Out[487]= e^(-t/8) Cos[4 t] + 25/32 e^(-t/8) Sin[4 t]
In[488]:= Plot[sol, {t, 0, 8π}]
```

We verify the calculation with `DSolve`.

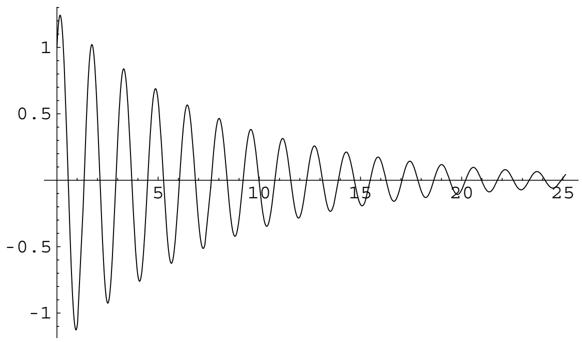


Figure 4-10 The solution to the initial-value problem tends to 0 as $t \to \infty$

```
In[489]:= DSolve[
            {64y''[t] + 16y'[t] + 1025y[t] == 0, y[0] == 1,
             y'[0] == 2}, y[t], t]
Out[489]= {{y[t] → e^(-t/8) (Cos[4 t] + 17/32 Sin[4 t])}}
```

■

4.2.2 Higher-Order Equations

As with second-order equations, solutions of any nth-order linear homogeneous differential equation with constant coefficients are determined by the solutions of the *characteristic equation*, which is obtained by assuming that $y = e^{kt}$.

Definition 15 (Characteristic Equation). *The equation*

$$a_n k^n + a_{n-1} k^{n-1} + \cdots + a_2 k^2 + a_1 k + a_0 = 0 \tag{4.11}$$

is called the ***characteristic equation*** *of the nth-order linear homogeneous differential equation with constant coefficients*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

In order to explain the process of finding a general solution of any nth-order linear homogeneous differential equations with constant coefficients, we state the following definition.

Definition 16 (Multiplicity). *Suppose that the characteristic equation* $a_n k^n + a_{n-1} k^{n-1} + \cdots + a_2 k^2 + a_1 k + a_0 = 0$ *can be written in factored form as* $(k - k_1)^{m_1} (k - k_2)^{m_2} \cdots (k - k_r)^{m_r}$, *where* $k_i \neq k_j$ *for* $i \neq j$ *and* $m_1 + m_2 + \cdots + m_r = n$. *Then the roots of the characteristic* *equation are $k = k_1$, $k = k_2$, ..., and $k = k_r$ where the roots have* ***multiplicity*** *m_1, m_2, ..., and m_r, respectively.*

In the same manner as in the case for a second-order homogeneous equation with real constant coefficients, a general solution of an nth-order linear homogeneous equation with real constant coefficients is determined by the solutions of its characteristic equation. Hence, we state the following rules for finding a general solution of an nth-order linear homogeneous equation for the many situations that may be encountered.

1. If a solution k of equation (4.11) has multiplicity m, m linearly independent solutions corresponding to k are
$$e^{kt}, te^{kt}, \ldots, t^{m-1}e^{kt}.$$
2. If a solution $k = \alpha + \beta i$, $\beta \neq 0$, of equation (4.11) has multiplicity m, $2m$ linearly independent solutions corresponding to $k = \alpha + \beta i$ (and $k = \alpha - \beta i$) are
$$e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, te^{\alpha t}\cos\beta t, te^{\alpha t}\sin\beta t, \ldots, t^{m-1}e^{\alpha t}\cos\beta t, t^{m-1}e^{\alpha t}\sin\beta t.$$

Notice that the key to the process is identifying each root of the characteristic equation and the associated solution(s).

EXAMPLE 4.2.3: Solve $12y''' - 5y'' - 6y' - y = 0$.

SOLUTION: The characteristic equation is

$$12k^3 - 5k^2 - 6k - 1 = (k-1)(3k+1)(4k+1) = 0$$

with solutions $k_1 = -1/3$, $k_2 = -1/4$, and $k_3 = 1$.

`Factor[expression]` attempts to factor `expression`.

```
In[490]:= Factor[12k^3 - 5k^2 - 6k - 1]
Out[490]= (-1 + k) (1 + 3 k) (1 + 4 k)
```

Thus, three linearly independent solutions of the equation are $y_1 = e^{-t/3}$, $y_2 = e^{-t/4}$, and $y_3 = e^t$; a general solution is $y = c_1e^{-t/3} + c_2e^{-t/4} + c_3e^t$. We check with `DSolve`.

```
In[491]:= DSolve[12y'''[t] - 5y''[t] - 6y'[t] - y[t] == 0,
            y[t], t]
Out[491]= {{y[t] → e^(-t/3) C[1] + e^(-t/4) C[2] + e^t C[3]}}
```

■

EXAMPLE 4.2.4: Solve $y''' + 4y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$.

Enter ?Solve to obtain basic help regarding the Solve function.

SOLUTION: The characteristic equation is $k^3 + 4k = k(k^2 + 4) = 0$ with solutions $k_1 = 0$ and $k_{2,3} = \pm 2i$ that are found with Solve.

```
In[492]:= Solve[k^3 + 4k == 0]
Out[492]= {{k → 0}, {k → -2 i}, {k → 2 i}}
```

Three linearly independent solutions of the equation are $y_1 = 1$, $y_2 = \cos 2t$, and $y_3 = \sin 2t$. A general solution is $y = c_1 + c_2 \sin 2t + c_3 \cos 2t$.

```
In[493]:= gensol = DSolve[y'''[t] + 4y'[t] == 0, y[t], t]
Out[493]= {{y[t] → C[3] + 1/2 C[1] Cos[2 t] + 1/2 C[2]
          Sin[2 t]}}
```

Application of the initial conditions shows us that $c_1 = -1/4$, $c_2 = 1/2$, and $c_3 = 1/4$ so the solution to the initial-value problem is $y = -\frac{1}{4} + \frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t$. We verify the computation with DSolve and graph the result with Plot in Figure 4-11.

```
In[494]:= e1 = y[t]/.gensol[[1]]/.t- > 0
Out[494]= C[1]/2 + C[3]

In[495]:= e2 = D[y[t]/.gensol[[1]], t]/.t- > 0

          e3 = D[y[t]/.gensol[[1]], {t, 2}]/.t- > 0
Out[495]= C[2]
Out[495]= -2 C[1]

In[496]:= cvals = Solve[{e1 == 0, e2 == 1, e3 == -1}]
Out[496]= {{C[1] → 1/2, C[2] → 1, C[3] → -1/4}}

In[497]:= partsol = DSolve[
             {y'''[t] + 4y'[t] == 0, y[0] == 0, y'[0] == 1,
              y''[0] == -1}, y[t], t]
Out[497]= {{y[t] → -1/4 + 1/4 Cos[2 t] + 1/2 Sin[2 t]}}

In[498]:= Plot[Evaluate[y[t]/.partsol], {t, 0, 2π},
            AspectRatio- > Automatic]
```

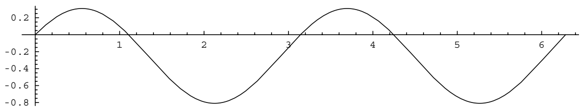


Figure 4-11 Graph of $y = -\frac{1}{4} + \frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t$

■

EXAMPLE 4.2.5: Solve (a) $4y^{(4)} + 12y''' + 49y'' + 42y' + 10y = 0$ and (b) $y^{(4)} + 4y''' + 24y'' + 40y' + 100y = 0$.

SOLUTION: (a) The characteristic equation of $4y^{(4)} + 12y''' + 49y'' + 42y' + 10y = 0$ is $4k^4 + 12k^3 + 49k^2 + 42k + 10 = 0$. We use `Factor` to try to factor the characteristic polynomial, but see that Mathematica does not completely factor the polynomial,

```
In[499]:= Factor[4k^4 + 12k^3 + 49k^2 + 42k + 10]
Out[499]= (1 + 2 k)^2 (10 + 2 k + k^2)
```

unless we include the option `GaussianIntegers->True` in the `Factor` command.

```
In[500]:= Factor[4k^4 + 12k^3 + 49k^2 + 42k + 10,
            GaussianIntegers → True]
Out[500]= ((1 - 3 i) + k) ((1 + 3 i) + k) (1 + 2 k)^2
```

From the results, we see that the solutions of the characteristic equation are $k = -1 \pm 3i$ and $k = -1/2$ with multiplicity 2. As you may suspect, we obtain the same results with `Solve`.

```
In[501]:= Solve[4k^4 + 12k^3 + 49k^2 + 42k + 10 == 0]
Out[501]= {{k → -1 - 3 i}, {k → -1 + 3 i}, {k → -1/2}, {k → -1/2}}
```

Four linearly independent solutions of the equation are then given by $y_1 = e^{-x}\cos 3x$, $y_2 = e^{-x}\sin 3x$, $y_3 = e^{-x/2}$, and $y_4 = xe^{-x/2}$. This tells us that a general solution is given by

$$y = e^{-x}\left(c_1 \cos 3x + c_2 \sin 3x\right) + e^{-x/2}\left(c_3 + c_4 x\right).$$

We obtain the same result with `DSolve`. The formula for the general solution is extracted from `gensolc` with `gensolc[[1,1,2]]`.

```
In[502]:= gensolc = DSolve[4y''''[x] + 12y'''[x] + 49 y''[x]
                  +42 y'[x] + 10 y[x] == 0, y[x], x]
Out[502]= {{y[x] → e^(-x/2) C[3] + e^(-x/2) x C[4]
                  +e^-x C[2] Cos[3 x] + e^-x C[1] Sin[3 x]}}
```

In this case, we will graph the general solution for (c_1, c_2, c_3, c_4) = (1, 0, 1, 0), (0, 1, 0, 1), (1, 1, 0, 1), (1, −1, 1, 2), (0, 2, 1, −2), and (1, −2, 1, 2). We accomplish this by applying the pure function

```
gensolc[[1,1,2]] /. {C[1]->#[[1]],C[2]->#[[2]],
   C[3]->#[[3]],C[4]->#[[4]]} &
```

to the set of ordered quadruples

```
{{1,0,1,0},{0,1,0,1},{1,1,0,1},{1,-1,1,2},{0,2,1,-2},
  {1,-2,1,2}}
```

with `Map`. Namely, given an argument `#`, the function

```
gensolc[[1,1,2]] /. {C[1]->#[[1]],C[2]->#[[2]],
   C[3]->#[[3]],C[4]->#[[4]]} &
```

replaces `C[1]` in `gensolc[[1,1,2]]` by the first part of the argument, `C[2]` by the second part, `C[3]` by the third part, and `C[4]` by the fourth part.

```
In[503]:= toplot = ((gensolc[[1, 1, 2]])/.
                    {C[1] → (#1[[1]]), C[2] → (#1[[2]]),
                     C[3] → (#1[[3]]), C[4] → (#1[[4]])}&)/@
                    {{1, 0, 1, 0}, {0, 1, 0, 1}, {1, 1, 0, 1},
                    {1, -1, 1, 2}, {0, 2, 1, -2},
                    {1, -2, 1, 2}};
```

We then graph the set of functions `toplot` on the interval [−1, 2] with `Plot` in Figure 4-12.

```
In[504]:= grays = Table[GrayLevel[i],
                  {i, 0, 0.7, 0.7/5}];

In[505]:= Plot[Evaluate[toplot], {x, -1, 2},
            PlotStyle → grays]
```

(b) The characteristic equation of $y^{(4)} + 4y''' + 24y'' + 40y' + 100y = 0$ is $k^4 + 4k^3 + 24k^2 + 40k + 100 = 0$ which we can solve by factoring

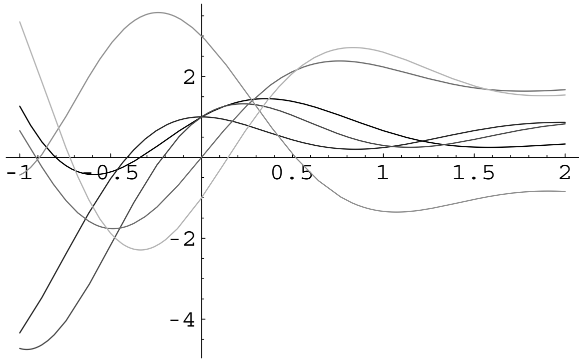


Figure 4-12 Various solutions of $4y^{(4)} + 12y''' + 49y'' + 42y' + 10y = 0$

$k^4 + 4k^3 + 24k^2 + 40k + 100$ using `Factor` together with the options `GaussianIntegers->True`

```
In[506]:= Factor[k^4 + 4k^3 + 24k^2 + 40k + 100,
             GaussianIntegers → True]
Out[506]= ((1 - 3 i) + k)^2 ((1 + 3 i) + k)^2
```

or using `Solve`.

```
In[507]:= Solve[k^4 + 4k^3 + 24k^2 + 40k + 100 == 0]
Out[507]= {{k → -1 - 3 i}, {k → -1 - 3 i},
             {k → -1 + 3 i}, {k → -1 + 3 i}}
```

Thus, we see that the solutions of the characteristic equation are $k = -1 + 3i$ and $k = -1 - 3i$, each with multiplicity 2, so the corresponding solutions are $y_1 = e^{-x}\cos 3x$, $y_2 = e^{-x}\sin 3x$, $y_3 = xe^{-x}\cos 3x$, and $y_4 = xe^{-x}\sin 3x$. This tells us that a general solution is given by

$$y = e^{-x}\left[(c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x\right].$$

We obtain the same result using `DSolve`.

```
In[508]:= gensold = DSolve[D[y[x], {x, 4}]
                     +4D[y[x], {x, 3}] + 24 y''[x]
                     +40y'[x] + 100 y[x] == 0, y[x], x]
Out[508]= {{y[x] → e^-x C[3] Cos[3 x] + e^-x x C[4] Cos[3 x]
                     +e^-x C[1] Sin[3 x] + e^-x x C[2] Sin[3 x]}}
```

To graph the solution for various values of the constants, we proceed in the same manner as in (a). First, we define a list of ordered quadruples, `vals`.

```
In[509]:= vals = {{5, 0, 1, 0}, {0, 1, 0, -3},
            {1, 3, 0, 1}, {1, -1, 1, 2}, {0, 2, 1, -2},
            {1, -2, 5, 2}, {0, -3, 0, 2}, {3, 0, 0, 2},
            {1, 1, 1, 1}};
```

We then use `Map` to replace `C[1]` in `gensold[[1,1,2]]`, which represents the formula for the solution, by the first part of each quadruple in `vals`, `C[2]` by the second part, `C[3]` by the third part, and `C[4]` by the fourth part.

```
In[510]:= toplot =
   Map[
    gensold⟦1, 1, 2⟧/.
     {C[1] → (#1⟦1⟧), C[2] → (#1⟦2⟧),
     C[3] → (#1⟦3⟧), C[4] → (#1⟦4⟧)}&,
     vals];
```

We then use `Table` and `Plot` to graph each function in `toplot` on the interval [−1/2, 3/2], naming the resulting list of nine graphics objects `ninegraphs`. The graphs are not displayed because the option `DisplayFunction->Identity` is included in the `Plot` command. (If you do not include this option, each graph is displayed as it is generated.)

```
In[511]:= ninegraphs =
   Table[Plot[toplot⟦i⟧, {x, -1/2, 3/2},
   DisplayFunction → Identity], {i, 1, 9}];
```

Now, we use `Partition` to partition the set `ninegraphs` into three element subsets, naming the resulting 3×3 array of graphics objects `todisplay`.

```
In[512]:= todisplay = Partition[ninegraphs, 3];
```

We then display the array of graphics objects `todisplay` using `Show` together with `GraphicsArray` in Figure 4-13.

```
In[513]:= Show[GraphicsArray[todisplay]]
```

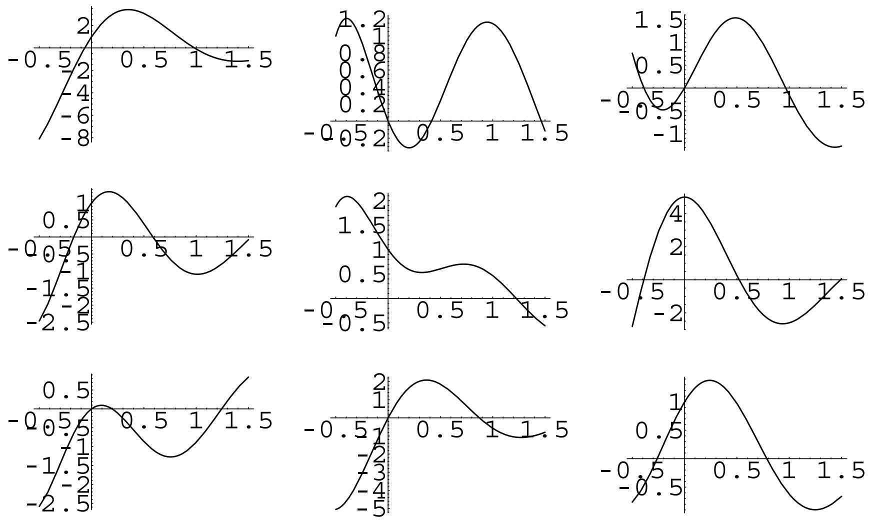


Figure 4-13 Various solutions of $y^{(4)} + 4y''' + 24y'' + 40y' + 100y = 0$

■

EXAMPLE 4.2.6: If a is a positive constant, find conditions on the constant b so that $y(x)$ satisfies

$$\begin{cases} y''' + 0.344425y'' + 12.4454y' - 4.50047y = 0 \\ y(0) = 0,\ y'(0) = a,\ y''(0) = b \end{cases}$$

and has the property that $\lim_{x\to\infty} y(x) = 0$. (b) For this function, find and classify the first critical point on the interval $[0, \infty)$.

SOLUTION: We use `Solve` to find (accurate approximations of) the solutions of the characteristic equation $k^3 + 0.344425k^2 + 12.4454k - 4.50047 = 0$.

```
In[514]:= Solve[k^3 + 0.344425 k^2 + 12.4454 k
             -4.50047 == 0]
Out[514]= {{k → -0.349491 - 3.54557 i},
             {k → -0.349491 + 3.54557 i}, {k → 0.354557}}
```

Then, a general solution of the equation is

$$y = c_1 e^{0.354557x} + e^{-0.349491x}\left(c_2 \cos 3.54557x + c_3 \sin 3.54557x\right).$$

```
In[515]:= Clear[y]

          y[x_] = c₁ Exp[0.354557 x]
                 + Exp[-0.349490878289872464 x]
                 × (c₂ Cos[3.54557256737378034 x]
                 +c₃ Sin[3.54557256737378034 x]);
```

Note how we use `Chop` to replace those numbers in `cvals` that are very close to zero by zero.

We now apply the initial conditions and solve for c_1, c_2, and c_3.

```
In[516]:= sys = {y[0] == 0, y'[0] == a, y''[0] == b}
Out[516]= {c₁ + c₂ == 0, 0.354557 c₁
          -0.349490878289872464 c₂
          +3.54557256737378034 c₃ == a,
           0.125711 c₁ - 12.4489409565056737 c₂
           -2.47829 c₃ == b}

In[517]:= cvals = Solve[sys, {c₁, c₂, c₃}]//Chop//
                   Simplify
Out[517]= {{c₁ → 0.0534931 a + 0.07653 b,
           c₂ → -0.0534931 a - 0.07653 b,
           c₃ → 0.27142 a - 0.0151966 b}}
```

We obtain the solution to the initial-value problem by substituting these values back into the general solution.

```
In[518]:= y[x_] = y[x] /. cvals[[1]]
Out[518]= (0.0534931 a + 0.07653 b) e^(0.354557 x)
          +e^(-0.349490878289872464 x) ((-0.0534931 a - 0.07653 b)
          × Cos[3.54557256737378034 x]
          +(0.27142 a - 0.0151966 b)
          × Sin[3.54557256737378034 x])
```

These results indicate that $\lim_{x\to\infty} y(x) = 0$ if $0.0534931a + 0.07653b = 0$ which leads to $b = -0.698982a$.

```
In[519]:= bval = Solve[0.0534930859299338479 a
                 +0.0765300173093015523 b == 0, b]
Out[519]= {{b → -0.698982 a}}
```

Substituting back into the solution to the initial-value problem yields

$$y = 0.282042ae^{-0.349491x} \sin 3.54557x.$$

```
In[520]:= y[x_] = y[x]/.bval[[1]]//Chop
Out[520]= 0.282042 a e^(-0.349490878289872464 x)
          × Sin[3.54557256737378034 x]
```

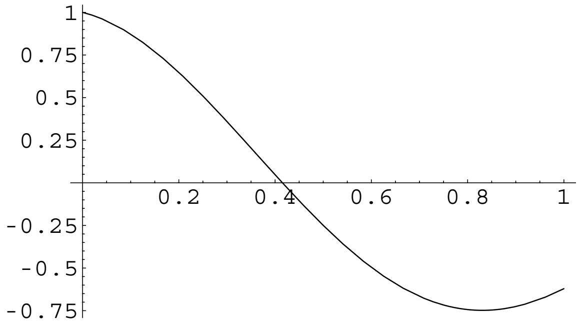


Figure 4-14 The first critical point of y occurs at the first 0 of y'

To find and classify the first critical point of $y = 0.282042ae^{-0.349491x}$ $\sin 3.54557x$, we compute y'

```
In[521]:= y'[x]
Out[521]= 1. a e^(-0.349490878289872464 x)
          × Cos[3.54557256737378034 x]
          -0.0985711 a e^(-0.349490878289872464 x)
          × Sin[3.54557256737378034 x]
```

and graph y'/a in Figure 4-14 to locate the first zero of y'.

```
In[522]:= Plot[y'[x]/a, {x, 0, 1}]
```

From the graph, we see that the first zero occurs near 0.4 and with `FindRoot` we obtain the critical number $x = 0.415319$.

```
In[523]:= critval = FindRoot[y'[x]/a == 0, {x, 0.4}]
Out[523]= {x → 0.415319}
```

At this critical number, we use `ReplaceAll` (`/.`) to find that $y(0.415319) = 0.242759a$. Because y' makes a "simple change in sign" from positive to negative at $x = 0.415319$, by the first derivative test $(0.415319, 0.242759a)$ is a relative (or local) maximum.

```
In[524]:= y[x] /. critval
Out[524]= 0.242759 a
```

To see that $(0.415319, 0.242759a)$ is the *absolute* maximum, we graph y for various values of a with `Plot` in Figure 4-15.

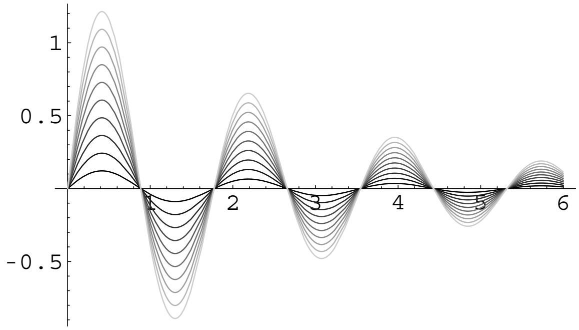


Figure 4-15 The maximum value of y is $0.242759a$

```
In[525]:= toplot = Table[y[x], {a, 0.5, 5, 0.5}];

In[526]:= grays = Table[GrayLevel[i],
                {i, 0, 0.8, 0.8/9}];

In[527]:= Plot[Evaluate[toplot], {x, 0, 6},
            PlotStyle → grays]
```

■

EXAMPLE 4.2.7: Find a differential equation with general solution $y = c_1 e^{-2t/3} + c_2 t e^{-2t/3} + c_3 t^2 e^{-2t/3} + c_4 \cos t + c_5 \sin t + c_6 t \cos t + c_7 t \sin t + c_8 t^2 \cos t + c_9 t^2 \sin t$.

SOLUTION: A linear homogeneous differential equation with constant coefficients that has this general solution has fundamental set of solutions

$$S = \left\{e^{-2t/3}, te^{-2t/3}, t^2 e^{-2t/3}, \cos t, \sin t, t\cos t, t\sin t, t^2 \cos t, t^2 \sin t\right\}.$$

Hence, in the characteristic equation $k = -2/3$ has multiplicity 3 while $k = \pm i$ has multiplicity 3. The characteristic equation is

$$\left(k + \frac{2}{3}\right)^3 (k-i)^3 (k+i)^3 = k^9 + 2k^8 + \frac{13}{3}k^7 + \frac{170}{27}k^6 + 7k^5 + \frac{62}{9}k^4 + 5k^3 + \frac{26}{9}k^2 + \frac{4}{3}k + \frac{8}{27},$$

where we use Mathematica to compute the multiplication with `Expand`.

```
In[528]:= Expand[27 (k + 2/3)^3 (k^2 + 1)^3]
Out[528]= 8 + 36 k + 78 k^2 + 135 k^3 + 186 k^4 + 189 k^5 + 170 k^6
          +117 k^7 + 54 k^8 + 27 k^9
```

Thus, a differential equation obtained with the indicated general solution is

$$\frac{d^9y}{dt^9} + 2\frac{d^8y}{dt^8} + \frac{13}{3}\frac{d^7y}{dt^7} + \frac{170}{27}\frac{d^6y}{dt^6} + 7\frac{d^5y}{dt^5} + \frac{62}{9}\frac{d^4y}{dt^4} + 5\frac{d^3y}{dt^3} + \frac{26}{9}\frac{d^2y}{dt^2} + \frac{4}{3}\frac{dy}{dt} + \frac{8}{27}y = 0.$$

■

Application: Testing for Diabetes

Sources: D. N. Burghess and M. S. Borrie, *Modeling with Differential Equations*, Ellis Horwood Limited, pp. 113–116. Joyce M. Black and Esther Matassarin–Jacobs, *Luckman and Sorensen's Medical–Surgical Nursing: A Psychophysiologic Approach*, Fourth Edition, W. B. Saunders Company (1993), pp. 1775–1808.

Diabetes mellitus affects approximately 12 million Americans; approximately one-half of these people are unaware that they have diabetes. Diabetes is a serious disease: it is the leading cause of blindness in adults, the leading cause of renal failure, responsible for approximately one-half of all nontraumatic amputations in the United States. In addition, people with diabetes have an increased rate of coronary artery disease and strokes. People at risk for developing diabetes include those who are obese; those suffering from excessive thirst, hunger, urination, and weight loss; women who have given birth to a baby with weight greater than nine pounds; those with a family history of diabetes; those who are over 40 years of age. People with diabetes cannot metabolize glucose because their pancreas produces an inadequate or ineffective supply of insulin. Subsequently, glucose levels rise. The body attempts to remove the excess glucose through the kidneys: the glucose acts as a diuretic, resulting in increased water consumption. Since some cells require energy, which is not being provided by glucose, fat and protein is broken down and ketone levels rise. Although there is no cure for diabetes at this time, many cases can be effectively managed by a balanced diet and insulin therapy in addition to maintaining an optimal weight. Diabetes can be diagnosed by several tests. In the **fasting blood sugar test**, a patient fasts for at least four hours, and then the glucose level is measured. In a fasting state, the glucose level in normal adults ranges from 70 to 110 milligrams per milliliter. An adult in a fasting state with consistent readings of over 150 milligrams probably has diabetes. However, people with mild cases of diabetes might have fasting state glucose levels within the normal range because individuals vary greatly. In these cases, a highly accurate test which is frequently used to diagnose mild diabetes is the **glucose tolerance test** (GTT), which was developed by Drs. Rosevear and Molnar of the

Mayo Clinic and Drs. Ackerman and Gatewood of the University of Minnesota. During the GTT, a blood and urine sample are taken from a patient in a fasting state to measure the glucose, G_0, hormone, H_0, and glycosuria levels, respectively. We assume that these values are equilibrium values. The patient is then given 100 grams of glucose. Blood and urine samples are then taken at 1, 2, 3, and 4 hour intervals. In a person without diabetes, glucose levels return to normal after two hours; in diabetics their blood sugar levels either take longer or never return to normal levels.

Let G denote the cumulative level of glucose in the blood, $g = G - G_0$, H the cumulative level of hormones that affect insulin production (like glucagon, epinephrine, cortisone, and thyroxin), and $h = H - H_0$. Notice that g and h represent the fluctuation of the cumulative levels of glucose and hormones from their equilibrium values. The relationship between the rate of change of glucose in the blood and the rate of change of the cumulative levels of the hormones in the blood that affects insulin production is

$$\begin{cases} g' = f_1(g,h) + J(t) \\ h' = f_2(g,h) \end{cases}$$

where $J(t)$ represents the **external** rate at which the blood glucose concentration is being increased. If we assume that f_1 and f_2 are linear functions, then this system of equations becomes

$$\begin{cases} g' = -ag - bh + J(t) \\ h' = -ch + dg \end{cases}$$

where a, b, c, and d represent positive numbers. We define these equations in `eq1` and `eq2`.

```
In[529]:= eq1 = g′[t] == -a g[t] - b h[t] + j[t];
          eq2 = h′[t] == -c h[t] + d g[t];
```

Next, we solve the first equation for $h(t)$

```
In[530]:= step1 = Solve[eq1, h[t]]
```

$$\text{Out[530]= } \left\{\left\{h[t] \to \frac{-a\,g[t] + j[t] - g'[t]}{b}\right\}\right\}$$

and differentiate this result with respect to t to obtain $h'(t)$.

```
In[531]:= step2 = ∂t step1[[1, 1, 2]]
```

$$\text{Out[531]= } \frac{-a\,g'[t] + j'[t] - g''[t]}{b}$$

Substituting these results into the second equation yields the second-order equation with constant coefficients

$$\frac{1}{b}\left(-g'' - ag' + J'\right) = -\frac{c}{b}\left(-g' - ag + J\right) + dg$$
$$g'' + (a+c)g' + (ac+bd)g = J' + cJ.$$

```
In[532] := step3 = eq2/.{step1[[1, 1]], h'[t] → step2}
Out[532] = (-a g'[t] + j'[t] - g''[t])/b ==
     d g[t] - (c (-a g[t] + j[t] - g'[t]))/b
```

For $t > 0$ we have that $J(t) = 0$ and $J'(t) = 0$ because the glucose solution is consumed at $t = 0$, so for $t > 0$ we can rewrite the equation as

$$g'' + (a+c)g' + (ac+bd)g = 0.$$

```
In[533] := step4 = step3/.{j'[t] → 0, j[t] → 0}
Out[533] = (-a g'[t] - g''[t])/b == d g[t] - (c (-a g[t] - g'[t]))/b
```

We now use `DSolve` to solve this second-order equation.

```
In[534] := sol = DSolve[step4, g[t], t]
Out[534] = {{g[t] → e^(1/2 (-a-c-Sqrt[a^2-2 a c+c^2-4 b d]) t) C[1]
     + e^(1/2 (-a-c+Sqrt[a^2-2 a c+c^2-4 b d]) t) C[2]}}
```

It might be reasonable to assume that glucose levels fluctuate in a periodic fashion so that the solutions to the equation involve periodic functions. In order to have periodic functions in the solution (like sine and cosine), we must have that $(a+c)^2 - 4(ac+bd) < 0$. We now replace $(a+c)^2 - 4(ac+bd)$ with $-4\omega^2$ and $-a-c$ with -2α using `ReplaceRepeated` (`//.`)

```
In[535] := step5 = (sol[[1, 1, 2]])//.
              {a^2 - 2a c + c^2 - 4b d → -4ω^2, -a - c → -2α}
Out[535] = e^(1/2 t (-2 α-2 Sqrt[-ω^2])) C[1] + e^(1/2 t (-2 α+2 Sqrt[-ω^2])) C[2]
```

We then simplify the result with `PowerExpand`.

```
In[536] := step6 = PowerExpand[step5]
Out[536] = e^(1/2 t (-2 α-2 i ω)) C[1] + e^(1/2 t (-2 α+2 i ω)) C[2]
```

Use `ComplexExpand` to rewrite `step6` in terms of trigonometric functions.

```
In[537]:= step7 = ComplexExpand[step6]
Out[537]= e^(-t α) C[1] Cos[t ω] + e^(-t α) C[2] Cos[t ω]
          +i (-e^(-t α) C[1] Sin[t ω] + e^(-t α) C[2] Sin[t ω])
```

We want to choose the constants `C[1]` and `C[2]` so that the result is a real-valued function. We begin by using `Collect` to collect together the terms involving $\cos \omega t$ and $\sin \omega t$.

```
In[538]:= step8 = Collect[step7, {Cos[ω t], Sin[ω t]}]
Out[538]= (e^(-t α) C[1] + e^(-t α) C[2]) Cos[t ω]
          +(-i e^(-t α) C[1] + i e^(-t α) C[2]) Sin[t ω]
```

If possible, we would like to choose `C[1]` and `C[2]` so that `C[1] + C[2]` can be replaced by an arbitrary real constant `c1` and `-I C[1]+ I C[2]` can be replaced by an arbitrary real constant `c2`. To see that this is possible, we solve this system of equations for `C[1]` and `C[2]` with `Solve`.

```
In[539]:= toapply = Solve[{C[1] + C[2] == c1,
                    -I C[1] + I C[2] == c2}, {C[1], C[2]}]
Out[539]= {{C[1] → -1/2 (-c1 - i c2), C[2] → -1/2 (-c1 + i c2)}}
```

Replacing `C[1]` and `C[2]` by the values obtained in `toapply` yields our model.

```
In[540]:= model = Simplify[step8/.(toapply[[1]])]
Out[540]= e^(-t α) (c1 Cos[t ω] + c2 Sin[t ω])
```

Thus, $g(t) = e^{-\alpha t}(c_1 \cos \omega t + c_2 \sin \omega t)$ and $G(t) = G_0 + e^{-\alpha t}(c_1 \cos \omega t + c_2 \sin \omega t)$.

Research has shown that lab results of $2\pi/\omega > 4$ indicate a mild case of diabetes. For example, suppose that we have given the GTT to four patients we suspect of having a mild case of diabetes. The results for each patient are shown in the following table. Which patients, if any, have a mild case of diabetes?

	Patient 1	Patient 2	Patient 3	Patient 4
G_0	80.00	90.00	100.00	110.00
$t = 1$	85.32	91.77	103.35	114.64
$t = 2$	82.54	85.69	98.26	105.89
$t = 3$	78.25	92.39	96.59	108.14
$t = 4$	76.61	91.13	99.47	113.76

In each case, we must find α, ω, c_1, and c_2 so that $G(t) = G_0 + e^{-\alpha t}(c_1 \cos \omega t + c_2 \sin \omega t)$ agrees with the data as closely as possible. To accomplish this, we take advantage of the `NonlinearFit` command that is contained in the **NonlinearFit** package which is located in the **Statistics** folder (or directory). First, we load the **NonlinearFit** package.

```
In[541]:= << Statistics`NonlinearFit`
```

For the first patient, we use `NonlinearFit` to find values of α, ω, c_1, and c_2

```
In[542]:= p1 = NonlinearFit[{{1, 85.32}, {2, 82.54}, {3, 78.25},
              {4, 76.61}}, model + 80, t, {c1, c2, ω, α}]
```

Out[542]= $80 + e^{-0.150145\,t}$ (2.7639 Cos[1.04611 t]
+5.54293 Sin[1.04611 t])

and then evaluate $2\pi/\omega$ for the value of ω obtained to see that the first patient probably has diabetes.

In[543]:= $N\left[\frac{2\pi}{1.04610632215009347}\right]$

Out[543]= 6.00626

Similarly, we use `NonlinearFit` to see that Patients 2 and 4 probably do not have diabetes while Patient 3 probably has diabetes.

```
In[544]:= p2 = NonlinearFit[{{1, 91.77}, {2, 85.69}, {3, 92.39},
              {4, 91.13}}, model + 90, t, {c1, c2, ω, α}]
```

Out[544]= $90 + e^{-0.152132\,t}$ (3.78531 Cos[2.09345 t]
+4.55901 Sin[2.09345 t])

In[545]:= $N\left[\frac{2\pi}{8.37663218680484966}\right]$

Out[545]= 0.750085

```
In[546]:= p3 = NonlinearFit[{{1, 103.35}, {2, 98.26}, {3, 96.59},
              {4, 99.47}}, model + 100, t, {c1, c2, ω, α}]
```

Out[546]= $100 + e^{-0.149988\,t}$ (4.7609 Cos[1.25572 t]
+2.54189 Sin[1.25572 t])

In[547]:= $N\left[\frac{2\pi}{1.25572438795247531}\right]$

Out[547]= 5.00363

```
In[548]:= p4 = NonlinearFit[{{1, 114.64}, {2, 105.89},
              {3, 108.14}, {4, 113.76}}, model + 110, t,
              {c1, c2, ω, α}]
Out[548]= 110 + e^(-0.14985 t) (3.14157 Cos[8.07894 t]
            +6.24831 Sin[8.07894 t])

In[549]:= N[2π / 1.79575405057308117]
Out[549]= 3.49891
```

4.3 Introduction to Solving Nonhomogeneous Equations with Constant Coefficients

In the previous section, we learned how to solve the nth-order linear homogeneous equation with real constant coefficients. These techniques are also useful in solving nonhomogeneous equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x), \tag{4.12}$$

where the a_i's are constant and $a_n \neq 0$. Before describing how to obtain solutions of some nonhomogeneous equations, we need to describe what is meant by a *general solution of a linear nonhomogeneous equation*.

Definition 17 (Particular Solution). *A* ***particular solution***, $y_p(x)$, *of the linear differential equation*

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots a_2(x) y'' + a_1(x) y' + a_0(x) y = f(x)$$

is a specific function that contains no arbitrary constants and satisfies the differential equation.

EXAMPLE 4.3.1: Verify that $y_p(x) = -\frac{3}{2}\sin x$ is a particular solution of $y'' - 2y' + y = 3\cos x$.

SOLUTION: After defining $y_p(x) = -\frac{3}{2}\sin x$,

```
In[550]:= yp[x_] = -3 Sin[x]/2
Out[550]= -3 Sin[x]/2
```

we compute and simplify $y_p'' - 2y_p' + y_p$

```
In[551]:= yp''[x] - 2yp'[x] + yp[x]
Out[551]= 3 Cos[x]
```

and see that the result is identically equal to $3\cos x$.

■

Suppose that y is *any* solution and that y_p is a particular solution of the nonhomogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

and that y_h is a general solution of the corresponding homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

Then, $y - y_p$ is a solution of the corresponding homogeneous equation so

$$y - y_p = y_h \qquad \text{or} \qquad y = y_h + y_p.$$

Thus, *any* solution of the nonhomogeneous equation can be written as the sum of a particular solution to the nonhomogeneous equation added to the general solution of the corresponding homogeneous equation.

Definition 18 (General Solution of a Nonhomogeneous Equation). *A **general solution** of the nonhomogeneous equation*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

is

$$y = y_h + y_p$$

where y_h is a general solution of the corresponding homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

and y_p is a particular solution to the nonhomogeneous equation.

EXAMPLE 4.3.2: Find a general solution of $y'' + 6y' + 13y = 2e^{-2x}\sin x$ if

$$y_p = e^{-2x}\left(-\frac{1}{5}\cos x + \frac{2}{5}\sin x\right)$$

is a particular solution to the nonhomogeneous equation and

$$y_h = e^{-3x}\left(c_1\cos 2x + c_2\sin 2x\right)$$

is a general solution of the corresponding homogeneous equation.

SOLUTION: We first show that $y_p = e^{-2x}\left(-\frac{1}{5}\cos x + \frac{2}{5}\sin x\right)$ is a particular solution of $y'' + 6y' + 13y = 2e^{-2x}\sin x$. After defining y_p, we calculate $y_p'' + 6y_p' + 13y_p$

```
In[552]:= yp[x_] = -1/5 Exp[-2 x] Cos[x]
                 +2/5 Exp[-2 x] Sin[x];

In[553]:= yp''[x] + 6yp'[x] + 13yp[x]//Simplify
Out[553]= 2 e^(-2 x) Sin[x]
```

and see that the result is $2e^{-2x}\sin x$. We see that $y_h = e^{-3x}(c_1\cos 2x + c_2\sin 2x)$ is a general solution of the corresponding homogeneous equation $y'' + 6y' + 13y = 0$ with `DSolve`.

```
In[554]:= Clear[y]

          DSolve[y''[x] + 6 y'[x] + 13 y[x] == 0,
            y[x], x]
Out[554]= {{y[x] → e^(-3 x) C[2] Cos[2 x] + e^(-3 x) C[1] Sin[2 x]}}
```

Thus, a general solution of the equation is $y = y_h + y_p = e^{-3x}(c_1\cos 2x + c_2\sin 2x) + e^{-2x}\left(-\frac{1}{5}\cos x + \frac{2}{5}\sin x\right)$. We now graph the general solution for various values of the arbitrary constant. To do so, we define $y_h = e^{-3x}(c_1\cos 2x + c_2\sin 2x)$ and $y(x) = y_h(x) + y_p(x)$.

```
In[555]:= yh[x_] = Exp[-3 x] (c1 Cos[2x] + c2 Sin[2x]);

In[556]:= y[x_] = yh[x] + yp[x];
```

Then, we use `Table` to create a list of functions obtained by replacing c_1 in $y(x)$ by -1, 0, and 1 and c_2 by -1, 0, and 1. The resulting list of functions `toplot` is graphed with `Plot` in Figure 4-16. The option

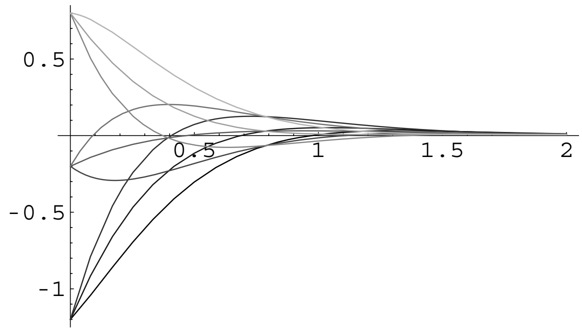


Figure 4-16 Various solutions to a nonhomogeneous equation

`PlotStyle->grays` specifies that the functions are graphed in shades of gray according to the `GrayLevels` in the list `grays`.

```
In[557]:= toplot = Table[y[x], {c1, -1, 1},
                   {c2, -1, 1}];

In[558]:= grays = Table[GrayLevel[i],
                  {i, 0, 0.7, 0.7/8}];

In[559]:= Plot[Evaluate[toplot], {x, 0, 2},
            PlotRange → All, PlotStyle → grays]
```

■

Techniques for solving nonhomogeneous equations with constant coefficients are discussed in the next two sections. In addition, you can often use `DSolve` to find a general solution of a linear nonhomogeneous equation.

EXAMPLE 4.3.3: Solve the initial-value problem

$$\begin{cases} y'' + y = \cos \omega x \\ y(0) = y'(0) = 0. \end{cases}$$

Graph the solution for various values of ω, including $\omega = 1$.

SOLUTION: We first use `DSolve` to solve the initial-value problem. Note that the result is not valid if $\omega = 1$.

```
In[560]:= Clear[y]

          DSolve[{y''[x] + y[x] == Cos[ωx],
            y[0] == 0, y'[0] == 0}, y[x], x]//
            Simplify
```

Out[560]= $\left\{\left\{y[x] \to \frac{\text{Cos}[x] - \text{Cos}[x\,\omega]}{-1+\omega^2}\right\}\right\}$

In fact, when we graph this solution for various values of ω, Mathematica generates several error messages (not all of which are displayed here) because $\omega = 1$ is included in the `Table` command. Notice that the empty graph corresponds to $\omega = 1$.

```
In[561]:= graphs =
            Table[Plot[(Cos[x] - Cos[xω])/(-1 + ω^2), {x, 0, 12π},
              Ticks- > {{0, 12π}, {-1, 1}},
              DisplayFunction → Identity],
              {ω, 0, 2, 2/8}];
```

```
Power :: infy : Infinite expression 1/0 encountered.
∞ :: indet : Indeterminate expression
    0 ComplexInfinity encountered.
```

From the graphs shown in Figure 4-17, we see that the solution to the initial-value problem is bounded and periodic if $\omega \neq 1$.

```
In[562]:= toshow = Partition[graphs, 3];

In[563]:= Show[GraphicsArray[toshow]]
```

We consider $\omega = 1$ separately.

```
In[564]:= DSolve[{y''[x] + y[x] == Cos[x], y[0] == 0,
              y'[0] == 0}, y[x], x]
```

Out[564]= $\left\{\left\{y[x] \to \frac{1}{4}\left(-2\ \text{Cos}[x] + 2\ \text{Cos}[x]^3 + 2\,x\ \text{Sin}[x] + \text{Sin}[x]\ \text{Sin}[2\,x]\right)\right\}\right\}$

```
In[565]:= Plot[(x Sin[x])/2, {x, 0, 12π}]
```

In Figure 4-18, we see that if $\omega = 1$ the solution is unbounded and not periodic.

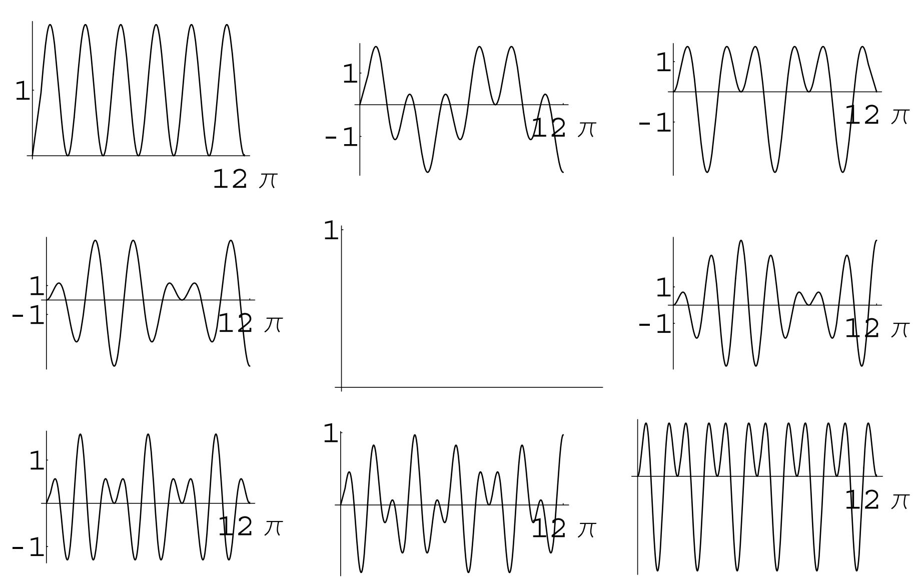


Figure 4-17 Solutions of $\begin{cases} y'' + y = \cos \omega x \\ y(0) = y'(0) = 0 \end{cases}$ for $\omega \neq 1$

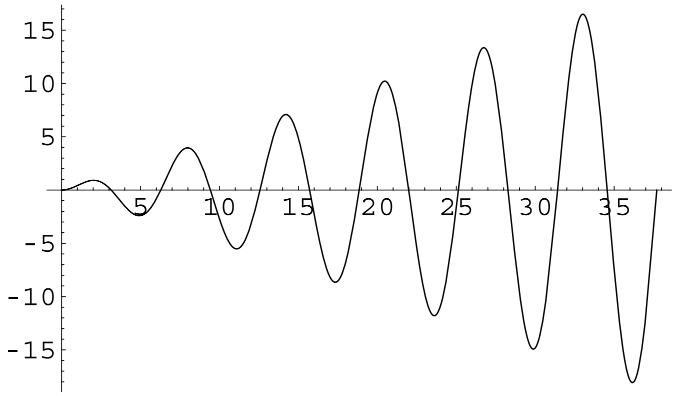


Figure 4-18 Solution of $\begin{cases} y'' + y = \cos \omega x \\ y(0) = y'(0) = 0 \end{cases}$ for $\omega = 1$

■

4.4 Nonhomogeneous Equations with Constant Coefficients: The Method of Undetermined Coefficients

Consider the nth-order linear differential equation with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x).$$

We know that a general solution of this differential equation is given by $y = y_h + y_p$ where y_p is a particular solution of the nonhomogeneous equation and y_h is a solution of the corresponding homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

If $f(x)$ is a linear combination of the functions $1, x, x^2, \ldots, e^{kx}, xe^{kx}, x^2e^{kx}, \ldots,$ $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$, $xe^{\alpha x}\cos\beta x$, $xe^{\alpha x}\sin\beta x$, $x^2e^{\alpha x}\cos\beta x$, $x^2e^{\alpha x}\sin\beta x, \ldots$ the *Method of Undetermined Coefficients* provides a method that we can use to determine a particular solution of the nonhomogeneous equation.

Outline of the Method of Undetermined Coefficients

1. Solve the corresponding homogeneous equation for $y_h(x)$.
2. Determine the form of a particular solution $y_p(x)$. (See **Determining the Form** of $y_p(x)$ next.)
3. Determine the unknown coefficients in $y_p(x)$ by substituting $y_p(x)$ into the nonhomogeneous equation and equating the coefficients of like terms.
4. Form a general solution with $y(x) = y_h(x) + y_p(x)$.

Determining the Form of $y_p(x)$ (Step 2):

Suppose that $f(x) = b_1 f_1(x) + b_2 f_2(x) + \cdots + b_j f_j(x)$, where $b_1, b_2, \ldots, b_j$ are constants and each $f_i(x)$, $i = 1, 2, \ldots, j$, is a function of the form x^m, $x^m e^{kx}$, $x^m e^{\alpha x}\cos\beta x$, or $x^m e^{\alpha x}\sin\beta x$.

1. If $f_i(x) = x^m$, the associated set of functions is

$$S = \left\{1, x, x^2, \ldots, x^m\right\}.$$

2. If $f_i(x) = x^m e^{kx}$, the associated set of functions is

$$S = \left\{e^{kx}, xe^{kx}, x^2e^{kx}, \ldots, x^m e^{kx}\right\}.$$

3. If $f_i(x) = x^m e^{\alpha x} \cos\beta x$ or $f_i(x) = x^m e^{\alpha x} \sin\beta x$, the associated set of functions is

$$S = \{e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, x^2 e^{\alpha x}\cos\beta x, \dots, x^m e^{\alpha x}\cos\beta x, \\ e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, x^2 e^{\alpha x}\sin\beta x, \dots, x^m e^{\alpha x}\sin\beta x\}.$$

For each function $f_i(x)$ in $f(x)$, determine the associated set of functions S. If any of the functions in S appears in the general solution to the corresponding homogeneous equation, $y_h(x)$, multiply each function in S by x^r to obtain a new set $x^r S$, where r is the smallest positive integer so that each function in $x^r S$ is not a solution of the corresponding homogeneous equation. A particular solution is obtained by taking the linear combination of all functions in the associated sets where repeated functions should appear only once in the particular solution.

4.4.1 Second-Order Equations

EXAMPLE 4.4.1: Solve the nonhomogeneous equations (a) $y'' + 5y' + 6y = 2e^x$ and (b) $y'' + 5y' + 6y = 3e^{-2x}$.

SOLUTION: (a) The corresponding homogeneous equation $y'' + 5y' + 6y = 0$ has general solution $y_h = c_1 e^{-2x} + c_2 e^{-3x}$.

```
In[566]:= homsol = DSolve[y''[x] + 5 y'[x] + 6 y[x]
                  == 0, y[x], x]
Out[566]= {{y[x] → e^(-3 x) C[1] + e^(-2 x) C[2]}}
```

Next, we determine the form of $y_p(x)$. We choose $S = \{e^x\}$ because $f(x) = 2e^x$. Notice that e^x is not a solution to the homogeneous equation, so we take $y_p(x)$ to be the linear combination of the functions in S. Therefore,

$$y_p(x) = Ae^x.$$

```
In[567]:= yp[x_] = a Exp[x]
Out[567]= a e^x
```

Substituting this solution into $y'' + 5y' + 6y = 2e^x$, we have

$$Ae^x + 5Ae^x + 6Ae^x = 12Ae^x = 2e^x.$$

```
In[568]:= eqn = yp''[x] + 5yp'[x] + 6yp[x] == 2 Exp[x]
Out[568]= 12 a e^x == 2 e^x
```

Equating the coefficients of e^x then gives us $A = 1/6$.

```
In[569]:= aval = SolveAlways[eqn, x]
Out[569]= {{a → 1/6}, {x → -∞}}
```

Hence, a particular solution is $y_p(x) = \frac{1}{6}e^x$,

```
In[570]:= yp[x_] = a Exp[x]/. a → 1/6
Out[570]= e^x/6
```

and a general solution of the nonhomogeneous equation is

$$y = y_h + y_p = c_1e^{-2x} + c_2e^{-3x} + \frac{1}{6}e^x.$$

```
In[571]:= gensol = e^(-3 x) C[1] + e^(-2 x) C[2] + yp[x]
Out[571]= e^x/6 + e^(-3 x) C[1] + e^(-2 x) C[2]
```

In this case, we find the same general solution with `DSolve`.

```
In[572]:= gensol =
           DSolve[y''[x] + 5 y'[x] + 6 y[x] == 2 Exp[x],
            y[x], x, GeneratedParameters → c]
Out[572]= {{y[x] → e^x/6 + e^(-3 x) c[1] + e^(-2 x) c[2]}}
```

We then graph the general solution for various values of the arbitrary constants in the same way as in other examples. See Figure 4-19.

```
In[573]:= toplot = Table[gensol[[1, 1, 2]],
                 {c[1], -1, 1}, {c[2], -1, 1}];

          grays = Table[GrayLevel[i],
                 {i, 0, 0.7, 0.7/8}];

          Plot[Evaluate[toplot], {x, -3, 5},
           PlotStyle → grays, PlotRange- > {-3, 5},
           AspectRatio- > 1]
```

(b) In this case, we see that $f(x) = 3e^{-2x}$ so the associated set is $S = \{e^{-2x}\}$. However, because $y = e^{-2x}$ is a solution to the corresponding homogeneous equation, we must multiply each element of this set by x^r so that no element is a solution of the corresponding homogeneous equation. We multiply the element of S by x to obtain $xS = \{xe^{-2x}\}$ because xe^{-2x} is not a solution of $y'' + 5y' + 6y = 0$. Hence, $y_p(x) = Axe^{-2x}$. Differentiating $y_p(x)$ twice

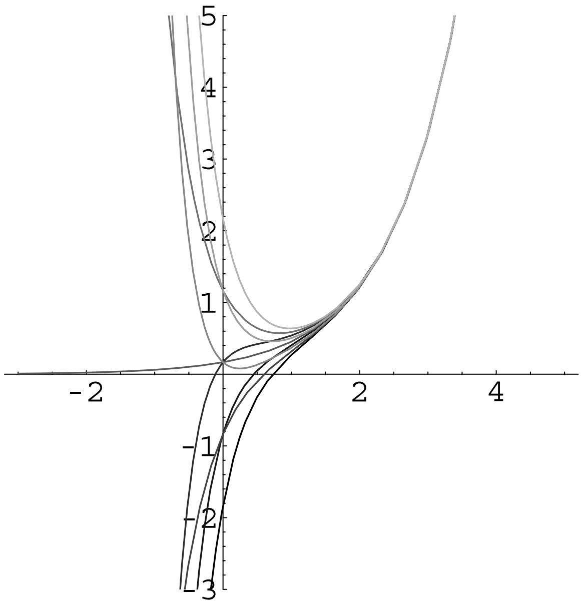


Figure 4-19 Various solutions of $y'' + 5y' + 6y = 2e^x$

```
In[574]:= yp[x_] = a x Exp[-2 x];

          yp'[x]

          yp''[x]
Out[574]= a e^(-2 x) - 2 a e^(-2 x) x
Out[574]= -4 a e^(-2 x) + 4 a e^(-2 x) x
```

and substituting into the nonhomogeneous equation yields

$$\begin{aligned} y'' + 5y' + 6y &= -4Ae^{-2x} + 4Axe^{-2x} + 5\left(Ae^{-2x} - 2Axe^{-2x}\right) + 6Axe^{-2x} \\ &= Ae^{-2x} = 3e^{-2x} \end{aligned}$$

```
In[575]:= eqn = yp''[x] + 5yp'[x] + 6yp[x] == 3 Exp[-2 x]
Out[575]= -4 a e^(-2 x) + 10 a e^(-2 x) x + 5 (a e^(-2 x) - 2 a e^(-2 x) x) ==
            3 e^(-2 x)
```

so $A = 3$ and $y_p(x) = 3xe^{-2x}$.

```
In[576]:= aval = SolveAlways[eqn, x]
Out[576]= {{a → 3}}
```

```
In[577]:= yp[x_] = yp[x]/.aval[[1]]

Out[577]= 3 e^(-2 x) x
```

A general solution of $y'' + 5y' + 6y = 3e^{-2x}$ is

$$y = y_h + y_p = c_1 e^{-2x} + c_2 e^{-3x} + 3xe^{-2x}.$$

As in (a), we can use `DSolve` to obtain equivalent results. For example, entering

```
In[578]:= gensol =
            DSolve[{y''[x] + 5 y'[x] + 6 y[x] == 3 Exp[-2 x],
               y[0] == a, y'[0] == b}, y[x], x]

Out[578]= {{y[x] →
               e^(-3 x) (3 - 2 a - b - 3 e^x + 3 a e^x + b e^x + 3 e^x x)}}
```

solves the equation subject to the initial conditions $y(0) = a$ and $y'(0) = b$ and names the resulting output `gensol`. Thus entering

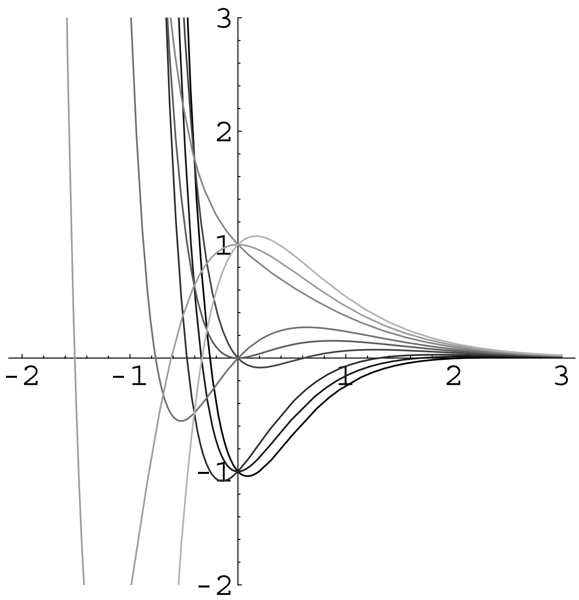


Figure 4-20 Various solutions of $y'' + 5y' + 6y = 3e^{-2x}$

```
In[579]:= toplot = Table[gensol[[1, 1, 2]], {a, -1, 1},
                {b, -1, 1}];

          grays = Table[GrayLevel[i],
                {i, 0, 0.7, 0.7/8}];

          Plot[Evaluate[toplot], {x, -2, 3},
            PlotStyle → grays, PlotRange- > {-2, 3},
            AspectRatio- > 1]
```

defines `toplot` to be the set consisting of nine functions corresponding to the solutions of $y'' + 5y' + 6y = 3e^{-2x}$ that satisfy the initial conditions $y(0) = a$ and $y'(0) = b$ for $a = -1, 0$, and 1 and $b = -1, 0$, and 1; and then graphs the set of functions `toplot` on the interval $[-2, 3]$ in Figure 4-20.

■

EXAMPLE 4.4.2: Solve

$$4\frac{d^2y}{dt^2} - y = t - 2 - 5\cos t - e^{-t/2}.$$

SOLUTION: The corresponding homogeneous equation is $4y'' - y = 0$ with general solution $y_h = c_1e^{-t/2} + c_2e^{t/2}$.

```
In[580]:= DSolve[4y''[t] - y[t] == 0, y[t], t]
Out[580]= {{y[t] → e^(-t/2) C[1] + e^(t/2) C[2]}}
```

A fundamental set of solutions for the corresponding homogeneous equation is $S = \{e^{-t/2}, e^{t/2}\}$. The associated set of functions for $t - 2$ is $F_1 = \{1, t\}$, the associated set of functions for $-5\cos t$ is $F_2 = \{\cos t, \sin t\}$, and the associated set of functions for $-e^{-t/2}$ is $F_3 = \{e^{-t/2}\}$. Note that $e^{-t/2}$ is an element of S so we multiply F_3 by t resulting in $tF_3 = \{te^{-t/2}\}$.

No element of F_1 is contained in S and no element of F_2 is contained in S.

Then, we search for a particular solution of the form

$$y_p = A + Bt + C\cos t + D\sin t + Ete^{-t/2},$$

where A, B, C, D, and E are constants to be determined.

```
In[581]:= yp[t_] = a + b t + c Cos[t] + d Sin[t]
                +e t Exp[-t/2]
Out[581]= a + b t + e e^(-t/2) t + c Cos[t] + d Sin[t]
```

Computing y'_p and y''_p

```
In[582]:= dyp = yp′[t]

          d2yp = yp″[t]
```

$$\text{Out[582]= } b + e\, e^{-t/2} - \frac{1}{2}\, e\, e^{-t/2}\, t + d\, \text{Cos}[t] - c\, \text{Sin}[t]$$

$$\text{Out[582]= } -e\, e^{-t/2} + \frac{1}{4}\, e\, e^{-t/2}\, t - c\, \text{Cos}[t] - d\, \text{Sin}[t]$$

and substituting into the nonhomogeneous equation results in

$$-A - Bt - 5C\cos t - 5D\sin t - 4Ee^{-t/2} = t - 2 - 5\cos t - e^{-t/2}.$$

```
In[583]:= eqn = 4 yp″[t] - yp[t] == t - 2 - 5 Cos[t]
             - Exp[-t/2]
```

$$\text{Out[583]= } -a - b\, t - e\, e^{-t/2}\, t - c\, \text{Cos}[t] - d\, \text{Sin}[t] + 4\left(-e\, e^{-t/2} + \frac{1}{4}\, e\, e^{-t/2}\, t - c\, \text{Cos}[t] - d\,\text{Sin}[t]\right) == -2 - e^{-t/2} + t - 5\, \text{Cos}[t]$$

Equating coefficients results in

$$-A = -2 \qquad -B = 1 \qquad -5C = -5 \qquad -5D = 0 \qquad -4E = -1$$

so $A = 2$, $B = -1$, $C = 1$, $D = 0$, and $E = 1/4$.

```
In[584]:= cvals =
            Solve[{-a == -2, -b == 1, -5c == -5, -5d == 0,
            -4e == -1}]
```

$$\text{Out[584]= } \left\{\left\{a \to 2, b \to -1, c \to 1, d \to 0, e \to \frac{1}{4}\right\}\right\}$$

y_p is then given by $y_p = 2 - t + \cos t + \frac{1}{4}te^{-t/2}$

```
In[585]:= yp[t]/.cvals[[1]]
```

$$\text{Out[585]= } 2 - t + \frac{1}{4}\, e^{-t/2}\, t + \text{Cos}[t]$$

and a general solution is given by

$$y = y_h + y_p = c_1 e^{-t/2} + c_2 e^{t/2} + 2 - t + \cos t + \frac{1}{4}te^{-t/2}.$$

Note that $-A - Bt - 5C\cos t - 5D\sin t - 4Ee^{-t/2} = t - 2 - 5\cos t - e^{-t/2}$ is true for *all* values of t. Evaluating for five different values of t gives us five equations that we then solve for A, B, C, D, and E, resulting in the same solutions as already obtained.

```
In[586]:= e1 = eqn/.t- > 0
```

Out[586]= $-a - c + 4\ (-c - e) == -8$

```
In[587]:= e2 = eqn/.t- > π/2

          e3 = eqn/.t- > π

          e4 = eqn/.t- > 1

          e5 = eqn/.t- > 2
```

Out[587]= $-a - d - \frac{b\ \pi}{2} - \frac{1}{2}\ e\ e^{-\pi/4}\ \pi + 4\ \left(-d - e\ e^{-\pi/4} + \frac{1}{8}\ e\ e^{-\pi/4}\ \pi\right) == -2 - e^{-\pi/4} + \frac{\pi}{2}$

Out[587]= $-a + c - b\ \pi - e\ e^{-\pi/2}\ \pi + 4\ \left(c - e\ e^{-\pi/2} + \frac{1}{4}\ e\ e^{-\pi/2}\ \pi\right) == 3 - e^{-\pi/2} + \pi$

Out[587]= $-a - b - \frac{e}{\sqrt{e}} - c\ \text{Cos}[1] - d\ \text{Sin}[1] + 4\ \left(-\frac{3\ e}{4\ \sqrt{e}} - c\ \text{Cos}[1] - d\ \text{Sin}[1]\right) == -1 - \frac{1}{\sqrt{e}} - 5\ \text{Cos}[1]$

Out[587]= $-a - 2\ b - \frac{2\ e}{e} - c\ \text{Cos}[2] - d\ \text{Sin}[2] + 4\ \left(-\frac{e}{2\ e} - c\ \text{Cos}[2] - d\ \text{Sin}[2]\right) == -\frac{1}{e} - 5\ \text{Cos}[2]$

```
In[588]:= Solve[{e1, e2, e3, e4, e5},
            {a, b, c, d, e}]//Simplify
```

Out[588]= $\left\{\left\{d \to 0, b \to -1, a \to 2, c \to 1, e \to \frac{1}{4}\right\}\right\}$

Last, we check our calculation with `DSolve` and `simplify`.

```
In[589]:= sol2 =
            DSolve[4y''[t] - y[t] == t - 2 - 5 Cos[t]
            - Exp[-t/2], y[t], t]
```

Out[589]= $\left\{\left\{y[t] \to e^{-t/2}\ C[1] + e^{t/2}\ C[2] + \frac{1}{4}\ \left(e^{-t/2} - 2\ t + 2\ \text{Cos}[t] - 4\ \text{Sin}[t]\right) + e^{-t/2}\ \left(2\ e^{t/2} + \frac{t}{4} - \frac{1}{2}\ e^{t/2}\ t + \frac{1}{2}\ e^{t/2}\ \text{Cos}[t] + e^{t/2}\ \text{Sin}[t]\right)\right\}\right\}$

```
In[590]:= Simplify[sol2]
Out[590]= {{y[t] →
            1/4 e^(-t/2) (1 + 8 e^(t/2) + t - 4 e^(t/2) t + 4 C[1]
            +4 e^t C[2]) + Cos[t]}}
```

■

In order to solve an initial-value problem, first determine a general solution and then use the initial conditions to solve for the unknown constants in the general solution.

EXAMPLE 4.4.3: Solve $y'' + 4y = \cos 2t$, $y(0) = 0$, $y'(0) = 0$.

SOLUTION: A general solution of the corresponding homogeneous equation is $y_h = c_1 \cos 2t + c_2 \sin 2t$. For this equation, $F = \{\cos 2t, \sin 2t\}$. Because elements of F are solutions to the corresponding homogeneous equation, we multiply each element of F by t resulting in $tF = \{t \cos 2t, t \sin 2t\}$. Therefore, we assume that a particular solution has the form

$$y_p = At \cos 2t + Bt \sin 2t,$$

where A and B are constants to be determined. Proceeding in the same manner as before, we compute y_p' and y_p''

```
In[591]:= yp[t_] = a t Cos[2t] + b t Sin[2 t];
          yp'[t]

          yp''[t]
Out[591]= a Cos[2 t] + 2 b t Cos[2 t] + b Sin[2 t]
          -2 a t Sin[2 t]
Out[591]= 4 b Cos[2 t] - 4 a t Cos[2 t] - 4 a Sin[2 t]
          -4 b t Sin[2 t]
```

and then substitute into the nonhomogeneous equation.

```
In[592]:= eqn = yp''[t] + 4yp[t] == Cos[2t]
Out[592]= 4 b Cos[2 t] - 4 a t Cos[2 t] - 4 a Sin[2 t]
          -4 b t Sin[2 t] + 4 (a t Cos[2 t]
          +b t Sin[2 t]) == Cos[2 t]
```

Equating coefficients readily yields $A = 0$ and $B = 1/4$. Alternatively, remember that $-4A \sin 2t + 4B \cos 2t = \cos 2t$ is true for *all* values of t.

Evaluating for two values of t and then solving for A and B gives the same result.

```
In[593]:= e1 = eqn/.t- > 0

          e2 = eqn/.t- > π/4

          cvals = Solve[{e1, e2}]
Out[593]= 4 b == 1
Out[593]= -4 a == 0
Out[593]= {{a → 0, b → 1/4}}
```

It follows that $y_p = \frac{1}{4}t\sin 2t$ and $y = c_1\cos 2t + c_2\sin 2t + \frac{1}{4}t\sin 2t$.

```
In[594]:= yp[t]/.cvals[[1]]
Out[594]= 1/4 t Sin[2 t]

In[595]:= y[t_] = c1 Cos[2t] + c2 Sin[2t] + 1/4 t Sin[2t]
Out[595]= c1 Cos[2 t] + c2 Sin[2 t] + 1/4 t Sin[2 t]
```

Applying the initial conditions

```
In[596]:= y'[t]
Out[596]= 2 c2 Cos[2 t] + 1/2 t Cos[2 t] + 1/4 Sin[2 t]
          -2 c1 Sin[2 t]

In[597]:= cvals = Solve[{y[0] == 0, y'[0] == 0}]
Out[597]= {{c1 → 0, c2 → 0}}
```

results in $y = \frac{1}{4}t\sin 2t$, which we graph with `Plot` in Figure 4-21.

```
In[598]:= y[t]/.cvals[[1]]
Out[598]= 1/4 t Sin[2 t]

In[599]:= Plot[Evaluate[y[t]/.cvals[[1]]], {t, 0, 16π}]
```

We verify the calculation with `DSolve`.

```
In[600]:= Clear[y]

          DSolve[
            y''[t] + 4y[t] == Cos[2t], y[0] == 0,
            y'[0], == 0}y[t], t]
Out[600]= {{y[t] → 1/4 t Sin[2 t]}}
```

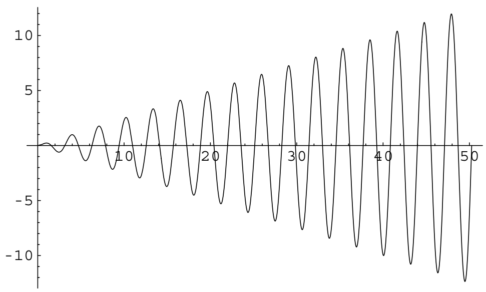


Figure 4-21 The forcing function causes the solution to become unbounded as $t \to \infty$

■

Initial-value problems and boundary-value problems can exhibit dramatically different behavior.

EXAMPLE 4.4.4: Show that the boundary-value problem

$$\begin{cases} 4y'' + 4y' + 37y = \cos 3x \\ y(0) = y(\pi) \end{cases}$$

has infinitely many solutions.

SOLUTION: First, we find a general solution of the corresponding homogeneous equation.

```
In[601]:= Clear[x, y]

          homsol =
            DSolve[4y''[x] + 4y'[x] + 37y[x] == 0,
              y[x], x]
Out[601]= {{y[x] → e^(-x/2) C[2] Cos[3 x] + e^(-x/2) C[1] Sin[3 x]}}
```

Using the Method of Undetermined Coefficients, we find a particular solution to the nonhomogeneous equation of the form $y_p = A \cos 3x + B \sin 3x$. Substitution into the nonhomogeneous equation yields.

```
In[602]:= yp[x_] = capa Cos[3x] + capb Sin[3x];

          step1 =
            4yp''[x] + 4yp'[x] + 37yp[x] == Cos[3x]//
              Simplify
```

```
Out[602]= (-1 + capa + 12 capb) Cos[3 x]
          + (-12 capa + capb) Sin[3 x] == 0
```

This equation is true for all values of x. In particular, substituting $x = 0$ and $x = \pi/6$ yields two equations

```
In[603]:= eq1 = step1 /. x- > 0
Out[603]= -1 + capa + 12 capb == 0

In[604]:= eq2 = step1 /. x- > π/6
Out[604]= -12 capa + capb == 0
```

that we then solve for A and B

```
In[605]:= vals = Solve[{eq1, eq2}]
Out[605]= {{capa → 1/145, capb → 12/145}}
```

to see that $A = 1/145$ and $B = 12/145$.

```
In[606]:= yp[x_] = yp[x] /. vals[[1]];

In[607]:= y[x_] = e^(-x/2) C[2] Cos[3x] - e^(-x/2) C[1] Sin[3x]
                +yp[x]
Out[607]= 1/145 Cos[3 x] + e^(-x/2) C[2] Cos[3 x]
          + 12/145 Sin[3 x] - e^(-x/2) C[1] Sin[3 x]
```

Applying the boundary conditions indicates that $\frac{1}{145} + c_2 = -\frac{1}{145} - e^{-\pi/2} c_2$

```
In[608]:= y[0]
Out[608]= 1/145 + C[2]

In[609]:= y[π]
Out[609]= -1/145 - e^(-π/2) C[2]
```

so $c_2 = \frac{2}{145(1+e^{-\pi/2})}$; c_1 is arbitrary.

```
In[610]:= cval = Solve[y[0] == y[π]]
Out[610]= {{C[2] → -(2 e^(π/2))/(145 (1 + e^(π/2)))}}

In[611]:= N[cval]
Out[611]= {{C[2] → -0.0114193}}

In[612]:= y[x_] = y[x] /. cval[[1]]
Out[612]= 1/145 Cos[3 x] - (2 e^(π/2 - x/2) Cos[3 x])/(145 (1 + e^(π/2)))
          + 12/145 Sin[3 x] - e^(-x/2) C[1] Sin[3 x]
```

Note that DSolve is able to solve this boundary-value problem as well.

```
In[613]:= Clear[x, y]

          sol =
            DSolve[
              {4y''[x] + 4y'[x] + 37y[x] == Cos[3x],
                y[0] == y[π]}, y[x], x,
              GeneratedParameters- > c]
```

Out[613]= $\{\{y[x] \to (e^{-x/2}(-24 e^{\pi/2} \text{Cos}[3x] + 12 e^{\frac{\pi}{2}+\frac{x}{2}} \text{Cos}[3x] \times \text{Cos}[6x] + 12 e^{x/2} \text{Cos}[3x] \text{Cos}[6x] + 145 e^{\frac{\pi}{2}+\frac{x}{2}} \text{Sin}[3x] + 145 e^{x/2} \text{Sin}[3x] + 1740 c[1] \text{Sin}[3x] + 1740 e^{\pi/2} c[1] \times \text{Sin}[3x] + e^{\frac{\pi}{2}+\frac{x}{2}} \text{Cos}[6x] \text{Sin}[3x] + e^{x/2} \text{Cos}[6x] \text{Sin}[3x] - e^{\frac{\pi}{2}+\frac{x}{2}} \text{Cos}[3x] \times \text{Sin}[6x] - e^{x/2} \text{Cos}[3x] \text{Sin}[6x] + 12 e^{\frac{\pi}{2}+\frac{x}{2}} \text{Sin}[3x] \text{Sin}[6x] + 12 e^{x/2} \text{Sin}[3x] \text{Sin}[6x])) / (1740 (1 + e^{\pi/2}))\}\}$

Several solutions are then graphed with Plot in Figure 4-22.

```
In[614]:= toplot = Table[sol[[1, 1, 2]], {c[1], -5, 5}];
```

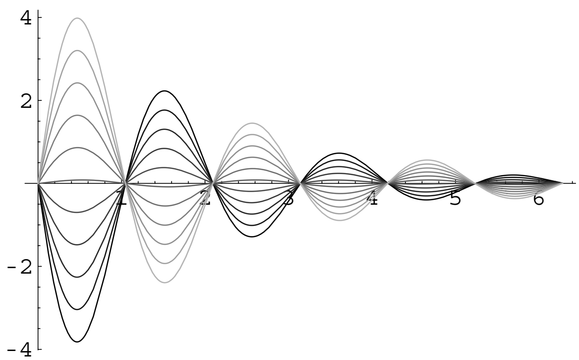


Figure 4-22 The boundary-value problem has infinitely many solutions

```
In[615]:= grays = Table[GrayLevel[i],
                {i, 0, 0.7, 0.7/10}];

          Plot[Evaluate[toplot], {x, 0, 2π},
            PlotStyle- > grays]
```

■

EXAMPLE 4.4.5: Graph the solution to the initial-value problem

$$\begin{cases} x'' + 4x = \sin \omega t \\ x(0) = 1, \ x'(0) = 0 \end{cases}$$

for various values of ω, including $\omega = 2$.

SOLUTION: First, we find a general solution of the corresponding homogeneous equation.

```
In[616]:= Clear[x, t]

          homsol =
            DSolve[x''[t] + 4x[t] == 0, x[t], t]//
              Simplify
Out[616]= {{x[t] → C[1] Cos[2 t] + C[2] Sin[2 t]}}
```

If $\omega \neq 2$, we can find a particular solution to the nonhomogeneous equation of the form $x_p = A\cos \omega t + B \sin \omega t$. We substitute this function into the nonhomogeneous equation and simplify the result.

```
In[617]:= xp[t_] = a Cos[ωt] + b Sin[ωt];

In[618]:= step1 = xp''[t] + 4xp[t] == Sin[ωt]//
                Simplify
Out[618]= -(-4 + ω^2) (a Cos[t ω] + b Sin[t ω]) == Sin[t ω]
```

This equation is true for all values of t. In particular, substituting $t = 0$ and $t = \pi/(2\omega)$ yields two equations

```
In[619]:= eqn1 = step1 /. t- > 0
Out[619]= -a (-4 + ω^2) == 0

In[620]:= eqn2 = step1 /. t- > π/(2ω)
Out[620]= -b (-4 + ω^2) == 1
```

that we then solve to determine A and B.

```
In[621]:= coeffs = Solve[{eqn1, eqn2}, {a, b}]
```

Out[621]= $\left\{\left\{a \to 0, b \to \frac{1}{4 - \omega^2}\right\}\right\}$

We then form a particular solution, x_p, to the nonhomogeneous equation and a general solution to the nonhomogeneous equation, $x = x_h + x_p$.

```
In[622]:= xp[t_] = xp[t]/.coeffs[[1]];

          x[t_] = homsol[[1, 1, 2]] + xp[t]
```

Out[622]= $C[1]\,\mathrm{Cos}[2\,t] + C[2]\,\mathrm{Sin}[2\,t] + \frac{\mathrm{Sin}[t\,\omega]}{4 - \omega^2}$

The solution to the initial-value problem is found by applying the initial conditions

```
In[623]:= cvals = Solve[{x[0] == 1, x'[0] == 0},
                  {C[1], C[2]}]
```

Out[623]= $\left\{\left\{C[1] \to 1, C[2] \to \frac{\omega}{2\,(-4 + \omega^2)}\right\}\right\}$

and substituting back into the general solution.

```
In[624]:= sol = x[t] /. cvals[[1]]
```

Out[624]= $\mathrm{Cos}[2\,t] + \frac{\omega\,\mathrm{Sin}[2\,t]}{2\,(-4 + \omega^2)} + \frac{\mathrm{Sin}[t\,\omega]}{4 - \omega^2}$

If $\omega = 2$, we can find a particular solution to the nonhomogeneous equation of the form $x_p = t\,(A \cos 2t + B \sin 2t)$. We proceed in the same manner as before.

```
In[625]:= xp[t_] = t(a Cos[2t] + b Sin[2t]);

In[626]:= step1 = xp''[t] + 4xp[t] == Sin[2t]//
                  Simplify
```

Out[626]= $4\,b\,\mathrm{Cos}[2\,t] == (1 + 4\,a)\,\mathrm{Sin}[2\,t]$

```
In[627]:= eqn1 = step1 /. t- > 0
```

Out[627]= $4\,b == 0$

```
In[628]:= eqn2 = step1 /. t- > π/12
```

Out[628]= $2\sqrt{3}\,b == \frac{1}{2}\,(1 + 4\,a)$

```
In[629]:= coeffs = Solve[{eqn1, eqn2}, {a, b}]
```

Out[629]= $\left\{\left\{a \to -\frac{1}{4}, b \to 0\right\}\right\}$

```
In[630]:= xp[t_] = xp[t]/.coeffs[[1]];

          x[t_] = homsol[[1, 1, 2]] + xp[t]
```

Out[630]= $-\frac{1}{4}\,t\,\mathrm{Cos}[2\,t] + C[1]\,\mathrm{Cos}[2\,t] + C[2]\,\mathrm{Sin}[2\,t]$

```
In[631]:= cvals = Solve[{x[0] == 1, x'[0] == 0},
                  {C[1], C[2]}]
```

Out[631]= $\{\{C[1] \to 1, C[2] \to \frac{1}{8}\}\}$

```
In[632]:= sol = x[t] /. cvals[[1]]
```

Out[632]= $\text{Cos}[2\,t] - \frac{1}{4}\,t\,\text{Cos}[2\,t] + \frac{1}{8}\,\text{Sin}[2\,t]$

We see that `DSolve` is able to solve the initial-value problem as well. Note that the result returned is valid for $\omega \neq 2$.

```
In[633]:= Clear[x, t]

          sola =
            DSolve[{x''[t] + 4x[t] == Sin[ωt],
                x[0] == 1, x'[0] == 0}, x[t], t]//
              Simplify
```

Out[633]= $\{\{x[t] \to \frac{2\,(-4+\omega^2)\,\text{Cos}[2\,t] + \omega\,\text{Sin}[2\,t] - 2\,\text{Sin}[t\,\omega]}{2\,(-4+\omega^2)}\}\}$

We use this result to graph the solution for various values of ω in Figure 4-23. Of course, Mathematica generates several error messages, which are not all displayed here, and an empty `Plot` when it encounters $\omega = 2$ because the solution obtained in `sola` is undefined if $\omega = 2$.

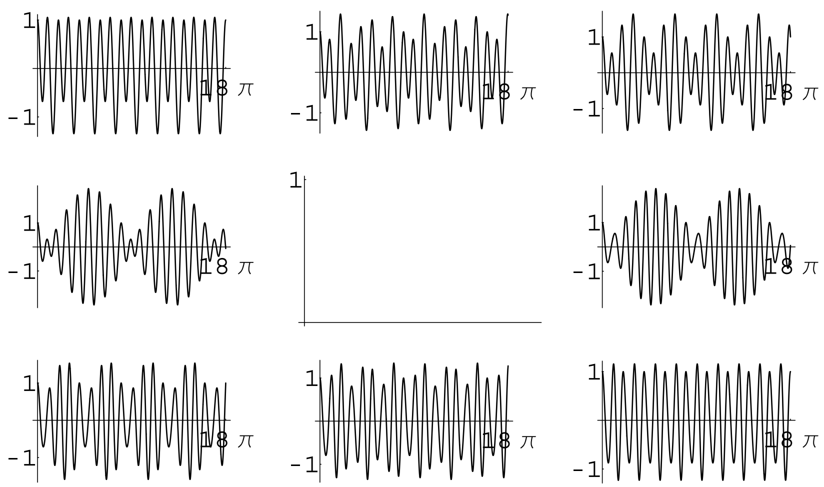


Figure 4-23 The empty plot corresponds to $\omega = 2$

```
In[634]:= graphs =
            Table[Plot[sola[[1,1,2]],{t,0,18π},
            DisplayFunction->Identity,
            Ticks->{{0,18π},{-1,1}},
            PlotPoints->200],{ω,1,3,2/8}];

          toshow = Partition[graphs,3];

          Show[GraphicsArray[toshow]]
Power::infy : Infinite expression 1/0 encountered.
∞::indet : Indeterminate expression
    0 ComplexInfinity encountered.
```

We also use DSolve to find the solution to the initial-value problem if $\omega = 2$ and then graph the result in Figure 4-24.

```
In[635]:= solb =
            DSolve[{x''[t]+4x[t]==Sin[2t],
                 x[0]==1,x'[0]==0},x[t],t]//
             Simplify
```

Out[635]= $\left\{\left\{x[t] \rightarrow \frac{1}{8}\,(-2\,(-4+t)\,\mathrm{Cos}[2\,t]+\mathrm{Sin}[2\,t])\right\}\right\}$

```
In[636]:= Plot[x[t]/.solb,{t,0,18π}]
```

The graphs indicate that if $\omega \neq 2$ the solution to the initial-value problem is bounded and periodic; if $\omega = 2$ the solution is unbounded. We investigate this type of behavior further in Chapter 5.

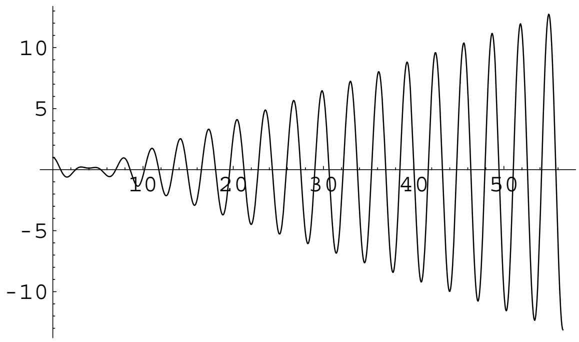


Figure 4-24 If $\omega = 2$, the solution becomes unbounded as $t \rightarrow \infty$

■

4.4.2 Higher-Order Equations

Higher-order nonhomogeneous equations are solved in the same way as second-order equations provided that the forcing function involves appropriate terms, although the calculations can be more complicated.

EXAMPLE 4.4.6: Solve

$$\frac{d^3y}{dt^3} + \frac{2}{3}\frac{d^2y}{dt^2} + \frac{145}{9}\frac{dy}{dt} = e^{-t},\ y(0) = 1,\ \frac{dy}{dt}(0) = 2,\ \frac{d^2y}{dt^2}(0) = -1.$$

SOLUTION: The corresponding homogeneous equation, $y''' + \frac{2}{3}y'' + \frac{145}{9}y' = 0$, has general solution $y_h = c_1 + (c_2 \sin 4t + c_3 \cos 4t)\, e^{-t/3}$ and a fundamental set of solutions for the corresponding homogeneous equation is $S = \left\{1, e^{-t/3}\cos 4t, e^{-t/3}\sin 4t\right\}$.

```
In[637]:= DSolve[y'''[t] + 2/3y''[t] + 145/9y'[t] == 0,
            y[t], t]//Simplify
Out[637]= {{y[t] → C[3] + 3/145 e^(-t/3)
            ×((12 C[1] - C[2]) Cos[4 t] + (C[1]
            +12 C[2]) Sin[4 t])}}
```

For e^{-t}, the associated set of functions is $F = \left\{e^{-t}\right\}$. Because no element of F is an element of S, we assume that $y_p = Ae^{-t}$, where A is a constant to be determined. After defining y_p, we compute the necessary derivatives

```
In[638]:= yp[t_] = a Exp[-t];
          yp'[t]

          yp''[t]

          yp'''[t]
Out[638]= -a e^-t
Out[638]= a e^-t
Out[638]= -a e^-t
```

and substitute into the nonhomogeneous equation.

```
In[639]:= eqn = yp'''[t] + 2/3yp''[t] + 145/9yp'[t] == Exp[-t]
Out[639]= -148/9 a e^-t == e^-t
```

Equating coefficients and solving for A gives us $A = -9/148$ so $y_p = -\frac{9}{148}e^{-t}$ and a general solution is $y = y_h + y_p$.

Remark. `SolveAlways[equation,variable]` attempts to solve `equation` so that it is true for all values of `variable`.

```
In[640]:= SolveAlways[eqn, t]
Out[640]= {{a → -9/148}}
```

We verify the result with `DSolve`.

```
In[641]:= gensol = DSolve[y'''[t] + 2/3y''[t]
                  +145/9y'[t] == Exp[-t], y[t], t]
Out[641]= {{y[t] → -(9 e^-t)/148 - (3/145 - (36 i)/145) e^((-1/3-4 i) t) C[1]
                 -(9/290 - (3 i)/1160)
                 × e^((-1/3+4 i) t) C[2] + C[3]}}
```

To obtain a real-valued solution, we use `ComplexExpand`:

```
In[642]:= ?ComplexExpand
"ComplexExpand[expr]expandsexprassuming
   thatallvariablesarereal.ComplexExpand[
   expr,x1,x2,...]expandsexprassuming
   thatvariablesmatchinganyofthexiarecomplex."

In[643]:= s1 = ComplexExpand[y[t]/.gensol[[1]]]
Out[643]= -(9 e^-t)/148 + C[3] - (3/145 - (36 i)/145)
          × e^(-t/3) C[1] Cos[4 t] - (9/290 - (3 i)/1160)
          × e^(-t/3) C[2] Cos[4 t] + (36/145 + (3 i)/145)
          × e^(-t/3) C[1] Sin[4 t] - (3/1160 + (9 i)/290)
          × e^(-t/3) C[2] Sin[4 t]

In[644]:= t1 = Coefficient[s1, Exp[-t/3] Cos[4t]]
Out[644]= (-3/145 + (36 i)/145) C[1] - (9/290 - (3 i)/1160) C[2]

In[645]:= t2 = Coefficient[s1, Exp[-t/3] Sin[4t]]
Out[645]= (36/145 + (3 i)/145) C[1] - (3/1160 + (9 i)/290) C[2]

In[646]:= t3 = C[3]
Out[646]= C[3]
```

```
In[647]:= Clear[c1, c2, c3]

          s2 = Solve[{t1 == c1, t2 == c2, t3 == c3},
               {C[1], C[2], C[3]}]
```

$$\text{Out[647]=}\ \left\{\left\{C[1]\to\left(-\frac{1}{6}-2\,i\right)(c1+i\ c2),\ C[2]\to\left(-16-\frac{4\,i}{3}\right)(c1-i\ c2),\ C[3]\to c3\right\}\right\}$$

The result indicates that the form returned by `DSolve` is equivalent to

```
In[648]:= s3 = s1/.s2[[1]]//Simplify
```

$$\text{Out[648]=}\ c3-\frac{9\,e^{-t}}{148}+c1\ e^{-t/3}\ \text{Cos}[4\ t]+c2\ e^{-t/3}\ \text{Sin}[4\ t]$$

To apply the initial conditions, we compute $y(0) = 1$, $y'(0) = 2$, and $y''(0) = -1$

```
In[649]:= e1 = (s3/.t- > 0) == 1

          e2 = (D[s3, t]/.t- > 0) == 2

          e3 = (D[s3, {t, 2}]/.t- > 0) == -1
```

$$\text{Out[649]=}\ -\frac{9}{148}+c1+c3==1$$

$$\text{Out[649]=}\ \frac{9}{148}-\frac{c1}{3}+4\ c2==2$$

$$\text{Out[649]=}\ -\frac{9}{148}-\frac{143\ c1}{9}-\frac{8\ c2}{3}==-1$$

and solve for c_1, c_2, and c_3.

```
In[650]:= cvals = Solve[{e1, e2, e3}]
```

$$\text{Out[650]=}\ \left\{\left\{c1\to-\frac{471}{21460},\ c2\to\frac{20729}{42920},\ c3\to\frac{157}{145}\right\}\right\}$$

The solution of the initial-value problem is obtained by substituting these values into the general solution.

```
In[651]:= s3/.cvals[[1]]
```

$$\text{Out[651]=}\ \frac{157}{145}-\frac{9\ e^{-t}}{148}-\frac{471\ e^{-t/3}\ \text{Cos}[4\ t]}{21460}+\frac{20729\ e^{-t/3}\ \text{Sin}[4\ t]}{42920}$$

We check by using `DSolve` to solve the initial-value problem and graph the result with `Plot` in Figure 4-25.

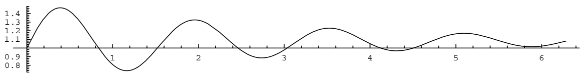


Figure 4-25 The solution of the equation that satisfies $y(0) = 1$, $y'(0) = 2$, and $y''(0) = -1$

```
In[652]:= sol = DSolve[{y'''[t] + 2/3y''[t] + 145/9y'[t] ==
              Exp[-t], y[0] == 1, y'[0] == 2,
              y''[0] == -1}, y[t], t]
```

Out[652]= $\left\{\left\{y[t] \to \frac{157}{145} - \frac{9\,e^{-t}}{148} - \left(\frac{471}{42920} - \frac{20729\,i}{85840}\right) \times e^{\left(-\frac{1}{3}-4\,i\right)\,t} - \left(\frac{471}{42920} + \frac{20729\,i}{85840}\right) e^{\left(-\frac{1}{3}+4\,i\right)\,t}\right\}\right\}$

```
In[653]:= realsol = ComplexExpand[y[t]/.sol[[1]]]
```

Out[653]= $\frac{157}{145} - \frac{9\,e^{-t}}{148} - \frac{471\,e^{-t/3}\,\text{Cos}[4\,t]}{21460} + \frac{20729\,e^{-t/3}\,\text{Sin}[4\,t]}{42920}$

```
In[654]:= Plot[realsol, {t, 0, 2π},
           AspectRatio- > Automatic]
```

■

EXAMPLE 4.4.7: Solve

$$\frac{d^8y}{dt^8} + \frac{7}{2}\frac{d^7y}{dt^7} + \frac{73}{2}\frac{d^6y}{dt^6} + \frac{229}{2}\frac{d^5y}{dt^5} + \frac{801}{2}\frac{d^4y}{dt^4} + 976\frac{d^3y}{dt^3} + 1168\frac{d^2y}{dt^2} + 640\frac{dy}{dt} + 128y = te^{-t} + \sin 4t + t.$$

SOLUTION: Solving the characteristic equation

```
In[655]:= Solve[k^8 + 7/2k^7 + 73/2k^6 + 229/2k^5
          +801/2k^4 + 976k^3 + 1168k^2
          +640k + 128 == 0]
```

Out[655]= $\left\{\{k \to -1\}, \{k \to -1\}, \{k \to -1\}, \left\{k \to -\frac{1}{2}\right\}, \{k \to -4\,i\}, \{k \to -4\,i\}, \{k \to 4\,i\}, \{k \to 4\,i\}\right\}$

shows us that the solutions are $k_1 = -1/2$, $k_2 = -1$ with multiplicity 3, and $k_{3,4} = \pm 4i$, each with multiplicity 2. A fundamental set of solutions for the corresponding homogeneous equation is

$$S = \left\{e^{-t/2}, e^{-t}, te^{-t}, t^2e^{-t}, \cos 4t, t\cos 4t, \sin 4t, t\sin 4t\right\}.$$

A general solution of the corresponding homogeneous equation is

$$y_h = c_1e^{-t/2} + \left(c_2 + c_3t + c_4t^2\right)e^{-t} + (c_5 + c_7t)\sin 4t + (c_6 + c_8t)\cos 4t.$$

```
In[656]:= gensol = DSolve[D[y[t], {t, 8}] + 7/2D[y[t],
                {t, 7}] + 73/2D[y[t], {t, 6}]
                +229/2D[y[t], {t, 5}] + 801/2D[y[t],
                {t, 4}] + 976D[y[t], {t, 3}]
                +1168D[y[t], {t, 2}] + 640D[y[t], t]
                +128y[t] == 0, y[t], t]

Out[656]= {{y[t] →
            e^-t C[1] + e^-t t C[2] + e^-t t^2 C[3]
            +e^(-t/2) C[4] + C[6] Cos[4 t]
            +t C[8] Cos[4 t] - C[5] Sin[4 t]
            -t C[7] Sin[4 t]}}
```

The associated set of functions for te^{-t} is $F_1 = \left\{e^{-t}, te^{-t}\right\}$. We multiply F_1 by t^r, where r is the smallest nonnegative integer so that no element of t^rF_1 is an element of S: $t^3F_1 = \left\{t^3e^{-t}, t^4e^{-t}\right\}$. The associated set of functions for $\sin 4t$ is $F_2 = \{\cos 4t, \sin 4t\}$. We multiply F_2 by t^r, where r is the smallest nonnegative integer so that no element of t^rF_2 is an element of S: $t^2F_2 = \left\{t^2\cos 4t, t^2\sin 4t\right\}$. The associated set of functions for t is $F_3 = \{1, t\}$. No element of F_3 is an element of S.

Thus, we search for a particular solution of the form

$$y_p = A_1t^3e^{-t} + A_2t^4e^{-t} + A_3t^2\cos 4t + A_4t^2\sin 4t + A_5 + A_6t,$$

where the A_i are constants to be determined.

After defining y_p, we compute the necessary derivatives

Remark. We have used `Table` twice for typesetting purposes. You can compute the derivatives using `Table[{n,D[yp[t],{t,n}]}, {n,1,8}]`.

```
In[657]:= yp[t_] = a[1]t^3 Exp[-t] + a[2]t^4 Exp[-t]
                   +a[3]t^2 Cos[4t] + a[4]t^2 Sin[4t]
                   +a[5] + a[6]t

Out[657]= e^-t t^3 a[1] + e^-t t^4 a[2] + a[5] + t a[6]
          +t^2 a[3] Cos[4 t] + t^2 a[4] Sin[4 t]


In[658]:= Table[{n,D[yp[t],{t,n}]},{n,1,4}]

Out[658]= {{1, 3 e^-t t^2 a[1] - e^-t t^3 a[1]
            +4 e^-t t^3 a[2] - e^-t t^4 a[2] + a[6]
            +2 t a[3] Cos[4 t] + 4 t^2 a[4] Cos[4 t]
            -4 t^2 a[3] Sin[4 t]
            +2 t a[4] Sin[4 t]},
           {2, 6 e^-t t a[1] - 6 e^-t t^2 a[1]
            +e^-t t^3 a[1] + 12 e^-t t^2 a[2] - 8 e^-t t^3 a[2]
            +e^-t t^4 a[2] + 2 a[3] Cos[4 t]
            -16 t^2 a[3] Cos[4 t]
            +16 t a[4] Cos[4 t]
            -16 t a[3] Sin[4 t] + 2 a[4] Sin[4 t]
            -16 t^2 a[4] Sin[4 t]},
           {3, 6 e^-t a[1] - 18 e^-t t a[1] + 9 e^-t t^2 a[1]
            -e^-t t^3 a[1] + 24 e^-t t a[2]
            -36 e^-t t^2 a[2] + 12 e^-t t^3 a[2]
            -e^-t t^4 a[2] - 96 t a[3] Cos[4 t]
            +24 a[4] Cos[4 t] - 64 t^2 a[4] Cos[4 t]
            -24 a[3] Sin[4 t] + 64 t^2 a[3] Sin[4 t]
            -96 t a[4] Sin[4 t]},
           {4, -24 e^-t a[1] + 36 e^-t t a[1]
            -12 e^-t t^2 a[1] + e^-t t^3 a[1] + 24 e^-t a[2]
            -96 e^-t t a[2] + 72 e^-t t^2 a[2]
            -16 e^-t t^3 a[2] + e^-t t^4 a[2]
            -192 a[3] Cos[4 t]
            +256 t^2 a[3] Cos[4 t]
            -512 t a[4] Cos[4 t]
            +512 t a[3] Sin[4 t]
            -192 a[4] Sin[4 t]
            +256 t^2 a[4] Sin[4 t]}}


In[659]:= Table[{n,D[yp[t],{t,n}]},{n,5,8}]
```

```
Out[659]= {{5, 60 e^-t a[1] - 60 e^-t t a[1] + 15 e^-t t^2 a[1]
          -e^-t t^3 a[1] - 120 e^-t a[2] + 240 e^-t t a[2]
          -120 e^-t t^2 a[2] + 20 e^-t t^3 a[2] - e^-t t^4 a[2]
          +2560 t a[3] Cos[4 t] - 1280 a[4]
          × Cos[4 t] + 1024 t^2 a[4] Cos[4 t]
          +1280 a[3] Sin[4 t] - 1024 t^2 a[3]
          × Sin[4 t] + 2560 t a[4] Sin[4 t]},
          {6, -120 e^-t a[1] + 90 e^-t t a[1] - 18 e^-t
          × t^2 a[1] + e^-t t^3 a[1] + 360 e^-t a[2]
          -480 e^-t t a[2] + 180 e^-t t^2 a[2] - 24 e^-t
          × t^3 a[2] + e^-t t^4 a[2] + 7680 a[3] Cos[4 t]
          -4096 t^2 a[3] Cos[4 t] + 12288 t a[4]
          × Cos[4 t] - 12288 t a[3] Sin[4 t]
          +7680 a[4] Sin[4 t] - 4096 t^2 a[4] Sin[4 t]},
          {7, 210 e^-t a[1] - 126 e^-t t a[1]
          +21 e^-t t^2 a[1] - e^-t t^3 a[1] - 840 e^-t a[2]
          +840 e^-t t a[2] - 252 e^-t t^2 a[2]
          +28 e^-t t^3 a[2] - e^-t t^4 a[2]
          -57344 t a[3] Cos[4 t] + 43008 a[4]
          × Cos[4 t] - 16384 t^2 a[4] Cos[4 t]
          -43008 a[3] Sin[4 t] + 16384 t^2 a[3]
          × Sin[4 t] - 57344 t a[4] Sin[4 t]},
          {8, -336 e^-t a[1] + 168 e^-t t a[1] - 24 e^-t t^2
          ×a[1] + e^-t t^3 a[1] + 1680 e^-t a[2]
          -1344 e^-t t a[2] + 336 e^-t t^2 a[2]
          -32 e^-t t^3 a[2] + e^-t t^4 a[2] - 229376 a[3]
          × Cos[4 t] + 65536 t^2 a[3] Cos[4 t]
          -262144 t a[4] Cos[4 t] + 262144 t a[3]
          ×Sin[4 t] - 229376 a[4] Sin[4 t]
          +65536 t^2 a[4] Sin[4 t]}}
```

and substitute into the nonhomogeneous equation, naming the result eqn. At this point we can either equate coefficients and solve for A_i or use the fact that eqn is true for *all* values of t.

```
In[660]:= eqn = D[yp[t], {t, 8}] + 7/2D[yp[t], {t, 7}]
              +73/2D[yp[t], {t, 6}] + 229/2D[yp[t],
                {t, 5}] + 801/2D[yp[t], {t, 4}]
                +976D[yp[t], {t, 3}] + 1168D[yp[t],
                {t, 2}] + 640D[yp[t], t] + 128yp[t] ==
                t Exp[-t] + Sin[4t] + t//
             Simplify
```

```
Out[660]= e^-t (-867 a[1] + 7752 a[2] - 3468 t a[2]+
          +128 e^t a[5] + 640 e^t a[6]
          +128 e^t t a[6])
          -64 (369 a[3] - 428 a[4]) Cos[4 t]
          -64 (428 a[3] + 369 a[4]) Sin[4 t] ==
            t + e^-t t + Sin[4 t]
```

We substitute in six values of t

```
In[661]:= sysofeqs = Table[eqn/.t- > n//N, {n, 0, 5}]

Out[661]= {-867. a[1.]
          +7752. a[2.]
          -64. (369. a[3.]
          -428. a[4.])
          +128. a[5.]
          +640. a[6.] == 0,
            41.8332 (369. a[3.]
          -428. a[4.])
          +48.4354 (428. a[3.]
          +369. a[4.])
          +0.367879 (-867. a[1.]
          +4284. a[2.]
          +347.94 a[5.]
          +2087.64 a[6.]) ==
            0.611077,
          9.312 (369. a[3.]
          -428. a[4.])
          -63.3189 (428. a[3.]
          +369. a[4.])
          +0.135335 (-867. a[1.]
          +816. a[2.]
          +945.799 a[5.]
          +6620.59 a[6.]) ==
            3.26003,
          -54.0067 (369. a[3.]
          -428. a[4.])
          +34.3407 (428. a[3.]
          +369. a[4.])
          +0.0497871 (-867. a[1.]
          -2652. a[2.]
          +2570.95 a[5.]
          +20567.6 a[6.]) ==
            2.61279,
```

```
Out[661]= 61.2902 (369. a[3.]
          -428. a[4.])
          +18.4258 (428. a[3.]
          +369. a[4.])
          +0.0183156 (-867. a[1.]
          -6120. a[2.]
          +6988.56 a[5.]
          +62897.1 a[6.]) ==
            3.78536,
          -26.1173 (369. a[3.]
          -428. a[4.])
          -58.4285 (428. a[3.]
          +369. a[4.])
          +0.00673795
          (-867. a[1.]
          -9588. a[2.]
          +18996.9 a[5.]
          +189969. a[6.]) ==
            5.94663}
```

and then solve for A_i.

```
In[662]:= coeffs =
            Solve[sysofeqs, {a[1.], a[2.], a[3.], a[4.],
              a[5.], a[6.]}]

Out[662]= {{a[1.] → -0.00257819,
              a[2.] → -0.000288351,
              a[3.] → -0.0000209413,
              a[4.] → -0.0000180545,
              a[5.] → -0.0390625,
              a[6.] → 0.0078125}}
```

y_p is obtained by substituting the values for A_i into y_p and a general solution is $y = y_h + y_p$. `DSolve` is able to find an exact solution.

```
In[663]:= gensol = DSolve[D[y[t], {t, 8}] + 7/2D[y[t],
                    {t, 7}] + 73/2D[y[t], {t, 6}]
                    +229/2D[y[t], {t, 5}]
                    +801/2D[y[t], {t, 4}] + 976D[y[t],
                    {t, 3}] + 1168D[y[t], {t, 2}]
                    +640D[y[t], t] + 128y[t] ==
                    t Exp[-t] + Sin[4t] + t, y[t], t]//
              Simplify
```

$$\text{Out[663]=}\ \Big\{\Big\{y[t]\to -\frac{5}{128}-\frac{2924806\ e^{-t}}{24137569}+\frac{t}{128}-\frac{86016\ e^{-t}\ t}{1419857}-\frac{1270\ e^{-t}\ t^2}{83521}-\frac{38\ e^{-t}\ t^3}{14739}-\frac{e^{-t}\ t^4}{3468}+e^{-t}\ C[1]+e^{-t}\ t\ C[2]+e^{-t}\ t^2 C[3]+e^{-t/2}\ C[4]+\Big(\frac{9041976373}{199643253056000}-\frac{107\ t^2}{5109520}+C[6]+t\ \Big(-\frac{1568449}{45168156800}+C[8]\Big)\Big)\ \text{Cos}[4\ t]+\Big(\frac{13794625331}{798573012224000}+\frac{20406\ t}{352876225}-\frac{369\ t^2}{20438080}-C[5]-t\ C[7]\Big)\ \text{Sin}[4\ t]\Big\}\Big\}$$

■

4.5 Nonhomogeneous Equations with Constant Coefficients: Variation of Parameters

4.5.1 Second-Order Equations

A **particular solution**, y_p, is a solution that does not contain any arbitrary constants.

Let $S = \{y_1, y_2\}$ be a fundamental set of solutions for equation $y'' + p(t)y' + q(t)y = 0$. To solve the nonhomogeneous equation $y'' + p(t)y' + q(t)y = f(t)$, we need to find a particular solution, y_p of equation $y''+p(t)y'+q(t)y = f(t)$. We search for a particular solution of the form

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t), \tag{4.13}$$

where u_1 and u_2 are functions of t. Differentiating equation (4.13) gives us

$$y_p{}' = u_1{}'y_1 + u_1y_1{}' + u_2{}'y_2 + u_2y_2{}'.$$

Assuming that

$$y_1 u_1{}' + y_2 u_2{}' = 0 \tag{4.14}$$

results in $y_p{}' = u_1 y_1{}' + u_2 y_2{}'$. Computing the second derivative then yields

$$y_p{}'' = u_1{}' y_1{}' + u_1 y_1{}'' + u_2{}' y_2{}' + u_2 y_2{}''.$$

Observe that it is pointless to search for solutions of the form $y_p = c_1 y_1 + c_2 y_2$ where c_1 and c_2 are constants because for every choice of c_1 and c_2, $c_1 y_1 + c_2 y_2$ is a solution to the corresponding homogeneous equation.

Substituting y_p, $y_p{}'$, and $y_p{}''$ into the equation $y'' + p(t)y' + q(t)y = f(t)$ and using the facts that

$$u_1 \left(y_1{}'' + p\, y_1{}' + q\, y_1\right) = 0 \quad \text{and} \quad u_2 \left(y_2{}'' + p\, y_2{}' + q\, y_2\right) = 0$$

(because y_1 and y_2 are solutions to the corresponding homogeneous equation, $y'' + p(t)y' + q(t)y = 0$) results in

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + p(t)\frac{dy_p}{dt} + q(t)y_p &= u_1{}' y_1{}' + u_1 y_1{}'' \\ &\quad + u_2{}' y_2{}' + u_2 y_2{}'' + p(t)\left(u_1 y_1{}' + u_2 y_2{}'\right) \\ &\quad + q(t)\left(u_1 y_1 + u_2 y_2\right) \\ &= y_1{}' u_1{}' + y_2{}' u_2{}' = f(t). \end{aligned} \tag{4.15}$$

Observe that equation (4.14) and equation (4.5.1) form a system of two linear equations in the unknowns $u_1{}'$ and $u_2{}'$:

$$\begin{aligned} y_1 u_1{}' + y_2 u_2{}' &= 0 \\ y_1{}' u_1{}' + y_2{}' u_2{}' &= f(t). \end{aligned} \tag{4.16}$$

Applying Cramer's Rule gives us

$$u_1{}' = \frac{\begin{vmatrix} 0 & y_2 \\ f(t) & y_2{}' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}} = -\frac{y_2(t)f(t)}{W(S)} \quad \text{and} \quad u_2{}' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1{}' & f(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}} = \frac{y_1(t)f(t)}{W(S)}, \tag{4.17}$$

where $W(S)$ is the Wronskian, $W(S) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}$. After integrating to obtain u_1 and u_2, we form y_p and then a general solution, $y = y_h + y_p$.

Summary of Variation of Parameters for Second-Order Equations
Given the second-order equation $a_2(t)y'' + a_2(t)y' + a_0(t)y = g(t)$.

1. Divide by $a_1(t)$ to rewrite the equation in standard form, $y'' + p(t)y' + q(t)y = f(t)$.

2. Find a general solution, y_h, of the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$.
3. Let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.
4. Let $u_1' = -\frac{y_2 f(t)}{W}$ and $u_2' = \frac{y_1 f(t)}{W}$.
5. Integrate to obtain u_1 and u_2.
6. A particular solution of $a_1(t)y'' + a_2(t)y' + a_0(t)y = g(t)$ is given by $y_p = u_1 y_1 + u_2 y_2$.
7. A general solution of $a_1(t)y'' + a_2(t)y' + a_0(t)y = g(t)$ is given by $y = y_h + y_p$.

EXAMPLE 4.5.1: Solve $y'' + 9y = \sec 3t$, $y(0) = 0$, $y'(0) = 0$, $0 \le t < \pi/6$.

SOLUTION: The corresponding homogeneous equation is $y'' + 9y = 0$ with general solution $y_h = c_1 \cos 3t + c_2 \sin 3t$. Then, a fundamental set of solutions is $S = \{\cos 3t, \sin 3t\}$ and $W(S) = 3$, as we see using `Det`, and `Simplify`.

```
In[664]:= fs = {Cos[3t], Sin[3t]};
          wm = {fs, D[fs, t]};
          wm//MatrixForm

          wd = Simplify[Det[wm]]
Out[664]= (  Cos[3 t]      Sin[3 t]  )
          (-3 Sin[3 t]  3 Cos[3 t]   )
Out[664]= 3
```

We use equation (4.17) to find $u_1 = \frac{1}{9} \ln \cos 3t$ and $u_2 = \frac{1}{3}t$.

```
In[665]:= u1 = Integrate[- Sin[3t] Sec[3t]/3, t]

          u2 = Integrate[Cos[3t] Sec[3t]/3, t]
          1
Out[665]= - Log[Cos[3 t]]
          9
          t
Out[665]= -
          3
```

It follows that a particular solution of the nonhomogeneous equation is

$$y_p = \frac{1}{9} \cos 3t \ \ln \cos 3t + \frac{1}{3} t \sin 3t$$

and a general solution is

$$y = y_h + y_p = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{9} \cos 3t \ \ln \cos 3t + \frac{1}{3} t \sin 3t.$$

```
In[666]:= yp = u1  Cos[3t] + u2  Sin[3t]
```

Out[666]= $\frac{1}{9}$ Cos[3 t] Log[Cos[3 t]] + $\frac{1}{3}$ t Sin[3 t]

Identical results are obtained using DSolve.

The negative sign in the output does not affect the result because C[1] is arbitrary.

```
In[667]:= DSolve[y''[t] + 9y[t] == Sec[3t], y[t], t]
```

Out[667]= {{y[t] → C[2] Cos[3 t] + $\frac{1}{9}$ Cos[3 t] × Log[Cos[3 t]] + $\frac{1}{3}$ t Sin[3 t] − C[1] × Sin[3 t]}}

Applying the initial conditions gives us $c_1 = c_2 = 0$ so we conclude that the solution to the initial-value problem is

$$y = \frac{1}{9}\cos 3t \,\ln\cos 3t + \frac{1}{3}t\sin 3t.$$

```
In[668]:= sol = DSolve[
            {y''[t] + 9y[t] == Sec[3t], y[0] == 0, y'[0]
            == 0}, y[t], t]
```

Out[668]= {{y[t] → $\frac{1}{9}$ (Cos[3 t] Log[Cos[3 t]] +3 t Sin[3 t])}}

We graph the solution with Plot in Figure 4-26.

```
In[669]:= Plot[Evaluate[y[t]/.sol], {t, 0, π/6}]
```

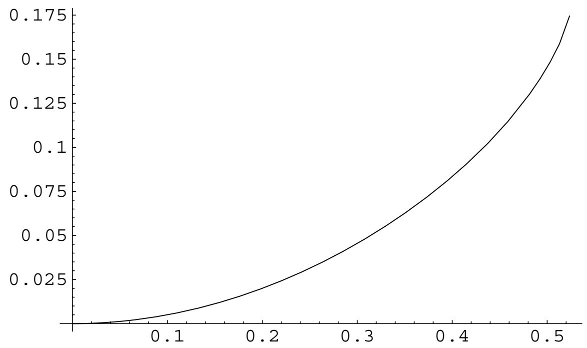


Figure 4-26 The domain of the solution is $-\pi/6 < t < \pi/6$

■

4.5.2 Higher-Order Nonhomogeneous Equations

In the same way as with second-order equations, we assume that a particular solution of the nth-order linear nonhomogeneous equation

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

has the form $y_p = u_1(t)y_1 + u_2(t)y_2 + \cdots + u_n(t)y_n$, where $S = \{y_1,\ y_2, \ldots, y_n\}$ is a fundamental set of solutions to the corresponding homogeneous equation

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_2(t)y'' + a_1(t)y' + a_0(t)y = 0.$$

With the assumptions

$$\begin{aligned} y_1u_1' + y_2u_2' + \cdots + y_nu_n' &= 0 \\ y_1'u_1' + y_2'u_2' + \cdots + y_n'u_n' &= 0 \\ &\vdots \\ y_1^{(n-2)}u_1' + y_2^{(n-2)}u_2' + \cdots + y_n^{(n-2)}u_n' &= 0 \end{aligned} \tag{4.18}$$

we obtain the equation

$$y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' = f(t). \tag{4.19}$$

Equations (4.18) and equation (4.19) form a system of n linear equations in the unknowns $u_1', u_2', \ldots, u_n'$. Applying Cramer's Rule,

$$u_i' = \frac{W_i(S)}{W(S)}, \tag{4.20}$$

where $W(S)$ is given by equation (4.6),

$$W(S) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix},$$

and $W_i(S)$ is the determinant of the matrix obtained by replacing the ith column of

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}.$$

EXAMPLE 4.5.2: Solve $y^{(3)} + 4y' = \sec 2t$.

SOLUTION: A general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2 \cos 2t + c_3 \sin 2t$; a fundamental set is $S = \{1, \cos 2t, \sin 2t\}$ with Wronskian $W(S) = 8$.

```
In[670]:= yh = DSolve[y'''[t] + 4y'[t] == 0, y[t], t]
Out[670]= {{y[t] → C[3] + 1/2 C[1] Cos[2 t]
            + 1/2 C[2] Sin[2 t]}}
In[671]:= s = {1, Cos[2t], Sin[2t]}; ws = {s, D[s, t],
              D[s, {t, 2}]}; MatrixForm[ws]
Out[671]= (1    Cos[2 t]      Sin[2 t]
           0   -2 Sin[2 t]   2 Cos[2 t]
           0   -4 Cos[2 t]  -4 Sin[2 t])
In[672]:= dws = Simplify[Det[ws]]
Out[672]= 8
```

Using variation of parameters to find a particular solution of the nonhomogeneous equation, we let $y_1 = 1$, $y_2 = \cos 2t$, and $y_3 = \sin 2t$ and assume that a particular solution has the form $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$. Using the variation of parameters formula, we obtain

$$u_1' = \frac{1}{8}\begin{vmatrix} 0 & \cos 2t & \sin 2t \\ 0 & -2\sin 2t & 2\cos 2t \\ \sec 2t & -4\cos 2t & -4\sin 2t \end{vmatrix} = \frac{1}{4}\sec 2t$$

so

$$u_1 = \frac{1}{8}\ln|\sec 2t + \tan 2t|,$$

$$u_2' = \frac{1}{8}\begin{vmatrix} 1 & 0 & \sin 2t \\ 0 & 0 & 2\cos 2t \\ 0 & \sec 2t & -4\sin 2t \end{vmatrix} = -\frac{1}{4} \quad \text{so} \quad u_2 = -\frac{1}{4}t,$$

and

$$u_3' = \frac{1}{8}\begin{vmatrix} 1 & \cos 2t & 0 \\ 0 & -2\sin 2t & 0 \\ 0 & -4\cos 2t & \sec 2t \end{vmatrix} = -\frac{1}{2}\tan 2t \quad \text{so} \quad u_3 = \frac{1}{8}\ln|\cos 2t|,$$

where we use `Det` and `Integrate` to evaluate the determinants and integrals. In the case of u_1, the output given by Mathematica looks different than the result we obtained by hand but using properties of logarithms ($\ln(a/b) = \ln a - \ln b$) and trigonometric identities ($\cos^2 x + \sin^2 x = 1$, $\sin 2x = 2\sin x\cos x$, $\cos^2 x - \sin^2 x = \cos 2x$, and the reciprocal identities) shows us that

$$\begin{aligned}\frac{1}{8}\left(\ln|\cos t + \sin t| - \ln|\cos t + \sin t|\right) &= \frac{1}{8}\ln\left|\frac{\cos t + \sin t}{\cos t - \sin t}\right| \\ &= \frac{1}{8}\ln\left|\frac{\cos t + \sin t}{\cos t - \sin t}\cdot\frac{\cos t + \sin t}{\cos t + \sin t}\right| \\ &= \frac{1}{8}\ln\left|\frac{\cos^2 t + 2\cos t\sin t + \sin^2 t}{\cos^2 t - \sin^2 t}\right| \\ &= \frac{1}{8}\ln\left|\frac{1+\sin 2t}{\cos 2t}\right| \\ &= \frac{1}{8}\ln\left|\frac{1}{\cos 2t} + \frac{\sin 2t}{\cos 2t}\right| \\ &= \frac{1}{8}\ln|\sec 2t + \tan 2t|\end{aligned}$$

so the results obtained by hand and with Mathematica are the same.

```
In[673]:= u1p = 1/8 Det[{{0, Cos[2t], Sin[2t]},
               {0, -2 Sin[2t], 2 Cos[2t]},
               {Sec[2t], -4 Cos[2t], -4 Sin[2t]}}]//
             Simplify
```

Out[673]= $\frac{1}{4}$ Sec[2 t]

```
In[674]:= Integrate[u1p, t]
```

Out[674]= $-\frac{1}{8}$ Log[Cos[t] - Sin[t]] + $\frac{1}{8}$ Log[Cos[t] + Sin[t]]

```
In[675]:= u2p = Simplify[1/8 Det[{{1, 0, Sin[2t]},
                 {0, 0, 2 Cos[2t]}, {0, Sec[2t],
                 -4 Sin[2t]}}]]
```

Out[675]= $-\frac{1}{4}$

```
In[676]:= Integrate[u2p, t]
```

Out[676]= $-\frac{t}{4}$

```
In[677]:= u3p = Simplify[1/8 Det[{{1, Cos[2t], 0},
                 {0, -2 Sin[2t], 0}, {0, -4 Cos[2t],
                 Sec[2t]}}]]
```

Out[677]= $-\frac{1}{4}$ Tan[2 t]

```
In[678]:= Integrate[u3p, t]
Out[678]= 1/8 Log[Cos[2 t]]
```

Thus, a particular solution of the nonhomogeneous equation is

$$y_p = \frac{1}{8}\ln|\sec 2t + \tan 2t| - \frac{1}{4}t\cos 2t + \frac{1}{8}\ln|\cos 2t|\sin 2t$$

and a general solution is $y = y_h + y_p$. We verify the calculations using `DSolve`, which returns an equivalent solution.

```
In[679]:= gensol =
            DSolve[y'''[t] + 4y'[t] == Sec[2t],
             y[t], t]//Simplify
Out[679]= {{y[t] → 1/8 (8 C[3] - 2 (t - 2 C[1]) Cos[2 t]
                  - Log[Cos[t] - Sin[t]] + Log[Cos[t]
                  + Sin[t]] + (4 C[2] + Log[Cos[2 t]])
                  × Sin[2 t])}}
```

■

4.6 Cauchy–Euler Equations

Generally, solving an arbitrary differential equation is a formidable, if not impossible task, particularly in the case when the coefficients are not constants. However, we are able to solve certain equations with variable coefficients using techniques similar to those discussed previously.

Definition 19 (Cauchy–Euler Equation). *A* ***Cauchy–Euler differential equation*** *is an equation of the form*

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 xy1 + a_0 y = f(x), \tag{4.21}$$

where $a_0, a_1, \ldots, a_n$ are constants.

4.6.1 Second-Order Cauchy–Euler Equations

Consider the second-order homogeneous Cauchy–Euler equation

$$ax^2y'' + bxy' + cy = 0, \tag{4.22}$$

where $a \neq 0$. Notice that because the coefficient of y'' is zero if $x = 0$, we must restrict our domain to either $x > 0$ or $x < 0$ in order to ensure that the theory of second-order equations stated in Section 4.1 holds.

Suppose that $y = x^m$, $x > 0$, for some constant m. Substitution of $y = x^m$ with derivatives $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$ into equation (4.22) yields

$$\begin{aligned} ax^2y'' + bxy' + cy &= am(m-1)x^m + bmx^m + cx^m \\ &= x^m\left[am(m-1) + bm + c\right] = 0. \end{aligned}$$

Then, $y = x^m$ is a solution of equation (4.22) if m satisfies

$$am(m-1) + bm + c = 0, \tag{4.23}$$

which is called the **characteristic equation** (or **auxiliary equation**) associated with the Cauchy–Euler equation of order two. The solutions of the characteristic equation completely determine the general solution of the homogeneous Cauchy–Euler equation of order two. Let m_1 and m_2 denote the two solutions of the characteristic (or auxiliary) equation (4.23):

$$m_{1,2} = \frac{1}{2a}\left[-(b-a) \pm \sqrt{(b-a)^2 - 4ac}\right].$$

Hence, we can obtain two real roots, one repeated real root, or a complex conjugate pair depending on the values of a, b, and c. We state a general solution that corresponds to the different types of roots.

Theorem 8. *Let m_1 and m_2 be the solutions of equation* (4.23).

1. *If $m_1 \neq m_2$ are real and distinct, two linearly independent solutions of equation* (4.22) *are $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$; a general solution of* (4.22) *is*

$$y = c_1x^{m_1} + c_2x^{m_2}, \ x > 0.$$

2. *If $m_1 = m_2$, two linearly independent solutions of equation* (4.22) *are $y_1 = x^{m_1}$ and $y_2 = x^{m_1}\ln x$; a general solution of* (4.22) *is*

$$y = c_1x^{m_1} + c_2x^{m_1}\ln x, \ x > 0.$$

3. *If $m_{1,2} = \alpha \pm \beta i$, $\beta \neq 0$, two linearly independent solutions of equation* (4.22) *are $y_1 = x^{\alpha}\cos(\beta\ln x)$ and $y_2 = x^{\alpha}\sin(\beta\ln x)$; a general solution of* (4.22) *is*

$$y = x^{\alpha}\left[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)\right], \ x > 0.$$

```
In[680]:= Clear[x, y]

          gensol = DSolve[ax^2 y''[x] + bx y'[x] + c y[x] == 0,
                   y[x], x]
Out[680]= {{y[x] →
          x^(-(-a+b-Sqrt[a^2-2 a b+b^2-4 a c])/(2 a)) C[1] + x^(-(-a+b+Sqrt[a^2-2 a b+b^2-4 a c])/(2 a)) C[2]}}
```

EXAMPLE 4.6.1: Solve each of the following equations: (a) $3x^2y''-2xy'+2y=0$, $x>0$; (b) $x^2y''-xy'+y=0$, $x>0$; (c) $x^2y''-5xy'+10y=0$, $x>0$.

SOLUTION: If $y=x^m$, $y'=mx^{m-1}$ and $y''=m(m-1)x^{m-2}$, substitution into the differential equation yields

$$3x^2y''-2xy'+2y=3x^2\cdot m(m-1)x^{m-2}-2x\cdot mx^{m-1}+2x^m$$
$$=x^m\left[3m(m-1)-2m+2\right]=0.$$

Hence, the auxiliary equation is

$$3m(m-1)-2m+2=3m(m-1)-2(m-1)=(3m-2)(m-1)=0$$

with roots $m_1=2/3$ and $m_2=1$. Therefore, a general solution is $y=c_1x^{2/3}+c_2x$. We obtain the same results with `DSolve`. Entering

```
In[681]:= Clear[x, y]

          gensol = DSolve[3x^2 y''[x] - 2x y'[x] + 2 y[x] == 0,
                   y[x], x]
Out[681]= {{y[x] → x^(2/3) C[1] + x C[2]}}
```

finds a general solution of the equation, naming the result `gensol`, and then entering

```
In[682]:= toplot =
            Table[gensol[[1, 1, 2]] /. {C[1]- > i,
              C[2]- > j}, {i, -2, 2, 2}, {j, -2, 2, 2}];

          grays = Table[GrayLevel[i],
                  {i, 0, 0.7, 0.7/8}];

          Plot[Evaluate[toplot], {x, 0, 12},
            PlotStyle → grays, PlotRange- > {-6, 6},
            AspectRatio- > 1]
```

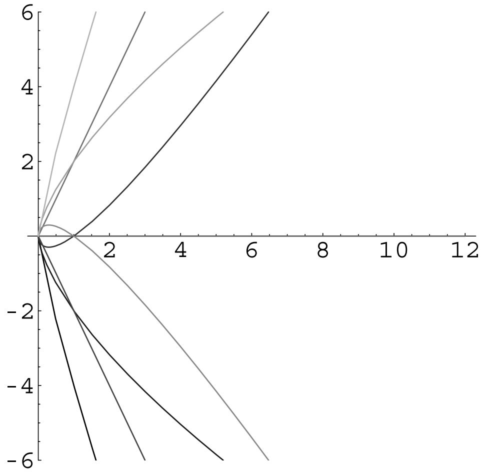


Figure 4-27 Various solutions of $3x^2y'' - 2xy' + 2y = 0$, $x > 0$

defines `toplot` to be the list of functions obtained by replacing `c[1]` in `gensol[[1,1,2]]` by −2, 0, and 2 and `C[2]` in gensol[[1,1,2]] by −2, 0, and 2, and graphs the set of functions `toplot` on the interval [0, 12]. See Figure 4-27.

(b) In this case, the auxiliary equation is

$$m(m-1) - m + 1 = m(m-1) - (m-1) = (m-1)^2 = 0$$

with root $m = 1$ of multiplicity 2. Hence, a general solution is $y = c_x + c_2 x \ln x$. As in the previous example, we see that we obtain the same results with `DSolve`. See Figure 4-28.

```
In[683]:= Clear[x, y]

          gensol = DSolve[x^2 y''[x] - x y'[x] + y[x] == 0,
                     y[x], x]
Out[683]= {{y[x] → x C[1] + x C[2] Log[x]}}

In[684]:= toplot =
            Table[gensol[[1, 1, 2]]/.{C[1]- > i, C[2]- > j},
              {i, -1, 1}, {j, -1, 1}];

          grays = Table[GrayLevel[i],
                    {i, 0, 0.7, 0.7/8}];

          Plot[Evaluate[toplot], {x, 0.01, 10},
            PlotStyle → grays, PlotRange- > {-5, 5},
            AspectRatio- > 1]
```

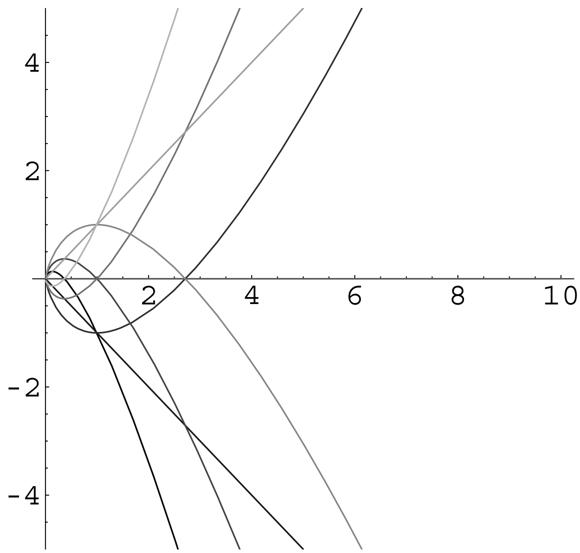


Figure 4-28 Various solutions of $x^2y'' - xy' + y = 0, x > 0$

(c) The auxiliary equation is given by

$$m(m-1) - 5m + 10 = m^2 - 6m + 10 = 0$$

with complex conjugate roots $m_{1,2} = \frac{1}{2}\left(6 \pm \sqrt{36-40}\right) = 3 \pm i$. Thus, a general solution is $y = x^3\,[c_1 \cos(\ln x) + c_2 \sin(\ln x)]$.

Again, we see that we obtain equivalent results with `DSolve`. First, we find a general solution of the equation, naming the resulting output `gensol`.

```
In[685]:= Clear[x, y]

          gensol = DSolve[x^2 y''[x] - 5x y'[x]
                   +10 y[x] == 0, y[x], x]
Out[685]= {{y[x] → x^3 C[2] Cos[Log[x]]
                   +x^3 C[1] Sin[Log[x]]}}
```

Now, we define $y(x)$ to be the general solution obtained in `gensol`. (The same result is obtained with `Part` by entering `y[x_] =gensol [[1,1,2]]`.)

```
In[686]:= y[x_] = x^3 C[2] Cos[Log[x]] -x^3 C[1] Sin[Log[x]];
```

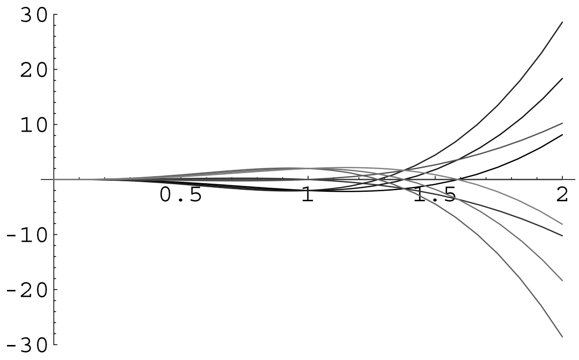


Figure 4-29 Various solutions of $x^2y'' - 5xy' + 10y = 0$, $x > 0$

To find the values of `C[1]` and `C[2]` so that the solution satisfies the initial conditions $y(1) = a$ and $y'(1) = b$, we use `Solve` and name the resulting list `cvals`.

```
In[687]:= cvals = Solve[{y[1] == a, y'[1] == b},
            {C[1], C[2]}]
Out[687]= {{C[1] → 3 a - b, C[2] → a}}
```

The solution to the initial-value problem

$$\begin{cases} x^2y'' - 5xy' + 10y = 0 \\ y(1) = a,\ y'(1) = b \end{cases}$$

is obtained by replacing `C[1]` and `C[2]` in $y(x)$ by the values found in `cvals`.

```
In[688]:= y[x_] = y[x]/.cvals[[1]]
Out[688]= a x^3 Cos[Log[x]] - (3 a - b) x^3 Sin[Log[x]]
```

This solution is then graphed for various initial conditions in Figure 4-29.

Note that when you enter the following `Plot` command, Mathematica may display several error messages because each solution is undefined if $x = 0$. Nevertheless, the resulting graphs are displayed correctly.

```
In[689]:= toplot = Table[y[x], {a, -2, 2, 2}, {b, -2, 2, 2}];

          grays = Table[GrayLevel[i], {i, 0, 0.5, 0.5/8}];

          Plot[Evaluate[toplot], {x, 0, 2},
            PlotStyle → grays, PlotRange → All]
```

■

4.6.2 Higher-Order Cauchy–Euler Equations

The auxiliary equation of higher-order Cauchy–Euler equations is defined in the same way and solutions of higher-order homogeneous Cauchy–Euler equations are determined in the same manner as solutions of higher-order homogeneous differential equations with constant coefficients. In the case of higher-order Cauchy–Euler equations, note that if a real root r of the auxiliary equation is repeated m times, m linearly independent solutions that correspond to r are x^r, $x^r \ln x$, $x^r (\ln x)^2$, $\ldots$, $x^r (\ln x)^{m-1}$; solutions corresponding to repeated complex roots are generated similarly.

EXAMPLE 4.6.2: Solve $2x^3y''' - 4x^2y'' - 20xy' = 0$, $x > 0$.

SOLUTION: In this case, if we assume that $y = x^m$ for $x > 0$, we have the derivatives $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$, and $y''' = m(m-1)(m-2)x^{m-3}$. Substitution into the differential equation and simplification then yields $\left(2m^3 - 10m^2 - 12m\right)x^m = 0$.

```
In[690]:= Clear[x, y]

          eq = 2 x^3 y^(3)[x] - 4x^2 y''[x] - 20x y'[x] == 0
Out[690]= -20 x y'[x] - 4 x^2 y''[x] + 2 x^3 y^(3)[x] == 0

In[691]:= y[x_] = x^m
Out[691]= x^m

In[692]:= eq
Out[692]= -20 m x^m - 4 (-1 + m) m x^m + 2 (-2 + m) (-1 + m) m x^m == 0

In[693]:= Factor[eq[[1]]]
Out[693]= 2 (-6 + m) m (1 + m) x^m
```

We must solve $2m^3 - 10m^2 - 12m = 2m(m+1)(m-6) = 0$ for m because $x^m \neq 0$.

```
In[694]:= mvals = Solve[eq, m]
Out[694]= {{m → -1}, {m → 0}, {m → 6}}
```

We see that the solutions are $m_1 = 0$, $m_2 = -1$, and $m_3 = 6$, so a general solution of the equation is $y = c_1 + c_2x^{-1} + c_3x^6$. As in the previous examples, we see that we obtain the same results with `DSolve`.

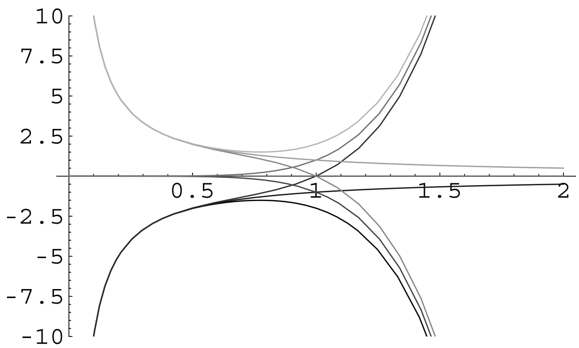


Figure 4-30 Various solutions of $2x^3y''' - 4x^2y'' - 20xy' = 0$, $x > 0$

```
In[695]:= Clear[x, y]

          gensol =
            DSolve[2 x^3 y^(3)[x] - 4x^2 y''[x] - 20x y'[x] == 0,
              y[x], x]
Out[695]= {{y[x] → 1/6 x^6 C[1] - C[2]/x + C[3]}}
```

We graph this solutions for various values of the arbitrary constants in the same way as we graph solutions of other equations. See Figure 4-30.

```
In[696]:= toplot = Table[c[1]/x + c[2] + x^6 c[3]/.c[2] → 0,
            {c[1], -1, 1}, {c[3], -1, 1}];

          grays = Table[GrayLevel[i], {i, 0, 0.7, 0.7/8}];

          Plot[Evaluate[toplot], {x, 0, 2},
            PlotRange → {-10, 10}, PlotStyle → grays]
```

■

EXAMPLE 4.6.3: Solve the initial-value problem

$$\begin{cases} x^4y^{(4)} + 4x^3y''' + 11x^2y'' - 9xy' + 9y = 0, \ x > 0 \\ y(1) = 1, \ y'(1) = -9, \ y''(1) = 27, \ y'''(1) = 1. \end{cases}$$

SOLUTION: Substitution of $y = x^m$ into the differential equation $x^4 y^{(4)} + 4x^3 y''' + 11x^2 y'' - 9xy' + 9y = 0$ and simplification leads to the equation

$$\left(m^4 - 2m^3 + 10m^2 - 18m + 9\right) x^m = 0.$$

```
In[697]:= eq = x^4 ∂{x,4}y[x] + 4 x^3 ∂{x,3}y[x] + 11 x^2 (y′)′[x]
             -9 x y′[x] + 9 y[x] == 0;

In[698]:= y[x_] = x^m;

In[699]:= Factor[eq[[1]]]
Out[699]= (-1 + m)^2 (9 + m^2) x^m
```

We solve

$$m^4 - 2m^3 + 10m^2 - 18m + 9 = \left(m^2 + 9\right)(m - 1)^2 = 0$$

for m because $x^m \neq 0$.

```
In[700]:= Solve[eq, m]
Out[700]= {{m → -3 i}, {m → 3 i}, {m → 1}, {m → 1}}
```

Hence, $m_{1,2} = \pm 3i$, and $m_{3,4} = 1$ is a root of multiplicity 2, so a general solution of the differential equation is

$$y = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x) + c_3 x + c_4 x \ln x$$

with first, second, and third derivatives computed as follows.

```
In[701]:= y[x_] = c1 Cos[3 Log[x]] + c2 Sin[3 Log[x]]
              +c3 x + c4 x Log[x];

In[702]:= Simplify[y′[x]]

          Simplify[y″[x]]

          Simplify[y^(3)[x]]

Out[702]= 1/x (-3 Sin[3 Log[x]] c1
          +3 Cos[3 Log[x]] c2 + x (c3 + (1 + Log[x]) c4))
Out[702]= 1/x^2 ((-9 Cos[3 Log[x]] + 3 Sin[3 Log[x]]) c1
          -3 (Cos[3 Log[x]] + 3 Sin[3 Log[x]]) c2 + x c4)
Out[702]= 1/x^3 (3 (9 Cos[3 Log[x]] + 7 Sin[3 Log[x]]) c1
          -3 (7 Cos[3 Log[x]] - 9 Sin[3 Log[x]]) c2
          -x c4)
```

Substitution of the initial conditions then yields the system of equations,

$$\begin{cases} c_1 + c_3 = 1 \\ 3c_2 + c_3 + c_4 = -9 \\ -9c_1 - 3c_2 + c_4 = 27 \\ 27c_1 - 21c_2 - c_4 = 1 \end{cases}$$

which has the solution $(c_1, c_2, c_3, c_4) = (-12/5, -89/30, 17/5, -7/2)$.

```
In[703]:= cvals =
            Solve[{y[1] == 1, y'[1] == -9, y''[1] == 27,
              y^(3)[1] == 1}]
```

$$\text{Out[703]=}\ \left\{\left\{c_1 \to -\frac{12}{5},\ c_3 \to \frac{17}{5},\ c_2 \to -\frac{89}{30},\ c_4 \to -\frac{7}{2}\right\}\right\}$$

Therefore, the solution to the initial-value problem is

$$y = -\frac{12}{5}\cos(3\ln x) - \frac{89}{30}\sin(3\ln x) + \frac{17}{5}x - \frac{7}{2}x\ln x.$$

```
In[704]:= y[x_] = y[x]/.cvals[[1]]
```

$$\text{Out[704]=}\ \frac{17\,x}{5} - \frac{12}{5}\,\text{Cos}[3\,\text{Log}[x]] - \frac{7}{2}\,x\,\text{Log}[x] - \frac{89}{30}\,\text{Sin}[3\,\text{Log}[x]]$$

Mathematica may display several error messages because the solution is undefined if $x = 0$.

We graph this solution with `Plot` in Figure 4-31. From the graph shown in Figure 4-31, we see that it *might appear* to be the case that $\lim_{x \to 0^+} y(x)$ exists.

```
In[705]:= Plot[y[x], {x, 0, 1}]
```

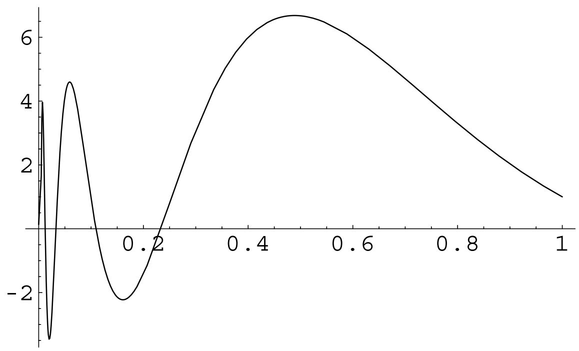


Figure 4-31 Plot of the solution to the initial-value problem

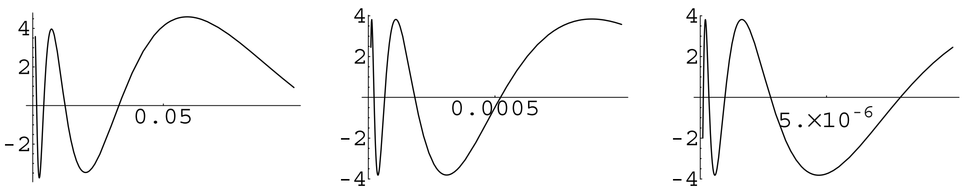


Figure 4-32 Zooming in near $x = 0$ helps convince us that $\lim_{x \to 0^+} y(x)$ does not exist

However, when we graph the solution on "small" intervals close to the origin as shown in Figure 4-32, we see that $\lim_{x \to 0^+} y(x)$ does not exist.

```
In[706]:= p1 = Plot[y[x], {x, 0.001, 0.1},
                Ticks → {{0.05}, Automatic},
                DisplayFunction → Identity];

           p2 = Plot[y[x], {x, 0.00001, 0.001},
                Ticks → {{0.0005}, Automatic},
                DisplayFunction → Identity];

           p3 = Plot[y[x], {x, 1. 10^-7, 0.00001},
                Ticks → {{5. 10^-6}, Automatic},
                DisplayFunction → Identity];

           Show[GraphicsArray[{p1, p2, p3}]]
```

As expected, we see that `DSolve` can be used to solve the initial-value problem directly.

```
In[707]:= Clear[x, y]

           partsol =
             DSolve[{eq, y[1] == 1, y'[1] == -9, y''[1] == 27,
                y^(3)[1] == 1}, y[x], x]
```

$$\text{Out[707]=}\ \left\{\left\{y[x] \to \frac{1}{30}\ (102\,x - 72\ \text{Cos}[3\ \text{Log}[x]] - 105\,x\ \text{Log}[x] - 89\ \text{Sin}[3\ \text{Log}[x]])\right\}\right\}$$

■

4.6.3 Variation of Parameters

Of course, Cauchy–Euler equations can be nonhomogeneous in which case the method of variation of parameters can be used to solve the problem.

EXAMPLE 4.6.4: Solve $x^2y'' - xy' + 5y = x$, $x > 0$.

SOLUTION: We first note that `DSolve` can be used to find a general solution of the equation directly.

```
In[708]:= Clear[x, y, gensol]

          gensol = DSolve[x^2 y''[x] - x y'[x] + 5 y[x] == x,
                   y[x], x]
Out[708]= {{y[x] → x C[2] Cos[2 Log[x]] + x C[1]
              × Sin[2 Log[x]] + 1/4 (2 x Cos[Log[x]]^2
              × Cos[2 Log[x]] + x Sin[2 Log[x]]^2)}}
```

Alternatively, we can use Mathematica to help us implement variation of parameters. We begin by finding a general solution to the corresponding homogeneous equation $x^2y'' - xy' + 5y = 0$ with `DSolve`.

```
In[709]:= homsol = DSolve[x^2 y''[x] - x y'[x] + 5 y[x] == 0,
                   y[x], x]
Out[709]= {{y[x] → x C[2] Cos[2 Log[x]]
             +x C[1] Sin[2 Log[x]]}}
```

We see that a general solution of the corresponding homogeneous equation is $y_h = x[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)]$. A fundamental set of solutions for the homogeneous equation is $S = \{x\cos(2\ln x),\ x\sin(2\ln x)\}$

```
In[710]:= y1[x_] = x Cos[2 Log[x]];

          y2[x_] = x Sin[2 Log[x]];

In[711]:= caps = {y1[x], y2[x]};
```

and the Wronskian is $W(S) = 2x$.

```
In[712]:= ws = Simplify[Det[{caps, ∂x caps}]]
Out[712]= 2 x
```

To implement variation of parameters, we rewrite the equation in the standard form

$$y'' - \frac{1}{x}y' + \frac{5}{x^2}y = \frac{1}{x}$$

by dividing by x^2 and identify $f(x) = 1/x$. We then use `Integrate` to compute

$$u_1 = \int \frac{-y_2(x)f(x)}{2x}dx \qquad \text{and} \qquad u_2 = \int \frac{y_1(x)f(x)}{2x}dx.$$

```
In[713]:= f[x_] = 1/x;

In[714]:= u1prime = -y2[x] f[x]/ws

          u2prime = y1[x] f[x]/ws
Out[714]= -Sin[2 Log[x]]/(2 x)
Out[714]= Cos[2 Log[x]]/(2 x)

In[715]:= u1[x_] = ∫ u1prime dx

          u2[x_] = ∫ u2prime dx
Out[715]= 1/2 Cos[Log[x]]^2
Out[715]= 1/4 Sin[2 Log[x]]
```

A particular solution of the nonhomogeneous equation is given by $y_p = y_1u_1 + y_2u_2$

```
In[716]:= yp[x_] = y1[x]u1[x] + y2[x]u2[x]//Simplify
Out[716]= 1/2 x Cos[Log[x]]^2
```

and a general solution is given by $y = y_h + y_p$.

```
In[717]:= y[x_] = c1 x Cos[2 Log[x]] + c2 x Sin[2 Log[x]]
                 +yp[x]
Out[717]= 1/2 x Cos[Log[x]]^2 + x Cos[2 Log[x]] c1
          +x Sin[2 Log[x]] c2
```

As in previous examples, we graph this general solution for various values of the arbitrary constants. See Figure 4-33.

```
In[718]:= toplot = Table[y[x], {c1 , -3, 3}, {c2, -3, 3}];

          grays = Table[GrayLevel[i], {i, 0, 0.7, 0.7/8}];

          Plot[Evaluate[toplot], {x, 0, 2},
            PlotStyle → grays]
```

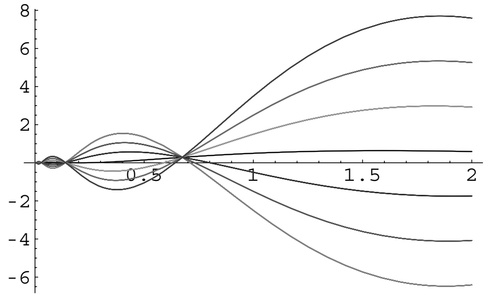


Figure 4-33 Various solutions of a nonhomogeneous Cauchy–Euler equation

■

4.7 Series Solutions

In calculus we learn that Maclaurin and Taylor polynomials can be used to approximate functions. This idea can be extended to approximating the solution of a differential equation. First, we introduce some necessary terminology.

4.7.1 Power Series Solutions about Ordinary Points

Definition 20 (Standard Form, Ordinary, and Singular Points). *Consider the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ and let $p(x) = a_1(x)/a_2(x)$ and $q(x) = a_0(x)/a_2(x)$. Then, the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ is equivalent to $y'' + p(x)y' + q(x)y = 0$, which is called the* ***standard form*** *of the equation. A number x_0 is an* ***ordinary point*** *of this differential equation if both $p(x)$ and $q(x)$ are analytic at x_0. If x_0 is not an ordinary point, x_0 is called a* ***singular point****.*

If x_0 is an ordinary point of the differential equation $y'' + p(x)y' + q(x)y = 0$, we can write $p(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$, where $b_n = p^{(n)}(x_0)/n!$, and $q(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, where $c_n = q^{(n)}(x_0)/n!$. Substitution into the equation $y'' + p(x)y' + q(x)y = 0$ results in

$$y'' + y' \sum_{n=0}^{\infty} b_n (x - x_0)^n + y \sum_{n=0}^{\infty} c_n (x - x_0)^n = 0.$$

If we assume that y is analytic at x_0, we can write $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$. Because a power series can be differentiated term-by-term, we can compute the first and second derivatives of y and substitute back into the equation to calculate the coefficients a_n. Thus, we obtain a power series solution of the equation.

Power Series Solution Method about an Ordinary Point

1. Assume that $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.
2. After taking the appropriate derivatives, substitute $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ into the differential equation.
3. Find the unknown series coefficients a_n.
4. When applicable, apply any given initial conditions.

Because the differentiation of power series is necessary in this method for solving differential equations, we should make a few observations about this procedure. Consider the Maclaurin series $y = \sum_{n=0}^{\infty} a_n x^n$. Term-by-term differentiation of this series yields $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$. Notice, however, that with the initial index value of $n = 0$, the first term of the series is 0 so we rewrite the series in its equivalent form

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Similarly,

$$y'' = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n.$$

We make use of these derivatives throughout the section.

EXAMPLE 4.7.1: (a) Find a general solution of $\left(4 - x^2\right) y' + y = 0$. (b) Solve the initial-value problem $\begin{cases} \left(4 - x^2\right) y' + y = 0 \\ y(0) = 1. \end{cases}$

SOLUTION: Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then term-by-term differentiation yields $y' = dy/dx = \sum_{n=0}^{\infty} n a_n x^{n-1}$ and substitution into the differential equation gives us

$$\begin{aligned}\left(4 - x^2\right) \frac{dy}{dx} + y &= \left(4 - x^2\right) \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} 4 n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0.\end{aligned}$$

Note that the first term in these three series involves x^0, x^2, and x^0, respectively. Thus, if we pull off the first two terms in the first and third series, all three series will begin with an x^2 term. Doing so, we have

$$(4a_1 + a_0) + (8a_2 + a_1)\,x + \sum_{n=3}^{\infty} 4na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=2}^{\infty} a_n x^n = 0.$$

Unfortunately, the indices of these three series do not match, so we must change two of the three to match the third. Substitution of $n+1$ for n in $\sum_{n=3}^{\infty} 4na_n x^{n-1}$ yields

$$\sum_{n+1=3}^{\infty} 4(n+1)a_{n+1}x^{n+1-1} = \sum_{n=2}^{\infty} 4(n+1)a_{n+1}x^n.$$

Similarly, substitution of $n-1$ for n in $\sum_{n=1}^{\infty} na_n x^{n+1}$ yields

$$\sum_{n-1=1}^{\infty} (n-1)a_{n-1}x^{n-1+1} = \sum_{n=2}^{\infty} (n-1)a_{n-1}x^n.$$

Therefore, after combining the three series, we have the equation

$$(4a_1 + a_0) + (8a_2 + a_1)\,x + \sum_{n=2}^{\infty} \left[a_n + 4(n+1)a_{n+1} - (n-1)a_{n-1}\right] x^n = 0.$$

Because the sum of the terms on the left-hand side of the equation is zero, each coefficient must be zero. Equating the coefficients of x^0 and x to zero yields $a_1 = -\frac{1}{4}a_0$ and $a_2 = -\frac{1}{8}a_1 = \frac{1}{32}a_0$. When the series coefficient $a_n + 4(n+1)a_{n+1} - (n-1)a_{n-1}$ is set to zero, we obtain the recurrence relation $a_{n+1} = \dfrac{(n-1)a_{n-1} - a_n}{4(n+1)}$ for the indices in the series, $n \geq 2$. After defining the recursively defined function a,

```
In[719]:= Clear[a, n]

          a[n_] := a[n] = ((n - 2) a[n - 2] - a[n - 1])/(4n);

          a[0] = a0;

          a[1] = -a0/4;
```

we use the formula to determine the values of a_n for $n = 2, 3, \ldots,$ 11, and give these values in the following table. In this case, note that we define a using the form `a[n_]:=a[n]=...` so that Mathematica "remembers" the values of `a[n]` computed. Thus, for particular values of n, Mathematica need not recompute `a[n-1]` and `a[n-2]` to compute `a[n]` if these values have previously been computed.

```
In[720]:= TableForm[Table[{n, a[n]}, {n, 0, 11}]]
```

Out[720]=

0	a0
1	$-\frac{a0}{4}$
2	$\frac{a0}{32}$
3	$-\frac{3\,a0}{128}$
4	$\frac{11\,a0}{2048}$
5	$-\frac{31\,a0}{8192}$
6	$\frac{69\,a0}{65536}$
7	$-\frac{187\,a0}{262144}$
8	$\frac{1843\,a0}{8388608}$
9	$-\frac{4859\,a0}{33554432}$
10	$\frac{12767\,a0}{268435456}$
11	$-\frac{32965\,a0}{1073741824}$

Therefore,

$$y = a_0 - \frac{1}{4}a_0x + \frac{1}{32}a_0x^2 - \frac{3}{128}a_0x^3 + \frac{11}{2048}a_0x^4 - \frac{31}{8192}a_0x^5 + \cdots$$

(b) When we apply the initial condition $y(0) = 1$, we substitute $x = 0$ into the general solution obtained in (a) and set the result equal to 1. Hence, $a_0 = 1$,

```
In[721]:= a0 = 1;

          TableForm[Table[{n, a[n]}, {n, 0, 11}]]
```

Out[721]=

0	1
1	$-\frac{1}{4}$
2	$\frac{1}{32}$
3	$-\frac{3}{128}$
4	$\frac{11}{2048}$
5	$-\frac{31}{8192}$
6	$\frac{69}{65536}$
7	$-\frac{187}{262144}$
8	$\frac{1843}{8388608}$
9	$-\frac{4859}{33554432}$
10	$\frac{12767}{268435456}$
11	$-\frac{32965}{1073741824}$

so the series solution of the initial-value problem is

$$y = 1 - \frac{1}{4}x + \frac{1}{32}x^2 - \frac{3}{128}x^3 + \frac{11}{2048}x^4 - \frac{31}{8192}x^5 + \cdots$$

Notice that the equation $\left(4 - x^2\right) y' + y = 0$ is separable, so we can compute the solution directly with separation of variables by rewriting the equation as $-\frac{1}{y}dy = \frac{1}{4 - x^2}dx$. Integrating yields $\ln y = \frac{1}{4}(\ln|x - 2| - \ln|x + 2|) = + C \ln\left|\frac{x-2}{x+2}\right|^{1/4} + C$. Applying the initial condition $y(0) = 1$ results in $y = \left|\frac{x-2}{x+2}\right|^{1/4}$. Nearly identical results are obtained with `DSolve`.

```
In[722]:= Clear[x, y]

exactsol =
  DSolve[{(4 - x^2) y'[x] + y[x] == 0, y[0] == 1},
   y[x], x]
```

Out[722]= $\left\{\left\{y[x] \to \frac{(1 - i)(-2 + x)^{1/4}}{\sqrt{2}\,(2 + x)^{1/4}}\right\}\right\}$

```
In[723]:= formula = Simplify[-((-1)^(3/4) (-2 + x)^(1/4))/(2 + x)^(1/4)]
```

Out[723]= $-(-1)^{3/4}\left(\frac{-2 + x}{2 + x}\right)^{1/4}$

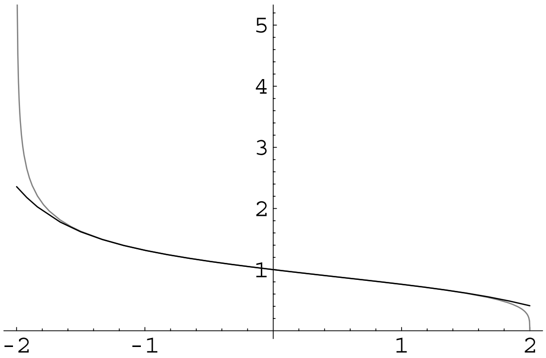


Figure 4-34 Comparison of exact solution to a polynomial approximation of the solution

We can approximate the solution of the problem by taking a finite number of terms of the series solution.

```
In[724]:= yapprox = Sum[a[i] x^i, {i, 0, 10}]
```

$$\text{Out[724]}= 1-\frac{x}{4}+\frac{x^2}{32}-\frac{3x^3}{128}+\frac{11x^4}{2048}-\frac{31x^5}{8192}+\frac{69x^6}{65536}-\frac{187x^7}{262144}+\frac{1843x^8}{8388608}-\frac{4859x^9}{33554432}+\frac{12767x^{10}}{268435456}$$

The graph of the polynomial approximation of degree 10 is shown in Figure 4-34 along with the solution obtained through separation of variables.

```
In[725]:= p1 = Plot[{formula, yapprox}, {x, -2, 2},
             PlotStyle → {GrayLevel[0.5],
             GrayLevel[0]}]
```

The graph shows that the accuracy of the approximation decreases near $x = \pm 2$, which are the singular points of the differential equation. (The reason for this is discussed in the theorem following this example.)

Alternatively, we can take advantage of `Series` to help us form a series solution of the problem.

First, we use `Series` to compute the first few terms of the power series expansion for the left-hand side of the equation about $x = 0$ and name the result `serapprox`.

```
In[726]:= Clear[x, y]

          serapprox = Series[(4 - x^2) y'[x] + y[x],
            {x, 0, 10}]
```

$$\begin{aligned}
\textit{Out[726]=}\ & (y[0]+4y'[0])+(y'[0]+4y''[0])\,x \\
&+\left(-y'[0]+\frac{y''[0]}{2}+2y^{(3)}[0]\right)x^2 \\
&+\left(-y''[0]+\frac{1}{6}y^{(3)}[0]+\frac{2}{3}y^{(4)}[0]\right)x^3 \\
&+\left(-\frac{1}{2}y^{(3)}[0]+\frac{1}{24}y^{(4)}[0]+\frac{1}{6}y^{(5)}[0]\right)x^4 \\
&+\left(-\frac{1}{6}y^{(4)}[0]+\frac{1}{120}y^{(5)}[0]+\frac{1}{30}y^{(6)}[0]\right)x^5 \\
&+\left(-\frac{1}{24}y^{(5)}[0]+\frac{1}{720}y^{(6)}[0]+\frac{1}{180}y^{(7)}[0]\right)x^6 \\
&+\left(-\frac{1}{120}y^{(6)}[0]+\frac{y^{(7)}[0]}{5040}+\frac{y^{(8)}[0]}{1260}\right)x^7 \\
&+\left(-\frac{1}{720}y^{(7)}[0]+\frac{y^{(8)}[0]}{40320}+\frac{y^{(9)}[0]}{10080}\right)x^8 \\
&+\left(-\frac{y^{(8)}[0]}{5040}+\frac{y^{(9)}[0]}{362880}+\frac{y^{(10)}[0]}{90720}\right)x^9 \\
&+\left(-\frac{y^{(9)}[0]}{40320}+\frac{y^{(10)}[0]}{3628800}+\frac{y^{(11)}[0]}{907200}\right)x^{10}+0[x]^{11}
\end{aligned}$$

Then, we use `LogicalExpand` to form the system of equations obtained by equating each coefficient in `serapprox` to zero.

```
In[727]:= sysofeqs = LogicalExpand[serapprox == 0]
```

$$\begin{aligned}
\textit{Out[727]=}\ & y[0]+4y'[0]==0\,\&\&\,y'[0]+4y''[0]==0\,\&\& \\
&-y'[0]+\frac{y''[0]}{2}+2y^{(3)}[0]==0\,\&\& \\
&-y''[0]+\frac{1}{6}y^{(3)}[0]+\frac{2}{3}y^{(4)}[0]==0\,\&\& \\
&-\frac{1}{2}y^{(3)}[0]+\frac{1}{24}y^{(4)}[0]+\frac{1}{6}y^{(5)}[0]==0\,\&\& \\
&-\frac{1}{6}y^{(4)}[0]+\frac{1}{120}y^{(5)}[0]+\frac{1}{30}y^{(6)}[0]==0\,\&\& \\
&-\frac{1}{24}y^{(5)}[0]+\frac{1}{720}y^{(6)}[0]+\frac{1}{180}y^{(7)}[0]==0\,\&\& \\
&-\frac{1}{120}y^{(6)}[0]+\frac{y^{(7)}[0]}{5040}+\frac{y^{(8)}[0]}{1260}==0\,\&\& \\
&-\frac{1}{720}y^{(7)}[0]+\frac{y^{(8)}[0]}{40320}+\frac{y^{(9)}[0]}{10080}==0\,\&\& \\
&-\frac{y^{(8)}[0]}{5040}+\frac{y^{(9)}[0]}{362880}+\frac{y^{(10)}[0]}{90720}==0\,\&\& \\
&-\frac{y^{(9)}[0]}{40320}+\frac{y^{(10)}[0]}{3628800}+\frac{y^{(11)}[0]}{907200}==0
\end{aligned}$$

We want to solve this system of equations for $y'(0)$, $y''(0)$, ..., $y^{(11)}(0)$ so that the results are in terms of $y(0)$. (Note that the symbol $\partial_{\{x,i\}}y[x]$

represents `D[y[x],{x,i}]` so the same result is obtained by entering `vars=Table[D[y[x],{x, i}]/.(x -> 0),{i,1,11}]`.)

In[728] := **vars = Table[$\partial_{\{x,i\}}$y[x]/.x → 0, {i, 1, 11}]**

Out[728]= $\{y'[0], y''[0], y^{(3)}[0], y^{(4)}[0], y^{(5)}[0], y^{(6)}[0], y^{(7)}[0], y^{(8)}[0], y^{(9)}[0], y^{(10)}[0], y^{(11)}[0]\}$

We then use `Solve` to solve the system of equations `sysofeqs` for the unknowns specified in `vars`.

In[729] := **sols = Solve[sysofeqs, vars]**

Out[729]= $$\Big\{\Big\{y'[0] \to -\frac{y[0]}{4}, y''[0] \to \frac{y[0]}{16}, y^{(3)}[0] \to -\frac{9\,y[0]}{64}, y^{(4)}[0] \to \frac{33\,y[0]}{256}, y^{(5)}[0] \to -\frac{465\,y[0]}{1024}, y^{(6)}[0] \to \frac{3105\,y[0]}{4096}, y^{(7)}[0] \to -\frac{58905\,y[0]}{16384}, y^{(8)}[0] \to \frac{580545\,y[0]}{65536}, y^{(9)}[0] \to -\frac{13775265\,y[0]}{262144}, y^{(10)}[0] \to \frac{180972225\,y[0]}{1048576}, y^{(11)}[0] \to -\frac{5140067625\,y[0]}{4194304}\Big\}\Big\}$$

The power series solution is formed by substituting these values into the power series for $y(x)$ about $x = 0$ with `ReplaceAll` (`/.`).

In[730] := **sersol = Series[y[x], {x, 0, 11}]/.sols[[1]]**

Out[730]= $$y[0] - \frac{1}{4}y[0]\,x + \frac{1}{32}y[0]\,x^2 - \frac{3}{128}y[0]\,x^3 + \frac{11\,y[0]\,x^4}{2048} - \frac{31\,y[0]\,x^5}{8192} + \frac{69\,y[0]\,x^6}{65536} - \frac{187\,y[0]\,x^7}{262144} + \frac{1843\,y[0]\,x^8}{8388608} - \frac{4859\,y[0]\,x^9}{33554432} + \frac{12767\,y[0]\,x^{10}}{268435456} - \frac{32965\,y[0]\,x^{11}}{1073741824} + O[x]^{12}$$

The solution to the initial-value problem is obtained by replacing each occurrence of $y(0)$ in `sersol` by 1.

In[731] := **sol = sersol/.y[0] → 1**

Out[731]= $$1 - \frac{x}{4} + \frac{x^2}{32} - \frac{3x^3}{128} + \frac{11x^4}{2048} - \frac{31x^5}{8192} + \frac{69x^6}{65536} - \frac{187x^7}{262144} + \frac{1843x^8}{8388608} - \frac{4859x^9}{33554432} + \frac{12767x^{10}}{268435456} - \frac{32965x^{11}}{1073741824} + O[x]^{12}$$

Remember that this result cannot be evaluated for particular values of x because of the `O`-term indicating the omitted higher-order terms of the series. To obtain an approximation of the solution that can be evaluated for particular values of x, use `Normal` to remove the `O`-term.

```
In[732]:= polyapprox = Normal[sol]
```

$$\text{Out[732]}= 1-\frac{x}{4}+\frac{x^2}{32}-\frac{3x^3}{128}+\frac{11x^4}{2048}-\frac{31x^5}{8192}+\frac{69x^6}{65536}-\frac{187x^7}{262144}+\frac{1843x^8}{8388608}-\frac{4859x^9}{33554432}+\frac{12767x^{10}}{268435456}-\frac{32965x^{11}}{1073741824}$$

■

The following theorem explains where the approximation of the solution of the differential equation by the series is valid.

A proof of this theorem can be found in more advanced texts, such as Rabenstein's *Introduction to Ordinary Differential Equations*, [22].

Theorem 9 (Convergence of a Power Series Solution). *Let $x = x_0$ be an ordinary point of the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ and suppose that R is the distance from $x = x_0$ to the closest singular point of the equation. Then the power series solution $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges at least on the interval $(x_0 - R, x_0 + R)$.*

The theorem indicates that the approximation may not be as accurate near singular points of the equation. Hence, we understand why the approximation in Example 4.7.1 breaks down near $x = \pm 2$, the closest singular point to the ordinary point $x = 0$. Of course, $x = 0$ is not an ordinary point for every differential equation. However, because the series $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is easier to work with if $x_0 = 0$, we can always make a transformation so that we can use $y = \sum_{n=0}^{\infty} a_n x^n$ to solve any linear equation. For example, suppose that $x = x_0$ is an ordinary point of a linear equation. Then, if we make the change of variable $t = x - x_0$, then $t = 0$ corresponds to $x = x_0$, so $t = 0$ is an ordinary point of the transformed equation.

EXAMPLE 4.7.2 (Legendre's Equation): Legendre's equation is the equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + k(k+1)y = 0, \tag{4.24}$$

where k is a constant, named after the French mathematician Adrien Marie Legendre (1752–1833). Find a general solution of Legendre's equation.

SOLUTION: In standard form, the equation is

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{k(k+1)}{1-x^2}y = 0.$$

There is a solution to the equation of the form $y = \sum_{n=0}^{\infty} a_n x^n$ because $x = 0$ is an ordinary point. This solution will converge at least on the interval $(-1, 1)$ because the closest singular points to $x = 0$ are $x = \pm 1$.

Substitution of this function and its derivatives

$$y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \qquad \text{and} \qquad y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$$

into Legendre's equation (4.24) and simplifying the results yields

$$[2a_2 + k(k+1)a_0] + [-2a_1 + k(k+1)a_1 + 6a_3]\,x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2}$$
$$- \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} 2na_n x^n + \sum_{n=2}^{\infty} k(k+1)a_n x^n = 0.$$

After substituting $n+2$ for each occurrence of n in the first series and simplifying, we have

$$[2a_2 + k(k+1)a_0] + [-2a_1 + k(k+1)a_1 + 6a_3]\,x + \sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2}$$
$$+ [-n(n-1) - 2n + k(k+1)]\,a_n\}\,x^n = 0.$$

Equating the coefficients to zero, we find a_2, a_3, and a_{n+2} with `Solve`.

```
In[733]:= Clear[a, k]

          Solve[2a₂ + k(k + 1) a₀ == 0, a₂]
```

$$\text{Out[733]= } \left\{\left\{a_2 \to -\frac{1}{2}\,k\,(1+k)\,a_0\right\}\right\}$$

```
In[734]:= Solve[-2a₁ + k(k + 1) a₁ + 6a₃ == 0, a₃]
```

$$\text{Out[734]= } \left\{\left\{a_3 \to \frac{1}{6}\,(2\,a_1 - k\,a_1 - k^2\,a_1)\right\}\right\}$$

```
In[735]:= genform =
            Solve[
              (n + 2) (n + 1) aₙ₊₂ + (-n (n - 1)
              -2n + k (k + 1)) aₙ == 0, aₙ₊₂]
```

$$\text{Out[735]= } \left\{\left\{a_{2+n} \to -\frac{(k + k^2 - n - n^2)\,a_n}{2 + 3\,n + n^2}\right\}\right\}$$

```
In[736]:= Factor[genform[[1, 1, 2]]]
```

$$\text{Out[736]= } -\frac{(k-n)\,(1+k+n)\,a_n}{(1+n)\,(2+n)}$$

We obtain a formula for a_n by replacing each occurrence of n in a_{n+2} by $n-2$.

```
In[737]:= genform[[1,1,2]]/.n- >n-2
```

$$\text{Out[737]}= -\frac{(2+k+k^2-(-2+n)^2-n)\,a_{-2+n}}{2+3\,(-2+n)+(-2+n)^2}$$

Using this formula, we find several coefficients with `Table`.

$$\text{In[738]}:= a_{n\_} := a_n = \frac{(-2-k-k^2+(-2+n)^2+n)\,a_{-2+n}}{(-1+n)\,n};$$

```
a1 = a1;
a0 = a0;
```

```
In[739]:= Table[{n,an},{n,2,10}]//TableForm
```

Out[739]=

2	$\frac{1}{2}\,a0\,(-k-k^2)$
3	$\frac{1}{6}\,a1\,(2-k-k^2)$
4	$\frac{1}{24}\,a0\,(-k-k^2)\,(6-k-k^2)$
5	$\frac{1}{120}\,a1\,(2-k-k^2)\,(12-k-k^2)$
6	$\frac{1}{720}\,a0\,(-k-k^2)\,(6-k-k^2)\,(20-k-k^2)$
7	$\frac{a1\,(2-k-k^2)\,(12-k-k^2)\,(30-k-k^2)}{5040}$
8	$\frac{a0\,(-k-k^2)\,(6-k-k^2)}{40320} + \frac{(20-k-k^2)\,(42-k-k^2)}{40320}$
9	$\frac{a1\,(2-k-k^2)\,(12-k-k^2)}{362880} + \frac{(30-k-k^2)\,(56-k-k^2)}{362880}$
10	$\frac{a0\,(-k-k^2)\,(6-k-k^2)\,(20-k-k^2)}{3628800} + \frac{(42-k-k^2)\,(72-k-k^2)}{3628800}$

Hence, we have the two linearly independent solutions

$$y_1 = a_0\left(1-\frac{k(k+1)}{2!}x^2+\frac{(2-k)(3+k)k(k+1)}{4!}x^4 - \frac{(4-k)(5+k)(2-k)(3+k)k(k+1)}{6!}x^6+\cdots\right)$$

and

$$y_2 = a_1\left(x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(3-k)(4+k)(k-1)(k+2)}{5!}x^5 - \frac{(5-k)(6+k)(3-k)(4+k)(k-1)(k+2)}{7!}x^7 + \cdots\right)$$

so a general solution of Legendre's equation (4.24) is

$$\begin{aligned} y &= y_1 + y_2 \\ y_1 &= a_0\left(1 - \frac{k(k+1)}{2!}x^2 + \frac{(2-k)(3+k)k(k+1)}{4!}x^4 - \frac{(4-k)(5+k)(2-k)(3+k)k(k+1)}{6!}x^6 + \cdots\right) \\ &\quad + a_1\left(x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(3-k)(4+k)(k-1)(k+2)}{5!}x^5 - \frac{(5-k)(6+k)(3-k)(4+k)(k-1)(k+2)}{7!}x^7 + \cdots\right). \end{aligned}$$

Note that `DSolve` is able to find a general solution as well—the result is given in terms of the functions `LegendreP` and `LegendreQ`, Mathematica's linearly independent solutions of Legendre's equation.

```
In[740]:= DSolve[(1 - x^2) y''[x] - 2 x y'[x]
           +k (k + 1) y[x] == 0, y[x], x]
Out[740]= {{y[x] → C[1] LegendreP[k, x]
           +C[2] LegendreQ[k, x]}}
```

An interesting observation from the general solution to Legendre's equation is that the series solutions terminate for integer values of k. If k is an even integer, the first series terminates while if k is an odd integer the second series terminates. Therefore, polynomial solutions are found for integer values of k. Because these polynomials are useful and are encountered in numerous applications, we have a special notation for them: $P_n(x)$ is called the **Legendre polynomial of degree** n and represents an nth degree polynomial solution to Legendre's equation. The Mathematica command `LegendreP[n,x]` returns $P_n(x)$.

We use `Table` together with `LegendreP` to list the first few Legendre polynomials.

```
In[741]:= toplot = Table[LegendreP[n, x], {n, 0, 5}];
```

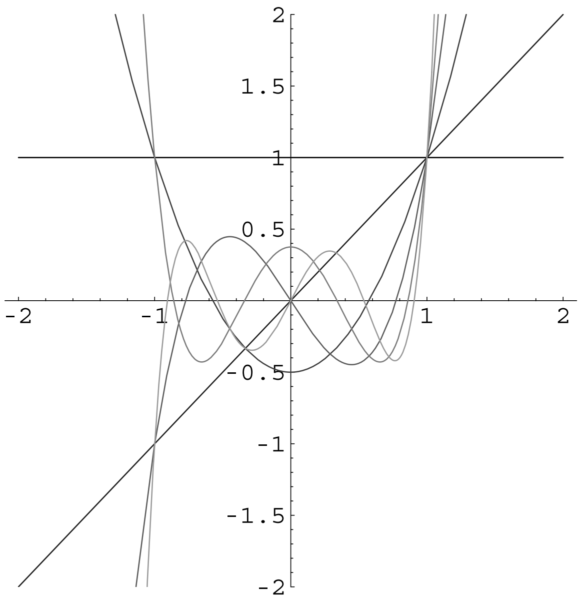


Figure 4-35 Plots of the first few Legendre polynomials

```
TableForm[toplot]
```

Out[741]=

$$\begin{array}{l} 1 \\ x \\ -\frac{1}{2}+\frac{3x^2}{2} \\ -\frac{3x}{2}+\frac{5x^3}{2} \\ \frac{3}{8}-\frac{15x^2}{4}+\frac{35x^4}{8} \\ \frac{15x}{8}-\frac{35x^3}{4}+\frac{63x^5}{8} \end{array}$$

We graph these polynomials for $-2 \le x \le 2$ in Figure 4-35.

```
In[742]:= grays = Table[GrayLevel[i], {i, 0, 0.6, 0.6/5}];
          Plot[Evaluate[toplot], {x, -2, 2},
            PlotRange → {-2, 2}, AspectRatio → 1,
            PlotStyle → grays]
```

Another interesting observation about the Legendre polynomials is that they satisfy the relationship $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$, $m \ne n$, called an *orthogonality condition*, which we verify with `Integrate` for $m, n = 0, 1, \ldots, 6$.

```
In[743]:= Table[∫_{-1}^{1} LegendreP[n, x] LegendreP[m, x] ⅆx,
              {n, 0, 6}, {m, 0, 6}]//TableForm
```

Out[743]=

$$\begin{matrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{13} \end{matrix}$$

Note that the entries down the diagonal of this result correspond to the value of $\int_{-1}^{1} [P_n(x)]^2 \, dx$ for $n = 0, 1, \ldots, 6$ and indicate that $\int_{-1}^{1} [P_n(x)]^2 \, dx = 2/(2n+1)$.

■

4.7.2 Series Solutions about Regular Singular Points

In the previous section, we used a power series expansion about an ordinary point to find (or approximate) the solution of a differential equation. We noted that these series solutions may not converge near the *singular points* of the equation.

In this section, we investigate the problem of obtaining a series expansion about a singular point. We begin with the following classification of singular points.

Definition 21 (Regular and Irregular Singular Points). *Let $x = x_0$ be a singular point of $y'' + p(x)y' + q(x)y = 0$. $x = x_0$ is a **regular singular point** of the equation if both $(x - x_0)\, p(x)$ and $(x - x_0)^2\, q(x)$ are analytic at $x = x_0$. If $x = x_0$ is not a regular singular point, then $x = x_0$ is called an **irregular singular point** of the equation.*

Sometimes this definition is difficult to apply. Therefore, we supply the following definition for polynomial coefficients $p(x)$ and $q(x)$ of the equation $y'' + p(x)y' + q(x)y = 0$.

Definition 22 (Singular Points of Equations with Polynomial Coefficients). *Suppose that $p(x)$ and $q(x)$ are polynomials with no common factors. If after reducing $p(x)$ and $q(x)$ to lowest terms, the highest power of $x - x_0$ in the denominator of $p(x)$ is 1 and the highest power of $x - x_0$ in the denominator of $q(x)$ is 2, then $x = x_0$ is a **regular singular point** of the equation. Otherwise, it is an **irregular singular point**.*

EXAMPLE 4.7.3: Classify the singular points of each of the following equations: (a) $x^2y'' + xy' + \left(x^2 - \mu^2\right)y = 0$ (**Bessel's equation**), and (b) $\left(x^2 - 16\right)^2 y'' + (x-4)y' + y = 0$.

SOLUTION: (a) In standard form, Bessel's equation is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\mu^2}{x^2}\right)y = 0$$

so $x = 0$ is a singular point of this equation because $p(x) = 1/x$ is not analytic at $x = 0$. Because $xp(x) = 1$ and $x^2\left(1 - \frac{\mu^2}{x^2}\right) = x^2 - \mu^2$, $x = 0$ is a regular singular point. We see that `DSolve` is able to find a general solution of Bessel's equation, although the result is given in terms of the Bessel functions, `BesselJ` and `BesselY`.

```
In[744]:= DSolve[x² y''[x] + x y'[x] + (x² - μ²) y[x] == 0,
            y[x], x]
Out[744]= {{y[x] → BesselJ[μ, x] C[1]
           +BesselY[μ, x] C[2]}}
```

(b) In standard form, the equation is

$$\frac{d^2y}{dx^2} + \frac{x-4}{\left(x^2-16\right)^2}\frac{dy}{dx} + \frac{1}{\left(x^2-16\right)^2}y = 0 \quad \text{or}$$

$$\frac{d^2y}{dx^2} + \frac{1}{(x-4)(x+4)^2}\frac{dy}{dx} + \frac{1}{(x-4)^2(x+4)^2}y = 0.$$

Thus, the singular points are $x = 4$ and $x = -4$. For $x = 4$, we have

$$(x-4)p(x) = (x-4)\frac{1}{(x-4)(x+4)^2} = \frac{1}{(x+4)^2}$$

and

$$(x-4)^2q(x) = (x-4)^2\frac{1}{(x-4)^2(x+4)^2} = \frac{1}{(x+4)^2}.$$

Both of these functions are analytic at $x = 4$, so $x = 4$ is a regular singular point.

For $x = -4$,

$$(x+4)p(x) = (x+4)\frac{1}{(x-4)(x+4)^2} = \frac{1}{(x-4)(x+4)},$$

which is not analytic at $x = -4$. Thus, $x = -4$ is an irregular singular point. `DSolve` is unable to find a general solution of this equation.

■

4.7.3 Method of Frobenius

Now we illustrate how a series expansion about a regular singular point can be used to solve an equation.

Theorem 10 (Method of Frobenius). *Let $x = x_0$ be a regular singular point of $y'' + p(x)y' + q(x)y = 0$. Then this differential equation has at least one solution of the form*

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r},$$

where r is a constant that must be determined. This solution is convergent at least on some interval $|x - x_0| < R$, $R > 0$.

EXAMPLE 4.7.4: Find a general solution of $xy'' + (1 + x)y' - \dfrac{1}{16x}y = 0$.

SOLUTION: First, we note that in standard form this equation is

$$\frac{d^2y}{dx^2} + \frac{1+x}{x}\frac{dy}{dx} - \frac{1}{16x^2}y = 0.$$

Thus, $x = 0$ is a singular point. Moreover, because $x\,p(x) = x \cdot \dfrac{1+x}{x} = 1+x$ and $x^2 q(x) = x^2 \cdot -\dfrac{1}{16x^2} = -16$ are both analytic at $x = 0$, we classify $x = 0$ as a regular singular point. By the Method of Frobenius, there is at least one solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Differentiating this function twice, we obtain

$$y' = \sum_{n=0}^{\infty} a_n (n + r) x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} a_n (n + r)(n + r - 1) x^{n+r-2}.$$

Substituting these series into the differential equation yields

$$x \sum_{n=0}^{\infty} a_n (n + r)(n + r - 1) x^{n+r-2} + (1 + x) \sum_{n=0}^{\infty} a_n (n + r) x^{n+r-1} - \frac{1}{16x} \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n (n + r)(n + r - 1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n + r) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n + r) x^{n+r} - \sum_{n=0}^{\infty} \frac{1}{16} a_n x^{n+r-1} = 0.$$

Notice that the first term in three of the four series begins with an x^{r-1} term while the first term in $\sum_{n=0}^{\infty} a_n(n+r)x^{n+r}$ begins with an x^r term, so we must pull off the first terms in the other three series so that they match. Hence,

$$\left[r(r-1)+r-\frac{1}{16}\right]a_0x^{r-1}+\sum_{n=1}^{\infty}a_n(n+r)(n+r-1)x^{n+r-1}$$
$$+\sum_{n=1}^{\infty}a_n(n+r)x^{n+r-1}+\sum_{n=0}^{\infty}a_n(n+r)x^{n+r}-\sum_{n=1}^{\infty}\frac{1}{16}a_nx^{n+r-1}=0.$$

Changing the index in the third series by substituting $n-1$ for each occurrence of n, we have

$$\sum_{n-1=0}^{\infty}a_{n-1}(n-1+r)x^{n-1+r}=\sum_{n=1}^{\infty}a_{n-1}(n+r-1)x^{n+r-1}.$$

After simplification, we have

$$\left[r(r-1)+r-\frac{1}{16}\right]a_0x^{r-1}+\sum_{n=1}^{\infty}\left\{\left[(n+r)(n+r-1)+(n+r)-\frac{1}{16}\right]a_n\right.$$
$$\left.+(n+r-1)\,a_{n-1}\right\}x^{n+r-1}=0.$$

We equate the coefficients to zero to find the coefficients and the value of r. Assuming that $a_0 \neq 0$ so that the first term of our series solution is not zero, we have from the first term the equation

$$r(r-1)+r-\frac{1}{16}=0,$$

called the **indicial equation**, because it yields the value of r. In this case,

```
In[745]:= Solve[r(r-1)+r-1/16 == 0]
Out[745]= {{r → -1/4}, {r → 1/4}}
```

the roots are $r_1 = 1/4$ and $r_2 = -1/4$. Starting with the *larger* of the two roots, $r_1 = 1/4$, we assume that $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1/4} = x^{1/4}\sum_{n=0}^{\infty} a_n x^n$. Equating the series coefficient to zero, we have

$$\left[\left(n+\frac{1}{4}\right)\left(n+\frac{1}{4}-1\right)+\left(n+\frac{1}{4}\right)-\frac{1}{16}\right]a_n+\left(n+\frac{1}{4}-1\right)a_{n-1}=0,$$

which we solve for a_n.

```
In[746]:= Solve[
            ((n + 1/4) (n + 1/4 - 1) + (n + 1/4) - 1/16) a_n
            + (n + 1/4 - 1) a_{n-1} == 0, a_n]
```

$$\text{Out[746]= } \left\{\left\{a_n \to -\frac{(-3+4n)\,a_{-1+n}}{2n(1+2n)}\right\}\right\}$$

Several of these coefficients are calculated using this formula with `Table`.

```
In[747]:= a_n_ := a_n = - ((-3 + 4 n) a_{-1+n}) / (2 (n + 2 n^2))

          a_0 = a0;

          TableForm[Table[{n, a_n}, {n, 0, 10}]]
```

Out[747]=

0	a0
1	$-\frac{a0}{6}$
2	$\frac{a0}{24}$
3	$-\frac{a0}{112}$
4	$\frac{13\,a0}{8064}$
5	$-\frac{221\,a0}{887040}$
6	$\frac{17\,a0}{506880}$
7	$-\frac{17\,a0}{4257792}$
8	$\frac{29\,a0}{68124672}$
9	$-\frac{29\,a0}{706019328}$
10	$\frac{1073\,a0}{296528117760}$

Therefore, one solution to the equation is

$$y_1 = a_0 x^{1/4}\left(1 - \frac{1}{6}x + \frac{1}{24}x^2 - \frac{1}{112}x^3 + \frac{13}{8064}x^4 + \cdots\right).$$

For $r_2 = -1/4$, we assume that $y_2 = \sum_{n=0}^{\infty} a_n x^{n-1/4} = x^{-1/4}\sum_{n=0}^{\infty} a_n x^n$. Then, we have

$$\left[\left(n-\frac{1}{4}\right)\left(n-\frac{1}{4}-1\right)+\left(n-\frac{1}{4}\right)-\frac{1}{16}\right]b_n + \left(n-\frac{1}{4}-1\right)b_{n-1} = 0,$$

which we solve for b_n with `Solve`.

In[748] := Solve[

$$\left(\left(n-\frac{1}{4}\right)\left(n-\frac{1}{4}-1\right)+\left(n-\frac{1}{4}\right)-\frac{1}{16}\right)b_n+\left(n-\frac{1}{4}-1\right)b_{n-1}==0,\ b_n\Big]$$

Out[748] = $\left\{\left\{b_n\to -\frac{(-5+4n)\,b_{-1+n}}{2n(-1+2n)}\right\}\right\}$

The value of several coefficients determined with this formula are computed as well.

In[749] := $b_{n\_} := b_n = -\frac{(-5+4n)\,b_{-1+n}}{2(-n+2n^2)}$

b_0 = b0;

TableForm[Table[{n, b_n}, {n, 0, 10}]]

Out[749] =

0	b0
1	$\frac{b0}{2}$
2	$-\frac{b0}{8}$
3	$\frac{7\,b0}{240}$
4	$-\frac{11\,b0}{1920}$
5	$\frac{11\,b0}{11520}$
6	$-\frac{19\,b0}{138240}$
7	$\frac{437\,b0}{25159680}$
8	$-\frac{437\,b0}{223641600}$
9	$\frac{13547\,b0}{68434329600}$
10	$-\frac{713\,b0}{39105331200}$

Therefore, a second linearly independent solution of the equation obtained with $r_2 = -1/4$ is

$$y_2 = b_0 x^{-1/4}\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{7}{240}x^3-\frac{11}{1920}x^4+\cdots\right)$$

and a general solution of the differential equation is $y = c_1y_1 + c_2y_2$ where c_1 and c_2 are arbitrary constants. Notice that these two solutions are linearly independent, because they are not scalar multiples of one another.

We see that `DSolve` is able to find a general solution of the equation as well, although the result is given in terms of the functions `HypergeometricU` and `LaguerreL`.

```
In[750]:= gensol1 = DSolve[x y''[x] + (1 + x) y'[x]
              - y[x]/(16 x) == 0, y[x], x]
Out[750]= {{y[x] → e^-x x^(1/4) C[1] HypergeometricU[5/4, 3/2, x]
              + e^-x x^(1/4) C[2] LaguerreL[-5/4, 1/2, x]}}
```

■

In the previous example, we found the **indicial equation** by direct substitution of the power series solution into the differential equation. In order to derive a general formula for the indicial equation, suppose that $x = 0$ is a regular singular point of the differential equation $y'' + p(x)y' + q(x)y = 0$. Then the functions $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$, which means that both of these functions have a power series in x with a positive radius of convergence. Hence,

$$xp(x) = p_0 + p_1x + p_2x^2 + \cdots \qquad \text{and} \qquad x^2q(x) = q_0 + q_1x + q_2x^2 + \cdots$$

and

$$p(x) = \frac{p_0}{x} + p_1 + p_2x + p_3x^2 + \cdots \qquad \text{and} \qquad q(x) = \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3x + \cdots$$

Substitution of these series into the differential equation $y'' + p(x)y' + q(x)y = 0$ and multiplying through by the first term in the series for $p(x)$ and $q(x)$, we see that the lowest term in the series involves x^{n+r-2}:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + \sum_{n=0}^{\infty} a_n p_0(n+r)x^{n+r-2}$$
$$+ \left(p_1 + p_2x + p_3x^2 + \cdots\right)\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} a_n q_0 x^{n+r-2}$$
$$+ \left(\frac{q_1}{x} + q_2 + q_3x^2 + \cdots\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Then, with $n = 0$, we find that the coefficient of x^{r-2} is

$$\begin{aligned} -ra_0 + r^2a_0 + ra_0p_0 + a_0q_0 &= a_0\left[r^2 + (p_0 - 1)\,r + q_0\right] \\ &= a_0\left[r(r-1) + p_0r + q_0\right] = 0. \end{aligned}$$

Thus, for any equation of the form $y'' + p(x)y' + q(x)y = 0$ with regular singular point $x = 0$, we have the **indicial equation**

$$r(r-1) + p_0r + q_0 = 0. \tag{4.25}$$

The values of r that satisfy the indicial equation are called the **exponents** or **indicial roots** and are

$$r_{1,2} = \frac{1}{2}\left(1 - p_0 \pm \sqrt{1 - 2p_0 + {p_0}^2 - 4q_0}\right). \tag{4.26}$$

Note that $r_1 \geq r_2$ and $r_1 - r_2 = \sqrt{1 - 2p_0 + {p_0}^2 - 4q_0}$.

Several situations can arise when finding the roots of the indicial equation.

1. If $r_1 \neq r_2$ and $\sqrt{1 - 2p_0 + {p_0}^2 - 4q_0}$ is not an integer, then there are two linearly independent solutions of the equation of the form
$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n.$$
2. If $r_1 \neq r_2$ and $\sqrt{1 - 2p_0 + {p_0}^2 - 4q_0}$ is an integer, then there are two linearly independent solutions of the equation of the form
$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad y_2 = cy_1 \ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n.$$
3. If $r_1 - r_2 = \sqrt{1 - 2p_0 + {p_0}^2 - 4q_0} = 0$, then there are two linearly independent solutions of the problem of the form
$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad y_2 = y_1 \ln x + x^{r_1}\sum_{n=0}^{\infty} b_n x^n.$$

In any case, if y_1 is a solution of the equation, a second linearly independent solution is given by

$$y_2 = y_1(x)\int \frac{1}{[y_1(x)]^2} e^{-\int p(x)\,dx}\,dx,$$

which can be obtained through reduction of order.

Note that when solving a differential equation in Case 2, first attempt to find a general solution using $y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n$, where r_2 is the *smaller* of the two roots. However, if the contradiction $a_0 = 0$ is reached, then find solutions of the form $y_1 = x^{r_1}\sum_{n=0}^{\infty} b_n x^n$ and $y_2 = cy_1 \ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n$.

The examples here do not illustrate the possibility of complex-valued roots of the indicial equation. When this occurs, the equation is solved using the procedures of Case 1. The solutions that are obtained are complex, so they can be transformed into real solutions by taking the appropriate linear combinations, such as those discussed for complex-valued roots of the characteristic equation of Cauchy–Euler differential equations.

Also, we have not mentioned if a solution can be found with an expansion about an irregular singular point. If $x = x_0$ is an irregular singular point of $y'' + p(x)y' +$

$q(x)y = 0$, there may or may not be a solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$ for some number r.

EXAMPLE 4.7.5 (Bessel's Equation): Bessel's equation (of order μ), named after the German astronomer Friedrich Wilhelm Bessel, is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \mu^2\right) y = 0, \tag{4.27}$$

where $\mu \geq 0$ is a constant. Solve Bessel's equation.

SOLUTION: To use a series method to solve Bessel's equation, we first write the equation in standard form as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{x^2 - \mu^2}{x^2} y = 0,$$

so $x = 0$ is a regular singular point. Using the Method of Frobenius, we assume that there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. We determine the value(s) of r with the indicial equation. Because $xp(x) = x \cdot 1/x = 1$ and $x^2 q(x) = x^2 \cdot \left(x^2 - \mu^2\right)/x^2 = x^2 - \mu^2$, $p_0 = 1$ and $q_0 = -\mu^2$. Hence, the indicial equation is

$$r(r-1) + p_0 r + q_0 = r(r-1) + r - \mu^2 = r^2 - \mu^2 = 0$$

with roots $r_{1,2} = \pm\mu$. Therefore, we assume that $y = \sum_{n=0}^{\infty} a_n x^{n+\mu}$ with derivatives $y' = \sum_{n=0}^{\infty}(n+\mu)a_n x^{n+\mu-1}$ and $y'' = \sum_{n=0}^{\infty}(n+\mu)(n+\mu-1)a_n x^{n+\mu-2}$. Substitution into Bessel's equation and simplifying the result yields

$$\left[\mu(\mu-1) + \mu - \mu^2\right] a_0 x^{\mu} + \left[(1+\mu)\mu + (1+\mu) - \mu^2\right] a_1 x^{\mu+1}$$
$$+ \sum_{n=2}^{\infty} \left\{\left[(n+\mu)(n+\mu-1) + (n+\mu) - \mu^2\right] a_n + a_{n-2}\right\} x^{n+\mu} = 0.$$

Notice that the coefficient of $a_0 x^{\mu}$ is zero. After simplifying the other coefficients and equating them to zero, we have $(1+2\mu)a_1 = 0$ and $\left[(n+\mu)(n+\mu-1) + (n+\mu) - \mu^2\right] a_n + a_{n-2} = 0$, which we solve for a_n.

```
In[751]:= Remove[a]

          Solve[((n + μ) (n + μ - 1)
          + (n + μ) - μ^2) a_n + a_{n-2} == 0, a_n]
```

Out[751]= $\left\{\left\{a_n \to -\frac{a_{-2+n}}{n\,(n + 2\,\mu)}\right\}\right\}$

From the first equation, $a_1 = 0$. Therefore, from $a_n = -\dfrac{a_{n-2}}{n(n+2\mu)}$, $n \geq 2$, so that $a_n = 0$ for all odd n. We use the formula for a_n to calculate several of the coefficients that correspond to even indices.

```
In[752]:= a[n_] := a[n] = -a[-2+n]/(n (n + 2 μ));

          a[0] = a0;

In[753]:= Table[{n, a[n]}, {n, 2, 10, 2}]//TableForm
```

Out[753]=

2	$-\dfrac{a0}{2\ (2+2\mu)}$
4	$\dfrac{a0}{8\ (2+2\mu)\ (4+2\mu)}$
6	$-\dfrac{a0}{48\ (2+2\mu)\ (4+2\mu)\ (6+2\mu)}$
8	$\dfrac{a0}{384\ (2+2\mu)\ (4+2\mu)\ (6+2\mu)\ (8+2\mu)}$
10	$-\dfrac{a0}{3840\ (2+2\mu)\ (4+2\mu)} \times \dfrac{1}{(6+2\mu)\ (8+2\mu)\ (10+2\mu)}$

A general formula for these coefficients is given by

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n!}(1+\mu)(2+\mu)\cdots(n+\mu)}, \quad n \geq 2.$$

Our solution can then be written as

$$y_1 = \sum_{n=0}^{\infty} a_{2n} x^{2n+\mu} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{\mu}}{(1+\mu)(2+\mu)\cdots(n+\mu)} \left(\frac{x}{2}\right)^{2n+\mu}.$$

If μ is an integer, then by using the gamma function, $\Gamma(x)$, we can write this solution as

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\mu+n)} \left(\frac{x}{2}\right)^{2n+\mu}. \tag{4.28}$$

This function, denoted $J_\mu(x)$, is called the **Bessel function of the first kind** of order μ. The command `BesselJ[μ, x]` returns $J_\mu(x)$. We use `BesselJ` to graph $J_\mu(x)$ for $\mu = 0, 1, 2, 3$, and 4 in Figure 4-36. Notice that these functions have numerous zeros. We will need to know these values in subsequent sections.

```
In[754]:= toplot = Table[BesselJ[μ, x], {μ, 0, 4}];

          grays = Table[GrayLevel[i],
                 {i, 0, 0.6, 0.6/4}];
```

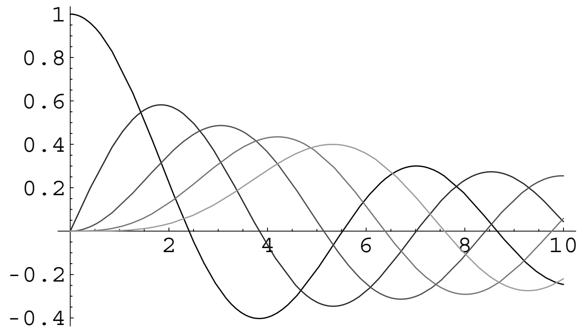


Figure 4-36 The first five Bessel functions of the first kind

```
In[755]:= Plot[Evaluate[toplot], {x, 0, 10},
            PlotStyle → grays]
```

For the other root, $r_2 = -\mu$, of the indicial equation, a similar derivation yields a second linearly independent solution of Bessel's equation,

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\mu+n)} \left(\frac{x}{2}\right)^{2n-\mu},$$

which is the **Bessel function of the first kind of order** $-\mu$ and is denoted $J_{-\mu}(x)$. Now, we must determine if the functions $J_\mu(x)$ and $J_{-\mu}(x)$ are linearly independent. Notice that if $\mu = 0$, then these two functions are the same. If $\mu > 0$, then $r_1 - r_2 = \mu - (-\mu) = 2\mu$. If 2μ is not an integer, then by the Method of Frobenius, the two solutions $J_\mu(x)$ and $J_{-\mu}(x)$ are linearly independent. Also, we can show that if 2μ is an odd integer, $J_\mu(x)$ and $J_{-\mu}(x)$ are linearly independent. In both of these cases, a general solution is given by $y = c_1 J_\mu(x) + c_2 J_{-\mu}(x)$.

If μ is not an integer, we define the **Bessel function of the second kind of order** μ, $Y_\mu(x)$, by the linear combination

$$Y_\mu(x) = \frac{1}{\sin \mu\pi} \left[\cos \mu\pi \, J_\mu(x) - J_{-\mu}(x)\right] \tag{4.29}$$

of the functions $J_\mu(x)$ and $J_{-\mu}(x)$. The command `BesselY[μ, x]` returns $Y_\mu(x)$. We can show that $J_\mu(x)$ and $Y_\mu(x)$ are linearly independent, so a general solution of Bessel's equation of order μ can be represented by

$$y = c_1 J_\mu(x) + c_2 Y_\mu(x),$$

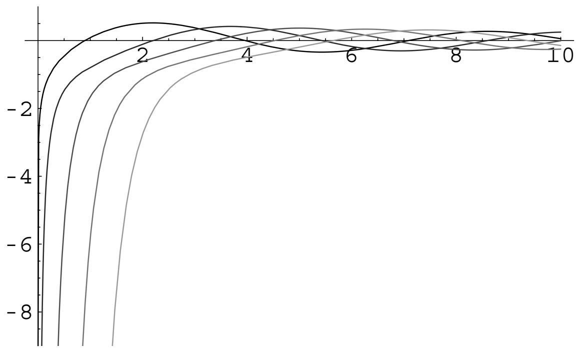


Figure 4-37 The Bessel functions of the second kind tend to $-\infty$ as $x \to 0^+$

which is the form of the general solution returned by using Mathematica's `DSolve` command to solve Bessel's equation.

```
In[756]:= gensol = DSolve[x^2 y''[x] + x y'[x]
                + (x^2 - μ^2) y[x] == 0, y[x], x]
Out[756]= {{y[x] → BesselJ[μ, x] C[1] + BesselY[μ, x] C[2]}}
```

We use `BesselY` to graph the functions $\mu = 0, 1, 2, 3$, and 4 in Figure 4-37. Notice that $\lim_{x \to 0^+} Y_\mu(x) = -\infty$. This property will be important in several applications in later chapters.

```
In[757]:= toplot = Table[BesselY[μ, x], {μ, 0, 4}];

          grays = Table[GrayLevel[i],
                  {i, 0, 0.6, 0.6/4}];

In[758]:= Plot[Evaluate[toplot], {x, 0, 10},
            PlotStyle → grays, PlotRange → {-9, 1}]
```

■

A more general form of Bessel's equation is expressed in the form

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(\lambda^2 x^2 - \mu^2\right) y = 0. \tag{4.30}$$

Through a change of variables, we can show that a general solution of this equation defined on the interval $0 < x < \infty$ is

$$y = c_1 J_\mu(\lambda x) + c_2 Y_\mu(\lambda x).$$

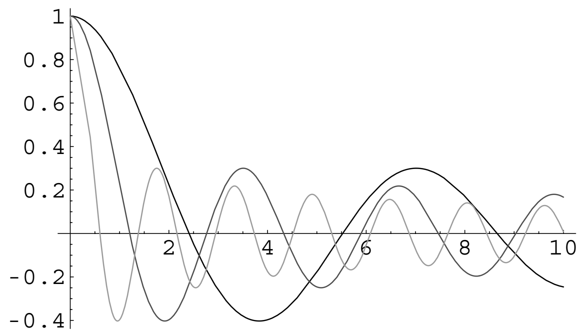


Figure 4-38 Plots of $J_0(x)$, $J_0(2x)$, and $J_0(4x)$

```
In[759]:= gensol =
            DSolve[x^2 y''[x] + x y'[x] + (λ^2 x^2 - μ^2) y[x] == 0,
              y[x], x]
Out[759]= {{y[x] → BesselJ[μ, x λ] C[1] + BesselY[μ, x λ] C[2]}}
```

We graph the functions $J_0(x)$, $J_0(2x)$, and $J_0(4x)$ in Figure 4-38. Notice that for larger values of the parameter λ, the graph of the function intersects the x-axis more often.

```
In[760]:= Plot[{BesselJ[0, x], BesselJ[0, 2 x],
              BesselJ[0, 4 x]}, {x, 0, 10},
            PlotStyle → {GrayLevel[0], GrayLevel[0.3],
              GrayLevel[0.6]}]
```

EXAMPLE 4.7.6: Find a general solution of each of the following equations: (a) $x^2y'' + xy' + \left(x^2 - 16\right)y = 0$ and (b) $x^2y'' + xy' + \left(9x^2 - 4\right)y = 0$.

SOLUTION: (a) In this case, $\mu = 4$ so a general solution is $y = c_1J_4(x) + c_2Y_4(x)$. We graph this solution for various choices of the arbitrary constants in Figure 4-39.

```
In[761]:= Clear[x, y]

          sol1 = DSolve[x^2 y''[x] + x y'[x]
                 + (x^2 - 16) y[x] == 0, y[x], x]
Out[761]= {{y[x] → BesselJ[4, x]C[1]+BesselY[4, x]C[2]}}
```

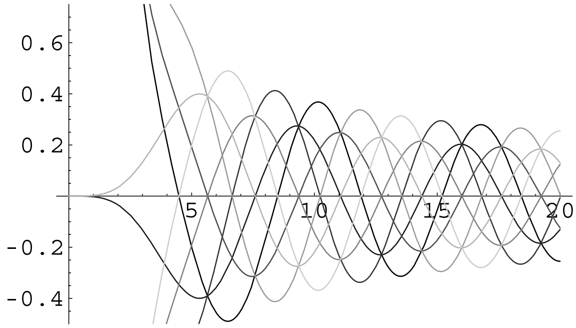


Figure 4-39 Solutions of $x^2y'' + xy' + \left(x^2 - 16\right)y = 0$

```
In[762]:= toplot =
            Table[sol1[[1,1,2]] /. {C[1]->i, C[2]->j},
              {i,-1,1}, {j,-1,1}];

          grays = Table[GrayLevel[i], {i,0,0.8,0.8/8}];

In[763]:= Plot[Evaluate[toplot], {x,0,20},
            PlotRange → {-1/2, 3/4}, PlotStyle → grays]
```

(b) Using the parametric Bessel's equation (4.30) with $\lambda = 3$ and $\mu = 2$, we have $y = c_1 J_2(3x) + c_2 Y_2(3x)$. We graph this solution for several choices of the arbitrary constants in Figure 4-40.

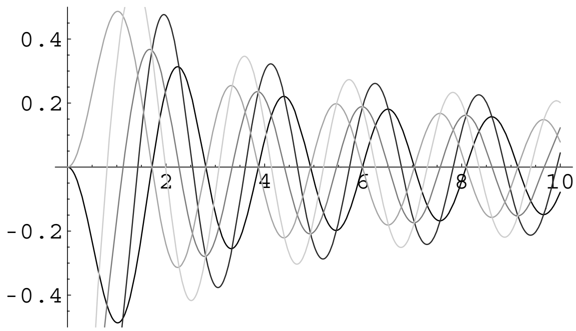


Figure 4-40 Solutions of $x^2y'' + xy' + \left(9x^2 - 4\right)y = 0$

```
In[764]:= Clear[x, y]

          sol2 = DSolve[x^2 y''[x] + x y'[x]
                + (9 x^2 - 4) y[x] == 0, y[x], x]

Out[764]= {{y[x] → BesselJ[2, 3 x] C[1]
           +BesselY[2, 3 x] C[2]}}
```

```
In[765]:= toplot =
           Table[sol2[[1, 1, 2]]/.{C[1]- > i, C[2]- > j},
            {i, -1, 1}, {j, 0, 1}];

          grays = Table[GrayLevel[i],
                 {i, 0, 0.8, 0.8/5}];
```

```
In[766]:= Plot[Evaluate[toplot], {x, 0, 10},
           PlotRange → {-1/2, 1/2}, PlotStyle → grays]
```

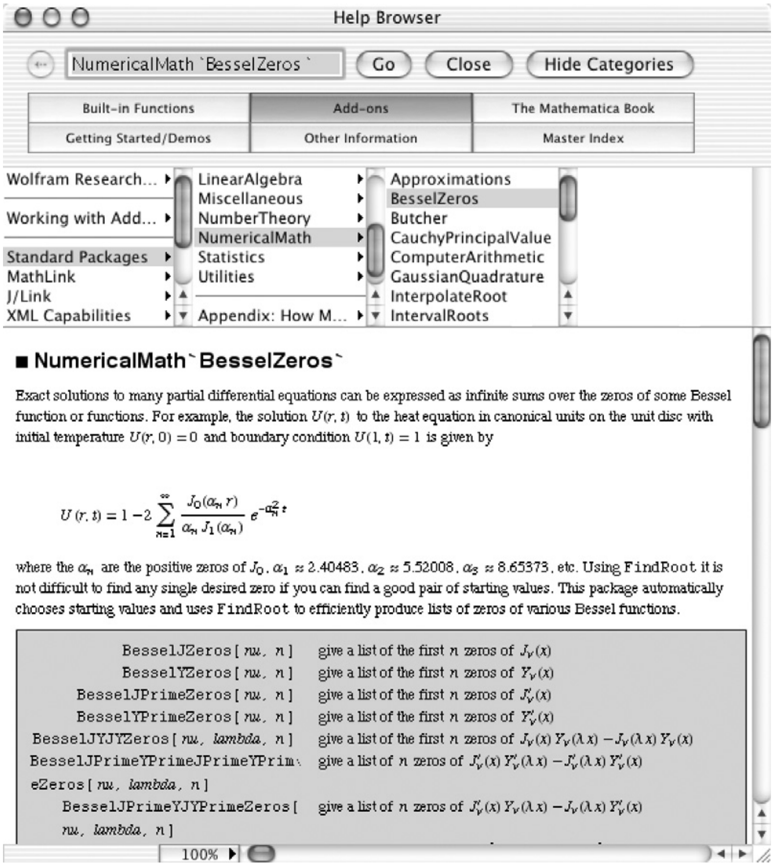
■

Application: Zeros of the Bessel Functions of the First Kind

As indicated earlier, zeros of the Bessel functions of the first kind will be used in applications in later chapters. Here, we graph the first nine Bessel functions of the first kind on the interval [0, 40] and show all nine graphs together as a `GraphicsArray` in Figure 4-41.

```
In[767]:= besselarray =
           Table[Plot[BesselJ[n, x], {x, 0, 40},
              DisplayFunction → Identity], {n, 0, 8}];

          toshow = Partition[besselarray, 3];

          Show[GraphicsArray[toshow]]
```

To approximate the zeros, we take advantage of the **BesselZeros** package that is contained in the **NumericalMath** folder (or directory). We obtain information about the **BesselZeros** package using Mathematica's help facility, as indicated in the following screen shot.



To use the package, we first load it by entering

```
<<NumericalMath`BesselZeros`.
```

Thus, entering

```
In[768]:= << NumericalMath`BesselZeros`

In[769]:= Table[BesselJZeros[μ, 5], {μ, 0, 4}]//TableForm
           2.40483 5.52008 8.65373 11.7915 14.9309
           3.83171 7.01559 10.1735 13.3237 16.4706
Out[769]=  5.13562 8.41724 11.6198 14.796  17.9598
           6.38016 9.76102 13.0152 16.2235 19.4094
           7.58834 11.0647 14.3725 17.616  20.8269
```

returns a table of the first five zeros of the Bessel functions $J_\mu(x)$ for $\mu = 0, 1, 2, 3,$ and 4. (The first row corresponds to the zeros of $J_0(x)$, the second row to the zeros

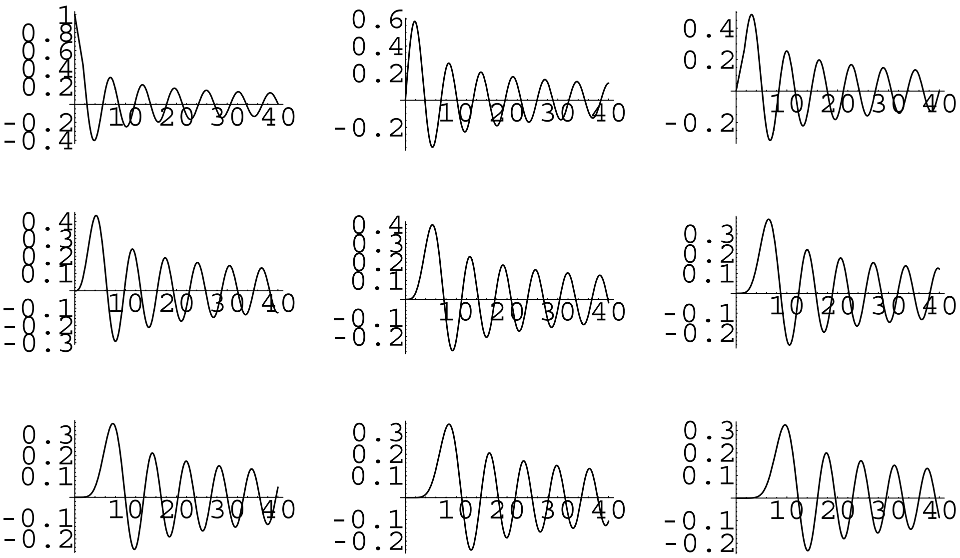


Figure 4-41 Plots of the first nine Bessel functions of the first kind

of $J_1(x)$, and so on.) Later, we will need to calculate particular zeros. Let α_n denote the nth zero of $J_0(x)$. We define α_n as follows and then calculate the 50th zero of $J_0(x)$. Defining α_n in this manner allows us to calculate particular zeros; not a list.

```
In[770]:= α_n_ := α_n = BesselJZeros[0, {n, n}][[1]]
```

```
In[771]:= α_50
Out[771]= 156.295
```

Next, we generate a list of the first 10 zeros of $J_0(x)$.

```
In[772]:= Table[α_n, {n, 1, 10}]
Out[772]= {2.40483, 5.52008, 8.65373, 11.7915, 14.9309,
           18.0711, 21.2116, 24.3525, 27.4935, 30.6346}
```

More generally, let $\alpha_{m,n}$ denote the nth zero of $J_m(x)$. We define $\alpha_{m,n}$ as follows.

```
In[773]:= α_m_,n_ := α_m,n = BesselJZeros[m, {n, n}][[1]]
```

Thus, entering

```
In[774]:= α_25,30
Out[774]= 130.328
```

returns the 30th zero of $J_{25}(x)$; entering

```
In[775]:= Table[α0,n, {n, 1, 5}]

Out[775]= {2.40483, 5.52008, 8.65373, 11.7915, 14.9309}
```

returns a list of the first five zeros of $J_0(x)$; and entering

```
In[776]:= Table[αm,n, {m, 0, 4}, {n, 1, 5}]//TableForm

           2.40483 5.52008 8.65373 11.7915 14.9309
           3.83171 7.01559 10.1735 13.3237 16.4706
Out[776]=  5.13562 8.41724 11.6198 14.796  17.9598
           6.38016 9.76102 13.0152 16.2235 19.4094
           7.58834 11.0647 14.3725 17.616  20.8269
```

returns a table of the first five zeros of the Bessel functions $J_\mu(x)$ for $\mu = 0, 1, 2, 3,$ and 4. (The first row corresponds to the zeros of $J_0(x)$, the second row to the zeros of $J_1(x)$, and so on.)

For a classic approach to the subject see Graff's *Wave Motion in Elastic Solids*, [13].

Application: The Wave Equation on a Circular Plate

The vibrations of a circular plate satisfy the equation

$$D\,\nabla^4 w(r,\theta,t) + \rho h \frac{\partial^2 w(r,\theta,t)}{\partial t^2} = q(r,\theta,t), \tag{4.31}$$

where $\nabla^4 w = \nabla^2\nabla^2 w$ and ∇^2 is the **Laplacian in polar coordinates**, which is defined by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Assuming no forcing so that $q(r,\theta,t) = 0$ and $w(r,\theta,t) = W(r,\theta)e^{-i\omega t}$, equation (4.31) can be written as

$$\nabla^4 W(r,\theta) - \beta^4 W(r,\theta) = 0, \qquad \beta^4 = \omega^2\rho h/D. \tag{4.32}$$

For a clamped plate, the boundary conditions are $W(a,\theta) = \partial W(a,\theta)/\partial r = 0$ and after *much work* (see [13]) the **normal modes** are found to be

$$W_{nm}(r,\theta) = \left[J_n(\beta_{nm}r) - \frac{J_n(\beta_{nm}a)}{I_n(\beta_{nm}a)} I_n(\beta_{nm}r)\right]\begin{pmatrix}\sin n\theta\\ \cos n\theta\end{pmatrix}. \tag{4.33}$$

In equation (4.33), $\beta_{nm} = \lambda_{nm}/a$ where λ_{nm} is the mth solution of

$$I_n(x)J_n{}'(x) - J_n(x)I_n{}'(x) = 0, \tag{4.34}$$

where $J_n(x)$ is the Bessel function of the first kind of order n and $I_n(x)$ is the **modified Bessel function of the first kind** of order n, related to $J_n(x)$ by $i^n I_n(x) = J_n(ix)$.

The Mathematica command `BesselI[n,x]` returns $I_n(x)$.

EXAMPLE 4.7.7: Graph the first few normal modes of the clamped circular plate.

SOLUTION: We must determine the value of λ_{nm} for several values of n and m so we begin by defining `eqn[n][x]` to be $I_n(x)J_n{}'(x) - J_n(x)I_n{}'(x)$. The mth solution of equation (4.34) corresponds to the mth zero of the graph of `eqn[n][x]` so we graph `eqn[n][x]` for $n = 0$, 1, 2, and 3 with `Plot` in Figure 4-42.

```
In[777]:= eqn[n_][x_] := BesselI[n,x]D[BesselJ[n,x],x]
          -BesselJ[n,x]D[BesselI[n,x],x]
```

The result of the `Table` and `Plot` command is a list of length four

```
In[778]:= p1 = Table[Plot[eqn[n][x],{x,0,25},
                PlotRange → {-10,10},
                DisplayFunction → Identity],{n,0,3}]
Out[778]= {-Graphics-,-Graphics-,-Graphics-,
           -Graphics-}
```

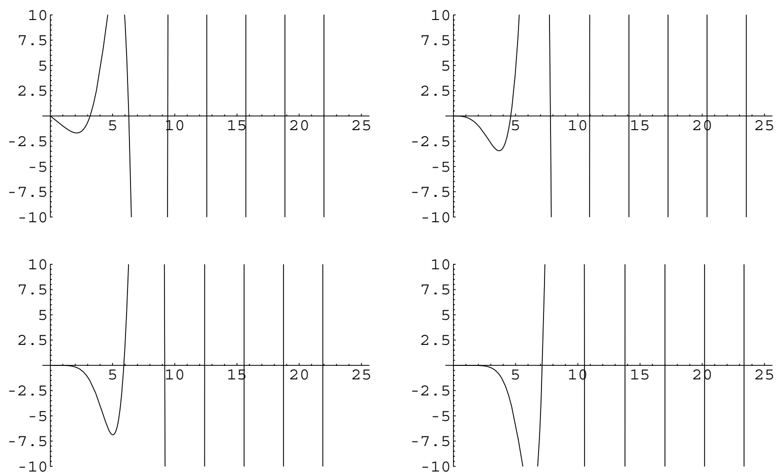


Figure 4-42 Plot of $I_n(x)J_n{}'(x) - J_n(x)I_n{}'(x)$ for $n = 0$ and 1 in the first row; $n = 2$ and 3 in the second row

so we use `Partition` to create a 2×2 array of graphics which is displayed using `Show` and `GraphicsArray`.

```
In[779]:= p2 = Show[GraphicsArray[Partition[p1, 2]]]
```

To determine λ_{nm} we use `FindRoot`. Recall that to use `FindRoot` to solve an equation an initial approximation of the solution must be given. For example,

```
In[780]:= lambda01 = FindRoot[eqn[0][x] == 0, {x, 3.04}]
Out[780]= {x → 3.19622}
```

approximates λ_{01}, the first solution of equation (4.34) if $n = 0$. However, the result of `FindRoot` is a list. The specific value of the solution is the second part of the first part of the list, `lambda01`, extracted from the list with `Part` (`[[...]]`).

```
In[781]:= lambda01[[1, 2]]
Out[781]= 3.19622
```

We use the graphs in Figure 4-42 to obtain initial approximations of each solution.

Thus,

```
In[782]:= λ0s = Map[FindRoot[eqn[0][x] == 0,
                    {x, #}][[1, 2]]&,
                    {3.04, 6.2, 9.36, 12.5, 15.7}]
Out[782]= {3.19622, 6.30644, 9.4395, 12.5771, 15.7164}
```

approximates the first five solutions of equation (4.34) if $n = 0$ and then returns the specific value of each solution. We use the same steps to approximate the first five solutions of equation (4.34) if $n = 1$, 2, and 3.

```
In[783]:= λ1s = Map[FindRoot[eqn[1][x] == 0,
                    {x, #}][[1, 2]]&,
                    {4.59, 7.75, 10.9, 14.1, 17.2}]
Out[783]= {4.6109, 7.79927, 10.9581, 14.1086, 17.2557}

In[784]:= λ2s = Map[FindRoot[eqn[2][x] == 0,
                    {x, #}][[1, 2]]&,
                    {5.78, 9.19, 12.4, 15.5, 18.7}]
Out[784]= {5.90568, 9.19688, 12.4022, 15.5795, 18.744}

In[785]:= λ3s = Map[FindRoot[eqn[3][x] == 0,
                    {x, #}][[1, 2]]&,
                    {7.14, 10.5, 13.8, 17, 20.2}]
Out[785]= {7.14353, 10.5367, 13.7951, 17.0053, 20.1923}
```

All four lists are combined together in λs.

```
In[786]:= λs = {λ0s, λ1s, λ2s, λ3s}
```

```
Out[786]= {{3.19622, 6.30644, 9.4395, 12.5771, 15.7164},
           {4.6109, 7.79927, 10.9581, 14.1086, 17.2557},
           {5.90568, 9.19688, 12.4022, 15.5795, 18.744},
           {7.14353, 10.5367, 13.7951,
           17.0053, 20.1923}}
```

For $n = 0, 1, 2$, and 3 and $m = 1, 2, 3, 4$, and 5, λ_{nm} is the mth part of the $(n + 1)$st part of λs.

Observe that the value of a does not affect the shape of the graphs of the normal modes so we use $a = 1$ and then define β_{nm}.

```
In[787]:= a = 1;
```

```
In[788]:= β[n_, m_] := λs[[n + 1, m]]/a
```

ws is defined to be the sine part of equation (4.33)

```
In[789]:= ws[n_, m_][r, θ] := (BesselJ[n, β[n, m] r]
                              -BesselJ[n, β[n, m] a]
                              /BesselI[n, β[n, m] a]
                              BesselI[n, β[n, m] r])
                              × Sin[n θ]
```

and wc to be the cosine part.

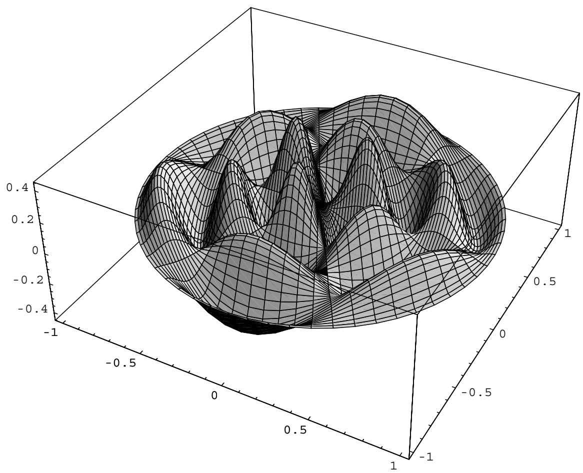


Figure 4-43 The sine part of $W_{34}(r, \theta)$

```
In[790]:= wc[n_,m_][r,θ] := (BesselJ[n,β[n,m] r]
                             -BesselJ[n,β[n,m] a]
                             /BesselI[n,β[n,m] a]
                             BesselI[n,β[n,m] r])
                             × Cos[n θ]
```

We use `ParametricPlot3D` to plot `ws` and `wc`. For example,

```
In[791]:= ParametricPlot3D[{r Cos[θ],r Sin[θ],
            ws[3,4][r,θ]},{r,0,1},{θ,-π,π},
            PlotPoints → 60]
```

graphs the sine part of $W_{34}(r, \theta)$ shown in Figure 4-43. We use `Table` together with `ParametricPlot3D` followed by `Show` and `GraphicsArray` to graph the sine part of $W_{nm}(r, \theta)$ for $n = 0, 1, 2$, and 3 and $m = 1, 2, 3$, and 4 shown in Figure 4-44.

```
In[792]:= ms = Table[ParametricPlot3D[{r Cos[θ],
                 r Sin[θ],ws[n,m][r,θ]},{r,0,1},
                 {θ,-π,π},DisplayFunction → Identity,
                 PlotPoints → 30,BoxRatios → {1,1,1}],
                 {n,0,3},{m,1,4}]

Out[792]= {{-Graphics3D-,-Graphics3D-,-Graphics3D-,
            -Graphics3D-},{-Graphics3D-,-Graphics3D-,
            -Graphics3D-,-Graphics3D-},{-Graphics3D-,
            -Graphics3D-,-Graphics3D-,-Graphics3D-},
            {-Graphics3D-,-Graphics3D-,-Graphics3D-,
            -Graphics3D-}}
```

```
In[793]:= Show[GraphicsArray[ms]]
```

Identical steps are followed to graph the cosine part shown in Figure 4-45.

```
In[794]:= mc = Table[ParametricPlot3D[{r Cos[θ],
                 r Sin[θ],wc[n,m][r,θ]},{r,0,1},
                 {θ,-π,π},DisplayFunction → Identity,
                 PlotPoints → 30,BoxRatios → {1,1,1}],
                 {n,0,3},{m,1,4}]
```

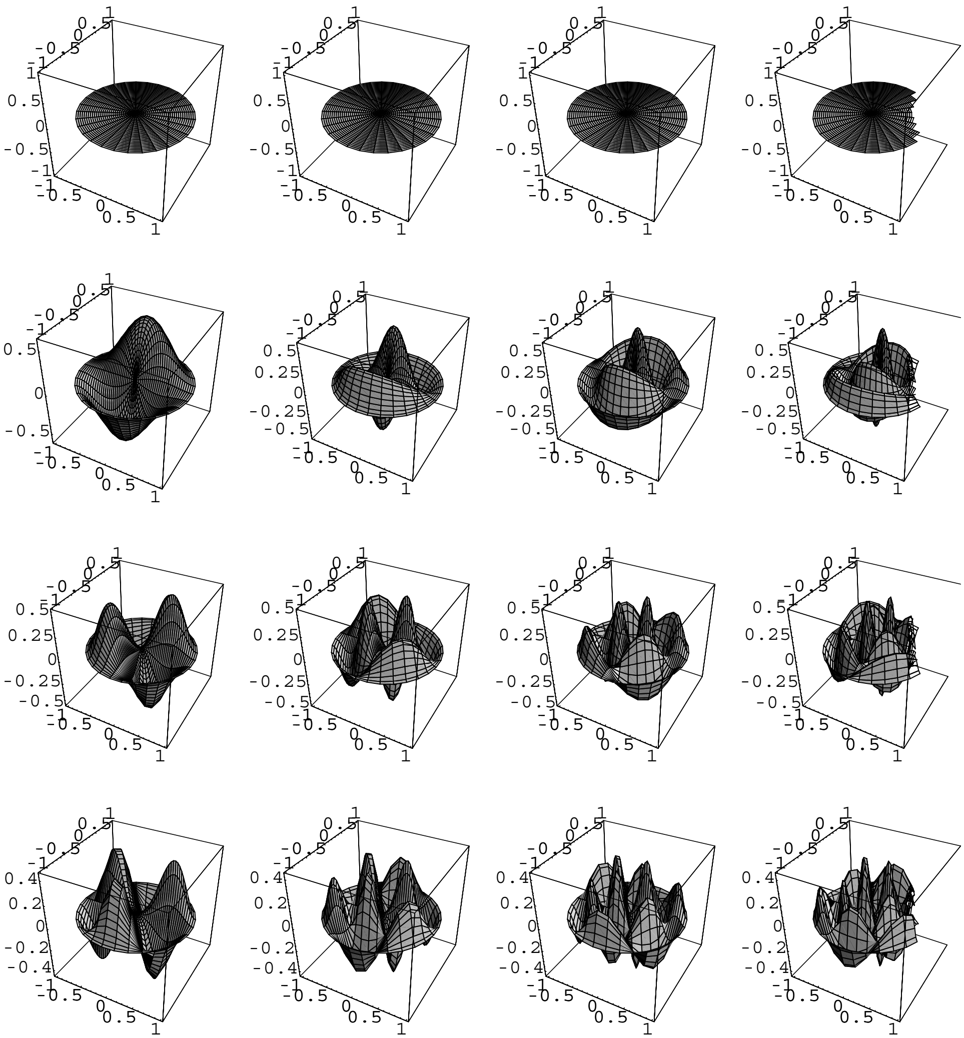


Figure 4-44 The sine part of $W_{nm}(r, \theta)$: $n = 0$ in row 1, $n = 1$ in row 2, $n = 2$ in row 3, and $n = 3$ in row 4 ($m = 1$ to 4 from left to right in each row)

```
Out[794]= {{-Graphics3D-, -Graphics3D-, -Graphics3D-,
           -Graphics3D-}, {-Graphics3D-, -Graphics3D-,
           -Graphics3D-, -Graphics3D-}, {-Graphics3D-,
           -Graphics3D-, -Graphics3D-, -Graphics3D-},
           {-Graphics3D-, -Graphics3D-, -Graphics3D-,
           -Graphics3D-}}

In[795]:= Show[GraphicsArray[mc]]
```

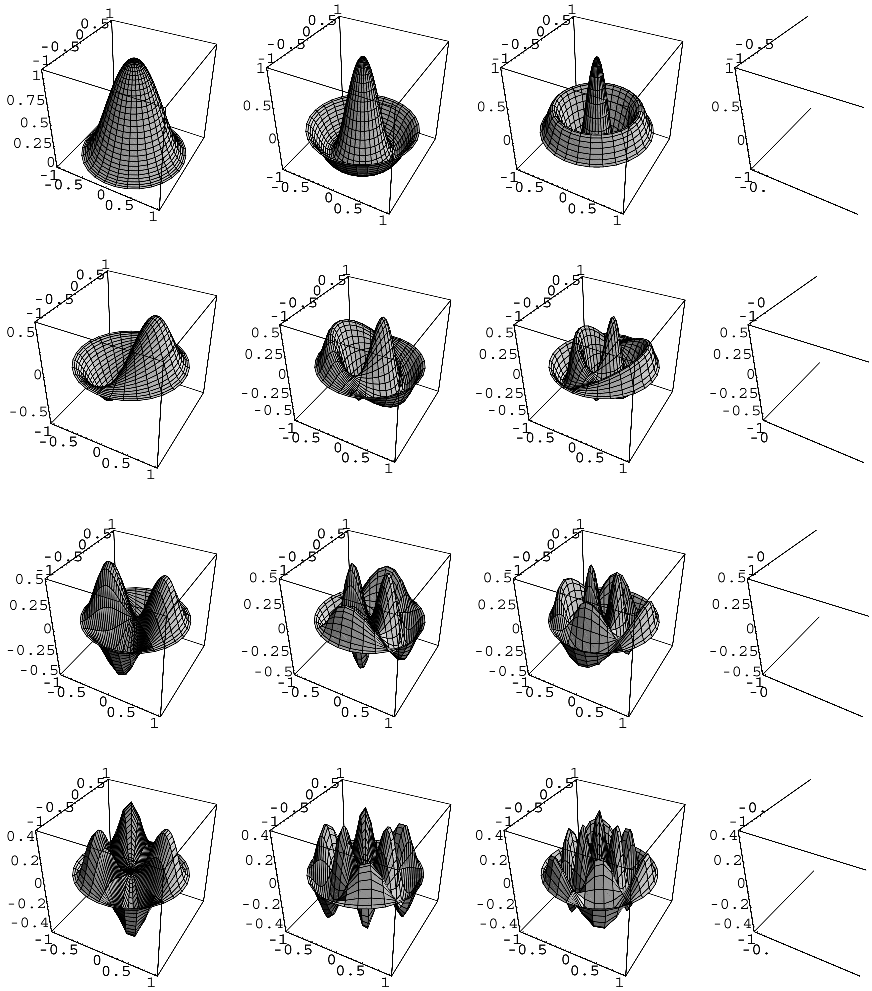


Figure 4-45 The cosine part of $W_{nm}(r, \theta)$: $n = 0$ in row 1, $n = 1$ in row 2, $n = 2$ in row 3, and $n = 3$ in row 4 ($m = 1$ to 4 from left to right in each row)

■

4.8 Nonlinear Equations

Generally, rigorous results regarding nonlinear equations are very difficult to obtain. In many cases, analysis is best carried out numerically and/or graphically. In other situations, rewriting the equation as a system can be of benefit, which is discussed in Chapter 6.

However, if a nonlinear equation can be solved with currently known techniques, Mathematica can often find a solution for you.

EXAMPLE 4.8.1: Solve $4y(y')^2 y'' = (y')^4 + 3$.

SOLUTION: Mathematica can solve this nonlinear equation with `DSolve`.

```
In[796] := DSolve[4 y[x]y′[x]^2 y″[x] == y′[x]^4 + 3,
            y[x], x]
```

Out[796] = $\left\{\left\{y[x] \to 3\, e^{-4C[1]} + \frac{3\, 3^{1/3}\, e^{\frac{4C[1]}{3}}\, (x + C[2])^{4/3}}{4\, 2^{2/3}}\right\}, \left\{y[x] \to 3\, e^{-4C[1]} - \frac{3\, 3^{1/3}\left(1 - i\sqrt{3}\right) e^{\frac{4C[1]}{3}}\, (x + C[2])^{4/3}}{8\, 2^{2/3}}\right\}, \left\{y[x] \to 3\, e^{-4C[1]} - \frac{3\, 3^{1/3}\left(1 + i\sqrt{3}\right) e^{\frac{4C[1]}{3}}\, (x + C[2])^{4/3}}{8\, 2^{2/3}}\right\}\right\}$

Proceeding by hand, let $p = y'$. Then,

$$y'' = p' = \frac{dp}{dx} = \frac{dy}{dx}\frac{dp}{dy} = p\frac{dp}{dy}.$$

With this substitution, we obtain a first-order separable equation.

$$\begin{aligned} 4p^3 y \frac{dp}{dy} &= 3 + p^4 \\ \frac{4p^3}{3 + p^4}\,dp &= \frac{1}{y}\,dy \\ \ln\left(3 + p^4\right) &= \ln|y| + c_1 \\ 3 + p^4 &= c_1 y \\ p &= \pm\left(c_1 y - 3\right)^{1/4}. \end{aligned}$$

Because $p = dy/dx$,

$$\pm\frac{1}{\left(c_1 y - 3\right)^{1/4}}\,dy = dx$$

and integrating and simplifying the result gives us

$$\begin{aligned} \frac{4}{3c_1}\left(c_1 y - 3\right)^{3/4} &= x + c_2 \\ \frac{256}{81c_4}\left(c_1 y - 3\right)^3 &= \left(x + c_2\right)^4. \end{aligned}$$

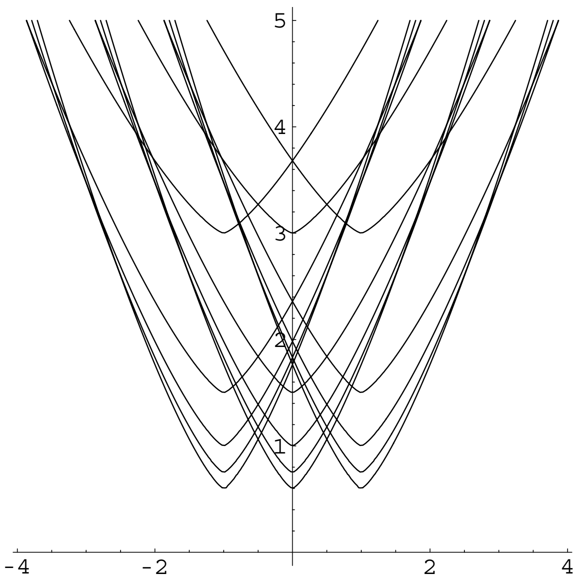


Figure 4-46 Various solutions of $4y(y')^2 y'' = (y')^4 + 3$ shown together

```
In[797]:= Integrate[4p^3/(3+p^4),p]
Out[797]= Log[3 + p^4]

In[798]:= Integrate[1/y,y]
Out[798]= Log[y]

In[799]:= Integrate[(c1 y - 3)^(-1/4),y]
          4 (-3 + c1 y)^(3/4)
Out[799]= -------------------
                3 c1
```

We plot various solutions by graphing level curves of

$$f(x, y) = \frac{256}{81c_4}(c_1 y - 3)^3 - (x + c_2)^4$$

corresponding to 0 for various values of c_1 and c_2 in Figures 4-46 and 4-47.

```
In[800]:= g1 = Table[ - (c2 + x)^4 + (256 (-3 + c1 y)^3)/(81 c1^4),
              {c1, 1, 5}, {c2, -1, 1}]//Flatten;

In[801]:= g2 =
           Map[ContourPlot[#, {x, -5, 5}, {y, -5, 5},
                ContourShading → False, Contours → {0},
                DisplayFunction → Identity,
                PlotPoints → 240]&, g1]
```

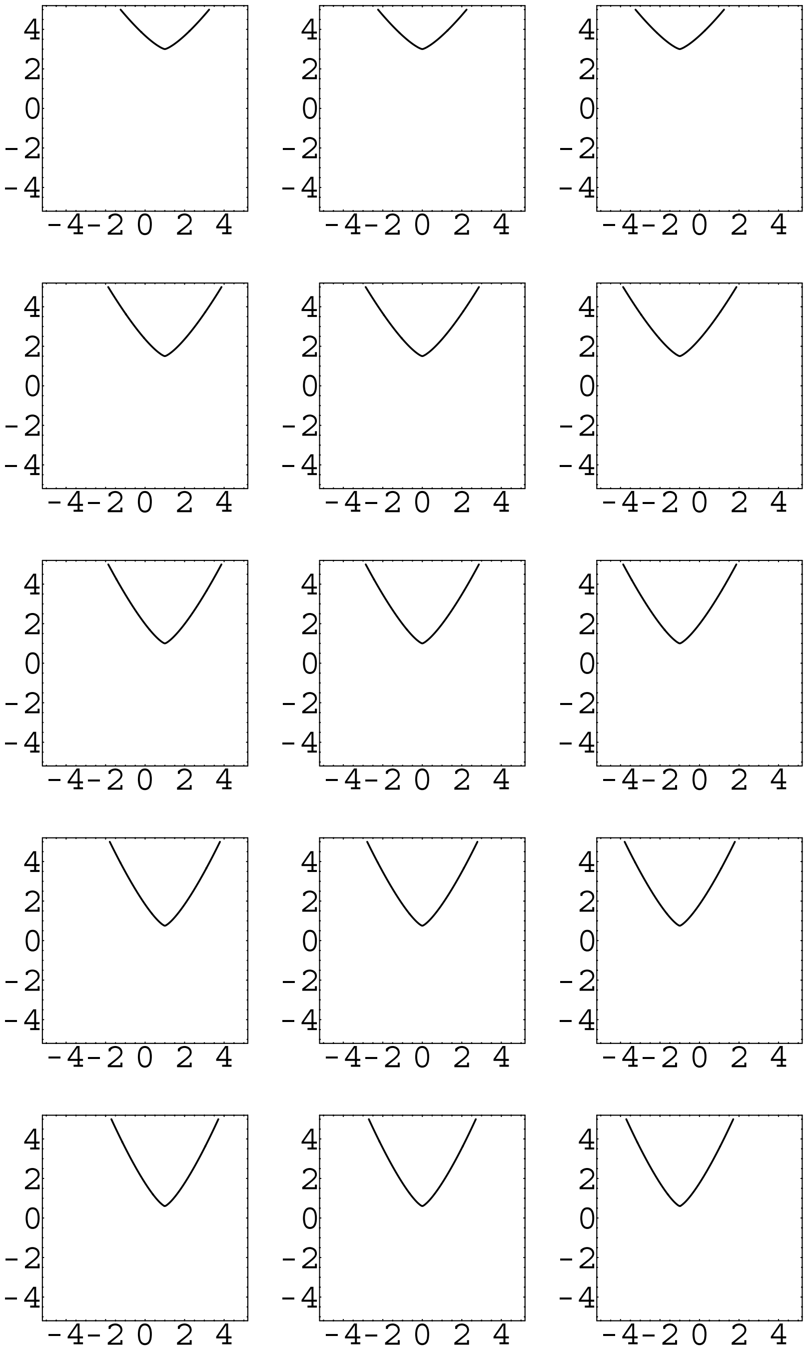


Figure 4-47 Various solutions of $4y(y')^2 y'' = (y')^4 + 3$ shown as an array

```
In[802]:= Show[g2, Frame → False, Axes → Automatic,
            AxesOrigin → {0, 0},
            DisplayFunction → $DisplayFunction]

In[803]:= Show[GraphicsArray[Partition[g2, 3]]]
```

■

EXAMPLE 4.8.2: Solve $\begin{cases} x^2 y'' + (y')^2 - 2xy' = 0 \\ y(2) = 5,\ y'(2) = 1. \end{cases}$

SOLUTION: Mathematica can find the solution to the initial-value problem, which we then graph with `Plot` in Figure 4-48.

```
In[804]:= sol = DSolve[{x^2 y''[x] + y'[x]^2 - 2x y'[x] == 0,
                 y[2] == 5, y'[2] == 1}, y[x], x]
```

Out[804]= $\left\{\left\{y[x] \to \frac{1}{2}\left(14 - 4x + x^2 - 8\,\mathrm{Log}[4] + 8\,\mathrm{Log}[2 + x]\right)\right\}\right\}$

```
In[805]:= Plot[y[x]/.sol, {x, 0, 10}]
```

By hand, we proceed as before by letting $p = y'$. Then $p' = y''$ and the equation becomes

$$x^2 \frac{dp}{dx} + p^2 - 2xp = 0$$
$$x^2\, dp + \left(p^2 - 2xp\right) dx = 0,$$

which is first-order homogeneous of degree 2. Solving for p

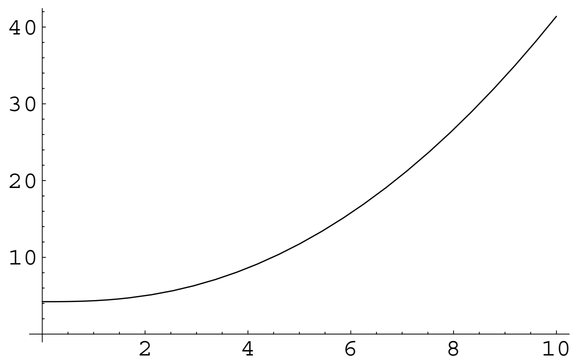


Figure 4-48 Plot of the solution to the initial-value problem

```
In[806]:= DSolve[x^2p'[x]+(p[x]^2-2xp[x]) == 0, p[x], x]
```

Out[806]= $\{\{p[x] \to \frac{x^2}{x + C[1]}\}\}$

and then integrating the result gives us

$$p = \frac{dy}{dx} = \frac{x^2}{x + c_1}$$

$$dy = \frac{x^2}{x + c_1} dx$$

$$y = \frac{1}{2}x^2 - c_1 x + c_1{}^2 \ln|x + c_1| + c_2.$$

```
In[807]:= y = Integrate[x^2/(x + c1), x] + c2
```

Out[807]= $c2 - c1\,x + \frac{x^2}{2} + c1^2\,\text{Log}[c1 + x]$

Applying the initial conditions gives us the nonlinear system

$$2 - 2c_1 + c_2 + c_1^2 \ln|2 + c_1| = 5$$

$$2 - c_1 + \frac{c_1{}^2}{2 + c_1} = 1.$$

```
In[808]:= f1 = y/.x → 2
```

Out[808]= $2 - 2\,c1 + c2 + c1^2\,\text{Log}[2 + c1]$

```
In[809]:= f2 = D[y, x]/.x → 2
```

Out[809]= $2 - c1 + \frac{c1^2}{2 + c1}$

We can see the solution to this system by graphing each equation with `ContourPlot` as shown in Figure 4-49.

```
In[810]:= cp1 = ContourPlot[f1, {c1, -5, 10}, {c2, -5, 10},
                 Contours → {5}, PlotPoints → 120,
                 ContourShading → False,
                 DisplayFunction → Identity];

          cp2 = ContourPlot[f2, {c1, -5, 10}, {c2, -5, 10},
                 Contours → {1}, ContourShading → False,
                 PlotPoints → 120,
                 ContourStyle → GrayLevel[0.4],
                 DisplayFunction → Identity];

          Show[cp1, cp2, DisplayFunction →
                 $DisplayFunction, Frame → False,
                 Axes → Automatic,
                 AxesOrigin → {0, 0}]
```

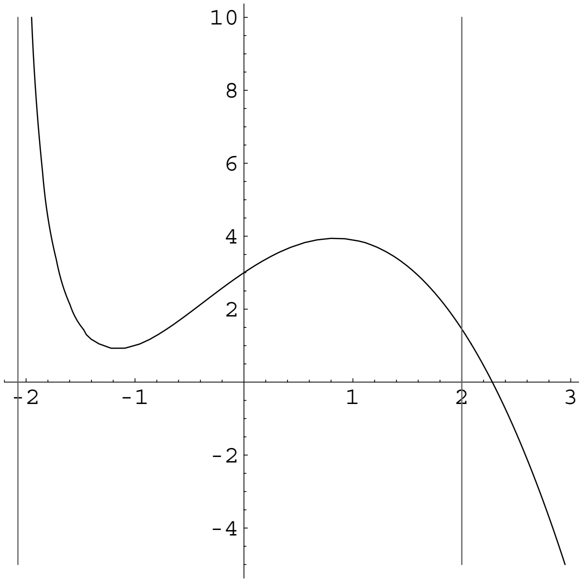


Figure 4-49 The nonlinear system of equations has a unique solution

By hand, solving the second equation for c_1 gives us $c_1 = 2$. Substituting into the first equation and solving for c_2 gives us $c_2 = 7 - 4\ln 4$. We confirm the result with `Solve`.

```
In[811]:= cvals = Solve[{f1 == 5, f2 == 1}]
Out[811]= {{c2 → 7 - 4 Log[4], c1 → 2}}
```

```
In[812]:= y/.cvals[[1]]
```

Out[812]= $7 - 2x + \frac{x^2}{2} - 4\,\text{Log}[4] + 4\,\text{Log}[2 + x]$

■

Of course, in many cases numerical results are most meaningful.

Sources: See texts like Jordan and Smith's *Nonlinear Ordinary Differential Equations*, [17].

EXAMPLE 4.8.3 (Duffing's Equation): Duffing's equation is the second-order nonlinear equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} - x + x^3 = \Gamma \cos \omega t, \tag{4.35}$$

where k, Γ, and ω are positive constants. Depending upon the values of the parameters, solutions to Duffing's equation can exhibit *very* interesting behavior.

SOLUTION: To investigate solutions we define the function duffing-plot. Given k, Γ, and ω,

duffingplot[k, Γ, ω][{x0, y0}, {t, a, b}]

graphs the solution to the initial-value problem

$$\begin{cases} x'' + kx' - x + x^3 = \Gamma \cos \omega t \\ x(0) = x_0,\ x'(0) = y_0 \end{cases} \tag{4.36}$$

for $a \le t \le b$. If {t,a,b} is omitted, the default is $900 \le t \le 1000$. Any options included are passed to the Plot command.

```
In[813]:= Clear[duffingplot]

          duffingplot[k_, capgamma_, ω_][{x0_, y0_},
             ts_:{t, 900, 1000}, opts___] :=
           Module[{numsol},
             numsol = NDSolve[{x''[t] + k x'[t] - x[t]
             +x[t]^3 == capgamma Cos[ω t],
                   x[0] == x0, x'[0] == y0}, x[t],
                   ts, MaxSteps → 100000];
             Plot[x[t]/.numsol, ts, opts]
           ]
```

For example, entering

```
In[814]:= duffingplot[0.3, 0.5, 1.2][{0, 0}]
```

plots the solution to the initial-value problem (4.36) shown in Figure 4-50 if $k = 0.3$, $\Gamma = 0.5$, $\omega = 1.2$, and $x_0 = y_0 = 0$. You can use duffingplot to see how varying the parameters affects the solutions. For example, suppose that $k = 0.3$, $\omega = 1.2$, $x_0 = 0$, and $y_0 = 1$. To see how the solutions vary depending on the value of Γ, we define kvals to be a list of 12 equally spaced numbers between 0 and 0.8 and then use Map to apply duffingplot to the list kvals. In this case, we generate a short-term plot for $0 \le t \le 50$. The resulting graphics are not displayed because we include the option DisplayFunction->Identity in the duffingplot command.

```
In[815]:= kvals = Table[k, {k, 0, 0.8, 0.8/11}];

In[816]:= toshow =
           Map[duffingplot[0.3, #, 1.2][{0, 1},
             {t, 0, 50}, PlotRange → {-3, 3},
             DisplayFunction → Identity]&, kvals]
```

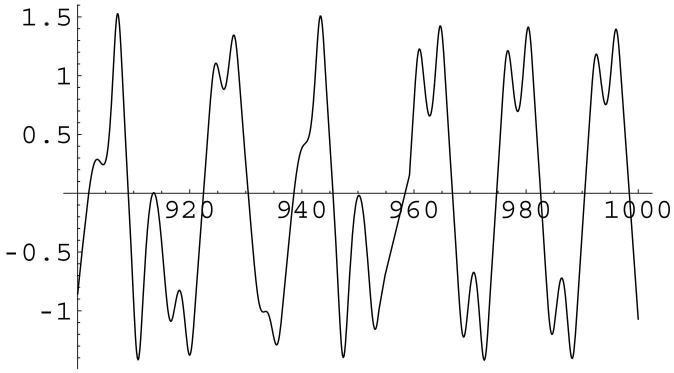


Figure 4-50 Solution to Duffing's equation if $k = 0.3$, $\Gamma = 0.5$, $\omega = 1.2$, and $x_0 = y_0 = 0$

```
Out[816]= {-Graphics-, -Graphics-, -Graphics-, ,
  -Graphics-, -Graphics-, -Graphics-
  -Graphics-, -Graphics-, -Graphics-, -
  Graphics-, -Graphics-, -Graphics-}
```

We then use `Partition`, `Show`, and `GraphicsArray` to display the list of graphics `toshow` in Figure 4-51.

```
In[817]:= Show[GraphicsArray[Partition[toshow, 3]]]
```

We enter nearly identical commands to generate the long-term plot shown in Figure 4-52.

```
In[818]:= toshow =
    Map[duffingplot[0.3, #, 1.2][{0, 1},
      {t, 900, 1000}, PlotRange → {-3, 3},
      DisplayFunction → Identity]&, kvals]
```

The **Fourier transform**, X_k ($k = 1, 2, \ldots, N$) of N equally spaced values of a time series `list` = $\{x_1, x_2, \ldots, x_N\}$ is

$$X_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} x_n e^{2\pi i(n-1)(k-1)/N}. \tag{4.37}$$

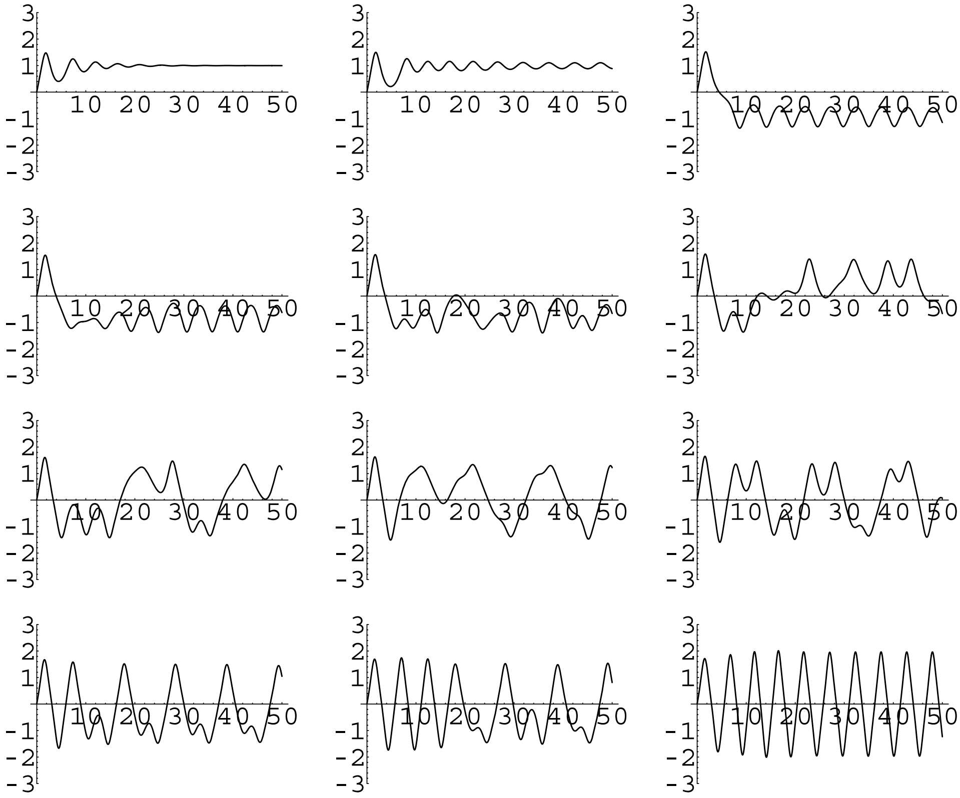


Figure 4-51 Short-term plot: depending upon the value of Γ, some solutions to Duffing's equation exhibit chaotic behavior

The Mathematica command `Fourier[list]` computes the Fourier transform of `list`. The **power spectrum**, $P(\omega_k)$ ($k = 1, 2, \ldots, N$), of the list $\{X_1, X_2, \ldots, X_N\}$ is

$$P(\omega_k) = X_k \bar{X}_k = |X_k|^2 . \tag{4.38}$$

The power spectrum helps detect dominant frequencies. See Jordan and Smith [17].

We define the function `duffingpower` to compute the power spectrum of Duffing's equation. Given the appropriate parameter values and initial conditions, `duffingpower` returns $P(\omega_{2000})$. The 2000 sample points are the value of $x(t_n)$ for $t_n = 0.5n$, $n = 1, \ldots, 2000$.

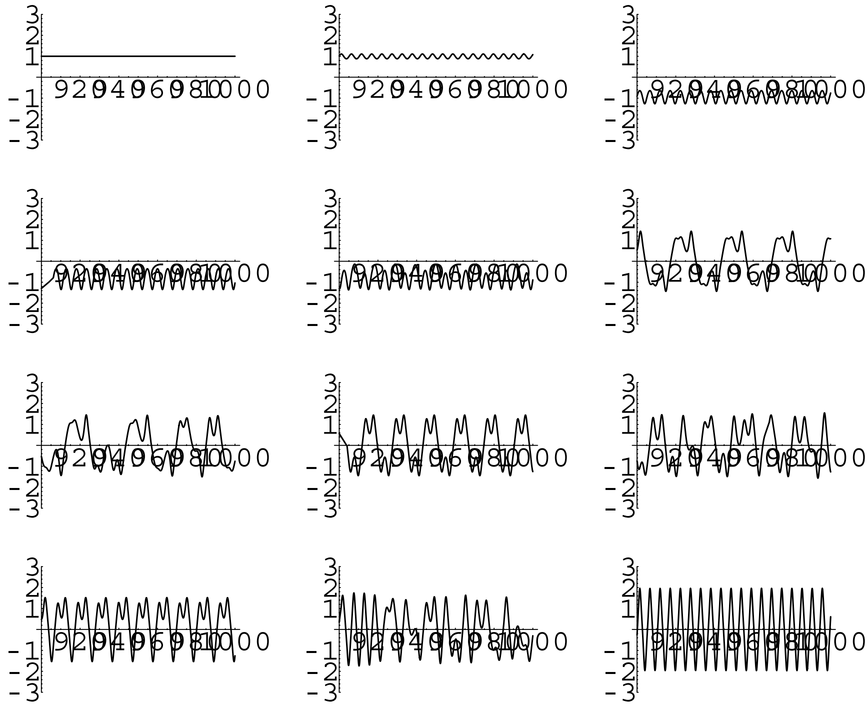


Figure 4-52 Long-term plot: depending upon the value of Γ, some solutions to Duffing's equation exhibit chaotic behavior

```
In[819]:= Show[GraphicsArray[Partition[toshow, 3]]]
```

```
In[820]:= Clear[duffingpower, s2, s3]

          duffingpower[k_, capgamma_, ω_][{x0_, y0_},
          omegak_ :2000] :=
            Module[{numsol, s2, s3},
              numsol = NDSolve[{x''[t] + k x'[t] - x[t]
              +x[t]^3 == capgamma Cos[ω t],
                    x[0] == x0, x'[0] == y0},
                    x[t], {t, 0, 1000},
                    MaxSteps → 100000];
              s2 = Table[x[t]/.numsol[[1]],
              {t, 0.5, 1000, 0.5}];
              s3 = Map[Abs[#]^2&,
              Fourier[s2]][[omegak]]
            ]
```

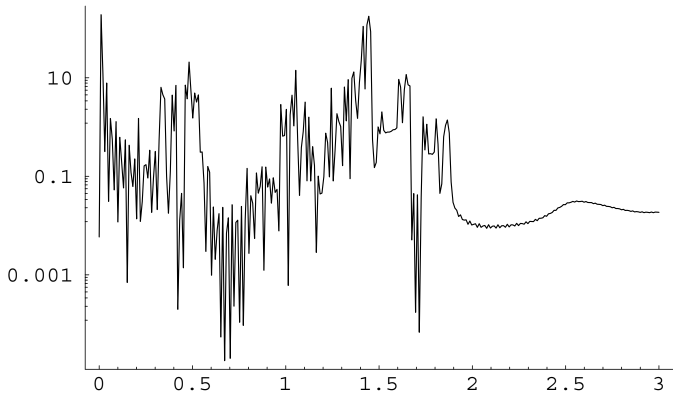


Figure 4-53 Power spectrum of Duffing's equation using $k = 0.3$, $\Gamma = 0.5$, and $x_0 = y_0 = 0$: the horizontal axis corresponds to ω; the vertical axis to the power spectrum $P(\omega_{2000})$

As an illustration, we set $k = 0.3$, $\Gamma = 0.5$, and $x_0 = y_0 = 0$ and then compute the power spectrum for 300 equally spaced values of ω between 0 and 3.

```
In[821]:= t1 = Table[{ω,
              duffingpower[0.3, 0.5, ω][{0, 0}]},
                {ω, 0, 3, 3./299}];
```

We use `LogListPlot`, which is contained in the **Graphics** package that is located in the **Graphics** folder (or directory), to plot the list of points `t1` so that Mathematica uses a logarithmic scale on the y-axis (the vertical axis). See Figure 4-53.

```
In[822]:= << Graphics`Graphics`
```

```
In[823]:= LogListPlot[t1, PlotJoined → True,
            PlotRange → All]
```

For a second-order equation like this, it is often desirable to generate a parametric plot of $x(t)$ versus $x'(t)$. To do so, we set $y = x'$. Then, $y' = x''$ and we see that Duffing's equation (4.35) can be rewritten as the nonlinear system

$$\begin{aligned} x' &= y \\ y' + ky - x + x^3 &= \Gamma \cos \omega t. \end{aligned} \tag{4.39}$$

We define the function `duffingparamplot` to graph solutions of the initial-value problem

$$\begin{cases} x' = y \\ y' + ky - x + x^3 = \Gamma \cos \omega t \\ x(0) = x_0,\ y(0) = y_0 \end{cases}$$

in the same way as we defined `duffingplot`.

```
In[824]:= Clear[duffingparamplot,x,y]

          duffingparamplot[k_,capgamma_,ω_][{x0_,y0_},
          ts_:{t,800,1000},
             opts___] :=
           Module[{numsol},
             numsol = NDSolve[{y′[t] + k y[t] - x[t]
             +x[t]^3 == capgamma Cos[ω t],
                  y[t] == x′[t],x[0] == x0,y[0] == y0},
                  {x[t],y[t]},ts,
                MaxSteps → 100000];
             ParametricPlot[Evaluate[{x[t],y[t]}
             /.numsol],ts,opts]
           ]
```

For example, entering

```
In[825]:= duffingparamplot[0.3,0.5, 0.2][{0,1},
           {t,800,1000}]
```

plots $x(t)$ versus $x'(t)$ if $k = 0.3$, $\Gamma = 0.5$, $\omega = 0.2$, $x(0) = 0$, and $y(0) = x'(0) = 1$ as shown in Figure 4-54.

With the following commands, we set $k = 0.3$, $\Gamma = 0.5$, $x(0) = 0$, and $y(0) = x'(0) = 1$. We then plot $x(t)$ versus $x'(t)$ for 12 equally spaced values of ω between 0 and 1.5.

```
In[826]:= kvals = Table[k,{k,0,1.5,1.5/11}];

In[827]:= toshow =
           Map[duffingplot[0.3,0.5,#][{0,1},
             {t,0,50},PlotRange → {-3,3},
             DisplayFunction → Identity]&,kvals]
```

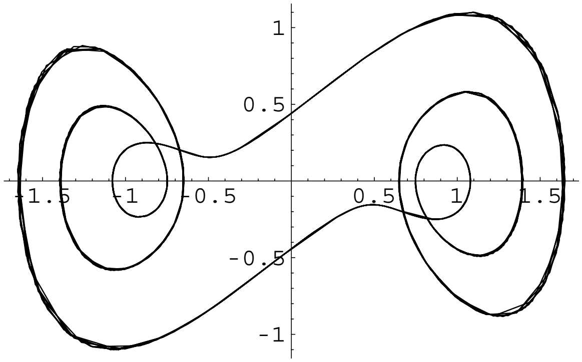


Figure 4-54 A parametric plot of $x(t)$ versus $x'(t)$ for a solution to Duffing's equation

The results are shown as an array in Figure 4-55.

```
In[828]:= Show[GraphicsArray[Partition[toshow, 3]]]
```

The long-term plot shown in Figure 4-56 is generated with nearly identical commands.

```
In[829]:= toshow =
            Map[duffingparamplot[0.3, 0.5, #][{0, 1},
            {t, 800, 1000},
                  PlotRange → {{-3, 3}, {-3, 3}},
                  AspectRatio → 1,
                  DisplayFunction → Identity]&,
                  kvals]
```

```
In[830]:= Show[GraphicsArray[Partition[toshow, 3]]]
```

The **Poincaré plots** (or **returns**) are obtained by plotting

$$x = x(2n\pi/\omega)$$
$$x' = y(2n\pi/\omega).$$

We define the function `duffingpoincareplot` to generate Poincaré plots for Duffing's equation.

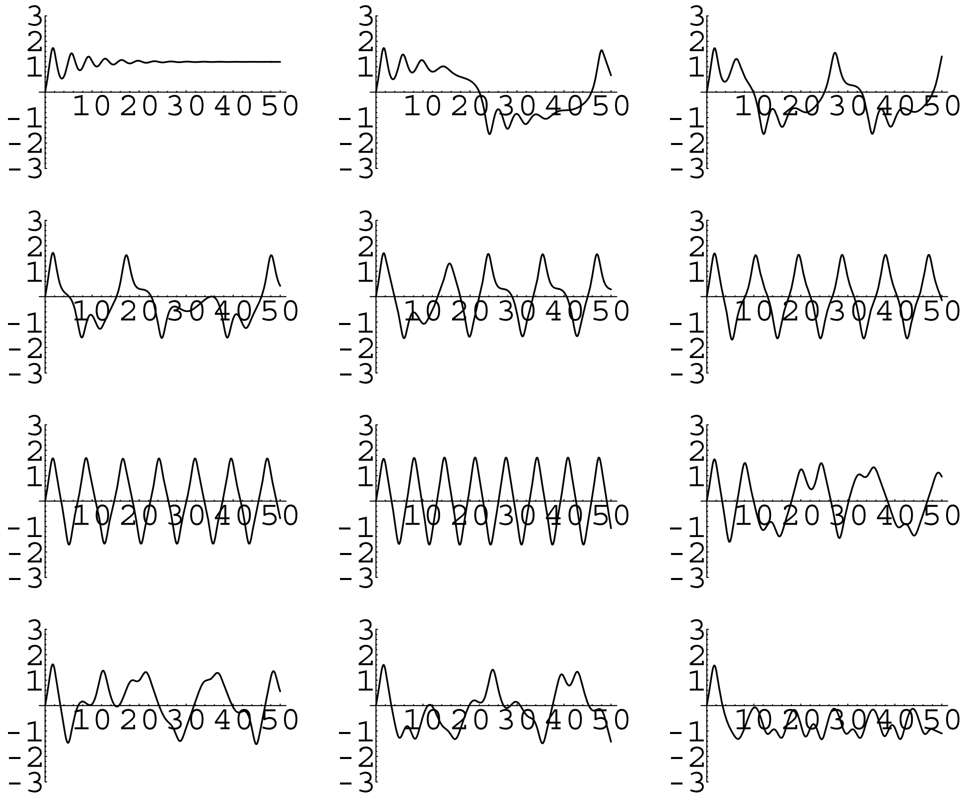


Figure 4-55 Short-term plot: sensitivity of solutions of Duffing's equation to ω

```
In[831]:= Clear[duffingpoincareplot, x, y, t1]

          duffingpoincareplot[k_,
          capgamma_, ω_][{x0_, y0_}, ns_ :{n, 1, 2000},
              opts___] := Module[{numsol, t1},
              numsol = NDSolve[{y'[t] + k x'[t] - x[t]
              +x[t]^3 == capgamma Cos[ω t],
                    y[t] == x'[t], x[0] == x0, y[0] == y0},
                    {x[t], y[t]}, {t, 0, 12000},
                  MaxSteps → 1000000];
              t1 = Table[{x[t], y[t]}
              /.numsol[[1]]/.t → 2 n π/ω, ns];
              ListPlot[t1]
            ]
```

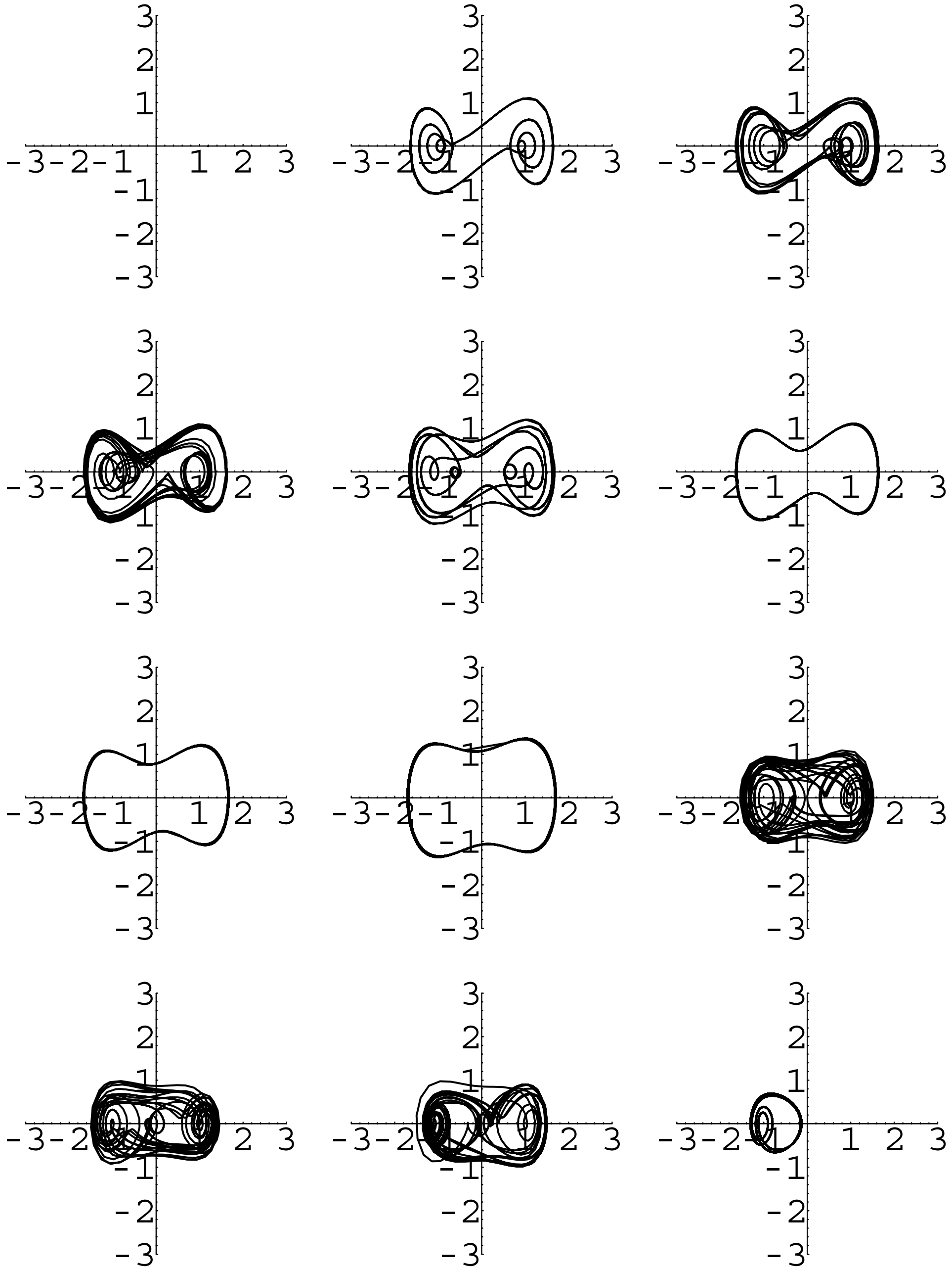


Figure 4-56 Long-term plot: sensitivity of solutions of Duffing's equation to ω

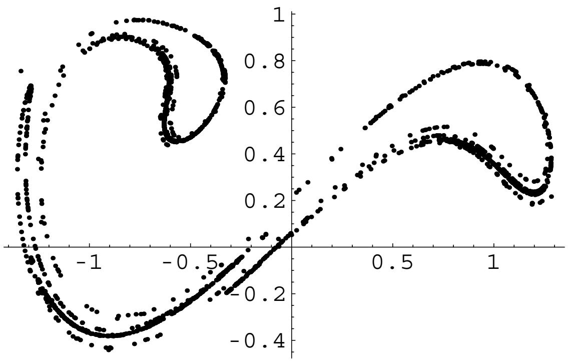


Figure 4-57

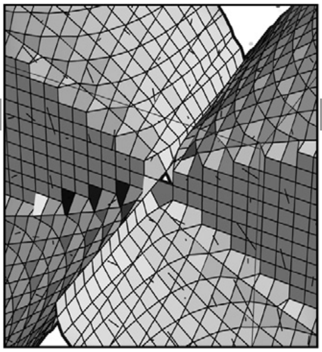
In Figure 4-57, we use `duffingpoincareplot` to generate a Poincaré plot for Duffing's equation if $k = 0.3$, $\Gamma = 0.4$, $\omega = 1.2$, $x(0) = 0$, and $y(0) = x'(0) = 1$.

```
In[832]:= f1 = duffingpoincareplot[0.3, 0.5, 1.2]
             [{0, 1}];
```

■

Applications of Higher-Order Differential Equations

5



In Chapter 4, we discussed several techniques for solving higher-order differential equations. In this chapter, we illustrate how some of these methods can be used to solve initial-value problems that model physical situations.

5.1 Harmonic Motion

5.1.1 Simple Harmonic Motion

Suppose that a mass is attached to an elastic spring that is suspended from a rigid support such as a ceiling. According to Hooke's law, the spring exerts a restoring force in the upward direction that is proportional to the displacement of the spring.

> **Hooke's Law:** $F = ks$, where $k > 0$ is the constant of proportionality or spring constant, and s is the displacement of the spring.

A spring has natural length b. When a mass is attached to the spring, it is stretched s units past its natural length to the equilibrium position $x = 0$. When the system is put into motion, the displacement from $x = 0$ at time t is given by $x(t)$.

By Newton's Second Law of Motion, $F = ma = m\,d^2x/dt^2$, where m represents mass and a represents acceleration. If we assume that there are no other forces acting on the mass, then we determine the differential equation that models this situation in the following way:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= \sum \left(\text{forces acting on the system}\right) \\ &= -k(s+x)+mg \\ &= -ks-kx+mg. \end{aligned}$$

At equilibrium $ks = mg$, so after simplification, we obtain the differential equation

$$m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx = 0.$$

The two initial conditions that are used with this problem are the initial displacement $x(0) = \alpha$ and initial velocity $dx/dt(0) = \beta$. Hence, the function $x(t)$ that describes the displacement of the mass with respect to the equilibrium position is found by solving the initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + kx = 0 \\ x(0) = \alpha, \ \dfrac{dx}{dt}(0) = \beta. \end{cases} \tag{5.1}$$

The differential equation in initial-value problem (5.1) disregards all retarding forces acting on the motion of the mass.

The solution $x(t)$ to this problem represents the displacement of the mass at time t. Based on the assumptions made in deriving the differential equation (the positive direction is down), positive values of $x(t)$ indicate that the mass is beneath the equilibrium position while negative values of $x(t)$ indicate that the mass is above the equilibrium position.

EXAMPLE 5.1.1: A mass weighing 60 lb stretches a spring 6 inches. Determine the function $x(t)$ that describes the displacement of the mass if the mass is released from rest 12 inches below the equilibrium position.

SOLUTION: First, the spring constant k must be determined from the given information. By Hooke's law, $F = ks$, so we have $60 = k \cdot 0.5$. Therefore, $k = 120$ lb/ft. Next, the mass, m, must be determined using $F = mg$. In this case, $60 = m \cdot 32$, so $m = 15/8$ slugs. Because $k/m = 64$

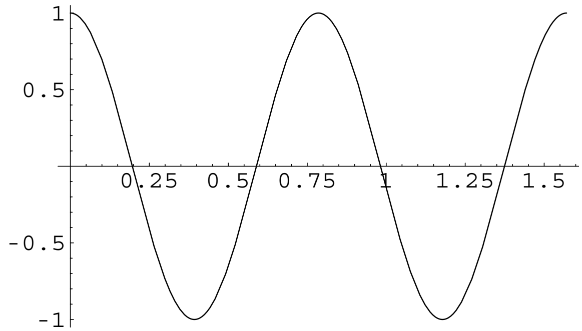


Figure 5-1 Simple harmonic motion

and 12 inches equals 1 foot, the initial-value problem that needs to be solved is

$$\begin{cases} x'' + 64x = 0 \\ x(0) = 1,\ x'(0) = 0. \end{cases}$$

This problem is now solved with `DSolve`, and the resulting output is named `de1`.

```
In[833]:= Clear[x, t, de1]

          de1 =
            DSolve[{x''[t] + 64 x[t] == 0, x[0] == 1,
              x'[0] == 0}, x[t], t]
Out[833]= {{x[t] → Cos[8 t]}}
```

We graph the solution with `Plot` in Figure 5-1.

```
In[834]:= Plot[x[t]/.de1, {t, 0, π/2}]
```

In order to better understand the relationship between the formula obtained in this example and the motion of the mass on the spring, an alternate approach is taken here. We begin by defining `sol` to be the solution to the initial-value problem: given t, `sol[t]` returns the value of $\cos 8t$.

```
In[835]:= Clear[sol]

In[836]:= sol[t_] = de1[[1, 1, 2]]
Out[836]= Cos[8 t]
```

Then, the function `zigzag` is defined to produce a list of points joined by line segments to represent the graphics of a spring. Given ordered pairs (a, b) and (c, d), a positive integer n, and a "small" number ϵ, `zigzag[{a,b},{c,d},n,eps]` connects the set of points

$$(a, b),\ \left(a-\epsilon, b+\frac{d-b}{n}\right),\ \left(a-\epsilon, b+2\frac{d-b}{n}\right), \ldots, \left(a+(-1)^i\epsilon, b+i\frac{d-b}{n}\right), \ldots$$
$$\left(a+(-1)^{n-1}\epsilon, b+(n-1)\frac{d-b}{n}\right),\ (c, d)$$

Note that we will always have $a = c$.

with line segments.

```
In[837]:= Clear[spring, zigzag, length, points, pairs]

          zigzag[{a_, b_}, {c_, d_}, n_, ε_] :=
            Module[{length, points, pairs},
              length = d - b;
              points = Table[b + (i length)/n, {i, 1, n - 1}];
              pairs = Table[{a + (-1)^i ε, points[[i]]},
                        {i, 1, n - 1}];
              PrependTo[pairs, {a, b}];
              AppendTo[pairs, {c, d}];
              Line[pairs]]
```

The function `spring` produces the graphics of a point (the mass attached to the end of the spring) as well as that of the spring obtained with `zigzag`. The result of entering `spring[t]` when displayed with `Show` looks like a spring with a mass attached.

```
In[838]:= spring[t_] :=
            Show[
              Graphics[{zigzag[{0, -sol[t]},
                  {0, 1}, 20, 0.05], PointSize[0.075],
              Point[{0, -sol[t]}]}], Axes → Automatic,
              AxesStyle → GrayLevel[0.5], Ticks → None,
              PlotRange → {{-1, 1}, {-3/2, 3/2}},
              AspectRatio → 1, DisplayFunction → Identity]
```

A list of graphics is produced in `somegraphs` for values of t from $t = 0$ to $t = \pi/2$ using increments of $\pi/16$.

```
In[839]:= somegraphs = Table[spring[t], {t, 0, π/2, π/16}]
Out[839]= {-Graphics-, -Graphics-, -Graphics-,
            -Graphics-, -Graphics-, -Graphics-,
            -Graphics-, -Graphics-, -Graphics-}
```

This list of nine graphics objects is then partitioned into groups of three with `Partition` in `toshow` for use with `GraphicsArray`.

```
In[840]:= toshow = Partition[somegraphs, 3]
Out[840]= {{-Graphics-, -Graphics-, -Graphics-},
           {-Graphics-, -Graphics-, -Graphics-},
           {-Graphics-, -Graphics-, -Graphics-}}
```

We then display the array of graphics objects `toshow` with `Show` and `GraphicsArray` in Figure 5-2. We see that the plots displayed show

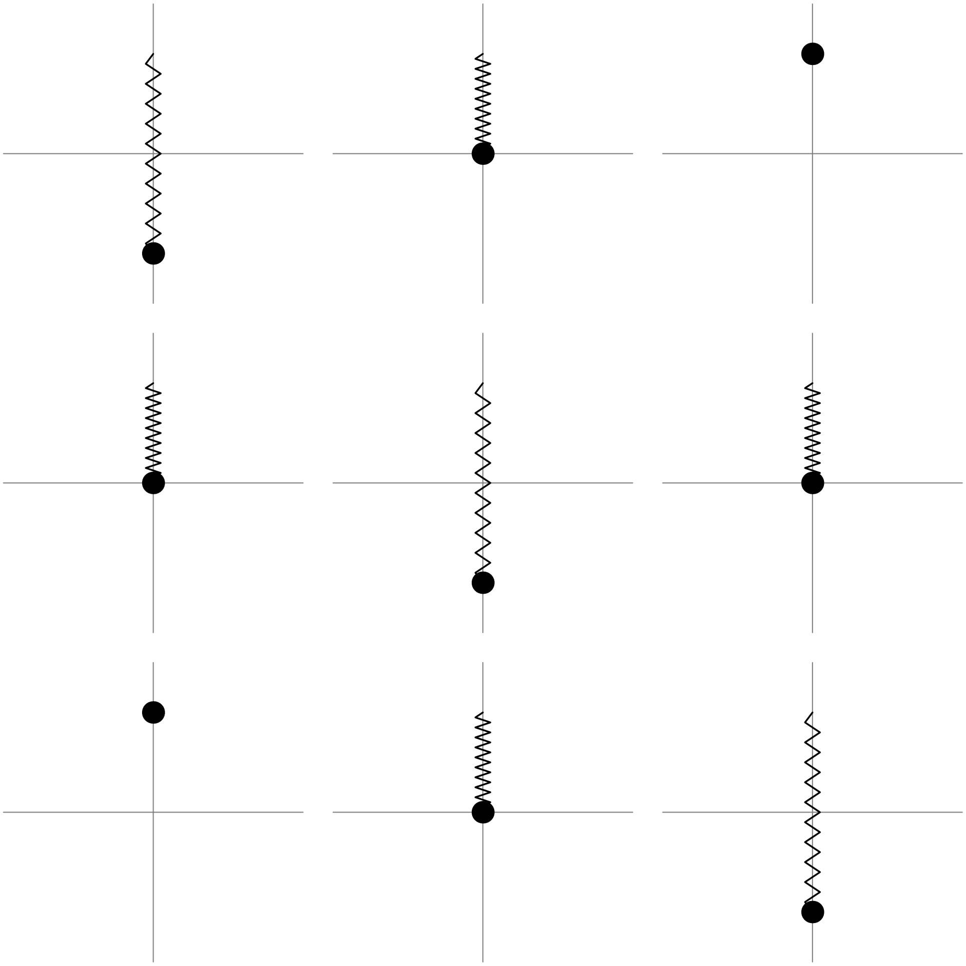


Figure 5-2 Simple harmonic motion: a spring

the displacement of the mass at the values of time from $t = 0$ to $t = \pi/2$ using increments of $\pi/16$.

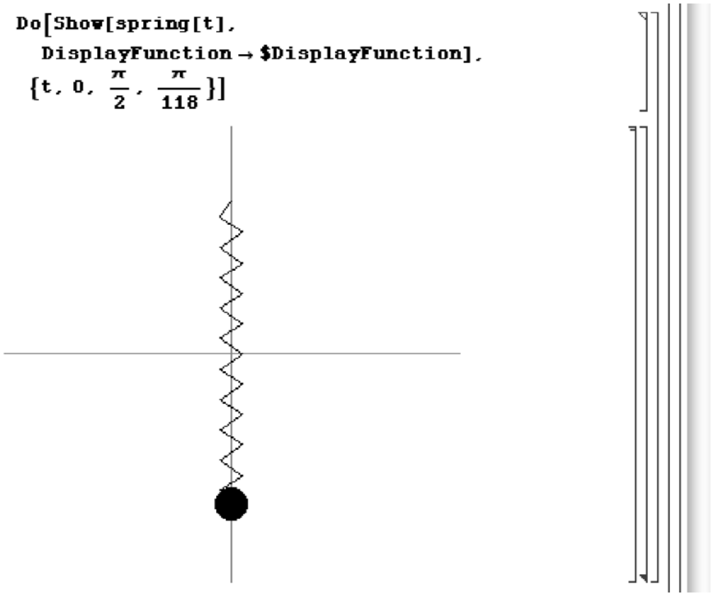
```
In[841]:= Show[GraphicsArray[toshow]]
```

In order to achieve an animation so that we can *see* the motion of the spring, we use a `Do` loop. For example, entering

```
In[842]:= Do[Show[spring[t],
              DisplayFunction → $DisplayFunction],
           {t, 0, π/2, π/118}]
```

displays `spring[t]` for t-values from $t = 0$ to $t = \pi/2$ using increments of $\pi/118$. To animate these graphs, select the cell bracket of the graphs to be animated, go to the menu under **Cell** and select **Animate Selected Graphics**. Alternatively, after selecting the graphs to be animated, you can use the keyboard shortcut **Command-Y** to animate the selected graphics.

When these graphs are animated, as indicated in the following screen shot, we can see the motion of the spring.



Remember that positive values of $x(t)$ indicate that the mass is beneath the equilibrium position while negative values of $x(t)$ indicate that the

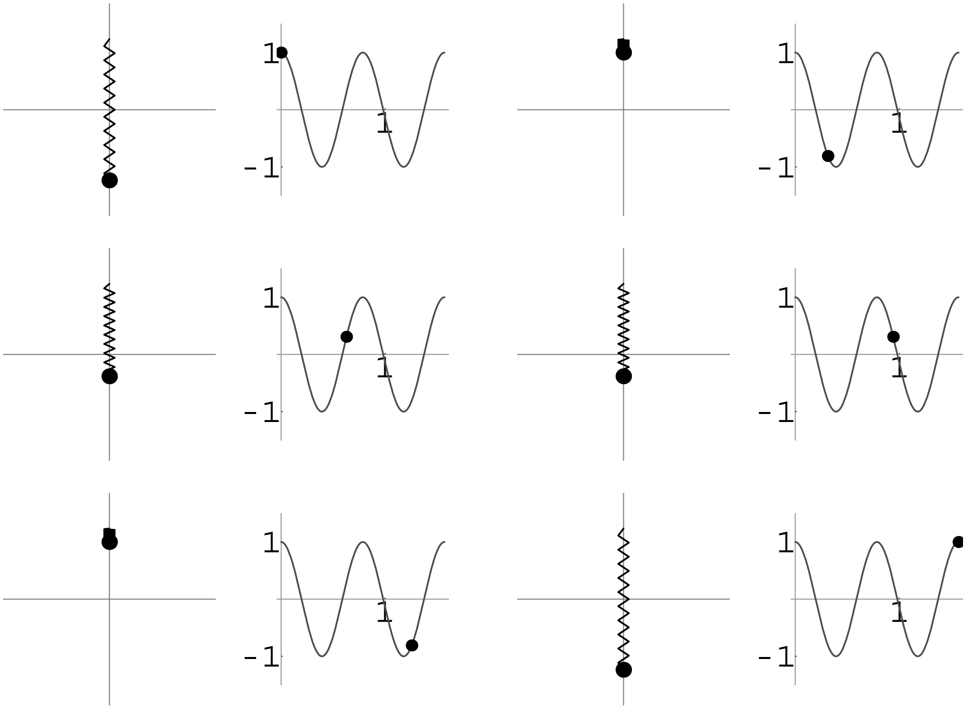


Figure 5-3 Simple harmonic motion illustrated with a spring and a plot

mass is above the equilibrium position. To see this, we graph $x(t)$ in Figure 5-3.

```
In[843]:= graph = Plot[sol[t], {t, 0, π/2},
              PlotStyle → GrayLevel[0.3],
              AxesStyle → GrayLevel[0.6],
              Ticks → {{1}, {-1, 1}},
              PlotRange → {-3/2, 3/2},
              AspectRatio → 1,
              DisplayFunction → Identity];
```

Then, we define p. Given t, p[t] generates a graphics object consisting of the graph of $x(t)$ on the interval $[0, \pi/2]$, which is named graph, and a "small" point placed at $(t, x(t))$.

```
In[844]:= p[t_] := Module[{dp},
            dp =
              Graphics[{PointSize[0.07],
              Point[{t, sol[t]}]}]; Show[graph,
              dp, DisplayFunction → Identity]]
```

We then use `Table` and `GraphicsArray` to generate a set of graphics objects consisting of the graphs of `spring[t]` and `p[t]`, shown side-by-side, for t-values from $t = 0$ to $t = \pi/2$ using increments of $\pi/10$.

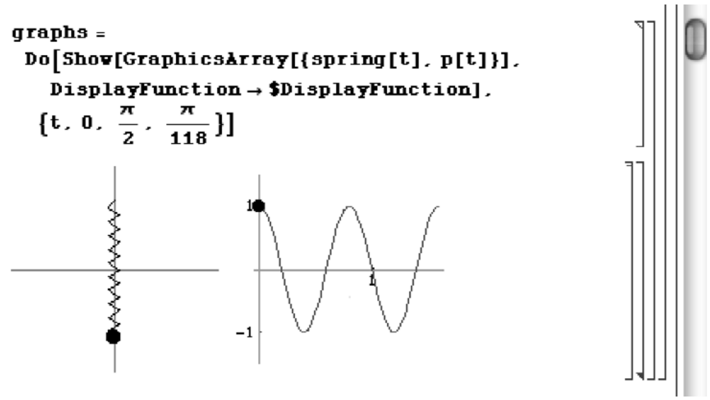
```
In[845]:= moregraphs = Table[GraphicsArray[{spring[t],
                         p[t]}], {t, 0, π/2, π/10}];
```

The list `moregraphs` is then partitioned into two element subsets and displayed using `Show` and `GraphicsArray` in Figure 5-3.

```
In[846]:= toshow = Partition[moregraphs, 2];

In[847]:= Show[GraphicsArray[toshow]]
```

As before, we can use a `Do` loop to generate several graphs and animate the result to see the motion of the spring, as indicated in the following screen shot.

```
In[848]:= graphs =
           Do[Show[GraphicsArray[{spring[t], p[t]}],
           DisplayFunction → $DisplayFunction],
             {t, 0, π/2, π/118}]
```



Notice that the displacement function $x(t) = \cos 8t$ indicates that the spring–mass system never comes to rest once it is set into motion. The solution is periodic, so the mass moves vertically, retracing its motion. Hence, motion of this type is called **simple harmonic motion**.

■

EXAMPLE 5.1.2: An object with mass $m = 1$ slug is attached to a spring with spring constant $k = 4$. (a) Determine the displacement function of the object if $x(0) = \alpha$ and $x'(0) = 0$. Plot the solution for $\alpha = 1, 4, -2$. How does varying the value of α affect the solution? Does it change the values of t at which the mass passes through the equilibrium position? (b) Determine the displacement function of the object if $x(0) = 0$ and $x'(0) = \beta$. Plot the solution for $\beta = 1, 4, -2$. How does varying the value of β affect the solution? Does it change the values of t at which the mass passes through the equilibrium position?

SOLUTION: For (a), the initial-value problem we need to solve is

$$\begin{cases} x'' + 4x = 0 \\ x(0) = \alpha, \ x'(0) = 0 \end{cases}$$

for $\alpha = 1, 4, -2$. We now determine the solution to each of the three problems with DSolve. For example, entering

```
In[849]:= Clear[x]

          de2 = DSolve[{x''[t] + 4 x[t] == 0, x[0] == 1,
                 x'[0] == 0}, x[t], t]
Out[849]= {{x[t] → Cos[2 t]}}
```

solves the initial-value problem if $\alpha = 1$ and names the result de2. Note that the formula for the solution is the second part of the first part of the first part of de2 and is extracted from de2 with Part ([[...]]) by entering de2[[1,1,2]]. Alternatively, if you are using Version 5 you can select and copy the formula in the output and paste it to any location. Similarly, entering

```
In[850]:= de3 = DSolve[{x''[t] + 4 x[t] == 0, x[0] == 4,
                 x'[0] == 0}, x[t], t]
Out[850]= {{x[t] → 4 Cos[2 t]}}
```

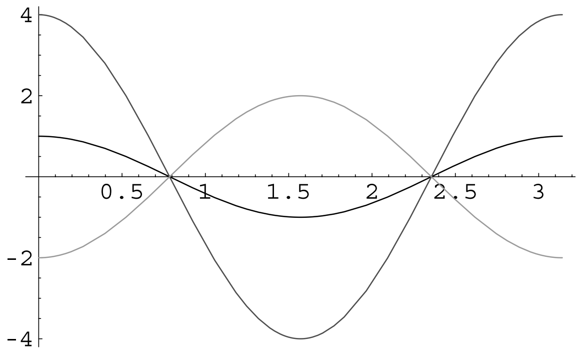


Figure 5-4 Simple harmonic motion: varying the initial displacement

```
In[851]:= de4 = DSolve[{x''[t] + 4 x[t] == 0, x[0] == -2,
                x'[0] == 0}, x[t], t]
Out[851]= {{x[t] → -2 Cos[2 t]}}
```

solves

$$\begin{cases} x'' + 4x = 0 \\ x(0) = 4,\ x'(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} x'' + 4x = 0 \\ x(0) = -2,\ x'(0) = 0 \end{cases}$$

naming the results de3 and de4, respectively. We graph the solutions on the interval $[0, \pi]$ with Plot in Figure 5-4. Note how we use Map to extract the formula for each solution from de2, de3, and de4.

```
In[852]:= toplot = Map[#[[1, 1, 2]]&, {de2, de3, de4}]
Out[852]= {Cos[2 t], 4 Cos[2 t], -2 Cos[2 t]}

In[853]:= Plot[Evaluate[toplot], {t, 0, π},
            PlotStyle → {GrayLevel[0], GrayLevel[0.3],
            GrayLevel[0.6]}]
```

We see that the initial position affects only the amplitude of the function (and direction in the case of the negative initial position). The mass passes through the equilibrium position ($x = 0$) at the same time in all three cases.

For (b), we need to solve the initial-value problem

$$\begin{cases} x'' + 4x = 0 \\ x(0) = 0,\ x'(0) = \beta \end{cases}$$

for $\beta = 1, 4, -2$. In this case, we define a procedure d that, given β, returns the solution to the initial-value problem.

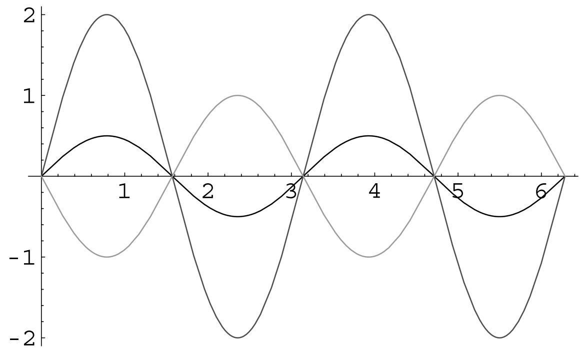


Figure 5-5 Simple harmonic motion: varying the initial velocity

```
In[854]:= d[β_] := Module[{},
                    DSolve[{x''[t] + 4 x[t] == 0,
                    x[0] == 0, x'[0] == β}, x[t], t]]
```

We then use `Map` to apply `d` to the list of numbers `{1,4,-2}` and name the resulting output `solutions`. (Note that the same result is obtained by using the keyboard shortcut for `Map`, `/@`, and entering `solutions=d/@{1,4,-2}`.)

```
In[855]:= solutions = Map[d, {1, 4, -2}]
```

Out[855]= $\{\{\{x[t] \to \frac{1}{2}\, Sin[2\,t]\}\},$
$\{\{x[t] \to 2\, Sin[2\,t]\}\}, \{\{x[t] \to -Sin[2\,t]\}\}\}$

We see that `solutions` consists of three lists. For example, the solution to the initial-value problem when $\beta = -2$ is contained in the third list in `solutions`. We now extract the formula for the `solution` with `Part` (`[[...]]`).

```
In[856]:= solutions[[3, 1, 1, 2]]
Out[856]= - Sin[2 t]
```

All three solutions are graphed together on $[0, 2\pi]$ with `Plot` in Figure 5-5.

```
In[857]:= Plot[Evaluate[x[t]/.solutions], {t, 0, 2π},
           PlotStyle → {GrayLevel[0], GrayLevel[0.3],
           GrayLevel[0.6]}]
```

Notice that varying the initial velocity affects the amplitude (and direction in the case of the negative initial velocity) of each function. The mass passes through the equilibrium position at the same time in all three cases.

■

5.1.2 Damped Motion

Equation (5.1) disregards all retarding forces acting on the motion of the mass and a more realistic model which takes these forces into account is needed.

Studies in mechanics reveal that resistive forces due to damping are functions of the velocity of the motion. Hence, for $c > 0$, $F_R = c\,dx/dt$, $F_R = c\,(dx/dt)^3$, or $F_R = c\,\text{sgn}\,(dx/dt)$, where

$$\text{sgn}\left(\frac{dx}{dt}\right) = \begin{cases} 1, & dx/dt > 0 \\ 0, & dx/dt = 0 \\ -1, & dx/dt < 0 \end{cases}$$

are typically used to represent the damping force. Incorporating damping into equation (5.1) and assuming that $F_R = c\,dx/dt$, the displacement function, $x(t)$, is found by solving the initial-value problem

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0 \\ x(0) = \alpha,\ \dfrac{dx}{dt}(0) = \beta. \end{cases} \tag{5.2}$$

From our experience with second-order ordinary differential equations with constant coefficients in Chapter 4, the solutions to initial-value problems of this type greatly depend on the values of m, k, and c.

Suppose we assume that solutions of the differential equation have the form $x(t) = e^{rt}$. Because $x' = re^{rt}$ and $x'' = r^2e^{rt}$, we have by substitution into the differential equation $mr^2e^{kt} + cre^{rt} + ke^{rt} = 0$, so $e^{rt}\left(mr^2 + cr + k\right) = 0$. The solutions to the characteristic equation are

This calculation is identical to those followed in Chapter 4 for second-order linear homogeneous equations with constant coefficients.

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2a}.$$

Hence, the solution depends on the value of the quantity $c^2 - 4mk$. In fact, problems of this type are characterized by the value of $c^2 - 4mk$ as follows.

1. $c^2 - 4mk > 0$. This situation is said to be **overdamped** because the damping coefficient c is large in comparison to the spring constant k.

2. $c^2 - 4mk = 0$. This situation is described as **critically damped** because the resulting motion is oscillatory with a slight decrease in the damping coefficient c.
3. $c^2 - 4mk > 0$. This situation is called **underdamped** because the damping coefficient c is small in comparison with the spring constant k.

EXAMPLE 5.1.3: Classify the following differential equations as overdamped, underdamped, or critically damped. Also, solve the corresponding initial-value problem using the given initial conditions and investigate the behavior of the solutions.
(a) $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 16x = 0$ subject to $x(0) = 0$ and $\dfrac{dx}{dt}(0) = 1$; and
(b) $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 4x = 0$ subject to $x(0) = 1$ and $\dfrac{dx}{dt}(0) = 1$.

SOLUTION: For (a), we identify $m = 1$, $c = 8$, and $k = 16$ so that $c^2 - 4mk = 0$, which means that the differential equation $x'' + 8x' + 16x = 0$ is critically damped. After defining `de1`, we solve the equation subject to the initial conditions and name the resulting output `sol1`. We then graph the solution shown in Figure 5-6 by selecting and copying the result given in `sol1` to the subsequent `Plot` command. If you prefer working with **InputForm**, the formula for the solution to the initial-value problem is extracted from `sol1` with `sol1[[1,1,2]]`.

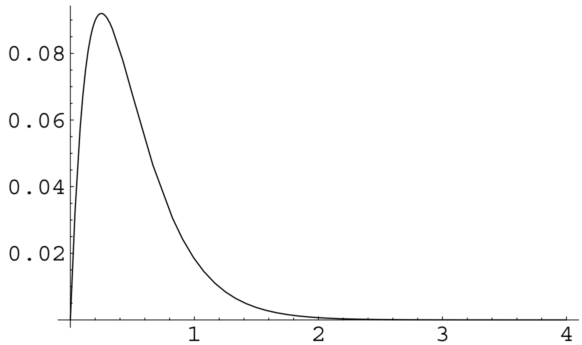


Figure 5-6 Critically damped motion

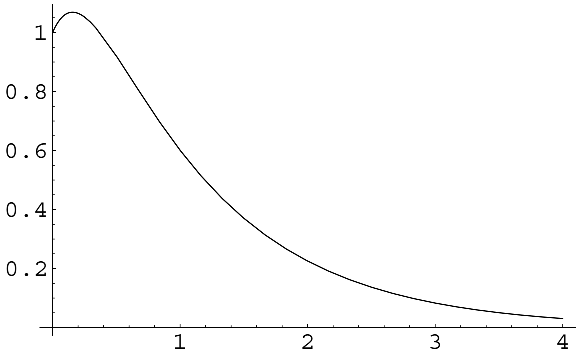


Figure 5-7 Overdamped motion

Thus, entering `Plot[sol[[1,1,2]],{t,0,4}]` displays the same graph as that obtained with the following `Plot` command. Note that replacing `sol1[[1,1,2]]` with `Evaluate[x[t]/.sol1]` in the `Plot` command also produces the same result.

```
In[858]:= Clear[de1, x, t]

          de1 = x''[t] + 8 x'[t] + 16 x[t] == 0;
          sol1 = DSolve[{de1, x[0] == 0, x'[0] == 1},
             x[t], t]
Out[858]= {{x[t] → e^(-4 t) t}}

In[859]:= Plot[e^(-4 t) t, {t, 0, 4}]
```

For (b), we proceed in the same manner. We identify $m = 1$, $c = 5$, and $k = 4$ so that $c^2 - 4mk = 9$ and the equation $x'' + 5x' + 4x = 0$ is overdamped. We then define `de2` to be the equation and the solution to the initial-value problem obtained with `DSolve`, `sol2` and then graph $x(t)$ on the interval [0, 4] in Figure 5-7.

```
In[860]:= Clear[de2, x, t]

          de2 = x''[t] + 5 x'[t] + 4 x[t] == 0;
          sol2 = DSolve[{de2, x[0] == 1, x'[0] == 1},
             x[t], t]
```

Out[860]= $\left\{\left\{x[t] \to \frac{1}{3} e^{-4t} (-2 + 5 e^{3t})\right\}\right\}$

```
In[861]:= Plot[sol2[[1, 1, 2]], {t, 0, 4}]
```

■

EXAMPLE 5.1.4: A 16-lb weight stretches a spring 2 feet. Determine the displacement function if the resistive force due to damping is $F_R = \frac{1}{2}dx/dt$ and the mass is released from the equilibrium position with a downward velocity of 1 ft/sec.

SOLUTION: Because $F = 16$ lb, the spring constant is determined with $16 = k \cdot 2$. Hence, $k = 8$ lb/ft. Also, $m = 16/32 = 1/2$ slug. Therefore, the differential equation is $\frac{1}{2}x'' + \frac{1}{2}x' + 8x = 0$ or $x'' + x' + 16x = 0$. The initial position is $x(0) = 0$ and the initial velocity is $x'(0) = 1$. Thus, we must solve the initial-value problem

$$\begin{cases} x'' + x' + 16x = 0 \\ x(0) = 0,\ x'(0) = 1 \end{cases}$$

which is now solved with `DSolve`.

```
In[862]:= Clear[x, t, deq, sol]

          deq =
           DSolve[{x''[t] + x'[t] + 16 x[t] == 0, x[0] == 0,
              x'[0] == 1}, x[t], t]
```

Out[862]= $\left\{\left\{x[t] \to \frac{2\, e^{-t/2}\, \text{Sin}\left[\frac{3\sqrt{7}\,t}{2}\right]}{3\sqrt{7}}\right\}\right\}$

In[863]:= $\texttt{sol[t\_]} = \frac{2e^{-\frac{t}{2}}\, \text{Sin}\left[\frac{3}{2}\sqrt{7}\,t\right]}{3\sqrt{7}};$

Solutions of this type have several interesting properties. First, the trigonometric component of the solution causes the motion to oscillate. Also, the exponential portion forces the solution to approach zero as t approaches infinity. These qualities are illustrated in the plot of the solution shown in Figure 5-8.

```
In[864]:= Plot[sol[t], {t, 0, 2π}]
```

Physically, the displacement of the mass in this case oscillates about the equilibrium position and eventually comes to rest in the equilibrium position. Of course, with our model the displacement function $x(t) \to 0$

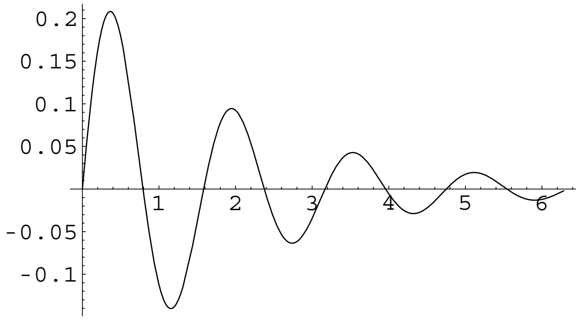


Figure 5-8 Underdamped motion

as $t \to \infty$, but there is no number T such that $x(t) = 0$ for $t > T$ as we might expect from the physical situation. Hence, our model only approximates the behavior of the mass. Notice also that the solution is bounded above and below by the exponential term of the solution $e^{-t/2}$ and its reflection through the horizontal axis, $-e^{-t/2}$. This is illustrated with the simultaneous display of these functions in Figure 5-9.

```
In[865]:= Plot[{sol[t], 2/(3 √7 Exp[t/2]), -2/(3 √7 Exp[t/2])},
     {t, 0, 2π},
     PlotStyle → {GrayLevel[0], GrayLevel[0.5],
     GrayLevel[0.5]}]
```

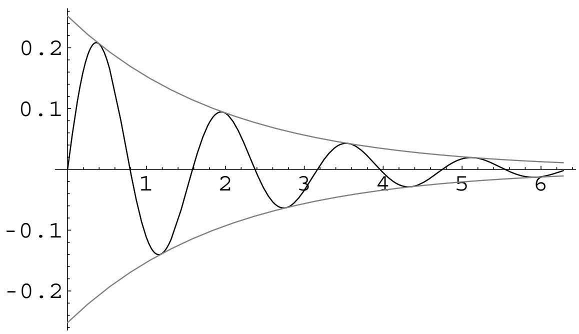


Figure 5-9 The solution shown with its envelope functions

Other questions of interest include: (1) When does the mass first pass through its equilibrium point? (2) What is the maximum displacement of the spring?

The time at which the mass passes through $x = 0$ can be determined in several ways. The solution equals zero at the time that $\sin\left(\frac{3}{2}\sqrt{7}t\right)$ first equals zero after $t = 0$ which occurs when $\frac{3}{2}\sqrt{7}t = \pi$. We use `Solve` to solve this equation for t and then use `N` to approximate the time. (Note that `%` refers to the most recent output.)

```
In[866]:= Solve[3 √7 t / 2 == π, t]

          N[%]

Out[866]= {{t → 2 π / (3 √7)}}

Out[866]= {{t → 0.791607}}
```

Alternatively, we can approximate the time with `FindRoot`.

```
In[867]:= FindRoot[sol[t] == 0, {t, 0.7}]

Out[867]= {t → 0.791607}
```

Similarly, the maximum displacement of the spring is found by finding the first value of t for which the derivative of the solution is equal to zero as done here with `FindRoot`.

```
In[868]:= cp1 = FindRoot[sol′[t] == 0, {t, 0.4}]

Out[868]= {t → 0.364224}
```

The maximum displacement is then given by evaluating the solution for the value of t obtained with `FindRoot`.

```
In[869]:= N[sol[t]/.cp1]

Out[869]= 0.208377
```

Another interesting characteristic of solutions to undamped problems is the time between successive maxima and minima of the solution, called the **quasiperiod**. This quantity is found by first determining the time at which the second maximum occurs with `FindRoot`. Then, the difference between these values of t is taken to obtain the value 1.58321.

```
In[870]:= cp2 = FindRoot[sol′[t] == 0, {t, 2}]

Out[870]= {t → 1.94744}
```

```
In[871]:= cp2[[1, 2]] - cp1[[1, 2]]

Out[871]= 1.58321
```

To investigate the solution further, an animation can be created with the `zigzag` and `spring` commands, which were defined previously. We redefine `zigzag` and `spring`.

```
In[872]:= Clear[spring, zigzag, length, points, pairs]

          zigzag[{a_, b_}, {c_, d_}, n_, ε_] :=
            Module[{length, points, pairs},
              length = d - b;
              points = Table[b + (i length)/n, {i, 1, n - 1}];
              pairs = Table[{a + (-1)^i ε, points[[i]]},
              {i, 1, n - 1}];
              PrependTo[pairs, {a, b}];
              AppendTo[pairs, {c, d}];
              Line[pairs]]

In[873]:= spring[t_] :=
            Show[
              Graphics[{zigzag[{0, -sol[t]}, {0, 0.25},
              20, 0.05], PointSize[0.075],
              Point[{0, -sol[t]}]}], Axes → Automatic,
              AxesStyle → GrayLevel[0.5], Ticks → None,
              PlotRange → {{-1, 1},
              {-0.25, 0.25}}, AspectRatio → 1,
              DisplayFunction → Identity]
```

Next, we display a graphics array consisting of `spring[t]` for t-values from $t = 0$ to $t = 4$ using increments of 4/15 in Figure 5-10.

```
In[874]:= somegraphs = Table[spring[t], {t, 0, 4, 4/15}];

          toshow = Partition[somegraphs, 4];

          Show[GraphicsArray[toshow]]
```

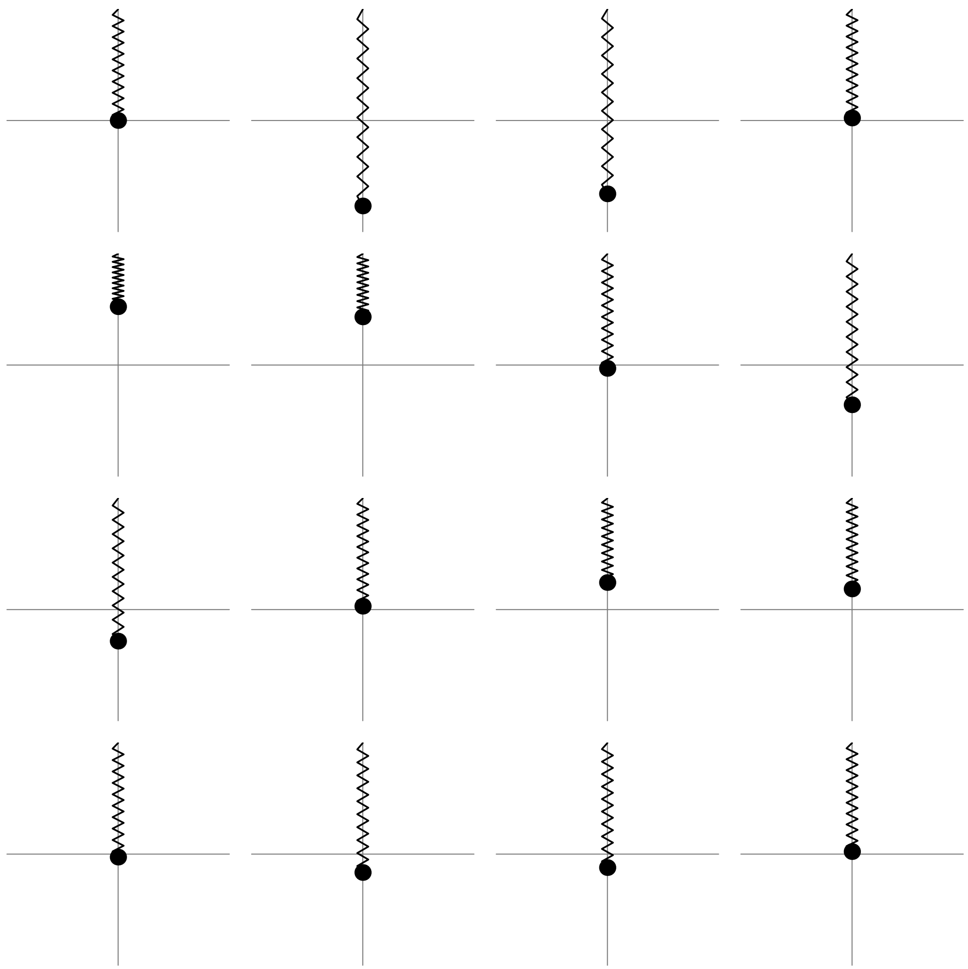
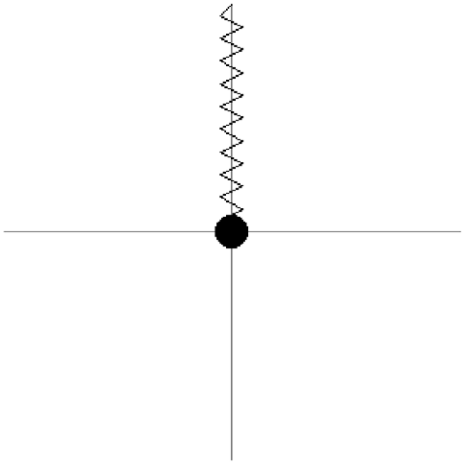


Figure 5-10 The motion of an underdamped spring

To generate an animation, we use a Do loop. We show a screen shot of the resulting animation.

```
In[875] := Do[Show[spring[t],
             DisplayFunction → $DisplayFunction],
            {t, 0, 6, 6/99}]
```

```
Do[Show[spring[t],
  DisplayFunction → $DisplayFunction],
 {t, 0, 6, 6/99}]
```



We can also compare the motion of the spring to the graph of the solution as shown in Figure 5-11.

```
In[876]:= graph = Plot[sol[t], {t, 0, 2π},
                PlotStyle → GrayLevel[0.3],
                AxesStyle → GrayLevel[0.6],
                Ticks → {{2, 4, 6}, {-0.2, 0.2}},
                PlotRange → {-0.25, 0.25},
                AspectRatio → 1,
                DisplayFunction- > Identity];

In[877]:= p[t_] := Module[{dp},
            dp =
            Graphics[{PointSize[0.07],
            Point[{t, sol[t]}]}]; Show[graph,
            dp, DisplayFunction → Identity]]

In[878]:= toshow =
           Partition[Table[GraphicsArray[{spring[t],
            p[t]}], {t, 0, 4, 4/9}], 2];

          Show[GraphicsArray[toshow]]
```

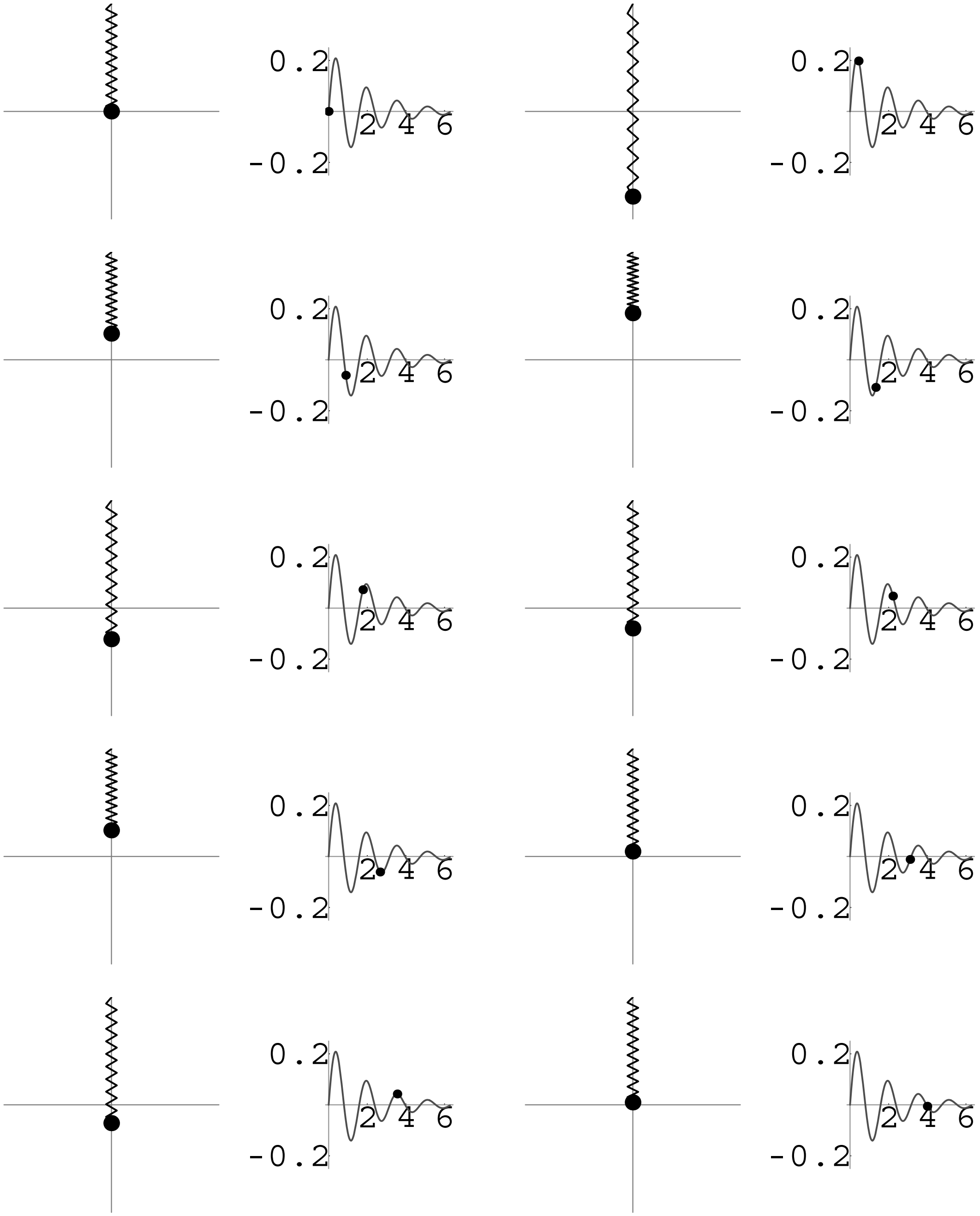
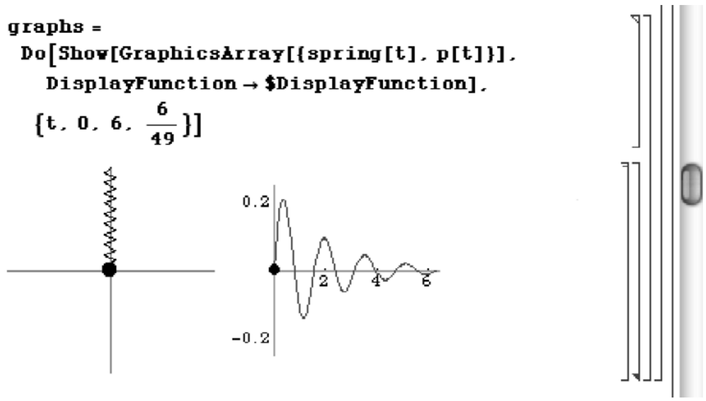


Figure 5-11 Visualizing underdamped motion with a spring and a plot

Alternatively, we can use a Do loop to generate graphics and animate the result. We show a screen shot obtained from animating the resulting graphics from the following Do loop.

```
In[879]:= graphs =
            Do[Show[GraphicsArray[{spring[t],p[t]}],
               DisplayFunction → $DisplayFunction],
             {t, 0, 6, 6/49}]
```



■

EXAMPLE 5.1.5: Suppose that we have the initial-value problem

$$\begin{cases} x'' + cx' + 6x = 0 \\ x(0) = 0,\ x'(0) = 1 \end{cases} \tag{5.3}$$

where $c = 2\sqrt{6}$, $4\sqrt{6}$, and $\sqrt{6}$. Determine how the value of c affects the solution of the initial-value problem.

Be sure to use (lower-case) d instead of (upper-case) D to avoid conflict with the built-in function D.

SOLUTION: We begin by defining the function d. Given c, d[c] solves the initial-value problem (5.3).

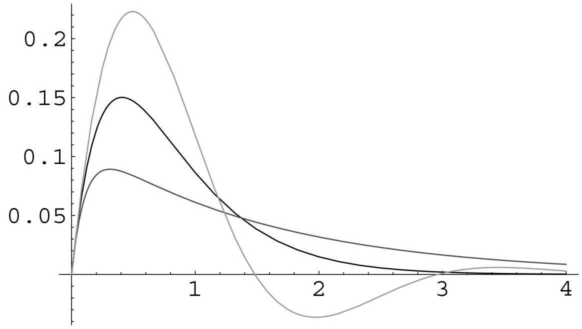


Figure 5-12 Depending on the value of $c > 0$, the motion can be critically damped, overdamped, or underdamped

```
In[880]:= Clear[x, t, d]

          d[c_] :=
            DSolve[{x''[t] + c x'[t] + 6 x[t] == 0, x[0] == 0,
              x'[0] == 1}, x[t], t]
```

We then use `Map` and `d` to find the solution of the initial-value problem for each value of c, naming the resulting list `somesols`.

```
In[881]:= somesols = d/@{2 √6, 4 √6, √6}
```

Out[881]= $\left\{\left\{\left\{x[t]\to e^{-\sqrt{6}\,t}\,t\right\}\right\},\ \left\{\left\{x[t]\to -\frac{e^{\left(-3\sqrt{2}-2\sqrt{6}\right)t}-e^{\left(3\sqrt{2}-2\sqrt{6}\right)t}}{6\sqrt{2}}\right\}\right\},\ \left\{\left\{x[t]\to \frac{1}{3}\sqrt{2}\,e^{-\sqrt{\frac{3}{2}}\,t}\,\mathrm{Sin}\left[\frac{3\,t}{\sqrt{2}}\right]\right\}\right\}\right\}$

Note that each case results in a different classification: $c = 2\sqrt{6}$, critically damped; $c = 4\sqrt{6}$, overdamped; and $c = \sqrt{6}$, underdamped.

All three solutions are graphed together on the interval [0, 4] in Figure 5-12 using different `GrayLevel` settings.

```
In[882]:= Plot[Evaluate[x[t]/.somesols], {t, 0, 4},
            PlotStyle → {GrayLevel[0], GrayLevel[0.3],
            GrayLevel[0.6]}]
```

■

EXAMPLE 5.1.6: Consider the system

$$\begin{cases} x'' + f(t)x' + \frac{5}{4}x = 0, \ t > 0 \\ x(0) = 0, \ x'(0) = 1 \end{cases}$$

where

$$f(t) = \begin{cases} 1, \ 0 \le t < \pi \\ 3, \ \pi \le t < 2\pi \end{cases} \qquad f(t) = f(t - 2\pi), \ t \ge 2\pi$$

in which damping oscillates periodically: the rate at which energy is taken away from the system fluctuates periodically. Find the displacement $x(t)$ for $0 \le t \le 4\pi$.

SOLUTION: For $0 < t < \pi$, the solution to the initial-value problem is found by solving

$$\begin{cases} x'' + x' + \frac{5}{4}x = 0 \\ x(0) = 0, \ x'(0) = 1 \end{cases}$$

which is found with `DSolve` and named `y1`. Similarly, for $\pi < t < 2\pi$, we solve

$$\begin{cases} x'' + 3x' + \frac{5}{4}x = 0 \\ x(\pi) = a, \ x'(\pi) = b \end{cases}$$

where `a=x1[Pi]` and `b=x1'[Pi]` and name the result `y2`.

```
In[883]:= Clear[sol]

          sol =
            DSolve[{x''[t] + x'[t] + 5 x[t]/4 == 0, x[0] == 0,
              x'[0] == 1}, x[t], t]
Out[883]= {{x[t] → e^(-t/2) Sin[t]}}

In[884]:= Clear[x1, x, sol2, a, b]

          x1[t_] = sol[[1, 1, 2]];
          a = x1[π];
          b = x1'[π];

          sol2 =
            DSolve[{x''[t] + 3 x'[t] + 5 x[t]/4 == 0,
              x[π] == a, x'[π] == b}, x[t], t]
```

```
Out[884]= {{x[t] → -1/2 e^(-5 t/2) (-e^(2 π) + e^(2 t))}}
```

In a similar way, we find the solution for $2\pi < t < 3\pi$ in `y3` and the solution for $3\pi < t < 4\pi$ in `y4`.

```
In[885]:= Clear[x2, x, sol3, a, b]

          x2[t_] = sol2[[1, 1, 2]];
          a = x2[2 π];
          b = x2'[2 π];

          sol3 =
            DSolve[{x''[t] - x'[t] + 5 x[t]/4 == 0,
              x[2 π] == a, x'[2 π] == b}, x[t], t]

Out[885]= {{x[t] →
              -1/2 e^(-4 π + t/2) (- Cos[t] + e^(2 π) Cos[t]
              +3 Sin[t] - e^(2 π) Sin[t])}}

In[886]:= Clear[x3, x, sol4, a, b]

          x3[t_] = sol3[[1, 1, 2]];
          a = x3[3 π];
          b = x3'[3 π];

          sol4 =
            DSolve[{x''[t] + 3 x'[t] + 5 x[t]/4 == 0,
              x[3 π] == a, x'[3 π] == b}, x[t], t]

Out[886]= {{x[t] → 1/2 e^(π - 5 t/2) (-e^(4 π) + e^(2 t))}}

In[887]:= x4[t_] = sol4[[1, 1, 2]];
```

We see the damped motion of the system by graphing the pieces of the solution individually and then displaying them together with `Show` in Figure 5-13.

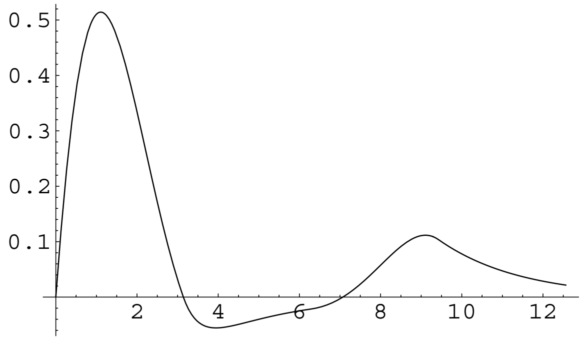


Figure 5-13 Harmonic motion with periodic damping

```
In[888]:= plot1 = Plot[x1[t], {t, 0, π},
                DisplayFunction → Identity];

          plot2 = Plot[x2[t], {t, π, 2 π},
                DisplayFunction → Identity];

          plot3 = Plot[x3[t], {t, 2 π, 3 π},
                DisplayFunction → Identity];

          plot4 = Plot[x4[t], {t, 3 π, 4 π},
                DisplayFunction → Identity];

          show4 = Show[plot1, plot2, plot3, plot4,
                DisplayFunction → $DisplayFunction]
```

■

5.1.3 Forced Motion

In some cases, the motion of the spring is influenced by an external driving force, $f(t)$. Mathematically, this force is included in the differential equation that models the situation as follows:

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + f(t).$$

The resulting initial-value problem is

$$\begin{cases} mx'' + cx' + kx = f(t) \\ x(0) = \alpha, \ x'(0) = \beta. \end{cases} \tag{5.4}$$

Therefore, differential equations modeling forced motion are nonhomogeneous and require the Method of Undetermined Coefficients or Variation of Parameters for solution. We first consider forced motion that is undamped.

EXAMPLE 5.1.7: An object of mass $m = 1$ slug is attached to a spring with spring constant $k = 4$. Assuming there is no damping and that the object is released from rest in the equilibrium position, determine the position function of the object if it is subjected to an external force of (a) $f(t) = 0$, (b) $f(t) = 1$, (c) $f(t) = \cos t$, and (d) $f(t) = \sin t$.

SOLUTION: First, we note that we must solve the initial-value problem

$$\begin{cases} x'' + 4x = f(t) \\ x(0) = 0, \; x'(0) = 0 \end{cases}$$

for each of the forcing functions in (a), (b), (c), and (d). Because we will be solving this initial-value problem for various forcing functions, we begin by defining the function `fm`. Given a function `f` $= f(t)$, `fm[f]` returns the formula for the solution to this initial-value problem.

```
In[889]:= Clear[x, t]

          fm[f_] :=
            DSolve[{x''[t] + 4 x[t] == f, x[0] == 0,
              x'[0] == 0}, x[t], t][[1, 1, 2]]
```

Next, we define `fs` to be the forcing functions in (a)–(d).

```
In[890]:= fs = {0, 1, Cos[t], Sin[t]};
```

We then use `Map` to apply `fm` to `fs` and name the resulting list of functions `somesols`.

```
In[891]:= somesols = Map[fm, fs]
```

$$\text{Out[891]}= \left\{0, \frac{1}{4}(1 - \text{Cos}[2\,t]), \frac{1}{12}(-4\,\text{Cos}[2\,t] + 3\,\text{Cos}[t]\,\text{Cos}[2\,t] + \text{Cos}[2\,t]\,\text{Cos}[3\,t] + 3\,\text{Sin}[t]\,\text{Sin}[2\,t] + \text{Sin}[2\,t]\,\text{Sin}[3\,t]), \frac{1}{12}(-3\,\text{Cos}[2\,t]\,\text{Sin}[t] - 2\,\text{Sin}[2\,t] + 3\,\text{Cos}[t]\,\text{Sin}[2\,t] - \text{Cos}[3\,t]\,\text{Sin}[2\,t] + \text{Cos}[2\,t]\,\text{Sin}[3\,t])\right\}$$

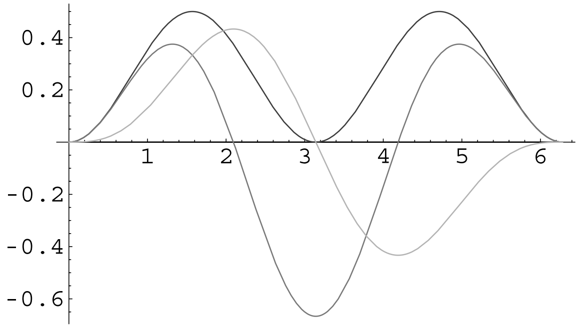


Figure 5-14 Forced motion without damping

From the result, we see that for (a) the solution is $x(t) = 0$. Physically, this solution indicates that because there is no forcing function, no initial displacement from the equilibrium position, and no initial velocity, the object does not move from the equilibrium position.

The nontrivial solutions in (b), (c), and (d) are then graphed on the interval $[0, 2\pi]$ with `Plot` in Figure 5-14.

```
In[892]:= grays = Table[GrayLevel[i], {i, 0, 0.7,
              0.7/3}];
          Plot[Evaluate[somesols],
              {t, 0, 2π}, PlotStyle → grays,
              PlotRange → All]
```

Negative values of x indicate that the mass is *above* the equilibrium position; positive values indicate that the mass is *below* the equilibrium position.

From the graph, we see that for (b) the object never moves above the equilibrium position. This makes sense because $0 \le \cos 2t \le 1$: $x(t) = \frac{1}{4}(1 - \cos 2t)$ for all t. For (c), we see that the mass passes through the equilibrium position twice (near $t = 2$ and $t = 4$) over the period. For (d), we again see that the resulting motion is periodic, although different from that observed in (c).

■

When we studied nonhomogeneous equations, we considered equations in which the nonhomogeneous function was a solution of the corresponding homogeneous equation. This situation is modeled by the initial-value problem

$$\begin{cases} x'' + \omega^2 x = F_1 \cos \omega t + F_2 \sin \omega t + G(t) \\ x(0) = \alpha, \; x'(0) = \beta \end{cases} \tag{5.5}$$

where $\omega > 0$, F_1 and F_2 are constants, and $G = G(t)$ is a function of t. In this case, we say that ω is the **natural frequency** of the system because the solution of the corresponding homogeneous equation, $x'' + \omega^2 x = 0$, is $x_h = c_1 \cos \omega t + c_2 \sin \omega t$.

Note that one of the constants F_1 or F_2 can equal zero and $G = G(t)$ can be identically the zero function.

EXAMPLE 5.1.8: Investigate the effect that the forcing functions (a) $f(t) = \cos 2t$ and (b) $f(t) = \sin 2t$ have on the solution of the initial-value problem

$$\begin{cases} x'' + 4x = f(t) \\ x(0) = 0, \ x'(0) = 0. \end{cases}$$

SOLUTION: We take advantage of the function `fm` defined in Example 5.1.7. In the same manner as in Example 5.1.7, we use `Map` to apply `fm` to each of the forcing functions in (a) and (b). (Note that entering

```
moresols=fm/@{Cos[2 t],Sin[2 t]}
```

produces the same result.)

```
In[893]:= moresols = Map[fm, {Cos[2 t], Sin[2 t]}]

Out[893]= { 1/16 (- Cos[2 t] + Cos[2 t] Cos[4 t]
           +4 t Sin[2 t] + Sin[2 t] Sin[4 t]),
            1/16 (-4 t Cos[2 t] + Sin[2 t]
            - Cos[4 t] Sin[2 t] + Cos[2 t] Sin[4 t])}
```

From the result, we see that the nonperiodic function $y = t \sin 2t$ appears in the result for (a) while the nonperiodic function $y = t \cos 2t$ appears in the result for (b). In each case, we see that the amplitude increases without bound as t increases, as illustrated in Figure 5-15. This indicates that the spring–mass system will encounter a serious problem in that the mass will eventually hit its support (like a ceiling or beam) or its lower boundary (like the ground or floor).

```
In[894]:= Plot[Evaluate[moresols], {t, 0, 2π},
            PlotStyle → {GrayLevel[0], GrayLevel[0.4]}]
```

■

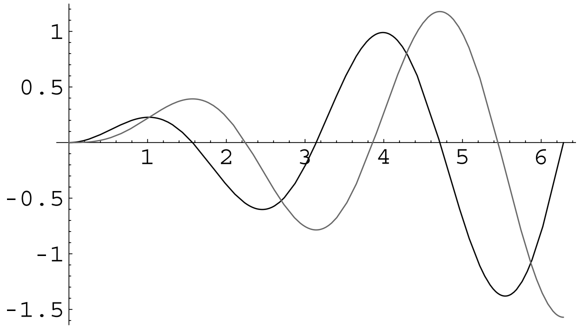


Figure 5-15 Resonance

The phenomenon illustrated in Example 5.1.8 is called **resonance** and can be extended to other situations such as vibrations of an aircraft wing, skyscraper, glass, or bridge. Some of the sources of excitation that lead to the vibration of these structures include unbalanced rotating devices, vortex shedding, strong winds, rough surfaces, and moving vehicles. Therefore, the engineer has to overcome many problems when structures and machines are subjected to forced vibrations.

EXAMPLE 5.1.9: How does slightly changing the value of the argument of the forcing function change the solution of the initial-value problem given in Example 5.1.8? Use the functions (a) $f(t) = \cos 1.9t$ and (b) $f(t) = \cos 2.1t$ with the initial-value problem.

SOLUTION: As in Example 5.1.8, we take advantage of the function `fm` defined in Example 5.1.7. (Note that entering

```
moresols=Map[fm,{Cos[1.9 t],Sin[2.1 t]}]
```

produces the same result as that obtained using `/@`, the keyboard shortcut for `Map`.)

```
In[895]:= moresols = fm/@{Cos[1.9 t], Sin[2.1 t]}
```

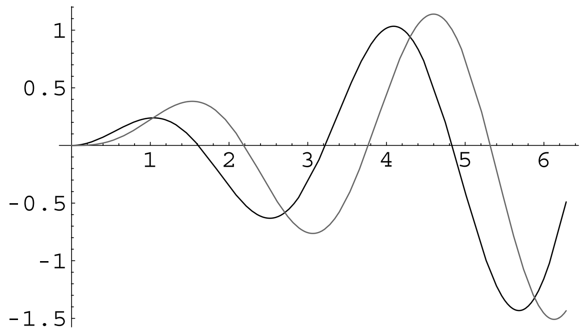


Figure 5-16 Resonance?

```
Out[895]= {-2.5641 e^(0. t) Cos[2. t]
          +2.5 e^(0. t) Cos[0.1 t] Cos[2. t]
          +0.0641026 e^(0. t) Cos[2. t] Cos[3.9 t]
          +0. e^(0. t) Sin[2. t]
          +2.5 e^(0. t) Sin[0.1 t] Sin[2. t]
          +0.0641026 e^(0. t) Sin[2. t] Sin[3.9 t],
           0. e^(0. t) Cos[2. t]
          -2.5 e^(0. t) Cos[2. t] Sin[0.1 t]
          +2.56098 e^(0. t) Sin[2. t]
          -2.5 e^(0. t) Cos[0.1 t] Sin[2. t]
          -0.0609756 e^(0. t) Cos[4.1 t] Sin[2. t]
          +0.0609756 e^(0. t) Cos[2. t] Sin[4.1 t]}
```

The result shows that each solution is periodic and bounded. These solutions are then graphed in Figure 5-16 to reveal the behavior of the curves. If the solutions are plotted over only a small interval, however, resonance *seems* to be present.

Compare Figure 5-16 to the graph generated in Example 5.1.8.

```
In[896]:= Plot[Evaluate[moresols], {t, 0, 2π},
            PlotStyle → {GrayLevel[0], GrayLevel[0.4]}]
```

However, the functions obtained with `fm` clearly indicate that there is no resonance. This is further indicated by graphing the solutions over a longer time interval in Figure 5-17.

```
In[897]:= Plot[moresols[[1]], {t, 0, 40π}, PlotPoints → 200]

In[898]:= Plot[moresols[[2]], {t, 0, 40π}, PlotPoints → 200]
```

■

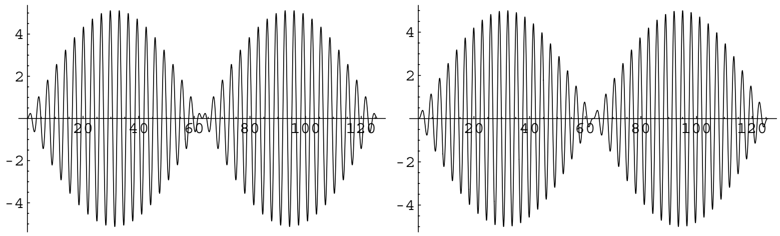


Figure 5-17 No resonance: the solution is periodic

Let us investigate in detail initial-value problems of the form

$$\begin{cases} x'' + \omega^2 x = F\cos\beta t,\ \omega > 0,\ \omega \neq \beta \\ x(0) = 0,\ x'(0) = 0. \end{cases} \tag{5.6}$$

A general solution of the corresponding homogeneous equation is $x_h = c_1 \cos\omega t + c_2 \sin\omega t$. Using the Method of Undetermined Coefficients, we assume that there is a particular solution to the nonhomogeneous equation of the form $x_p = A\cos\beta t + B\sin\beta t$.

In[899] := $\mathbf{x_p[t\_] = a\ Cos[\beta t] + b\ Sin[\beta t];}$

Next, we calculate the corresponding derivatives of this solution

In[900] := $\mathbf{x_p'[t]}$

Out[900]= $\text{b}\,\beta\ \text{Cos}[\text{t}\,\beta] - \text{a}\,\beta\ \text{Sin}[\text{t}\,\beta]$

In[901] := $\mathbf{x_p''[t]}$

Out[901]= $-\text{a}\,\beta^2\ \text{Cos}[\text{t}\,\beta] - \text{b}\,\beta^2\ \text{Sin}[\text{t}\,\beta]$

and substitute into the nonhomogeneous equation $x'' + \omega^2 x = F\cos\beta t$.

In[902] := $\mathbf{step1 = Simplify\left[x_p''[t] + \omega^2 x_p[t]\right] == f\ Cos[\beta t]}$

Out[902]= $-\frac{1}{4}\,\mathbb{e}^{-5\pi/2}\,(\beta^2 - \omega^2)$
$(2\,(-1 + \mathbb{e}^{2\pi})\ \text{Cos}[\text{t}\,\beta] - (-5 + \mathbb{e}^{2\pi})\ \text{Sin}[\text{t}\,\beta]) == \text{f}\ \text{Cos}[\text{t}\,\beta]$

This equation is true for all values of t. In particular, substituting $t = 0$ and $t = \pi/(2\beta)$ yields two equations

```
In[903]:= eq1 = step1 /. t- > 0

Out[903]= -a (β² - ω²) == f

In[904]:= eq2 = step1 /. t- > π/(2β)

Out[904]= -b (β² - ω²) == 0
```

that we then solve for A and B to see that $A = \dfrac{F}{\omega^2 - \beta^2}$ and $B = 0$ and a general solution of the nonhomogeneous equation is

$$x = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F}{\omega^2 - \beta^2} \cos \beta t.$$

Application of the initial conditions yields the solution

$$x = \frac{F}{\omega^2 - \beta^2} (\cos \beta t - \cos \omega t) = \frac{F}{\beta^2 - \omega^2} (\cos \omega t - \cos \beta t).$$

We can use `DSolve` and `Simplify` to solve the initial-value problem (5.6) as well.

```
In[905]:= DSolve[{x''[t] + ω²x[t] == f Cos[βt], x[0] == 0,
              x'[0] == 0}, x[t], t]//Simplify

Out[905]= {{x[t] → f (Cos[t β] - Cos[t ω]) / (-β² + ω²)}}
```

Using the trigonometric identity $\frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] = \sin\alpha \sin\beta$, we have

$$x = \frac{2F}{\omega^2 - \beta^2} \sin\left(\frac{\omega + \beta}{2} t\right) \sin\left(\frac{\omega - \beta}{2} t\right).$$

These solutions are of interest because of what they indicate about the motion of the spring under consideration. Notice that the solution can be represented as

$$x = A(t) \sin\left(\frac{\omega + \beta}{2} t\right), \quad \text{where} \quad A(t) = \frac{2F}{\omega^2 - \beta^2} \sin\left(\frac{\omega - \beta}{2} t\right).$$

Therefore, if the quantity $\omega - \beta$ is small, $\omega + \beta$ is relatively large in comparison. Hence, the function $\sin\left(\frac{\omega+\beta}{2} t\right)$ oscillates quite frequently because it has period

$\pi/(\omega+\beta)$. Meanwhile, the function $\sin\left(\frac{\omega-\beta}{2}t\right)$ oscillates relatively slowly because it has period $\pi/|\omega-\beta|$, so the functions $\pm\frac{2F}{\omega^2-\beta^2}\sin\left(\frac{\omega-\beta}{2}t\right)$ form an **envelope** for the solution.

EXAMPLE 5.1.10: Solve the initial-value problem

$$\begin{cases} x'' + 4x = f(t) \\ x(0) = 0,\ x'(0) = 0 \end{cases}$$

with (a) $f(t) = \cos 3t$ and (b) $f(t) = \cos 5t$.

SOLUTION: Again, we use `fm`, defined in Example 5.1.7, to solve the initial-value problem in each case.

```
In[906]:= Clear[x, t]

          fm[f_] :=
            DSolve[{x''[t] + 4 x[t] == f, x[0] == 0,
              x'[0] == 0}, x[t], t][[1, 1, 2]]

In[907]:= fs = {Cos[3 t], Cos[5 t]};

In[908]:= somesols = Map[fm, fs]
Out[908]= {1/20 (4 Cos[2 t] - 5 Cos[t] Cos[2 t]
          + Cos[2 t] Cos[5 t] + 5 Sin[t] Sin[2 t]
          + Sin[2 t] Sin[5 t]),
          1/84 (4 Cos[2 t] - 7 Cos[2 t] Cos[3 t]
          +3 Cos[2 t] Cos[7 t]
          +7 Sin[2 t] Sin[3 t]
          +3 Sin[2 t] Sin[7 t])}
```

The solution for (a) is graphed in Figure 5-18 and named `p1` for later use.

```
In[909]:= p1 = Plot[somesols[[1]], {t, 0, 6 π}]
```

Using the formula obtained earlier for the functions that "envelope" the solution, we have $x(t) = \pm\sin\left(\frac{1}{2}t\right)$. These functions are graphed in `p2` and displayed with `p1` with `Show` in Figure 5-19.

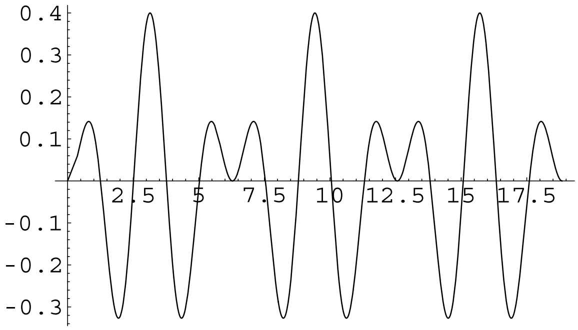


Figure 5-18 The forcing function causes *beats*

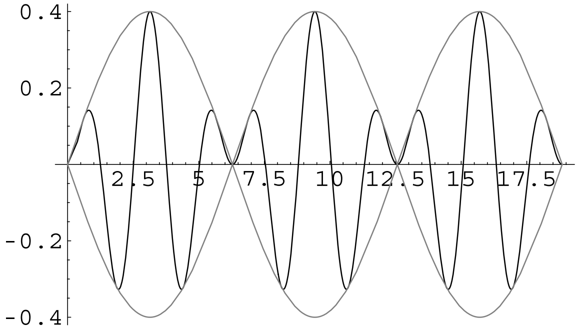


Figure 5-19 The envelope functions show the beats more clearly

```
In[910]:= p2 = Plot[{2/5 Sin[t/2], -2/5 Sin[t/2]}, {t, 0, 6π},
              PlotStyle → GrayLevel[0.5],
              DisplayFunction → Identity];

          Show[p1, p2]
```

For (b), the graph of the solution with the envelope functions $x(t) = \pm\frac{2}{21}\sin\left(\frac{3}{2}t\right)$ is as follows. See Figure 5-20.

```
In[911]:= Plot[{somesols[[2]], 2/21 Sin[3t/2],
            -2/21 Sin[3t/2]}, {t, 0, 4π},
            PlotStyle → {GrayLevel[0], GrayLevel[0.5],
            GrayLevel[0.5]}]
```

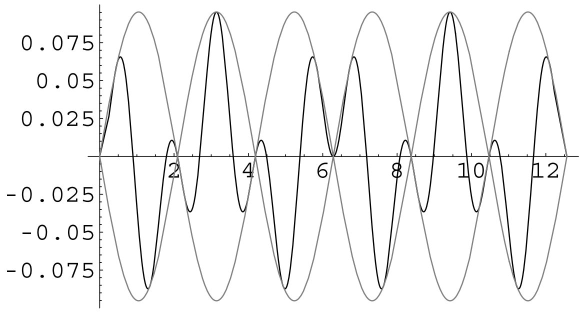
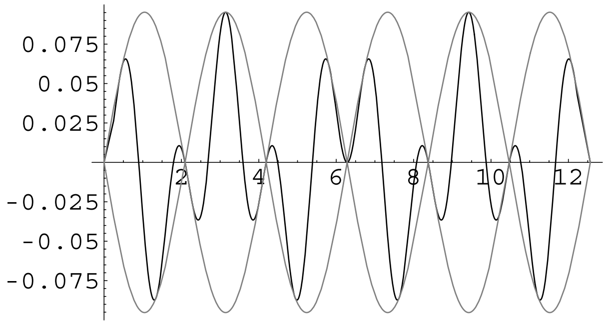


Figure 5-20 Beats are seen more clearly when shown with the envelope functions




```
Out[911]= -Graphics-
```

■

Oscillations like those illustrated in the previous example are called **beats** because of the periodic variation of amplitude. This phenomenon is commonly encountered when two musicians (especially bad ones) try to simultaneously tune their instruments or when two tuning forks with almost equivalent frequencies are played at the same time.

See the *Application* at the end of the section for a discussion of how you can listen to beats and resonance with Mathematica.

We now consider spring problems that involve forces due to damping as well as external forces. In particular, consider the following initial-value problem:

$$\begin{cases} mx'' + cx' + kx = \rho\cos\lambda t \\ x(0) = \alpha,\ x'(0) = \beta. \end{cases} \tag{5.7}$$

Problems of this nature have solutions of the form $x(t) = h(t) + s(t)$, where $\lim_{t\to\infty} h(t) = 0$ and $s(t) = c_1 \cos \lambda t + c_2 \sin \lambda t$.

The function $h(t)$ is called the **transient solution** while $s(t)$ is called the **steady-state solution**. Therefore, as t approaches infinity, the solution $x(t)$ approaches the steady-state solution. Note that the steady-state solution corresponds to the particular solution obtained through the Method of Undetermined Coefficients or Variation of Parameters.

EXAMPLE 5.1.11: Solve the initial-value problem

$$\begin{cases} x'' + 4x' + 13x = \cos t \\ x(0) = 0,\ x'(0) = 1 \end{cases}$$

that models the motion of an object of mass $m = 1$ attached to a spring with spring constant $k = 13$ that is subjected to a resistive force of $F_R = 4x'$ and an external force of $f(t) = \cos t$. Identify the transient and steady-state solutions.

SOLUTION: First, `DSolve` is used to obtain the solution of this nonhomogeneous problem.

The Method of Undetermined Coefficients could be used to find this solution as well.

```
In[912]:= deq =
            Simplify[
              DSolve[{x''[t] + 4 x'[t] + 13 x[t] == Cos[t],
                  x[0] == 0, x'[0] == 1}, x[t], t]]
```

$$\text{Out[912]=} \left\{\left\{x[t] \to \frac{1}{40} e^{-2t} (3 e^{2t} \text{Cos}[t] - 3 \text{Cos}[3t] + e^{2t} \text{Sin}[t] + 11 \text{Sin}[3t])\right\}\right\}$$

The solution is then graphed over the interval $[0, 5\pi]$ in `plot1` to illustrate the behavior of this solution. See Figure 5-21.

```
In[913]:= plot1 = Plot[x[t]/.deq, {t, 0, 5 π}]
```

The transient solution is $h = e^{-2t}\left(-\frac{3}{40}\cos 3t + \frac{11}{40}\sin 3t\right)$ and the steady-state solution is $s = \frac{3}{40}\cos t + \frac{1}{40}\sin t$. We graph the steady-state solution

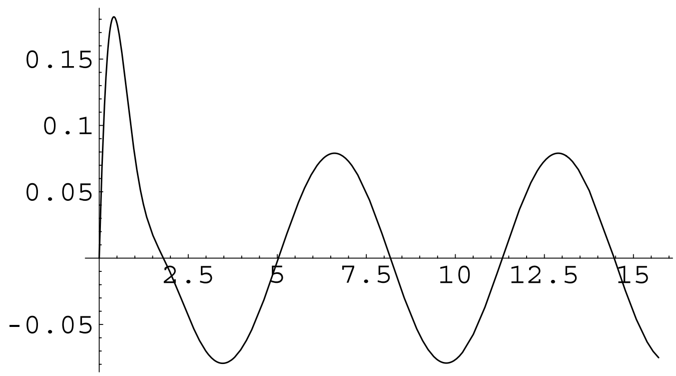


Figure 5-21 Forced motion with damping

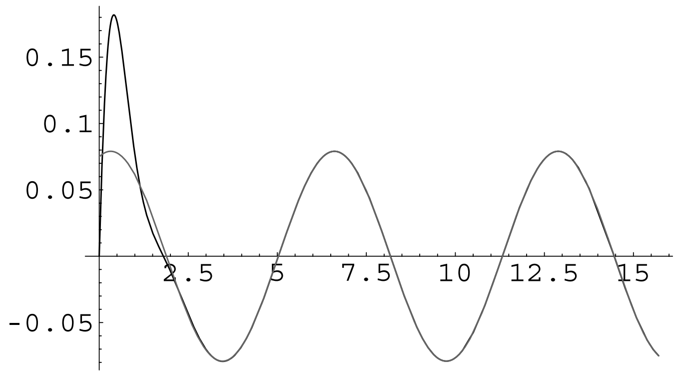


Figure 5-22 Forced motion with damping shown with its steady-state solution

over the same interval so that it can be compared to `plot1` and then we show the two graphs together with `Show` in Figure 5-22.

In[914]:= **ss[t_] = $\frac{1}{40}$ (3 Cos[t] + Sin[t]);**

```
In[915]:= ssplot = Plot[ss[t], {t, 0, 5π},
               PlotStyle → GrayLevel[0.4],
               DisplayFunction → Identity];

          Show[plot1, ssplot]
```

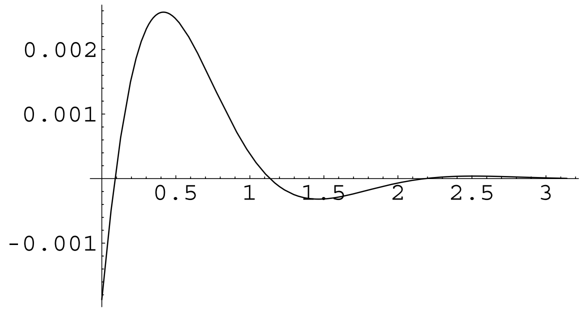


Figure 5-23 The transient solution quickly tends to 0

Notice that the two curves appear identical for $t > 2.5$. The reason for this is shown in the plot of the transient solution in Figure 5-23, which becomes quite small near $t = 2.5$.

```
In[916]:= Plot[1/40 Exp[-2 t] (-3 Cos[3t]/40 + 11 Sin[3t]/40),
    {t, 0, π}]
```

Notice also that the steady-state solution corresponds to a particular solution to the nonhomogeneous differential equation as verified here with `Simplify`.

```
In[917]:= Simplify[ss''[t] + 4 ss'[t] + 13 ss[t]]
Out[917]= Cos[t]
```

■

Instead of solving initial value problems that model the motion of damped and undamped systems as functions of time only, we can consider problems that involve an arbitrary parameter. In doing this, we can obtain a new understanding of the phenomena of resonance and beats.

EXAMPLE 5.1.12: Solve (a) $\begin{cases} x'' + 4x' + 13x = \cos\omega t \\ x(0) = 0,\ x'(0) = 0 \end{cases}$;

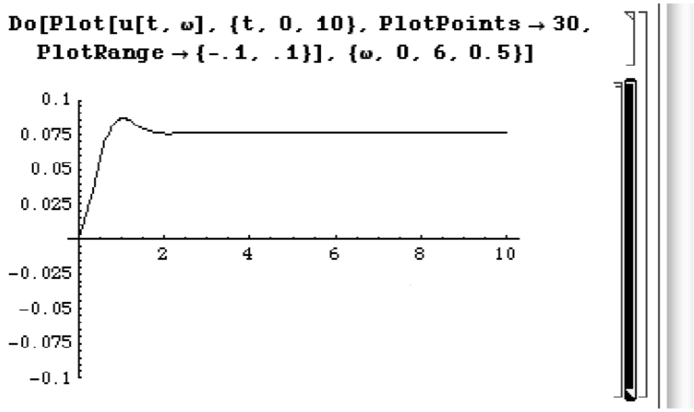
(b) $\begin{cases} x'' + 4x = \cos\omega t \\ x(0) = 0,\ x'(0) = 0 \end{cases}$. Plot the solution for various values of ω near the natural frequency of the system.

SOLUTION: (a) We solve the initial-value problem and simplify the result with `Simplify` for arbitrary ω in `sol`, extract the solution with `Part` (`[[...]]`), and define it to be u[t, ω].

```
In[918]:= Clear[sol]

          sol =
            DSolve[{x''[t] + 4 x'[t] + 13 x[t] == Cos[ω t],
               x[0] == 0, x'[0] == 0}, x[t], t]//Simplify

Out[918]= {{x[t] → (e^(-2 t) (6 (-13 + ω^2) Cos[3 t]
                    -6 e^(2 t) (-13 + ω^2) Cos[t ω]
                    -4 ((13 + ω^2) Sin[3 t]
                    -6 e^(2 t) ω Sin[t ω])))
                    /(6 (169 - 10 ω^2 + ω^4))}}

In[919]:= u[t_, ω_] = sol[[1, 1, 2]];
```

We can graph the solution for $0 \le \omega \le 6$ using increments of 0.25 in a `Do` command in order to animate the resulting plots. We show a screen shot from the resulting animation.

```
In[920]:= Do[Plot[u[t, ω], {t, 0, 10}, PlotPoints → 30,
            PlotRange → {-0.1, 0.1}], {ω, 0, 6, 0.5}]
```



We can also observe how the motion approaches and then moves away from resonance using a `GraphicsArray` as shown in Figure 5-24.

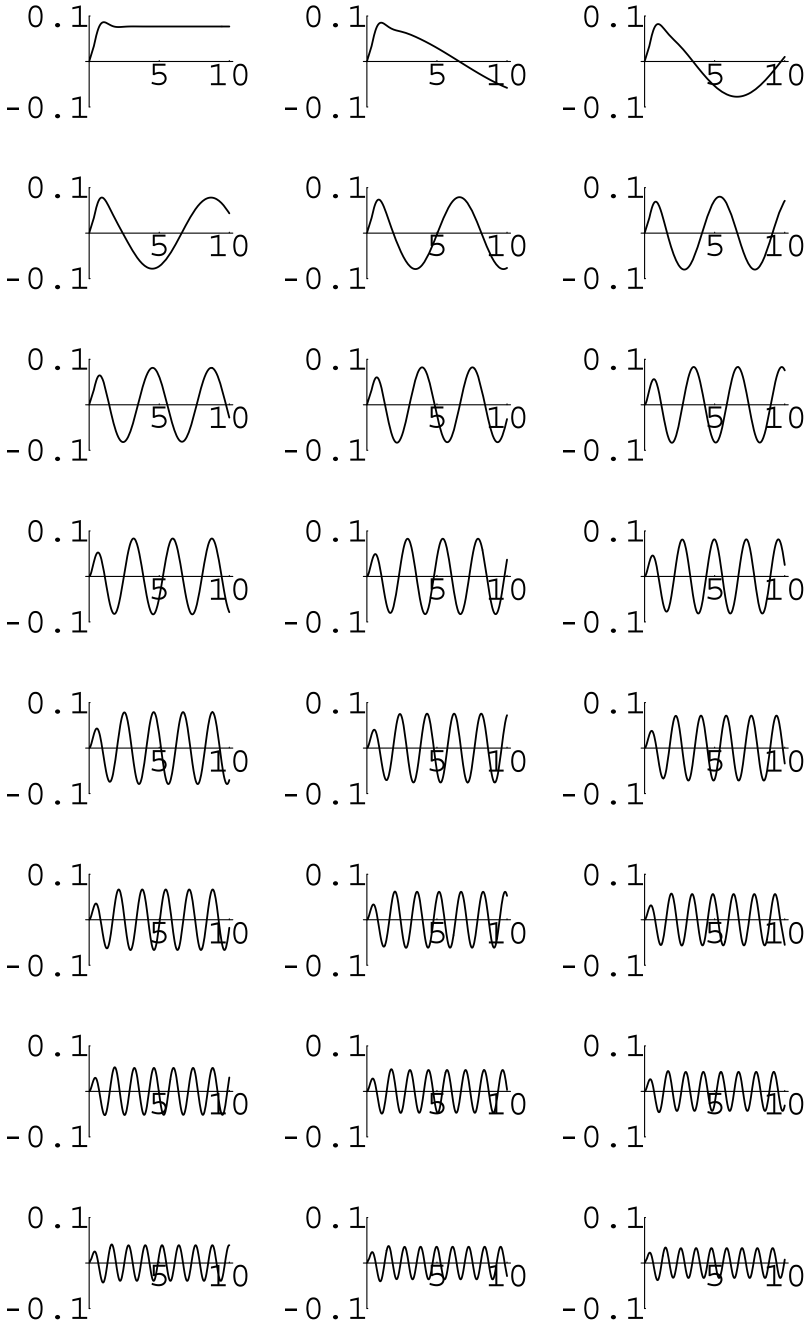


Figure 5-24 Varying ω

```
In[921]:= ωgraph[ω_] := Plot[u[t, ω], {t, 0, 10},
            PlotPoints → 30, PlotRange → {-0.1, 0.1},
            Ticks → {{5, 10}, {-0.1, 0.1}},
            DisplayFunction → Identity]

In[922]:= graphs = Table[ωgraph[ω], {ω, 0, 6, 0.25}];

In[923]:= toshow = Partition[graphs, 3];

          Show[GraphicsArray[toshow]]
```

On the other hand, we can graph the three-dimensional surface u[t, ω] to see how the motion depends on the value of ω. See Figure 5-25.

```
In[924]:= Plot3D[u[t, ω], {t, 0, 10}, {ω, 0, 6},
            PlotPoints → 30]
```

(b) In a similar way, we solve $\begin{cases} x'' + 4x = \cos \omega t \\ x(0) = 0,\ x'(0) = 0 \end{cases}$ for arbitrary ω in sol.

```
In[925]:= Clear[u, sol]

          sol =
            DSolve[{x''[t] + 4 x[t] == Cos[ω t], x[0] == 0,
              x'[0] == 0}, x[t], t]//Simplify
```

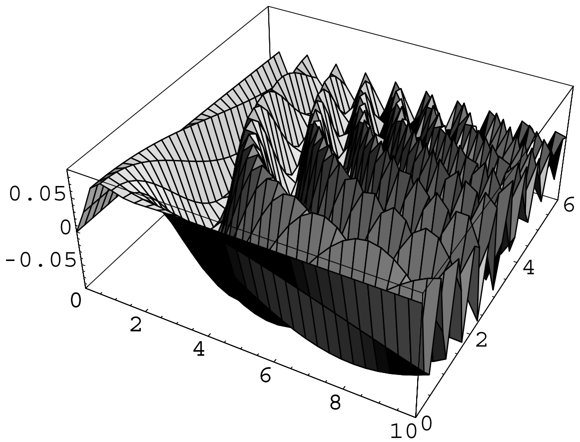


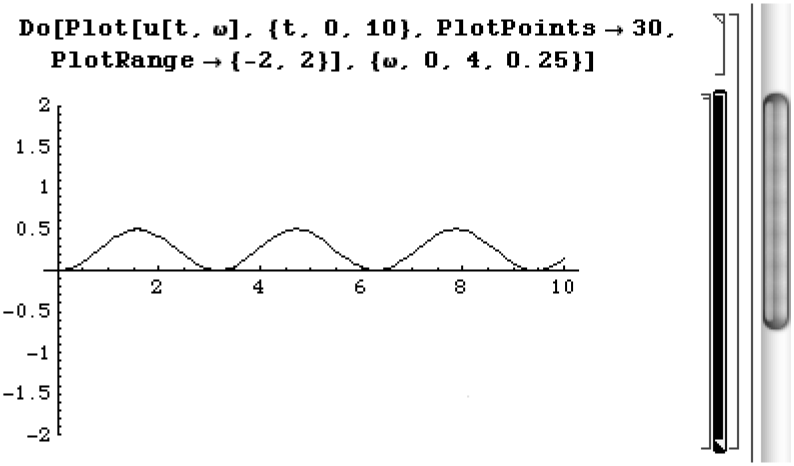
Figure 5-25 Cross-sections of the three-dimensional plot are solutions of the initial-value problem

Out[925]= $\left\{\left\{x[t] \to \frac{\text{Cos}[2\,t] - \text{Cos}[t\,\omega]}{-4 + \omega^2}\right\}\right\}$

```
In[926]:= u[t_, ω_] = sol[[1, 1, 2]];
```

Using the following `Do` loop, we can animate the solution for $0 \leq \omega \leq 3$ using a stepsize of 0.1 to observe how the solution behaves as ω approaches the natural frequency of the system, 2. We show a screen shot from the resulting animation. Note that Mathematica generates several error messages when it encounters $\omega = 2$ because the solution obtained with `DSolve` is not defined if $\omega = 2$. Nevertheless, Mathematica accurately displays the graphs of the solutions for $\omega \neq 2$.

```
In[927]:= Do[Plot[u[t, ω], {t, 0, 10}, PlotPoints → 30,
            PlotRange → {-2, 2}], {ω, 0, 4, 0.25}]
```



In addition, we can use a `GraphicsArray` to observe the behavior of the function as shown in Figure 5-26.

```
In[928]:= Clear[ωgraph]

          ωgraph[ω_] := Plot[u[t, ω], {t, 0, 10},
             PlotPoints → 30, PlotRange → {-2, 2},
             Ticks → {{5, 10}, {-2, 2}},
             DisplayFunction → Identity]
```

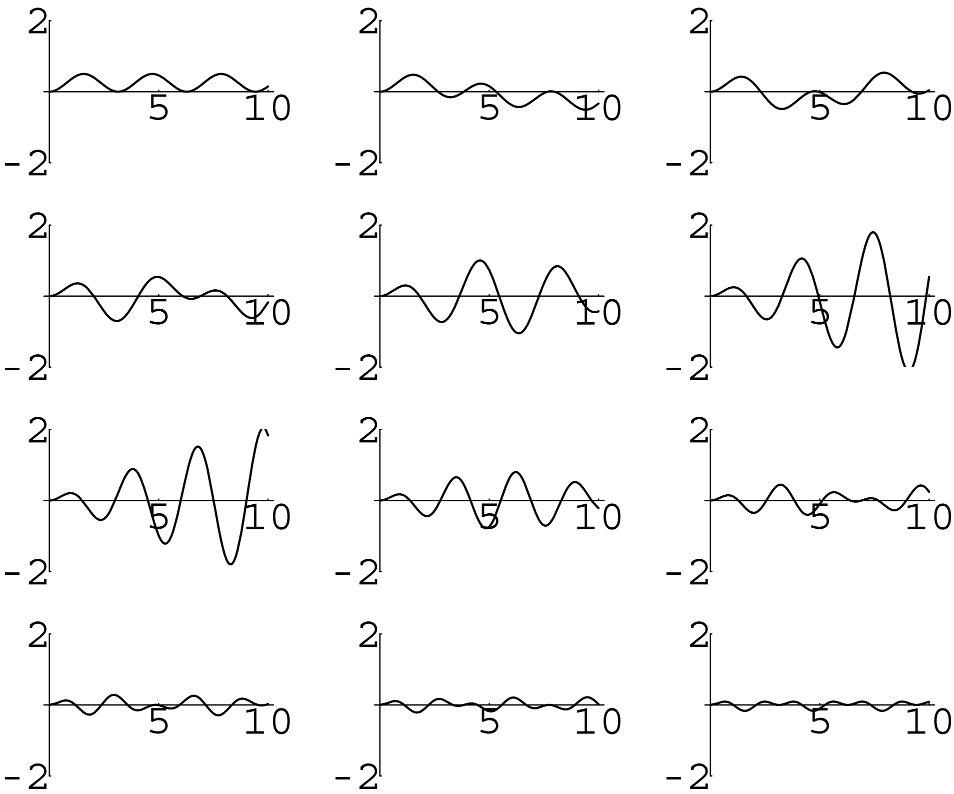


Figure 5-26 The solution is periodic unless $\omega = 2$

```
In[929]:= graphs = Table[ωgraph[ω], {ω, 0, 4, 4/11}];

In[930]:= toshow = Partition[graphs, 3];

          Show[GraphicsArray[toshow]]
```

We can see this behavior in the three-dimensional graph of `u[t,w]` in Figure 5-27 as well.

```
In[931]:= Plot3D[u[t, ω], {t, 0, 10}, {ω, 0, 3},
            PlotPoints → 30]
```

■

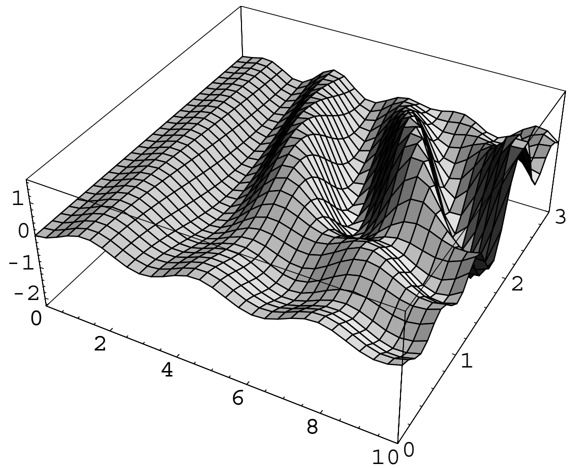


Figure 5-27 Cross-sections of the three-dimensional plot are solutions of the initial-value problem for various values of ω

5.1.4 Soft Springs

In the case of a soft spring, the spring force weakens with compression or extension. For springs of this type, we model the physical system with

$$\begin{cases} x'' + cx' + kx - jx^3 = f(t) \\ x(0) = \alpha,\ x'(0) = \beta \end{cases} \tag{5.8}$$

where j is a positive constant.

EXAMPLE 5.1.13: Approximate the solution to

$$\begin{cases} x'' + 0.2x' + 10kx - 0.2x^3 = -9.8 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

for various values of α and β in the initial conditions.

SOLUTION: After stating this nonlinear differential equation in `eq`, we define the function `s[`α, β`]` to approximate the solution to the initial-value problem with `NDSolve` for specified values of α and β.

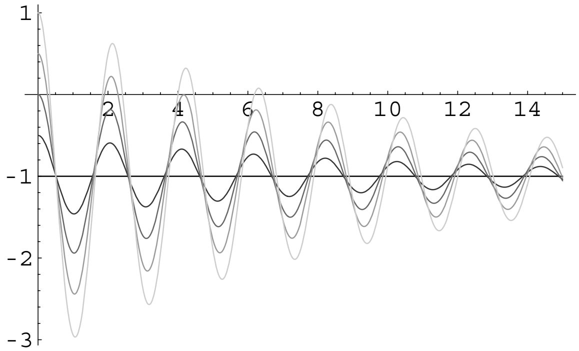


Figure 5-28 The spring does not converge to its equilibrium position

```
In[932]:= Clear[eq]

          eq = x''[t] + 0.2 x'[t] + 10 x[t] - 0.2 x[t]^3 ==
             -9.8;

In[933]:= s[α_, β_] := NDSolve[{eq, x[0] == α, x'[0] == β},
           x[t], {t, 0, 15}];
```

We then define values for α in `vals` so that we can solve the initial-value problem using $x(0) = \alpha$ for the numbers in `vals` and $x'(0) = 0$ in `sols`. The results are graphed in Figure 5-28. We notice that $x(t) \to -1$ as $t \to \infty$.

```
In[934]:= vals = {-1, -0.5, 0, 0.5, 1};

In[935]:= grays = Table[GrayLevel[i],
                 {i, 0, 0.8, 0.8/4}];

In[936]:= sols = Map[s[#, 0]&, vals];

In[937]:= one = Plot[Evaluate[x[t]/.sols], {t, 0, 15},
                PlotStyle → grays]
```

Similarly in `sols2`, we use the numbers in `vals` as the initial velocity in the initial-value problem. We graph these approximate solutions in Figure 5-29.

```
In[938]:= sols2 = Map[s[0, #1]&, vals];

In[939]:= two = Plot[Evaluate[x[t]/.sols2],
                {t, 0, 15}, PlotStyle → grays]
```

In each of the two previous sets of initial conditions, we see that $x(t)$ approaches a limit as $t \to \infty$. However, this is not `always` the case.

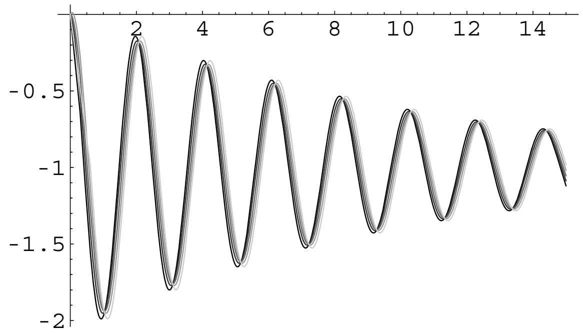


Figure 5-29 Varying the initial velocity in a soft spring

If we consider larger values of α as defined in vals2, we find that solutions are unbounded. Because of this, we must use a smaller interval for t in the NDSolve command in s2[α,β]. Otherwise, we do not obtain meaningful results. The approximate results are graphed in Figure 5-30.

```
In[940]:= s2[α_, β_] := NDSolve[{eq, x[0] == α,
                          x'[0] == β}, x[t], {t, 0, 0.4}];

In[941]:= vals2 = {-10, -9, -8, 8, 9, 10};

In[942]:= sols3 = s2[#1, 0]& /@ vals2;

In[943]:= three = Plot[Evaluate[x[t]/.sols3],
                   {t, 0, 0.4}, PlotStyle → grays]
```

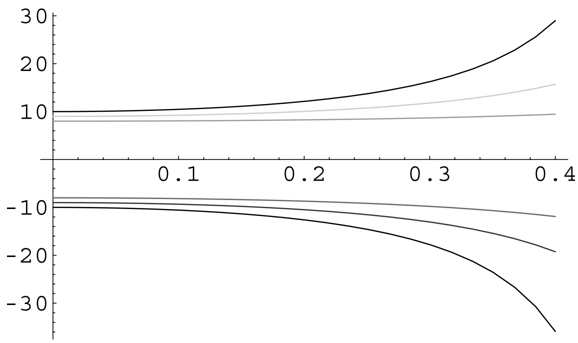


Figure 5-30 The spring becomes weak

■

5.1.5 Hard Springs

In the case of a hard spring, the spring force strengthens with compression or extension. For springs of this type, we model the physical system with

$$\begin{cases} x'' + cx' + kx + jx^3 = f(t) \\ x(0) = \alpha,\ x'(0) = \beta \end{cases} \tag{5.9}$$

where j is a positive constant.

EXAMPLE 5.1.14: Approximate the solution to

$$\begin{cases} x'' + 0.3x + 0.04x^3 = 0 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

for various values of α and β in the initial conditions.

SOLUTION: First, we define the undamped nonlinear differential equation in `eq`. Then, we define `s[α, β]` to numerically approximate the solution to the initial-value problem for given values of α and β.

```
In[944]:= Clear[eq]

          eq = x''[t] + 0.3 x[t] + 0.04 x[t]^3 == 0;

In[945]:= Clear[s]

          s[α_, β_] := NDSolve[{eq, x[0] == α, x'[0] == β},
                         x[t], {t, 0, 15}];
```

We approximate the solution using the constants defined in `vals4` as the initial displacement, $x(0) = \alpha$. These numerical solutions are then graphed in Figure 5-31. Notice that solutions with larger amplitudes have smaller periods as expected with a hard spring.

```
In[946]:= vals4 = {1, 2, 3, 4, 5};

          sols4 = Table[s[vals4[[i]], 0][[1, 1, 2]],
                    {i, 1, 5}];

In[947]:= grays = Table[GrayLevel[i],
                    {i, 0, 0.8, 0.8/4}];

In[948]:= five = Plot[Evaluate[sols4], {t, 0, 15},
                    PlotStyle → grays]
```

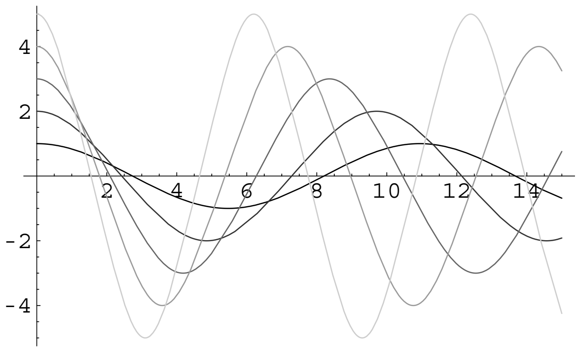


Figure 5-31 Varying the initial displacement of a hard spring

In a similar manner, we use the values in `vals4` as the initial velocity $x'(0) = \beta$. In Figure 5-32, we see that when the amplitude is large, the spring strengthens so that the period of the motion is decreased.

```
In[949]:= sols5 = Table[s[0, vals4[[i]]][[1, 1, 2]],
                  {i, 1, 5}];

In[950]:= six = Plot[Evaluate[sols5], {t, 0, 15},
                PlotStyle → grays]
```

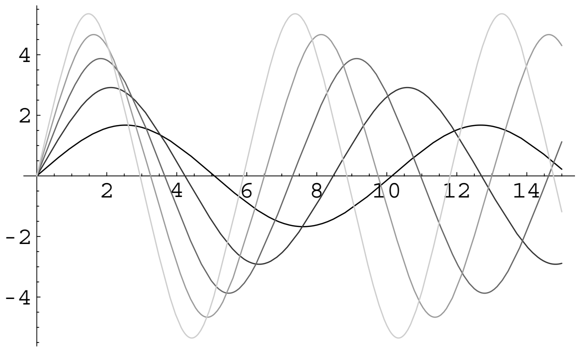


Figure 5-32 Varying the initial velocity of a hard spring

■

5.1.6 Aging Springs

In the case of an **aging spring**, the spring constant weakens with time. For springs of this type, we model the physical system with

$$\begin{cases} x'' + cx' + k(t)x = f(t) \\ x(0) = \alpha,\ x'(0) = \beta \end{cases} \tag{5.10}$$

where $k(t) \to 0$ as $t \to \infty$.

EXAMPLE 5.1.15: Approximate the solution to

$$\begin{cases} x'' + 4e^{-t/4}x = 0 \\ x(0) = \alpha,\ x'(0) = \beta \end{cases}$$

for various values of α and β in the initial conditions.

SOLUTION: First, we state the differential equation in `eq` and then we define s[α,β] to solve the initial value problem for given values of α and β. Using the numbers in `vals4` as the initial displacement and using 0 as the initial velocity, we approximate the solution to five initial value problems in `sols6`. We graph these numerical solutions in Figure 5-33. Notice that the period of the oscillations increases over time due to the diminishing value of the spring constant.

```
In[951]:= Clear[eq]

          eq = x''[t] + 4 Exp[-t/4] x[t] == 0;

In[952]:= Clear[s]

          s[α_, β_] := NDSolve[{eq, x[0] == α, x'[0] == β},
                    x[t], {t, 0, 30}];

In[953]:= vals4 = {1, 2, 3, 4, 5};

          sols6 = Table[s[vals4[[i]], 0][[1, 1, 2]],
                 {i, 1, 5}];

In[954]:= grays = Table[GrayLevel[i],
                 {i, 0, 0.8, 0.8/4}];

In[955]:= seven = Plot[Evaluate[sols6], {t, 0, 30},
                 PlotStyle → grays]
```

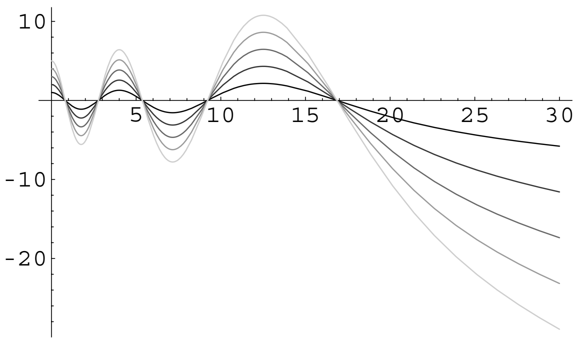


Figure 5-33 Varying the initial displacement of an aging spring

```
In[956]:= Clear[s2]

          s2[α_, β_] :=
            NDSolve[{eq, x[0] == α, x'[0] == β}, x[t],
            {t, 0, 100}];

In[957]:= vals4 = {1, 2, 3, 4, 5};

          sols7 = Table[s2[vals4[[i]], 0][[1, 1, 2]],
                  {i, 1, 5}];

In[958]:= eight = Plot[Evaluate[sols7], {t, 0, 100},
                  PlotStyle → grays]
```

Choosing a longer time interval in the NDSolve command as we do in s2[α,β], we see that eventually the motion is not oscillatory. See Figure 5-34.

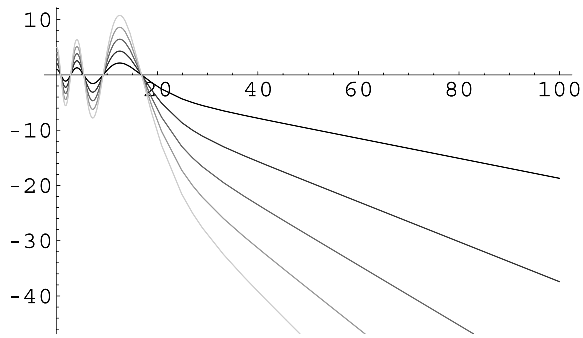


Figure 5-34 An aging spring eventually stops working

■

Application: Hearing Beats and Resonance

In order to *hear* beats and resonance, we solve the initial-value problem

$$\begin{cases} x'' + \omega^2 x = F \cos \beta t \\ x(0) = \alpha,\ x'(0) = \beta \end{cases} \tag{5.11}$$

for each of the following parameter values: (a) $\omega^2 = 6000^2$, $\beta = 5991.62$, $F = 2$; and (b) $\omega^2 = 6000^2$, $\beta = 6000$, $F = 2$.

First, we define the function `sol` which given the parameters, solves the initial-value problem (5.11).

```
In[959]:= Clear[x, t, f, sol]
```

```
In[960]:= sol[ω_, β_, f_] :=
            DSolve[{x''[t] + ω² x[t] == f Cos[β t], x[0] == 0,
              x'[0] == 0}, x[t], t][[1, 1, 2]]
```

Thus, our solution for (a) is obtained by entering

```
In[961]:= a = sol[6000, 5991.62, 2]

Out[961]= -0.0000199025 e^(0.t) Cos[6000. t]
          +0.0000198886 e^(0.t) Cos[8.38 t] Cos[6000. t]
          +1.38986 × 10^-8 e^(0.t) Cos[6000. t] Cos[11991.6 t]
          +0. e^(0.t) Sin[6000. t]
          +0.0000198886 e^(0.t) Sin[8.38 t] Sin[6000. t]
          +1.38986 × 10^-8 e^(0.t) Sin[6000. t] Sin[11991.6 t]
```

To *hear* the function we use `Play` in the same way that we use `Plot` to *see* functions. The values of `a` correspond to the amplitude of the sound as a function of time. See Figure 5-35.

```
In[962]:= Play[a, {t, 0, 6}]
```

Similarly, the solution for (b) is obtained by entering

```
In[963]:= b = sol[6000, 6000, 2]

Out[963]= 1/72000000 (-2 Cos[6000 t] + 2 Cos[6000 t]^3
          +12000 t Sin[6000 t] + Sin[6000 t] Sin[12000 t])
```

We hear resonance with `Play`. See Figure 5-36.

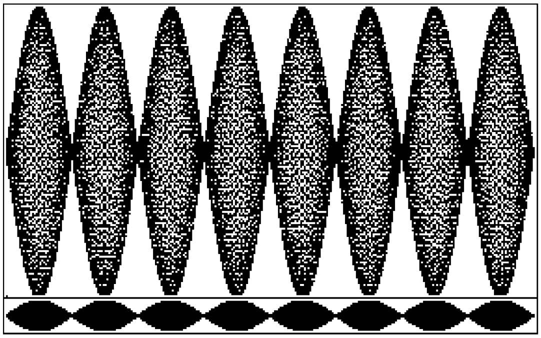


Figure 5-35 Hearing and seeing beats

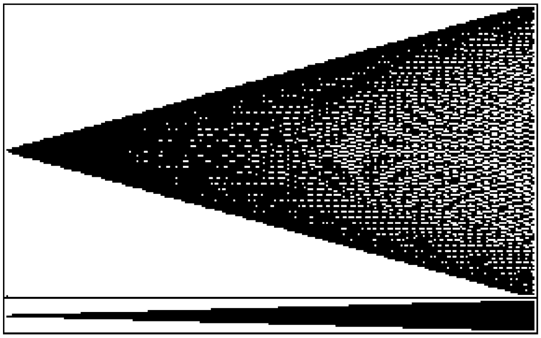
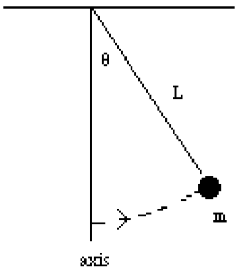


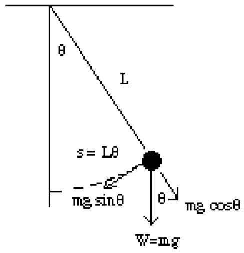
Figure 5-36 Hearing and seeing resonance

5.2 The Pendulum Problem

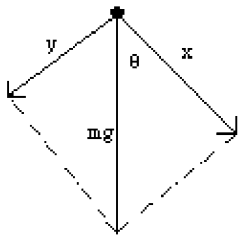
Suppose that a mass m is attached to the end of a rod of length L, the weight of which is negligible.



We want to determine the equation that describes the motion of the mass in terms of the displacement $\theta(t)$ which is measured counterclockwise in radians from the axis shown above. This is possible if we are given an initial displacement and an initial velocity of the mass. A force diagram for this situation is shown as follows.



Notice that the forces are determined with trigonometry using the diagram. Here, $\cos\theta = mg/x$ and $\sin\theta = mg/y$, so we obtain the forces $x = mg\cos\theta$ and $y = mg\sin\theta$, indicated as follows.



Because the momentum of the mass is given by $m\,ds/dt$, the rate of change of the momentum is

$$\frac{d}{dt}\left(m\frac{ds}{dt}\right) = m\frac{d^2s}{dt^2}$$

where s represents the length of the arc formed by the motion of the mass. Then, because the force $y = mg\sin\theta$ acts in the opposite direction of the motion of the mass, we have the equation

$$m\frac{d^2s}{dt^2} = -mg\sin\theta. \tag{5.12}$$

Using the relationship from geometry between the length of the arc, the length of the rod, and the angle θ, $s = L\theta$, we have the relationship

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(L\theta) = L\frac{d^2\theta}{dt^2}.$$

Hence, the displacement $\theta(t)$ satisfies $mL\, d^2\theta/dt^2 = -mg\sin\theta$ or

$$mL\frac{d^2\theta}{dt^2} + mg\sin\theta = 0, \tag{5.13}$$

which is a nonlinear equation. However, because we are only concerned with small displacements, we note from the Maclaurin series for $\sin\theta$, $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots$, that for small values of θ, $\sin\theta \approx \theta$. Therefore, with this approximation, we obtain the linear equation

$$mL\frac{d^2\theta}{dt^2} + mg\theta = 0 \qquad \text{or} \qquad \frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \tag{5.14}$$

which approximates the original equation (5.13). If the initial displacement is given by $\theta(0) = \theta_0$ and the initial velocity is given by $\theta'(0) = v_0$, then we have the initial-value problem

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0 \\ \theta(0) = \theta_0, \ \dfrac{d\theta}{dt}(0) = v_0 \end{cases} \tag{5.15}$$

to find the displacement function $\theta(t)$.

Suppose that $\omega^2 = g/L$ so that the differential equation becomes $\theta'' + \omega^2\theta = 0$, which has general solution

$$\theta(t) = c_1\cos\omega t + c_2\sin\omega t.$$

Application of the initial conditions $\theta(0) = \theta_0$ and $\theta'(0) = v_0$ shows us that

$$\theta(t) = \theta_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t \tag{5.16}$$

is the solution of equation (5.15), where $\omega = \sqrt{g/L}$. We can write this function solely in terms of a cosine function that includes a phase shift with

$$\theta(t) = \sqrt{\theta_0^2 + \frac{v_0^2}{\omega^2}}\cos(\omega t - \phi), \tag{5.17}$$

where

$$\phi = \cos^{-1}\left(\frac{\theta_0}{\sqrt{\theta_0^2 + \dfrac{v_0^2}{\omega^2}}}\right) \qquad \text{and} \qquad \omega = \sqrt{\frac{g}{L}}.$$

Note that the approximate period of $\theta(t)$ is $T = 2\dfrac{\pi}{\omega} = 2\pi\sqrt{\dfrac{L}{g}}$.

EXAMPLE 5.2.1: Determine the displacement of a pendulum of length $L = 32$ feet if $\theta(0) = 0$ and $\theta'(0) = 1/2$ using both the linear and nonlinear models. What is the period? If the pendulum is part of a clock that ticks once for each time the pendulum makes a complete swing, how many ticks does the clock make in one minute?

SOLUTION: The linear initial-value problem that models this situation is

$$\begin{cases} \theta'' + \theta = 0 \\ \theta(0) = 0, \ \theta'(0) = 1/2 \end{cases}$$

because $g/L = 32/32 = 1$. We use `DSolve` to find a general solution of the equation

```
In[964]:= gensol = DSolve[x''[t] + x[t] == 0, x[t], t]
Out[964]= {{x[t] → C[1] Cos[t] + C[2] Sin[t]}}
```

and the solution to the initial-value problem

$$\begin{cases} \theta'' + \theta = 0 \\ \theta(0) = a, \ \theta'(0) = b. \end{cases}$$

```
In[965]:= BoxData(eq = DSolve[{x''[t] + x[t] == 0,
            x[0] == a, x'[0] == b}, x[t], t])
Out[965]= {{x[t] → a Cos[t] + b Sin[t]}}
```

In this case, we have that $a = 0$ and $b = 1/2$ so substituting these values into `eq[[1,1,2]]` results in the solution to the initial-value problem.

```
In[966]:= pen = eq[[1, 1, 2]]/.{a → 0, b → 1/2};
```

The period of this function is

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{32 \text{ ft}}{32 \text{ ft/s}^2}} = 2\pi \text{ s}.$$

Therefore, the number of ticks made by the clock per minute is calculated with the conversion

$$\frac{1 \text{ rev}}{2\pi \text{ s}} \times \frac{1 \text{ tick}}{1 \text{ rev}} \times \frac{60 \text{ s}}{1 \text{ min}} \approx 9.55 \text{ ticks/min}.$$

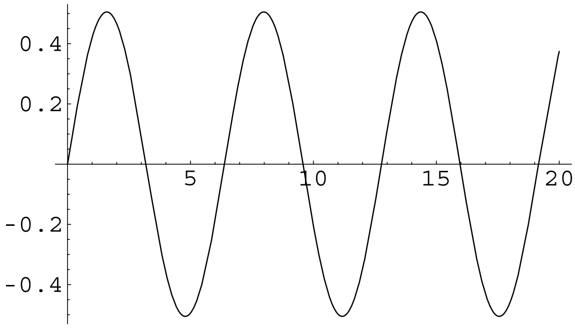


Figure 5-37 Plot of the solution to the nonlinear initial-value problem

Hence, the clock makes approximately 9.55 ticks in one minute. To solve the nonlinear equation, we use `NDSolve` to generate a numerical solution to the initial-value problem valid for $0 \le t \le 20$.

```
In[967]:= BoxData(numsol = NDSolve[{x''[t]
          + Sin[x[t]] == 0, x[0] == 0, x'[0] == 1/2},
            x[t], {t, 0, 20}])
Out[967]= BoxData({{x[t] →
            InterpolatingFunction[{{0., 20.}},
            " <> "][t]}})
```

We then graph this solution on the interval [0, 20] in Figure 5-37.

```
In[968]:= plot1 = Plot[x[t]/.numsol, {t, 0, 20},
                PlotRange → All]
```

The solution `pen` is also graphed on the interval [0, 20], the resulting graph is named `plot2`, and then `plot1` and `plot2` are displayed together with `Show` in Figure 5-38.

```
In[969]:= plot2 = Plot[pen, {t, 0, 20},
                PlotStyle → GrayLevel[0.4],
                DisplayFunction → Identity];
          Show[plot1, plot2]
```

The graphs indicate that the error between the two functions increases as t increases, which is confirmed by graphing the absolute value of the difference of the two functions shown in Figure 5-39.

```
In[970]:= plot3 = Plot[Abs[pen-x[t]]/.numsol, {t, 0, 20}]
```

■

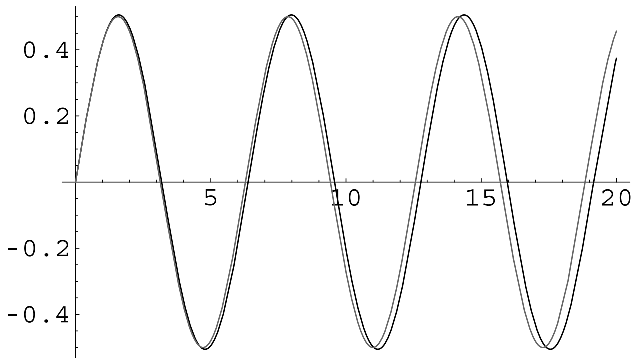


Figure 5-38 Solution of the linear (in gray) and nonlinear (in black) initial-value problems shown together

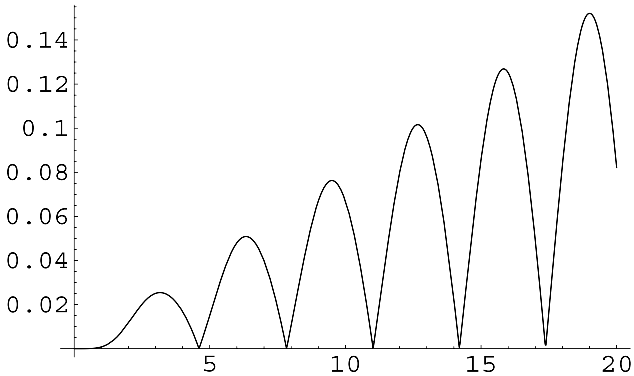


Figure 5-39 The linear approximation approximates the nonlinear solution very well until t becomes large

Suppose that the pendulum undergoes a damping force that is proportional to the instantaneous velocity. Then, the force due to damping is given as $F_R = b\, d\theta/dt$. Incorporating this force into the sum of the forces acting on the pendulum, we have the nonlinear equation $L\theta'' + b\theta' + g\sin\theta = 0$. Again, using the approximation $\sin\theta \approx \theta$ for small values of t, we obtain the linear equation $L\theta'' + b\theta' + g\theta = 0$ which approximates the situation. Thus, we solve the initial-value problem

$$\begin{cases} L\dfrac{d^2\theta}{dt^2} + b\dfrac{d\theta}{dt} + g\theta = 0 \\ \theta(0) = \theta_0,\ \dfrac{d\theta}{dt}(0) = v_0 \end{cases} \tag{5.18}$$

to find the displacement function $\theta(t)$.

EXAMPLE 5.2.2: A pendulum of length $L = 8/5$ ft is subjected to the resistive force $F_R = 32/5\, d\theta/dt$ due to damping. Determine the displacement function if $\theta(0) = 1$ and $\theta'(0) = 2$.

SOLUTION: The initial-value problem that models this situation is

$$\begin{cases} \frac{8}{5}\frac{d^2\theta}{dt^2} + \frac{32}{5}\frac{d\theta}{dt} + 32\theta = 0 \\ \theta(0) = 0,\ \frac{d\theta}{dt}(0) = 2. \end{cases}$$

Simplifying the differential equation, we obtain $\theta'' + 4\theta' + 20\theta = 0$, and then using `DSolve`, we find the solution to the initial-value problem,

```
In[971]:= sol = DSolve[{θ''[t] + 4 θ'[t] + 20 θ[t] == 0,
                  θ[0] == 1, θ'[0] == 2}, θ[t], t]
Out[971]= {{θ[t] → e^(-2 t) (Cos[4 t] + Sin[4 t])}}
```

which is then graphed with `Plot` in Figure 5-40.

```
In[972]:= θ[t_] = e^(-2 t) (Cos[4 t] + Sin[4 t]);

In[973]:= Plot[θ[t], {t, 0, 2}]
```

Notice that the damping causes the displacement of the pendulum to decrease over time.

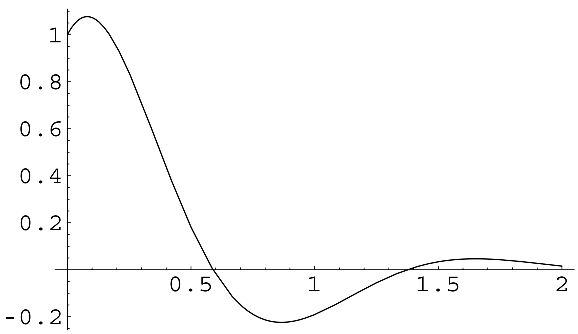


Figure 5-40 A solution to the damped pendulum equation

To see the pendulum move, we define the procedure pen. Given t, len, and opts, where opts are any options of the Show command, pen[t,len,opts] declares the variable pt1 to be local to the procedure pen, defines pt1 to be the point

$$\left(\texttt{len}\,\cos\left(\frac{3}{2}\pi t+\theta(t)\right),\texttt{len}\,\sin\left(\frac{3}{2}\pi t+\theta(t)\right)\right),$$

and connects the points pt1 and (0, 0) with a line segment. Note that PointSize is used so that pt1 is slightly enlarged in the resulting graphics object. The resulting graphics object *looks* like the pendulum of length $L =$ len at time t.

```
In[974]:= Clear[pen]

          pen[t_, len_, opts___] := Module[{pt1},
            pt1 = {len Cos[3 π/2 + θ[t]],
            len Sin[3 π/2 + θ[t]]};
            Show[
             Graphics[{Line[{{0, 0}, pt1}],
             PointSize[0.05], Point[pt1]}],
             Axes → Automatic, Ticks → None,
             AxesStyle → GrayLevel[0.5],
             PlotRange → {{-2, 2}, {-2, 0}}, opts]]
```

For example, entering

```
In[975]:= pen[1, 8/5, DisplayFunction → Identity]
Out[975]= -Graphics-
```

produces a graphics object corresponding to a pendulum of length $L = 1$ at time $t = 1$. The resulting graphics object is not displayed because the option DisplayFunction->Identity, which is an option of Show, is included in the pen command. On the other hand,

```
In[976]:= pen[1, 8/5]
```

You can view a list of the options associated with the Show command by entering Options[Show].

produces and displays a graphics object corresponding to a pendulum of length $L = 8/5$ at time $t = 1$ as shown in Figure 5-41.

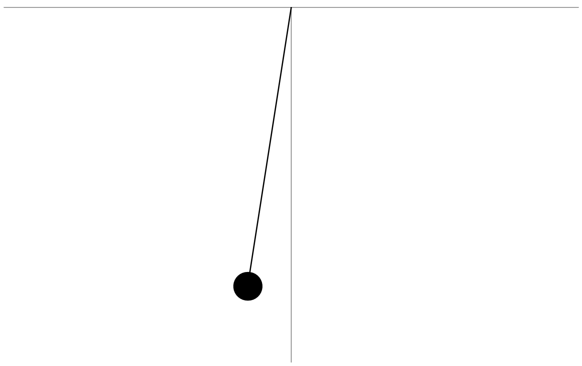


Figure 5-41 A pendulum

To see the pendulum at various times, we use `Table` and `pen` to generate a table consisting of graphics corresponding to a pendulum of length $L = 8/5$ at time t from $t = 0$ to $t = \pi/2$ using increments of 2/15. The resulting list of 16 graphics objects is then partitioned into four element subsets with `Partition` and the array of graphics objects `toshow` is displayed with `Show` and `GraphicsArray` in Figure 5-42.

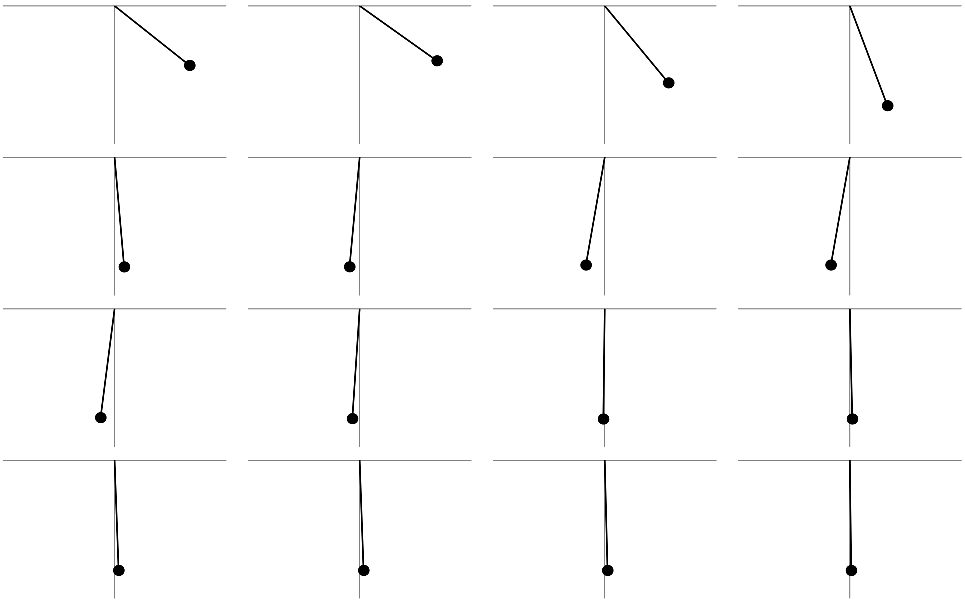
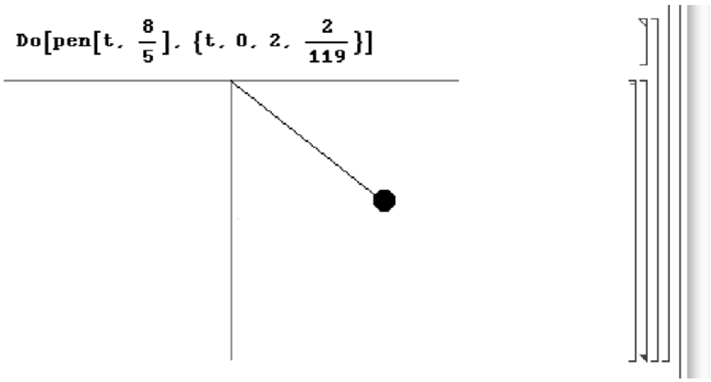


Figure 5-42 A damped pendulum comes to rest

```
In[977]:= somegraphs =
            Table[pen[t, 8/5,
              DisplayFunction → Identity],
              {t, 0, 2, 2/15}];

          toshow = Partition[somegraphs, 4];

          Show[GraphicsArray[toshow]]
```

On the other hand, to see the pendulum move, we can use a `Do` loop to generate several graphs and then animate the result. We show a screen shot of one frame of the resulting animation.

```
In[978]:= Do[pen[t, 8/5], {t, 0, 2, 2/119}]
```



Notice that from our approximate solution, the displacement of the pendulum becomes very close to zero near $t = 2$, which was our observation from the graph of $\theta(t) = e^{-2t}(\cos 4t + 2\sin 4t)$ in Figure 5-40.

■

Our last example investigates the properties of the nonlinear differential equation.

EXAMPLE 5.2.3: Graph the solution to the initial-value problem

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + 0.5\dfrac{d\theta}{dt} + \theta = 0 \\ \theta(0) = \theta_0, \ \dfrac{d\theta}{dt}(0) = v_0 \end{cases} \tag{5.19}$$

subject to the following initial conditions.

θ_0	v_0	θ_0	v_0	θ_0	v_0	θ_0	v_0
−1	0	−0.5	0	0.5	0	1	0
0	−2	0	−1	0	1	0	2
1	1	1	−1	−1	1	−1	−1
1	2	1	3	−1	4	−1	5
−1	2	−1	3	1	−4	1	−5

SOLUTION: We begin by defining eq to be $\theta'' + 0.5\theta' + \sin\theta = 0$.

```
In[979]:= Clear[eq, t, θ, s]

          eq = θ''[t] + 0.5 θ'[t] + Sin[θ[t]] == 0;
```

To avoid retyping the same commands, we define the procedure s. Given an ordered pair (θ_0, v_0) and any options opts of the Show command, s[{theta0,v0},opts] first declares the variables numsol and numgraph local to the procedure s, uses NDSolve to define numsol to be a numerical solution of the initial-value problem (5.19) valid for $0 \leq t \leq 15$, generates, but does not display, a graph of the resulting numerical solution on the interval [0, 15], and then displays the result with Show using any options opts passed through the s command.

```
In[980]:= s[{theta0_, v0_}, opts___] := Module[{numsol},
            numsol = NDSolve[{eq, θ[0] ==
            theta0, θ'[0] == v0}, θ[t], {t, 0, 15}];
            numgraph = Plot[θ[t]/.numsol, {t, 0, 15},
              DisplayFunction → Identity];
            Show[numgraph, opts]]
```

Thus, we see that entering

```
In[981]:= s[{-1, 0}]
Out[981]= -Graphics-
```

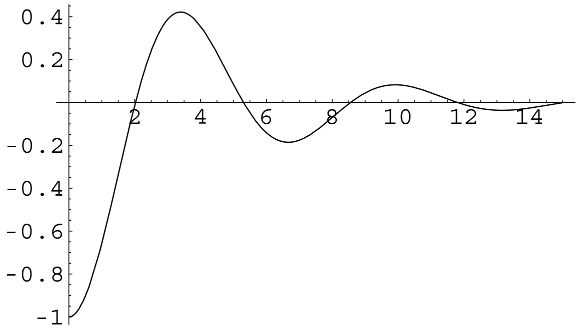


Figure 5-43 Plot of the solution of equation (5.19) that satisfies $\theta_0 = -1$ and $v_0 = 0$

does not display the graph of the solution to equation (5.19) if $\theta_0 = -1$ and $v_0 = 0$ but entering

```
In[982]:= s[{-1,0},DisplayFunction → $DisplayFunction]
```

displays the graph of the solution shown in Figure 5-43. Thus, to graph the solutions that satisfy the initial conditions

θ_0	v_0	θ_0	v_0	θ_0	v_0	θ_0	v_0
−1	0	−0.5	0	0.5	0	1	0

we first define `t1` to be the initial conditions, use `Map` to apply `s` to `t1`, and then use `Show` together with the option `DisplayFunction->` `\$DisplayFunction` to display the resulting graphs in Figure 5-44.

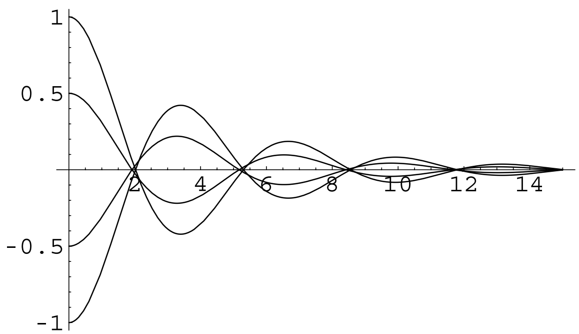


Figure 5-44 Varying the initial displacement in the pendulum equation

```
In[983]:= t1 = {{-1, 0}, {-0.5, 0}, {0.5, 0}, {1, 0}};
          toshow1 = Map[s, t1]; Show[toshow1,
          DisplayFunction → $DisplayFunction]
```

Similarly, entering

```
In[984]:= t2 = {{0, -2}, {0, -1}, {0, 1}, {0, 2}};
          toshow2 = s/@t2; Show[toshow2,
          DisplayFunction → $DisplayFunction]
```

defines `t2` to be the list of ordered pairs corresponding to the initial conditions

θ_0	v_0	θ_0	v_0	θ_0	v_0	θ_0	v_0
0	−2	0	−1	0	1	0	2

`toshow2` to be the resulting list of graphics objects obtained by applying `s` to each ordered pair in `t2`, and then displays the list of graphics `toshow2` together in Figure 5-45. The solutions that satisfy the remaining initial conditions

θ_0	v_0	θ_0	v_0	θ_0	v_0	θ_0	v_0
1	1	1	−1	−1	1	−1	−1
1	2	1	3	−1	4	−1	5
−1	2	−1	3	1	−4	1	−5

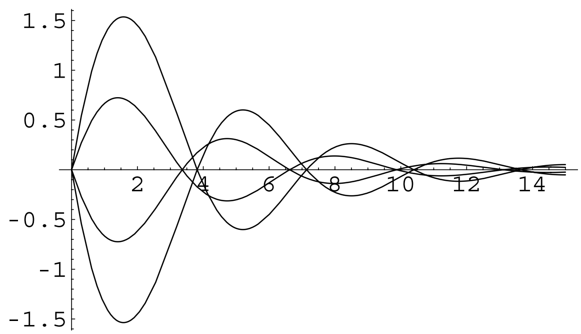


Figure 5-45 Varying the initial velocity in the pendulum equation

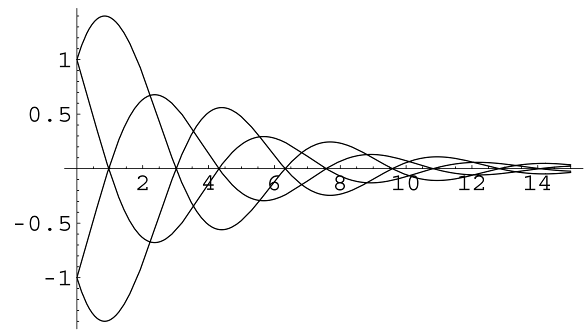


Figure 5-46 Varying the initial displacements and velocity in the pendulum equation

are graphed in the same manner in Figures 5-46, 5-47, and 5-48.

```
In[985]:= t3 = {{1, 1}, {1, -1}, {-1, 1}, {-1, -1}};
          toshow3 = s/@t3; Show[toshow3,
          DisplayFunction → $DisplayFunction]

In[986]:= t4 = {{1, 2}, {1, 3}, {-1, 4}, {-1, 5}};
          toshow4 = s/@t4; Show[toshow4,
          DisplayFunction → $DisplayFunction]
```

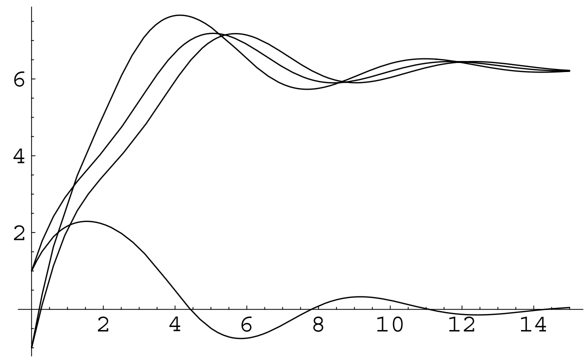


Figure 5-47 Varying the initial displacements and velocity in the pendulum equation

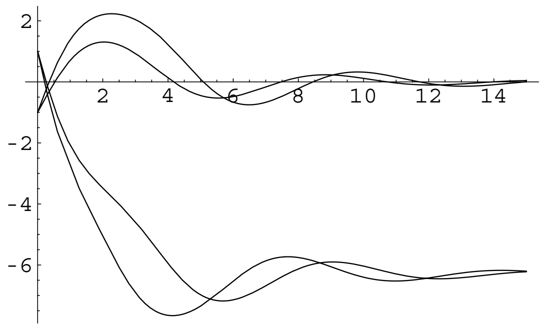


Figure 5-48 Varying the initial displacements and velocity in the pendulum equation

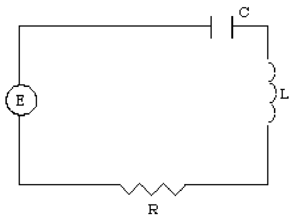
```
In[987]:= t5 = {{-1, 2}, {-1, 3}, {1, -4}, {1, -5}};
          toshow5 = s/@t5; Show[toshow5,
          DisplayFunction → $DisplayFunction]
```

■

5.3 Other Applications

5.3.1 *L–R–C* Circuits

Second-order nonhomogeneous linear ordinary differential equations arise in the study of electrical circuits after the application of *Kirchhoff's law*. Suppose that $I(t)$ is the current in the *L–R–C* series electrical circuit where L, R, and C represent the inductance, resistance, and capacitance of the circuit, respectively.



The voltage drops across the circuit elements shown in the following table have been obtained from experimental data where Q is the charge of the capacitor and $dQ/dt = I$.

Circuit Element	Voltage Drop
Inductor	$L\dfrac{dI}{dt}$
Resistor	RI
Capacitor	$\dfrac{1}{C}Q$

Our goal is to model this physical situation with an initial-value problem so that we can determine the current and charge in the circuit. For convenience, the terminology used in this section is summarized in the following table.

Electrical Quantities	Units
Inductance (L)	Henrys (H)
Resistance (R)	Ohms (Ω)
Capacitance (C)	Farads (F)
Charge (Q)	Coulombs (C)
Current (I)	Amperes (A)

The physical principle needed to derive the differential equation that models the L-R-C series circuit is stated as follows.

> **Kirchhoff's Law:** The sum of the voltage drops across the circuit elements is equivalent to the voltage $E(t)$ impressed on the circuit.

Applying Kirchhoff's law, therefore, yields the differential equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

Using the fact that $dQ/dt = I$, we also have $d^2Q/dt^2 = dI/dt$. Therefore, the equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t),$$

which can be solved by the Method of Undetermined Coefficients or the Method of Variation of Parameters. Hence, if the initial charge and current are $Q(0) = Q_0$ and $I(0) = Q'(0) = I_0$, then we must solve the initial-value problem

$$\begin{cases} L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t) \\ Q(0) = Q_0,\ I(0) = \dfrac{dQ}{dt}(0) = I_0 \end{cases} \tag{5.20}$$

for the charge $Q(t)$. This solution can then be differentiated to find the current $I(t)$.

EXAMPLE 5.3.1: Consider the *L-R-C* circuit with $L = 1$ Henry, $R = 40$ Ohms, $C = 4000$ Farads, and $E(t) = 24$ volts. Determine the current in this circuit if there is zero initial current and zero initial charge.

SOLUTION: Using the indicated values, the initial-value problem that we must solve is

$$\begin{cases} Q'' + 40Q' + 4000Q = 24 \\ Q(0) = 0,\ I(0) = Q'(0) = 0. \end{cases}$$

`DSolve` is used to obtain the solution to the nonhomogeneous problem in `cir1`.

Note that we use lower-case letters to avoid any possible ambiguity with built-in Mathematica functions.

```
In[988]:= Clear[q]

          cir1 = DSolve[{q''[t]
                +40 q'[t] + 4000 q[t] == 24, q[0] == 0,
                  q'[0] == 0}, q[t], t]
```

These results indicate that in time the charge approaches the constant value of 3/500, which is known as the **steady-state charge**. Also, due to the exponential term, the current approaches zero as t increases. This limit is indicated by the graph of $Q(t)$ in Figure 5-49, as well.

Out[988]= $\left\{\left\{q[t] \to \frac{1}{500} e^{-20t}(3e^{20t} - 3\,\text{Cos}[60t] - \text{Sin}[60t])\right\}\right\}$

```
In[989]:= q[t_] = cir1[[1, 1, 2]];

In[990]:= Plot[q[t], {t, 0, 0.35}, PlotRange → All]
```

The current, $I(t)$, is obtained by differentiating the charge, $Q(t)$, which is graphed in Figure 5-50.

```
In[991]:= q'[t]//Simplify
```

Out[991]= $\frac{2}{5} e^{-20t}\,\text{Sin}[60\,t]$

```
In[992]:= Plot[q'[t], {t, 0, 0.35}, PlotRange → All]
```

■

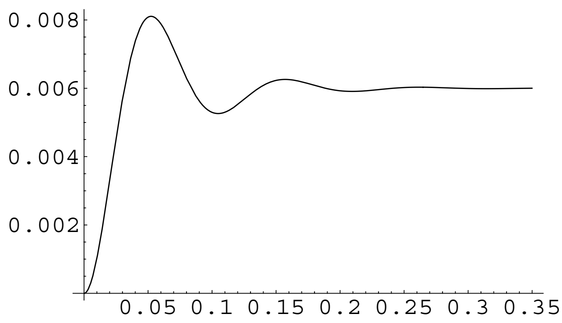


Figure 5-49 Plot of the charge

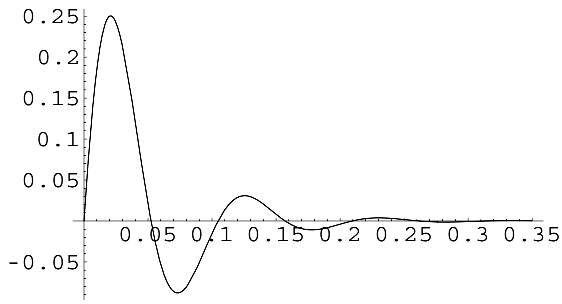
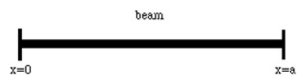


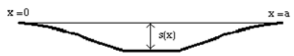
Figure 5-50 Plot of the current

5.3.2 Deflection of a Beam

An important mechanical model involves the deflection of a long beam that is supported at one or both ends as shown in the following figure.



Assuming that in its undeflected form the beam is horizontal, then the deflection of the beam can be expressed as a function of x.



Suppose that the shape of the beam when it is deflected is given by the graph of the function $y(x) = -s(x)$, where x is the distance from one end of the beam and s the measurement of the vertical deflection from the equilibrium position. The boundary value problem that models this situation is derived as follows.

Let $m(x)$ equal the turning moment of the force relative to the point x and $w(x)$ represent the weight distribution of the beam. These two functions are related by the equation

$$\frac{d^2m}{dx^2} = w(x). \tag{5.21}$$

Also, the turning moment is proportional to the curvature of the beam. Hence,

$$m(x) = \frac{EI}{\left[\sqrt{1 + \left(\frac{ds}{dx}\right)^2}\right]^3} \frac{d^2s}{dx^2}, \tag{5.22}$$

where E and I are constants related to the composition of the beam and the shape and size of a cross-section of the beam, respectively. Notice that this equation is, unfortunately, nonlinear. However, this difficulty is overcome with an approximation. For small values of s, the denominator of the right-hand side of equation (5.22) can be approximated by the constant 1. Therefore, equation (5.22) is simplified to

$$m(x) = EI\frac{d^2s}{dx^2}. \tag{5.23}$$

Equation (5.23) is linear and can be differentiated twice to obtain

$$\frac{d^2m}{dx^2} = EI\frac{d^4s}{dx^2}. \tag{5.24}$$

Equation (5.24) can then be used with equation (5.21) relating $m(x)$ and $w(x)$ to obtain the single fourth-order linear nonhomogeneous differential equation

$$EI\frac{d^4s}{dx^4} = w(x). \tag{5.25}$$

Boundary conditions for this problem may vary. In most cases, two conditions are given for each end of the beam. Some of these conditions are specified in pairs. For example, at $x = a$ these include: $s(a) = 0$, $s'(a) = 0$ (fixed end); $s''(a) = 0$, $s'''(a) = 0$ (free end); $s(a) = 0$, $s''(a) = 0$ (simple support); and $s'(a) = 0$, $s'''(a) = 0$ (sliding clamped end).

The following example investigates the effects that a constant weight distribution function $w(x)$ has on the solution to these boundary-value problems.

EXAMPLE 5.3.2: Solve the beam equation over the interval $0 \le x \le 1$ if $E = I = 1$, $w(x) = 48$, and the following boundary conditions are used: $s(0) = 0$, $s'(0) = 0$ (fixed end at $x = 0$); and
(a) $s(1) = 0$, $s''(1) = 0$ (simple support at $x = 1$);
(b) $s''(1) = 0$, $s'''(1) = 0$ (free end at $x = 1$);
(c) $s'(1) = 0$, $s'''(1) = 0$ (sliding clamped end at $x = 1$); and
(d) $s(1) = 0$, $s'(1) = 0$ (fixed end at $x = 1$).

SOLUTION: `DSolve` is used to obtain the solution to this nonhomogeneous problem. In `de1`, the solution that depends on E, I, and w is given.

Note that we use (lower-case) `e` to represent E to avoid conflict with the built-in constant `E` and (lower-case) `i` to represent I to avoid conflict with the built-in constant `I`.

```
In[993]:= Clear[e, i, w, s]

          de1 =
            DSolve[{e i D[s[x], {x, 4}] == w, s[0] == 0,
            s'[0] == 0, s[1] == 0, s''[1] == 0}, s[x], x]
```

Out[993]= $\left\{\left\{s[x] \to \frac{3\,w\,x^2 - 5\,w\,x^3 + 2\,w\,x^4}{48\,e\,i}\right\}\right\}$

We can visualize the shape of the beam by graphing $y = -s(x)$. Thus, we define `toplot1` to be the negative of the solution obtained in `de1`.

```
In[994]:= toplot1 = -de1[[1, 1, 2]]/.{e → 1, i → 1,
            w → 48};
```

Similar steps are followed to determine the solution to each of the other three boundary value problems. The corresponding functions to be graphed are named `toplot2`, `toplot3`, and `toplot4`. (Note that $\partial_{\{x,4\}}$s[x] represents `D[s[x], {x, 4}]`, the fourth derivative of $s(x)$.)

```
In[995]:= de2 =
            DSolve[{e i ∂{x,4} s[x] == w, s[0] == 0,
            s'[0] == 0, s(3)[1] == 0, s''[1] == 0}, s[x], x]
```

Out[995]= $\left\{\left\{s[x] \to \frac{6\,w\,x^2 - 4\,w\,x^3 + w\,x^4}{24\,e\,i}\right\}\right\}$

```
In[996]:= toplot2 = -de2[[1, 1, 2]]/.{e → 1, i → 1,
            w → 48};

In[997]:= de3 =
            DSolve[{e i ∂{x,4} s[x] == w, s[0] == 0,
            s'[0] == 0, s(3)[1] == 0, s'[1] == 0}, s[x], x]
```

Out[997]= $\left\{\left\{s[x] \to \frac{4\,w\,x^2 - 4\,w\,x^3 + w\,x^4}{24\,e\,i}\right\}\right\}$

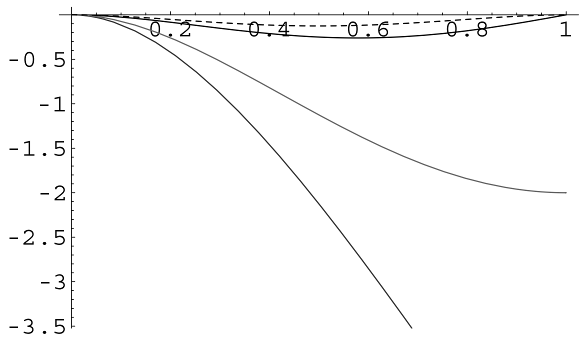


Figure 5-51 Solutions to the beam equation

```
In[998]:= toplot3 = -de3[[1, 1, 2]]/.{e → 1, i → 1,
                w → 48};

In[999]:= de4 = DSolve[{e i ∂{x,4}s[x] == w, s[0] == 0,
              s′[0] == 0, s[1] == 0, s′[1] == 0}, s[x], x]
```

Out[999]= $\left\{\left\{s[x] \rightarrow \frac{w x^2 - 2 w x^3 + w x^4}{24 e i}\right\}\right\}$

```
In[1000]:= toplot4 = -de4[[1, 1, 2]]/.{e → 1, i → 1,
            w → 48};
```

In order to compare the effects that the varying boundary conditions have on the resulting solution, all four functions are graphed together with `Plot` on the interval [0, 1] in Figure 5-51.

```
In[1001]:= Plot[{toplot1, toplot2, toplot3, toplot4},
            {x, 0, 1},
            PlotStyle → {GrayLevel[0], GrayLevel[0.2],
            GrayLevel[0.4], Dashing[{0.01}]}]
```

■

5.3.3 Bodé Plots

Consider the differential equation

$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + k^2x = F_0 \sin \omega t, \tag{5.26}$$

where c and k are positive constants with $c < k$ so that the equation $x'' + 2cx' + k^2x = 0$ is underdamped. To find a particular solution, we can consider the complex exponential form of the forcing function, $F_0 e^{i\omega t}$, which has imaginary part $F_0 \sin \omega t$. Assuming a solution of the form $z_p(t) = Ae^{i\omega t}$, substitution into the differential equation yields $A\left(-\omega^2 + 2ic\omega + k^2\right) = F_0$. Because $k^2 - \omega^2 + 2ic\omega = 0$ only when $k = \omega$ and $c = 0$, we find that

$$A = \frac{F_0}{k^2 - \omega^2 + 2ic\omega}$$

or

$$A = \frac{F_0}{k^2 - \omega^2 + 2ic\omega} \cdot \frac{k^2 - \omega^2 - 2ic\omega}{k^2 - \omega^2 - 2ic\omega} = \frac{k^2 - \omega^2 - 2ic\omega}{\left(k^2 - \omega^2\right)^2 + 4c^2\omega^2} F_0 = H(i\omega)F_0.$$

Therefore, a particular solution is $z_p(t) = H(i\omega)F_0 e^{i\omega t}$. Now, we can write $H(i\omega)$ in polar form as $H(i\omega) = M(\omega)e^{i\phi(\omega)}$, where

$$M(\omega) = \frac{1}{\sqrt{\left(k^2 - \omega^2\right) + 4c^2\omega^2}} \qquad \text{and} \qquad \phi(\omega) = \cot^{-1}\left(\frac{\omega^2 - k^2}{2c\omega}\right), \ -\pi \le \phi \le 0.$$

A particular solution can then be written as

$$z_p(t) = M(\omega)F_0 e^{i\omega t} e^{i\phi(\omega)} = M(\omega)F_0 e^{i(\omega t + \phi(\omega))}$$

with imaginary part $M(\omega)F_0 \sin(\omega t + \phi(\omega))$, so we take the particular solution to be $x_p(t) = M(\omega)F_0 \sin(\omega t + \phi(\omega))$. Comparing the forcing function to x_p, we see that the two functions have the same form but with differing amplitudes and phase shifts. The ratio of the amplitude of the particular solution (or steady-state), $M(\omega)F_0$, to that of the forcing function, F_0, is $M(\omega)$ and is called the **gain**. Also, x_p is shifted in time by $|\phi(\omega)|/\omega$ radians to the right, so $\phi(\omega)$ is called the **phase shift**. When we graph the gain and the phase shift against ω (using a $\log_{10}$ scale on the ω-axis) we obtain the **Bodé plots**. Engineers refer to the value of $20 \log_{10} M(\omega)$ as the gain in **decibels**.

EXAMPLE 5.3.3: Solve the initial-value problem

$$\begin{cases} x'' + 2x' + 4x = \sin 2t \\ x(0) = 1/2, \ x'(0) = 1. \end{cases}$$

(a) Graph the solution simultaneously with the forcing function $f(t) = \sin 2t$. Approximate $M(2)$ and $\phi(2)$ using this graph. (b) Graph the corresponding Bodé plots. Compare the values of $M(2)$ and $\phi(2)$ with those obtained in (a).

SOLUTION: First, we define the nonhomogeneous differential equation in eq. Next, we solve the initial-value problem in sol.

```
In[1002]:= Clear[eq]
           eq = x''[t] + 2 x'[t] + 4 x[t] == Sin[2 t];
           sol = DSolve[{eq,
             x[0] == 1/2, x'[0] == 1}, x[t], t]//Simplify
Out[1002]= {{x[t] → 1/12 e^-t (- 3 e^t Cos[2 t]
           +9 Cos[√3 t] + 7 √3 Sin[√3 t])}}
```

We extract the formula for the solution with sol[[1,1,2]] and graph it simultaneously with $f(t) = \sin 2t$ using a lighter level of gray for the graph of $f(t) = \sin 2t$ in Figure 5-52. Clicking inside the graphics cell and holding down the **Command** key, we use the cursor to see that a minimum value of the forcing function occurs near 5.49 and a minimum value of sol[[1,1,2]] happens near 6.26. Therefore, the solution is shifted approximately $6.26 - 5.49 = 0.77$ units to the right. Returning to the solution containing $\omega t + \phi(\omega) = 2\left(t + \frac{1}{2}\phi(2)\right)$, we see that $\frac{1}{2}\phi(2) \approx -0.77$, so $\phi(2) \approx -1.54$. Using a similar technique (with the **Command** key and cursor), we approximate the amplitude of the steady-state solution (after it dies down) to be 0.255. Therefore from the graph, $M(2) \approx 0.255$.

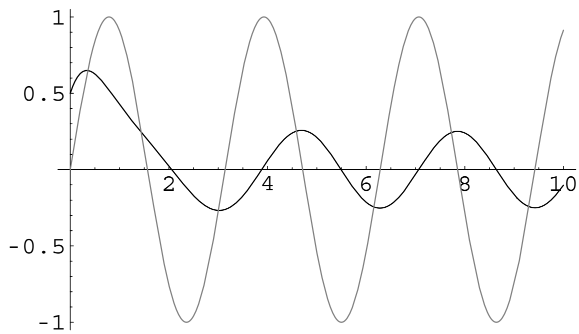


Figure 5-52 Plots of $x(t)$ and $f(t) = \sin 2t$

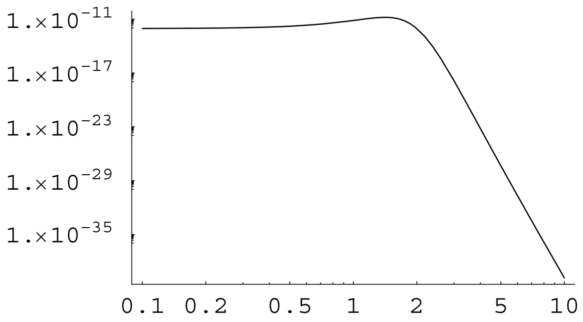


Figure 5-53 A log–log plot of $M(\omega)^{20}$

```
In[1003]:= Plot[{sol[[1,1,2]],Sin[2 t]},{t,0,10},
              PlotStyle → {GrayLevel[0],
              GrayLevel[0.5]}]

In[1004]:= 6.26 - 5.49

Out[1004]= 0.77
```

(b) In the equation $x'' + 2x' + 4x = \sin 2t$, $2c = 2$ and $k^2 = 4$. Therefore, $c = 1$ and $k = 2$. We define the gain function based on these constants in `m[w]`. Because the graph of $M(\omega)$ is a log–log graph, we load the **Graphics** package to take advantage of the `LogLogPlot` command. We graph `m[w]^20` because engineers are interested in $20 \log_{10} M(\omega) = \log_{10} M(\omega)^{20}$. See Figure 5-53. In (a), we obtained $M(2) \approx 0.255$. With the formula for $M(\omega)$, we find that $M(2) = 0.25$.

```
In[1005]:= k = 2;

           c = 1;

           m[w_] := 1/Sqrt[(k^2 - w^2)^2 + 4 c^2 w^2]

In[1006]:= << Graphics`Graphics`

In[1007]:= LogLogPlot[m[w]^20, {w, 0.1, 10}]

In[1008]:= N[m[2]]

Out[1008]= 0.25
```

The branch of $y = \cot^{-1} x$ used by Mathematica is not continuous at $x = 0$ as seen in Figure 5-54.

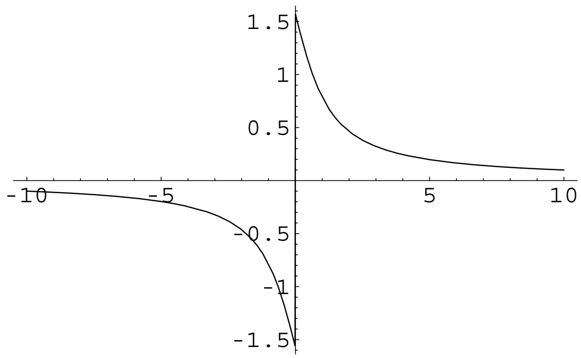


Figure 5-54 Plot of Mathematica's inverse cotangent function

```
In[1009]:= Plot[ArcCot[x], {x, -10, 10}]
```

However, we can construct a function continuous at $x = 0$ as we do in `newarccot`. See Figure 5-55.

```
In[1010]:= Clear[newarccot]

           newarccot[x_] := ArcCot[x]/; x ≥ 0

           newarccot[x_] := ArcCot[x] + π/; x < 0

In[1011]:= Plot[newarccot[x], {x, -10, 10}]
```

Using `newarccot`, we are able to graph $\phi(\omega)$ in Figure 5-56. We define $\phi(\omega)$ so that it `returns` an angle between -180^o and 0^o.

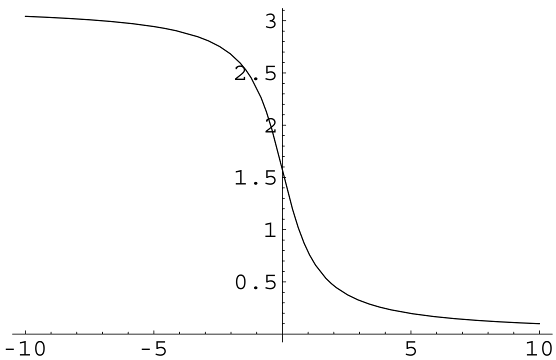


Figure 5-55 The branch is continuous at $x = 0$

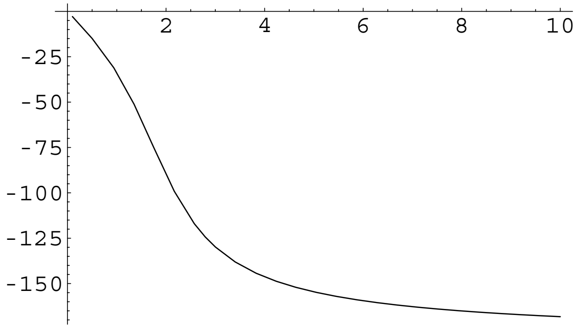


Figure 5-56 Plot of $\phi(\omega)$

In (a), we found $\phi(2) \approx -1.54$ (radians). Here, we see that $\phi(2) = -90^{\circ}$. However, $-\pi/2 \approx -1.57$, so the approximations of $M(2)$ and $\phi(2)$ obtained in (a) are quite accurate.

```
In[1012]:= Clear[ϕ]

            ϕ[w_] := (180 newarccot[(w^2 - k^2)/(2 c w)])/π - 180

In[1013]:= Plot[ϕ[w], {w, 0.1, 10}]

In[1014]:= N[ϕ[2]]

Out[1014]= -90.
```

■

5.3.4 The Catenary

The solution of the second-order nonlinear equation

$$\begin{cases} \dfrac{d^2y}{dx^2} = \dfrac{1}{a}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} \\ y(0) = a,\ \dfrac{dy}{dx}(0) = 0 \end{cases}$$

is called a **catenary**.

```
In[1015]:= DSolve[{y''[x] == 1/a Sqrt[1 + y'[x]^2], y[0] == a,
              y'[0] == 0}, y[x], x]
```

Out[1015]= $\left\{\left\{y[x] \to a \operatorname{Cosh}\left[\frac{x}{a}\right]\right\}, \left\{y[x] \to 2a + a \operatorname{Cosh}\left[\frac{-i a \pi + x}{a}\right]\right\}, \left\{y[x] \to 2a + a \operatorname{Cosh}\left[\frac{i a \pi + x}{a}\right]\right\}\right\}$

A flexible wire or cable suspended between two poles of the same height takes the shape of the **catenary**,

$$y = c + a\cosh\left(\frac{x}{a}\right), \quad a > 0. \tag{5.27}$$

$y = \cosh x$ is defined by $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

EXAMPLE 5.3.4: A flexible cable with length 150 feet is to be suspended between two poles with height 100 feet. How far apart must the poles be spaced so that the bottom of the cable is 50 feet off the ground?

SOLUTION: Let $2s$ denote the distance the poles must be separated and $f(x, c, a) = c + a\cosh\left(\frac{x}{a}\right)$.

```
In[1016]:= f[x_, c_, a_] = c + a Cosh[x/a]
```

Out[1016]= $c + a \operatorname{Cosh}\left[\frac{x}{a}\right]$

At the endpoints, $x = -s$ and $x = s$,

$$f(-s, c, a) = f(s, c, a) = c + a\cosh\left(\frac{s}{a}\right) = 100 \text{ or } \cosh^2\left(\frac{s}{a}\right) = \left(\frac{100 - c}{a}\right)^2. \tag{5.28}$$

The minimum of f is attained at $x = 0$ and must be 50:

$$f(0, c, a) = a + c = 50. \tag{5.29}$$

The length of the wire is 150 feet so by the arc length formula

$$\int_{-s}^{s} \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = 2a\sinh\left(\frac{s}{a}\right) = 150 \quad \text{or} \quad \sinh^2\left(\frac{s}{a}\right) = \left(\frac{75}{a}\right)^2. \tag{5.30}$$

The **length**, L, of the *smooth* curve $y = f(x)$ from $x = a$ to $x = b$ is $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

```
In[1017]:= df = D[f[x, c, a], x]
```

Out[1017]= $\operatorname{Sinh}\left[\frac{x}{a}\right]$

```
In[1018]:= length = Integrate[Sqrt[1 + df^2],
                     {x, -s, s}]//PowerExpand
```

Out[1018]= $2\, a \operatorname{Sinh}\left[\frac{s}{a}\right]$

```
In[1019]:= f[-s, c, a] == 100

           eq1 = f[s, c, a] == 100

           eq2 = f[0, c, a] == 50

           eq3 = length == 150
Out[1019]= c + a Cosh[s/a] == 100
Out[1019]= c + a Cosh[s/a] == 100
Out[1019]= a + c == 50
Out[1019]= 2 a Sinh[s/a] == 150
```

Mathematica can solve equations (5.28), (5.29), and (5.30) for s, a, and c as they are written.

```
In[1020]:= vals = Solve[{eq1, eq2, eq3}, {s, a, c}]
Out[1020]= {{c → 75/4, s → 125/4 ArcCosh[13/5], a → 125/4}}

In[1021]:= vals//N

Out[1021]= {{c → 18.75, s → 50.2949, a → 31.25}}
```

The system can also be solved by hand if you use the identity $\cosh^2 x - \sinh^2 x = 1$. Subtracting equation (5.30) from equation (5.28) gives us

$$1 = \cosh^2\left(\frac{s}{a}\right) - \sinh^2\left(\frac{s}{a}\right) = \left(\frac{100-c}{a}\right)^2 - \left(\frac{75}{a}\right)^2. \tag{5.31}$$

We use `ContourPlot` to graph equations (5.29) and (5.31) together in Figure 5-57. The coordinates of the intersection point, (a, c) are the solutions to the system {(5.29), (5.31)}.

```
In[1022]:= p1 = ContourPlot[(4375 - 200 c + c^2)/a^2,
                {a, 0.01, 50}, {c, 0, 50}, Contours- > {1},
                ContourShading- > False,
                DisplayFunction- > Identity];
           p2 = ContourPlot[a + c,
                {a, 0.01, 50}, {c, 0, 50}, Contours- > {50},
                ContourShading- > False,
                DisplayFunction- > Identity];
                Show[p1,
           p2, DisplayFunction- > $DisplayFunction];
```

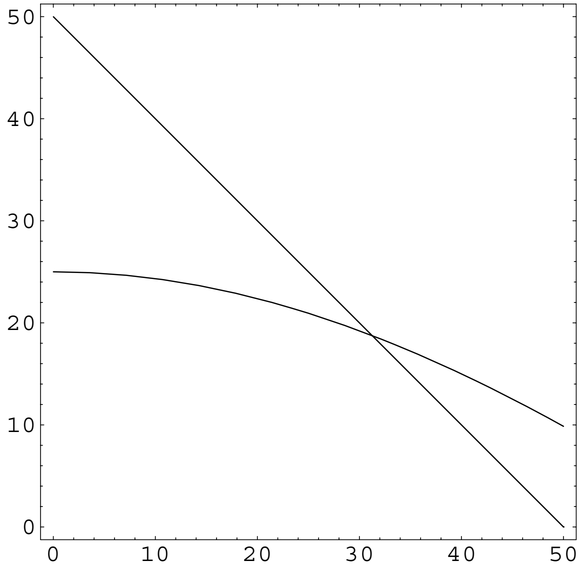


Figure 5-57 Graph of equations (5.29) and (5.31) together

Solving equations (5.29) and (5.31) for a and c with `Solve` gives us $a = 125/4$ and $c = 75/4$.

```
In[1023]:= acvals = Solve[{((100 - c)/a)^2
            -(75/a)^2 == 1, a + c == 50}, {a, c}]
```

Out[1023]= $\left\{\left\{c \to \frac{75}{4}, a \to \frac{125}{4}\right\}\right\}$

Substituting these values into equation (5.28) and solving for s gives us $s = \frac{125}{5} \cosh^{-1}(13/5) \approx 50.2949$.

```
In[1024]:= eq1b = eq1/.acvals[[1]]
```

Out[1024]= $\frac{75}{4} + \frac{125}{4} \text{Cosh}\left[\frac{4\ s}{125}\right] == 100$

```
In[1025]:= Solve[eq1b, s]
```

Out[1025]= $\left\{\left\{s \to -\frac{125}{4} \text{ArcCosh}\left[\frac{13}{5}\right]\right\}, \left\{s \to \frac{125}{4} \text{ArcCosh}\left[\frac{13}{5}\right]\right\}\right\}$

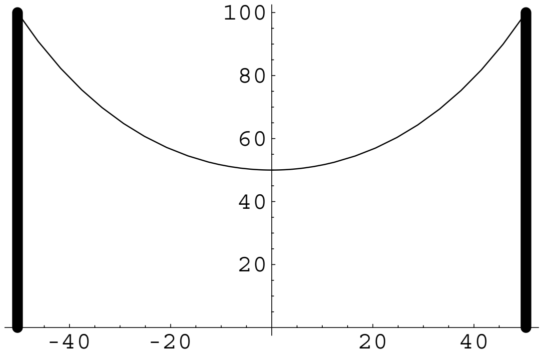


Figure 5-58 The two poles and cable using the *s*, *c*, and *a* values obtained in Example 5.3.4

```
In[1026]:= N[%]

Out[1026]= {{s → -50.2949}, {s → 50.2949}}
```

Using these values, we visualize the cable and poles in Figure 5-58.

```
In[1027]:= p1 = Graphics[{Thickness[0.02],
              Line[{{-50.2949, 0}, {-50.2949, 100}}],
              Line[{{50.2949, 0}, {50.2949, 100}}]}];
            p2 = Plot[f[x, 75/4, 125/4], {x, -50.2949,
              50.2949}, DisplayFunction- > Identity];
            Show[p1, p2, Axes- > Automatic,
             AxesOrigin- > {0, 0}]
```

■

Using the same notation as Example 5.3.4, if a flexible cable with length 150 feet is suspended between two poles with height 100 feet, the distance between the two poles, $2s$, satisfies $0 < s < 75$. Let h denote the distance from the bottom of the cable to the ground. Then,

$$\begin{gathered} f(-s, c, a) = f(s, c, a) = 100, \\ f(0, c, a) = h, \text{ and} \\ \int_{-s}^{s} \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx = 2a \sinh\left(\frac{s}{a}\right) = 150. \end{gathered} \tag{5.32}$$

```
In[1028]:= f[-s, c, a] == 100

            eq1 = f[s, c, a] == 100

            eq2 = f[0, c, a] == h

            eq3 = length == 150
```

Out[1028]= $c + a \text{ Cosh}\left[\frac{s}{a}\right] == 100$

Out[1028]= $c + a \text{ Cosh}\left[\frac{s}{a}\right] == 100$

Out[1028]= $a + c == h$

Out[1028]= $2\ a\ \text{Sinh}\left[\frac{s}{a}\right] == 150$

We use `Solve` to solve system (5.32) for *s*, *c*, and *a*. Mathematica returns two solutions.

```
In[1029]:= posheights = Solve[{eq1, eq2, eq3}, {s, c, a}]
```

Out[1029]=
$$\left\{\left\{c \to \frac{-4375 + h^2}{2\ (-100 + h)},\ s \to -\frac{(4375 - 200\ h + h^2)\ \text{ArcCosh}\left[\frac{-15625+200\ h-h^2}{4375-200\ h+h^2}\right]}{2\ (-100 + h)},\ a \to \frac{4375 - 200\ h + h^2}{-200 + 2\ h}\right\},\ \left\{c \to \frac{-4375 + h^2}{2\ (-100 + h)},\ s \to \frac{(4375 - 200\ h + h^2)\ \text{ArcCosh}\left[\frac{-15625+200\ h-h^2}{4375-200\ h+h^2}\right]}{2\ (-100 + h)},\ a \to \frac{4375 - 200\ h + h^2}{-200 + 2\ h}\right\}\right\}$$

We are assuming that *a* is positive so the meaningful solution is the one for which *a* is positive. We graph each *a*, *s*, and *c* for each component given in `posheights` in Figure 5-59.

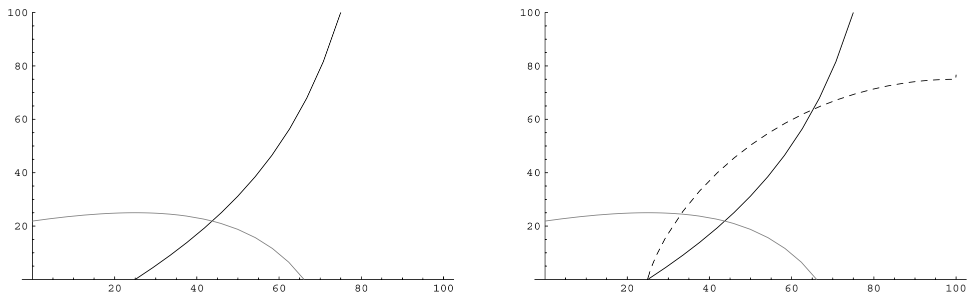


Figure 5-59 Mathematica gives two solutions to system (5.32); the second solution is the meaningful one because $a > 0$

```
In[1030]:= p1 = Plot[Evaluate[{a, s, c}/.posheights[[1]]],
                {h, 0, 100}, PlotStyle- >
                {GrayLevel[0], Dashing[{0.01}],
                GrayLevel[0.5]}, PlotRange- > {0, 100},
                DisplayFunction- > Identity];
            p2 = Plot[Evaluate[{a, s, c}/.posheights[[2]]],
                {h, 0, 100}, PlotStyle- >
                {GrayLevel[0], Dashing[{0.01}],
                GrayLevel[0.5]}, PlotRange- > {0, 100},
                DisplayFunction- > Identity];
            Show[GraphicsArray[{p1, p2}]]
```

Using the results of the second solution given by Mathematica, we are able to generate a graphics array illustrating the position of the poles and the cable for various heights, h, in Figure 5-60.

```
In[1031]:= Clear[wire]
            wire[h_, opts___] := Module[{a, s, c, p1, p2},
            a = (4375 - 200 h + h^2)/(-200 + 2 h);
            s = -((4375 - 200 h + h^2) ArcCosh[(-15625+200 h-h^2)/(4375-200 h+h^2)])/(2 (-100 + h));
            c = (-4375 + h^2)/(2 (-100 + h));
            p1 = Graphics[{Thickness[0.02], Line[{{-s, 0},
                {-s, 100}}], Line[{{s, 0}, {s, 100}}]}];
            p2 = Plot[c + a Cosh[x/a], {x, -s, s},
                DisplayFunction- > Identity];
            Show[p1, p2, Axes- > Automatic, AxesOrigin- > {0, 0},
            PlotRange- > {{-75, 75}, {0, 110}}, opts,
            DisplayFunction- > Identity]  ]})
```

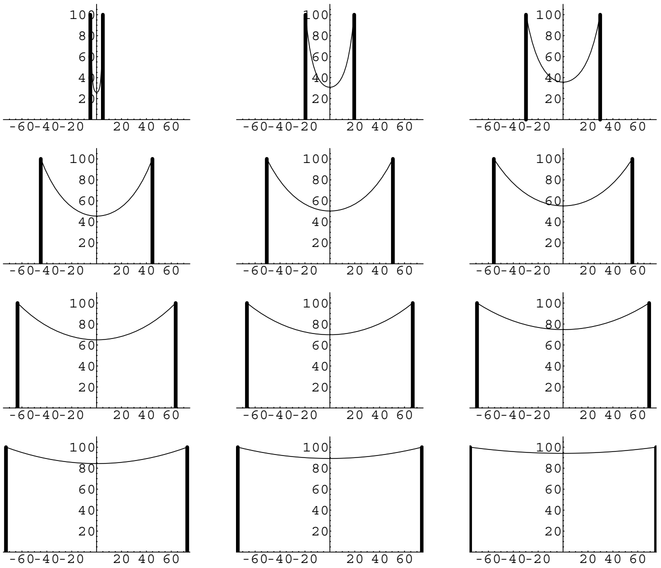


Figure 5-60 An array illustrating how two poles of height 100 feet can be connected by a flexible cable of length 150 feet

```
In[1032]:= graphs = Table[wire[n], {n, 26, 99, (99 - 26)/15}];
           toshow = Partition[graphs, 4];
           Show[GraphicsArray[toshow]]
```

EXAMPLE 5.3.5: According to our electric utility, *Excelsior Electric Membership Corp* (*EMC*), Metter, Georgia, due to terrain, easements, and so on, the average distance between utility poles ranges from 325 to 340 feet. Each pole is approximately 40 feet long with 6 feet buried so that the length of the pole from the ground to the top of the pole is 34 feet. The *Georgia Department of Transportation* states that the maximum height of a truck using interstates, national, and state routes is 13′ 6″. However, special permits may be granted by the *DOT* for heights up to 18′ 0″. With these restrictions in mind, *EMC* maintains a minimum clearance of 20′ under those lines it installs during cooler months because expansion causes lines to sag during warmer months. For the obvious reasons, *EMC* prefers that the distance from its lines to the ground is greater than 18′ 6″ at all times.

Find c and a so that $f(x, c, a) = c + a\cosh\left(\frac{x}{a}\right)$ models this situation.

SOLUTION: Centering $f(x, c, a) = c + a\cosh\left(\frac{x}{a}\right)$ at $x = 0$, we require that

$$f(-170, c, a) = f(170, c, a) = 34, \text{ and} \qquad f(0, c, a) = 20. \tag{5.33}$$

```
In[1033]:= f[x_, c_, a_] = c + a Cosh[x/a]

Out[1033]= c + a  Cosh[x/a]

In[1034]:= df = D[f[x, c, a], x]

Out[1034]= Sinh[x/a]

In[1035]:= f[170, c, a]

Out[1035]= c + a  Cosh[170/a]

In[1036]:= f[0, c, a]

Out[1036]= a + c
```

Mathematica cannot solve system (5.33) exactly with `Solve`

```
In[1037]:= Solve[{c + a  Cosh[170/a] == 34,
               a + c == 20}, {a, c}]

Solve :: "tdep" :  "Theequationsappeartoinvolve
     transcendentalfunctionsofthevariablesin
     anessentiallynon - algebraicway."

Out[1037]= Solve[{c + a  Cosh[170/a] == 40,
               a + c == 20}, {a, c}]
```

but using `ContourPlot` to graph the equations $f(170, c, a) = 34$ and $f(0, c, a) = 20$ together as shown in Figure 5-61 shows us that the system does have a solution.

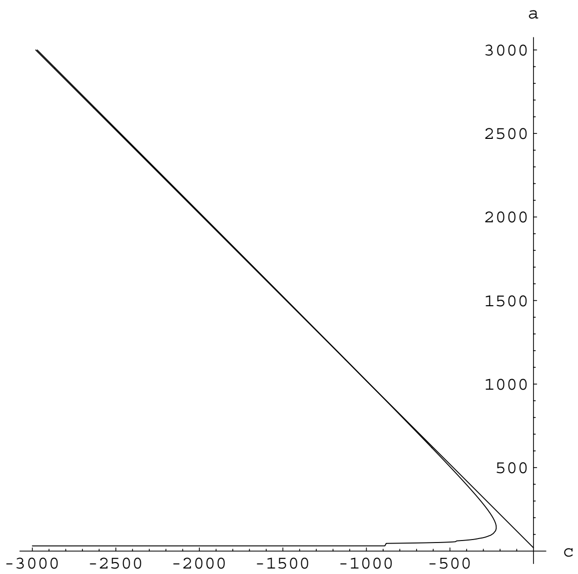


Figure 5-61 Graphs of equations $f(170, c, a) = 40$ and $f(0, c, a) = 20$: the intersection point is difficult to see

```
In[1038]:= cp1 = ContourPlot[f[170, c, a],
               {c, -3000, 0}, {a, 1, 3000},
               Contours- > {34},
               ContourShading- > False,
               PlotPoints- > 200,
               AxesOrigin- > {0, 0}, Frame- > False,
               Axes- > Automatic,
               DisplayFunction- > Identity,
               AxesLabel- > {"c", "a"}];
           cp2 = ContourPlot[a + c, {c, -3000, 0},
               {a, 1, 3000}, Contours- > {20},
               ContourShading- > False,
               PlotPoints- > 200,
               AxesOrigin- > {0, 0}, Frame- > False,
               Axes- > Automatic,
               DisplayFunction- > Identity,
               AxesLabel- > {"c", "a"}];
           g1 = Show[cp1, cp2,
               DisplayFunction- > $DisplayFunction]
```

We use `FindRoot` to find the solution.

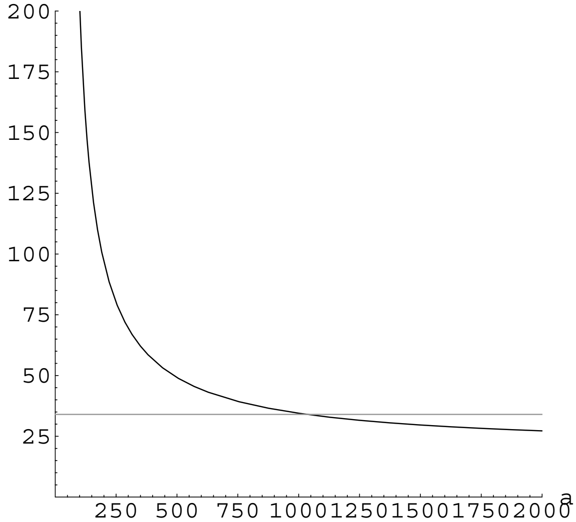


Figure 5-62 Plot of $f(170, 20 - a, a)$ and $a = 34$ together: the solution for a is much easier to see

```
In[1039]:= FindRoot[{c + a Cosh[170/a] == 34, a + c == 20},
              {a, 1500}, {c, -1500}]
Out[1039]= {a → 1034.47, c → -1014.47}
```

However, using $c = 20 - a$ we graph $f(170, 20 - a, a)$ and $a = 34$ together in Figure 5-62. The solution is much easier to see in Figure 5-62 than in Figure 5-61.

```
In[1040]:= Plot[{f[170, 20 - a, a], 34}, {a, 0, 3000},
              PlotStyle- > {GrayLevel[0],
              GrayLevel[0.6]},
              PlotRange- > {{0, 2000}, {0, 200}},
              AspectRatio- > 1, AxesLabel- > {"a", ""}]
```

Now, we obtain the same results with `FindRoot` and `Solve` as we did previously with `FindRoot`.

```
In[1041]:= aval = FindRoot[f[170, 20 - a,
                  a] == 34, {a, 1100}]
Out[1041]= {a → 1034.47}

In[1042]:= aval[[1, 2]]
Out[1042]= 1034.47
```

Figure 5-63 A graphic illustrating how a utility line like an electrical cable may be connected between two poles of equal height

```
In[1043]:= cval = 20 - a/.aval[[1]]
Out[1043]= -1014.47

In[1044]:= length/.{s- > 170, aval[[1]]}
Out[1044]= 341.532

In[1045]:= Solve[a + c == 20, c]
Out[1045]= {{c → 20 - a}}
```

With the results obtained above, we generate a plot illustrating the hanging wire in Figure 5-63.

```
In[1046]:= p1b = Graphics[{Thickness[0.01],
                   Line[{{-170, 0}, {-170, 34}}],
                   Line[{{170, 0}, {170, 34}}]}];
            p2b = Plot[Evaluate[f[x, cval,
                   aval[[1, 2]]]], {x, -170, 170},
                   DisplayFunction- > Identity];
            Show[p1b, p2b, Axes- > None,
                 AxesOrigin- > {0, 0},
                 AspectRatio- > Automatic]
```

■

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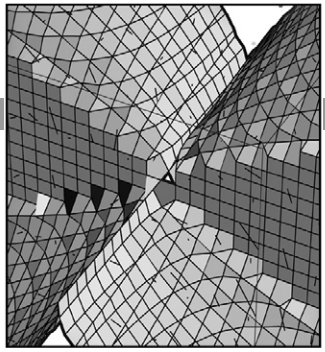
Systems of Ordinary Differential Equations

6



6.1 Review of Matrix Algebra and Calculus

Because of their importance in the study of systems of linear equations, we now review matrices and the operations associated with them.

6.1.1 Defining Nested Lists, Matrices, and Vectors

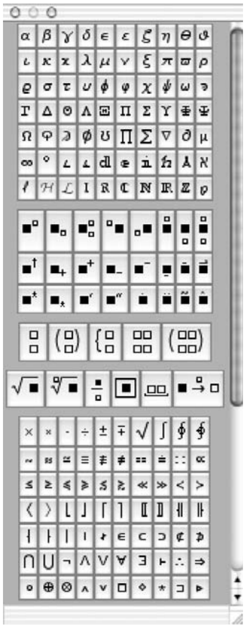
In Mathematica, a **matrix** is a list of lists where each list represents a row of the matrix. Therefore, the $m \times n$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

is entered with `A={{a11,a12,...,a1n},{a21,a22,...,a2n},...,{am1, am2,...amn}}`. For example, to use Mathematica to define `m` to be the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ enter the command

```
m={{a11,a12},{a21,a22}}.
```

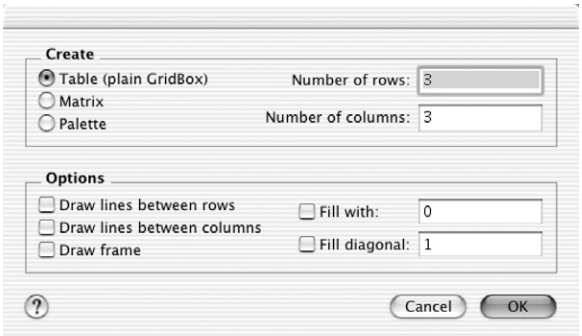
The command `m=Array[a,{2,2}]` produces a result equivalent to this. Once a matrix `A` has been entered, it can be viewed in the traditional row-and-column form using the command `MatrixForm[A]`. You can quickly construct 2×2 matrices by clicking on the (□□ □□) button from the **BasicTypesetting** palette, which is accessed by going to **File** under the Mathematica menu, followed by **Palettes** and then **BasicTypesetting**.



Alternatively, you can construct matrices of any dimension by going to the Mathematica menu under **Input** and selecting **Create Table/Matrix/Palette...**

As when using `TableForm`, the result of using `MatrixForm` is no longer a list that can be manipulated using Mathematica commands. Use `MatrixForm` to view a matrix in traditional row-and-column form. Do not attempt to perform matrix operations on a `MatrixForm` object.

The resulting pop-up window allows you to create tables, matrices, and palettes. To create a matrix, select **Matrix**, enter the number of rows and columns of the matrix, and select any other options. Pressing the **OK** button places the desired matrix at the position of the cursor in the Mathematica notebook.



EXAMPLE 6.1.1: Use Mathematica to define the matrices $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix}$.

SOLUTION: In this case, both `Table[a_{i,j}, {i, 1, 3}, {j, 1, 3}]` and `Array[a,{3,3}]` produce equivalent results when we define `matrixa` to be the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The commands `MatrixForm` or `TableForm` are used to display the results in traditional matrix form.

```
In[1047]:= Clear[a,b,matrixa,matrixb]

In[1048]:= matrixa = Table[a_{i,j}, {i, 1, 3}, {j, 1, 3}]

Out[1048]= {{a_{1,1}, a_{1,2}, a_{1,3}},
            {a_{2,1}, a_{2,2}, a_{2,3}}, {a_{3,1}, a_{3,2}, a_{3,3}}}

In[1049]:= MatrixForm[matrixa]
```

```
Out[1049]= (a1,1 a1,2 a1,3
            a2,1 a2,2 a2,3
            a3,1 a3,2 a3,3)

In[1050]:= matrixa = Array[a, {3, 3}]
Out[1050]= {{a[1, 1], a[1, 2], a[1, 3]},
  {a[2, 1], a[2, 2], a[2, 3]},
  {a[3, 1], a[3, 2], a[3, 3]}}

In[1051]:= MatrixForm[matrixa]
Out[1051]= (a[1, 1] a[1, 2] a[1, 3]
            a[2, 1] a[2, 2] a[2, 3]
            a[3, 1] a[3, 2] a[3, 3])
```

We may also use Mathematica to define nonsquare matrices.

```
In[1052]:= matrixb = Array[b, {2, 4}]
Out[1052]= {{b[1, 1], b[1, 2], b[1, 3], b[1, 4]},
  {b[2, 1], b[2, 2], b[2, 3], b[2, 4]}}

In[1053]:= MatrixForm[matrixb]
Out[1053]= (b[1, 1] b[1, 2] b[1, 3] b[1, 4]
            b[2, 1] b[2, 2] b[2, 3] b[2, 4])
```

Equivalent results would have been obtained by entering `Table[`$b_{i,j}$, {i, 1, 2}, {j, 1, 4}].

■

More generally the commands `Table[f[i,j],{i,imax},{j,jmax}]` and `Array[f,{imax,jmax}]` yield nested lists corresponding to the `imax` × `jmax` matrix

$$\begin{pmatrix} f(1,1) & f(1,2) & \cdots & f(1,\mathtt{jmax}) \\ f(2,1) & f(2,2) & \cdots & f(2,\mathtt{jmax}) \\ \vdots & \vdots & \vdots & \vdots \\ f(\mathtt{imax},1) & f(\mathtt{imax},2) & \cdots & f(\mathtt{imax},\mathtt{jmax}) \end{pmatrix}.$$

`Table[f[i,j],{i,imin,imax,istep},{j,jmin,jmax,jstep}]` returns the list of lists

```
{{f[imin,jmin],f[imin,jmin+jstep],...,f[imin,jmax]},
    {f[imin+istep,jmin],...,f[imin+istep,jmax]},
        ...,{f[imax,jmin],...,f[imax,jmax]}}
```

and the command

```
Table[f[i,j,k,...],{i,imin,imax,istep},{j,jmin,jmax,jstep},
  {k,kmin,kmax,kstep},...]
```

calculates a nested list; the list associated with i is outermost. If `istep` is omitted, the stepsize is one.

In Mathematica, a **vector** is a list of numbers and, thus, is entered in the same manner as lists. For example, to use Mathematica to define the row vector `vector v` to be $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ enter `vector v={v1,v2, v3}`. Similarly, to define the column vector `vector v` to be $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ enter `vector v={v1,v2,v3}` or `vector v={{v1},{v2},{v3}}`. For a 2×1 vector, you can use the  button on the **Basic-Typesetting** palette. Generally, with Mathematica you do not need to distinguish between row and column vectors: Mathematica performs computations with vectors and matrices correctly as long as the computations are well-defined.

With Mathematica, you do not need to distinguish between row and column vectors. Provided that computations are well-defined, Mathematica carries them out correctly. Mathematica warns of any ambiguities when they (rarely) occur.

EXAMPLE 6.1.2: Define the vector $\mathbf{w} = \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}$, `vectorv` to be the vector $\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}$, and `zerovec` to be the vector $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

SOLUTION: To define **w**, we enter

```
In[1054]:= w = {-4, -5, 2}
Out[1054]= {-4, -5, 2}
```

or

```
In[1055]:= w = {{-4}, {-5}, {2}};
            MatrixForm[w]
```

Out[1055]= $\begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}$

To define `vectorv`, we use `Array`.

```
In[1056]:= vectorv = Array[v, 4]
Out[1056]= {v[1], v[2], v[3], v[4]}
```

Equivalent results would have been obtained by entering `Table[`v_i`, {i, 1, 4}]`. To define `zerovec`, we use `Table`.

```
In[1057]:= zerovec = Table[0, {5}]
Out[1057]= {0, 0, 0, 0, 0}
```

The same result is obtained by going to **Input** under the Mathematica menu and selecting **Create Table/Matrix/Palette...**

```
In[1058]:= (0 0 0 0 0)
Out[1058]= {{0, 0, 0, 0, 0}}
```

■

6.1.2 Extracting Elements of Matrices

For the 2×2 matrix `m` = {{$a_{1,1}$, $a_{1,2}$}, {$a_{2,1}$, $a_{2,2}$}} defined earlier, `m[[1]]` yields the first element of matrix `m` which is the list {$a_{1,1}$, $a_{1,2}$} or the first row of `m`; `m[[2,1]]` yields the first element of the second element of matrix `m` which is $a_{2,1}$. In general, if **m** is an $i \times j$ matrix, `m[[i,j]]` or `Part[m,i,j]` returns the unique element in the ith row and jth column of **m**. More specifically, `m[[i,j]]` yields the jth part of the ith part of **m**; `list[[i]]` or `Part[list,i]` yields the ith part of `list`; `list[[i,j]]` or `Part[list,i,j]` yields the jth part of the ith part of list, and so on.

EXAMPLE 6.1.3: Define `mb` to be the matrix $\begin{pmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{pmatrix}$. (a) Extract the third row of `mb`. (b) Extract the element in the first row and third column of `mb`. (c) Display `mb` in traditional matrix form.

SOLUTION: We begin by defining the command `mb`. `mb[[i,j]]` yields the (unique) number in the ith row and jth column of `mb`. Observe how various components of `mb` (rows and elements) can be extracted and how `mb` is placed in `MatrixForm`.

```
In[1059]:= mb = {{10, -6, -9}, {6, -5, -7},
                 {-10, 9, 12}};

In[1060]:= MatrixForm[mb]
```

$$\text{Out[1060]=} \begin{pmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{pmatrix}$$

```
In[1061]:= mb[[3]]
Out[1061]= {-10, 9, 12}

In[1062]:= mb[[1, 3]]
Out[1062]= -9
```

■

If m is a matrix, the *i*th row of m is extracted with m[[i]]. The command Transpose[m] yields the transpose of the matrix m, the matrix obtained by interchanging the rows and columns of m. We extract columns of m by computing Transpose[m] and then using Part to extract rows from the transpose. Namely, if m is a matrix, Transpose[m][[i]] extracts the *i*th row from the transpose of m which is the same as the *i*th column of m.

EXAMPLE 6.1.4: Extract the second and third columns from **A** if **A** = $\begin{pmatrix} 0 & -2 & 2 \\ -1 & 1 & -3 \\ 2 & -4 & 1 \end{pmatrix}$.

SOLUTION: We first define matrixa and then use Transpose to compute the transpose of matrixa, naming the result ta, and then displaying ta in MatrixForm.

```
In[1063]:= matrixa = {{0, -2, 2}, {-1, 1, -3},
              {2, -4, 1}};

In[1064]:= ta = Transpose[matrixa];

            MatrixForm[ta]
```

Out[1064]= $\begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & -4 \\ 2 & -3 & 1 \end{pmatrix}$

Next, we extract the second column of matrixa using Transpose together with Part ([[...]]). Because we have already defined ta to be the transpose of matrixa, entering ta[[2]] would produce the same result.

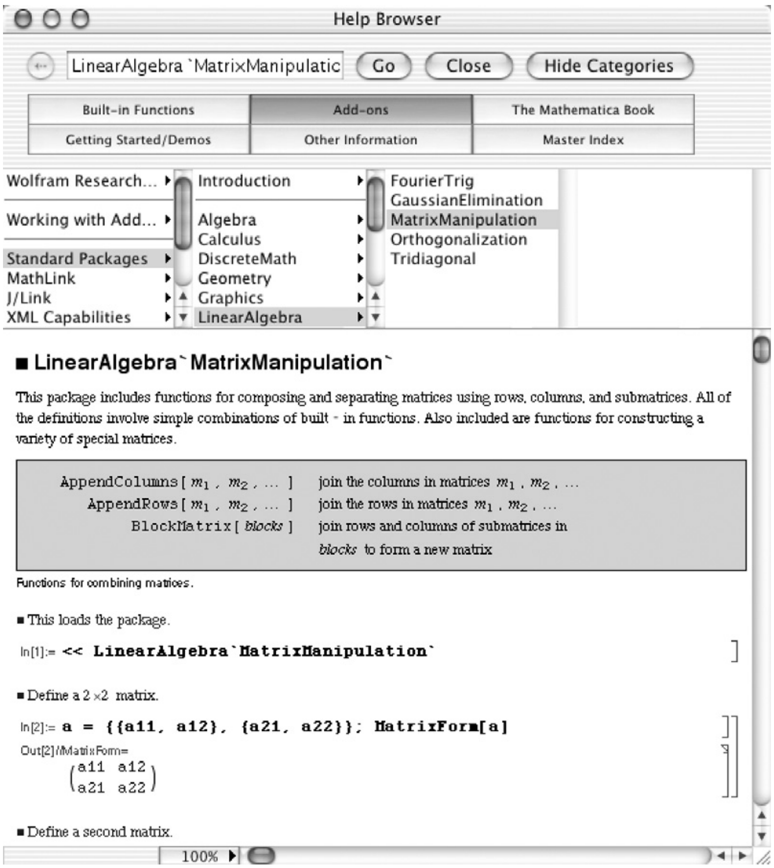
```
In[1065]:= Transpose[matrixa][[2]]
Out[1065]= {-2, 1, -4}
```

To extract the third column, we take advantage of the fact that we have already defined `ta` to be the transpose of `matrixa`. Entering `Transpose[matrixa][[3]]` produces the same result.

```
In[1066]:= ta[[3]]

Out[1066]= {2,-3,1}
```

■

Other commands that can be used to manipulate matrices are included in the **MatrixManipulation** package, which is contained in the **Linear Algebra** folder (or directory).



After this package has been loaded,

```
In[1067]:= << LinearAlgebra`MatrixManipulation`
```

we can use commands like `TakeColumns` and `TakeRows` to extract columns and rows from a given matrix. For example, entering

```
In[1068]:= TakeColumns[matrixa, {2}]//MatrixForm
```

$$\text{Out[1068]=} \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

extracts the second column of the matrix **A** defined in the previous example and displays the result in `MatrixForm` while entering

```
In[1069]:= TakeColumns[matrixa, {2, 3}]//
              MatrixForm
```

$$\text{Out[1069]=} \begin{pmatrix} -2 & 2 \\ 1 & -3 \\ -4 & 1 \end{pmatrix}$$

extracts the second and third columns of **A** and displays the result in `Matrix Form`.

6.1.3 Basic Computations with Matrices

Mathematica performs all of the usual operations on matrices. Matrix addition ($\mathbf{A} + \mathbf{B}$), scalar multiplication ($k\mathbf{A}$), matrix multiplication (when defined) ($\mathbf{AB}$), and combinations of these operations are all possible. The **transpose** of $\mathbf{A}$, $\mathbf{A}^t$, is obtained by interchanging the rows and columns of **A** and is computed with the command `Transpose[A]`. If **A** is a square matrix, the determinant of **A** is obtained with `Det[A]`.

If **A** and **B** are $n \times n$ matrices satisfying $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, where **I** is the $n \times n$ matrix with 1s on the diagonal and 0s elsewhere (the $n \times n$ identity matrix), **B** is called the **inverse** of **A** and is denoted by $\mathbf{A}^{-1}$. If the inverse of a matrix **A** exists, the inverse is found with `Inverse[A]`. Thus, assuming that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse ($ad - bc \neq 0$), the inverse is

```
In[1070]:= Inverse[{{a, b}, {c, d}}]
```

$$\text{Out[1070]=} \left\{\left\{\frac{d}{-b\ c + a\ d}, -\frac{b}{-b\ c + a\ d}\right\}, \left\{-\frac{c}{-b\ c + a\ d}, \frac{a}{-b\ c + a\ d}\right\}\right\}$$

EXAMPLE 6.1.5: Let $A = \begin{pmatrix} 3 & -4 & 5 \\ 8 & 0 & -3 \\ 5 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \end{pmatrix}$. Compute (a)$\mathbf{A} + \mathbf{B}$; (b) $\mathbf{B} - 4\mathbf{A}$; (c) the inverse of $\mathbf{AB}$; (d) the transpose of $(\mathbf{A} - 2\mathbf{B})\,\mathbf{B}$; and (e) $\det \mathbf{A} = |\mathbf{A}|$.

SOLUTION: We enter `ma` (corresponding to **A**) and `mb` (corresponding to **B**) as nested lists where each element corresponds to a row of the matrix. We suppress the output by ending each command with a semi-colon.

```
In[1071]:= ma = {{3, -4, 5}, {8, 0, -3}, {5, 2, 1}};

            mb = {{10, -6, -9}, {6, -5, -7},
                  {-10, 9, 12}};
```

Entering

```
In[1072]:= ma + mb//MatrixForm
```

$$\text{Out[1072]=} \begin{pmatrix} 13 & -10 & -4 \\ 14 & -5 & -10 \\ -5 & 11 & 13 \end{pmatrix}$$

adds matrix `ma` to `mb` and expresses the result in traditional matrix form. Entering

```
In[1073]:= mb - 4ma//MatrixForm
```

$$\text{Out[1073]=} \begin{pmatrix} -2 & 10 & -29 \\ -26 & -5 & 5 \\ -30 & 1 & 8 \end{pmatrix}$$

subtracts four times matrix `ma` from `mb` and expresses the result in traditional matrix form. Entering

```
In[1074]:= Inverse[ma.mb]//MatrixForm
```

$$\text{Out[1074]=} \begin{pmatrix} \frac{59}{380} & \frac{53}{190} & -\frac{167}{380} \\ -\frac{223}{570} & -\frac{92}{95} & \frac{979}{570} \\ \frac{49}{114} & \frac{18}{19} & -\frac{187}{114} \end{pmatrix}$$

Matrix products, when defined, are computed by placing a period (.) between the matrices being multiplied. Note that a period is also used to compute the dot product of two vectors, when the dot product is defined.

computes the inverse of the matrix product **AB**. Similarly, entering

```
In[1075]:= Transpose[(ma - 2mb).mb]//MatrixForm
```

$$\text{Out[1075]=} \begin{pmatrix} -352 & -90 & 384 \\ 269 & 73 & -277 \\ 373 & 98 & -389 \end{pmatrix}$$

computes the transpose of $(\mathbf{A} - 2\mathbf{B})\,\mathbf{B}$ and entering

```
In[1076]:= Det[ma]
Out[1076]= 190
```

computes the determinant of `ma`.
■

EXAMPLE 6.1.6: Compute $\mathbf{AB}$ and $\mathbf{BA}$ if $\mathbf{A} = \begin{pmatrix} -1 & -5 & -5 & -4 \\ -3 & 5 & 3 & -2 \\ -4 & 4 & 2 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ -4 & 3 \\ 4 & -4 \\ -5 & -3 \end{pmatrix}$.

SOLUTION: Because $\mathbf{A}$ is a 3×4 matrix and $\mathbf{B}$ is a 4×2 matrix, $\mathbf{AB}$ is defined and is a 3×2 matrix. We define `matrixa` and `matrixb` with the following commands.

Remember that you can also define matrices by going to **Input** under the Mathematica menu and selecting **Create Table/Matrix/Palette...**. After entering the desired number of rows and columns and pressing the **OK** button, a matrix template is placed at the location of the cursor that you can fill in.

```
In[1077]:= matrixa = (-1 -5 -5 -4
                      -3  5  3 -2
                      -4  4  2 -3);

In[1078]:= matrixb = ( 1 -2
                      -4  3
                       4 -4
                      -5 -3);
```

We then compute the product, naming the result `ab`, and display `ab` in `MatrixForm`.

```
In[1079]:= ab = matrixa.matrixb;
           MatrixForm[ab]

Out[1079]= (19 19
            -1 15
             3 21)
```

However, the matrix product $\mathbf{BA}$ is not defined and Mathematica produces error messages when we attempt to compute it.

```
In[1080]:= matrixb.matrixa
Dot::dotsh :
  Tensors {{1,-2},{-4,3},{4,-4},{-5,-3}}
    and {{-1,-5,-5,-4},{-3,5,3,-2},
      {-4,4,2,-3}} have incompatible shapes.
Out[1080]= {{1,-2},{-4,3},{4,-4},{-5,-3}}.
           {{-1,-5,-5,-4},
             {-3,5,3,-2},{-4,4,2,-3}}
```

■

6.1.4 Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be an $n \times n$ matrix. The number λ is an **eigenvalue** of $\mathbf{A}$ if there is a *nonzero* vector, $\mathbf{v}$, called an **eigenvector**, satisfying

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}. \tag{6.1}$$

We find the eigenvalues of $\mathbf{A}$ by solving the **characteristic polynomial**

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{6.2}$$

for λ. Once we find the eigenvalues, the corresponding eigenvectors are found by solving

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0} \tag{6.3}$$

for $\mathbf{v}$.

If $\mathbf{A}$ is a square matrix,

```
Eigenvalues[A]
```

finds the eigenvalues of $\mathbf{A}$,

```
Eigenvectors[A]
```

finds the eigenvectors, and

```
Eigensystem[A]
```

finds the eigenvalues and corresponding eigenvectors.

```
CharacteristicPolynomial[A,lambda]
```

finds the characteristic polynomial of $\mathbf{A}$ as a function of λ.

EXAMPLE 6.1.7: Find the eigenvalues and corresponding eigenvectors for each of the following matrices. (a) $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$; (b) $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$; (c) $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$; (d) $\mathbf{A} = \begin{pmatrix} -1/4 & 2 \\ -8 & -1/4 \end{pmatrix}$.

SOLUTION: (a) We begin by finding the eigenvalues. Solving

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 5 = 0$$

gives us $\lambda_1 = -5$ and $\lambda_2 = -1$.

Observe that the same results are obtained using `CharacteristicPolynomial` and `Eigenvalues`.

```
In[1081]:= capa = {{-3, 2}, {2, -3}};
            CharacteristicPolynomial[capa, λ]//Factor

            e1 = Eigenvalues[capa]
Out[1081]= (1 + λ) (5 + λ)
Out[1081]= {-5, -1}
```

We now find the corresponding eigenvectors. Let $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ be an eigenvector corresponding to λ_1, then

$$(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$$

$$\left[\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} - (-5)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which row reduces to

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That is, $x_1 + y_1 = 0$ or $x_1 = -y_1$. Hence, for any value of $y_1 \neq 0$,

$$\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_1 \end{pmatrix} = y_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to λ_1. Of course, this represents infinitely many vectors. But, they are all linearly dependent. Choosing $y_1 = 1$ yields $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Note that you might have chosen $y_1 = -1$ and obtained $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. However, both of our results are "correct" because these vectors are linearly dependent.

Similarly, letting $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be an eigenvector corresponding to λ_2 we solve $(\mathbf{A} - \lambda_2 \mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $x_2 - y_2 = 0$ or $x_2 = y_2$. Hence, for any value of $y_2 \neq 0$,

$$\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_2 \end{pmatrix} = y_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to λ_2. Choosing $y_2 = 1$ yields $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We confirm these results using `RowReduce`.

```
In[1082]:= i2 = {{1, 0}, {0, 1}};
            ev1 = capa - e1[[1]] i2
Out[1082]= {{2, 2}, {2, 2}}

In[1083]:= RowReduce[ev1]
Out[1083]= {{1, 1}, {0, 0}}

In[1084]:= ev2 = capa - e1[[2]] i2

            RowReduce[ev2]
Out[1084]= {{-2, 2}, {2, -2}}
Out[1084]= {{1, -1}, {0, 0}}
```

We obtain the same results using `Eigenvectors` and `Eigensystem`.

```
In[1085]:= Eigenvectors[capa]

            Eigensystem[capa]
Out[1085]= {{-1, 1}, {1, 1}}
Out[1085]= {{-5, -1}, {{-1, 1}, {1, 1}}}
```

(b) In this case, we see that $\lambda = 2$ has multiplicity 2. There is only one linearly independent eigenvector, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, corresponding to λ.

```
In[1086]:= capa = {{1, -1}, {1, 3}};
           Factor[CharacteristicPolynomial[capa, λ]]

           Eigenvectors[capa]

           Eigensystem[capa]
Out[1086]= (-2 + λ)^2
Out[1086]= {{-1, 1}, {0, 0}}
Out[1086]= {{2, 2}, {{-1, 1}, {0, 0}}}
```

(c) The eigenvalue $\lambda_1 = 2$ has corresponding eigenvector $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The eigenvalue $\lambda_{2,3} = -1$ has multiplicity 2. In this case, there are two linearly independent eigenvectors corresponding to this eigenvalue: $\mathbf{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{v_3} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

```
In[1087]:= capa = {{0, 1, 1}, {1, 0, 1}, {1, 1, 0}};
           Factor[CharacteristicPolynomial[capa, λ]]

           Eigenvectors[capa]

           Eigensystem[capa]
Out[1087]= -(-2 + λ) (1 + λ)^2
Out[1087]= {{-1, 0, 1}, {-1, 1, 0}, {1, 1, 1}}
Out[1087]= {{-1, -1, 2}, {{-1, 0, 1}, {-1, 1, 0}, {1, 1, 1}}}
```

(d) In this case, the eigenvalues $\lambda_{1,2} = -\frac{1}{4} \pm 4i$ are complex conjugates. We see that the eigenvectors $\mathbf{v}_{1,2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$ are complex conjugates as well.

```
In[1088]:= capa = {{-1/4, 2}, {-8, -1/4}};
           Eigenvectors[capa]

           Eigensystem[capa]
Out[1088]= {{i, 2}, {-i, 2}}
Out[1088]= {{-1/4 - 4 i, -1/4 + 4 i}, {{i, 2}, {-i, 2}}}
```

■

6.1.5 Matrix Calculus

Definition 23 (Derivative and Integral of a Matrix). *The **derivative** of the $m \times n$ matrix*

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \cdots & a_{2n}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) & \cdots & a_{3n}(t) \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1}(t) & a_{m2}(t) & a_{m3}(t) & \cdots & a_{mn}(t) \end{pmatrix},$$

where $a_{ij}(t)$ is differentiable for all values of i and j, is

$$\frac{d}{dt}\mathbf{A}(t) = \begin{pmatrix} \frac{d}{dt}a_{11}(t) & \frac{d}{dt}a_{12}(t) & \frac{d}{dt}a_{13}(t) & \cdots & \frac{d}{dt}a_{1n}(t) \\ \frac{d}{dt}a_{21}(t) & \frac{d}{dt}a_{22}(t) & \frac{d}{dt}a_{23}(t) & \cdots & \frac{d}{dt}a_{2n}(t) \\ \frac{d}{dt}a_{31}(t) & \frac{d}{dt}a_{32}(t) & \frac{d}{dt}a_{33}(t) & \cdots & \frac{d}{dt}a_{3n}(t) \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{d}{dt}a_{m1}(t) & \frac{d}{dt}a_{m2}(t) & \frac{d}{dt}a_{m3}(t) & \cdots & \frac{d}{dt}a_{mn}(t) \end{pmatrix}.$$

*The **integral** of $\mathbf{A}(t)$, where $a_{ij}(t)$ is integrable for all values of i and j, is*

$$\int \mathbf{A}(t)\,dt = \begin{pmatrix} \int a_{11}(t)\,dt & \int a_{12}(t)\,dt & \int a_{13}(t)\,dt & \cdots & \int a_{1n}(t)\,dt \\ \int a_{21}(t)\,dt & \int a_{22}(t)\,dt & \int a_{23}(t)\,dt & \cdots & \int a_{2n}(t)\,dt \\ \int a_{31}(t)\,dt & \int a_{32}(t)\,dt & \int a_{33}(t)\,dt & \cdots & \int a_{3n}(t)\,dt \\ \vdots & \vdots & \vdots & & \vdots \\ \int a_{m1}(t)\,dt & \int a_{m2}(t)\,dt & \int a_{m3}(t)\,dt & \cdots & \int a_{mn}(t)\,dt \end{pmatrix}.$$

EXAMPLE 6.1.8: Find $\frac{d}{dt}\mathbf{A}(t)$ and $\int \mathbf{A}(t)\,dt$ if $\mathbf{A}(t) = \begin{pmatrix} \cos 3t & \sin 3t & e^{-t} \\ t & t\sin t^2 & e^t \end{pmatrix}$.

SOLUTION: We find $\frac{d}{dt}\mathbf{A}(t)$ by differentiating each element of $\mathbf{A}(t)$ with `D`.

```
In[1089]:= a = {{Cos[3t], Sin[3t], Exp[-t]},
              {t, t Sin[t^2], Exp[t]}};

            D[a, t]//MatrixForm
```

$$\text{Out[1089]=} \begin{pmatrix} -3\,\text{Sin}[3t] & 3\,\text{Cos}[3t] & -e^{-t} \\ 1 & 2t^2\,\text{Cos}[t^2] + \text{Sin}[t^2] & e^t \end{pmatrix}$$

Similarly, we find $\int \mathbf{A}(t)\,dt$ by integrating each element of $\mathbf{A}(t)$ with `Integrate`.

```
In[1090]:= Integrate[a,t]//MatrixForm
```

$$\text{Out[1090]=} \begin{pmatrix} \frac{\text{Sin[3t]}}{3} & -\frac{\text{Cos[3t]}}{3} & -e^{-t} \\ \frac{t^2}{2} & -\frac{\text{Cos}\left[t^2\right]}{2} & e^{t} \end{pmatrix}$$

Note that Mathematica does not include an arbitrary constant of integration with each anti-derivative.

■

6.2 Systems of Equations: Preliminary Definitions and Theory

Up to this point, we have focused our attention on solving differential equations that involve one dependent variable. However, many physical situations are modeled with more than one equation and involve more than one dependent variable. For example, if we want to determine the population of two interacting populations such as foxes and rabbits, we would have two dependent variables which represent the two populations where these populations depend on one independent variable which represents time. Situations like this lead to systems of differential equations which we study in this chapter. For example, we encountered a nonlinear initial-value problem like this Van-der-Pol initial-value problem,

$$\begin{cases} x'' + \left(x^2 - 1\right)x' + x = 0 \\ x(0) = 1, \; x'(0) = 1 \end{cases}$$

in Chapter 4. If we let $x' = y$, then

$$y' = x'' = -\left[\left(x^2 - 1\right)x' + x\right] = \left(1 - x^2\right)y - x,$$

so the second-order equation $x'' + \left(x^2 - 1\right)x' + x = 0$ is equivalent to the system of first-order differential equations

$$\begin{cases} x' = y \\ y' = \left(1 - x^2\right)y - x \end{cases}$$

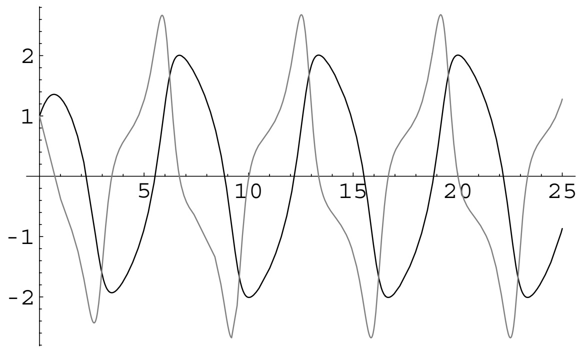


Figure 6-1 In the limit as $t \to \infty$, the solution is periodic

and the initial-value problem is equivalent to the initial-value problem

$$\begin{cases} x' = y \\ y' = \left(1 - x^2\right) y - x \\ x(0) = 1,\ y(0) = 1. \end{cases}$$

We use `NDSolve` to generate a numerical solution to this initial-value problem valid for $0 \le t \le 25$.

```
In[1091]:= numsol =
             NDSolve[{x'[t] == y[t],
                y'[t] == (1 - x[t]^2) y[t] - x[t],
                x[0] == 1, y[0] == 1}, {x[t], y[t]}, {t, 0, 25}]
Out[1091]= {{x[t] →
                InterpolatingFunction[{{0., 25.}}, <>][t],
                 y[t] → InterpolatingFunction[
                      {{0., 25.}}, <>][t]}}
```

We can use this result to approximate the solution for various values of t. For example, entering

```
In[1092]:= {x[t], y[t]} /. numsol /. t- > 1
Out[1092]= {{1.29848, -0.367035}}
```

shows us that $x(1) \approx 1.29848$ and $x'(1) = y(1) \approx -0.367035$. We use `Plot` to graph $x(t)$ and $y(t)$ (the graph of $y(t)$ is in gray) in Figure 6-1.

```
In[1093]:= Plot[Evaluate[{x[t], y[t]} /. numsol], {t, 0, 25},
             PlotStyle- > {GrayLevel[0], GrayLevel[0.5]}]
```

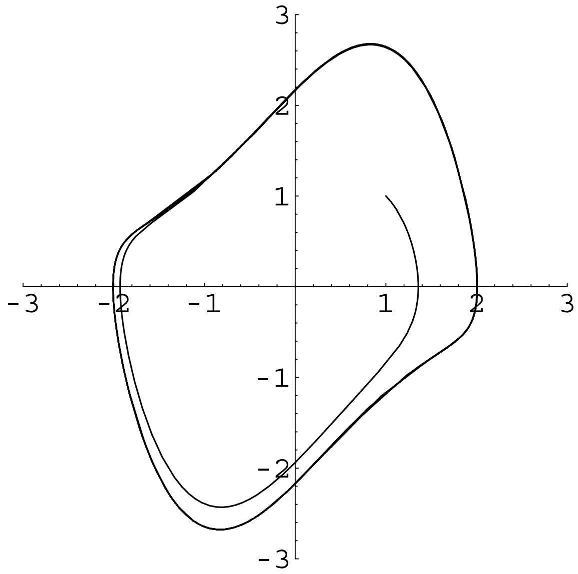


Figure 6-2 The solution approaches a *limit cycle*

Because we let $x' = y$, notice that $y(t) > 0$ when $x(t)$ is increasing and $y(t) < 0$ when $x(t)$ is decreasing. The observation that these solutions are periodic is further confirmed by a graph of $x(t)$ (the horizontal axis) versus $y(t)$ (the vertical axis) generated with `ParametricPlot` in Figure 6-2. We see that as t increases, the solution approaches a certain fixed path, called a *limit cycle*.

```
In[1094]:= ParametricPlot[{x[t],y[t]}/.numsol,{t,0,25},
              PlotRange->{{-3,3},{-3,3}},
              AspectRatio->1,Compiled->False]
```

We will find that nonlinear equations are more easily studied when they are written as a system of equations.

6.2.1 Preliminary Theory

Definition 24 (System of Ordinary Differential Equations). *A **system** of ordinary differential equations is a simultaneous set of equations that involves two or more dependent variables that depend on one independent variable. A **solution** of the system is a set of functions that satisfies each equation on some interval I.*

If the differential equations in the system of differential equations are linear equations, we say that the system is a **linear system of differential equations** or a **linear system**.

EXAMPLE 6.2.1: Show that $\begin{cases} x = \frac{1}{5}e^{-t}\left(e^{t} - \cos 2t - 3\sin 2t\right) \\ y = -e^{-t}\left(\cos 2t - \sin 2t\right) \end{cases}$ is a solution to the system $\begin{cases} x' - y = 0 \\ y' + 5x + 2y = 1. \end{cases}$

SOLUTION: The set of functions is a solution to the system of equations because

```
In[1095]:= Clear[x, y, t]

           x[t_] = 1/5 Exp[-t] (Exp[t] - Cos[2t] - 3 Sin[2t]);

           y[t_] = -Exp[-t] (Cos[2t] - Sin[2t]);

In[1096]:= x'[t] - y[t]//Simplify
Out[1096]= 0
```

and

```
In[1097]:= y'[t] + 5x[t] + 2y[t]//Simplify
Out[1097]= 1
```

We graph this solution in several different ways. First, we graph the solution $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ parametrically with `ParametricPlot` in Figure 6-3. Then, we graph $x(t)$ and $y(t)$ together as functions of t in Figure 6-4.

```
In[1098]:= ParametricPlot[{x[t], y[t]}, {t, 0, 3π},
             PlotRange → {{-0.5, 1},
             {-1, 0.5}}, AspectRatio → 1]

           Plot[{x[t], y[t]}, {t, 0, 3π},
             PlotStyle → {GrayLevel[0], GrayLevel[0.5]},
             PlotRange → {-1, 0.5}]
```

Notice that $\lim_{t\to\infty} x(t) = \frac{1}{5}$ and $\lim_{t\to\infty} y(t) = 0$. Therefore, in the parametric plot, the points on the curve approach (1/5, 0) as t increases.

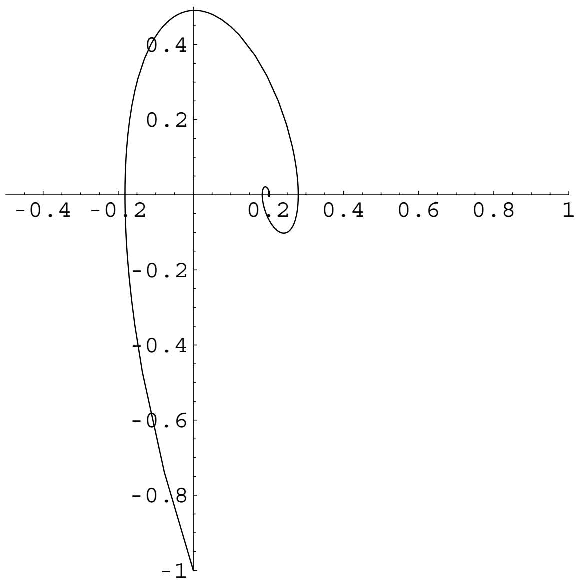


Figure 6-3 *x* (on the horizontal axis) versus *y* (on the vertical axis)

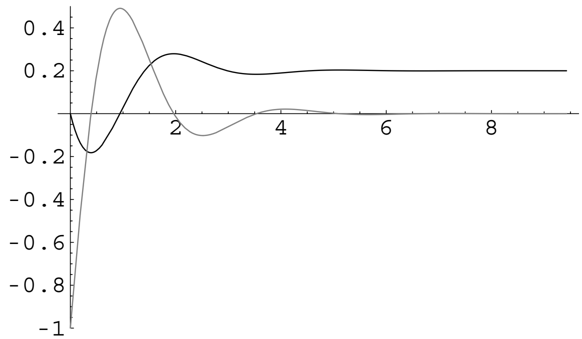


Figure 6-4 *x* (in black) and *y* (in gray) as functions of *t*

■

We will discuss techiques for solving systems in the following sections. For now, we make the following remarks. First, we saw previously that you can often use `NDSolve` to generate a numerical solution of a system, which is of particular benefit with nonlinear systems. `DSolve` can also often be used to find solutions of linear systems and, in a few special cases, nonlinear systems.

EXAMPLE 6.2.2: Solve $\begin{cases} x' = 2y \\ y' = -\frac{1}{4}x \end{cases}$ and $\begin{cases} x' = 2y \\ y' = -\frac{1}{4}x \\ x(0) = 2,\ y(0) = 1. \end{cases}$

SOLUTION: `DSolve` can find a general solution of this linear system.

```
In[1099]:= gensol = DSolve[{x'[t] == 2y[t], y'[t] ==
                -1/4 x[t]}, {x[t], y[t]}, t]
```

$$\text{Out[1099]=}\ \left\{\left\{x[t]\to C[1]\ \text{Cos}\left[\frac{t}{\sqrt{2}}\right]+2\sqrt{2}\,C[2]\ \text{Sin}\left[\frac{t}{\sqrt{2}}\right],\ y[t]\to C[2]\ \text{Cos}\left[\frac{t}{\sqrt{2}}\right]-\frac{C[1]\ \text{Sin}\left[\frac{t}{\sqrt{2}}\right]}{2\sqrt{2}}\right\}\right\}$$

Similarly, `DSolve` can solve the initial-value problem. The resulting list is named `partsol`.

```
In[1100]:= partsol = DSolve[{x'[t] == 2y[t],
                y'[t] == -1/4 x[t], x[0] == 2,
                y[0] == 1}, {x[t], y[t]}, t]
```

$$\text{Out[1100]=}\ \left\{\left\{x[t]\to 2\left(\text{Cos}\left[\frac{t}{\sqrt{2}}\right]+\sqrt{2}\ \text{Sin}\left[\frac{t}{\sqrt{2}}\right]\right),\ y[t]\to\frac{1}{2}\left(2\ \text{Cos}\left[\frac{t}{\sqrt{2}}\right]-\sqrt{2}\ \text{Sin}\left[\frac{t}{\sqrt{2}}\right]\right)\right\}\right\}$$

We use `Plot` to graph the x and y components of the solution individually and `ParametricPlot` to graph them parametrically. See Figure 6-5.

```
In[1101]:= p1 = Plot[{x[t], y[t]}/.partsol,
            {t, 0, 4 Sqrt[2]π},
            PlotStyle → {GrayLevel[0],
            GrayLevel[0.3]},
            AxesLabel → {"t", "x, y"},
            DisplayFunction → Identity];

           p2 = ParametricPlot[{x[t], y[t]}/.partsol,
            {t, 0, 2 Sqrt[2]π}, AxesLabel → {"x", "y"},
            AspectRatio → Automatic,
            DisplayFunction → Identity];

           Show[GraphicsArray[{p1, p2}]]
```

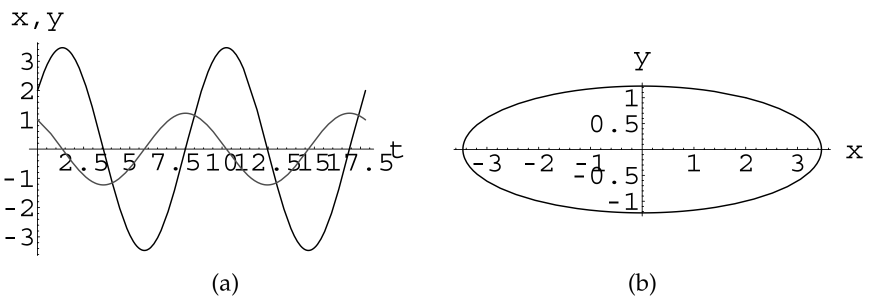


Figure 6-5 (a) Plots of x (in black) and y (in gray). (b) Parametric plot of x versus y

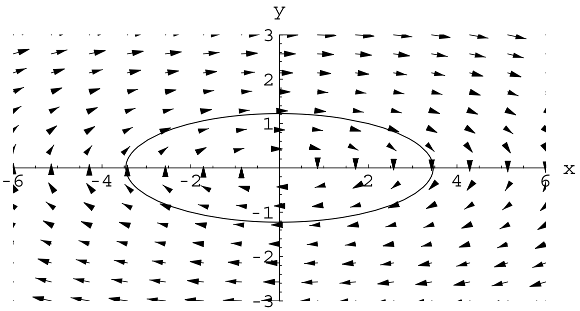


Figure 6-6 Direction field associated with the system together with the solution to the initial-value problem

For an autonomous system like this, we use `PlotVectorField` to graph the direction field associated with the system. `PlotVectorField` is contained in the **PlotField** package that is located in the **Graphics** folder (or directory). After loading the **PlotField** package,

An **autonomous system** is one for which the independent variable does not explicitly occur in the equations.

```
In[1102]:= << Graphics`PlotField`
```

we use `PlotVectorField` to graph the direction field associated with the system and then display the direction field together with the solution to the initial-value problem in Figure 6-6.

```
In[1103]:= pvf = PlotVectorField[{2 y, -1/4x}, {x, -6, 6},
                 {y, -3, 3}, DisplayFunction → Identity];

            Show[pvf, p2, PlotRange → {{-6, 6}, {-3, 3}},
              AspectRatio → Automatic,
              DisplayFunction → $DisplayFunction,
              Axes → Automatic, AxesLabel → {"x", "y"}];
```

In fact, we can show the direction field together with several solutions. With the following command, we use `Map` to apply a pure function to the list `{0.5,1,1.5,2,2.5}` that solves the system if $x(0) = 0$ and $y(0) = i$ for $i = 0.5, 1.0, \ldots, 2.5$.

```
In[1104]:= severalsols =
             Map[DSolve[{x'[t] == 2y[t],
             y'[t] == -1/4 x[t], x[0] == 2, y[0] == #},
             {x[t], y[t]}, t]&, {0.5, 1, 1.5, 2, 2.5}]
```

Out[1104]=

$$\left\{\left\{\left\{x[t] \to 2.\ \mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] + 1.41421\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right],\ y[t] \to 0.5\ \mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] - 0.707107\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right]\right\}\right\},\right.$$

$$\left\{\left\{x[t] \to 2\left(\mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] + \sqrt{2}\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right]\right),\ y[t] \to \frac{1}{2}\left(2\ \mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] - \sqrt{2}\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right]\right)\right\}\right\},$$

$$\left\{\left\{x[t] \to 2.\ \mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] + 4.24264\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right],\ y[t] \to 1.5\ \mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] - 0.707107\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right]\right\}\right\},$$

$$\left\{\left\{x[t] \to 2\left(\mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] + 2\sqrt{2}\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right]\right),\ y[t] \to \frac{1}{2}\left(4\ \mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] - \sqrt{2}\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right]\right)\right\}\right\},$$

$$\left.\left\{\left\{x[t] \to 2.\ \mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] + 7.07107\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right],\ y[t] \to 2.5\ \mathrm{Cos}\left[\frac{t}{\sqrt{2}}\right] - 0.707107\ \mathrm{Sin}\left[\frac{t}{\sqrt{2}}\right]\right\}\right\}\right\}$$

We then use `ParametricPlot` to graph the solutions obtained in `severalsols` together and display them with the direction field, named `pvf`, in Figure 6-7.

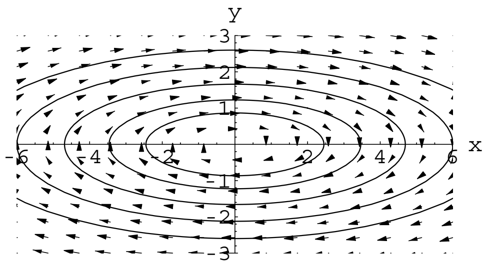


Figure 6-7 Direction field associated with the system together with several solutions of the system

```
In[1105]:= p3 = ParametricPlot[{x[t],
                   y[t]}/.severalsols,
                   {t, 0, 2Sqrt[2]π}, Compiled → False,
                   DisplayFunction → Identity];

            Show[pvf, p3, PlotRange → {{-6, 6}, {-3, 3}},
              AspectRatio → Automatic,
              DisplayFunction → $DisplayFunction,
              Axes → Automatic, AxesLabel → {"x", "y"}];
```

■

EXAMPLE 6.2.3: The **Jacobi elliptic functions** satisfy the nonlinear system

$$\begin{cases} du/dt = vw \\ dv/dt = -uw \\ dw/dt = -k^2 uv. \end{cases}$$

Use Mathematica to solve this system.

The system is nonlinear because of the products of the dependent variables u, v, and w.
For this system, t is the *independent* variable; $u = u(t)$, $v = v(t)$, and $w = w(t)$ are the *dependent* variables.

SOLUTION: Although Mathematica generates several error messages, we see that Mathematica is able to find a general solution of the system, although the result is given in terms of the *Jacobi elliptic function*, JacobiSN.

```
In[1106]:= gensol = DSolve[{u'[t] ==
                     v[t]w[t], v'[t] == -u[t]w[t],
                     w'[t] == -k^2 u[t]v[t]},
                     {u[t], v[t], w[t]}, t]
```

```
Solve :: ifun : Inverse functions are being used by
    Solve, so some solutions may not be found.

Solve :: ifun : Inverse functions are being used by
    Solve, so some solutions may not be found.

Solve :: ifun : Inverse functions are being used by
    Solve, so some solutions may not be found.

General :: stop : Further output of
    Solve :: ifun will be suppressed during this calculation.
```

Out[1106]=

$$\left\{\left\{u[t]\to\sqrt{2}\,\sqrt{C[1]}\,\text{JacobiSN}\left[\sqrt{2}\,t\,\sqrt{C[2]}-\sqrt{2}\,\sqrt{C[2]}\,C[3],\ \frac{k^2\,C[1]}{C[2]}\right],\right.\right.$$

$$v[t]\to-\sqrt{\left(2\,C[1]-2\,C[1]\,\text{JacobiSN}\left[\sqrt{2}\,t\,\sqrt{C[2]}-\sqrt{2}\,\sqrt{C[2]}\,C[3],\ \frac{k^2\,C[1]}{C[2]}\right]^2\right)},$$

$$\left.w[t]\to-\sqrt{\left(2\,C[2]-2\,k^2\,C[1]\,\text{JacobiSN}\left[\sqrt{2}\,t\,\sqrt{C[2]}-\sqrt{2}\,\sqrt{C[2]}\,C[3],\ \frac{k^2\,C[1]}{C[2]}\right]^2\right)}\right\},$$

$$\left\{u[t]\to\sqrt{2}\,\sqrt{C[1]}\,\text{JacobiSN}\left[\sqrt{2}\,t\,\sqrt{C[2]}-\sqrt{2}\,\sqrt{C[2]}\,C[3],\ \frac{k^2\,C[1]}{C[2]}\right],\right.$$

$$v[t]\to\sqrt{\left(2\,C[1]-2\,C[1]\,\text{JacobiSN}\left[\sqrt{2}\,t\,\sqrt{C[2]}-\sqrt{2}\,\sqrt{C[2]}\,C[3],\ \frac{k^2\,C[1]}{C[2]}\right]^2\right)},$$

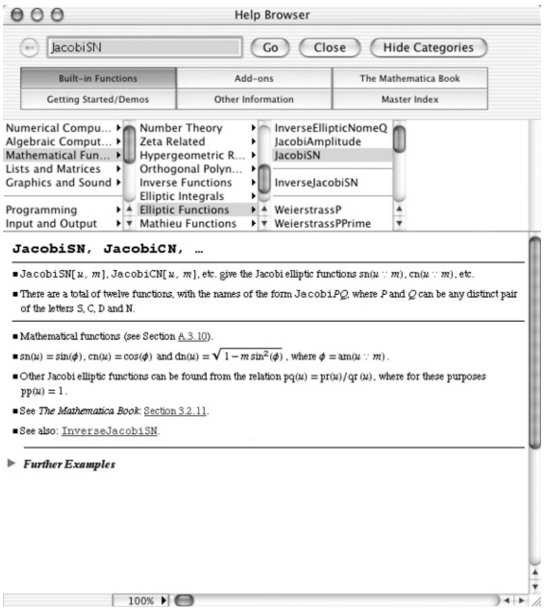
$$\left.w[t]\to\sqrt{\left(2\,C[2]-2\,k^2\,C[1]\,\text{JacobiSN}\left[\sqrt{2}\,t\,\sqrt{C[2]}-\sqrt{2}\,\sqrt{C[2]}\,C[3],\ \frac{k^2\,C[1]}{C[2]}\right]^2\right)}\right\},$$

$$\left\{u[t]\to\sqrt{2}\,\sqrt{C[1]}\,\text{JacobiSN}\left[-\sqrt{2}\,t\,\sqrt{C[2]}+\sqrt{2}\,\sqrt{C[2]}\,C[3],\ \frac{k^2\,C[1]}{C[2]}\right],\right.$$

$$v[t]\to-\sqrt{\left(2\,C[1]-2\,C[1]\,\text{JacobiSN}\left[-\sqrt{2}\,t\,\sqrt{C[2]}+\sqrt{2}\,\sqrt{C[2]}\,C[3],\ \frac{k^2\,C[1]}{C[2]}\right]^2\right)},$$

```
Out[1106]= w[t] → √(2 C[2] - 2 k² C[1]
                SuperscriptBox(JacobiSN[ - √2 t √C[2]+)
                ×√2 √C[2] C[3], k² C[1]/C[2]]²)},
         {u[t] → √2 √C[1] JacobiSN[ - √2 t √C[2]
                +√2 √C[2] C[3], k² C[1]/C[2]],
          v[t] → √(2 C[1] - 2 C[1] JacobiSN[ - √2 t √C[2]
                +√2 √C[2] C[3], k² C[1]/C[2]]²),
          w[t] → -√(2 C[2] - 2 k² C[1] JacobiSN[ - √2 t √C[2]
                +√2 √C[2] C[3], k² C[1]/C[2]]²)}}
```

We use the **Help Browser** to obtain information regarding the `JacobiSN` function as indicated in the following screen shot.



■

As with other equations, under reasonable conditions, a solution to a system of differential equations always exists.

Theorem 11 (Existence and Uniqueness). *Assume that each of the functions*

$$f_1(t, x_1, x_2, \dots, x_n), \; f_2(t, x_1, x_2, \dots, x_n), \; \dots, \; f_n(t, x_1, x_2, \dots, x_n)$$

and the partial derivatives $\partial f_1/\partial x_1$, $\partial f_2/\partial x_2$, $\dots$, $\partial f_n/\partial x_n$ *are continuous in a region* R *containing the point* $(t_0, y_1, y_2, \dots, y_n)$. *Then, the initial-value problem*

$$\begin{cases} x_1' = f_1(t, x_1, x_2, \dots, x_n) \\ x_2' = f_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ x_n' = f_n(t, x_1, x_2, \dots, x_n) \\ x_1(t_0) = y_1, \; x_2(t_0) = y_2, \dots \; x_n(t_0) = y_n \end{cases} \tag{6.4}$$

has a unique solution

$$\begin{cases} x_1 = \phi_1(t) \\ x_2 = \phi_2(t) \\ \vdots \\ x_n = \phi_n(t) \end{cases} \tag{6.5}$$

on an interval I *containing* $t = t_0$.

EXAMPLE 6.2.4: Show that the initial-value problem

$$\begin{cases} dx/dt = 2x - xy \\ dy/dt = -3y + xy \\ x(0) = 2, \; y(0) = 3/2 \end{cases}$$

has a unique solution.

SOLUTION: In this case, we identify $f_1(t, x, y) = 2x - xy$ and $f_2(t, x, y) = -3y + xy$ with $\partial f_1/\partial x = 2 - y$ and $\partial f_2/\partial y = -3 + x$. All four of these functions are continuous on a region containing $(0, 2, 3/2)$. Thus, by the Existence and Uniqueness Theorem, a unique solution to the initial-value problem exists. In this case, we use `NDSolve` to approximate the solution to this nonlinear problem valid for $0 \le t \le 10$.

`DSolve` is not able to find an explicit solution to this nonlinear system.

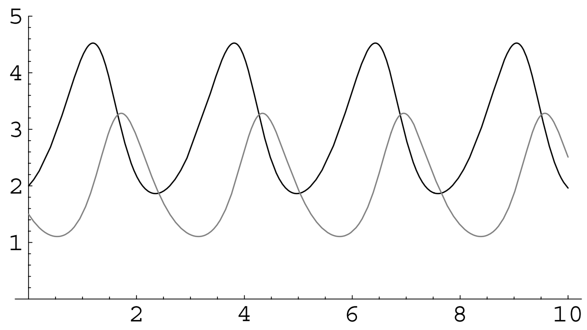


Figure 6-8 $x(t)$ (in black) and $y(t)$ (in gray) as functions of t

```
In[1107]:= Clear[x,y]

           numsol =
            NDSolve[{x'[t] == 2 x[t] - x[t] y[t], y'[t] ==
              -3 y[t] + x[t] y[t], x[0] == 2, y[0] == 3/2},
              {x[t], y[t]}, {t, 0, 10}]
Out[1107]= {{x[t] → InterpolatingFunction[
                 {{0., 10.}}, <>][t],
            y[t] → InterpolatingFunction[
                 {{0., 10.}}, <>][t]}}
```

As illustrated previously, we can use this result to approximate $x(t)$ and $y(t)$ for various values of t. For example,

```
In[1108]:= {x[t], y[t]} /. numsol /. t- > 4
Out[1108]= {{4.26901, 2.62469}}
```

shows us that $x(4) \approx 4.26901$ and $y(4) \approx 2.62469$. Next, we use `Plot` to graph $x(t)$ and $y(t)$ individually in Figure 6-8 and `ParametricPlot` to graph the parametric equations $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in Figure 6-9 for $0 \le t \le 10$.

```
In[1109]:= p1 = Plot[Evaluate[{x[t], y[t]}/.(numsol[[1]])],
              {t, 0, 10}, PlotStyle → {GrayLevel[0],
              GrayLevel[0.5]}, AspectRatio → 1/2,
              PlotRange → {0, 5}]
```

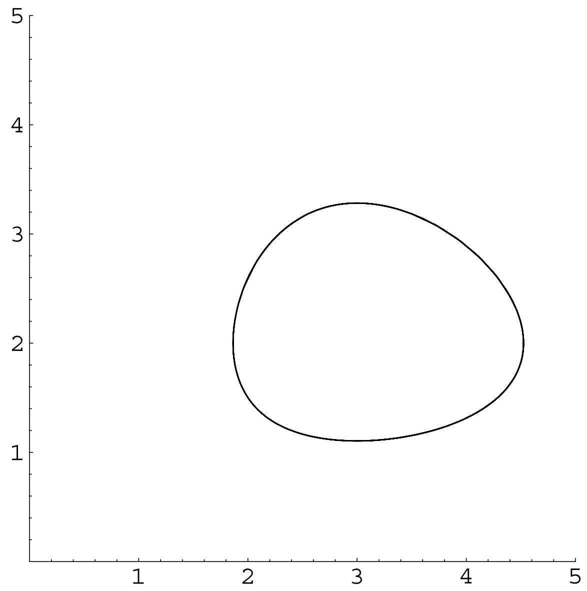


Figure 6-9 $x(t)$ and $y(t)$ are periodic

The graphs illustrate that the solution to the initial-value problem is periodic.

In fact, all *meaningful* (or *interesting*) solutions to the equation are periodic. *Meaningful* (or *interesting*) solutions are ones for which both x and y are greater than 0 and neither is constant. To see this, we use `PlotVectorField` to graph the direction field associated with the system. We display the direction field together with the solution to the initial-value problem in Figure 6-10.

Later, we will see that a system like this is used to model a basic predator–prey relationship. In such a model, x and y represent population sizes (or ratios) so we are only interested in solutions where both of these quantities are greater than or equal to 0.

```
In[1110]:= << Graphics`PlotField`

           pvf = PlotVectorField[{2x - x y,
                   -3y + x y}, {x, 0, 5}, {y, 0, 5},
                   DisplayFunction → Identity];

           Show[p2, pvf]
```

■

Although the Existence and Uniqueness Theorem, Theorem 11, guarantees the existence and uniqueness of a solution, the behavior of the solutions of a system can be remarkably complicated, even for systems that appear quite simple.

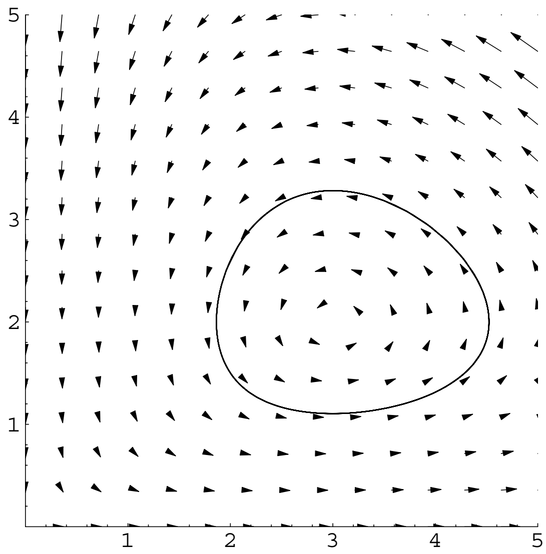


Figure 6-10 The direction field indicates that all meaningful solutions are periodic

See texts like Jordan and Smith's *Nonlinear Ordinary Differential Equations* [17] for discussions of ways to analyze systems like the Rössler attractor and the Lorenz equations.

EXAMPLE 6.2.5 (Rössler Attractor): The **Rössler attractor** is the system

$$\begin{cases} x' = -y - z \\ y' = x + ay \\ z' = bx - cz + xz \,. \end{cases} \tag{6.6}$$

Observe that system (6.6) is nonlinear because of the product of the x and z terms in the z' equation.

If $a = 0.4$, $b = 0.3$, $x_0 = 1$, $y_0 = 0.4$, and $z_0 = 0.7$, how does the value of c affect solutions to the initial-value problem

$$\begin{cases} x' = -y - z \\ y' = x + ay \\ z' = bx - cz + xz \\ x(0) = x_0, \ y(0) = y_0, \ z(0) = z_0 ? \end{cases} \tag{6.7}$$

SOLUTION: By the Existence and Uniqueness Theorem, initial-value problem (6.7) will *always* have a unique solution.

We define the function `rosslerplot`:

```
rosslerplot[a,b,c][{x0,y0,z0},{t,a,b}]
```

1. solves the initial-value problem (6.7) for $a \leq t \leq b$,
2. generates parametric plots of $x(t)$ versus $y(t)$, $y(t)$ versus $z(t)$, $x(t)$ versus $z(t)$, and $x(t)$ versus $y(t)$ versus $z(t)$, and displays the four graphics as a graphics array, and
3. returns a numerical solution to the initial-value problem (6.7) valid for $a \leq t \leq b$.

If `{t,a,b}` is omitted from the `rosslerplot` function, the default is $950 \leq t \leq 1000$. Any options included are passed to the `Show` command.

```
In[1111]:= rosslerplot[a_, b_, c_][{x0_, y0_,
               z0_}, ts_ :{t, 950, 1000}, opts___] :=
            Module[{numsol}, numsol =
               NDSolve[{x′[t] == -y[t] - z[t],
                 y′[t] == x[t] + a y[t],
                 z′[t] == b x[t] - c z[t]
                 +x[t]z[t], x[0] == x0,
                 y[0] == y0, z[0] == z0},
                 {x[t], y[t], z[t]},
                 ts, MaxSteps → 100000];
            p1a = ParametricPlot[
                  Evaluate[{x[t], y[t]}/.numsol], ts,
                  PlotPoints → 1000, AspectRatio → 1,
                  AxesLabel → {"x", "y"},
                  DisplayFunction → Identity];
            p1b = ParametricPlot[
                  Evaluate[{x[t], z[t]}/.numsol], ts,
                  PlotPoints → 1000, AspectRatio → 1,
                  AxesLabel → {"x", "z"},
                  PlotRange → All,
                  DisplayFunction → Identity];
            p1c = ParametricPlot[
                  Evaluate[{y[t], z[t]}/.numsol], ts,
                  PlotPoints → 1000, AspectRatio → 1,
                  AxesLabel → {"y", "z"}, PlotRange → All,
                  DisplayFunction → Identity];
```

```
In[1111]:= p1d = ParametricPlot3D[
                   Evaluate[{x[t], y[t], z[t]}/.numsol],
                   ts, PlotPoints → 3000,
                   BoxRatios → {1, 1, 1},
                   AxesLabel → {"x", "y", "z"},
                   PlotRange → All,
                   DisplayFunction → Identity];
           Show[GraphicsArray[{{p1a, p1b},
                   {p1c, p1d}}],
                   opts]; numsol]
```

For example, entering

```
In[1112]:= rosslerplot[0.4, 0.3, 4.44][{1, 0.4, 0.7},
               {t, 800, 1000}]
Out[1112]= {{x[t] → InterpolatingFunction[{{800.,
                   1000.}}, <>][t],
             y[t] → InterpolatingFunction[{{800.,
                   1000.}}, <>][t], z[t] →
                   InterpolatingFunction[{{800.,
                   1000.}}, <>][t]}}
```

generates the plots shown in Figure 6-11, which corresponds to plots for our problem if $c = 4.44$.

For the given values of a, b, x_0, y_0, and z_0, we will vary c by using $c = 1.4, 2.4, 2.6, 3.4$. We then use `Map` to apply `rosslerplot` to the list `{1.4, 2.4, 2.6, 3.4}`. The resulting list, which corresponds to the numerical solutions to the initial-value problems is named `r1`; the resulting graphs are shown in Figure 6-12.

```
In[1113]:= r1 =
             Map[rosslerplot[0.4, 0.3, #][{1, 0.4, 0.7},
                   {t, 800, 1000}]&, {1.4, 2.4, 2.6, 3.4}]
Out[1113]= {{{x[t] → InterpolatingFunction[{{800.,
                    1000.}}, <>][t],
              y[t] → InterpolatingFunction[{{800.,
                    1000.}}, <>][t],
              z[t] → InterpolatingFunction[{{800.,
                    1000.}}, <>][t]}},
             {{x[t] → InterpolatingFunction[{{800.,
                    1000.}}, <>][t],
              y[t] → InterpolatingFunction[{{800.,
                    1000.}}, <>][t],
              z[t] → InterpolatingFunction[{{800.,
                    1000.}}, <>][t]}},
```

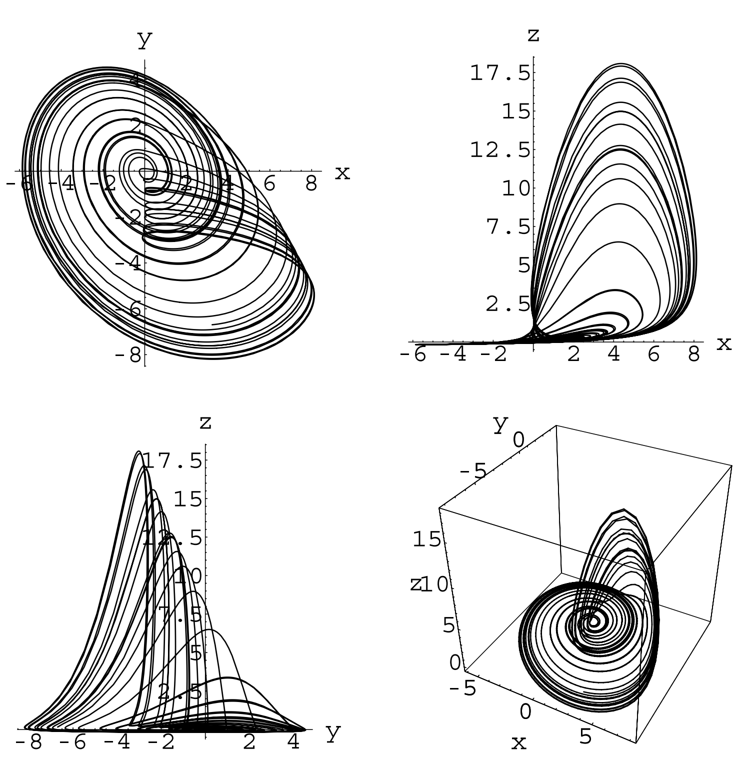


Figure 6-11 Solutions to the Rössler attractor if $a = 0.4$, $b = 0.3$, and $c = 4.44$

```
Out[1113]= {{x[t] → InterpolatingFunction[{{800.,
                1000.}}, <>][t],
             y[t] → InterpolatingFunction[{{800.,
                1000.}}, <>][t],
             z[t] → InterpolatingFunction[{{800.,
                1000.}}, <>][t]}},
            {{x[t] → InterpolatingFunction[{{800.,
                1000.}}, <>][t],
             y[t] → InterpolatingFunction[{{800.,
                1000.}}, <>][t],
             z[t] → InterpolatingFunction[{{800.,
                1000.}}, <>][t]}}}
```

In Figure 6-12, we see that the value of c dramatically affects the long-term behavior of the solutions: $c = 1.4$ results in a single limit cycle, $c = 2.4$ results in a 2-cycle, $c = 2.6$ results in a 4-cycle, and $c = 3.4$ and $c = 4.44$ (see Figure 6-12) appear to be "chaotic."

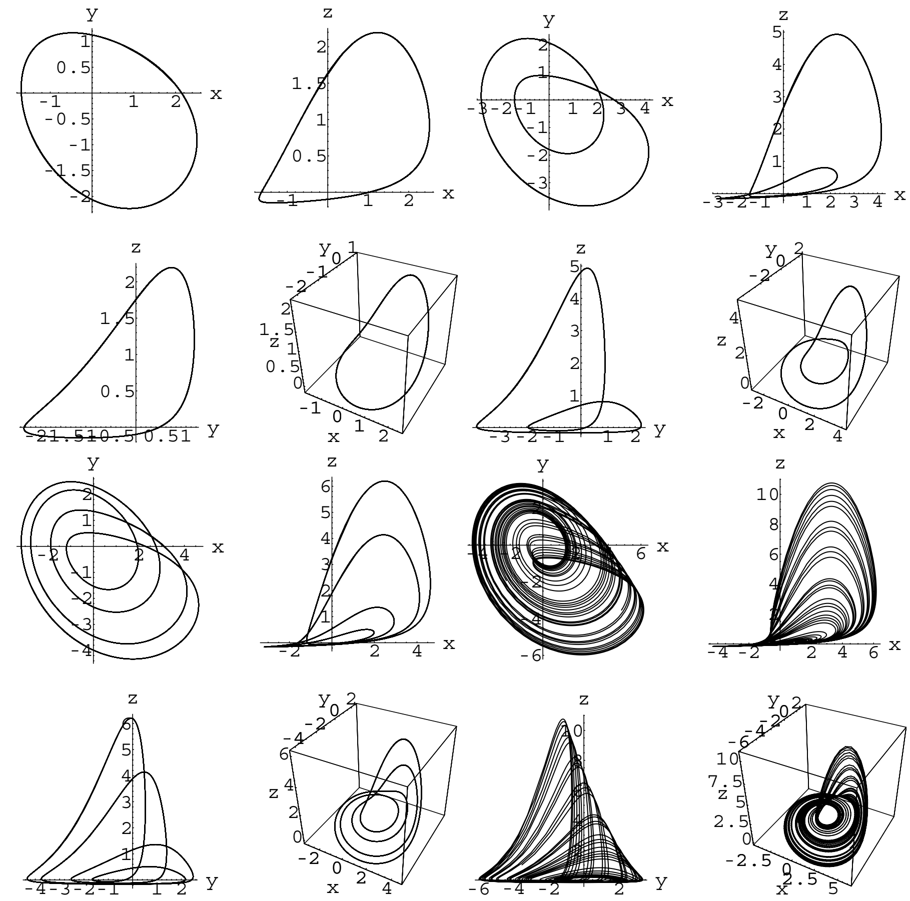


Figure 6-12 Solutions to the Rössler attractor if $a = 0.4$, $b = 0.3$, and $c = 1.4, 2.4, 2.6, 3.4$

We designed the `rosslerplot` function to return the numerical solutions instead of the graphics in case further manipulation of the numerical solutions is needed. For example, entering

```
In[1114]:= r2 = Map[Plot[Evaluate[{x[t], y[t],
              z[t]}/.#], {t, 950, 1000},
              PlotStyle → {GrayLevel[0],
              GrayLevel[0.3], Dashing[{0.01}]}},
              PlotPoints → 1000,
              DisplayFunction → Identity]&, r1];

           Show[GraphicsArray[Partition[r2, 2]]]
```

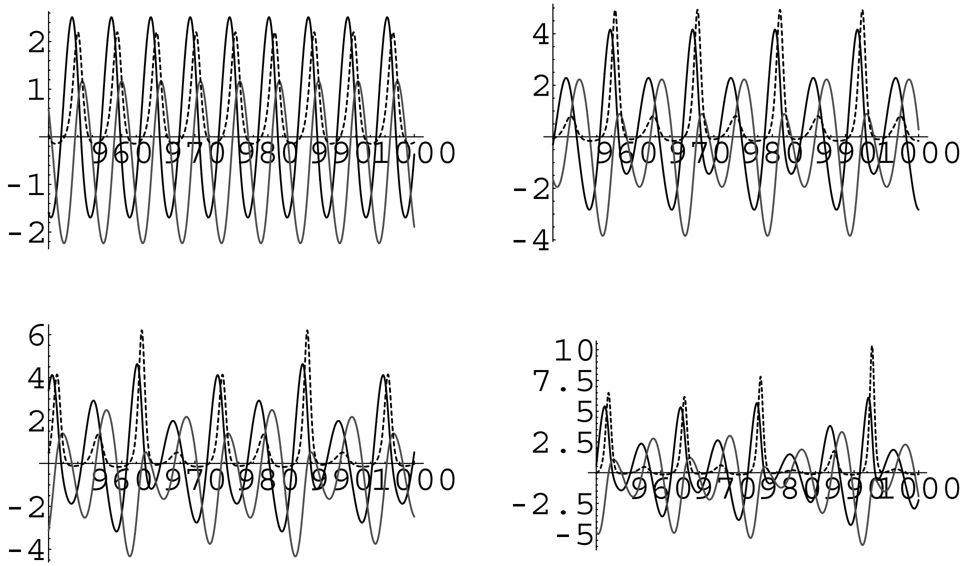


Figure 6-13 Plots of $x(t)$ (in black), $y(t)$ (in gray), and $z(t)$ (dashed) if $a = 0.4$, $b = 0.3$, and $c = 1.4, 2.4, 2.6, 3.4$

graphs each of the solutions $x(t)$, $y(t)$, and $z(t)$ in `r1`. The resulting array is shown in Figure 6-13.

■

6.2.2 Linear Systems

We now turn our attention to linear systems.

We begin our study of linear systems of ordinary differential equations by introducing several definitions along with some convenient notation.

Let

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

Then, the homogeneous system of first-order linear differential equations

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}x_n(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}x_n(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}x_n(t) \end{cases} \tag{6.8}$$

is equivalent to

$$\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) \tag{6.9}$$

and the nonhomogeneous system

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}x_n(t) + f_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}x_n(t) + f_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}x_n(t) + f_n(t) \end{cases} \tag{6.10}$$

is equivalent to

$$\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t). \tag{6.11}$$

For the nonhomogeneous system (6.11), the **corresponding homogeneous system** is system (6.9).

EXAMPLE 6.2.6: (a) Write the homogeneous system $\begin{cases} x' = -5x + 5y \\ y' = -5x + y \end{cases}$ in matrix form. (b) Write the nonhomogeneous system $\begin{cases} x' = x + 2y - \sin t \\ y' = 4x - 3y + t^2 \end{cases}$ in matrix form.

SOLUTION: (a) The homogeneous system $\begin{cases} x' = -5x + 5y \\ y' = -5x + y \end{cases}$ is equivalent to the system $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -5 & 5 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. (b) The nonhomogeneous system $\begin{cases} x' = x + 2y - \sin t \\ y' = 4x - 3y + t^2 \end{cases}$ is equivalent to $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\sin t \\ t^2 \end{pmatrix}$.

With our notation, $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$

■

The nth-order linear equation

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)} + \cdots + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t), \tag{6.12}$$

The nth-order linear equation is discussed in Chapter 4.

discussed in previous chapters, can be written as a system of first-order equations as well. Let $x_1 = y$, $x_2 = dx_1/dt = y'$, $x_3 = dx_2/dt = y''$, $\ldots$, $x_{n-1} = dx_{n-2}/dt = y^{(n-2)}$, $x_n = dx_{n-1}/dt = y^{(n-1)}$. Then, equation (6.12) is equivalent to the system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = -a_{n-1}x_n - \cdots - a_2x_3 - a_1x_2 - a_0x_1 + f(t) \end{cases} \tag{6.13}$$

which can be written in matrix form as

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}. \tag{6.14}$$

EXAMPLE 6.2.7: Write the equation $y'' + 5y' + 6y = \cos t$ as a system of first-order differential equations.

SOLUTION: We let $x_1 = y$ and $x_2 = x_1' = y'$. Then,

$$x_2' = y'' = \cos t - 6y - 5y' = \cos t - 6x_1 - 5x_2$$

so the second-order equation $y'' + 5y' + 6y = \cos t$ is equivalent to the system

$$\begin{cases} x_1' = x_2 \\ x_2' = \cos t - 6x_1 - 5x_2 \end{cases}$$

which can be written in matrix form as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}.$$

■

At this point, given a system of ordinary differential equations, our goal is to construct either an explicit, numerical, or graphical solution of the system of equations.

We now state the following theorems and terminology which are used in establishing the fundamentals of solving systems of differential equations. In each case, we assume that the matrix $\mathbf{A} = \mathbf{A}(t)$ in the systems $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ (equation (6.9)) and $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t)$ (equation (6.11)) is an $n \times n$ matrix.

Definition 25 (Solution Vector). *A* ***solution vector*** *(or* ***solution****) of the system* $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)+\mathbf{F}(t)$ *(equation* (6.11)*) on the interval I is an* $n\times 1$ *matrix (or vector) of the form*

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

where the $x_i(t)$ *are differentiable functions that satisfies* $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{F}(t)$ *on I.*

Consider the homogeneous linear system $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$, where

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \text{and} \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}$$

for which $a_{ij}(t)$ is continuous for all $1 \le i \le n$ and $1 \le j \le n$.

Let $\{\mathbf{\Phi}_i\}_{i=1}^m = \left\{\begin{pmatrix} \Phi_{1i} \\ \Phi_{2i} \\ \vdots \\ \Phi_{mi} \end{pmatrix}\right\}_{i=1}^m$ be a set of m solutions of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$. We define *linear dependence* and *independence* of the set of vectors $\{\mathbf{\Phi}_i\}_{i=1}^m$ in the same way as we define linear dependence and independence of sets of functions. The set $\{\mathbf{\Phi}_i\}_{i=1}^m$ is **linearly dependent** on an interval I means that there is a set of constants $\{c_i\}_{i=1}^m$ not all zero such that $\sum_{i=1}^m c_i\mathbf{\Phi}_i = \mathbf{0}$; otherwise, the set is **linearly independent**.

Definition 26 (Fundamental Set of Solutions). *Any set* $\{\mathbf{\Phi}_i\}_{i=1}^n = \left\{\begin{pmatrix} \Phi_{1i} \\ \Phi_{2i} \\ \vdots \\ \Phi_{ni} \end{pmatrix}\right\}_{i=1}^n$ *of n linearly independent solution vectors of* $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ *on an interval I is called a* ***fundamental set of solutions*** *of* $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ *on I.*

We can determine if a set of vectors is linearly independent or linearly dependent by computing the *Wronskian*.

Theorem 12. *The set* $\{\mathbf{\Phi}_i\}_{i=1}^n = \left\{\begin{pmatrix}\Phi_{1i}\\ \Phi_{2i}\\ \vdots \\ \Phi_{ni}\end{pmatrix}\right\}_{i=1}^n$ *is linearly independent if and only if the Wronskian*

$$W\left(\{\mathbf{\Phi}_i\}_{i=1}^n\right) = \begin{vmatrix}\mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \cdots & \mathbf{\Phi}_n\end{vmatrix} = \begin{vmatrix}\Phi_{11} & \Phi_{12} & \dots & \Phi_{1n}\\ \Phi_{21} & \Phi_{22} & \dots & \Phi_{2n}\\ \vdots & \vdots & \dots & \vdots \\ \Phi_{n1} & \Phi_{n2} & \dots & \Phi_{nn}\end{vmatrix} \neq 0.$$

EXAMPLE 6.2.8: Which of the following is a fundamental set of solutions for

$$\begin{pmatrix}x\\ y\end{pmatrix}' = \begin{pmatrix}-2 & -8\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}?$$

(a) $S_1 = \left\{\begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix}, \begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix}\right\}$; (b) $S_2 \left\{\begin{pmatrix}-2\sin 2t + 2\cos 2t\\ \sin 2t\end{pmatrix}, \begin{pmatrix}4\cos 2t\\ \sin 2t - \cos 2t\end{pmatrix}\right\}$.

SOLUTION: We first remark that the equation $\begin{pmatrix}x\\ y\end{pmatrix}' = \begin{pmatrix}-2 & -8\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ is equivalent to the system $\begin{cases}x' = -2x - 8y\\ y' = x + 2y\end{cases}$. (a) Differentiating we see that

$$\begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix}' = \begin{pmatrix}-2\sin 2t\\ 2\cos 2t\end{pmatrix} \neq \begin{pmatrix}-2\cos 2t - 8\sin 2t\\ \cos 2t + 2\sin 2t\end{pmatrix},$$

which shows us that $\begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix}$ is not a solution of the system.

```
In[1115]:= a = {{-2, -8}, {1, 2}};

In[1116]:= v1 = {Cos[2t], Sin[2t]};

In[1117]:= ∂tv1
Out[1117]= {-2 Sin[2t], 2 Cos[2t]}

In[1118]:= a.v1
Out[1118]= {-2 Cos[2t] - 8 Sin[2t], Cos[2t] + 2 Sin[2t]}
```

Therefore, S_1 is not a fundamental set of solutions.

(b) First we verify that $\begin{pmatrix} -2\sin 2t + 2\cos 2t \\ \sin 2t \end{pmatrix}$ is a solution of the system.

```
In[1119]:= v2 = {-2 Sin[2t] + 2 Cos[2t], Sin[2t]};

In[1120]:= ∂tv2
Out[1120]= {-4 Cos[2t] - 4 Sin[2t], 2 Cos[2t]}

In[1121]:= Simplify[a.v2]
Out[1121]= {-4 (Cos[2t] + Sin[2t]), 2 Cos[2t]}

In[1122]:= Simplify[∂tv2 == a.v2]
Out[1122]= True
```

Next, we see that $\begin{pmatrix} 4\cos 2t \\ \sin 2t - \cos 2t \end{pmatrix}$ is a solution of the system.

```
In[1123]:= v3 = {4 Cos[2t], Sin[2t] - Cos[2t]}
Out[1123]= {4 Cos[2t], -Cos[2t] + Sin[2t]}

In[1124]:= ∂tv3
Out[1124]= {-8 Sin[2t], 2 Cos[2t] + 2 Sin[2t]}

In[1125]:= Simplify[a.v3]
Out[1125]= {-8 Sin[2t], 2 (Cos[2t] + Sin[2t])}

In[1126]:= Simplify[∂tv3 == a.v3]
Out[1126]= True
```

To see that these vectors are linearly independent, we compute the Wronskian.

```
In[1127]:= m1 = Transpose[{v2, v3}];

           MatrixForm[m1]

In[1128]:= Simplify[Det[m1]]
Out[1128]= -2
```

Thus, the set S_2 is a set of two linearly independent solutions of the system and, consequently, a fundamental set of solutions.

■

The following theorem implies that a fundamental set of solutions cannot contain more than n vectors, because the solutions would not be linearly independent.

Theorem 13. *Any $n+1$ nontrivial solutions of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ are linearly dependent.*

Finally, we state the following theorems, which state that a fundamental set of solutions of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ exists and a general solution can (theoretically) be constructed.

Theorem 14. *There is a set of n nontrivial linearly independent solutions of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$.*

Theorem 15 (General Solution). *Let $S = \{\mathbf{\Phi}_i\}_{i=1}^n = \left\{\begin{pmatrix}\Phi_{1i}\\ \Phi_{2i}\\ \vdots \\ \Phi_{ni}\end{pmatrix}\right\}_{i=1}^n$ be a set of n linearly independent solutions of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$. Then every solution of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ is a linear combination of these solutions.*

In this case, S is said to be a **fundamental set of solutions** of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$; a **general solution** of $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ is

$$\mathbf{X}(t) = c_1\mathbf{\Phi}_1(t) + c_2\mathbf{\Phi}_2(t) + \cdots + c_n\mathbf{\Phi}_n(t).$$

Definition 27 (Fundamental Matrix). *Let $\{\mathbf{\Phi}_i\}_{i=1}^n = \left\{\begin{pmatrix}\Phi_{1i}\\ \Phi_{2i}\\ \vdots \\ \Phi_{ni}\end{pmatrix}\right\}_{i=1}^n$ be a fundamental set of solutions for $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$. Then*

$$\mathbf{\Phi}(t) = \begin{pmatrix}\mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \cdots & \mathbf{\Phi}_n\end{pmatrix} = \begin{pmatrix}\Phi_{11} & \Phi_{12} & \dots & \Phi_{1n}\\ \Phi_{21} & \Phi_{22} & \dots & \Phi_{2n}\\ \vdots & \vdots & \dots & \vdots \\ \Phi_{n1} & \Phi_{n2} & \dots & \Phi_{nn}\end{pmatrix}$$

is called a ***fundamental matrix*** *of the system $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$. Thus, a general solution of the system $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ can be written as $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C}$, where $\mathbf{C} = \begin{pmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix}$ is a constant vector.*

If $\mathbf{\Phi} = \begin{pmatrix}\mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \cdots & \mathbf{\Phi}_n\end{pmatrix}$ is a fundamental matrix for $\mathbf{X}' = \mathbf{AX}$, $\mathbf{\Phi}' = \mathbf{A\Phi}$:

$$\begin{aligned}\mathbf{\Phi}' &= \begin{pmatrix}\mathbf{\Phi}_1' & \mathbf{\Phi}_2' & \cdots & \mathbf{\Phi}_n'\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{A\Phi}_1 & \mathbf{A\Phi}_2 & \cdots & \mathbf{A\Phi}_n\end{pmatrix}\\ &= \mathbf{A}\begin{pmatrix}\mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \cdots & \mathbf{\Phi}_n\end{pmatrix}\\ &= \mathbf{A\Phi}.\end{aligned}$$

EXAMPLE 6.2.9: Show that $\Phi = \begin{pmatrix} e^{-2t} & -3e^{5t} \\ 2e^{-2t} & e^{5t} \end{pmatrix}$ is a fundamental matrix for the system $\mathbf{X}'(t) = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\mathbf{X}(t)$. Use the matrix to find a general solution of $\mathbf{X}'(t) = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\mathbf{X}(t)$.

SOLUTION: Because $\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}' = \begin{pmatrix} -2e^{-2t} \\ -4e^{-2t} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}$ and $\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}'$ $= \begin{pmatrix} -15e^{5t} \\ 5e^{5t} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}$, both $\mathbf{X}_1 = \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}$ are solutions of the system $\mathbf{X}'(t) = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\mathbf{X}(t)$. Alternatively, we show that $\Phi'(t)$ and $\begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}\Phi(t)$ are the same.

```
In[1129]:= a = {{4, -3}, {-2, -1}};
            Ψ[t_] = {{Exp[-2t], -3 Exp[5t]},
                      {2 Exp[-2t], Exp[5t]}};

            MatrixForm[Ψ[t]]
```

$$\text{Out[1129]=} \begin{pmatrix} e^{-2t} & -3\,e^{5t} \\ 2e^{-2t} & e^{5t} \end{pmatrix}$$

```
In[1130]:= Ψ'[t]//MatrixForm
```

$$\text{Out[1130]=} \begin{pmatrix} -2\,e^{-2t} & -15\,e^{5t} \\ -4\,e^{-2t} & 5e^{5t} \end{pmatrix}$$

```
In[1131]:= a.Ψ[t]//MatrixForm
```

$$\text{Out[1131]=} \begin{pmatrix} -2\,e^{-2t} & -15\,e^{5t} \\ -4\,e^{-2t} & 5e^{5t} \end{pmatrix}$$

The solutions are linearly independent because the Wronskian is not the zero function.

```
In[1132]:= Det[Ψ[t]]
```

$$\text{Out[1132]=}\ 7e^{3t}$$

A general solution is given by

$$\mathbf{X}(t) = \Phi(t)\mathbf{C} = \begin{pmatrix} e^{-2t} & -3e^{5t} \\ 2e^{-2t} & e^{5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1e^{-2t} - 3c_2e^{5t} \\ 2c_1e^{-2t} + c_2e^{5t} \end{pmatrix}$$
$$= c_1\begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} -3e^{5t} \\ e^{5t} \end{pmatrix}.$$

■

6.3 Homogeneous Linear Systems with Constant Coefficients

Now that we have covered the necessary terminology, we can turn our attention to solving linear systems with constant coefficients. Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

be an $n \times n$ matrix with real-valued entries and let $\{\lambda_k\}$ be the eigenvalues and $\{\mathbf{v}_k\}$ the corresponding eigenvectors of $\mathbf{A}$. Then a general solution of the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is determined by the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. For the moment, we consider the cases when the eigenvalues of $\mathbf{A}$ are distinct and real or the eigenvalues of $\mathbf{A}$ are distinct and complex. We will consider the case when $\mathbf{A}$ has repeated eigenvalues (eigenvalues of multiplicity greater than one) separately.

6.3.1 Distinct Real Eigenvalues

Let λ be an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$. Then,

$$\begin{aligned} \lambda\mathbf{v} &= \mathbf{A}\mathbf{v} \\ \lambda\mathbf{v}e^{\lambda t} &= \mathbf{A}\mathbf{v}e^{\lambda t} \\ \frac{d}{dt}\left(\mathbf{v}e^{\lambda t}\right) &= \mathbf{A}\left(\mathbf{v}e^{\lambda t}\right), \end{aligned}$$

which shows that $\mathbf{v}e^{\lambda t}$ is a solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

If the eigenvalues $\{\lambda_k\}_{k=1}^{n}$ of $\mathbf{A}$ are distinct with corresponding eigenvectors $\{\mathbf{v}_k\}_{k=1}^{n}$,

$$S = \left\{\mathbf{v}_1 e^{\lambda_1 t},\ \mathbf{v}_2 e^{\lambda_2 t},\ \dots,\ \mathbf{v}_n e^{\lambda_n t}\right\}$$

is a fundamental set of solutions for $\mathbf{X}' = \mathbf{A}\mathbf{X}$ because

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

$$\begin{aligned} W(S) &= \left|\mathbf{v}_1 e^{\lambda_1 t}\ \ \mathbf{v}_2 e^{\lambda_2 t}\ \dots\ \mathbf{v}_n e^{\lambda_n t}\right| \\ &= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)t}\left|\mathbf{v}_1\ \ \mathbf{v}_2\ \dots\ \mathbf{v}_n\right| \neq 0. \end{aligned}$$

Therefore, a general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is

$$\mathbf{X} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$$

and a fundamental matrix for $\mathbf{X}' = \mathbf{AX}$ is

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{v}_1 e^{\lambda_1 t} & \mathbf{v}_2 e^{\lambda_2 t} & \cdots & \mathbf{v}_n e^{\lambda_n t} \end{pmatrix}.$$

Remark. After you have loaded the `PlotField` package,

```
PlotVectorField[{f[x,y],g[x,y]},{x,a,b},{y,c,d}]
```

generates a basic direction field for the system $\{x' = f(x, y),\ y' = g(x, y)\}$ for $a \le x \le b$ and $c \le y \le d$.

EXAMPLE 6.3.1: Solve (a) $\begin{cases} x' = 5x - y \\ y' = 3y \end{cases}$ and (b) $\mathbf{X}' = \begin{pmatrix} -1/2 & -1/3 \\ -1/3 & -1/2 \end{pmatrix}\mathbf{X}$.

SOLUTION: (a) In matrix form the system is $\mathbf{X}' = \begin{pmatrix} 5 & -1 \\ 0 & 3 \end{pmatrix}\mathbf{X}$. We find the eigenvalues of $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 0 & 3 \end{pmatrix}$ with `Eigensystem`. The results indicate that the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, respectively.

```
In[1133]:= a = {{5, -1}, {0, 3}};

In[1134]:= Eigensystem[a]
Out[1134]= {{3, 5}, {{1, 2}, {1, 0}}}
```

Therefore, $S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{5t} \right\}$ is a fundamental set of solutions of the system, a general solution is

$$\mathbf{X} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{5t},$$

and a fundamental matrix is $\mathbf{\Phi} = \begin{pmatrix} e^{3t} & e^{5t} \\ 2e^{3t} & 0 \end{pmatrix}$. We can write the general solution as $\begin{cases} x = c_1 e^{3t} + c_2 e^{5t} \\ y = 2c_1 e^{3t} \end{cases}$ or as $\mathbf{X} = \begin{pmatrix} e^{3t} & e^{5t} \\ 2e^{3t} & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

We can use DSolve to find a general solution as well.

```
In[1135]:= DSolve[
              Thread[{x′[t], y′[t]} == a.{x[t], y[t]}],
              {x[t], y[t]}, t]

Out[1135]= {{x[t] → e^(5 t) C[1] - 1/2 e^(3 t) (-1 + e^(2 t)) C[2],
              y[t] → e^(3 t) C[2]}}
```

We can graph the solution parametrically for various values of c_1 and c_2 with ParametricPlot. First, we use Table to generate a list corresponding to replacing c_1 and c_2 in $\begin{cases} x = c_1 e^{3t} + c_2 e^{5t} \\ y = 2c_1 e^{3t} \end{cases}$ by −2, −1, 0, 1, and 2. The result in step1, however, is *not* a list of ordered pairs of functions corresponding to $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$; it is a nested list.

```
In[1136]:= step1 = Table[{c1 e^(3t) + c2 e^(5t), 2c1 e^(3t)},
                         {c1, -2, 2}, {c2, -2, 2}]

Out[1136]= {{{-2 e^(3 t) - 2 e^(5 t), -4 e^(3 t)}, {-2 e^(3 t) - e^(5 t), -4 e^(3 t)},
              {-2 e^(3 t), -4 e^(3 t)}, {-2 e^(3 t) + e^(5 t), -4 e^(3 t)},
              {-2 e^(3 t) + 2 e^(5 t), -4 e^(3 t)}},
            {{-e^(3 t) - 2 e^(5 t), -2 e^(3 t)}, {-e^(3 t) - e^(5 t), -2 e^(3 t)},
              {-e^(3 t), -2 e^(3 t)}, {-e^(3 t) + e^(5 t), -2 e^(3 t)},
              {-e^(3 t) + 2 e^(5 t), -2 e^(3 t)}}, {{-2 e^(5 t), 0},
              {-e^(5 t), 0}, {0, 0}, {e^(5 t), 0}, {2 e^(5 t), 0}},
            {{e^(3 t) - 2 e^(5 t), 2 e^(3 t)}, {e^(3 t) - e^(5 t), 2 e^(3 t)},
              {e^(3 t), 2 e^(3 t)}, {e^(3 t) + e^(5 t), 2 e^(3 t)},
              {e^(3 t) + 2 e^(5 t), 2 e^(3 t)}}, {{2 e^(3 t) - 2 e^(5 t), 4 e^(3 t)},
              {2 e^(3 t) - e^(5 t), 4 e^(3 t)}, {2 e^(3 t), 4 e^(3 t)},
              {2 e^(3 t) + e^(5 t), 4 e^(3 t)}, {2 e^(3 t) + 2 e^(5 t), 4 e^(3 t)}}}
```

Flatten[list,n] flattens list to level *n*.

To create a list of ordered pairs of functions that we can graph with ParametricPlot, we use Flatten.

```
In[1137]:= toplot = Flatten[step1, 1]
```

```
Out[1137]= {{-2 e^(3t) - 2 e^(5t), -4 e^(3t)},
  {-2 e^(3t) - e^(5t), -4 e^(3t)}, {-2 e^(3t), -4 e^(3t)},
  {-2 e^(3t) + e^(5t), -4 e^(3t)}, {-2 e^(3t) + 2 e^(5t), -4 e^(3t)},
  {-e^(3t) - 2 e^(5t), -2 e^(3t)}, {-e^(3t) - e^(5t), -2 e^(3t)},
  {-e^(3t), -2 e^(3t)}, {-e^(3t) + e^(5t), -2 e^(3t)},
  {-e^(3t) + 2 e^(5t), -2 e^(3t)}, {-2 e^(5t), 0},
  {-e^(5t), 0}, {0, 0}, {e^(5t), 0}, {2 e^(5t), 0},
  {e^(3t) - 2 e^(5t), 2 e^(3t)}, {e^(3t) - e^(5t), 2 e^(3t)},
  {e^(3t), 2 e^(3t)}, {e^(3t) + e^(5t), 2 e^(3t)},
  {e^(3t) + 2 e^(5t), 2 e^(3t)}, {2 e^(3t) - 2 e^(5t), 4 e^(3t)},
  {2 e^(3t) - e^(5t), 4 e^(3t)}, {2 e^(3t), 4 e^(3t)},
  {2 e^(3t) + e^(5t), 4 e^(3t)}, {2 e^(3t) + 2 e^(5t), 4 e^(3t)}}
```

Next, we use `ParametricPlot` to graph the list of parametric functions in `toplot` and name the resulting graphics object `pp1`.

```
In[1138]:= pp1 = ParametricPlot[Evaluate[toplot],
              {t, -1, 1}, PlotStyle- > GrayLevel[0],
              DisplayFunction- > Identity];
```

To show the graphs of the solutions together with the direction field associated with the system, we first load the **PlotField** package

```
In[1139]:= << Graphics`PlotField`
```

and then use `PlotVectorField` to graph the direction field associated with the system on the rectangle $[-5, 5] \times [-5, 5]$, naming the resulting graphics object `pvf`.

```
In[1140]:= pvf = PlotVectorField[{5x - y, 3y},
              {x, -5, 5}, {y, -5, 5},
              ScaleFunction- > (1&),
              DefaultColor- > GrayLevel[0.5],
              DisplayFunction- > Identity];
```

`Show` is then used to display the graphs together in Figure 6-14.

```
In[1141]:= Show[pvf, pp1,
   PlotRange- > {{-5, 5}, {-5, 5}},
   AspectRatio- > 1, Axes- > Automatic,
   DisplayFunction- > $DisplayFunction]
```

Notice that each curve corresponds to the parametric plot of the pair $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$. Because both eigenvalues are positive, all solutions move

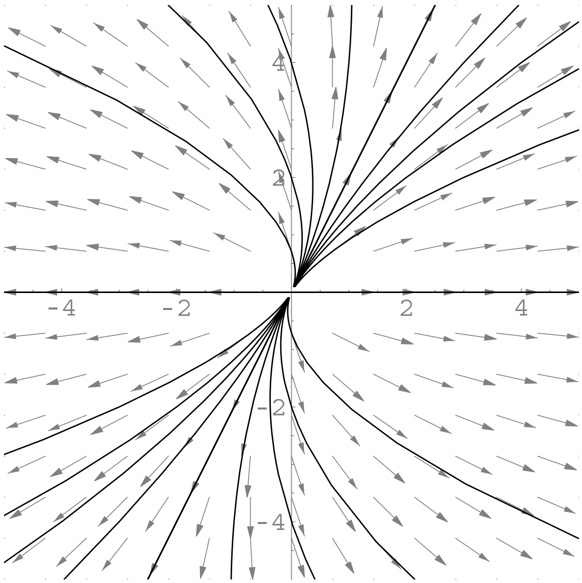


Figure 6-14 All nontrivial solutions move away from the origin as t increases

away from the origin as t increases. The arrows on the vectors in the direction field show this behavior.

(b) With `Eigensystem`, we see that the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} -1/2 & -1/3 \\ -1/3 & -1/2 \end{pmatrix}$ are $\lambda_1 = -1/6$ and $\lambda_2 = -5/6$ and $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively.

```
In[1142]:= capa = {{-1/2,-1/3},{-1/3,-1/2}};
            Eigensystem[capa]
```

Out[1142]= $\left\{\left\{-\frac{5}{6}, -\frac{1}{6}\right\}, \{\{1, 1\}, \{-1, 1\}\}\right\}$

Then $\mathbf{X}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/6}$ and $\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t/6}$ are two linearly independent solutions of the system so a general solution is $\mathbf{X} = \begin{pmatrix} -e^{-t/6} & e^{-5t/6} \\ e^{-t/6} & e^{-5t/6} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$; a fundamental matrix is $\mathbf{\Phi} = \begin{pmatrix} -e^{-t/6} & e^{-5t/6} \\ e^{-t/6} & e^{-5t/6} \end{pmatrix}$.

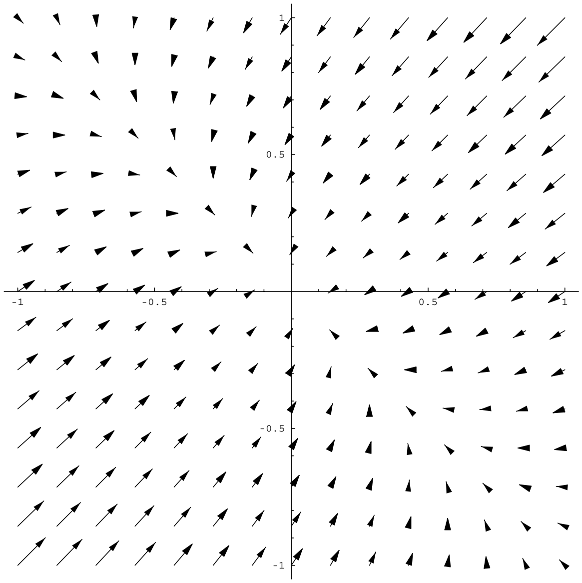


Figure 6-15 Direction field for $\mathbf{X}' = \mathbf{A}\mathbf{X}$

We use DSolve to find a general solution of the system by entering

```
In[1143]:= gensol = DSolve[{x'[t] == -1/2x[t] - 1/3y[t],
                 y'[t] == -1/3x[t] - 1/2y[t]},
                 {x[t], y[t]}, t]
Out[1143]= {{x[t] → e^(-5 t/6) C[1] - e^(-t/6) C[2],
     y[t] → e^(-5 t/6) C[1] + e^(-t/6) C[2]}}
```

We graph the direction field with PlotVectorField, which is contained in the PlotField package located in the **Graphics** directory, in Figure 6-15.

You do not need to reload the **PlotField** package if you have already loaded it during your *current* Mathematica session.

```
In[1144]:= << Graphics`PlotField`

In[1145]:= pvf =
     PlotVectorField[{-1/2x - 1/3y, -1/3x - 1/2y},
     {x, -1, 1}, {y, -1, 1}, Axes- > Automatic]
```

Several solutions are also graphed with ParametricPlot and shown together with the direction field in Figure 6-16.

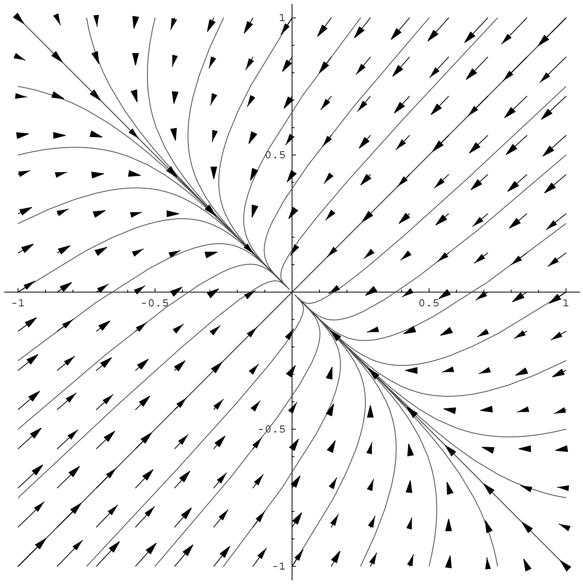


Figure 6-16 Direction field for $\mathbf{X}' = \mathbf{A}\mathbf{X}$ along with various solution curves

```
In[1146]:= initsol = DSolve[{x'[t] == -1/2x[t] - 1/3y[t],
                   y'[t] == -1/3x[t] - 1/2y[t],
                   x[0] == x0, y[0] == y0},
                   {x[t], y[t]}, t]
```

Out[1146]= $\left\{\left\{x[t] \to -e^{-5\ t/6}\left(\frac{1}{2}\ (-x0 - y0) + \frac{1}{2}\ e^{2\ t/3}\ (-x0 + y0)\right), y[t] \to e^{-5\ t/6}\left(\frac{1}{2}\ e^{2\ t/3}\ (-x0 + y0) + \frac{x0 + y0}{2}\right)\right\}\right\}$

```
In[1147]:= t1 = Table[ParametricPlot[Evaluate[{x[t],
                y[t]}/.initsol/.{x0- > 1, y0- > i}],
                {t, 0, 15}, DisplayFunction- > Identity,
                PlotStyle- > GrayLevel[0.3]],
                {i, -1, 1, 2/8}];
           t2 = Table[ParametricPlot[Evaluate[{x[t],
                y[t]}/.initsol/.{x0- > -1, y0- > i}],
                {t, 0, 15}, DisplayFunction- > Identity,
                PlotStyle- > GrayLevel[0.3]],
                {i, -1, 1, 2/8}];
```

```
In[1147]:= t3 = Table[ParametricPlot[Evaluate[{x[t],
                y[t]}/.initsol/.{x0- > i, y0- > 1}],
                {t, 0, 15}, DisplayFunction- > Identity,
                PlotStyle- > GrayLevel[0.3]],
                {i, -1, 1, 2/8}];
            t4 = Table[ParametricPlot[Evaluate[{x[t],
                y[t]}/.initsol/.{x0- > i, y0- > -1}],
                {t, 0, 15}, DisplayFunction- > Identity,
                PlotStyle- > GrayLevel[0.3]],
                {i, -1, 1, 2/8}];
```

```
In[1148]:= Show[t1, t2, t3, t4,
              pvf, DisplayFunction- > $DisplayFunction,
              AspectRatio- > Automatic]
```

■

6.3.2 Complex Conjugate Eigenvalues

If $\mathbf{A}$ has complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm \beta i$, $\beta \neq 0$, and corresponding eigenvectors $\mathbf{v}_{1,2} = \mathbf{a} \pm \mathbf{b}i$, then one solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is

We use Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$.

$$\begin{aligned}\mathbf{X} = \mathbf{v}_1 e^{\lambda_1 t} &= (\mathbf{a} + \mathbf{b}i)e^{(\alpha+\beta i)t} = e^{\alpha t}(\mathbf{a} + \mathbf{b}i)e^{i\beta t} = e^{\alpha t}(\mathbf{a} + \mathbf{b}i)(\cos\beta t + i\sin\beta t)\\ &= e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) + ie^{\alpha t}(\mathbf{b}\cos\beta t + \mathbf{a}\sin\beta t)\\ &= \mathbf{X}_1(t) + i\mathbf{X}_2(t).\end{aligned}$$

Now, because $\mathbf{X}$ is a solution of the system, $\mathbf{X}' = \mathbf{A}\mathbf{X}$, we have $\mathbf{X}_1' + i\mathbf{X}_2' = \mathbf{A}\mathbf{X}_1 + i\mathbf{A}\mathbf{X}_2$. Equating the real and imaginary parts of this equation yields $\mathbf{X}_1' = \mathbf{A}\mathbf{X}_1$ and $\mathbf{X}_2' = \mathbf{A}\mathbf{X}_2$. Therefore, $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions of $\mathbf{X}' = \mathbf{A}\mathbf{X}$, so any linear combination of $\mathbf{X}_1$ and $\mathbf{X}_2$ is also a solution. We can show that $\mathbf{X}_1$ and $\mathbf{X}_2$ are linearly independent, so this linear combination forms a portion of a general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

Theorem 16. *If $\mathbf{A}$ has complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm \beta i$, $\beta \neq 0$, and corresponding eigenvectors $\mathbf{v}_{1,2} = \mathbf{a} \pm \mathbf{b}i$, then two linearly independent solutions of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ are*

$$\mathbf{X}_1 = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) \quad \textit{and} \quad \mathbf{X}_2 = e^{\alpha t}(\mathbf{b}\cos\beta t + \mathbf{a}\sin\beta t).$$

Notice that in the case of complex conjugate eigenvalues, we are able to obtain two linearly independent solutions from knowing one of the eigenvalues and an eigenvector that corresponds to it.

Observe that our chosen eigenvectors are scalar multiples of the eigenvectors found with Mathematica.

EXAMPLE 6.3.2: Solve (a) $\begin{cases} x' = \frac{1}{2}y \\ y' = -\frac{1}{8}x \end{cases}$ and (b) $\begin{cases} dx/dt = -\frac{1}{4}x + 2y \\ dy/dt = -8x - \frac{1}{4}y. \end{cases}$

SOLUTION: (a) In matrix form the system is equivalent to the system $\mathbf{X}' = \begin{pmatrix} 0 & 1/2 \\ -1/8 & 0 \end{pmatrix}\mathbf{X}$. As in (a), we use `Eigensystem` to see that the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 0 & 1/2 \\ -1/8 & 0 \end{pmatrix}$ are $\lambda_{1,2} = 0 \pm \frac{1}{4}i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} i$.

```
In[1149]:= capa = {{0, 1/2}, {-1/8, 0}};
            Eigensystem[capa]

Out[1149]= {{-i/4, i/4}, {{2 i, 1}, {-2 i, 1}}}
```

Two linearly independent solutions are then $\mathbf{X}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos \frac{1}{4}t - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \sin \frac{1}{4}t = \begin{pmatrix} \cos \frac{1}{4}t \\ -\frac{1}{2}\sin \frac{1}{4}t \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin \frac{1}{4}t + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \cos \frac{1}{4}t = \begin{pmatrix} \sin \frac{1}{4}t \\ \frac{1}{2}\cos \frac{1}{4}t \end{pmatrix}$ and a general solution is $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = \begin{pmatrix} \cos \frac{1}{4}t & \sin \frac{1}{4}t \\ -\frac{1}{2}\sin \frac{1}{4}t & \frac{1}{2}\cos \frac{1}{4}t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ or $x = c_1 \cos \frac{1}{4}t + c_2 \sin \frac{1}{4}t$ and $y = -c_1 \frac{1}{2} \sin \frac{1}{4}t + \frac{1}{2}c_2 \cos \frac{1}{4}t$.

As before, we use `DSolve` to find a general solution.

```
In[1150]:= gensol = DSolve[{x'[t] == 1/2y[t], y'[t] ==
                  -1/8x[t]}, {x[t], y[t]}, t]

Out[1150]= {{x[t] → -2 C[1] Cos[t/4] + 2 C[2] Sin[t/4],
             y[t] → C[2] Cos[t/4] + C[1] Sin[t/4]}}
```

Initial-value problems for systems are solved in the same way as for other equations. For example, entering

```
In[1151]:= partsol = DSolve[{x'[t] == 1/2y[t],
                   y'[t] == -1/8x[t], x[0] == 1,
                   y[0] == -1}, {x[t], y[t]}, t]
```

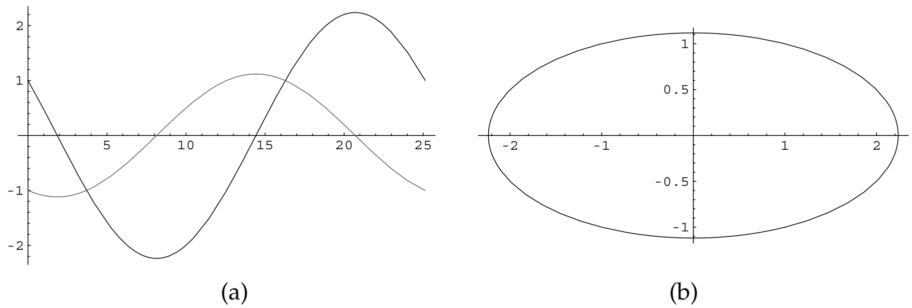


Figure 6-17 (a) Graph of $x(t)$ and $y(t)$. (b) Parametric plot of $x(t)$ versus $y(t)$

```
Out[1151]= {{x[t] → -2 (-1/2 Cos[t/4] + Sin[t/4]),
            y[t] → -Cos[t/4] - 1/2 Sin[t/4]}}
```

finds the solution that satisfies $x(0) = 1$ and $y(0) = -1$.

We graph $x(t)$ and $y(t)$ together as well as parametrically with `Plot` and `ParametricPlot`, respectively, in Figure 6-17.

```
In[1152]:= p1 = Plot[Evaluate[{x[t], y[t]}/.partsol],
                 {t, 0, 8π}, PlotStyle- > {GrayLevel[0],
                 GrayLevel[0.4]},
                 DisplayFunction- > Identity];
            p2 = ParametricPlot[
                 Evaluate[{x[t], y[t]}/.partsol],
                 {t, 0, 8π}, DisplayFunction- > Identity,
                 AspectRatio- > Automatic];
            Show[GraphicsArray[{p1, p2}]]
```

We can also use `PlotVectorField` and `ParametricPlot` to graph the direction field and/or various solutions as we do next in Figure 6-18.

```
In[1153]:= pvf = PlotVectorField[{1/2y, -1/8x},
                   {x, -2, 2}, {y, -1, 1},
                   DisplayFunction- > Identity];

In[1154]:= initsol = DSolve[{x′[t] == 1/2y[t],
                       y′[t] == -1/8x[t], x[0] == x0,
                       y[0] == y0}, {x[t], y[t]}, t]
Out[1154]= {{x[t] → -2 (-1/2 x0 Cos[t/4] - y0 Sin[t/4]),
            y[t] → y0 Cos[t/4] - 1/2 x0 Sin[t/4]}}
```

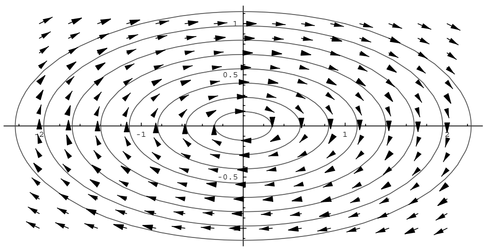


Figure 6-18 Notice that all nontrivial solutions are periodic

```
In[1155] := t1 = Table[ParametricPlot[Evaluate[{x[t],
                y[t]}/.initsol/.{x0- > i,
                y0- > i}], {t, 0, 8π},
                DisplayFunction- > Identity,
                PlotStyle- > GrayLevel[0.3]],
                {i, 0, 1, 1/8}];

In[1156] := Show[t1, pvf,
              DisplayFunction- > $DisplayFunction,
              AspectRatio- > Automatic]
```

(b) In matrix form, the system is equivalent to the system $\mathbf{X}' = \begin{pmatrix} -\frac{1}{4} & 2 \\ -8 & -\frac{1}{4} \end{pmatrix}\mathbf{X}$. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} -\frac{1}{4} & 2 \\ -8 & -\frac{1}{4} \end{pmatrix}$ are found to be $\lambda_{1,2} = -\frac{1}{4} \pm 4i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$ with `Eigensystem`.

```
In[1157] := capa = {{-1/4, 2}, {-8, -1/4}};
            Eigensystem[capa]
```

Out[1157] = $\left\{\left\{-\frac{1}{4} - 4\ i, -\frac{1}{4} + 4\ i\right\}, \{\{i, 2\}, \{-i, 2\}\}\right\}$

A general solution is then

$$\begin{aligned}\mathbf{X} &= c_1\mathbf{X}_1 + c_2\mathbf{X}_2 \\ &= c_1 e^{-t/4}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin 4t\right) + c_2 e^{-t/4}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 4t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos 4t\right) \\ &= e^{-t/4}\left[c_1\begin{pmatrix} \cos 4t \\ -2\sin 4t \end{pmatrix} + c_2\begin{pmatrix} \sin 4t \\ 2\cos 4t \end{pmatrix}\right] = e^{-t/4}\begin{pmatrix} \cos 4t & \sin 4t \\ -2\sin 4t & 2\cos 4t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\end{aligned}$$

or $x = e^{-t/4}(c_1 \cos 4t + c_2 \sin 4t)$ and $y = e^{-t/4}(2c_2 \cos 4t - 2c_1 \sin 4t)$. We confirm this result using `DSolve`. Notice that the result returned by Mathematica contains the hyperbolic trigonometric functions.

```
In[1158]:= gensol = DSolve[{x′[t] ==
                -1/4x[t] + 2y[t], y′[t] ==
                -8x[t] - 1/4y[t]}, {x[t], y[t]}, t]
```

Out[1158]= $\{\{x[t] \to C[2]\ \left(-\frac{1}{2}\ i\ Cos\left[\left(4+\frac{i}{4}\right)\ t\right] + \frac{1}{2}\ i\ Cosh\left[\left(\frac{1}{4}+4\ i\right)\ t\right] + \frac{1}{2}\ Sin\left[\left(4+\frac{i}{4}\right)\ t\right] - \frac{1}{2}\ i\ Sinh\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right) + C[1]\ \left(-\frac{1}{2}\ Cos\left[\left(4+\frac{i}{4}\right)\ t\right] - \frac{1}{2}\ Cosh\left[\left(\frac{1}{4}+4\ i\right)\ t\right] - \frac{1}{2}\ i\ Sin\left[\left(4+\frac{i}{4}\right)\ t\right] + \frac{1}{2}\ Sinh\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right), y[t] \to C[2]\ \left(Cos\left[\left(4+\frac{i}{4}\right)\ t\right] + Cosh\left[\left(\frac{1}{4}+4\ i\right)\ t\right] + i\ Sin\left[\left(4+\frac{i}{4}\right)\ t\right] - Sinh\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right) + C[1]\ \left(-i\ Cos\left[\left(4+\frac{i}{4}\right)\ t\right] + i\ Cosh\left[\left(\frac{1}{4}+4\ i\right)\ t\right] + Sin\left[\left(4+\frac{i}{4}\right)\ t\right] - i\ Sinh\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right)\}\}$

```
In[1159]:= gensol[[1, 1, 2]]
```

Out[1159]= $C[2]\ \left(-\frac{1}{2}\ i\ Cos\left[\left(4+\frac{i}{4}\right)\ t\right] + \frac{1}{2}\ i\ Cosh\left[\left(\frac{1}{4}+4\ i\right)\ t\right] + \frac{1}{2}\ Sin\left[\left(4+\frac{i}{4}\right)\ t\right] - \frac{1}{2}\ i\ Sinh\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right) + C[1]\ \left(-\frac{1}{2}\ Cos\left[\left(4+\frac{i}{4}\right)\ t\right] - \frac{1}{2}\ Cosh\left[\left(\frac{1}{4}+4\ i\right)\ t\right] - \frac{1}{2}\ i\ Sin\left[\left(4+\frac{i}{4}\right)\ t\right] + \frac{1}{2}\ Sinh\left[\left(\frac{1}{4}+4\ i\right)\ t\right]\right)$

```
In[1160]:= ComplexExpand[gensol[[1, 1, 2]]]//Simplify
```

Out[1160]= $(C[1]\ Cos[4\ t] - C[2]\ Sin[4\ t])\ \left(-Cosh\left[\frac{t}{4}\right] + Sinh\left[\frac{t}{4}\right]\right)$

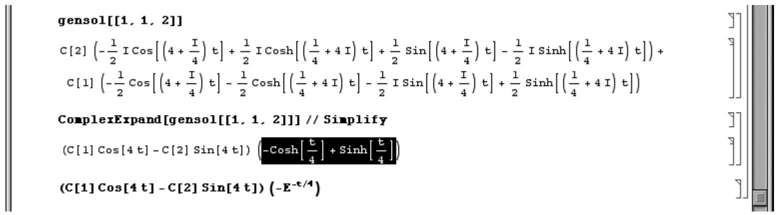
In[1161]:= **(C[1] Cos[4 t] - C[2] Sin[4 t])** $\left(-e^{-t/4}\right)$

In[1162]:= **ComplexExpand[gensol[[1,2,2]]]//Simplify**

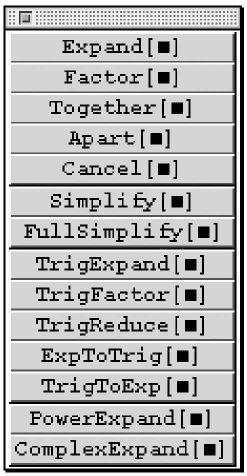
Out[1162]= $2\ (C[2]\ \text{Cos}[4\ t] + C[1]\ \text{Sin}[4\ t])\ \left(\text{Cosh}\left[\frac{t}{4}\right] - \text{Sinh}\left[\frac{t}{4}\right]\right)$

In[1163]:= $\mathbf{2\ (C[2]\ Cos[4\ t] + C[1]\ Sin[4\ t])\ (e^{-t/4})}$

In this case, we obtained the real form of the solution by selecting the portion of the expression that we wanted to write in terms of exponential functions



and then accessed TrigToExp from the **Algebraic Manipulation** palette



to obtain the result.

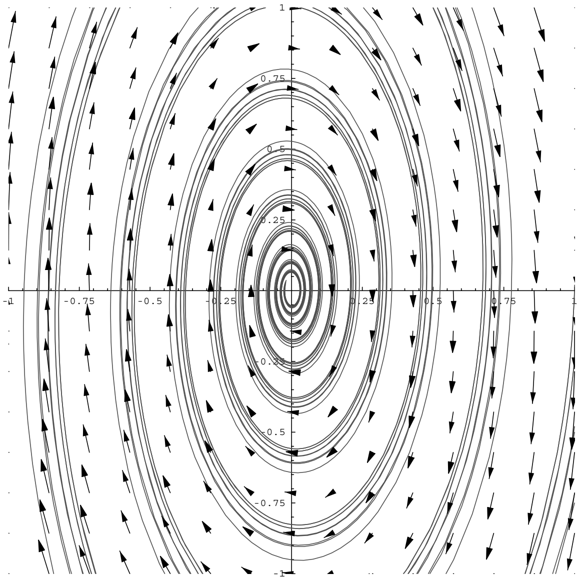


Figure 6-19 Various solutions and direction field associated with the system

```
ComplexExpand[gensol[[1, 2, 2]]] // Simplify
2 (C[2] Cos[4 t] + C[1] Sin[4 t]) (Cosh[t/4] - Sinh[t/4])
2 (C[2] Cos[4 t] + C[1] Sin[4 t]) (E^(-t/4))
```

We use `PlotVectorField` and `ParametricPlot` to graph the direction field associated with the system along with various solutions in Figure 6-19.

```
In[1164]:= pvf = PlotVectorField[{-1/4x + 2y, -8x - 1/4y},
                  {x, -1, 1}, {y, -1, 1}, Axes- > Automatic,
                  DisplayFunction- > Identity];

In[1165]:= initsol = DSolve[{x'[t] == -1/4x[t] + 2y[t],
                   y'[t] == -8x[t] - 1/4y[t], x[0] == x0,
                   y[0] == y0}, {x[t], y[t]}, t]
```

Out[1165]= $\{\{x[t] \to x0\ Cos[4\ t]\ Cosh[\frac{t}{4}] + \frac{1}{2}\ y0 \times Cosh[\frac{t}{4}]\ Sin[4\ t] - x0\ Cos[4\ t] \times Sinh[\frac{t}{4}] - \frac{1}{2}\ y0\ Sin[4\ t]\ Sinh[\frac{t}{4}],$

```
Out[1165]= y[t] → 2 (1/2 y0 Cos[4 t] Cosh[t/4]
              -x0 Cosh[t/4] Sin[4 t]
              -1/2 y0 Cos[4 t] Sinh[t/4]
              +x0 Sin[4 t]sinh[t/4])}}

In[1166]:= t1 = Table[ParametricPlot[
              Evaluate[{x[t],y[t]}/.initsol/.
              {x0- > 1,y0- > i}],{t,0,15},
              DisplayFunction- > Identity,
              PlotStyle- > GrayLevel[0.3]],
              {i,-1,1,2/8}];

In[1167]:= Show[t1,pvf,DisplayFunction- >
            $DisplayFunction,PlotRange- > {{-1,1},
            {-1,1}},AspectRatio- > Automatic]
```

Last, we illustrate how to solve an initial-value problem and graph the resulting solutions by finding the solution that satisfies the initial conditions $x(0) = 100$ and $y(0) = 10$ and then graphing the results with `Plot` and `ParametricPlot` in Figure 6-20.

```
In[1168]:= partsol = DSolve[{x′[t] == -1/4x[t] + 2y[t],
                  y′[t] == -8x[t] - 1/4y[t],x[0] == 100,
                  y[0] == 10},{x[t],y[t]},t]
Out[1168]= {{x[t] → 100 Cos[4 t] Cosh[t/4]
              +5 Cosh[t/4] Sin[4 t] - 100 Cos[4 t]
              Sinh[t/4] - 5 Sin[4 t] Sinh[t/4],
           y[t] → 2 (5 Cos[4 t] Cosh[t/4]
              -100 Cosh[t/4] Sin[4 t]
              -5 Cos[4 t] Sinh[t/4]
              +100 Sin[4 t] Sinh[t/4])}}

In[1169]:= p1 = Plot[Evaluate[{x[t],y[t]}/.partsol],
              {t,0,20},PlotStyle- > {GrayLevel[0],
              GrayLevel[0.4]},
              DisplayFunction- > Identity,
              PlotRange- > All];
```

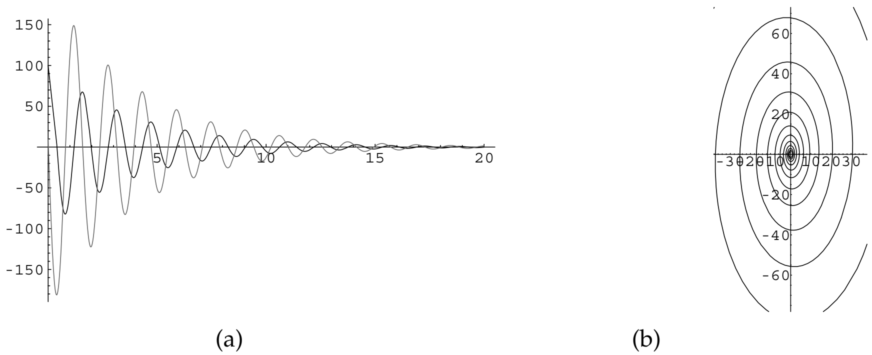


Figure 6-20 (a) Graph of $x(t)$ and $y(t)$. (b) Parametric plot of $x(t)$ versus $y(t)$. (For help with `Show` and `GraphicsArray` use the **Help Browser**)

```
In[1169]:=
        p2 = ParametricPlot[
                Evaluate[{x[t],y[t]}/.partsol],
                {t,0,20},DisplayFunction->Identity,
                AspectRatio->Automatic];
        Show[GraphicsArray[{p1,p2}]]
```

Notice the spiraling motion of the vectors in the direction field. This is due to terms in the solution formed by a product of exponential and trigonometric functions.

■

Initial-value problems can be solved through the use of eigenvalues and eigenvectors as well.

EXAMPLE 6.3.3: Solve $\begin{cases} x' = -\frac{1}{2}x - y + 64z \\ y' = -\frac{1}{4}y - 16z \\ z' = y - \frac{1}{4}z \\ x(0) = 1,\ y(0) = -1,\ z(0) = 0. \end{cases}$

SOLUTION: In matrix form, the system is equivalent to $\mathbf{X}' = \mathbf{AX}$, where $\mathbf{A} = \begin{pmatrix} -1/2 & -1 & 64 \\ 0 & -1/4 & -16 \\ 0 & 1 & -1/4 \end{pmatrix}$. The eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are found with `Eigensystem`.

```
In[1170]:= Clear[a,b,c,d]
In[1171]:= a = {{-1/2, -1, 64}, {0, -1/4, -16},
              {0, 1, -1/4}};
In[1172]:= Eigensystem[a]
Out[1172]= {{-1/2, -1/4 - 4 i, -1/4 + 4 i},
             {{1, 0, 0}, {16 i, -4 i, 1}, {-16 i, 4 i, 1}}}
```

These results mean that the eigenvalue $\lambda_1 = -1/2$ has corresponding eigenvector $\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ so one solution of the system is $\mathbf{X}_1 = \mathbf{v}_1 e^{\lambda_1 t} = \begin{pmatrix}1\\0\\0\end{pmatrix} e^{-t/2}$. An eigenvector corresponding to $\lambda_2 = -1/4 + 4i$ is $\mathbf{v}_2 = \begin{pmatrix}-16i\\4i\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}-16\\4\\0\end{pmatrix} i$. Thus, two linearly independent solutions that correspond to the complex conjugate pair of eigenvalues $\lambda_{2,3} = -1/4 \pm 4i$ are

$$\mathbf{X}_2 = e^{-t/4}\left[\begin{pmatrix}0\\0\\1\end{pmatrix}\cos 4t - \begin{pmatrix}-16\\4\\0\end{pmatrix}\sin 4t\right] = \begin{pmatrix}16\sin 4t\\-4\sin 4t\\\cos 4t\end{pmatrix} e^{-t/4}$$

and

$$\mathbf{X}_3 = e^{-t/4}\left[\begin{pmatrix}-16\\4\\0\end{pmatrix}\cos 4t + \begin{pmatrix}0\\0\\1\end{pmatrix}\sin 4t\right] = \begin{pmatrix}-16\cos 4t\\4\cos 4t\\\sin 4t\end{pmatrix} e^{-t/4}.$$

Hence, a general solution of the system is

$$\begin{aligned}\mathbf{X} &= c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3\\ &= c_1\begin{pmatrix}1\\0\\0\end{pmatrix} e^{-t/2} + c_2\begin{pmatrix}16\sin 4t\\-4\sin 4t\\\cos 4t\end{pmatrix} e^{-t/4} + c_3\begin{pmatrix}-16\cos 4t\\4\cos 4t\\\sin 4t\end{pmatrix} e^{-t/4}\\ &= \begin{pmatrix}c_1 e^{-t/2} + 16e^{-t/4}\left(-c_3\cos 4t + c_2\sin 4t\right)\\ 4e^{-t/4}\left(c_3\cos 4t - c_2\sin 4t\right)\\ e^{-t/4}\left(c_2\cos 4t + c_3\sin 4t\right)\end{pmatrix};\end{aligned}$$

a fundamental matrix is

$$\mathbf{\Phi} = \begin{pmatrix}e^{-t/2} & 16e^{-t/4}\sin 4t & -16e^{-t/4}\cos 4t\\ 0 & -4e^{-t/4}\sin 4t & 4e^{-t/4}\cos 4t\\ 0 & e^{-t/4}\cos 4t & e^{-t/4}\sin 4t\end{pmatrix}.$$

```
In[1173]:= x[t_] = c1 Exp[-t/2]
                +16 Exp[-t/4] (c2 Sin[4t] - c3 Cos[4t]);

            y[t_] = 4 Exp[-t/4]
                (-c2 Sin[4t] + c3 Cos[4t]);

            z[t_] =
              Exp[-t/4] (c3 Sin[4t] + c2 Cos[4t]);
```

We solve the initial-value problem by applying the initial condition $\mathbf{X}(0) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

```
In[1174]:= sysofeqs = {x[0] == 1, y[0] == -1, z[0] == 0}

Out[1174]= {c1 - 16 c3 == 1, 4 c3 == -1, c2 == 0}
```

and solving the resulting system of equations for c_1, c_2, and c_3.

```
In[1175]:= cvals = Solve[sysofeqs]
```

Out[1175]= $\left\{\left\{c_2 \to 0, c_1 \to -3, c_3 \to -\frac{1}{4}\right\}\right\}$

Substitution of these values into the general solution yields the solution to the initial-value problem.

```
In[1176]:= x[t_] = x[t] /. cvals[[1]]

            y[t_] = y[t] /. cvals[[1]]

            z[t_] = z[t] /. cvals[[1]]
```

Out[1176]= $-3\, e^{-t/2} + 4\, e^{-t/4}$ Cos[4 t]

Out[1176]= $-e^{-t/4}$ Cos[4 t]

Out[1176]= $-\frac{1}{4}\, e^{-t/4}$ Sin[4 t]

We graph $x(t)$, $y(t)$, and $z(t)$ with `Plot` in Figure 6-21 and a parametric plot of $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ in three dimensions with `ParametricPlot3D` in Figure 6-22.

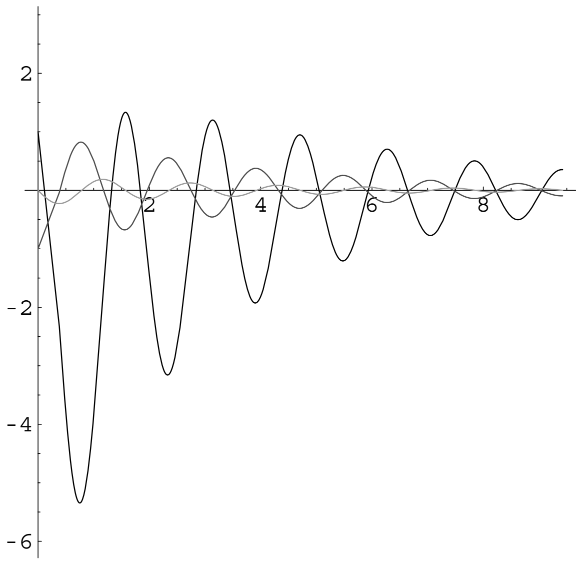


Figure 6-21 $x(t)$ (in black), $y(t)$ (in gray), and $z(t)$ in light gray

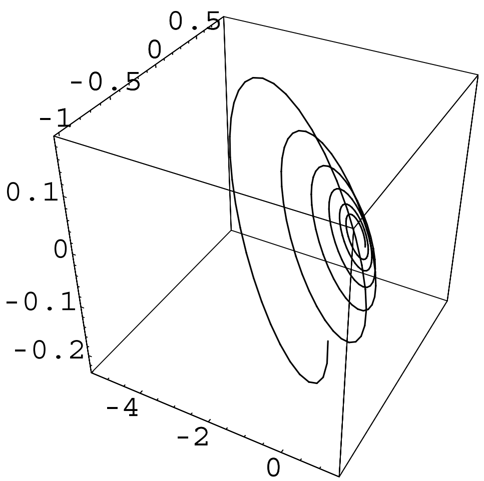


Figure 6-22 The solution to the initial-value problem tends to (0, 0, 0) as $t \to \infty$

```
In[1177]:= Plot[{x[t],y[t],z[t]},{t,0,3π},
              PlotRange→{-2π,π},AspectRatio→1,
              PlotStyle→{GrayLevel[0],GrayLevel[0.3],
              GrayLevel[0.6]}]
```

```
In[1178]:= ParametricPlot3D[{x[t],y[t],z[t]},
              {t,0,3π},BoxRatios→{1,1,1},
              PlotPoints→200]
```

As in previous examples, we see that DSolve is able to find a general solution of the system as well as solve the initial-value problem, although the results are given in terms of complex exponentials.

```
In[1179]:= Clear[x,y,z]

           gensol =
             DSolve[{x'[t] == -x[t]/2 - y[t] + 64 z[t],
                y'[t] == -y[t]/4 - 16 z[t],
                z'[t] == y[t] - z[t]/4},
                {x[t],y[t],z[t]},t]
```

Out[1179]=

$$\left\{\left\{x[t] \to e^{-t/2}\, C[1] - 2\, e^{\left(-\frac{1}{2}-4\, i\right) t} \left(-2\, e^{4\, i\, t} + e^{t/4} + e^{\left(\frac{1}{4}+8\, i\right) t}\right) C[2] - 8\, i\, e^{\left(-\frac{1}{4}-4\, i\right) t} \left(-1 + e^{8\, i\, t}\right) C[3],\right.\right.$$

$$y[t] \to \frac{1}{2}\, e^{\left(-\frac{1}{4}-4\, i\right) t} \left(1 + e^{8\, i\, t}\right) C[2] + 2\, i\, e^{\left(-\frac{1}{4}-4\, i\right) t} \left(-1 + e^{8\, i\, t}\right) C[3],$$

$$\left.\left. z[t] \to -\frac{1}{8}\, i\, e^{\left(-\frac{1}{4}-4\, i\right) t} \left(-1 + e^{8\, i\, t}\right) C[2] + \frac{1}{2}\, e^{\left(-\frac{1}{4}-4\, i\right) t} \left(1 + e^{8\, i\, t}\right) C[3]\right\}\right\}$$

```
In[1180]:= Clear[x,y,z]

           partsol =
             DSolve[{x'[t] == -x[t]/2 - y[t] + 64 z[t],
                y'[t] == -y[t]/4 - 16 z[t],
                z'[t] == y[t] - z[t]/4, x[0] == 1,
                y[0] == -1, z[0] == 0},
                {x[t],y[t],z[t]},t]
```

```
Out[1180]= {{x[t] → e^((-1/2-4 i) t) (-3 e^(4 i t) + 2 e^(t/4) + 2 e^((1/4+8 i) t)),
             y[t] → -1/2 e^((-1/4-4 i) t) (1 + e^(8 i t)),
             z[t] → 1/8 i e^((-1/4-4 i) t) (-1 + e^(8 i t))}}
```

To see that $x(t)$, $y(t)$, and $z(t)$ are real-valued functions, we use `ComplexExpand` together with `Simplify` or `ExpToTrig` together with `Simplify` as follows.

```
In[1181]:= x[t_] =
             ExpToTrig[
               e^((-1/2-4 I) t) (-3 e^(4 I t) + 2 e^(t/4) + 2 e^((1/4+8 I) t))]
              //Simplify
Out[1181]= ((-3 + 4 Cos[4 t]) Cosh[t/8]
             + (3 + 4 Cos[4 t]) Sinh[t/8])
             (Cosh[3 t/8] - Sinh[3 t/8])

In[1182]:= y[t_] =
             ComplexExpand[
               4 I e^((-1/4-4 I) t) (I/8 + 1/8 I e^(8 I t))]//Simplify
Out[1182]= -e^(-t/4) Cos[4 t]

In[1183]:= z[t_] =
             ComplexExpand[
               e^((-1/4-4 I) t) (-I/8 + 1/8 I e^(8 I t))]//Simplify
Out[1183]= -1/4 e^(-t/4) Sin[4 t]
```

■

6.3.3 Alternate Method for Solving Initial-Value Problems

An alternate method can be used to solve initial-value problems.

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the system of equations $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$. Then, a general solution is $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C}$, where $\mathbf{C}$ is a constant vector. If the initial condition $\mathbf{X}(0) = \mathbf{X}_0$ is given, then

$$\begin{aligned} \mathbf{X}(0) &= \mathbf{\Phi}(0)\mathbf{C} \\ \mathbf{X}_0 &= \mathbf{\Phi}(0)\mathbf{C} \\ \mathbf{C} &= \mathbf{\Phi}^{-1}(0)\mathbf{X}_0. \end{aligned}$$

Therefore, the solution to the initial-value problem $\begin{cases} \mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) \\ \mathbf{X}(0) = \mathbf{X}_0 \end{cases}$ is $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0)\mathbf{X}_0$.

EXAMPLE 6.3.4: Use a fundamental matrix to solve the initial-value problem $\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{X}$ subject to $\mathbf{X}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

SOLUTION: We first remark that you can use `DSolve` to solve the initial-value problem directly with the command

```
In[1184]:= Clear[x, y]

           DSolve[{x′[t] == x[t] + y[t],
              y′[t] == 4 x[t] - 2 y[t], x[0] == 1,
              y[0] == -2}, {x[t], y[t]}, t]
```

Out[1184]= $\left\{\left\{x[t] \to \frac{1}{5} e^{-3t} (3 + 2 e^{5t}), y[t] \to \frac{2}{5} e^{-3t} (-6 + e^{5t})\right\}\right\}$

The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ are found with `Eigensystem`.

```
In[1185]:= a = {{1, 1}, {4, -2}};

           Eigensystem[a]
Out[1185]= {{-3, 2}, {{-1, 4}, {1, 1}}}
```

Hence, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. A fundamental matrix is then given by $\mathbf{\Phi}(t) = \begin{pmatrix} -e^{3t} & e^{2t} \\ 4e^{3t} & e^{2t} \end{pmatrix}$.

```
In[1186]:= Ψ[t_] = {{-Exp[-3t], Exp[2t]},
                    {4 Exp[-3t], Exp[2t]}};

            MatrixForm[Ψ[t]]
```

Out[1186]= $\begin{pmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{pmatrix}$

`Inverse[A]` finds the inverse of the square matrix $\mathbf{A}$, if $\mathbf{A}$ is invertible.

We calculate $\mathbf{\Phi}^{-1}(0)$ with `Inverse`.

```
In[1187]:= Inverse[Ψ[0]]//MatrixForm
```

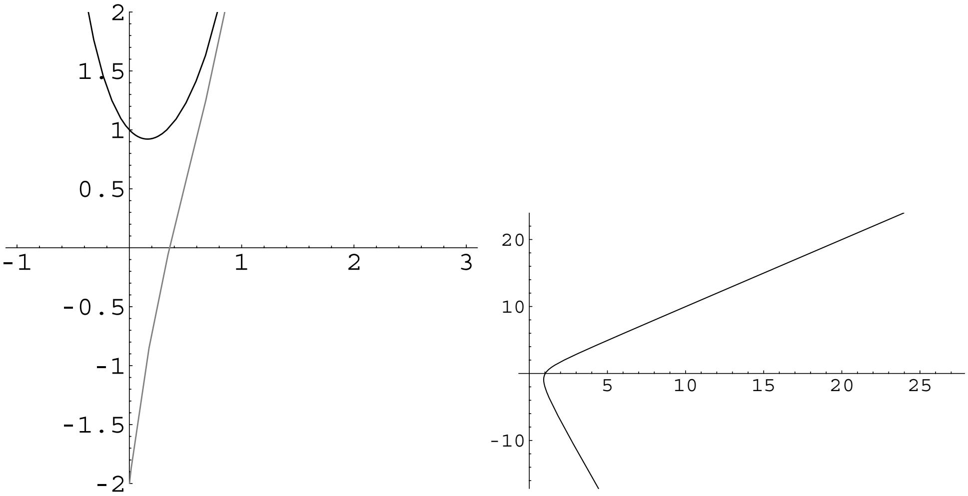
Out[1187]= $\begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$

Hence, the solution to the initial-value problem is $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0)\mathbf{X}_0$.

```
In[1188]:= sol = Ψ[t] .Inverse[Ψ[0]].{1, -2}//
                 Simplify
```

Out[1188]= $\left\{\frac{1}{5} e^{-3t} (3 + 2 e^{5t}), \frac{2}{5} e^{-3t} (-6 + e^{5t})\right\}$

As in the previous examples, we graph $x(t)$ and $y(t)$ together in Figure 6-23 (a) and parametrically in (b).



(a) (b)

Figure 6-23 (a) $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of $x(t)$ versus $y(t)$

```
In[1189]:= Plot[Evaluate[sol],{t,-1,3},
             PlotStyle->{GrayLevel[0],GrayLevel[0.5]},
             PlotRange->{-2,2},AspectRatio->1]
In[1190]:= ParametricPlot[sol,{t,-1,3}]
```

■

6.3.4 Repeated Eigenvalues

We now consider the case of repeated eigenvalues, which is more complicated than the other cases because two situations can arise. An eigenvalue of multiplicity m may have m corresponding linearly independent eigenvectors or it can have fewer than m corresponding linearly independent eigenvectors. In the case of m linearly independent eigenvectors, a general solution is found in the same manner as the case of n distinct eigenvalues.

EXAMPLE 6.3.5: Solve $\mathbf{X}' = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \mathbf{X}$.

SOLUTION: The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$ are found with `Eigensystem`.

```
In[1191]:= Clear[x,y,z,a]

           a = {{1,-3,3},{3,-5,3},{6,-6,4}};

           Eigensystem[a]
Out[1191]= {{-2,-2,4},
             {{-1,0,1},{1,1,0},{1,1,2}}}
```

From the results, we see that the eigenvalue $\lambda_{1,2} = -2$ of multiplicity 2 has two corresponding linearly independent eigenvectors, $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. An eigenvector corresponding to $\lambda_3 = 4$ is $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ so a fundamental set of solutions for the system is $S = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-2t}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-2t}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} e^{4t} \right\}$. A general solution is then

$$\begin{aligned} \mathbf{X} &= c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + c_3 \mathbf{v}_3 e^{\lambda_3 t} \\ &= c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} e^{4t} \\ &= \begin{pmatrix} (c_1 - c_2) e^{-2t} + c_3 e^{4t} \\ c_1 e^{-2t} + c_3 e^{4t} \\ c_2 e^{-2t} + 2c_3 e^{4t} \end{pmatrix} \end{aligned}$$

and a fundamental matrix is

$$\mathbf{\Phi} = \begin{pmatrix} e^{-2t} & -e^{-2t} & e^{4t} \\ e^{-2t} & 0 & e^{4t} \\ 0 & e^{-2t} & 2e^{4t} \end{pmatrix}.$$

Of course, `DSolve` can be used to find a general solution of the system as well, although the form is slightly different than that obtained above.

```
In[1192]:= DSolve[
              Thread[{x'[t],y'[t],z'[t]} ==
               a.{x[t],y[t],z[t]}],
              {x[t],y[t],z[t]},t]//Simplify
```

Out[1192]= $\left\{\left\{x[t] \to \frac{1}{2} e^{-2t} \left((1 + e^{6t}) C[1] - (-1 + e^{6t}) (C[2] - C[3])\right),\right.\right.$
$y[t] \to \frac{1}{2} e^{-2t} \left((-1 + e^{6t}) C[1] - (-3 + e^{6t}) C[2] + (-1 + e^{6t}) C[3]\right),$
$\left.\left. z[t] \to e^{-2t} \left((-1 + e^{6t}) C[1] + C[2] - e^{6t} C[2] + e^{6t} C[3]\right)\right\}\right\}$

■

Because an eigenvalue of multiplicity 2 can have only one corresponding eigenvector, let us first restrict our attention to a system where the repeated eigenvalue $\lambda_1 = \lambda_2$ of $\mathbf{A}$ has only one corresponding eigenvector $\mathbf{v}_1$. We obtain one solution, $\mathbf{X}_1 = \mathbf{v}_1 e^{\lambda_1 t}$, to the system $\mathbf{X}' = \mathbf{AX}$ that corresponds to the eigenvalue λ_1 of $\mathbf{A}$. We now seek a second linearly independent solution corresponding to λ_1 in a manner similar to that considered in the case of repeated characteristic roots of higher-order equations. In this case, however, we suppose that the second linearly independent solution corresponding to λ_1 is of the form

$$\mathbf{X}_2 = (\mathbf{v}_2 t + \mathbf{w}_2)\, e^{\lambda_1 t}.$$

In order to find $\mathbf{v}_2$ and $\mathbf{w}_2$, we substitute $\mathbf{X}_2$ into $\mathbf{X}' = \mathbf{AX}$. Because $\mathbf{X}_2' = \lambda_1 (\mathbf{v}_2 t + \mathbf{w}_2)\, e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_1 t}$, we have

$$\begin{aligned} \mathbf{X}_2' &= \mathbf{A}\mathbf{X}_2 \\ \lambda_1 (\mathbf{v}_2 t + \mathbf{w}_2)\, e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_1 t} &= \mathbf{A}(\mathbf{v}_2 t + \mathbf{w}_2)\, e^{\lambda_1 t} \\ \lambda_1 \mathbf{v}_2 t + (\lambda_1 \mathbf{w}_2 + \mathbf{v}_2) &= \mathbf{A}\mathbf{v}_2 t + \mathbf{A}\mathbf{w}_2. \end{aligned}$$

Equating coefficients yields $\lambda_1 \mathbf{v}_2 = \mathbf{A}\mathbf{v}_2$ and $\lambda_1 \mathbf{w}_2 + \mathbf{v}_2 = \mathbf{A}\mathbf{w}_2$. The equation $\lambda_1 \mathbf{v}_2 = \mathbf{A}\mathbf{v}_2$ indicates that $\mathbf{v}_2$ is an eigenvector of $\mathbf{A}$ that corresponds to λ_1, so we choose $\mathbf{v}_2 = \mathbf{v}_1$. We simplify the equation $\lambda_1 \mathbf{w}_2 + \mathbf{v}_2 = \mathbf{A}\mathbf{w}_2$:

$$\begin{aligned} \lambda_1 \mathbf{w}_2 + \mathbf{v}_2 &= \mathbf{A}\mathbf{w}_2 \\ \mathbf{v}_2 &= \mathbf{A}\mathbf{w}_2 - \lambda_1 \mathbf{w}_2 \\ \mathbf{v}_2 &= (\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{w}_2. \end{aligned}$$

Because $\mathbf{v}_2 = \mathbf{v}_1$, $\mathbf{w}_2$ satisfies the equation

$$(\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{w}_2 = \mathbf{v}_1.$$

Therefore, a second linearly independent solution corresponding to the eigenvalue λ_2 has the form

$$\mathbf{X}_2 = (\mathbf{v}_1 t + \mathbf{w}_2)\, e^{\lambda_1 t},$$

where $\mathbf{w}_2$ satisfies $(\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{w}_2 = \mathbf{v}_1$.

EXAMPLE 6.3.6: Find a general solution of $\mathbf{X}' = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix} \mathbf{X}$.

SOLUTION: We first note that `DSolve` can find a general solution of the system.

```
In[1193]:= DSolve[{x'[t] == -8 x[t] - y[t],
              y'[t] == 16 x[t]}, {x[t], y[t]}, t]
Out[1193]= {{x[t] → -e^(-4 t) (-1 + 4 t) C[1] - e^(-4 t) t C[2],
             y[t] → 16 e^(-4 t) t C[1] + e^(-4 t) (1 + 4 t) C[2]}}
```

We find the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$ with `Eigensystem`.

```
In[1194]:= a = (-8 -1
                16  0);
            Eigensystem[a]
Out[1194]= {{-4, -4}, {{-1, 4}, {0, 0}}}
```

Hence, $\lambda_{1,2} = -4$ and an eigenvector that corresponds to $\lambda_1 = -4$ is $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and one solution to the system is $\mathbf{X}_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-4t}$; there is not a second linearly independent eigenvector corresponding to this repeated eigenvalue.

Therefore, to find $\mathbf{w}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ in a second linearly independent solution $\mathbf{X}_2 = (\mathbf{v}_1 t + \mathbf{w}_2)\, e^{\lambda_1 t}$, we solve $(\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{w}_2 = \mathbf{v}_1$, which in this case is

$$\begin{pmatrix} -4 & -1 \\ 16 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix},$$

with `LinearSolve`.

`LinearSolve[A,b]` solves $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$.

```
In[1195]:= LinearSolve[a + 4 IdentityMatrix[2],
             {-1, 4}]
Out[1195]= {1/4, 0}
```

We can use `Solve` to solve the system as well,

```
In[1196]:= Solve[{-4x2 - y2 == -1, 16x2 + 4y2 == 4}]
Out[1196]= {{x2 → 1/4 - y2/4}}
```

which indicates that $x_2 = \frac{1}{4}(1 - y_2)$. Choosing $y_2 = 0$, $x_2 = 1/4$. With $\mathbf{w}_2 = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$, a second linearly independent solution is

$$\mathbf{X}_2 = \left[\begin{pmatrix} -1 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right] e^{-4t}$$

and a general solution is

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-4t} + c_2 \left[\begin{pmatrix} -1 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right] e^{-4t};$$

a fundamental matrix for the system is

$$\mathbf{\Phi} = \begin{pmatrix} -1 & -t + 1/4 \\ 4 & 4t \end{pmatrix} e^{-4t}.$$

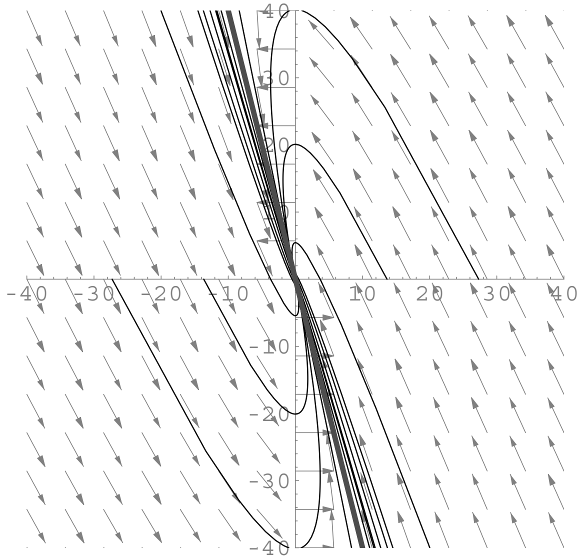


Figure 6-24 All solutions tend to (0, 0) as $t \to \infty$

We now graph several solutions of the system as well as the direction field associated with the system in Figure 6-24.

```
In[1197]:= sol =
             c1 (-1) Exp[-4t]
                ( 4)
             + c2 ((-1) t + (1/4)) Exp[-4t]//Flatten;
                  ( 4)      ( 0 )

In[1198]:= step1 = Table[sol, {c1, -2, 2},
                  {c2, -2, 2}];

In[1199]:= toplot = Flatten[step1, 1];

In[1200]:= pp1 = ParametricPlot[Evaluate[toplot],
                {t, -1, 1}, PlotStyle- > GrayLevel[0],
                DisplayFunction- > Identity];

In[1201]:= << Graphics`PlotField`

In[1202]:= pvf = PlotVectorField[{-8x - y, 16x},
                {x, -40, 40}, {y, -40, 40},
                DisplayFunction- > Identity,
                DefaultColor- > GrayLevel[0.5],
                ScaleFunction- > (1&)];
```

```
In[1203]:= line = Plot[-4x, {x, -30, 40},
                PlotStyle- >
                {{GrayLevel[0.3], Thickness[0.01]}},
                DisplayFunction- > Identity];

In[1204]:= Show[pvf, pp1, line,
             PlotRange- > {{-40, 40}, {-40, 40}},
             Axes- > Automatic, AspectRatio- > 1,
             DisplayFunction- > $DisplayFunction]
```

In Figure 6-24, notice that the behavior of these solutions differs from those of the other systems solved earlier in the section. This is due to the repeated eigenvalues.

■

A similar method is carried out in the case that an eigenvalue of $\mathbf{A}$ has multiplicity 3. Suppose that $\lambda_1 = \lambda_2 = \lambda_3$ has only one linearly independent corresponding eigenvector $\mathbf{v}_1$. In this situation, one solution of $\mathbf{X}' = \mathbf{AX}$ is $\mathbf{X}_1 = \mathbf{v}_1 e^{\lambda_1 t}$. We assume that two other linearly independent solutions have the form

$$\mathbf{X}_2 = (\mathbf{v}_2 t + \mathbf{w}_2)\, e^{\lambda_1 t} \quad \text{and} \quad \mathbf{X}_3 = \left(\frac{1}{2}\mathbf{v}_3 t^2 + \mathbf{w}_3 t + \mathbf{u}_3\right) e^{\lambda_1 t}.$$

Substitution of these solutions into the system of differential equations $\mathbf{X}' = \mathbf{AX}$ yields the following system of equations that is solved for the unknown vectors $\mathbf{v}_2$, $\mathbf{w}_2$, $\mathbf{v}_3$, $\mathbf{w}_3$, and $\mathbf{u}_3$:

$$\begin{cases} \lambda_1 \mathbf{v}_2 = \mathbf{A}\mathbf{v}_2 \\ (\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{w}_2 = \mathbf{v}_2 \\ \lambda_1 \mathbf{v}_3 = \mathbf{A}\mathbf{v}_3 \\ (\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{w}_3 = \mathbf{v}_3 \\ (\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{u}_3 = \mathbf{w}_3. \end{cases}$$

Similar to the previous case, $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{w}_3$, and the vector $\mathbf{u}_3$ is found by solving the system

$$(\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{u}_3 = \mathbf{w}_2.$$

Hence, the three solutions have the form

$$\mathbf{X}_1 = \mathbf{v}_1 e^{\lambda_1 t}, \quad \mathbf{X}_2 = (\mathbf{v}_1 t + \mathbf{w}_2)\, e^{\lambda_1 t}, \quad \text{and} \quad \mathbf{X}_3 = \left(\frac{1}{2}\mathbf{v}_1 t^2 + \mathbf{w}_2 t + \mathbf{u}_3\right) e^{\lambda_1 t}.$$

Notice that this method is generalized for instances when the multiplicity of the repeated eigenvalue is greater than 3.

EXAMPLE 6.3.7: Solve $\mathbf{X}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{X}$.

SOLUTION: After defining **A**, we can also use `DSolve` to find a general solution of the system.

```
In[1205]:= DSolve[
              Thread[{x'[t], y'[t], z'[t]} ==
              a.{x[t], y[t], z[t]}],
              {x[t], y[t], z[t]}, t]
```

$$\text{Out[1205]=}\ \Big\{\Big\{x[t] \to -e^{2t}(-1+t)\,C[1] + e^{2t}\,t\,C[2] + e^{2t}\,t\,C[3],$$
$$y[t] \to -\frac{1}{2}e^{2t}(-4+t)\,t\,C[1] + \frac{1}{2}e^{2t}(2-2t+t^2)\,C[2] + \frac{1}{2}e^{2t}(-2+t)\,t\,C[3],$$
$$z[t] \to \frac{1}{2}e^{2t}(-6+t)\,t\,C[1] - \frac{1}{2}e^{2t}(-4+t)\,t\,C[2] - \frac{1}{2}e^{2t}(-2-4t+t^2)\,C[3]\Big\}\Big\}$$

Alternatively, we can use the eigenvalues and corresponding eigenvectors to construct a general solution.

The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$ are found with `Eigensystem`.

```
In[1206]:= a = {{1, 1, 1}, {2, 1, -1}, {-3, 2, 4}};

           Eigensystem[a]
Out[1206]= {{2, 2, 2}, {{0, -1, 1}, {0, 0, 0}, {0, 0, 0}}}
```

Here, $\lambda_{1,2,3} = 2$ has multiplicity 3 and has one eigenvector, $\mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, that corresponds to it; there are not 1 or 2 other linearly independent eigenvectors corresponding to $\lambda = 2$. One solution of the system is

$\mathbf{X}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t}$. The vector $\mathbf{w}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ in the second linearly independent solution of the form $\mathbf{X}_2 = (\mathbf{v}_1 t + \mathbf{w}_2) e^{2t}$ is found by solving the system $(\mathbf{A} - \lambda_1 \mathbf{I}) \mathbf{w}_2 = \mathbf{v}_1$. We can solve this system with `Solve` or `LinearSolve`. Here we use `LinearSolve`

```
In[1207]:= LinearSolve[a - 2 IdentityMatrix[3],
              {0, -1, 1}]
Out[1207]= {-1, -1, 0}
```

to see that $\mathbf{w}_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$, so $\mathbf{X}_2 = \left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right) e^{2t}$. Finally, we must determine the vector $\mathbf{u}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ in the third linearly independent solution $\mathbf{X}_3 = \left(\frac{1}{2} \mathbf{v}_1 t^2 + \mathbf{w}_2 t + \mathbf{u}_3 \right) e^{\lambda_1 t}$ by solving the system $(\mathbf{A} - \lambda_1 \mathbf{I}) \mathbf{u}_3 = \mathbf{w}_2$.

```
In[1208]:= Solve[
              (a - 2 IdentityMatrix[3]).{x3, y3, z3} ==
               {-1, -1, 0}]
Solve::svars :
  Equations may not give solutions
     for all "solve" variables.
Out[1208]= {{x3 → -2, y3 → -3 - z3}}
```

Therefore, $x_3 = -2$ and $y_3 = -3 - z_3$. We select $z_3 = 0$ so $y_3 = -3$. Hence, $\mathbf{u}_3 = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$ and a third linearly independent solution is $\mathbf{X}_3 = \left(\frac{1}{2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} \right) e^{2t}$. A general solution is then given by

$$
\begin{aligned}
\mathbf{X} &= c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + c_3 \mathbf{X}_3 \\
&= c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right) e^{2t} + c_3 \left(\frac{1}{2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} \right) e^{2t} \\
&= \begin{pmatrix} -c_2 + c_3(-t - 2) \\ -c_1 + c_2(-t - 1) + c_3 \left(-\frac{1}{2} t^2 - t - 3 \right) \\ c_1 + c_2 t + \frac{1}{2} c_3 t^2 \end{pmatrix} e^{2t}.
\end{aligned}
$$

■

6.4 Nonhomogeneous First-Order Systems: Undetermined Coefficients, Variation of Parameters, and the Matrix Exponential

In Chapter 4, we learned how to solve nonhomogeneous differential equations through the use of Undetermined Coefficients and Variation of Parameters. Here we approach the solution of systems of nonhomogeneous equations using those methods.

Let

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \dots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

A general solution of the homogeneous system $\mathbf{X}' = \mathbf{AX}$ is $\mathbf{X} = \mathbf{\Phi}(t)\mathbf{C}$, where $\mathbf{\Phi}(t) = \begin{pmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \cdots & \mathbf{\Phi}_n \end{pmatrix}$ is a fundamental matrix for the system $\mathbf{X}' = \mathbf{AX}$ and $\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is an $n \times 1$ constant vector.

Let $\mathbf{X}$ be *any* solution of $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, $\mathbf{X}_h = \mathbf{\Phi}(t)\mathbf{C}$ a general solution of the corresponding homogeneous system, $\mathbf{X}' = \mathbf{AX}$, and $\mathbf{X}_p$ a particular solution of the nonhomogeneous system.

Then, $\mathbf{X} - \mathbf{X}_p$ is a solution of the corresponding homogeneous system, $\mathbf{X}' = \mathbf{AX}$, so $\mathbf{X} - \mathbf{X}_p = \mathbf{X}_h$ and, consequently, $\mathbf{X} = \mathbf{X}_h + \mathbf{X}_p$.

Thus, to find a general solution of $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, we note that if $\mathbf{X}_p$ is a particular solution of the equation then all other solutions to the equation can be written in the form

$$\mathbf{X} = \underbrace{\mathbf{\Phi}(t)\mathbf{C}}_{\mathbf{X}_h} + \mathbf{X}_p.$$

A **particular solution** to a system of ordinary differential equations is a set of functions that satisfy the system but do not contain any arbitrary constants. That is, a particular solution to a system is a set of specific functions, *containing no arbitrary constants*, that satisfy the system.

$\mathbf{X}_h = \mathbf{\Phi C}$ is a general solution of the corresponding homogeneous equation, $\mathbf{X}' = \mathbf{AX}$.

6.4.1 Undetermined Coefficients

We use the method of undetermined coefficients to find a particular solution of a nonhomogeneous system in much the same way as we approached nonhomogeneous higher-order equations in Chapter 4. The main difference is that the coefficients are *constant vectors* when we work with systems.

EXAMPLE 6.4.1: Solve $\begin{cases} x' = 2x + y + \sin 3t \\ y' = -8x - 2y \\ x(0) = 0,\ y(0) = 1. \end{cases}$

SOLUTION: In matrix form, the system is equivalent to $\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \sin 3t \\ 0 \end{pmatrix}$. We find a general solution of the corresponding homogeneous system $\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix}\mathbf{X}$ with `DSolve`.

```
In[1209]:= homsol =
            DSolve[{x'[t] == 2 x[t] + y[t],
              y'[t] == -8 x[t] - 2 y[t]}, {x[t], y[t]},
              t]//Simplify
Out[1209]= {{x[t] → C[1] Cos[2 t] + (2 C[1]
                 +C[2]) Cos[t] Sin[t],
            y[t] → C[2] Cos[2 t] - (4 C[1]
                 +C[2]) Sin[2 t]}}
```

These results indicate that a general solution of the corresponding homogeneous system is

$$\mathbf{X}_h = \begin{pmatrix} -\cos 2t - \sin 2t & \sin 2t - \cos 2t \\ 4\sin 2t & 4\cos 2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

```
In[1210]:= xh[t_] = (x[t]
                     y[t]) /.homsol[[1]];
```

Thus, we search for a particular solution of the nonhomogeneous system of the form $\mathbf{X}_p = \mathbf{a}\sin 3t + \mathbf{b}\cos 3t$, where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. After defining $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix}$ and $\mathbf{X}_p = \mathbf{a}\sin 3t + \mathbf{b}\cos 3t$, we substitute $\mathbf{X}_p$ into the nonhomogeneous system.

```
In[1211]:= capa = ( 2  1
                   -8 -2 );
```

```
In[1212]:= xp[t_] = (a1
                     a2) Sin[3t] + (b1
                                    b2) Cos[3t];
```

```
In[1213]:= step1 = xp'[t] == capa.xp[t] + (Sin[3t]) //
                                          (  0   )
                              Simplify
Out[1213]= {{-3 Sin[3 t] b1 + 3 Cos[3 t] a1},
             {-3 Sin[3 t] b2
             +3 Cos[3 t] a2}} ==
             {{2 Cos[3 t] b1 + Cos[3 t] b2
             + Sin[3 t] (1 + 2 a1 + a2)},
             {-2 (4 Cos[3 t] b1 + Cos[3 t] b2
             + Sin[3 t] (4 a1 + a2))}}
```

The result represents a system of equations that is true for all values of t. In particular, substituting $t = 0$ yields

```
In[1214]:= eq1 = step1 /. t- > 0
Out[1214]= {{3 {{1, 1}, {4, -2}}1},
            {3 {{1, 1}, {4, -2}}2}} ==
            {{2 b1 + b2}, {-2 (4 b1 + b2)}}
```

which is equivalent to the system of equations

$$\begin{cases} 3a_1 = 2b_1 + b_2 \\ 3a_2 = -2\,(4b_1 + b_2)\,. \end{cases}$$

Similarly, substituting $t = \pi/2$ results in

```
In[1215]:= eq2 = step1 /. t- > π/2
Out[1215]= {{3 b1}, {3 b2}} ==
            {{-1 - 2 a1 - a2},
            {-2 (-4 a1 - a2)}}
```

which is equivalent to the system of equations

$$\begin{cases} 3b_1 = -1 - 2a_1 - a_2 \\ 3b_2 - 2\,(-4a_1 - a_2)\,. \end{cases}$$

We now use `Solve` to solve these four equations for a_1, a_2, b_1, and b_2

```
In[1216]:= coeffs = Solve[{eq1, eq2}]
Out[1216]= {{b1 → -3/5, b2 → 0, a1 → -2/5, a2 → 8/5}}
```

and substitute into $\mathbf{X}_p$ to obtain a particular solution to the nonhomogeneous system.

```
In[1217]:= xp[t_] = xp[t]/.coeffs[[1]]
```

Out[1217]= $\left\{\left\{-\frac{3}{5}\text{Cos}[3t]-\frac{2}{5}\text{Sin}[3t]\right\},\left\{\frac{8}{5}\text{Sin}[3t]\right\}\right\}$

A general solution to the nonhomogeneous system is then given by $\mathbf{X}=\mathbf{X}_h+\mathbf{X}_p$.

```
In[1218]:= x[t_] = xh[t] + xp[t]
```

Out[1218]= $\left\{\left\{C[1]\,\text{Cos}[2t]-\frac{3}{5}\text{Cos}[3t]+(2C[1]+C[2])\,\text{Cos}[t]\,\text{Sin}[t]-\frac{2}{5}\text{Sin}[3t]\right\},\left\{C[2]\,\text{Cos}[2t]-(4C[1]+C[2])\,\text{Sin}[2t]+\frac{8}{5}\text{Sin}[3t]\right\}\right\}$

To solve the initial-value problem, we apply the initial condition and solve for the unknown constants.

```
In[1219]:= x[0]
```

Out[1219]= $\left\{\left\{-\frac{3}{5}+C[1]\right\},\{C[2]\}\right\}$

```
In[1220]:= cvals = Solve[x[0] == {{0}, {1}}]
```

Out[1220]= $\left\{\left\{C[1]\to\frac{3}{5},C[2]\to 1\right\}\right\}$

We obtain the solution by substituting these values back into the general solution.

```
In[1221]:= x[t_] = x[t] /. cvals[[1]] //Flatten//
              Simplify
```

Out[1221]= $\left\{\frac{1}{5}(3\,\text{Cos}[2t]-3\,\text{Cos}[3t]+11\,\text{Cos}[t]\,\text{Sin}[t]-2\,\text{Sin}[3t]),\ \text{Cos}[2t]-\frac{17}{5}\text{Sin}[2t]+\frac{8}{5}\text{Sin}[3t]\right\}$

We confirm this result by graphing $x(t)$ (in black) and $y(t)$ (in gray) together in Figure 6-25 (a) as well as parametrically in (b).

```
In[1222]:= Plot[Evaluate[x[t]], {t, 0, 4π},
             PlotStyle →
             {GrayLevel[0], GrayLevel[0.5]},
             PlotRange → {-2 π, 2π}, AspectRatio → 1]

In[1223]:= ParametricPlot[x[t], {t, 0, 4π},
             PlotRange → {{-6, 5}, {-5, 6}},
             AspectRatio → 1, Compiled → False]
```

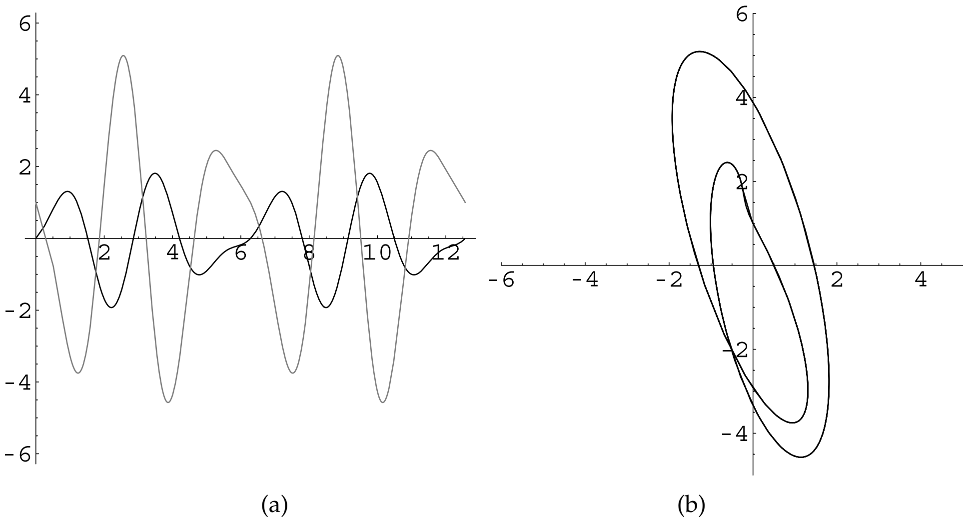


Figure 6-25 (a) $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of $x(t)$ versus $y(t)$

Finally, we note that DSolve is able to find a general solution of the nonhomogeneous system

```
In[1224]:= Clear[x, y, t]

           gensol =
             Simplify[
               DSolve[{x'[t] == 2 x[t] + y[t] + Sin[3 t],
                   y'[t] == -8 x[t] - 2 y[t]}, {x[t],
                 y[t]}, t]]
Out[1224]= {{x[t] → C[1] Cos[2 t] - 3/5 Cos[3 t]
                 +C[2] Cos[t] Sin[t] + C[1] Sin[2 t]
                 -2/5 Sin[3 t], y[t] → C[2] Cos[2 t]
                 -(4 C[1] + C[2]) Sin[2 t]
                 +8/5 Sin[3 t]}}
```

as well as solve the initial-value problem.

```
In[1225]:= partsol =
             Simplify[
               DSolve[{x'[t] == 2 x[t] + y[t] + Sin[3 t],
                   y'[t] == -8 x[t] - 2 y[t], x[0] == 0,
                   y[0] == 1}, {x[t], y[t]}, t]]
```

```
Out[1225]= {{x[t] → 1/10 (6 Cos[2 t]
                 -6 Cos[3 t] + 11 Sin[2 t]
                 -4 Sin[3 t]),
              y[t] → Cos[2 t] - 17/5 Sin[2 t]
                 + 8/5 Sin[3 t]}}

In[1226]:= partsol[[1, 1, 2]]//ExpToTrig//
             Simplify

Out[1226]= 1/10 (6 Cos[2 t]
            -6 Cos[3 t] + 11 Sin[2 t] - 4 Sin[3 t])

In[1227]:= partsol[[1, 2, 2]]//ExpToTrig//
             Simplify

Out[1227]= Cos[2 t] - 17/5 Sin[2 t] + 8/5 Sin[3 t]
```

■

6.4.2 Variation of Parameters

Generally, the method of undetermined coefficients is difficult to implement for nonhomogeneous linear systems as the choice for the particular solution must be very carefully made.

Variation of parameters is implemented in much the same way as for first-order linear equations.

Let $\mathbf{\Phi}$ be a fundamental matrix for the corresponding homogeneous system. We assume that a particular solution has the form $\mathbf{X}_p = \mathbf{\Phi}\mathbf{U}(t)$. Differentiating $\mathbf{X}_p$ gives us

$$\mathbf{X}_p' = \mathbf{\Phi}'\mathbf{U} + \mathbf{\Phi}\mathbf{U}'.$$

Substituting into equation $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$ results in

$$\begin{aligned} \mathbf{\Phi}'\mathbf{U} + \mathbf{\Phi}\mathbf{U}' &= \mathbf{A}\mathbf{\Phi}\mathbf{U} + \mathbf{F} \\ \mathbf{\Phi}\mathbf{U}' &= \mathbf{F} \\ \mathbf{U}' &= \mathbf{\Phi}^{-1}\mathbf{F} \\ \mathbf{U} &= \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt, \end{aligned}$$

where we have used the fact that $\mathbf{\Phi}'\mathbf{U} - \mathbf{A}\mathbf{\Phi}\mathbf{U} = (\mathbf{\Phi}' - \mathbf{A}\mathbf{\Phi})\,\mathbf{U} = \mathbf{0}$. It follows that

$$\mathbf{X}_p = \mathbf{\Phi} \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt. \tag{6.15}$$

A general solution is then

$$\begin{aligned}\mathbf{X} &= \mathbf{X}_h + \mathbf{X}_p \\ &= \mathbf{\Phi C} + \mathbf{\Phi} \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt \\ &= \mathbf{\Phi}\left(\mathbf{C} + \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt\right) = \mathbf{\Phi} \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt,\end{aligned}$$

where we have incorporated the constant vector $\mathbf{C}$ into the indefinite integral $\int \mathbf{\Phi}^{-1}\mathbf{F}\,dt$.

EXAMPLE 6.4.2: Solve the initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}\mathbf{X} - \begin{pmatrix} t\cos 3t \\ t\sin t + t\cos 3t \end{pmatrix}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Remark. In traditional form, the system is equivalent to

$$\begin{cases} x' = x - y - t\cos 3t \\ y' = 10x - y - t\sin t - t\cos 3t, \end{cases} \qquad x(0) = 1,\ y(0) = -1.$$

SOLUTION: The corresponding homogeneous system is $\mathbf{X}_h' = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}\mathbf{X}_h$. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 10 & -1 \end{pmatrix}$ are $\lambda_{1,2} = \pm 3i$ and $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 10 \end{pmatrix} \pm \begin{pmatrix} 3 \\ 0 \end{pmatrix} i$, respectively.

```
In[1228]:= capa = {{1, -1}, {10, -1}};
           Eigensystem[capa]

Out[1228]= {{-3 i, 3 i}, {{1 - 3 i, 10}, {1 + 3 i, 10}}}
```

A fundamental matrix is $\mathbf{\Phi} = \begin{pmatrix} \sin 3t & \cos 3t \\ \sin 3t - 3\cos 3t & \cos 3t + 3\sin 3t \end{pmatrix}$ with inverse $\mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{3}\cos 3t + \sin 3t & -\frac{1}{3}\cos 3t \\ -\frac{1}{3}\sin 3t + \cos 3t & \frac{1}{3}\sin 3t \end{pmatrix}$.

```
In[1229]:= fm = {{Sin[3t], Sin[3t] - 3 Cos[3t]},
                  {Cos[3t], Cos[3t] + 3 Sin[3t]}}//Transpose;
            fminv = Inverse[fm]//Simplify
```

$$\text{Out[1229]=}\ \left\{\left\{\frac{1}{3}\ \text{Cos[3 t]} + \text{Sin[3 t]}, \text{Cos[3 t]} - \frac{1}{3}\ \text{Sin[3 t]}\right\}, \left\{-\frac{1}{3}\ \text{Cos[3 t]}, \frac{1}{3}\ \text{Sin[3 t]}\right\}\right\}$$

We now compute $\mathbf{\Phi}^{-1}\mathbf{F}(t)$

```
In[1230]:= ft = {-t Cos[3t], -t Sin[t] - t Cos[3t]};
            step1 = fminv.ft
```

$$\text{Out[1230]=}\ \left\{(-\text{t Cos[3 t]} - \text{t Sin[t]})\left(\text{Cos[3 t]} - \frac{1}{3}\ \text{Sin[3 t]}\right) - \text{t Cos[3 t]}\left(\frac{1}{3}\ \text{Cos[3 t]} + \text{Sin[3 t]}\right), \frac{1}{3}\ \text{t Cos[3 t]}^2 + \frac{1}{3}\ (-\text{t Cos[3 t]} - \text{t Sin[t]})\ \text{Sin[3 t]}\right\}$$

and $\int \mathbf{\Phi}^{-1}\mathbf{F}(t)\,dt$.

```
In[1231]:= step2 = Integrate[step1, t]
```

$$\text{Out[1231]=}\ \left\{\frac{1}{864}\ \left(-288\ \text{t}^2 + 36\ \text{Cos[2 t]} - 216\ \text{t Cos[2 t]} - 9\ \text{Cos[4 t]} + 108\ \text{t Cos[4 t]} - 16\ \text{Cos[6 t]} + 48\ \text{t Cos[6 t]} + 108\ \text{Sin[2 t]} + 72\ \text{t Sin[2 t]} - 27\ \text{Sin[4 t]} - 36\ \text{t Sin[4 t]} - 8\ \text{Sin[6 t]} - 96\ \text{t Sin[6 t]}\right), \frac{1}{864}\left(72\ \text{t}^2 - 36\ \text{Cos[2 t]} + 9\ \text{Cos[4 t]} + 4\ \text{Cos[6 t]} + 24\ \text{t Cos[6 t]} - 72\ \text{t Sin[2 t]} + 36\ \text{t Sin[4 t]} - 4\ \text{Sin[6 t]} + 24\ \text{t sin[6 t]}\right)\right\}$$

A general solution of the nonhomogeneous system is then $\mathbf{\Phi}\left(\int \mathbf{\Phi}^{-1}\mathbf{F}(t)\,dt + \mathbf{C}\right)$.

```
In[1232]:= Simplify[fm.step2]
```

```
Out[1232]= {1/288 (27 Cos[t] - 4 ((1 + 6 t + 18 t^2)
      × Cos[3 t] + 27 t Sin[t] - Sin[3 t]
      +6 t Sin[3 t] + 18 t^2 Sin[3 t])),
    1/288 (- 36 t Cos[t] - 4 (1 - 6 t + 18 t^2)
      × Cos[3 t] - 45 Sin[t] - 4 Sin[3 t]
      -24 t Sin[3 t] + 72 t^2 Sin[3 t])}
```

It is easiest to use DSolve to solve the initial-value problem directly as we do next.

```
In[1233]:= check = DSolve[{x'[t] == x[t]
               -y[t] - t Cos[3t], y'[t] == 10x[t] - y[t]
               -t Sin[t] - t Cos[3t], x[0] == 1,
                y[0] == -1}, {x[t], y[t]}, t]
Out[1233]= {{x[t] → 1/288 (- 9 Cos[t] + 297 Cos[3 t]
               -72 t^2 Cos[3 t] + 36 t Sin[t]
               +192 Sin[3 t] - 24 t Sin[3 t]),
            y[t] → 1/288 (- 9 Cos[t] - 36 t Cos[t]
               -279 Cos[3 t] - 72 t Cos[3 t]
               -72 t^2 Cos[3 t] - 45 Sin[t]
               +36 t Sin[t] + 1107 Sin[3 t]
               -24 t Sin[3 t] - 216 t^2 Sin[3 t])}}
```

After using ?Evaluate to obtain basic information regarding the Evaluate function, the solutions are graphed with Plot and ParametricPlot in Figure 6-26.

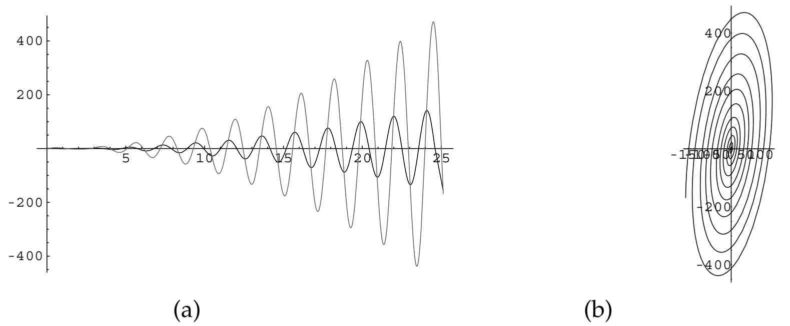


Figure 6-26 (a) Graph of $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of $x(t)$ versus $y(t)$

```
In[1234]:= ?Evaluate

"Evaluate[expr]causesexprtobeevaluatedeven
   ifitappearsastheargumentofafunction
   whoseattributesspecifythatitshouldbe
   heldunevaluated."

In[1235]:= p1 = Plot[Evaluate[{x[t], y[t]}/.check],
                 {t, 0, 8π}, PlotStyle- > {GrayLevel[0],
                 GrayLevel[0.4]}, DisplayFunction- >
                 Identity];
            p2 = ParametricPlot[Evaluate[{x[t],
                 y[t]}/.check], {t, 0, 8π},
                 DisplayFunction- > Identity,
                 AspectRatio- > Automatic];
            Show[GraphicsArray[{p1, p2}]]
```

■

EXAMPLE 6.4.3: Solve $\begin{cases} x' = -3x + 2y + e^t \sec t \\ y' = -10x + 5y + e^t \csc t, \quad 0 < t < \pi. \\ x(\pi/4) = 3,\ y(\pi/4) = -1 \end{cases}$

SOLUTION: To implement the method of Variation of Parameters, we proceed in the same manner as before. First, we find a general solution of the corresponding homogeneous system.

```
In[1236]:= Clear[x, y, a]

           a = ( -3  2
                 -10 5 );

In[1237]:= homsol = DSolve[
                      Thread[{x'[t], y'[t]} ==
                      a.{x[t], y[t]}],
                      {x[t], y[t]}, t]//
                      FullSimplify;
```

This result means that a general solution of the corresponding homogeneous system is

$$\mathbf{X}_h = \begin{pmatrix} e^t(-\cos 2t + 2\sin 2t) & e^t(2\cos 2t + \sin 2t) \\ 5e^t \sin 2t & 5e^t \cos 2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

and a fundamental matrix is given by

$$\Phi = \begin{pmatrix} e^t(-\cos 2t + 2\sin 2t) & e^t(2\cos 2t + \sin 2t) \\ 5e^t \sin 2t & 5e^t \cos 2t \end{pmatrix}.$$

```
In[1238] := Ψ[t_] =
  ( Exp[t] (- Cos[2t] + 2 Sin[2t])   Exp[t] (2 Cos[2t] + Sin[2t]) );
  (      5 Exp[t] Sin[2t]                 5 Exp[t] Cos[2t]        )
```

Next, we compute $\mathbf{X}_p(t) = \Phi(t) \int \Phi^{-1}(t)\mathbf{F}(t)\,dt$. The result is *very* lengthy so we suppress the resulting output by including a semi-colon at the end of the command.

```
In[1239] := inverseΨ = Inverse[Ψ[t]]//Simplify;
            MatrixForm[inverseΨ]
```

$$Out[1239] = \begin{pmatrix} -e^{-t}\,\mathrm{Cos}[2\,t] & \frac{1}{5}\,e^{-t}\,(2\,\mathrm{Cos}[2\,t] + \mathrm{Sin}[2\,t]) \\ e^{-t}\,\mathrm{Sin}[2\,t] & \frac{1}{5}\,e^{-t}\,(\mathrm{Cos}[2\,t] - 2\,\mathrm{Sin}[2\,t]) \end{pmatrix}$$

```
In[1240] := f[t_] = ( Exp[t] Sec[t] );
                    ( Exp[t] Csc[t] )
            xp[t_] =
              Ψ[t].Integrate[Inverse[Ψ[t]].f[t],t]//
              Simplify;
```

However, we view abbreviations of $x(t)$ and $y(t)$ with `Short`.

```
In[1241] := xp[t][[1]]//Short
Out[1241] = {e^t (<<1>>)}

In[1242] := xp[t][[2]]//Short
Out[1242] = {-e^t (<<1>>)}
```

Finally, we form a general solution of the nonhomogeneous system.

```
In[1243] := x[t_] = Ψ[t].( c1 ) + xp[t]//Simplify;
                          ( c2 )
```

To solve the initial-value problem, we substitute $t = \pi/4$ into the general solution

```
In[1244] := x[π/4]//FullSimplify
```

Out[1244]= $\left\{\left\{e^{\pi/4}\left(-2\sqrt{2}-2\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{\pi}{8}\right]-\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]+2\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{\pi}{8}\right]+\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]+\mathrm{Log}\left[\mathrm{Tan}\left[\frac{\pi}{8}\right]\right]+2\,c_1+c_2\right)\right\},\right.$
$\left.\left\{e^{\pi/4}\left(-2\sqrt{2}+2\,\mathrm{Log}\left[\mathrm{Sec}\left[\frac{\pi}{8}\right]\right]-5\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{\pi}{8}\right]-\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]+2\,\mathrm{Log}\left[\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]+5\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{\pi}{8}\right]+\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]+5\,c_1\right)\right\}\right\}$

and solve $\begin{cases} x(\pi/4) = 3 \\ y(\pi/4) = -1 \end{cases}$ for c_1 and c_2.

In[1245]:= **cvals = Solve[x[π/4] == $\begin{pmatrix}3\\-1\end{pmatrix}$]**

Out[1245]= $\left\{\left\{c_2\to\frac{1}{5}e^{-\pi/4}\left(17+6\sqrt{2}\,e^{\pi/4}+e^{\pi/4}\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{\pi}{8}\right]\right]-e^{\pi/4}\,\mathrm{Log}\left[\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]\right),\right.\right.$
$\left.\left.c_1\to\frac{1}{5}e^{-\pi/4}\left(-1+2\sqrt{2}\,e^{\pi/4}+2\,e^{\pi/4}\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{\pi}{8}\right]\right]+5\,e^{\pi/4}\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{\pi}{8}\right]-\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]-2\,e^{\pi/4}\,\mathrm{Log}\left[\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]-5\,e^{\pi/4}\,\mathrm{Log}\left[\mathrm{Cos}\left[\frac{\pi}{8}\right]+\mathrm{Sin}\left[\frac{\pi}{8}\right]\right]\right)\right\}\right\}$

This result is rather complicated so we compute more meaningful approximations with N.

In[1246]:= **numcvals = N[cvals]**

Out[1246]= $\{\{c_{2.}\to 3.42352,\ c_{1.}\to -0.0543264\}\}$

The solution to the initial-value problem is obtained by substituting these numbers back into the general solution.

In[1247]:= **x[t_] = x[t] /.cvals[[1]]//N**

```
Out[1247]= {{2.71828^t (-4. Cos[t] + Cos[2. t]
            × (Log[Cos[0.5 t] - 1. Sin[0.5 t]]
            -1. Log[Cos[0.5 t] + Sin[0.5 t]])
            +0.0543264 (Cos[2. t] - 2. Sin[2. t])
            + (-1. Log[Cos[0.5 t]] - 2. Log[Cos[0.5 t]
            -1. Sin[0.5 t]] + Log[Sin[0.5 t]]
            +2. Log[Cos[0.5 t] + Sin[0.5 t]]) Sin[2. t]
            +3.42352 (2. Cos[2. t] + Sin[2. t]))},
            {2.71828^t (17.1176 Cos[2. t] - 1. Cos[2. t]
            × (Log[Cos[0.5 t]] - 1. Log[Sin[0.5 t]])
            +4. Sin[t] - 2. Cos[t] (4.
            + (2. Log[Cos[0.5 t]] + 5. Log[Cos[0.5 t]
            -1. Sin[0.5 t]] - 2. Log[Sin[0.5 t]]
            -5. Log[Cos[0.5 t] + Sin[0.5 t]]) Sin[t])
            -0.271632 Sin[2. t])}}
```

We confirm that the initial conditions are satisfied by graphing $x(t)$ and $y(t)$ on the interval $(0, \pi/2)$ in Figure 6-27.

```
In[1248]:= Plot[Evaluate[x[t]], {t, 0, π/2},
              PlotStyle → {GrayLevel[0], GrayLevel[0.5]}]
```

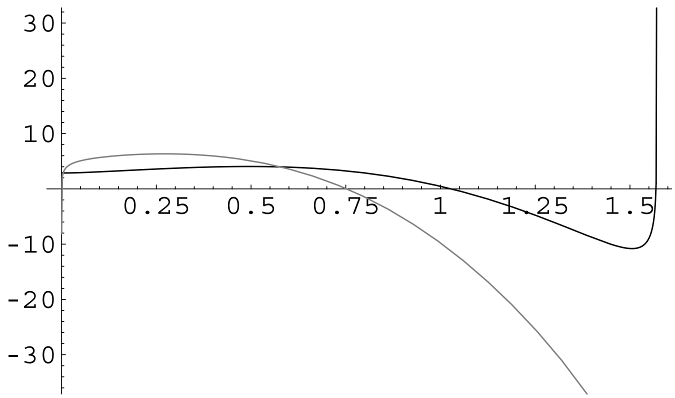


Figure 6-27 $x(t)$ (in black) and $y(t)$ (in gray)

■

6.4.3 The Matrix Exponential

Definition 28 (Matrix Exponential). *If* $\mathbf{A}t$ *is* $n \times n$*, the* ***matrix exponential*** *is defined by*

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n t^n. \tag{6.16}$$

Use the command `MatrixExp` to compute the matrix exponential of a matrix. For example, here we use `MatrixExp` to calculate $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$.

```
In[1249]:= MatrixExp[{{t,0},{-2t,3t}}]
Out[1249]= {{e^t, 0}, {e^t - e^(3t), e^(3t)}}
```

Differentiating the series (6.16) term-by-term shows us that $\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \mathbf{A}e^{\mathbf{A}t}$ so $e^{\mathbf{A}t}$ satisfies the differential equation $\mathbf{X}' = \mathbf{A}\mathbf{X}$. We can use the matrix exponential $e^{\mathbf{A}t}$ to solve the linear first-order system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$ in much the same way that we used the integrating factor $e^{\int p(x)\,dx}$ to solve the linear first-order equation $y' + p(x)y = q(x)$. Moreover, $e^{\mathbf{A}t}$ is a fundamental matrix for the homogeneous system; $\left(e^{\mathbf{A}t}\right)^{-1} = e^{-\mathbf{A}t}$; and if $t = 0$, $e^{\mathbf{A}t} = \mathbf{I}$.

To solve the system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$, we first rewrite it as $\mathbf{X}' - \mathbf{A}\mathbf{X} = \mathbf{F}(t)$. Now, multiply both sides of the equation by $e^{-\mathbf{A}t}$ and integrate:

$$\begin{aligned} e^{-\mathbf{A}t}\left(\mathbf{X}' - \mathbf{A}\mathbf{X}\right) &= e^{-\mathbf{A}t}\mathbf{F}(t) \\ \frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{X}\right) &= e^{-\mathbf{A}t}\mathbf{F}(t) \\ e^{-\mathbf{A}t}\mathbf{X} &= \int e^{-\mathbf{A}t}\mathbf{F}(t)\,dt + \mathbf{C} \\ \mathbf{X} &= e^{\mathbf{A}t}\int e^{-\mathbf{A}t}\mathbf{F}(t)\,dt + e^{\mathbf{A}t}\mathbf{C}, \end{aligned}$$

where C is an arbitrary constant vector.

If, in addition, we are given the initial condition $\mathbf{X}(t_0) = \mathbf{X}_0$, the solution to the initial-value problem $\begin{cases} \mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t) \\ \mathbf{X}(t_0) = \mathbf{X}_0 \end{cases}$ is

$$\mathbf{X} = \int_{t_0}^{t} e^{\mathbf{A}(t-s)}\mathbf{F}(s)\,ds + e^{\mathbf{A}(t-t_0)}\mathbf{X}_0.$$

EXAMPLE 6.4.4: Solve $\mathbf{X}' = \begin{pmatrix} 2 & 5 \\ -4 & -2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \cos 4t \\ \sin 4t \end{pmatrix}$.

SOLUTION: Here, $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -4 & -2 \end{pmatrix}$. We compute $e^{\mathbf{A}t}$ with MatrixExp.

```
In[1250]:= a = ( 2   5
                -4  -2 );

In[1251]:= expa = MatrixExp[at]//ExpToTrig//
              FullSimplify;

           MatrixForm[expa]
```

$$\text{Out[1251]=} \begin{pmatrix} \text{Cos}[4\,t] + \frac{1}{2}\,\text{Sin}[4\,t] & \frac{5}{4}\,\text{Sin}[4\,t] \\ -\text{Sin}[4\,t] & \text{Cos}[4\,t] - \frac{1}{2}\,\text{Sin}[4\,t] \end{pmatrix}$$

A general solution of the system is then given by $\mathbf{X} = e^{\mathbf{A}t} \int e^{-\mathbf{A}t}\mathbf{F}(t)\,dt$ $+e^{\mathbf{A}t}\mathbf{C}$. We compute $e^{-\mathbf{A}t}$, $\int e^{-\mathbf{A}t}\mathbf{F}(t)\,dt$, and $e^{\mathbf{A}t} \int e^{-\mathbf{A}t}\mathbf{F}(t)\,dt$.

```
In[1252]:= invexpa = Inverse[expa]//Simplify;

           MatrixForm[invexpa]
```

$$\text{Out[1252]=} \begin{pmatrix} \text{Cos}[4\,t] - \frac{1}{2}\,\text{Sin}[4\,t] & -\frac{5}{4}\,\text{Sin}[4\,t] \\ \text{Sin}[4\,t] & \text{Cos}[4\,t] + \frac{1}{2}\,\text{Sin}[4\,t] \end{pmatrix}$$

```
In[1253]:= f[t_] = ( Cos[4t]
                     Sin[4t] );

In[1254]:= step1 = invexpa.f[t]//Simplify;

           MatrixForm[step1]
```

$$\text{Out[1254]=} \begin{pmatrix} \frac{1}{8}\,(-1 + 9\,\text{Cos}[8\,t] - 2\,\text{Sin}[8\,t]) \\ \frac{1}{4} - \frac{1}{4}\,\text{Cos}[8\,t] + \text{Sin}[8\,t] \end{pmatrix}$$

```
In[1255]:= step2 = Integrate[step1, t]//Simplify;

           MatrixForm[step2]
```

$$\text{Out[1255]=} \begin{pmatrix} \frac{1}{64}\,(-8\,t + 2\,\text{Cos}[8\,t] + 9\,\text{Sin}[8\,t]) \\ \frac{1}{32}\,(8\,t - 4\,\text{Cos}[8\,t] - \text{Sin}[8\,t]) \end{pmatrix}$$

```
In[1256]:= step3 = expa.step2//Simplify;

           MatrixForm[step3]
```

$$\text{Out[1256]=} \begin{pmatrix} \frac{1}{64}\,((2 - 8\,t)\,\text{Cos}[4\,t] + (9 + 16\,t)\,\text{Sin}[4\,t]) \\ \frac{1}{32}\,((-4 + 8\,t)\,\text{Cos}[4\,t] - \text{Sin}[4\,t]) \end{pmatrix}$$

Then, we form our general solution.

```
In[1257]:= gensol = step3 + expa.(c1
                                  c2)//Simplify
```

$$Out[1257]= \left\{\left\{\frac{1}{64}((2-8t)\,Cos[4t]+(9+16t)\,Sin[4t]) + \left(Cos[4t]+\frac{1}{2}Sin[4t]\right)c_1+\frac{5}{4}Sin[4t]\,c_2\right\}, \left\{\frac{1}{32}((-4+8t)\,Cos[4t]-Sin[4t]) - Sin[4t]\,c_1+\left(Cos[4t]-\frac{1}{2}Sin[4t]\right)c_2\right\}\right\}$$

To graph the solution parametrically for various values of the arbitrary constant, we use `Flatten` to convert `gensol` to a list of the form $\{x(t), y(t)\}$.

```
In[1258]:= step1 = Flatten[gensol]
```

$$Out[1258]= \left\{\frac{1}{64}((2-8t)\,Cos[4t]+(9+16t)\,Sin[4t]) + \left(Cos[4t]+\frac{1}{2}Sin[4t]\right)c_1+\frac{5}{4}Sin[4t]\,c_2, \frac{1}{32}((-4+8t)\,Cos[4t]-Sin[4t]) - Sin[4t]\,c_1+\left(Cos[4t]-\frac{1}{2}Sin[4t]\right)c_2\right\}$$

Next, we use `Table` together with `Flatten` to create a set of parametric functions that we will graph with `ParametricPlot`.

```
In[1259]:= toplot =
             Flatten[Table[step1, {c1, -1, 1},
               {c2, -1, 1}], 1];
```

Now, we define `paramgraph`. Given a list of the form $\{x(t), y(t)\}$, `paramgraph` parametrically graphs $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ for $-3\pi \le t \le 3\pi$. The resulting graphics object is not displayed.

```
In[1260]:= paramgraph[list_] :=
             ParametricPlot[list, {t, -3 π, 3π},
               PlotRange → {{-5, 5}, {-5, 5}},
               Ticks → {{-5, 5}, {-5, 5}},
               AspectRatio → 1,
               DisplayFunction → Identity]
```

We then use `Map` to apply `paramgraph` to the list of parametric functions `toplot`.

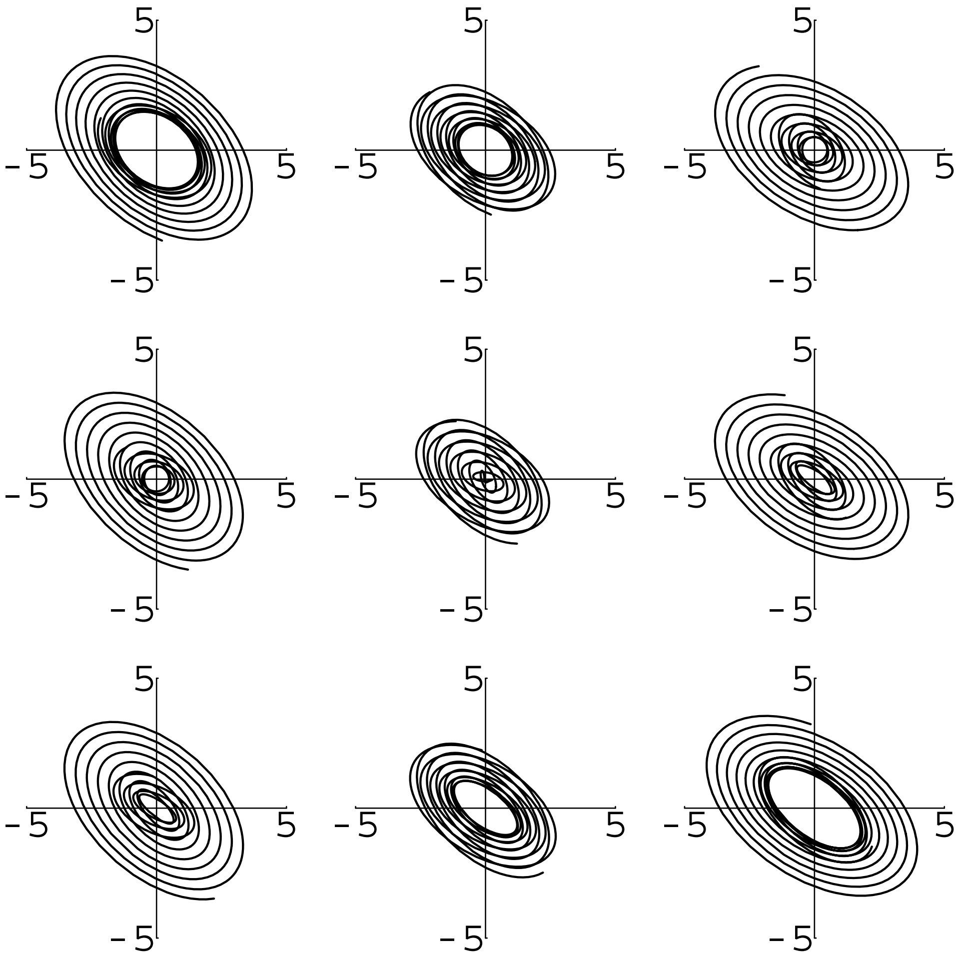


Figure 6-28 Parametric plots of various solutions to the nonhomogeneous system

```
In[1261]:= somegraphs = Map[paramgraph, toplot]

Out[1261]= {-Graphics-, -Graphics-, -Graphics-,
            -Graphics-, -Graphics-, -Graphics-,
            -Graphics-, -Graphics-, -Graphics-}
```

The resulting list of nine graphics objects is partitioned into three element subsets with `Partition`. All nine graphs are then shown together using `Show` and `GraphicsArray` in Figure 6-28.

```
In[1262]:= toshow = Partition[somegraphs, 3];
            Show[GraphicsArray[toshow]]
```

■

If a system of differential equations contains derivatives of order greater than one, we can often rewrite it as a system of first-order equations.

In Chapter 8, we will also see that Laplace transforms can often be used to solve systems of this type.

EXAMPLE 6.4.5: Solve $\begin{cases} -2\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = 0 \\ \dfrac{d^2y}{dt^2} + y - \dfrac{dx}{dt} = \cos t \\ x(0) = 2,\ x'(0) = 1,\ y(0) = 1,\ y'(0) = 2. \end{cases}$

SOLUTION: To rewrite the system as a system of first-order equations, we let $z = dx/dt$ and $w = dy/dt$. Then, $dz/dt = d^2x/dt^2$ and $dw/dt = d^2y/dt^2$. Substituting into the first equation we have $-2\,dz/dt - 2w = 0$ so $dz/dt = -w$. Similarly, substituting into the second equation yields $dw/dt + y - z = \cos t$ so $dw/dt = -y + z + \cos t$. Therefore, the original system is equivalent to the system of first-order equations

$$\begin{cases} dx/dt = z \\ dy/dt = w \\ dz/dt = -w \\ dw/dt = -y + z + \cos t \\ x(0) = 2,\ y(0) = 1,\ z(0) = 1,\ w(0) = 2. \end{cases}$$

In matrix form, the initial-value problem is equivalent to $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, $\mathbf{X}(0) = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, where

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{F}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix}.$$

Using the exponential matrix, the solution to the initial-value problem is given by

$$\mathbf{X} = \int_0^t e^{\mathbf{A}(t-s)}\mathbf{F}(s)\,ds + e^{\mathbf{A}t}\begin{pmatrix}2\\1\\1\\2\end{pmatrix}.$$

First, we define $\mathbf{A}$ and then use `MatrixExp` together with `ExpToTrig` and `FullSimplify` to compute $e^{\mathbf{A}t}$.

```
In[1263]:= a = {{0, 0, 1, 0}, {0, 0, 0, 1},
               {0, 0, 0, -1}, {0, -1, 1, 0}};

In[1264]:= expa = MatrixExp[at]//ExpToTrig//
                  FullSimplify;

           MatrixForm[expa]
Out[1264]=
```

$$\begin{pmatrix} 1 & \frac{1}{4}\left(2t-\sqrt{2}\,\text{Sin}\left[\sqrt{2}\,t\right]\right) & \frac{1}{4}\left(2t+\sqrt{2}\,\text{Sin}\left[\sqrt{2}\,t\right]\right) & -\text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 \\ 0 & \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 & \frac{\text{Sin}\left[\sqrt{2}\,t\right]}{\sqrt{2}} \\ 0 & \text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2 & \text{Cos}\left[\frac{t}{\sqrt{2}}\right]^2 & -\frac{\text{Sin}\left[\sqrt{2}\,t\right]}{\sqrt{2}} \\ 0 & -\frac{\text{Sin}\left[\sqrt{2}\,t\right]}{\sqrt{2}} & \frac{\text{Sin}\left[\sqrt{2}\,t\right]}{\sqrt{2}} & \text{Cos}\left[\sqrt{2}\,t\right] \end{pmatrix}$$

The matrix $e^{\mathbf{A}(t-s)}$ is obtained by replacing each occurrence of t in $e^{\mathbf{A}t}$ by $t-s$.

```
In[1265]:= expats = expa /. t- > t - s;
```

Next, we compute $e^{\mathbf{A}(t-s)}\mathbf{F}(s)$ and integrate the result.

```
In[1266]:= f[t_] = {{0}, {0}, {0}, {Cos[t]}};

           MatrixForm[f[t]]
```

$$\text{Out[1266]=}\begin{pmatrix}0\\0\\0\\\text{Cos}[t]\end{pmatrix}$$

```
In[1267]:= tointegrate = expats.f[s]//Simplify;

           MatrixForm[tointegrate]
```

$$Out[1267]= \begin{pmatrix} -\text{Cos}[s]\,\text{Sin}\left[\frac{-s+t}{\sqrt{2}}\right]^2 \\ \frac{\text{Cos}[s]\,\text{Sin}\left[\sqrt{2}\,(-s+t)\right]}{\sqrt{2}} \\ -\frac{\text{Cos}[s]\,\text{Sin}\left[\sqrt{2}\,(-s+t)\right]}{\sqrt{2}} \\ \text{Cos}[s]\,\text{Cos}\left[\sqrt{2}\,(-s+t)\right] \end{pmatrix}$$

In[1268] := **step2 = Simplify[$\int_0^t$ tointegrated s];**

MatrixForm[step2]

$$Out[1268]= \begin{pmatrix} -\text{Sin}[t]+\frac{\text{Sin}\left[\sqrt{2}\,t\right]}{\sqrt{2}} \\ \text{Cos}[t]-\text{Cos}\left[\sqrt{2}\,t\right] \\ -\text{Cos}[t]+\text{Cos}\left[\sqrt{2}\,t\right] \\ -\text{Sin}[t]+\sqrt{2}\,\text{Sin}\left[\sqrt{2}\,t\right] \end{pmatrix}$$

Finally, we form the solution to the initial-value problem. Note that the first and second rows correspond to x and y, respectively.

```
In[1269] := x0 = {{2}, {1}, {1}, {2}};

In[1270] := sol = step2 + expa.x0//Simplify;

            MatrixForm[sol]
```

$$Out[1270]= \begin{pmatrix} 2+t-\text{Sin}[t]-2\,\text{Sin}\left[\frac{t}{\sqrt{2}}\right]^2+\frac{\text{Sin}\left[\sqrt{2}\,t\right]}{\sqrt{2}} \\ 1+\text{Cos}[t]-\text{Cos}\left[\sqrt{2}\,t\right]+\sqrt{2}\,\text{Sin}\left[\sqrt{2}\,t\right] \\ 1-\text{Cos}[t]+\text{Cos}\left[\sqrt{2}\,t\right]-\sqrt{2}\,\text{Sin}\left[\sqrt{2}\,t\right] \\ 2\,\text{Cos}\left[\sqrt{2}\,t\right]-\text{Sin}[t]+\sqrt{2}\,\text{Sin}\left[\sqrt{2}\,t\right] \end{pmatrix}$$

We confirm that the initial conditions are satisfied by graphing $x(t)$ and $y(t)$ together in Figure 6-29 and parametrically in Figure 6-30.

In[1271] := **Plot[**

$$\left\{1+t+\text{Cos}\left[\sqrt{2}\,t\right]-\text{Sin}[t]+\frac{\text{Sin}\left[\sqrt{2}\,t\right]}{\sqrt{2}},\right.$$

$$\left.1+\text{Cos}[t]-\text{Cos}\left[\sqrt{2}\,t\right]+\sqrt{2}\,\text{Sin}\left[\sqrt{2}\,t\right]\right\},$$

```
  {t, 0, 12}, PlotRange- > {-2, 10},
  PlotStyle- > {GrayLevel[0],
  GrayLevel[0.5]},
  AspectRatio- > 1]
```

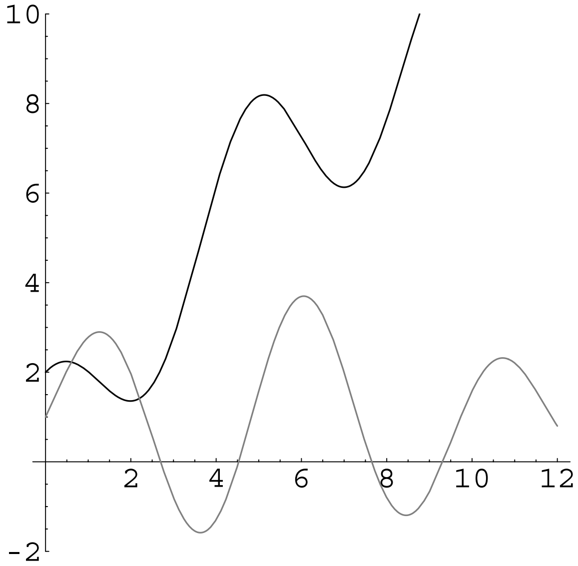


Figure 6-29 $x(t)$ (in black) and $y(t)$ (in gray)

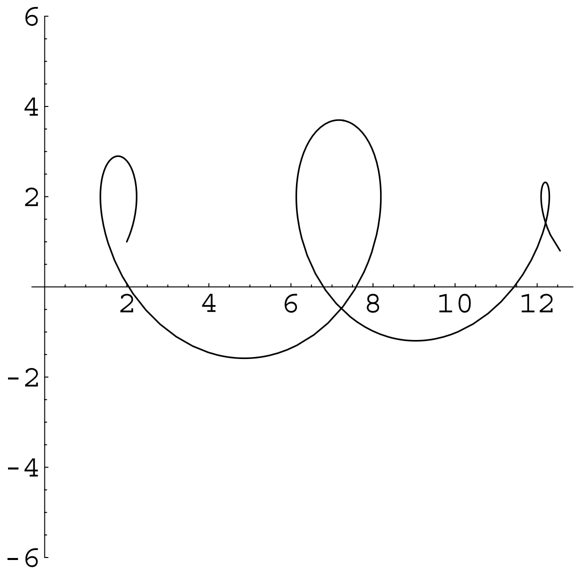


Figure 6-30 Parametric plot of $x(t)$ versus $y(t)$

```
In[1272]:= ParametricPlot[
              {1 + t + Cos[√2 t] - Sin[t] + Sin[√2 t]/√2,
              1 + Cos[t] - Cos[√2 t] + √2 Sin[√2 t]},
              {t, 0, 12}, PlotRange- > {-6, 6},
              AspectRatio- > 1]
```

■

6.5 Numerical Methods

Because it may be difficult or even impossible to construct an explicit solution to some systems of differential equations, we now turn our attention to discussing some numerical methods that are used to construct numerical solutions to systems of differential equations.

6.5.1 Built-In Methods

Numerical approximations of solutions to systems of ordinary differential equations can be obtained with `NDSolve`. This command is particularly useful when working with nonlinear systems of equations for which `DSolve` alone is unable to find an explicit or implicit solution.

EXAMPLE 6.5.1: Consider the nonlinear system of equations

$$\begin{cases} x' = \mu x + y - x\left(x^2 + y^2\right) \\ y' = \mu y - x - y\left(x^2 + y^2\right). \end{cases} \tag{6.17}$$

(a) Graph the direction field associated with the system for $\mu = 2$, 1, 1/4, and $-1/2$. (b) For each value of μ in (a), approximate the solution that satisfies the initial conditions $x(0) = 0$ and $y(0) = 1/2$. Use each numerical solution to approximate $x(5)$ and $y(5)$.

SOLUTION: After loading the **PlotField** package, we define the function `dfield`. Given μ, `dfield[`μ`]` graphs the direction field associated with the system (6.17) on the rectangle $[-2, 2] \times [-2, 2]$. The resulting graphics object is not displayed because we include the option `DisplayFunction->Identity` in the `PlotVectorField` command.

```
In[1273]:= << Graphics`PlotField`

In[1274]:= dfield[μ_] :=
              PlotVectorField[
                {μx + y - x (x² + y²), μy - x - y (x² + y²)},
                {x, -2, 2}, {y, -2, 2},
                ScaleFunction → (1&), Axes → Automatic,
                AxesOrigin → {0, 0}, PlotPoints → 20,
                DefaultColor → GrayLevel[0.5],
                DisplayFunction → Identity];
```

We use `dfield` to graph the direction field associated with the system for $\mu = 2, 1, 1/4$, and $-1/2$.

```
In[1275]:= pvfa = dfield[2];

           pvfb = dfield[1];

           pvfc = dfield[1/4];

           pvfd = dfield[-1/2];
```

`Show` together with `GraphicsArray` is used to display all four graphs together in Figure 6-31. The direction field indicates that the behavior of the solutions strongly depends on the value of μ.

```
In[1276]:= Show[GraphicsArray[
              {{pvfa, pvfb}, {pvfc, pvfd}}]]
```

Now, we use `NDSolve` to generate a numerical approximation to the initial-value problem if $\mu = 2$.

```
In[1277]:= sys = {x'[t] == μ x[t] + y[t]
              -x[t] (x[t]² + y[t]²),
                y'[t] == μ y[t] - x[t] - y[t] (x[t]²
                +y[t]²), x[0] == 0, y[0] == 1/2};

In[1278]:= μ = 2;

           sola = NDSolve[sys, {x[t], y[t]}, {t, 0, 10}]
```

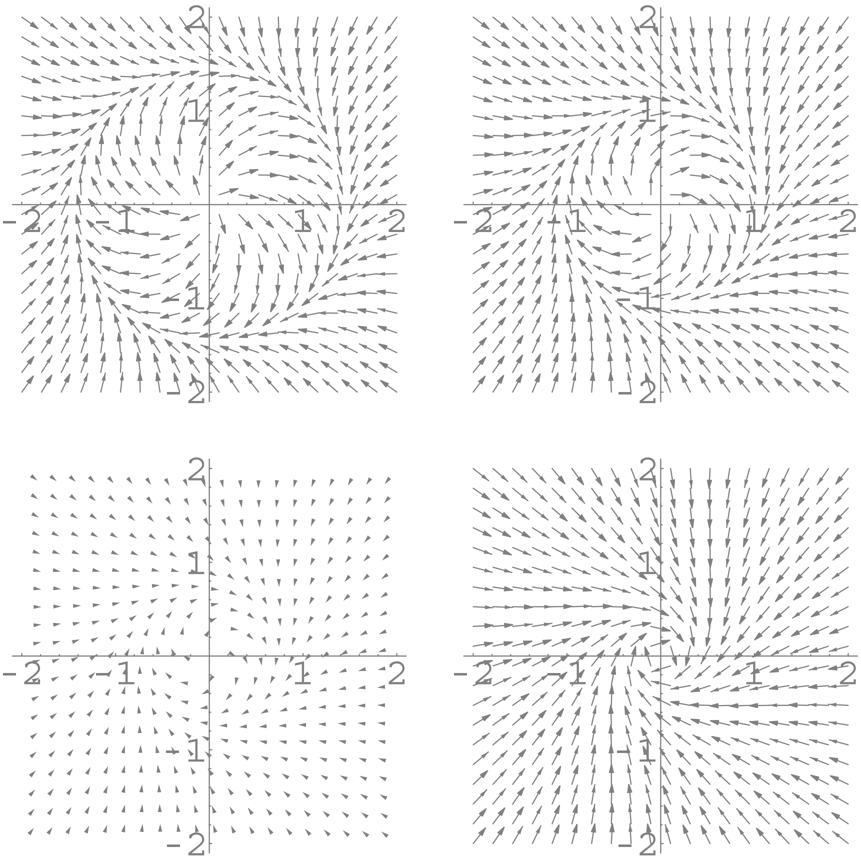


Figure 6-31 The behavior of solutions to the nonlinear system depends strongly on the value of μ

```
Out[1278]= {{x[t] → InterpolatingFunction[
                {{0., 10.}}, <>][t],
             y[t] → InterpolatingFunction[
                {{0., 10.}}, <>][t]}}
```

We use `ReplaceAll` (`/.`) to see that $x(5) \approx -1.35612$ and $y(5) \approx 0.401167$.

```
In[1279]:= sola /. t- > 5
```

```
Out[1279]= {{x[5] → -1.35612, y[5] → 0.401167}}
```

We use `Plot` to graph $x(t)$ and $y(t)$ for $0 \le t \le 10$ in Figure 6-32. Notice that the solution appears to become periodic.

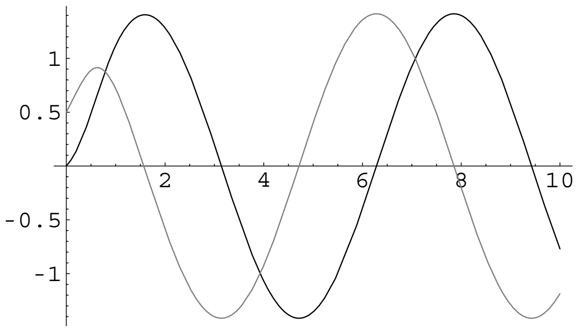


Figure 6-32 If $\mu = 2$, the solution to the initial-value problem approaches a limit cycle

```
In[1280]:= Plot[Evaluate[{x[t],y[t]} /. sola],
             {t,0,10},PlotStyle->{GrayLevel[0],
             GrayLevel[0.5]}]
```

This is further confirmed by graphing the solution parametrically and showing it together with the direction field in Figure 6-33.

```
In[1281]:= ppa = ParametricPlot[{x[t],y[t]} /. sola,
                   {t,0,10},Compiled->False,
                   PlotRange->{{-2,2},{-2,2}},
                   AspectRatio->1,
                   PlotStyle->GrayLevel[0],
                   DisplayFunction->Identity];

In[1282]:= Show[pvfa,ppa,
             DisplayFunction->$DisplayFunction]
```

For the remaining values of μ, we define the function numsol. Given μ, numsol[μ] generates a numerical solution to the initial-value problem.

```
In[1283]:= Clear[μ]

           numsol[μ_] := NDSolve[{x'[t] == μ x[t] + y[t]
                           -x[t] (x[t]^2 + y[t]^2),
                             y'[t] == μ y[t] - x[t] - y[t]
                             (x[t]^2 + y[t]^2), x[0] == 0,
                             y[0] == 1/2}, {x[t], y[t]},
                             {t, 0, 10}];
```

We use numsol to solve each initial-value problem. Note that Mathematica does not display any output because we have included a

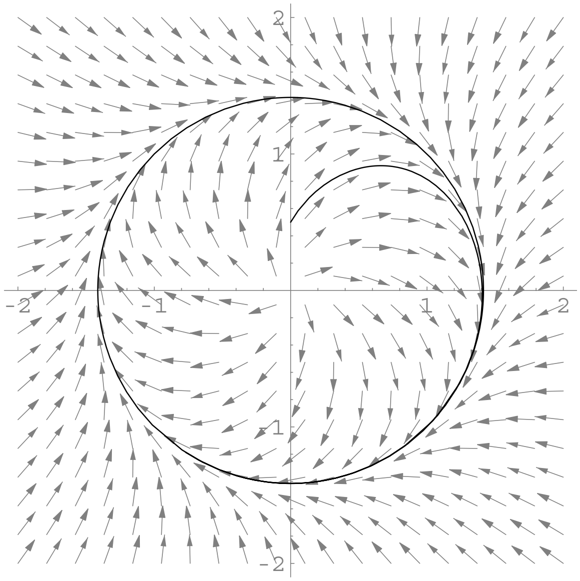


Figure 6-33 The direction field highlights the location of the limit cycle

semi-colon at the end of each command. (You could use `Map[sol[#]&, {1,1/4,-1/2}]` to obtain an equivalent result.)

```
In[1284]:= solb = numsol[1];

            solc = numsol[1/4];

            sold = numsol[-1/2];
```

As before, we use `ReplaceAll` (`/.`) to approximate the value of each solution if $t = 5$.

```
In[1285]:= {solb, solc, sold} /. t- > 5

Out[1285]= {{{x[5] → -0.958856, y[5] → 0.283647}},
             {{x[5] → -0.479456, y[5] → 0.141833}},
             {{x[5] → -0.032173, y[5] → 0.00951576}}}
```

For each numerical solution, we use `Plot` to graph $x(t)$ and $y(t)$ for $0 \leq t \leq 10$ in Figure 6-34. Notice that the solutions corresponding to positive values of μ appear to become periodic while the solution corresponding to the negative value of μ appears to tend towards zero.

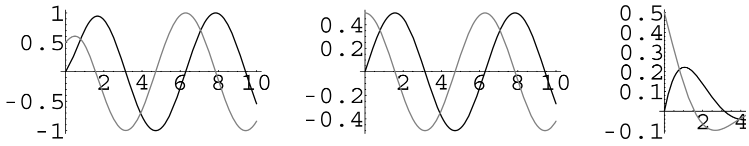


Figure 6-34 If μ is positive, the solutions approach a limit cycle; if μ is negative, the solutions tend to 0

```
In[1286]:= pb = Plot[Evaluate[{x[t],y[t]} /. solb],
                {t,0,10},PlotStyle- >
                {GrayLevel[0],GrayLevel[0.5]},
                DisplayFunction- > Identity];

            pc = Plot[Evaluate[{x[t],y[t]} /. solc],
                {t,0,10},PlotStyle- >
                {GrayLevel[0],GrayLevel[0.5]},
                DisplayFunction- > Identity];

            pd = Plot[Evaluate[{x[t],y[t]} /. sold],
                {t,0,10},PlotStyle- >
                {GrayLevel[0],GrayLevel[0.5]},
                PlotRange- > All,
                DisplayFunction- > Identity];

In[1287]:= Show[GraphicsArray[{pb,pc,pd}]]
```

These results are further confirmed when we graph each solution parametrically and display the graphs with the direction fields generated in (a) in Figure 6-35.

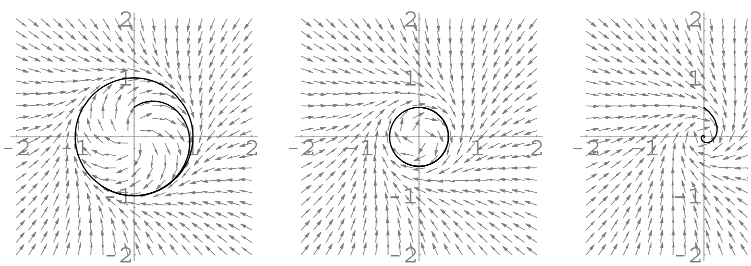


Figure 6-35 If μ is positive, the solutions approach a limit cycle; if μ is negative, the solutions tend to 0

```
In[1288]:= ppb = ParametricPlot[{x[t], y[t]} /. solb,
                {t, 0, 10}, Compiled- > False,
                PlotRange- > {{-2, 2}, {-2, 2}},
                AspectRatio- > 1,
                PlotStyle- > GrayLevel[0],
                DisplayFunction- > Identity];

            ppc = ParametricPlot[{x[t], y[t]} /. solc,
                {t, 0, 10}, Compiled- > False,
                PlotRange- > {{-2, 2}, {-2, 2}},
                AspectRatio- > 1,
                PlotStyle- > GrayLevel[0],
                DisplayFunction- > Identity];

            ppd = ParametricPlot[{x[t], y[t]} /. sold,
                {t, 0, 10}, Compiled- > False,
                PlotRange- > {{-2, 2}, {-2, 2}},
                AspectRatio- > 1,
                PlotStyle- > GrayLevel[0],
                DisplayFunction- > Identity];

In[1289]:= graphb = Show[pvfb, ppb];

            graphc = Show[pvfc, ppc];

            graphd = Show[pvfd, ppd];

In[1290]:= Show[GraphicsArray[
              {graphb, graphc, graphd}]]
```

In the cases corresponding to the positive values of μ, we see in the direction field that all solutions appear to tend to a closed curve. Can we find the curve in each case? If $\mu = 2$, we see that the solution that satisfies $x(0) = 0$ and $y(0) = 1.41$ will be periodic. Similarly, if $\mu = 1/4$, we see that the solution that satisfies $x(0) = 0$ and $y(0) = 0.482$ will be periodic. On the other hand, if $\mu = 1$, we need not approximate the solution. From the graph, we see that the solution that satisfies $x(0) = 0$ and $y(0) = 1$ will be periodic. It is relatively easy to verify that the solution that satisfies these initial conditions is $\begin{cases} x = \cos t \\ y = \sin t. \end{cases}$

■

Application: Controlling the Spread of a Disease

If a person becomes immune to a disease after recovering from it and births and deaths in the population are not taken into account, then the percent of persons susceptible to becoming infected with the disease, $S(t)$, the percent of people in the population infected with the disease, $I(t)$, and the percent of the population recovered and immune to the disease, $R(t)$, can be modeled by the system

Sources: Herbert W. Hethcote, "Three Basic Epidemiological Models," *Applied Mathematical Ecology*, edited by Simon A. Levin, Thomas G. Hallan, and Louis J. Gross, Springer-Verlag (1989), pp. 119–143. Roy M. Anderson and Robert M. May, "Directly Transmitted Infectious Diseases: Control by Vaccination," *Science*, Volume 215 (February 26, 1982), pp. 1053–1060. J. D. Murray, *Mathematical Biology*, Springer-Verlag (1990), pp. 611–618.

$$\begin{cases} S' = -\lambda SI \\ I' = \lambda SI - \gamma I \\ R' = \gamma I \\ S(0) = S_0,\ I(0) = I_0,\ R(0) = 0. \end{cases} \tag{6.18}$$

Because $S(t) + I(t) + R(t) = 1$, once we know $S(t)$ and $I(t)$, we can compute $R(t)$ with $R(t) = 1 - S(t) - I(t)$. This model is called an **SIR model without vital dynamics** because once a person has had the disease the person becomes immune to the disease and because births and deaths are not taken into consideration. This model might be used to model diseases that are **epidemic** to a population: those diseases that persist in a population for short periods of time (less than one year). Such diseases typically include influenza, measles, rubella, and chickenpox.

If $S_0 < \gamma/\lambda$, $I'(0) = \lambda S_0 I_0 - \gamma I_0 < \lambda \frac{\gamma}{\lambda} I_0 - \gamma I_0 = 0$. Thus, the rate of infection immediately begins to decrease; the disease dies out. On the other hand, if $S_0 > \gamma/\lambda$, $I'(0) = \lambda S_0 I_0 - \gamma I_0 > \lambda \frac{\gamma}{\lambda} I_0 - \gamma I_0 = 0$ so the rate of infection first increases; an epidemic results.

Although we cannot find explicit formulas for S, I, and R as functions of t, we can, for example, solve for I in terms of S. The equation $\frac{dI}{dS} = -\frac{(\lambda S - \gamma)I}{\lambda SI} = -1 + \frac{\rho}{S}$, $\rho = \gamma/\lambda$, is separable:

$$\frac{dI}{dS} = -1 + \frac{\rho}{S} \Longrightarrow dI = \left(-1 + \frac{\rho}{S}\right) dS \Longrightarrow I = -S + \rho \ln S + C$$

and applying the initial condition results in

$$I_0 = -S_0 + \rho \ln S_0 + C \Longrightarrow C = I_0 + S_0 - \rho \ln S_0$$

so $I = -S + \rho \ln S + I_0 + S_0 - \rho \ln S_0 \Longrightarrow I + S - \rho \ln S = I_0 + S_0 - \rho \ln S_0$.

When diseases persist in a population for long periods of time, births and deaths must be taken into consideration. If a person becomes immune to a disease after recovering from it and births and deaths in the population are taken into account, then the percent of persons susceptible to becoming infected with the disease, $S(t)$, and the percent of people in the population infected with the disease, $I(t)$, can be modeled by the system

$$\begin{cases} S' = -\lambda SI + \mu - \mu S \\ I' = \lambda SI - \gamma I - \mu I \\ S(0) = S_0,\ I(0) = I_0. \end{cases} \tag{6.19}$$

This model is called an **SIR model with vital dynamics** because once a person has had the disease the person becomes immune to the disease and because births and deaths are taken into consideration. This model might be used to model diseases that are **endemic** to a population: those diseases that persist in a population for long periods of time (10 or 20 years). Smallpox is an example of a disease that was endemic until it was eliminated in 1977. We use `Solve` to see that the solutions to the system of equations

$$\begin{cases} -\lambda SI + \mu - \mu S = 0 \\ \lambda SI - \gamma I - \mu I = 0 \end{cases}$$

are $S = 1$, $I = 0$ and $S = \dfrac{\gamma+\mu}{\lambda}$, $I = \dfrac{\mu\,[\lambda - (\gamma+\mu)]}{\lambda(\gamma+\mu)}$.

```
In[1291]:= eq1 = -λ s i + μ - μ s;
           eq2 = λ s i - γ i - μ i;
```

```
In[1292]:= eqpts = Solve[{eq1 == 0, eq2 == 0}, {s, i}]
```

$$\text{Out[1292]=}\ \left\{\{i \to 0, s \to 1\}, \left\{i \to -\frac{\mu\,(\gamma - \lambda + \mu)}{\lambda\,(\gamma + \mu)}, s \to \frac{\gamma + \mu}{\lambda}\right\}\right\}$$

These two points are called **equilibrium points** because they are constant solutions to the system.

Because $S(t) + I(t) + R(t) = 1$, it follows that $S(t) + I(t) \leq 1$. The following table shows the average infectious period, $1/\gamma$, γ, and typical contact numbers, σ, for several diseases during certain epidemics.

Disease	$1/\gamma$	γ	σ
Measles	6.5	0.153846	14.9667
Chickenpox	10.5	0.0952381	11.3
Mumps	19	0.0526316	8.1
Scarlet fever	17.5	0.0571429	8.5

Let us assume that the average lifetime, $1/\mu$, is 70 so that $\mu = 0.0142857$.

For each of the diseases listed in the previous table, we use the formula $\sigma = \lambda/(\gamma + \mu)$ to calculate the daily contact rate λ.

Disease	λ
Measles	2.51638
Chickenpox	1.23762
Mumps	0.54203
Scarlet fever	0.607143

Diseases like those listed above can be controlled once an effective and inexpensive vaccine has been developed. Since it is virtually impossible to vaccinate everybody against a disease, we would like to know what percentage of a population needs to be vaccinated to eliminate a disease. A population of people has **herd immunity** to a disease means that enough people are immune to the disease so that if it is introduced into the population, it will not spread throughout the population. In order to have herd immunity, an infected person must infect less than one uninfected person during the time the person is infectious. Thus, we must have

$$\sigma S < 1.$$

Since $I + S + R = 1$, when $I = 0$ we have that $S = 1 - R$ and, consequently, herd immunity is achieved when

$$\begin{aligned} \sigma(1 - R) &< 1 \\ \sigma - \sigma R &< 1 \\ -\sigma R &< 1 - \sigma \\ R &> \frac{\sigma - 1}{\sigma} = 1 - \frac{1}{\sigma}. \end{aligned}$$

For each of the diseases listed above, we estimate the minimum percentage of a population that needs to be vaccinated to achieve herd immunity.

Disease	**Minimum value of R to achieve herd immunity**
Measles	0.933186
Chickenpox	0.911505
Mumps	0.876544
Scarlet fever	0.882354

Using the values in the previous tables, for each disease we graph the direction field and several solutions $\begin{cases} S = S(t) \\ I = I(t) \end{cases}$ parametrically. For measles, we proceed as follows. After loading the **PlotField** package, we define μ, γ, σ, and λ. For these values, we graph the direction field associated with the system on the rectangle $[0, 1] \times [0, 1]$. Because $S(t) + I(t) \leq 1$, we are only concerned with solutions of the system that are below the line $S + I = 1$.

```
In[1293]:= << Graphics`PlotField`

           p1 = Plot[1 - x, {x, 0, 1},
                 PlotStyle → Thickness[0.0075],
                 DisplayFunction → Identity];

           μ = 0.0142857;

           γ = 0.153846;

           σ = 14.9667;

           λ = σ (γ + μ);

           eq1 = -λ si + μ - μs;

           eq2 = λsi - γi - μi;
           pvf1 = PlotVectorField[{eq1, eq2},
                   {s, 0, 1}, {i, 0, 1},
                   ScaleFunction → (1&), PlotPoints → 20,
                   DefaultColor → GrayLevel[0.5],
                   DisplayFunction → Identity];
```

Next, we define two lists of ordered pairs and use `Union` to join the two lists. The points in `initconds1` are "close" to the S axis while the points in `initconds2` are close to the I axis. We will graph the solutions that satisfy these initial conditions.

```
In[1294]:= initconds1 = Table[{i/10, 0.01}, {i, 1, 9}];

           initconds2 = Table[{1 - i/10, i/10}, {i, 1, 9}];

           initconds = Union[initconds1, initconds2];
```

Now we define the function `numgraph`. Given an ordered pair (S_0, I_0), `numgraph` generates a numerical solution to the initial-value problem (6.19) and graphs the result.

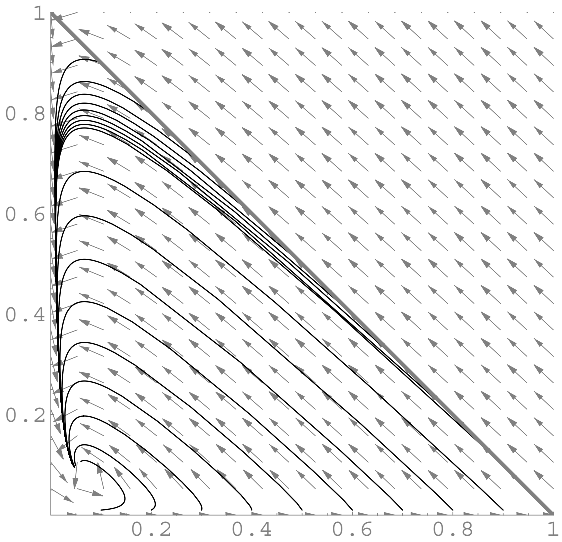


Figure 6-36 Using an SIR model to model measles

```
In[1295]:= numgraph[{s0_, i0_}] := Module[{numsol},
            numsol = NDSolve[{s′[t] == -λ s[t] i[t] + μ - μ s[t],
              i′[t] == λ s[t] i[t] - γ i[t] - μ i[t],
              s[0] == s0, i[0] == i0}, {s[t], i[t]}, {t, 0, 20}];
              ParametricPlot[{s[t], i[t]}/. numsol, {t, 0, 20},
              PlotStyle- > GrayLevel[0], Compiled- > False,
              DisplayFunction- > Identity]]
```

We then use `Map` to apply `numgraph` to the list `initconds`. `Show` is used to display all three graphics objects together in Figure 6-36. In the result, we see that the (nontrivial) solutions approach the equilibrium point.

```
In[1296]:= toshow = Map[numgraph, initconds];

            Show[pvf1, toshow, p1,
              PlotRange- > {{0, 1}, {0, 1}},
              AspectRatio- > 1,
              DisplayFunction- > $DisplayFunction,
              Axes- > Automatic]
```

For the remaining three diseases, we change the values of μ, γ, σ, and λ and reenter the code. Here are the results for chickenpox. See Figure 6-37.

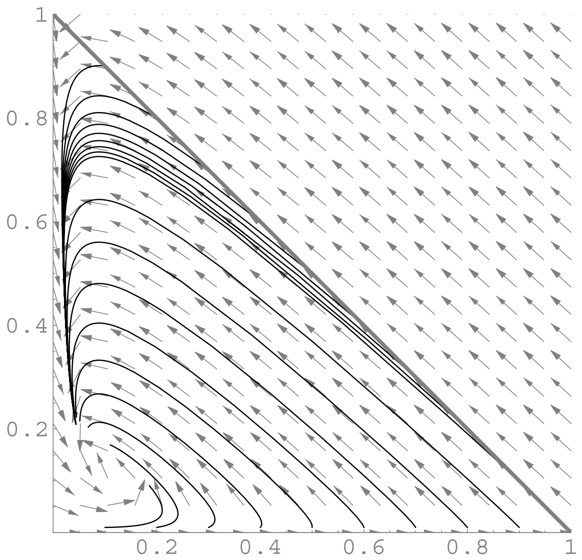


Figure 6-37 Using an SIR model to model chickenpox

```
In[1297]:= μ = 0.0142857;

γ = 0.0952381;

σ = 11.3;

λ = σ(γ + μ);

eq1 = -λ si + μ - μs;

eq2 = λsi - γi - μi;
pvf1 = PlotVectorField[{eq1, eq2},
        {s, 0, 1}, {i, 0, 1},
        ScaleFunction- > (1&), PlotPoints- > 20,
        DefaultColor- > GrayLevel[0.5],
        DisplayFunction- > Identity];

numgraph[{s0_, i0_}] := Module[{numsol},
  numsol = NDSolve[{s'[t] == -λ s[t] i[t]
         +μ - μ s[t], i'[t] == λ s[t] i[t]
         -γ i[t] - μ i[t], s[0] == s0, i[0] == i0},
         {s[t], i[t]}, {t, 0, 20}];
         ParametricPlot[{s[t], i[t]}/.
           numsol, {t, 0, 20},
```

```
In[1297]:= PlotStyle- > GrayLevel[0],
           Compiled- > False,
           DisplayFunction- > Identity]]

In[1298]:= toshow = Map[numgraph, initconds];

           Show[pvf1, toshow, p1,
             PlotRange- > {{0, 1}, {0, 1}},
             AspectRatio- > 1,
             DisplayFunction- > $DisplayFunction,
             Axes- > Automatic]
```

Similar results are obtained for mumps. See Figure 6-38.

```
In[1299]:= μ = 0.0142857;

           γ = 0.0526316;

           σ = 8.1;

           λ = σ(γ + μ);

           eq1 = -λ si + μ - μs;

           eq2 = λsi - γi - μi;
           pvf1 = PlotVectorField[{eq1, eq2},
                   {s, 0, 1}, {i, 0, 1},
                   ScaleFunction- > (1&), PlotPoints- > 20,
                   DefaultColor- > GrayLevel[0.5],
                   DisplayFunction- > Identity];

           numgraph[{s0_, i0_}] := Module[{numsol},
             numsol =
                 NDSolve[
                   {s′[t] == -λ s[t] i[t] + μ - μ s[t],
                   i′[t] == λ s[t] i[t] - γ i[t] - μ i[t],
                   s[0] == s0, i[0] == i0},
                   {s[t], i[t]}, {t, 0, 40}];
                 ParametricPlot[
                   {s[t], i[t]}/.    numsol,
                   {t, 0, 40}, PlotStyle- > GrayLevel[0],
                   Compiled- > False,
                   DisplayFunction- > Identity]]
```

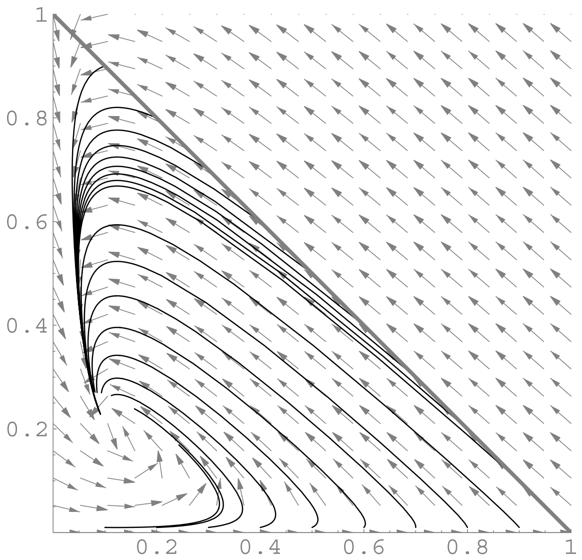


Figure 6-38 Using an SIR model to model mumps

```
In[1300]:= toshow = Map[numgraph, initconds];

           Show[pvf1, toshow, p1,
             PlotRange- > {{0, 1}, {0, 1}},
             AspectRatio- > 1,
             DisplayFunction- > $DisplayFunction,
             Axes- > Automatic]
```

Last, we generate graphs for scarlet fever. See Figure 6-39. In all four cases, we see that all solutions approach the equilibrium point, which indicates that although the epidemic runs its course, the disease is never completely removed from the population.

```
In[1301]:= μ = 0.0142857;

           γ = 0.0571429;

           σ = 8.5;

           λ = σ(γ + μ);

           eq1 = -λ si + μ - μs;

           eq2 = λsi - γi - μi;
```

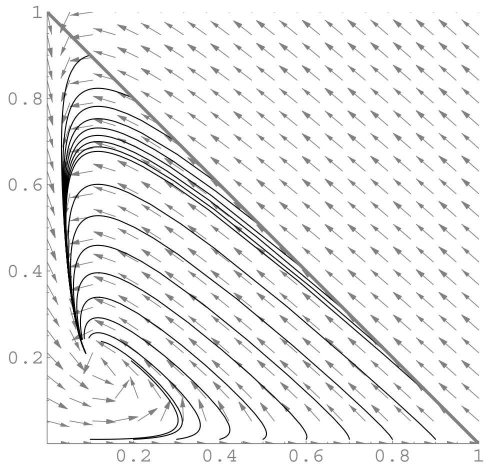


Figure 6-39 Using an SIR model to model scarlet fever

```
In[1301]:= pvf1 = PlotVectorField[{eq1, eq2},
                {s, 0, 1}, {i, 0, 1},
                ScaleFunction- > (1&), PlotPoints- > 20,
                DefaultColor- > GrayLevel[0.5],
                DisplayFunction- > Identity];

            numgraph[{s0_, i0_}] := Module[{numsol},
              numsol =
                NDSolve[
                  {s′[t] == -λ s[t] i[t] + μ - μ s[t],
                  i′[t] == λ s[t] i[t] - γ i[t] - μ i[t],
                  s[0] == s0, i[0] == i0},
                  {s[t], i[t]}, {t, 0, 40}];
                ParametricPlot[
                  {s[t], i[t]}/.   numsol,
                  {t, 0, 40}, PlotStyle- > GrayLevel[0],
                  Compiled- > False,
                  DisplayFunction- > Identity]]

In[1302]:= toshow = Map[numgraph, initconds];

            Show[pvf1, toshow, p1,
              PlotRange- > {{0, 1}, {0, 1}},
              AspectRatio- > 1,
              DisplayFunction- > $DisplayFunction,
              Axes- > Automatic]
```

NDSolve can be used to generate numerical solutions of systems that involve more than one differential equation as well.

EXAMPLE 6.5.2 (FitzHugh–Nagumo Equation): Under certain assumptions, the **FitzHugh–Nagumo equation** that arises in the study of the impulses in a nerve fiber can be written as the system of ordinary differential equations

$$\begin{cases} dV/d\xi = W \\ dW/d\xi = F(V) + R - uW \\ dR/d\xi = \dfrac{\epsilon}{u}(bR - V - a) \\ V(0) = v_0,\ W(0) = W_0,\ R(0) = R_0 \end{cases} \tag{6.20}$$

where $F(V) = \frac{1}{3}V^3 - V$. (a) Graph the solution to the FitzHugh–Nagumo equation that satisfies the initial conditions $V(0) = 1$, $W(0) = 0$, and $R(0) = 1$ if $\epsilon = 0.08$, $a = 0.7$, $b = 0$, and $u = 1$. (b) Graph the solution that satisfies the initial conditions $V(0) = 1$, $W(0) = 0.5$, and $R(0) = 0.5$ if $\epsilon = 0.08$, $a = 0.7$, $b = 0.8$, and $u = 0.6$.

SOLUTION: We begin by defining the function fnsol, which given the appropriate parameter values and initial conditions returns a numerical solution of system (6.20). If {ξ, a, b} is not included after the initial conditions, the default solution is valid for $0 \leq \xi \leq 100$; any options included are passed to the NDSolve command. In this case, we use lower-case letters to avoid any ambiguity with built-in Mathematica functions.

```
In[1303]:= Clear[fnsol]

           fnsol[ϵ_, a_, b_, u_][{v0_, w0_, r0_},
             ξs_ : {ξ, 0, 100}, opts___] :=
             NDSolve[{v′[ξ] == w[ξ], w′[ξ] ==
             1/3v[ξ]^3 - v[ξ] + r[ξ] - u w[ξ], r′[ξ] ==
             ϵ/u (b r[ξ] - v[ξ] - a), v[0] == v0, w[0] ==
             w0, r[0] == r0}, {v[ξ],
             w[ξ], r[ξ]}, ξs, opts]
```

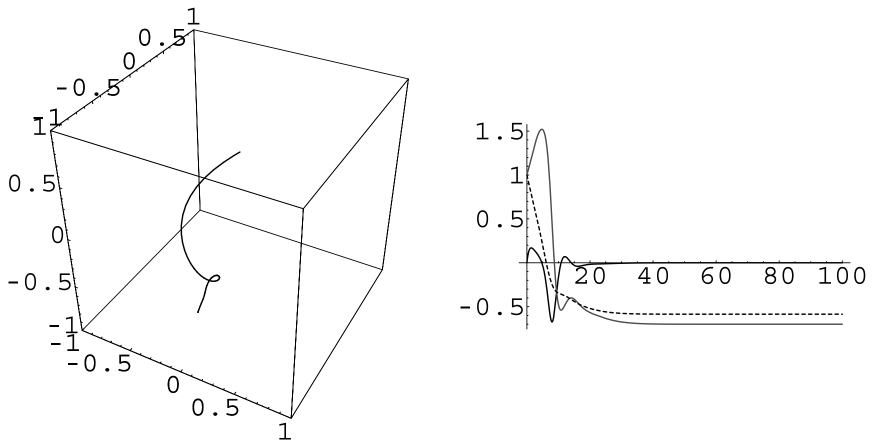
For (a), we enter

```
In[1304]:= sola = fnsol[0.08, 0.7, 0, 1][{1, 0, 1}]
```

```
Out[1304]= {{v[ξ] → InterpolatingFunction[{{0.,100.}},
                <>][ξ],
            w[ξ] → InterpolatingFunction[{{0.,100.}},
                <>][ξ],
            r[ξ] → InterpolatingFunction[{{0.,100.}},
                <>][ξ]}}
```

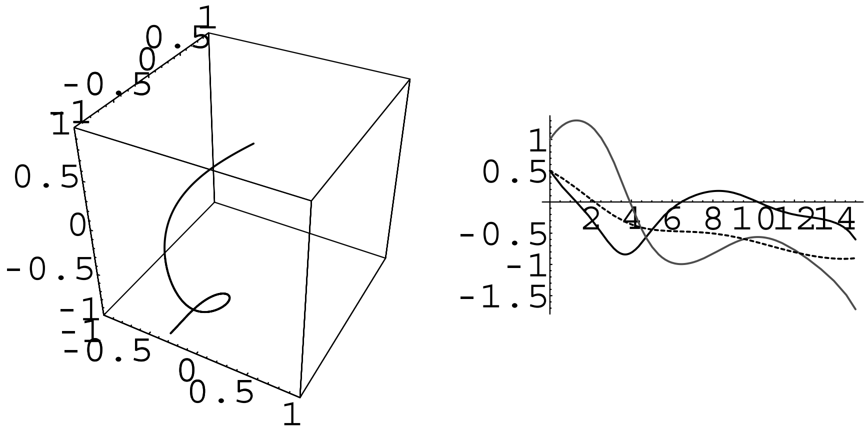
We then graph the solution functions parametrically with `ParametricPlot3D` and then individually with `Plot`. The option `PlotPoints->200` is included in the `ParametricPlot3D` to help assure that the resulting graph is smooth. Using `Show` and `GraphicsArray`, both plots are shown in Figure 6-40.

```
In[1305]:= pp1 = ParametricPlot3D[
                Evaluate[{w[ξ],v[ξ],r[ξ]}/.sola],
                {ξ,0,100},PlotRange → {{-1,1},
                {-1,1},{-1,1}},BoxRatios → {1,1,1},
                PlotPoints → 500,
                DisplayFunction → Identity];

In[1306]:= pa = Plot[Evaluate[{w[ξ],v[ξ],r[ξ]}/.sola],
               {ξ,0,100},PlotStyle → {GrayLevel[0],
               GrayLevel[0.3],Dashing[{0.01}]},
               DisplayFunction → Identity];

In[1307]:= Show[GraphicsArray[{pp1,pa}]]
```



(a) (b)

Figure 6-40 For $0 \leq \xi \leq 50$: (a) parametric plot of W versus V versus R; (b) W (in black), V (in gray), and R (dashed)



(a) (b)

Figure 6-41 For $0 \le \xi \le 15$: (a) parametric plot of W versus V versus R; (b) W (in black), V (in gray), and R (dashed)

For (b), we specify that we want the solution to be valid for $0 \le \xi \le 15$ so enter

```
In[1308]:= solb = fnsol[0.08, 0.7, 0.8, 0.6]
              [{1, 0.5, 0.5}, {ξ, 0, 15}]
Out[1308]= {{v[ξ] → InterpolatingFunction[{{0., 15.}},
                  <>][ξ],
             w[ξ] → InterpolatingFunction[
                  {{0., 15.}}, <>][ξ],
             r[ξ] → InterpolatingFunction[{{0., 15.}},
                  <>][ξ]}}
```

Parametric plots and individual plots are generated in the same way as in (a). See Figure 6-41.

```
In[1309]:= pp2 = ParametricPlot3D[Evaluate[
                {w[ξ], v[ξ], r[ξ]}/.solb], {ξ, 0, 15},
                PlotRange → {{-1, 1}, {-1, 1}, {-1, 1}},
                BoxRatios → {1, 1, 1}, PlotPoints → 500,
                DisplayFunction → Identity];

In[1310]:= pb = Plot[Evaluate[{w[ξ], v[ξ], r[ξ]}/.solb],
               {ξ, 0, 15}, PlotStyle → {GrayLevel[0],
               GrayLevel[0.3], Dashing[{0.01}]},
               DisplayFunction → Identity];

In[1311]:= Show[GraphicsArray[{pp2, pb}]]
```

■

In other cases, you may wish to implement your own numerical algorithms to approximate solutions of differential equations. We briefly discuss two familiar methods (Euler's method and the Runge–Kutta method) and illustrate how to implement these algorithms using Mathematica. Details regarding these and other algorithms, including discussions of the error involved in implementing them, can be found in most numerical analysis texts or other references like the Zwillinger's *Handbook of Differential Equations*, [29].

6.5.2 Euler's Method

Euler's method for approximation that was discussed for first-order equations may be extended to include systems of first-order equations. The initial-value problem

$$\begin{cases} dx/dt = f(t, x, y) \\ dy/dt = g(t, x, y) \\ x(t_0) = x_0, \; y(t_0) = y_0 \end{cases} \tag{6.21}$$

is approximated at each step by the recursive relationship based on the Taylor expansion of x and y:

$$\begin{cases} x_{n+1} = x_n + hf(t_n, x_n, x_n) \\ y_{n+1} = y_n + hg(t_n, x_n, x_n) \end{cases} \tag{6.22}$$

where $t_n = t_0 + nh$, $n = 0, 1, 2, \ldots$.

EXAMPLE 6.5.3: Use Euler's method with $h = 0.1$ to approximate the solution of the initial-value problem

$$\begin{cases} dx/dt = x - y + 1 \\ dy/dt = x + 3y + e^{-t} \\ x(0) = 0, \; y(0) = 1. \end{cases}$$

Compare these results to those of the exact solution of the system of equations.

SOLUTION: We use the same notation as in equations (6.21) and (6.22): $f(x, y) = x - y + 1$, $g(x, y) = dy/dt = x + 3y + e^{-t}$, $t_0 = 0$, $x_0 = 0$, and $y_0 = 1$, so we use the formulas

$$\begin{cases} x_{n+1} = x_n + h\,(x_n - y_n + 1) \\ y_{n+1} = y_n + h\left(x_n + 3y_n + e^{-t_n}\right) \end{cases}$$

where $t_n = 0.1n$, $n = 0, 1, 2, \ldots$.

For example, if $n = 0$, then

$$\begin{cases} x_1 = x_0 + h\,(x_0 - y_0 + 1) = 0 \\ y_1 = y_0 + h\left(x_0 + 3y_0 + e^{-t_0}\right) = 1.4. \end{cases}$$

The exact solution of this initial-value problem is found to be

$$\begin{cases} x(t) = -\frac{3}{4} - \frac{1}{9}e^{-t} + \frac{31}{36}e^{2t} - \frac{11}{6}te^{2t} \\ y(t) = \frac{1}{4} - \frac{2}{9}e^{-t} + \frac{35}{36}e^{2t} + \frac{11}{6}te^{2t} \end{cases}$$

with `DSolve`.

```
In[1312]:= partsol =
              DSolve[{x'[t] == x[t] - y[t] + 1,
                y'[t] == x[t] + 3 y[t] + Exp[-t],
                x[0] == 0, y[0] == 1}, {x[t], y[t]}, t]
```

$$\text{Out[1312]=}\ \left\{\left\{x[t] \to -\frac{1}{36}\, e^{-t}\,(4 + 27\,e^{t} - 31\,e^{3t} + 66\,e^{3t}\,t),\ y[t] \to \frac{1}{36}\, e^{-t}\,(-8 + 9\,e^{t} + 35\,e^{3t} + 66\,e^{3t}\,t)\right\}\right\}$$

$$\text{In[1313]:= xex[t\_]} = -\frac{3}{4} - \frac{\text{Exp[-t]}}{9} + \frac{31\,\text{Exp[2t]}}{36} - \frac{11}{6}\,\text{t Exp[2t];}$$

$$\text{In[1314]:= yex[t\_]} = \frac{1}{4} - \frac{2\,\text{Exp[-t]}}{9} + \frac{35\,\text{Exp[2t]}}{36} + \frac{11}{6}\,\text{t Exp[2t];}$$

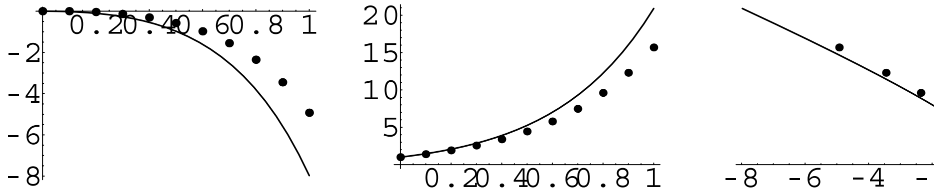
We display the results obtained with this method (in columns three and five) and compare them to the actual function values (in columns four and six).

```
In[1315]:= Clear[f, g, t, h, x, y]

           f[t_, x_, y_] = x - y + 1;

           g[t_, x_, y_] = x + 3y + Exp[-t];

           h = 0.1;

           t[n_] := t0 + nh;

           t0 = 0;

           xe[n_] :=
             xe[n] =
               xe[n - 1]
               +h f[t[n - 1], xe[n - 1], ye[n - 1]];

           ye[n_] :=
             ye[n] =
               ye[n - 1]
               +h g[t[n - 1], xe[n - 1], ye[n - 1]];

           xe[0] = 0;

           ye[0] = 1;


In[1316]:= Table[{n, t[n], xe[n], xex[t[n]],
             ye[n], yex[t[n]]}, {n, 0, 10}]//
             TableForm
```

Out[1316]=

0	0	0	0	1	1
1	0.1	0	-0.0226978	1.4	1.46032
2	0.2	-0.04	-0.103346	1.91048	2.06545
3	0.3	-0.135048	-0.265432	2.5615	2.85904
4	0.4	-0.304703	-0.540105	3.39053	3.89682
5	0.5	-0.574227	-0.968408	4.44425	5.24975
6	0.6	-0.976074	-1.60412	5.78076	7.00806
7	0.7	-1.55176	-2.51737	7.47226	9.28638
8	0.8	-2.35416	-3.79926	9.60842	12.23
9	0.9	-3.45042	-5.56767	12.3005	16.0232
10	1.	-4.9255	-7.97468	15.6862	20.8987

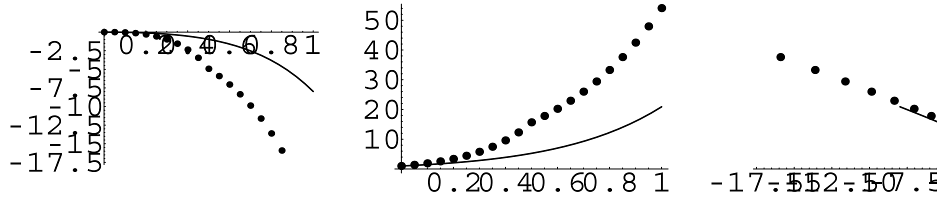
We also graph the approximation with the actual solution in Figure 6-42.



(a) (b) (c)

Figure 6-42 Euler's method using $h = 0.1$: (a) comparison of x_n to $x(t)$; (b) comparison of y_n to $y(t)$; (c) comparison of (x_n, y_n) to $(x(t), y(t))$

```
In[1317]:= xs = Table[{t[n], xe[n]}, {n, 0, 10}];

In[1318]:= p1 = ListPlot[xs,
                 PlotStyle- > PointSize[0.03],
                 DisplayFunction- > Identity];

            p2 = Plot[xex[t], {t, 0, 1},
                 DisplayFunction- > Identity];

            p3 = Show[p1, p2];

In[1319]:= ys = Table[{t[n], ye[n]}, {n, 0, 10}];


In[1320]:= p4 = ListPlot[ys,
                 PlotStyle- > PointSize[0.03],
                 DisplayFunction- > Identity];

            p5 = Plot[yex[t], {t, 0, 1},
                 DisplayFunction- > Identity];

            p6 = Show[p4, p5];

In[1321]:= both = Table[{xe[n], ye[n]}, {n, 0, 10}];

In[1322]:= p7 = ListPlot[both,
                 PlotStyle- > PointSize[0.03],
                 DisplayFunction- > Identity];

            p8 = ParametricPlot[{xex[t], yex[t]},
                 {t, 0, 1}, DisplayFunction- >
                 Identity];

            p9 = Show[p7, p8];

In[1323]:= Show[GraphicsArray[{p3, p6, p9}]]
```



(a) (b) (c)

Figure 6-43 Euler's method using $h = 0.05$: (a) comparison of x_n to $x(t)$; (b) comparison of y_n to $y(t)$; (c) comparison of (x_n, y_n) to $(x(t), y(t))$

Because the accuracy of this approximation diminishes as t increases, we attempt to improve the approximation by decreasing the increment size. We do this next by entering the value $h = 0.05$ and repeating the procedure which was followed above. See Figure 6-43.

```
In[1324]:= Clear[f, g, t, h, x, y]

           f[t_, x_, y_] = x - y + 1;

           g[t_, x_, y_] = x + 3y + Exp[-t];

           h = 0.05;

           t[n_] := t0 + nh;

           t0 = 0;

           xe[n_] :=
             xe[n] =
               xe[n - 1]
               +h f[t[n - 1], xe[n - 1], ye[n - 1]];

           ye[n_] :=
             ye[n] =
               ye[n - 1]
               +h g[t[n - 1], xe[n - 1], ye[n - 1]];

           xe[0] = 0;

           ye[0] = 1;

In[1325]:= Table[{t[n], xe[n], xex[t[n]], ye[n],
             yex[t[n]]}, {n, 0, 20}]//TableForm
```

Out[1325]=

0	0	0	1	1
0.05	0	-0.00532454	1.4	1.21439
0.1	-0.04	-0.0226978	1.91048	1.46032
0.15	-0.135048	-0.054467	2.5615	1.74231
0.2	-0.304703	-0.103346	3.39053	2.06545
0.25	-0.574227	-0.172465	4.44425	2.43552
0.3	-0.976074	-0.265432	5.78076	2.85904
0.35	-1.55176	-0.386392	7.47226	3.34338
0.4	-2.35416	-0.540105	9.60842	3.89682
0.45	-3.45042	-0.732029	12.3005	4.52876
0.5	-4.9255	-0.968408	15.6862	5.24975
0.55	-5.90609	-1.25639	17.8232	6.07171
0.6	-7.04255	-1.60412	20.2302	7.00806
0.65	-8.35619	-2.02091	22.9401	8.07394
0.7	-9.871	-2.51737	25.9894	9.28638
0.75	-11.614	-3.10558	29.419	10.6645
0.8	-13.6157	-3.79926	33.2748	12.23
0.85	-15.9102	-4.61405	37.6077	14.0071
0.9	-18.5361	-5.56767	42.4747	16.0232
0.95	-21.5366	-6.68027	47.9395	18.3088
1.	-24.9604	-7.97468	54.0729	20.8987

```
In[1326]:= xs = Table[{t[n], xe[n]}, {n, 0, 20}];

In[1327]:= p1 = ListPlot[xs,
                PlotStyle- > PointSize[0.03],
                DisplayFunction- > Identity];

            p2 = Plot[xex[t], {t, 0, 1},
                DisplayFunction- > Identity];

            p3 = Show[p1, p2];

In[1328]:= ys = Table[{t[n], ye[n]}, {n, 0, 20}];

In[1329]:= p4 = ListPlot[ys,
                PlotStyle- > PointSize[0.03],
                DisplayFunction- > Identity];

            p5 = Plot[yex[t], {t, 0, 1},
                DisplayFunction- > Identity];

            p6 = Show[p4, p5];

In[1330]:= both = Table[{xe[n], ye[n]}, {n, 0, 20}];

In[1331]:= p7 = ListPlot[both,
                PlotStyle- > PointSize[0.03],
                DisplayFunction- > Identity];
```

```
In[1331]:= p8 = ParametricPlot[{xex[t], yex[t]},
                {t, 0, 1}, DisplayFunction- >
                Identity];

            p9 = Show[p7, p8];
In[1332]:= Show[GraphicsArray[{p3, p6, p9}]]
```

Notice that the approximations are more accurate with the smaller value of h. We also see this in the graphs that compare the approximation with the exact solution.

■

6.5.3 Runge–Kutta Method

Because we would like to be able to improve the approximation without using such a small value for h, we seek to improve the method. As with first-order equations, the Runge–Kutta method can be extended to systems. In this case, the recursive formula at each step is

$$\begin{cases} x_{n+1} = x_n + \frac{1}{6}h\,(k_1 + 2k_2 + 2k_3 + k_4) \\ y_{n+1} = y_n + \frac{1}{6}h\,(m_1 + 2m_2 + 2m_3 + m_4) \end{cases} \tag{6.23}$$

where

$$k_1 = f\,(t_n, x_n, y_n) \qquad m_1 = g\,(t_n, x_n, y_n) \tag{6.24}$$

$$k_2 = f\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_1, y_n + \frac{1}{2}hm_1\right) \quad m_2 = g\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_1, y_n + \frac{1}{2}hm_1\right) \tag{6.25}$$

$$k_3 = f\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_2, y_n + \frac{1}{2}hm_2\right) \quad m_3 = g\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_2, y_n + \frac{1}{2}hm_2\right) \tag{6.26}$$

$$k_4 = f\,(t_n + h, x_n + hk_3, y_n + hm_3) \qquad m_4 = g\,(t_n + h, x_n + hk_3, y_n + hm_3) \tag{6.27}$$

EXAMPLE 6.5.4: Use the Runge–Kutta method to approximate the solution of the initial-value problem from Example 2

$$\begin{cases} dx/dt = x - y + 1 \\ dy/dt = x + 3y + e^{-t} \\ x\,(0) = 0, \; y\,(0) = 1 \end{cases}$$

using $h = 0.1$. Compare these results to those of the exact solution of the system of equations as well as those obtained with Euler's method.

SOLUTION: We use equations (6.23) and (6.24) with $f(x, y) = x - y + 1$, $g(x, y) = x + 3y + e^{-t}$, $t_0 = 0$, $x_0 = 0$, $y_0 = 1$, and $h = 0.1$.

We show the results obtained with this method and compare them to the exact values.

```
In[1333]:= Clear[t0, f, g, x, y, t, k1, k2, k3,
              k4, m1, m2, m3, m4, xr, yr]

            f[t_, x_, y_] = x - y + 1;

            g[t_, x_, y_] = x + 3y + Exp[-t];

            t0 = 0;

            h = 0.1;

            t[n_] := t0 + nh

            x[n_] :=
              x[n] =
                x[n - 1]
                 1
                +- h (k1[n - 1] + 2 k2[n - 1] + 2 k3[n - 1]
                 6
                +k4[n - 1]);

            x[0] = 0;

            y[n_] :=
              y[n] =
                y[n - 1]
                 1
                +- h (m1[n - 1] + 2 m2[n - 1] + 2 m3[n - 1]
                 6
                +m4[n - 1]);

            y[0] = 1;


In[1334]:= k1[n_] := k1[n] = f[t[n], x[n], y[n]];

            k2[n_] :=
                                  h            h k1[n]
              k2[n] = f[t[n] + -, x[n] + -------,
                                  2               2
                              h m1[n]
                     y[n] + -------];
                                2
```

```
In[1334]:= k3[n_] :=
              k3[n] = f[t[n] + h/2, x[n] + (h k2[n])/2,
                        y[n] + (h m2[n])/2];

            k4[n_] :=
              k4[n] = f[t[n] + h, x[n] + h k3[n],
                        y[n] + h m3[n]];

            m1[n_] := m1[n] = g[t[n], x[n], y[n]];

            m2[n_] :=
              m2[n] = g[t[n] + h/2, x[n] + (h k1[n])/2,
                        y[n] + (h m1[n])/2];

            m3[n_] :=
              m3[n] = g[t[n] + h/2, x[n] + (h k2[n])/2,
                        y[n] + (h m2[n])/2];

            m4[n_] :=
              m4[n] = g[t[n] + h, x[n] + h k3[n],
                        y[n] + h m3[n]];

In[1335]:= Table[{t[n], x[n], xex[t[n]], y[n],
              yex[t[n]]}, {n, 0, 10}]//TableForm
```

Out[1335]=

0	0	0	1	1
0.1	-0.0226878	-0.0226978	1.46031	1.46032
0.2	-0.10332	-0.103346	2.06541	2.06545
0.3	-0.265382	-0.265432	2.85897	2.85904
0.4	-0.540021	-0.540105	3.8967	3.89682
0.5	-0.968273	-0.968408	5.24956	5.24975
0.6	-1.60391	-1.60412	7.00778	7.00806
0.7	-2.51707	-2.51737	9.28596	9.28638
0.8	-3.79882	-3.79926	12.2294	12.23
0.9	-5.56704	-5.56767	16.0223	16.0232
1.	-7.97379	-7.97468	20.8975	20.8987

Notice that the Runge–Kutta method is much more accurate than Euler's method. In fact, the Runge–Kutta with $h = 0.1$ is more accurate than Euler's method with $h = 0.05$. We also observe the accuracy of the approximation in the graphs that compare the approximation to the exact solution in Figure 6-44.

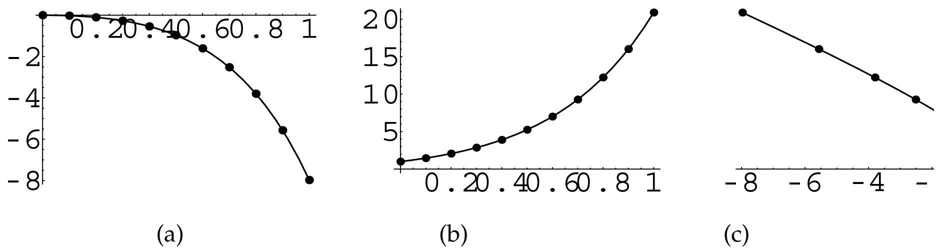


Figure 6-44 Runge–Kutta method using $h = 0.1$: (a) comparison of x_n to $x(t)$; (b) comparison of y_n to $y(t)$; (c) comparison of (x_n, y_n) to $(x(t), y(t))$

```
In[1336]:= xs = Table[{t[n],x[n]},{n,0,10}];

In[1337]:= p1 = ListPlot[xs,
              PlotStyle- > PointSize[0.03],
              DisplayFunction- > Identity];

           p2 = Plot[xex[t],{t,0,1},
              DisplayFunction- > Identity];

           p3 = Show[p1,p2];

In[1338]:= ys = Table[{t[n],y[n]},{n,0,10}];

In[1339]:= p4 = ListPlot[ys,
              PlotStyle- > PointSize[0.03],
              DisplayFunction- > Identity];

           p5 = Plot[yex[t],{t,0,1},
              DisplayFunction- > Identity];

           p6 = Show[p4,p5];

In[1340]:= both = Table[{x[n],y[n]},{n,0,10}];

In[1341]:= p7 = ListPlot[both,
              PlotStyle- > PointSize[0.03],
              DisplayFunction- > Identity];

           p8 = ParametricPlot[{xex[t],yex[t]},
              {t,0,1},DisplayFunction- >
              Identity];

           p9 = Show[p7,p8];

In[1342]:= Show[GraphicsArray[{p3,p6,p9}]]
```

■

6.6 Nonlinear Systems, Linearization, and Classification of Equilibrium Points

We now turn our attention to the systems of equations of the form

$$\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y). \end{cases} \tag{6.28}$$

This system is **autonomous**, because $f(x, y)$ and $g(x, y)$ do not depend explicitly on the independent variable t.

Definition 29 (Equilibrium Point). *A point* (x_0, y_0) *is an* ***equilibrium point*** *of system* (6.28) *if* $f(x_0, y_0) = 0$ *and* $g(x_0, y_0) = 0$.

Equilibrium points are also called **rest points**.

Before discussing nonlinear systems, we first investigate properties of systems of the form

$$\begin{cases} dx/dt = ax + by \\ dy/dt = cx + dy \end{cases} \tag{6.29}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, which have only one equilibrium point: (0, 0). We have solved many systems of this type by using the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

6.6.1 Real Distinct Eigenvalues

If λ_1 and λ_2 are real eigenvalues of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $\lambda_2 < \lambda_1$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, a general solution of system (6.29) is

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = e^{\lambda_1 t}\left[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 e^{(\lambda_2-\lambda_1)t}\right]. \tag{6.30}$$

1. Suppose that both eigenvalues are negative. If we assume that $\lambda_2 < \lambda_1 < 0$, then $\lambda_2 - \lambda_1 < 0$. Then $e^{(\lambda_2-\lambda_1)t}$ and $e^{\lambda_1 t}$ are very small for large values of t. If $c_1 \neq 0$, then $\lim_{t\to\infty} \mathbf{X} = \mathbf{0}$ in one of the directions determined by $\mathbf{v}_1$ or $-\mathbf{v}_1$. If $c_1 = 0$, then $\mathbf{X} = c_2\mathbf{v}_2 e^{\lambda_2 t}$. Again, because $\lambda_2 < 0$, $\lim_{t\to\infty} \mathbf{X} = \mathbf{0}$ in the directions determined by $\mathbf{v}_2$ or $-\mathbf{v}_2$. In this case, (0, 0) is a **stable node**.

2. Suppose that both eigenvalues are positive. If $0 < \lambda_2 < \lambda_1$, then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ both become unbounded as t increases. If $c_1 \neq 0$, then $\mathbf{X}$ becomes unbounded in either the direction of $\mathbf{v}_1$ or $-\mathbf{v}_1$. If $c_1 = 0$, then $\mathbf{X}$ becomes unbounded in the directions given by $\mathbf{v}_2$ or $-\mathbf{v}_2$. In this case, $(0, 0)$ is an **unstable node**.
3. Suppose that the eigenvalues have opposite sign. Then, if $\lambda_2 < 0 < \lambda_1$ and $c_1 \neq 0$, $\mathbf{X}$ becomes unbounded in either the direction of $\mathbf{v}_1$ or $-\mathbf{v}_1$ as it did in (2). However, if $c_1 = 0$, then due to the fact that $\lambda_2 < 0$, $\lim_{t\to\infty} \mathbf{X} = \mathbf{0}$ along the line determined by $\mathbf{v}_2$. If the initial point $\mathbf{X}(0)$ is not on the line determined by $\mathbf{v}_2$, then the line given by $\mathbf{v}_1$ is an asymptote for the solution. We say that $(0, 0)$ is a **saddle point** in this case.

EXAMPLE 6.6.1: Classify the equilibrium point $(0, 0)$ of the systems: (a) $\begin{cases} x' = 5x + 3y \\ y' = -4x - 3y \end{cases}$; (b) $\begin{cases} x' = x - 2y \\ y' = 3x - 4y \end{cases}$; (c) $\begin{cases} x' = -x - 2y \\ y' = 3x + 4y \end{cases}$.

SOLUTION: (a) We find the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ -4 & -3 \end{pmatrix}$ with `Eigensystem`.

```
In[1343]:= Clear[a, x, y]

            a = (5   3
                 -4 -3);

            Eigensystem[a]
Out[1343]= {{-1, 3}, {{-1, 2}, {-3, 2}}}
```

Because these eigenvalues have opposite sign, $(0, 0)$ is a saddle point. Eigenvectors corresponding to $\lambda_1 = -1$ and $\lambda_2 = 3$ are $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, respectively. Hence the solution becomes unbounded in the directions associated with the positive eigenvalue, $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $-\mathbf{v}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Along the line through $(0, 0)$ determined by $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, the solution approaches $(0, 0)$. We see this when we graph various solutions and display the results together with the direction field associated with the system. First, we use `DSolve` to find a general solution of the system.

```
In[1344]:= gensol = DSolve[{x'[t] == 5 x[t]
                  +3 y[t],
              y'[t] == -4 x[t] - 3 y[t]}, {x[t], y[t]}, t]
```

$$\text{Out[1344]=}\ \left\{\left\{x[t] \to \frac{1}{2} e^{-t} (-1+3e^{4t})\, C[1] + \frac{3}{4} e^{-t} (-1+e^{4t})\, C[2],\ y[t] \to -e^{-t}(-1+e^{4t})\, C[1] - \frac{1}{2} e^{-t}(-3+e^{4t})\, C[2]\right\}\right\}$$

Then, we use `Table` and `Flatten` to create a list of ordered pairs $\{x(t), y(t)\}$, corresponding to the solution for various values of the arbitrary constants. These functions are then graphed with `ParametricPlot`.

```
In[1345]:= toplot =
              Flatten[
                Table[
                  {- e^-t C[1] - 3e^3t C[2],
                   2e^-t C[1] + 2e^3t C[2]} /.
                   {C[1]- > i, C[2]- > j},
                   {i, -0.5, 0.5, 0.25},
                   {j, -0.5, 0.5, 0.25}], 1];

In[1346]:= somegraphs = ParametricPlot[
              Evaluate[toplot], {t, -3, 3},
              PlotRange → {{-1, 1}, {-1, 1}},
              AspectRatio → 1, PlotStyle → GrayLevel[0],
              DisplayFunction → Identity];

In[1347]:= p4 = Plot[{- 2 x, -2x/3}, {x, -1, 1},
                 PlotStyle →
                   {{GrayLevel[0], Dashing[{0.02}],
                   Thickness[0.01]},
                   {GrayLevel[0.2], Dashing[{0.02}],
                   Thickness[0.01]}},
                 DisplayFunction → Identity];
```

We graph the direction field associated with the system with `Plot-VectorField`. Last, all graphs are displayed together with `Show` in Figure 6-45.

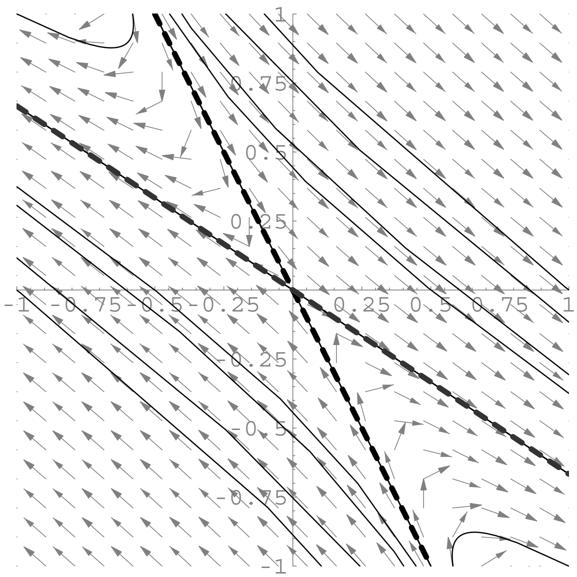


Figure 6-45 The origin is a saddle point

```
In[1348]:= << Graphics`PlotField`

In[1349]:= pvf1 = PlotVectorField[
              {5x + 3y, -4 x - 3y}, {x, -1, 1},
              {y, -1, 1},
              DefaultColor → GrayLevel[0.5],
              PlotPoints → 20,
              ScaleFunction → (0.05&),
              DisplayFunction → Identity];

In[1350]:= Show[pvf1, somegraphs, p4,
            DisplayFunction → $DisplayFunction,
            PlotRange → {{-1, 1}, {-1, 1}},
            AspectRatio → 1,
            Axes → Automatic, AxesOrigin → {0, 0},
            DisplayFunction → $DisplayFunction]
```

(b) In this case, the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$ are both negative.

```
In[1351]:= a = (1 -2
                3 -4);
            Eigensystem[a]
Out[1351]= {{-2, -1}, {{2, 3}, {1, 1}}}
```

Hence, (0, 0) is a stable node. Corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix}2\\3\end{pmatrix}$. Therefore, the solutions approach (0, 0) along the lines through the origin determined by these vectors, $y = x$ and $y = \frac{3}{2}x$. We see this in the graph of the direction field and graphs of several solutions to the system. First, we graph the direction field associated with the system.

```
In[1352]:= << Graphics`PlotField`

In[1353]:= Clear[x, y, x0, y0, sol]

           pvf1 = PlotVectorField[{x - 2y, 3x - 4y},
                  {x, -1, 1}, {y, -1, 1},
                  DefaultColor → GrayLevel[0.5],
                  PlotPoints → 20,
                  ScaleFunction → (0.5&),
                  DisplayFunction → Identity];
```

Then we use DSolve to solve the initial-value problem

$$\begin{cases} x' = x - 2y \\ y' = 3x - 4y \\ x(0) = x_0,\ y(0) = y_0. \end{cases}$$

```
In[1354]:= gensol =
             DSolve[{x'[t] == x[t] - 2 y[t],
               y'[t] == 3 x[t] - 4 y[t], x[0] == x0,
               y[0] == y0}, {x[t], y[t]}, t]//Simplify

Out[1354]= {{x[t] → e^(-2 t) ((-2 + 3 e^t) x0 - 2 (-1 + e^t) y0),
             y[t] → e^(-2 t) (3 (-1 + e^t) x0 + (3 - 2 e^t) y0)}}
```

Given an ordered pair $\{x_0, y_0\}$, sol[$\{x_0, y_0\}$] returns the solution that satisfies $x(0) = x_0$ and $y(0) = y_0$.

```
In[1355]:= sol[{x0_, y0_}] =
             {e^-t (3x0 - 2y0 + 2e^-t (-x0 + y0)),
              e^-t (3x0 - 2y0 + 3e^-t (-x0 + y0))};
```

We then generate several lists of ordered pairs with `Table`

```
In[1356]:= initconds1 =
             Table[{-1, i}, {i, -1, 1, 2/9}];

           initconds2 =
             Table[{1, i}, {i, -1, 1, 2/9}];

           initconds3 =
             Table[{i, 1}, {i, -1, 1, 2/9}];

           initconds4 =
             Table[{i, -1}, {i, -1, 1, 2/9}];
```

and use `Union` to join them together.

```
In[1357]:= initconds = initconds1 ∪ initconds2 ∪
             initconds3 ∪ initconds4;
```

`Map` is used to apply `sol` to the list `initconds`. The resulting list of parametric functions is graphed with `ParametricPlot`.

```
In[1358]:= toplot = Map[sol, initconds];

In[1359]:= somegraphs = ParametricPlot[Evaluate[toplot],
                          {t, -3, 3}, PlotRange → {{-1, 1},
                          {-1, 1}}, AspectRatio → 1,
                          PlotStyle → GrayLevel[0],
                          DisplayFunction → Identity];

In[1360]:= p4 = Plot[{x, 3x/2}, {x, -1, 1},
                  PlotStyle →
                    {{GrayLevel[0], Dashing[{0.02}],
                    Thickness[0.01]},
                    {GrayLevel[0.2], Dashing[{0.02}],
                    Thickness[0.01]}},
                  DisplayFunction → Identity];
```

Finally, all the graphics are displayed together with `Show` in Figure 6-46.

```
In[1361]:= Show[pvf1, somegraphs, p4,
             DisplayFunction → $DisplayFunction,
             PlotRange → {{-1, 1}, {-1, 1}},
             AspectRatio → 1,
             Axes → Automatic, AxesOrigin → {0, 0}]
```

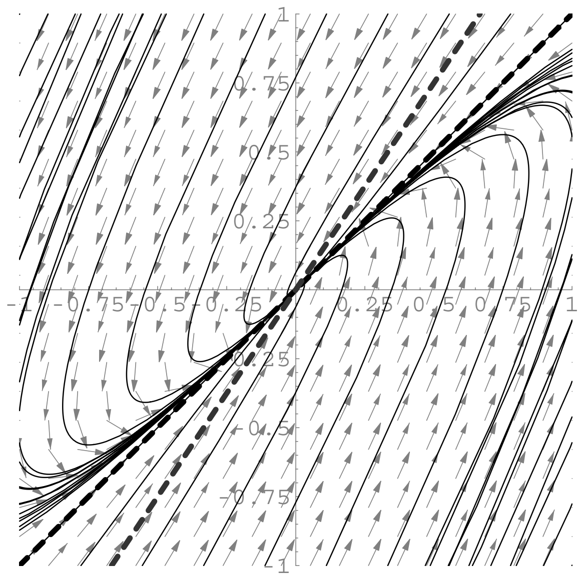


Figure 6-46 The origin is a stable node

(c) Because the eigenvalues $\lambda_1 = 2$ and $\lambda_1 = 1$ are both positive, (0, 0) is an unstable node.

```
In[1362]:= a = (-1 -2
                 3  4);
            Eigensystem[a]
Out[1362]= {{1, 2}, {{-1, 1}, {-2, 3}}}
```

Note that the corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively. Hence, the solutions become unbounded along the lines passing through the origin determined by these vectors, $y = -\frac{3}{2}x$ and $y = -x$. As before, we see this in the graph of the direction field and various solutions of the system. See Figure 6-47.

```
In[1363]:= pvf1 = PlotVectorField[{-x - 2y, 3x + 4y},
                  {x, -1, 1}, {y, -1, 1},
                  DefaultColor → GrayLevel[0.5],
                  PlotPoints → 20,
                  ScaleFunction → (0.05&),
                  DisplayFunction → Identity];
```

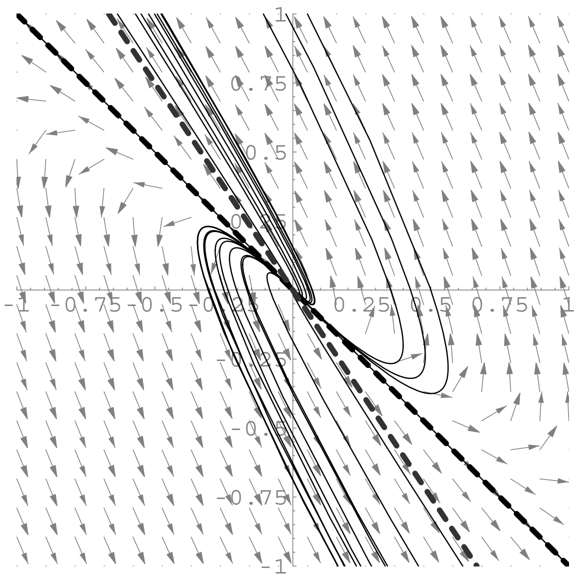


Figure 6-47 The origin is an unstable node

```
In[1364]:= gensol =
              DSolve[{x′[t] == -x[t] - 2 y[t],
              y′[t] == 3 x[t] + 4 y[t],
              x[0] == x0, y[0] == y0},
              {x[t], y[t]}, t]
Out[1364]= {{x[t] → -ⅇ^t (-3 x0 + 2 ⅇ^t x0 - 2 y0 + 2 ⅇ^t y0),
              y[t] → ⅇ^t (-3 x0 + 3 ⅇ^t x0 - 2 y0 + 3 ⅇ^t y0)}}

In[1365]:= sol[{x0_, y0_}] =
              {-e^t (-3 x0 - 2y0 + 2e^t (x0 + y0)),
               e^t (-3 x0 - 2y0 + 3e^t (x0 + y0))};

In[1366]:= initconds =
              Table[{0.5t Cos[2πt], 0.5t Sin[2πt]},
              {t, 0, 1, 1/24}];

In[1367]:= toplot = Map[sol, initconds];

In[1368]:= somegraphs =
              ParametricPlot[Evaluate[toplot],
              {t, -3, 3}, PlotRange → {{-1, 1}, {-1, 1}},
              AspectRatio → 1, PlotStyle → GrayLevel[0],
              DisplayFunction → Identity];
```

```
In[1369]:= p4 = Plot[{-x, -3x/2}, {x, -1, 1},
              PlotStyle →
                {{GrayLevel[0], Dashing[{0.02}],
                Thickness[0.01]},
                {GrayLevel[0.2], Dashing[{0.02}],
                Thickness[0.01]}},
              DisplayFunction → Identity];

In[1370]:= Show[pvf1, somegraphs, p4,
           DisplayFunction → $DisplayFunction,
           PlotRange → {{-1, 1}, {-1, 1}},
           AspectRatio → 1,
           Axes → Automatic, AxesOrigin → {0, 0}]
```

■

6.6.2 Repeated Eigenvalues

We recall from our previous experience with repeated eigenvalues of a 2×2 system that the eigenvalue can have two linearly independent eigenvectors associated with it or only one eigenvector associated with it. Hence, we investigate the behavior of solutions in this case by considering both of these possibilities.

1. Suppose that the eigenvalue $\lambda = \lambda_1 = \lambda_2$ has two corresponding linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then, a general solution is

$$\mathbf{X} = c_1\mathbf{v}_1 e^{\lambda t} + c_2\mathbf{v}_2 e^{\lambda t}.$$

Hence, if $\lambda > 0$, then $\mathbf{X}$ becomes unbounded along the line through the origin determined by the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ where c_1 and c_2 are arbitrary constants. In this case, we call the equilibrium point a **degenerate unstable node** (or an **unstable star**). On the other hand, if $\lambda < 0$, then $\mathbf{X}$ approaches $(0, 0)$ along these lines, and we call $(0, 0)$ a **degenerate stable node** (or **stable star**). Note that the name "star" was selected due to the shape of the solutions.

2. Suppose that $\lambda = \lambda_1 = \lambda_2$ has only one corresponding eigenvector $\mathbf{v}_1$. Hence, a general solution is

$$\mathbf{X} = c_1\mathbf{v}_1 e^{\lambda t} + c_2\left(\mathbf{v}_1 t + \mathbf{w}_2\right)e^{\lambda t} = \left(c_1\mathbf{v}_1 + c_2\mathbf{w}_2\right)e^{\lambda t} + c_2\mathbf{v}_1 t e^{\lambda t},$$

where $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$. We can more easily investigate the behavior of this solution if we write this solution as

$$\mathbf{X} = te^{\lambda t}\left[\frac{1}{t}\left(c_1\mathbf{v}_1 + c_2\mathbf{w}_2\right) + c_2\mathbf{v}_1\right].$$

If $\lambda < 0$, $\lim_{t\to\infty} te^{\lambda t} = 0$ and $\lim_{t\to\infty}\left[\frac{1}{t}(c_1\mathbf{v}_1 + c_2\mathbf{w}_2) + c_2\mathbf{v}_1\right] = c_2\mathbf{v}_1$. Hence, the solutions approach (0, 0) along the line determined by $\mathbf{v}_1$, and we call (0, 0) a **degenerate stable node**. If $\lambda > 0$, the solutions become unbounded along this line, and we say that (0, 0) is a **degenerate unstable node**.

EXAMPLE 6.6.2: Classify the equilibrium point (0, 0) in the systems: (a) $\begin{cases} x' = x + 9y \\ y' = -x - 5y \end{cases}$; (b) $\begin{cases} x' = 2x \\ y' = 2y \end{cases}$.

SOLUTION: (a) Using `Eigensystem`,

```
In[1371]:= a = (1  9
                -1 -5);
           Eigensystem[a]
Out[1371]= {{-2, -2}, {{-3, 1}, {0, 0}}}
```

we see that $\lambda_1 = \lambda_2 = -2$ and that there is only one corresponding eigenvector. Therefore, because $\lambda = -2 < 0$, (0, 0) is a degenerate stable node. Notice that in the graph of several members of the family of solutions of this system along with the direction field shown in Figure 6-48, which

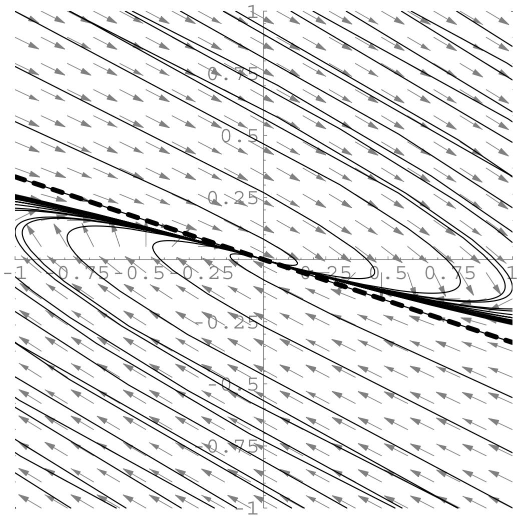


Figure 6-48 The origin is a degenerate stable node

we generate using the same technique as in part (b) of the previous example, the solutions approach (0, 0) along the line in the direction of $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $y = -\frac{1}{3}x$.

```
In[1372]:= << Graphics`PlotField`

In[1373]:= Clear[x, y]

           pvf1 = PlotVectorField[{x + 9y, -x - 5y},
                  {x, -1, 1}, {y, -1, 1},
                  DefaultColor → GrayLevel[0.5],
                  PlotPoints → 20,
                  ScaleFunction → (0.5&),
                  DisplayFunction → Identity];

In[1374]:= Simplify[
             DSolve[{x'[t] == x[t] + 9 y[t],
               y'[t] == -x[t] - 5 y[t], x[0] == x0,
               y[0] == y0}, {x[t], y[t]}, t]]
Out[1374]= {{x[t] → e^(-2 t) (x0 + 3 t x0 + 9 t y0),
             y[t] → e^(-2 t) (y0 - t (x0 + 3 y0))}}

In[1375]:= sol[{x0_, y0_}] =
             {(x0 + 3tx0 + 9ty0)/e^(2t), (-(tx0) + y0 - 3ty0)/e^(2t)};

In[1376]:= initconds1 =
             Table[{-1, i}, {i, -1, 1, 2/9}];

           initconds2 =
             Table[{1, i}, {i, -1, 1, 2/9}];

           initconds3 =
             Table[{i, 1}, {i, -1, 1, 2/9}];

           initconds4 =
             Table[{i, -1}, {i, -1, 1, 2/9}];

In[1377]:= initconds = initconds1 ∪ initconds2 ∪
               initconds3 ∪ initconds4;

In[1378]:= toplot = Map[sol, initconds];

In[1379]:= somegraphs =
               ParametricPlot[Evaluate[toplot],
               {t, -3, 3}, PlotRange → {{-1, 1}, {-1, 1}},
               AspectRatio → 1, PlotStyle → GrayLevel[0],
               DisplayFunction → Identity];
```

```
In[1380]:= p4 = Plot[-x/3, {x, -1, 1},
              PlotStyle →
                {{GrayLevel[0], Dashing[{0.02}],
                Thickness[0.01]}},
              DisplayFunction → Identity];

In[1381]:= Show[pvf1, somegraphs, p4,
             DisplayFunction → $DisplayFunction,
             PlotRange → {{-1, 1}, {-1, 1}},
             AspectRatio → 1,
             Axes → Automatic, AxesOrigin → {0, 0}]
```

(b) We have $\lambda_1 = \lambda_2 = 2$ and two linearly independent vectors, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (Note: The choice of these two vectors does not change the value of the solution, because of the form of the general solution in this case.)

```
In[1382]:= a = (2 0
                0 2);
           Eigensystem[a]
Out[1382]= {{2, 2}, {{0, 1}, {1, 0}}}
```

Because $\lambda = 2 > 0$, we classify (0, 0) as a degenerate unstable node (or star). Some of these solutions along with the direction field are graphed in Figure 6-49 in the same manner as in part (c) of the previous example. Notice that they become unbounded in the direction of any vector in the xy-plane because $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

```
In[1383]:= Clear[x, y]

           pvf1 = PlotVectorField[{2x, 2y}, {x, -1, 1},
                  {y, -1, 1}, DefaultColor →
                  GrayLevel[0.5], PlotPoints → 20,
                  ScaleFunction → (0.5&),
                  DisplayFunction → Identity];

In[1384]:= Simplify[DSolve[{x'[t] == 2 x[t],
              y'[t] == 2 y[t],
              x[0] == x0, y[0] == y0}, {x[t], y[t]}, t]]
Out[1384]= {{x[t] → e^(2 t) x0, y[t] → e^(2 t) y0}}
```

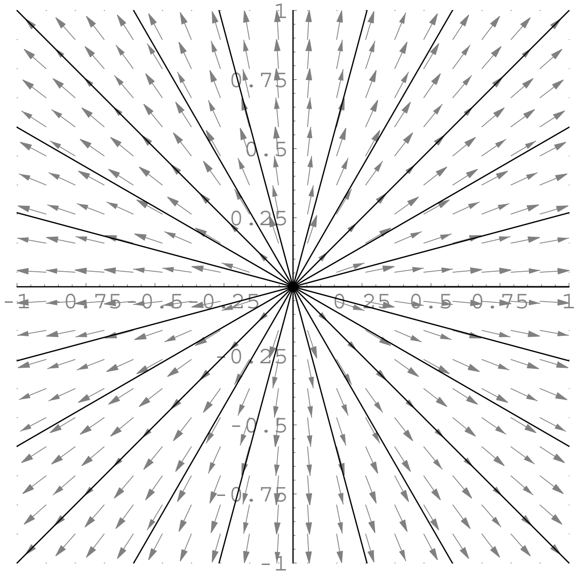


Figure 6-49 The origin is a degenerate unstable node

```
In[1385]:= sol[{x0_, y0_}] = {e^(2t) x0, e^(2t) y0};

In[1386]:= initconds = Table[{0.05 Cos[2πt],
                              0.05 Sin[2πt]},
                              {t, 0, 1, 1/24}];

In[1387]:= toplot = Map[sol, initconds];

In[1388]:= somegraphs =
              ParametricPlot[Evaluate[toplot],
              {t, -3, 3}, PlotRange → {{-1, 1}, {-1, 1}},
              AspectRatio → 1, PlotStyle → GrayLevel[0],
              DisplayFunction → Identity];

In[1389]:= Show[pvf1, somegraphs,
            DisplayFunction → $DisplayFunction,
            PlotRange → {{-1, 1}, {-1, 1}},
            AspectRatio → 1,
            Axes → Automatic, AxesOrigin → {0, 0}]
```

■

6.6.3 Complex Conjugate Eigenvalues

We have seen that if the eigenvalues of the system (6.29) are $\lambda_{1,2} = \alpha \pm \beta i$, $\beta \neq 0$, with corresponding eigenvectors $\mathbf{v}_{1,2} = \mathbf{a} \pm \mathbf{b}i$, two linearly independent solutions of the system are

$$\mathbf{X}_1 = e^{\alpha t}\left(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t\right) \quad \text{and} \quad \mathbf{X}_2 = e^{\alpha t}\left(\mathbf{b}\cos\beta t + \mathbf{a}\sin\beta t\right).$$

Hence, a general solution is $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$, so there are constants A_1, A_2, B_1, and B_2 so that x and y are given by

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_1 e^{\alpha t}\cos\beta t + A_2 e^{\alpha t}\sin\beta t \\ B_1 e^{\alpha t}\cos\beta t + B_2 e^{\alpha t}\sin\beta t \end{pmatrix}.$$

1. If $\alpha = 0$, the solution is

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_1\cos\beta t + A_2\sin\beta t \\ B_1\cos\beta t + B_2\sin\beta t \end{pmatrix}.$$

   Hence, both x and y are periodic and $(0, 0)$ is classified as a **center**. Note that the motion around these circles or ellipses is either clockwise or counterclockwise for all solutions.
2. If $\alpha \neq 0$, then $e^{\alpha t}$ is present in the solution. This term causes the solution to spiral around the equilibrium point. If $\alpha > 0$, then the solution spirals away from $(0, 0)$, so we classify $(0, 0)$ as an **unstable spiral**. Otherwise, if $\alpha < 0$, the solution spirals towards $(0, 0)$, so we say that $(0, 0)$ is a **stable spiral**.

EXAMPLE 6.6.3: Classify the equilibrium point $(0, 0)$ in each of the following systems: (a) $\begin{cases} x' = -y \\ y' = x \end{cases}$; (b) $\begin{cases} x' = \frac{1}{2}x - \frac{153}{32}y \\ y' = 2x - y \end{cases}$.

SOLUTION: (a) The eigenvalues are found to be $\lambda_{1,2} = \pm i$.

```
In[1390]:= a = (0 -1
                1  0);
           Eigensystem[a]
Out[1390]= {{-i, i}, {{-i, 1}, {i, 1}}}
```

Because these eigenvalues have zero real part (and, hence, are purely imaginary), $(0, 0)$ is a center. Several solutions along with the direction field are graphed in Figure 6-50.

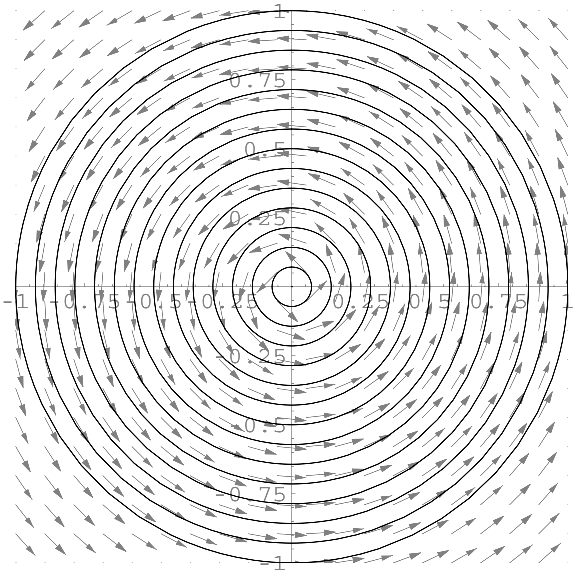


Figure 6-50 The origin is a center

```
In[1391]:= Clear[x,y]

           pvf1 = PlotVectorField[{-y,x},{x,-1,1},
                    {y,-1,1},DefaultColor →
                    GrayLevel[0.5],PlotPoints → 20,
                    ScaleFunction → (0.5&),
                    DisplayFunction → Identity];

In[1392]:= Simplify[DSolve[{x′[t] == - y[t],
                y′[t] == x[t],x[0] == x0,
                y[0] == y0},{x[t],y[t]},t]]

Out[1392]= {{x[t] → x0 Cos[t] - y0 Sin[t],
             y[t] → y0 Cos[t] + x0 Sin[t]}}

In[1393]:= sol[{x0_,y0_}] = {x0 Cos[t] - y0 Sin[t],
                              y0 Cos[t] + x0 Sin[t]};

In[1394]:= initconds = Table[{0,i},
                          {i,0,1,1/14}];

In[1395]:= toplot = Map[sol,initconds];
```

```
In[1396]:= somegraphs =
              ParametricPlot[Evaluate[toplot],
              {t, 0, 2π}, PlotRange → {{-1, 1}, {-1, 1}},
              AspectRatio → 1, PlotStyle → GrayLevel[0],
              DisplayFunction → Identity];

In[1397]:= Show[pvf1, somegraphs,
            DisplayFunction → $DisplayFunction,
            PlotRange → {{-1, 1}, {-1, 1}},
            AspectRatio → 1,
            Axes → Automatic, AxesOrigin → {0, 0}]
```

(b) The eigenvalues are found to be $\lambda_{1,2} = -1/4 \pm 3i$.

```
In[1398]:= a = (1/2  -153/32
                2      -1   );
           Eigensystem[a]
```

Out[1398]= $\left\{\left\{-\frac{1}{4}-3\,i,\ -\frac{1}{4}+3\,i\right\},\ \left\{\left\{\frac{3}{8}-\frac{3\,i}{2},\ 1\right\},\ \left\{\frac{3}{8}+\frac{3\,i}{2},\ 1\right\}\right\}\right\}$

Thus, (0, 0) is a stable spiral, because $\alpha = -1/4 < 0$. Several solutions along with the direction field are graphed in Figure 6-51 in the same way that we have done before.

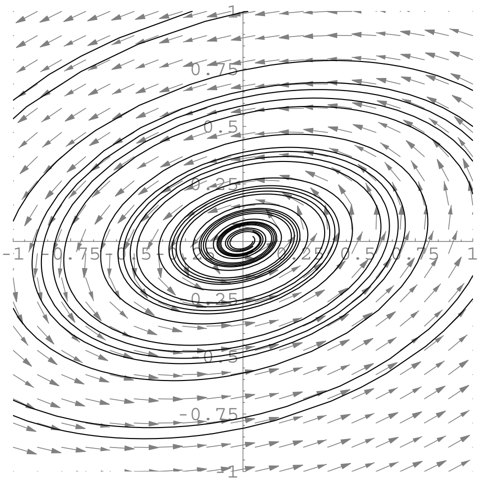


Figure 6-51 The origin is a stable spiral

```
In[1399]:= Clear[x, y]

           pvf1 = PlotVectorField[{1/2x - 153/32y,
                    2x - y}, {x, -1, 1}, {y, -1, 1},
                    DefaultColor → GrayLevel[0.5],
                    PlotPoints → 20, ScaleFunction →
                    (0.5&), DisplayFunction → Identity];

In[1400]:= FullSimplify[DSolve[{x′[t] ==
              1/2x[t] - 153/32y[t],
              y′[t] == 2x[t] - y[t], x[0] == x0,
              y[0] == y0}, {x[t], y[t]}, t]]
```

$$Out[1400]= \left\{\left\{x[t] \to \frac{1}{32} e^{-t/4} (32\, x0\, Cos[3\, t] + (8\, x0 - 51\, y0)\, Sin[3\, t]),\ y[t] \to \frac{1}{12} e^{-t/4} (12\, y0\, Cos[3\, t] + (8\, x0 - 3\, y0)\, Sin[3\, t])\right\}\right\}$$

In[1401]:= `sol[{x0_, y0_}] =`

$$\left\{\frac{1}{32} e^{-t/4} (32x0\ Cos[3t] + (8x0 - 51y0)\ Sin[3t]),\ \frac{1}{12} e^{-t/4} (12y0\ Cos[3t] + (8x0 - 3y0)\ Sin[3t])\right\};$$

```
In[1402]:= initconds =
             Table[{1, i}, {i, -1, 1, 2/4}];

In[1403]:= toplot = Map[sol, initconds];

In[1404]:= somegraphs =
              ParametricPlot[Evaluate[toplot],
              {t, 0, 4π}, PlotRange → {{-1, 1}, {-1, 1}},
              AspectRatio → 1, PlotStyle → GrayLevel[0],
              DisplayFunction → Identity];

In[1405]:= Show[pvf1, somegraphs,
             DisplayFunction → $DisplayFunction,
             PlotRange → {{-1, 1}, {-1, 1}},
             AspectRatio → 1,
             Axes → Automatic, AxesOrigin → {0, 0}]
```

■

6.6.4 Nonlinear Systems

When working with nonlinear systems, we can often gain a great deal of information concerning the system by making a linear approximation near each equilibrium point of the nonlinear system and solving the linear system. Although the solution to the linearized system only approximates the solution to the nonlinear system, the general behavior of solutions to the nonlinear system near each equilibrium is the same as that of the corresponding linear system in most cases. The first step towards approximating a nonlinear system near each equilibrium point is to find the equilibrium points of the system and the matrix for linearization near each point as defined below.

Recall from multivariate calculus that if $z = F(x, y)$ is a differentiable function, the tangent plane to the surface S given by the graph of $z = F(x, y)$ at the point (x_0, y_0) is

$$z = F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + F(x_0, y_0).$$

Hence, near each equilibrium point (x_0, y_0) of the nonlinear system

$$\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y) \end{cases}$$

the system can be approximated with

$$\begin{cases} dx/dt = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0) \\ dy/dt = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) + g(x_0, y_0). \end{cases}$$

Then, because $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$, the approximate system is

$$\begin{cases} dx/dt = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ dy/dt = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) \end{cases}$$

which can be written in matrix form as

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}. \tag{6.31}$$

Note that we often call system (6.31) the **linearized system corresponding to the nonlinear system** due to the fact that we have removed the nonlinear terms from the original system. Now that the system is approximated by a system of the form $\begin{cases} dx/dt = ax + by \\ dy/dt = cx + dy \end{cases}$ an equilibrium point (x_0, y_0) of the system $\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y) \end{cases}$ is classified by the eigenvalues of the matrix

$$\mathbf{J} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \tag{6.32}$$

which is called the **Jacobian matrix**. Of course, this linearization must be carried out for each equilibrium point. After determining the matrix for linearization for each equilibrium point, the eigenvalues for the matrix must be found. Then, we classify each equilibrium point according to the following criteria.

The Jacobian matrix is also called the **variational matrix**.

Classification of Equilibrium Points

Let (x_0, y_0) be an equilibrium point of the system $\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y) \end{cases}$ and let λ_1 and λ_2 be the eigenvalues of the Jacobian matrix, (6.32).

1. Suppose that λ_1 and λ_2 are real. If $\lambda_1 > \lambda_2 > 0$, then (x_0, y_0) is an unstable node; if $\lambda_2 < \lambda_1 < 0$, then (x_0, y_0) is a stable node; and if $\lambda_2 < 0 < \lambda_1$, then (x_0, y_0) is a saddle.
2. Suppose that $\lambda_{1,2} = \alpha \pm \beta i, \beta \neq 0$. If $\alpha < 0$, (x_0, y_0) is a stable spiral; if $\alpha > 0$, (x_0, y_0) is an unstable spiral; and if $\alpha = 0$, (x_0, y_0) may be a center, unstable spiral, or stable spiral. Hence, we can draw no conclusion.

We will not discuss the case if the eigenvalues are the same or one eigenvalue is zero. For analyzing nonlinear systems, we state the following useful theorem.

Theorem 17. *Suppose that (x_0, y_0) is an equilibrium point of the autonomous nonlinear system*

$$\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y). \end{cases}$$

Then, the relationships in the following table hold for the classification of (x_0, y_0) in the nonlinear system and that in the associated linearized system.

Associated Linearized System	*Nonlinear System*
Stable Node	*Stable Node*
Unstable Node	*Unstable Node*
Stable Spiral	*Stable Spiral*
Unstable Spiral	*Unstable Spiral*
Saddle	*Saddle*
Center	*No Conclusion*

An **autonomous system** does not explicitly depend on the independent variable, t. That is, if you write the system omitting all arguments, the independent variable (typically t) does not appear.

More generally, for the autonomous system of the form

$$\begin{aligned} x_1{}' &= f_1\left(x_1, x_2, \ldots, x_n\right) \\ x_2{}' &= f_2\left(x_1, x_2, \ldots, x_n\right) \\ &\vdots \\ x_n{}' &= f_n\left(x_1, x_2, \ldots, x_n\right) \end{aligned} \tag{6.33}$$

an **equilibrium** (or **rest**) **point**, $E = (x_1{}^*, x_2{}^*, \ldots, x_n{}^*)$, of equation (6.33) is a solution of the system

$$\begin{aligned} f_1\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ f_2\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ &\vdots \\ f_n\left(x_1, x_2, \ldots, x_n\right) &= 0. \end{aligned} \tag{6.34}$$

The **Jacobian** of equation (6.33) is

$$\mathbf{J}\left(x_1, x_2, \ldots, x_n\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

The rest point, E, is **locally stable** if and only if all the eigenvalues of $\mathbf{J}(E)$ have negative real part. If E is not locally stable, E is **unstable**.

EXAMPLE 6.6.4: Find and classify the equilibrium points of $\begin{cases} dx/dt = 1 - y \\ dy/dt = x^2 - y^2. \end{cases}$

SOLUTION: We begin by finding the equilibrium points of this non-linear system by solving $\begin{cases} 1 - y = 0 \\ x^2 - y^2 = 0. \end{cases}$

```
In[1406]:= Clear[f, g]

           f[x_, y_] = 1 - y;

           g[x_, y_] = x^2 - y^2;
```

```
In[1407]:= Solve[{f[x,y] == 0,g[x,y] == 0}]
Out[1407]= {{y → 1, x → -1}, {y → 1, x → 1}}
```

Because $f(x, y) = 1 - y$ and $g(x, y) = x^2 - y^2$, $f_x(x, y) = 0$, $f_y(x, y) = -1$, $g_x(x, y) = 2x$, and $g_y(x, y) = -2y$, so the Jacobian matrix is $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 2x & -2y \end{pmatrix}$.

```
In[1408]:= jac = {{D[f[x,y],x],D[f[x,y],y]},
                  {D[g[x,y],x],D[g[x,y],y]}};

            MatrixForm[jac]
```

Out[1408]= $\begin{pmatrix} 0 & -1 \\ 2\,x & -2\,y \end{pmatrix}$

Next, we obtain the linearized system about each equilibrium point.

For (1, 1), we obtain $\lambda_{1,2} = -1 \pm i$. Because these eigenvalues are complex-valued with negative real part, we classify (1, 1) as a stable spiral.

```
In[1409]:= jac /. {x- > 1,y- > 1}//Eigenvalues
Out[1409]= {-1 - ⅈ, -1 + ⅈ}
```

For (−1, 1), we obtain $\lambda_1 = -1 + \sqrt{3} > 0$ and $\lambda_2 = -1 - \sqrt{3} < 0$, so (−1, 1) is a saddle.

```
In[1410]:= jac /. {x- > -1,y- > 1}//Eigenvalues
```

Out[1410]= $\{-1 - \sqrt{3}, -1 + \sqrt{3}\}$

We graph solutions to this nonlinear system obtained using `NDSolve` together with the direction field associated with the nonlinear system in Figure 6-52. We can see how the solutions move towards and move away from the equilibrium points by observing the arrows on the vectors in the direction field.

```
In[1411]:= << Graphics`PlotField`

In[1412]:= pvf = PlotVectorField[{f[x,y],g[x,y]},
                  {x,-3/2,3/2},{y,-1,2},
                  Axes- > Automatic,AxesOrigin- > {0,0},
                  ScaleFunction- > (1&),
                  DefaultColor- > GrayLevel[0.5],
                  DisplayFunction- > Identity];
```

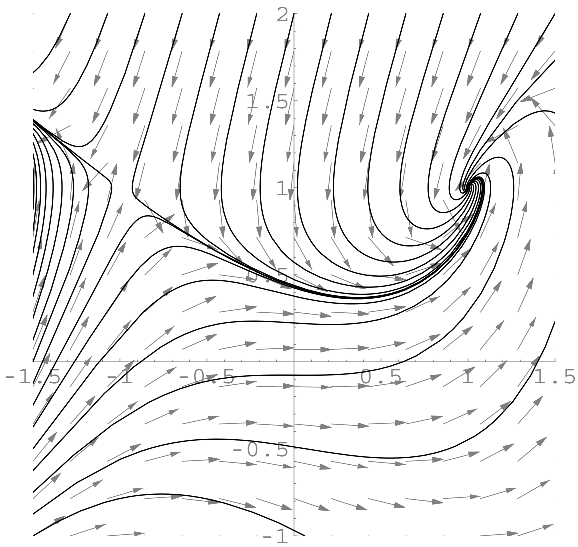


Figure 6-52 Linearization gives us information about the *local* behavior of solutions near equilibrium points but not information about the *global* behavior of the solution

```
In[1413]:= numgraph[{x0_, y0_}] := Module[{numsol},
              numsol = NDSolve[{x′[t] == f[x[t], y[t]],
                    y′[t] == g[x[t], y[t]], x[0] == x0,
                    y[0] == y0}, {x[t], y[t]},
                    {t, 0, 15}];
              ParametricPlot[
                {x[t], y[t]} /. numsol, {t, 0, 15},
                PlotStyle- > GrayLevel[0],
                Compiled- > False,
                DisplayFunction- > Identity]
              ]

In[1414]:= initconds1 =
             Table[{-3/2, i}, {i, -1, 2, 3/24}];

In[1415]:= initconds2 =
             Table[{i, 2}, {i, -3/2, 3/2, 3/14}];

In[1416]:= initconds = initconds1 ∪ initconds2;

In[1417]:= somegraphs = Map[numgraph, initconds];
```

```
In[1418]:= Show[pvf, somegraphs,
              PlotRange- > {{-3/2, 3/2}, {-1, 2}},
              AspectRatio → 1, Axes → Automatic,
              AxesOrigin → {0, 0},
              DisplayFunction- > $DisplayFunction]
```

■

EXAMPLE 6.6.5 (Duffing's Equation): Consider the forced **pendulum equation** with damping,

$$x'' + kx' + \omega \sin x = F(t). \tag{6.35}$$

Recall the Maclaurin series for $\sin x$: $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$. Using $\sin x \approx x$, equation (6.35) reduces to the linear equation $x'' + kx' + \omega x = F(t)$.

On the other hand, using the approximation $\sin x \approx x - \frac{1}{6}x^3$, we obtain $x'' + kx' + \omega\left(x - \frac{1}{6}x^3\right) = F(t)$. Adjusting the coefficients of x and x^3 and assuming that $F(t) = F \cos \omega t$ gives us **Duffing's equation**:

$$x'' + kx' + cx + \epsilon x^3 = F \cos \omega t, \tag{6.36}$$

where k and c are positive constants.

Let $y = x'$. Then, $y' = x'' = F \cos \omega t - kx' - cx - \epsilon x^3 = F \cos \omega t - ky - cx - \epsilon x^3$ and we can write equation (6.36) as the system

$$\begin{aligned} x' &= y \\ y' &= F \cos \omega t - ky - cx - \epsilon x^3. \end{aligned} \tag{6.37}$$

Assuming that $F = 0$ results in the autonomous system

$$\begin{aligned} x' &= y \\ y' &= -cx - \epsilon x^3 - ky. \end{aligned} \tag{6.38}$$

The rest points of system equation (6.38) are found by solving

$$\begin{aligned} y &= 0 \\ -cx - \epsilon x^3 - ky &= 0, \end{aligned}$$

resulting in $E_0 = (0, 0)$.

```
In[1419]:= Solve[{y == 0, -c x  - ϵ x^3 - k y == 0}, {x, y}]
```

```
Out[1419]= {{y → 0, x → 0}, {y → 0, x → -(i √c)/√ϵ},
            {y → 0, x → (i √c)/√ϵ}}
```

We find the Jacobian of equation (6.38) in `s1`, evaluate the Jacobian at E_0,

```
In[1420]:= s1 = {{0, 1}, {-c  - 3ϵ x^2, -k}};
            s2 = s1/.x- > 0
Out[1420]= {{0, 1}, {-c, -k}}
```

and then compute the eigenvalues with `Eigenvalues`.

```
In[1421]:= s3 = Eigenvalues[s2]
Out[1421]= {1/2 (-k - √(-4 c + k^2)), 1/2 (-k + √(-4 c + k^2))}
```

Because k and c are positive, $k^2 - 4c < k^2$ so the real part of each eigenvalue is always negative if $k^2 - 4c \neq 0$. Thus, E_0 is locally stable.

For the autonomous system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{6.39}$$

Bendixson's theorem states that if $f_x(x, y) + g_y(x, y)$ is a continuous function that is either always positive or always negative in a particular region R of the plane, then system (6.39) has no limit cycles in R. For equation (6.38) we have

$$\frac{d}{dx}(y) + \frac{d}{dy}\left(-cx - \epsilon x^3 - ky\right) = -k,$$

which is always negative. Hence, equation (6.38) has no limit cycles and it follows that E_0 is globally, asymptotically stable.

```
In[1422]:= D[y, x] + D[-c x  - ϵ x^3 - k y, y]
Out[1422]= -k
```

We use `PlotVectorField` and `ParametricPlot` to illustrate two situations that occur. In Figure 6-53 (a), we use $c = 1$, $\epsilon = 1/2$, and $k = 3$. In this case, E_0 is a *stable node*. On the other hand, in Figure 6-53 (b), we use $c = 10$, $\epsilon = 1/2$, and $k = 3$. In this case, E_0 is a *stable spiral*.

```
In[1423]:= << Graphics`PlotField`

            pvf1 = PlotVectorField[{y, -x - 1/2x^3 - 3y},
                    {x, -2.5, 2.5}, {y, -2.5, 2.5},
                    DisplayFunction- > Identity];
```

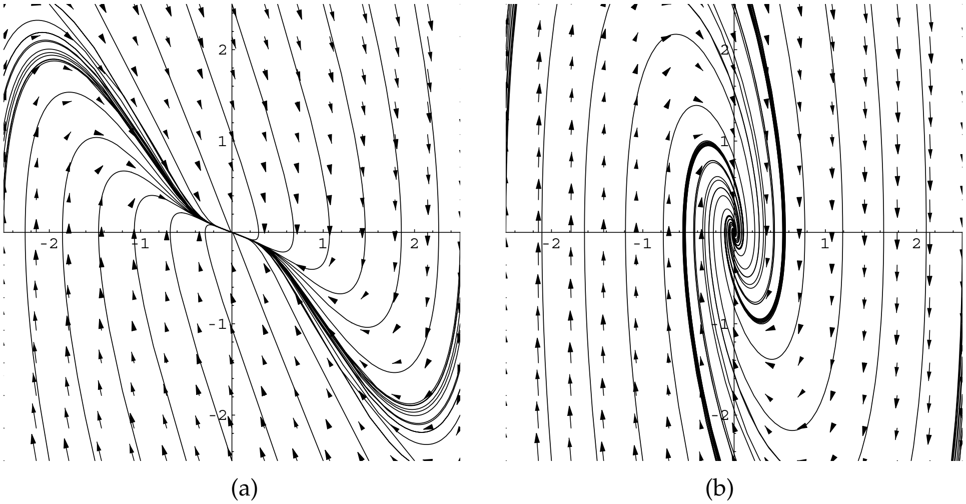


Figure 6-53 (a) The origin is a stable node. (b) The origin is a stable spiral

```
In[1424]:= numgraph[init_, c_, opts___] := Module
                [{numsol},
            numsol = NDSolve[{x′[t] == y[t],
                 y′[t] ==
                 -c x[t] - 1/2x[t]^3 - 3y[t],
                 x[0] == init[[1]],
                 y[0] == init[[2]]},
                 {x[t], y[t]}, {t, 0, 10}];
            ParametricPlot[Evaluate[{x[t],
                 y[t]}/.numsol], {t, 0, 10}, opts,
                 DisplayFunction- > Identity]]

In[1425]:= i1 = Table[numgraph[{2.5, i}, 1],
                {i, -2.5, 2.5, 1/2}];
            i2 = Table[numgraph[{-2.5, i}, 1],
                {i, -2.5, 2.5, 1/2}];
            i3 = Table[numgraph[{i, 2.5}, 1],
                {i, -2.5, 2.5, 1/2}];
            i4 = Table[numgraph[{i, -2.5}, 1],
                {i, -2.5, 2.5, 1/2}];

In[1426]:= c1 = Show[i1, i2, i3, i4, pvf1,
                PlotRange- > {{-2.5, 2.5}, {-2.5, 2.5}},
                AspectRatio- > Automatic];
```

```
In[1427]:= pvf2 = PlotVectorField[{y, -10x - 1/2x^3 - 3y},
                    {x, -2.5, 2.5}, {y, -2.5, 2.5},
                    DisplayFunction- > Identity];

In[1428]:= i1 = Table[numgraph[{2.5, i}, 10],
                 {i, -2.5, 2.5, 1/2}];
           i2 = Table[numgraph[{-2.5, i}, 10],
                 {i, -2.5, 2.5, 1/2}];
           i3 = Table[numgraph[{i, 2.5}, 10],
                 {i, -2.5, 2.5, 1/2}];
           i4 = Table[numgraph[{i, -2.5}, 10],
                 {i, -2.5, 2.5, 1/2}];

In[1429]:= c2 = Show[i1, i2, i3, i4, pvf2,
                  PlotRange- > {{-2.5, 2.5}, {-2.5, 2.5}},
                  AspectRatio- > Automatic];

In[1430]:= Show[GraphicsArray[{c1, c2}]]
```

Although linearization can help you determine local behavior near rest points, the long-term behavior of solutions to nonlinear systems can be quite complicated, even for deceptively simple looking systems.

See texts like Jordan and Smith's *Nonlinear Ordinary Differential Equations* [17] for discussions of ways to analyze systems like the Rössler attractor and the Lorenz equations.

EXAMPLE 6.6.6 (Lorenz Equations): The **Lorenz equations** are

$$\begin{cases} dx/dt = a(y-x) \\ dy/dt = bx - y - xz \\ dz/dt = xy - cz \end{cases} \tag{6.40}$$

Graph the solutions to the Lorenz equations if $a = 7$, $b = 27.2$, and $c = 3$ if the initial conditions are $x(0) = 3$, $y(0) = 4$, and $z(0) = 2$.

SOLUTION: So that you can experiment with different parameters and initial conditions, we define the function `lorenzsol`. Given the appropriate parameters and initial conditions,

```
lorenzsol[a,b,c][{x0,y0,z0}]
```

returns a numerical solution of the Lorenz equations (6.40) that satisfies $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$ and is valid for $0 \le t \le 1000$. Because the behavior of solutions can be quite intricate, we include the option `MaxSteps->100000` to help Mathematica capture the oscillatory behavior in the long-term solution.

```
In[1431]:= lorenzsol[a_, b_, c_][{x0_, y0_, z0_},
             ts_ : {t, 0, 1000}, opts___] :=
            Module[{numsol},
            numsol = NDSolve[{x'[t] ==
                -a x[t] + a y[t],
                y'[t] == b x[t] - y[t] - x[t] z[t],
                z'[t] == x[t] y[t] - c z[t], x[0] == x0,
                y[0] == y0, z[0] == z0}, {x[t], y[t], z[t]},
                ts, MaxSteps → 100000]
            ]
```

We then use `lorenzplot` to generate a numerical solution for our parameter values and initial conditions.

```
In[1432]:= n2 = lorenzsol[7, 27.2, 3][{3, 4, 2}]

Out[1432]= {{x[t] → InterpolatingFunction[{{0.,
             1000.}}, <>][t],
            y[t] → InterpolatingFunction[{{0.,
             1000.}}, <>][t],
            z[t] → InterpolatingFunction[{{0.,
             1000.}}, <>][t]}}
```

We generate a short-term plot of the solution in Figure 6-54

```
In[1433]:= Plot[Evaluate[{x[t], y[t], z[t]}/.n2],
            {t, 0, 25}, PlotStyle → {GrayLevel[0],
            GrayLevel[0.3],
            Dashing[{0.01}]}, PlotPoints → 1000]
```

and a long-term plot in Figure 6-55.

```
In[1434]:= Plot[Evaluate[{x[t], y[t], z[t]}/.n2],
            {t, 950, 1000}, PlotStyle → {GrayLevel[0],
            GrayLevel[0.3],
            Dashing[{0.01}]}, PlotPoints → 1000]
```

In Figures 6-54 and 6-55 the oscillatory nature of the solutions is very difficult to see. We use `ParametricPlot` and `ParametricPlot3D` to generate parametric plots of the solutions in Figure 6-56.

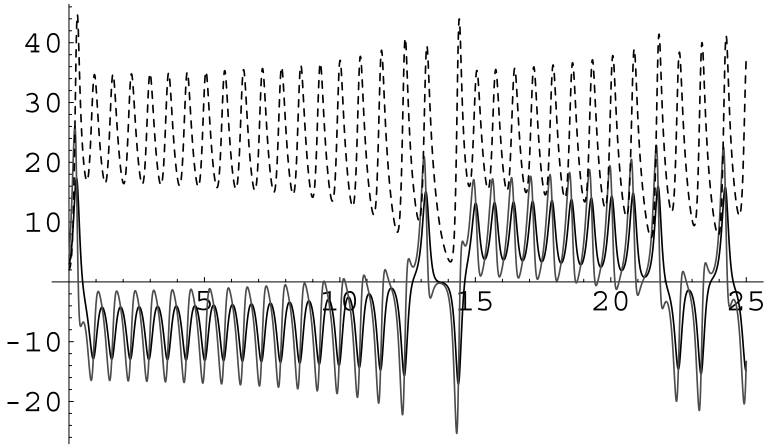


Figure 6-54 Plots of $x(t)$ (in black), $y(t)$ (in gray), and $z(t)$ (dashed) for $0 \le t \le 25$

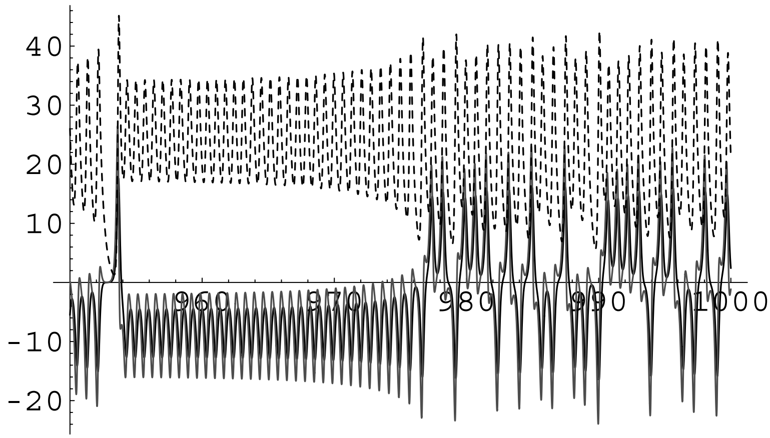


Figure 6-55 Plots of $x(t)$ (in black), $y(t)$ (in gray), and $z(t)$ (dashed) for $950 \le t \le 1000$

```
In[1435]:= p1a = ParametricPlot[Evaluate[{x[t],
                 y[t]}/.n2], {t, 0, 25}, PlotPoints → 1000,
                 AspectRatio → 1, AxesLabel → {"x", "y"},
                 DisplayFunction → Identity];

            p1b = ParametricPlot[Evaluate[{x[t],
                 z[t]}/.n2], {t, 0, 25}, PlotPoints → 1000,
                 AspectRatio → 1, AxesLabel → {"x", "z"},
                 DisplayFunction → Identity];
```

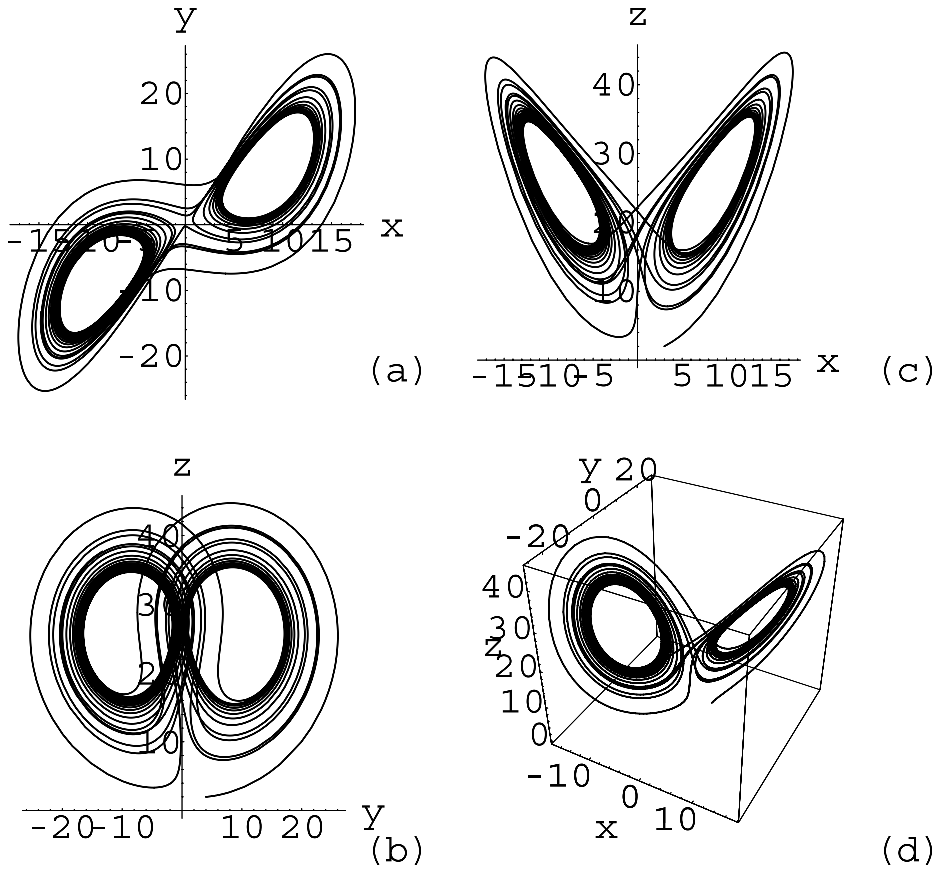


Figure 6-56 (a) *x* versus *y*; (b) *y* versus *z*; (c) *x* versus *z*; (d) *x* versus *y* versus *z*

```
In[1435]:= p1c = ParametricPlot[Evaluate[{y[t],
                   z[t]}/.n2], {t, 0, 25}, PlotPoints → 1000,
                   AspectRatio → 1, AxesLabel → {"y", "z"},
                   DisplayFunction → Identity];

            p1d = ParametricPlot3D[Evaluate[{x[t],
                   y[t], z[t]}/.n2], {t, 0, 25},
                   PlotPoints → 3000, BoxRatios → {1, 1, 1},
                   AxesLabel → {"x", "y", "z"},
                   DisplayFunction → Identity];

            Show[GraphicsArray[{{p1a, p1b}, {p1c, p1d}}]]
```

Of course, you could combine all of these commands into a single function. For example, given the appropriate parameter values

```
lorenzplot[a,b,c][{x0,y0,z0},{t,a,b}]
```

solves the Lorenz system using initial conditions $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$ for $a \le t \le b$, generates parametric plots of x versus y, y versus z, x versus z, and x versus y versus z, and displays the four resulting plots as a graphics array. If {t,a,b} is omitted, the default is $950 \le t \le 1000$.

The chaotic nature of the solutions to the Lorenz equations using these parameter values is seen more clearly in Figure 6-56 than in Figures 6-54 or 6-55.

```
In[1436]:= lorenzplot[a_, b_, c_][{x0_, y0_, z0_},
              ts_ : {t, 950, 1000}, opts___] :=
            Module[{numsol},
            numsol = NDSolve[{x'[t] ==
              -a x[t] + a y[t],
              y'[t] == b x[t] - y[t] - x[t] z[t],
              z'[t] == x[t] y[t] - c z[t], x[0] == x0,
              y[0] == y0, z[0] == z0}, {x[t],
              y[t], z[t]}, ts, MaxSteps → 100000];
          p1a = ParametricPlot[
                Evaluate[{x[t], y[t]}/.numsol], ts,
                PlotPoints → 1000, AspectRatio → 1,
                AxesLabel → {"x", "y"},
                DisplayFunction → Identity];
          p1b = ParametricPlot[
                Evaluate[{x[t], z[t]}/.numsol], ts,
                PlotPoints → 1000, AspectRatio → 1,
                AxesLabel → {"x", "z"},
                DisplayFunction → Identity];
          p1c = ParametricPlot[
                Evaluate[{y[t], z[t]}/.numsol], ts,
                PlotPoints → 1000, AspectRatio → 1,
                AxesLabel → {"y", "z"},
                DisplayFunction → Identity];
          p1d = ParametricPlot3D[
                Evaluate[{x[t], y[t], z[t]}/.numsol],
                ts, PlotPoints → 3000, BoxRatios → {1,
                1, 1}, AxesLabel → {"x", "y", "z"},
                DisplayFunction → Identity];
          Show[GraphicsArray[{{p1a, p1b},
              {p1c, p1d}}], opts]]
```

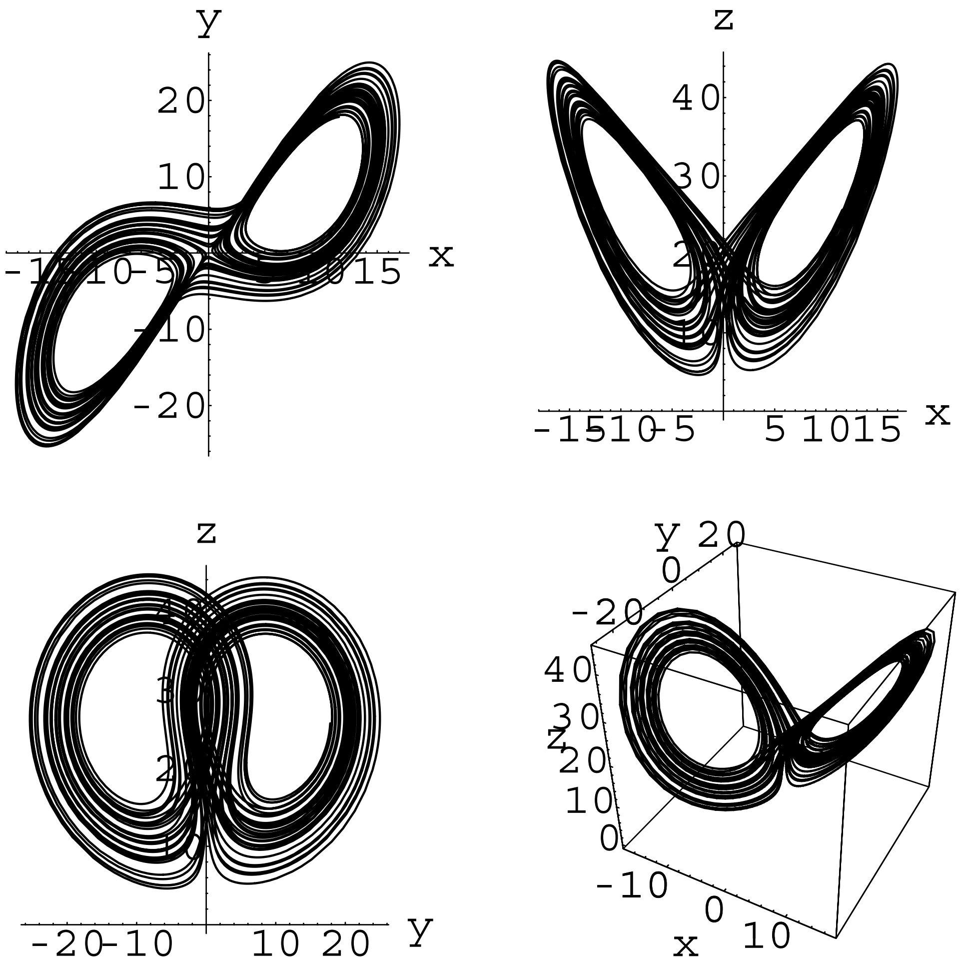


Figure 6-57 Changing b from 27.2 to 28: (a) x versus y; (b) y versus z; (c) x versus z; (d) x versus y versus z

For example, entering

```
In[1437]:= lorenzplot[7,28,3][{3,4,2}]
```

generates the four plots shown in Figure 6-57, corresponding to changing b from 27.2 to 28. Again, we obtain a chaotic solution.

■

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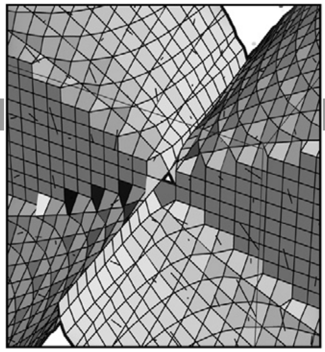
Applications of Systems of Ordinary Differential Equations

7



7.1 Mechanical and Electrical Problems with First-Order Linear Systems

7.1.1 *L–R–C* Circuits with Loops

As indicated in Chapter 5, an electrical circuit can be modeled with an ordinary differential equation with constant coefficients. In this section, we illustrate how a circuit involving loops can be described as a system of linear ordinary differential equations with constant coefficients. This derivation is based on the following principles.

> **Kirchhoff's Current Law:** The current entering a point of the circuit equals the current leaving the point.
>
> **Kirchhoff's Voltage Law:** The sum of the changes in voltage around each loop in the circuit is zero.

As was the case in Chapter 5, we use the following standard symbols for the components of the circuit:

$$I(t) = \text{current, where } I(t) = \frac{dQ}{dt}(t),$$

$$Q(t) = \text{charge,}$$

$$R = \text{resistance,}$$

C = capacitance,
V = voltage, and
L = inductance.

The relationships corresponding to the drops in voltage in the various components of the circuit that were stated in Chapter 5 are also given in the following table.

Circuit Element	Voltage Drop
Inductor	$L\frac{dI}{dt}$
Resistor	RI
Capacitor	$\frac{1}{C}Q$
Voltage Source	$-V(t)$

7.1.2 *L–R–C* Circuit with One Loop

In determining the drops in voltage around the circuit, we consistently add the voltages in the clockwise direction. The positive direction is directed from the negative symbol towards the positive symbol associated with the voltage source. In summing the voltage drops encountered in the circuit, a drop across a component is added to the sum if the positive direction through the component agrees with the clockwise direction. Otherwise, this drop is subtracted. In the case of the following *L–R–C* circuit with one loop involving each type of component, the current is equal around the circuit by Kirchhoff's Current Law as illustrated in Figure 7-1.

Also, by Kirchhoff's Voltage Law, we have the sum

$$RI + L\frac{dI}{dt} + \frac{1}{C}Q - V(t) = 0.$$

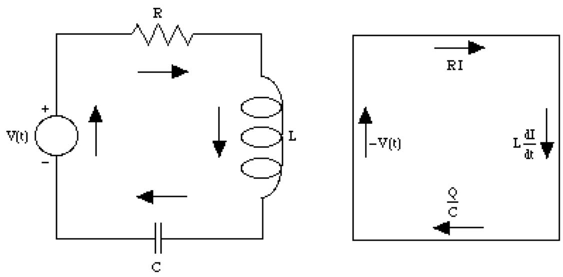


Figure 7-1 A simple *L–R–C* circuit

Solving this equation for dI/dt and using the relationship between I and Q, $dQ/dt = I$, we have the following system of differential equations with initial conditions on charge and current, respectively:

$$\begin{cases} dQ/dt = I \\ dI/dt = -\dfrac{1}{LC}Q - \dfrac{R}{L}I + \dfrac{V(t)}{L} \\ Q(0) = Q_0,\ I(0) = I_0. \end{cases} \tag{7.1}$$

EXAMPLE 7.1.1: Determine the charge and current in an L–R–C circuit with $L = 1$, $R = 2$, $C = 4/3$, and $V(t) = e^{-t}$ if $Q(0) = Q_0$ and $I(0) = I_0$.

SOLUTION: We begin by modeling the circuit with the system of differential equations

$$\begin{cases} dQ/dt = I \\ dI/dt = -\frac{3}{4}Q - 2I + e^{-t} \end{cases}$$

which can be written in matrix form as

$$\begin{pmatrix} dQ/dt \\ dI/dt \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3/4 & -2 \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}.$$

We solve the initial-value problem with `DSolve`, naming the result `sol`.

```
In[1438]:= Clear[q, i]

           sol = DSolve[{D[q[t], t] == i[t],
                   D[i[t], t] == -3/4q[t] - 2i[t] + Exp[-t],
                   q[0] == q0, i[0] == i0}, {q[t], i[t]}, t]
```

$$\text{Out[1438]=}\ \left\{\left\{q[t] \to \frac{1}{2} e^{-3t/2} (4 - 8 e^{t/2} + 4 e^{t} - 2\, i0 + 2 e^{t}\, i0 - q0 + 3 e^{t}\, q0),\right.\right.$$
$$\left.\left. i[t] \to -\frac{1}{4} e^{-3t/2} (12 - 16 e^{t/2} + 4 e^{t} - 6\, i0 + 2 e^{t}\, i0 - 3\, q0 + 3 e^{t}\, q0)\right\}\right\}$$

We now select, copy, and paste the formulas obtained in `sol` for Q and I, respectively, and then use `Expand` to distribute the $e^{-3t/2}$ term through the parentheses.

```
In[1439]:= Expand[1/2 e^(-3 t/2) (- 4 - 8 e^(t/2) - 2 (-4 + i0)
            +2 e^t (-4 + i0) - q0 + 3 e^t (4 + q0))]
```

$$\text{Out[1439]}= 2\,e^{-3t/2} - 4\,e^{-t} + 2\,e^{-t/2} - e^{-3t/2}\,\text{i0} + e^{-t/2}\,\text{i0} - \frac{1}{2}\,e^{-3t/2}\,\text{q0} + \frac{3}{2}\,e^{-t/2}\,\text{q0}$$

```
In[1440]:= Expand[- 1/4 e^(-3 t/2) (- 16 e^(t/2) - 6 (-4 + i0)
            +2 e^t (-4 + i0) - 3 (4 + q0) + 3 e^t (4 + q0))]
```

$$\text{Out[1440]}= -3\,e^{-3t/2} + 4\,e^{-t} - e^{-t/2} + \frac{3}{2}\,e^{-3t/2}\,\text{i0} - \frac{1}{2}\,e^{-t/2}\,\text{i0} + \frac{3}{4}\,e^{-3t/2}\,\text{q0} - \frac{3}{4}\,e^{-t/2}\,\text{q0}$$

The result indicates that $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} I(t) = 0$ regardless of the values of Q_0 and I_0. This is confirmed by graphing $Q(t)$ (in black) and $I(t)$ (in gray) together (choosing $Q(0) = I(0) = 1$) in Figure 7-2 as well as parametrically in Figure 7-3.

```
In[1441]:= Plot[Evaluate[{q[t],i[t]}/.sol/.
            {q0 → 1, i0 → 1}], {t, 0, 10},
            PlotStyle → {GrayLevel[0], GrayLevel[0.5]}]
```

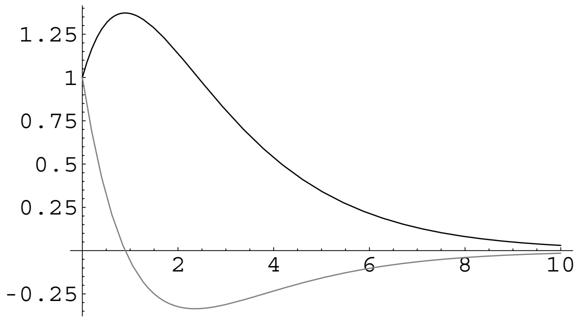


Figure 7-2 $Q(t)$ (in black) and $I(t)$ (in gray) for $0 \le t \le 10$

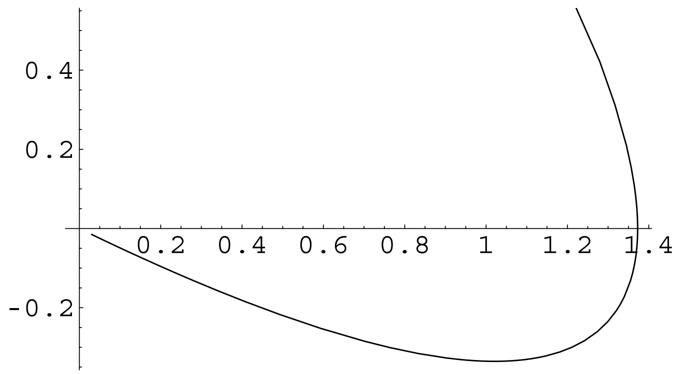


Figure 7-3 Parametric plot of Q versus I for $0 \le t \le 10$

```
In[1442]:= ParametricPlot[
            Evaluate[{q[t],i[t]}/.sol/.
            {q0 → 1,i0 → 1}],{t,0,10}]
```

■

7.1.3 *L–R–C* Circuit with Two Loops

The differential equations that model the circuit become more difficult to derive as the number of loops in the circuit increases. For example, consider the circuit in Figure 7-4 that contains two loops.

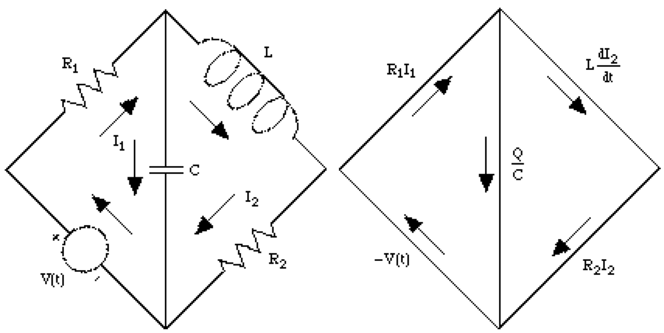


Figure 7-4 A two-loop circuit

In this case, the current through the capacitor is equivalent to $I_1 - I_2$. Summing the voltage drops around each loop, we have:

$$\begin{cases} R_1 I_1 + \dfrac{1}{C} Q - V(t) = 0 \\ L \dfrac{dI_2}{dt} + R_2 I_2 - \dfrac{1}{C} Q = 0. \end{cases} \tag{7.2}$$

Solving the first equation for I_1 we find that $I_1 = \dfrac{1}{R_1} V(t) - \dfrac{1}{R_1 C} Q$ and using the relationship $dQ/dt = I = I_1 - I_2$ we have the following system:

$$\begin{cases} \dfrac{dQ}{dt} = -\dfrac{1}{R_1 C} Q - I_2 + \dfrac{1}{R_1} V(t) \\ \dfrac{dI_2}{dt} = \dfrac{1}{LC} Q - \dfrac{R_2}{L} I_2 . \end{cases} \tag{7.3}$$

EXAMPLE 7.1.2: Find $Q(t)$, $I(t)$, $I_1(t)$, and $I_2(t)$ in the L–R–C circuit with two loops given that $R_1 = R_2 = C = 1$ and $V(t) = e^{-t}$ if $Q(0) = 3$ and $I_2(0) = 1$.

SOLUTION: The nonhomogeneous system that models this circuit is

$$\begin{cases} dQ/dt = -Q - I_2 + e^{-t} \\ dI_2/dt = Q - I_2 \end{cases}$$

with initial conditions $Q(0) = 3$ and $I_2(0) = 1$. We solve the initial-value problem with `DSolve` naming the result `sol`. We define $Q(t)$ and $I_2(t)$ to be the results.

```
In[1443]:= Clear[q, i]

           sol = DSolve[{D[q[t], t] == -q[t] - i2[t] + Exp[-t],
                  D[i2[t], t] == q[t] - i2[t], q[0] == 3,
                  i2[0] == 1}, {q[t], i2[t]}, t]

Out[1443]= {{q[t] → 3 e^-t Cos[t],
             i2[t] → e^-t (Cos[t]^2 + 3 Sin[t] + Sin[t]^2)}}

In[1444]:= q[t_] = sol[[1, 1, 2]];

In[1445]:= i2[t_] = sol[[1, 2, 2]];
```

We verify that these functions satisfy the system by substituting back into each equation and simplifying the result with `Simplify`.

```
In[1446]:= D[q[t],t] - (-q[t] - i2[t] + Exp[-t])//
             Simplify
Out[1446]= 0
```

```
In[1447]:= D[i2[t],t] - (q[t] - i2[t])//Simplify
Out[1447]= 0
```

We use the relationship $dQ/dt = I$ to find $I(t)$

```
In[1448]:= i[t_] = D[q[t],t]
Out[1448]= -3 e^-t Cos[t] - 3 e^-t Sin[t]
```

and then $I_1(t) = I(t) + I_2(t)$ to find $I_1(t)$.

```
In[1449]:= i1[t_] = i[t] + i2[t]
Out[1449]= e^-t - 3 e^-t Cos[t]
```

We graph $Q(t)$, $I(t)$, $I_1(t)$, and $I_2(t)$ with `Plot` and display the result using `Show` and `GraphicsArray` in Figure 7-5.

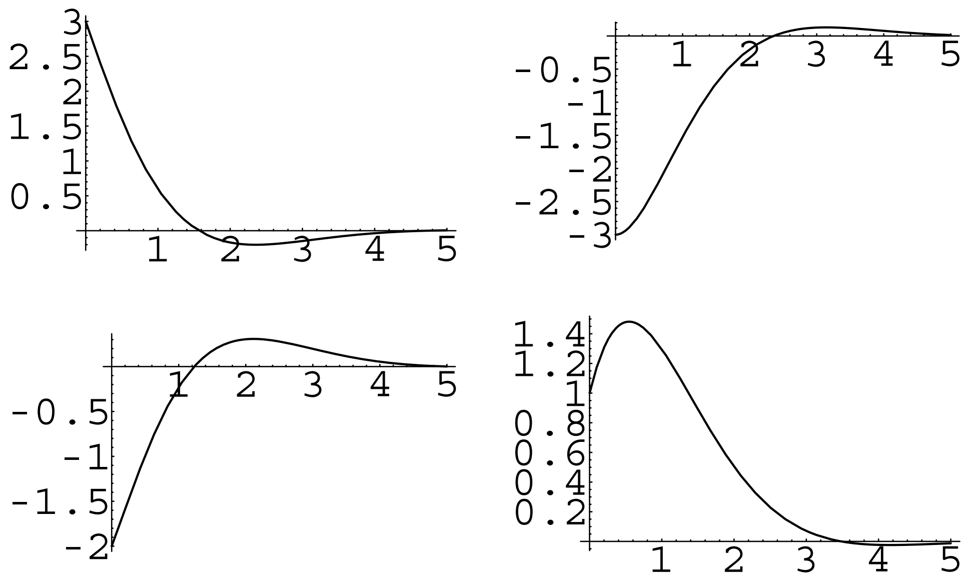


Figure 7-5 $Q(t)$, $I(t)$, $I_1(t)$, and $I_2(t)$ for $0 \le t \le 5$

```
In[1450]:= p1 = Plot[q[t], {t, 0, 5}, PlotRange → All,
                DisplayFunction → Identity];
            p2 = Plot[i[t], {t, 0, 5}, PlotRange → All,
                DisplayFunction → Identity];
            p3 = Plot[i1[t], {t, 0, 5}, PlotRange → All,
                DisplayFunction → Identity];
            p4 = Plot[i2[t], {t, 0, 5}, PlotRange → All,
                DisplayFunction → Identity];
            Show[GraphicsArray[{{p1, p2}, {p3, p4}}]]
```

■

7.1.4 Spring–Mass Systems

The displacement of a mass attached to the end of a spring was modeled with a second-order linear differential equation with constant coefficients in Chapter 5. This situation can then be expressed as a system of first-order ordinary differential equations as well. Recall that if there is no external forcing function, then the second-order differential equation that models this situation is $mx'' + cx' + kx = 0$, where m is the mass attached to the end of the spring, c is the damping coefficient, and k is the spring constant found with Hooke's law. This equation is transformed into a system of equations by letting $x' = y$ so that $y' = x'' = -\frac{k}{m}x - \frac{c}{m}x'$ and then solving the differential equation for x''. After substitution, we have the system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{k}{m}x - \dfrac{c}{m}y. \end{cases} \tag{7.4}$$

In previous chapters, the displacement of the spring was illustrated as a function of time. However, problems of this type may also be investigated using the phase plane.

EXAMPLE 7.1.3: Solve the system of differential equations to find the displacement of the mass if $m = 1$, $c = 0$, and $k = 1$.

SOLUTION: In this case, the system is $\begin{cases} dx/dt = y \\ dy/dt = -x \end{cases}$ which in matrix form is $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{X}$. A general solution is found with `DSolve` and named `gensol` for later use.

```
In[1451]:= Clear[x, y]

           gensol = DSolve[{D[x[t], t] == y[t],
                    D[y[t], t] == -x[t]}, {x[t], y[t]}, t]

Out[1451]= {{x[t] → C[1] Cos[t] + C[2] Sin[t],
             y[t] → C[2] Cos[t] - C[1] Sin[t]}}
```

Note that this system is equivalent to the second-order differential equation $x'' + x = 0$, which we solved in Chapters 4 and 5. At that time, we found a general solution to be $x(t) = c_1 \cos t + c_2 \sin t$ which is equivalent to the first component of $\mathbf{X} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, the result obtained with `DSolve`. Also notice that (0, 0) is the equilibrium point of the system. The eigenvalues of $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ are $\lambda = \pm i$,

```
In[1452]:= Eigenvalues[( 0  1
                        -1  0 )]

Out[1452]= {-i, i}
```

so we classify the origin as a center.

We graph several members of the phase plane for this system with `ParametricPlot` in Figure 7-6.

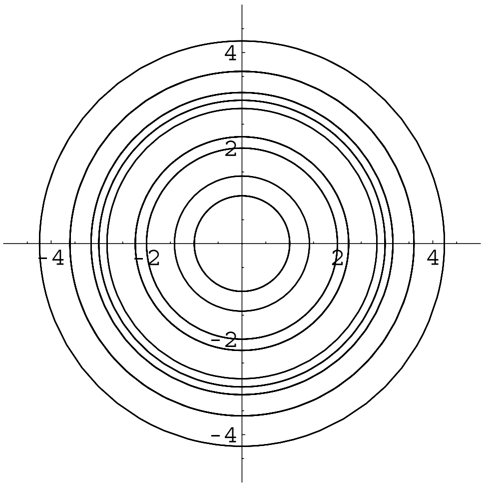


Figure 7-6 The origin is a center

```
In[1453]:= toplot = Flatten[
                      Table[{x[t],y[t]}/.gensol/.
                      {C[1] → i, C[2] → j}, {i, -3, 3},
                      {j, 0, 3}], 2];

In[1454]:= ParametricPlot[Evaluate[toplot],
              {t, 0, 2 π}, PlotRange → {{-5, 5}, {-5, 5}},
              AspectRatio → 1]
```

■

7.2 Diffusion and Population Problems with First-Order Linear Systems

7.2.1 Diffusion through a Membrane

Solving problems to determine the diffusion of a substance (such as glucose or salt) in a medium (like a blood cell) also leads to systems of first-order linear ordinary differential equations. For example, suppose that two solutions of a substance are separated by a membrane where the amount of the substance that passes through the membrane is proportional to the difference in the concentrations of the solutions. The constant of proportionality is called the **permeability**, P, of the membrane. Therefore, if we let x and y represent the concentration of each solution, and V_1 and V_2 represent the volume of each solution, respectively, then the system of differential equations is given by

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{P}{V_1}(y - x) \\ \dfrac{dy}{dt} = \dfrac{P}{V_2}(x - y) \end{cases} \tag{7.5}$$

where the initial concentrations of x and y are given.

EXAMPLE 7.2.1: Suppose that two salt concentrations of equal volume V are separated by a membrane of permeability P. Given that $P = V$, determine each concentration at time t if $x(0) = 2$ and $y(0) = 10$.

SOLUTION: In this case, the initial-value problem that models the situation is

$$\begin{cases} dx/dt = y - x \\ dy/dt = x - y \\ x(0) = 2,\ y(0) = 10. \end{cases}$$

A general solution of the system is found with `DSolve` and named `gensol`.

```
In[1455]:= Clear[x,y]

           gensol = DSolve[{D[x[t],t] == y[t] - x[t],
                   D[y[t],t] == x[t] - y[t]},
                   {x[t],y[t]},t]
```

$$\text{Out[1455]= } \left\{\left\{x[t] \to \frac{1}{2} e^{-2t} (1 + e^{2t})\, C[1] + \frac{1}{2} e^{-2t} (-1 + e^{2t})\, C[2],\ y[t] \to \frac{1}{2} e^{-2t} (-1 + e^{2t})\, C[1] + \frac{1}{2} e^{-2t} (1 + e^{2t})\, C[2]\right\}\right\}$$

We then apply the initial conditions and use `Solve` to determine the values of the arbitrary constants.

```
In[1456]:= cvals = Solve[{(- e^-2t C[1] + C[2]/.t- > 0) == 2,
                   (e^-2t C[1] + C[2]/.t- > 0) == 10}]
Out[1456]= {{C[1] → 4, C[2] → 6}}
```

The solution is obtained by substituting these values back into the general solution.

```
In[1457]:= sol = gensol/.cvals[[1]]
```

$$\text{Out[1457]= } \{\{x[t] \to 3 e^{-2t} (-1 + e^{2t}) + 2 e^{-2t} (1 + e^{2t}),\ y[t] \to 2 e^{-2t} (-1 + e^{2t}) + 3 e^{-2t} (1 + e^{2t})\}\}$$

Of course, `DSolve` can be used to solve the initial-value problem directly as well.

```
In[1458]:= sol = DSolve[{D[x[t],t] == y[t] - x[t],
                 D[y[t],t] == x[t] - y[t],x[0] == 2,
                 y[0] == 10},{x[t],y[t]},t]
```

$$\text{Out[1458]= } \{\{x[t] \to 2 e^{-2t} (-2 + 3 e^{2t}),\ y[t] \to 2 e^{-2t} (2 + 3 e^{2t})\}\}$$

We graph this solution parametrically with `ParametricPlot` in Figure 7-7(a). We then graph $x(t)$ and $y(t)$ together in Figure 7-7(b). Notice

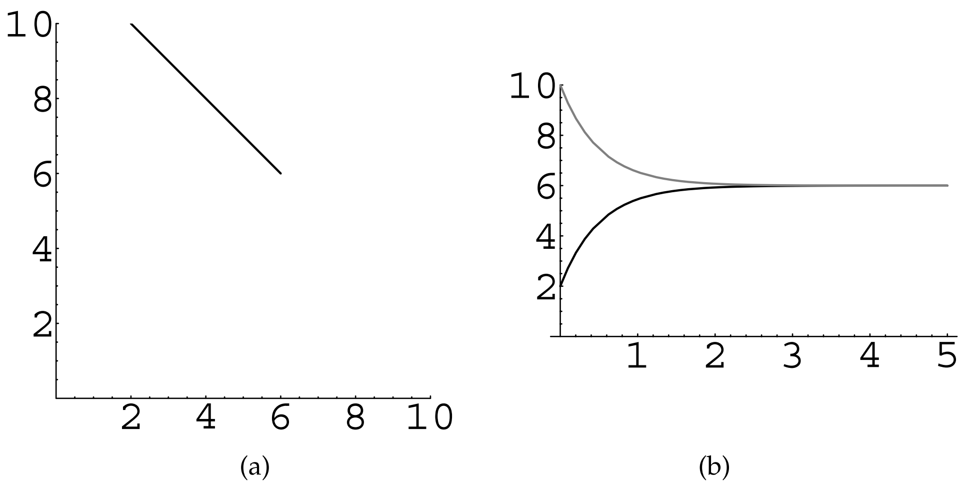


Figure 7-7 (a) Parametric plot of x versus y. (b) $x(t)$ (in black) and $y(t)$ (in gray)

that each concentration approaches 6 which is the average value of the two initial concentrations.

```
In[1459]:= p1 = ParametricPlot[{x[t], y[t]}/.sol,
                {t, 0, 5}, Compiled → False,
                PlotRange → {{0, 10}, {0, 10}},
                AspectRatio → 1, AxesOrigin → {0, 0},
                DisplayFunction → Identity];

In[1460]:= p2 = Plot[Evaluate[{x[t], y[t]}/.sol],
                {t, 0, 5}, PlotRange → {0, 10},
                PlotStyle → {GrayLevel[0],
                GrayLevel[0.5]},
                DisplayFunction → Identity];

In[1461]:= Show[GraphicsArray[{p1, p2}]]
```

■

7.2.2 Diffusion through a Double-Walled Membrane

Next, consider the situation in which two solutions are separated by a double-walled membrane, where the inner wall has permeability P_1 and the outer wall has permeability P_2 with $0 < P_1 < P_2$. Suppose that the volume of solution within the inner wall is V_1 and that between the two walls is V_2. Let x represent the concentration of the solution within the inner wall and y the concentration between the two walls. Assuming that the concentration of the solution outside the outer wall is constantly C, we have the following system of first-order ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{P_1}{V_1}(y-x) \\ \dfrac{dy}{dt} = \dfrac{1}{V_2}\left[P_2(C-y)+P_1(x-y)\right] \\ x(0)=x_0,\ y(0)=y_0\,. \end{cases} \tag{7.6}$$

EXAMPLE 7.2.2: Given that $P_1 = 3$, $P_2 = 8$, $V_1 = 2$, $V_2 = 10$, and $C = 10$, determine x and y if $x(0) = 2$ and $y(0) = 1$.

SOLUTION: In this case, we must solve the initial-value problem

$$\begin{cases} dx/dt = \frac{3}{2}(y-x) \\ dy/dt = -\frac{11}{10}y + \frac{3}{10}x + 8 \\ x(0)=2,\ y(0)=1\,. \end{cases}$$

A general solution of the corresponding homogeneous system is found with `DSolve`.

```
In[1462]:= Clear[x,y]

           homsol = DSolve[{x'[t] == 3/2 y[t] - 3/2 x[t],
                    y'[t] == -11/10 y[t] + 3/10 x[t]}, {x[t],
                    y[t]}, t]
Out[1462]= {{x[t] → 1/14 e^(-2 t) (9 + 5 e^(7 t/5)) C[1]
                    + 15/14 e^(-2 t) (-1 + e^(7 t/5)) C[2],
              y[t] → 3/14 e^(-2 t) (-1 + e^(7 t/5)) C[1]
                    + 1/14 e^(-2 t) (5 + 9 e^(7 t/5)) C[2]}}
```

The result indicates that a fundamental matrix for the corresponding homogeneous system is $\mathbf{\Phi}(t) = \begin{pmatrix} -3e^{-2t} & \frac{5}{3}e^{-3t/5} \\ e^{-2t} & e^{-3t/5} \end{pmatrix}$.

```
In[1463]:= Φ[t_] = ( -3 e^(-2 t)   5/3 e^(-3 t/5)
                      e^(-2 t)      e^(-3 t/5)    );
```

Therefore, using the method of variation of parameters, the solution to the initial-value problem is given by

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0)\mathbf{X}(0) + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(u)\mathbf{F}(u)\,du.$$

```
In[1464]:= sol = Φ[t].(Inverse[Φ[t]]/.t → 0).(2
                                              1)
                 +Φ[t].∫_0^t Inverse[Φ[u]].(0
                                            8) du//Simplify;
            MatrixForm[sol]
```

Out[1464]= $\begin{pmatrix} 10 + \frac{9\,e^{-2t}}{2} - \frac{25}{2}\,e^{-3t/5} \\ 10 - \frac{3\,e^{-2t}}{2} - \frac{15}{2}\,e^{-3t/5} \end{pmatrix}$

Of course, `DSolve` can be used to solve the initial-value problem directly, as well.

```
In[1465]:= sol = DSolve[{x'[t] == 3/2 (y[t] - x[t]),
                 y'[t] == -11 y[t]/10 + 3 x[t]/10 + 8, x[0] == 2,
                 y[0] == 1}, {x[t], y[t]}, t]
```

Out[1465]= $\left\{\left\{x[t] \to \frac{1}{2}\,e^{-2t}\,(9 - 25\,e^{7t/5} + 20\,e^{2t}),\ y[t] \to \frac{1}{2}\,e^{-2t}\,(-3 - 15\,e^{7t/5} + 20\,e^{2t})\right\}\right\}$

We graph this solution parametrically in addition to graphing the two functions simultaneously in Figure 7-8. Notice that initially $x(t) > y(t)$. However, the two graphs intersect at a value of t near $t \approx 0.2$ so that the value of the two functions approaches 10, which is the concentration of the solution outside the outer wall, as t increases.

```
In[1466]:= p1 = ParametricPlot[{x[t], y[t]}/.sol,
                {t, 0, 7}, Compiled → False,
                PlotRange → {{0, 10}, {0, 10}},
                AspectRatio → 1, AxesOrigin → {0, 0},
                DisplayFunction → Identity];

In[1467]:= p2 = Plot[Evaluate[{x[t], y[t]}/.sol],
                {t, 0, 7}, PlotRange → {0, 10},
                PlotStyle → {GrayLevel[0],
                GrayLevel[0.5]},
                DisplayFunction → Identity];
```

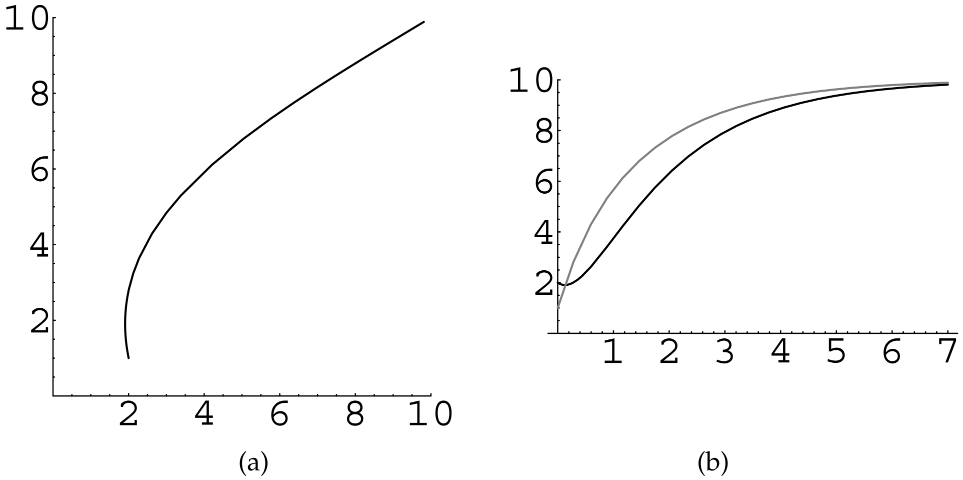


Figure 7-8 (a) Parametric plot of x versus y. (b) $x(t)$ (in black) and $y(t)$ (in gray) for $0 \le t \le 7$

```
In[1468]:= Show[GraphicsArray[{p1,p2}]]
```

Last, we plot the solution parametrically for various initial conditions.

```
In[1469]:= sol = DSolve[{x'[t] == 3/2 (y[t] - x[t]),
              y'[t] == -(11 y[t])/10 + (3 x[t])/10 + 8, x[0] == x0,
              y[0] == y0}, {x[t], y[t]}, t]
```

Out[1469]= $\left\{\left\{x[t] \to \frac{1}{14} e^{-2t} (60 - 200 e^{7t/5} + 140 e^{2t} + 9\,x0 + 5 e^{7t/5} x0 - 15\,y0 + 15 e^{7t/5} y0),\right.\right.$
$\left.\left. y[t] \to \frac{1}{14} e^{-2t} (-20 - 120 e^{7t/5} + 140 e^{2t} - 3\,x0 + 3 e^{7t/5} x0 + 5\,y0 + 9 e^{7t/5} y0)\right\}\right\}$

Notice how the formulas for $x(t)$ and $y(t)$ are extracted from `sol` with `Part([[...]])`. The formula for $x(t)$ is the second part of the first part of the first part of `sol`; the formula for $y(t)$ is the second part of the second part of the first part of `sol`.

```
In[1470]:= sol[[1,1,2]]

           sol[[1,2,2]]
```

Out[1470]= $\frac{1}{14} e^{-2t} (60 - 200 e^{7t/5} + 140 e^{2t} + 9\,x0 + 5 e^{7t/5} x0 - 15\,y0 + 15 e^{7t/5} y0)$

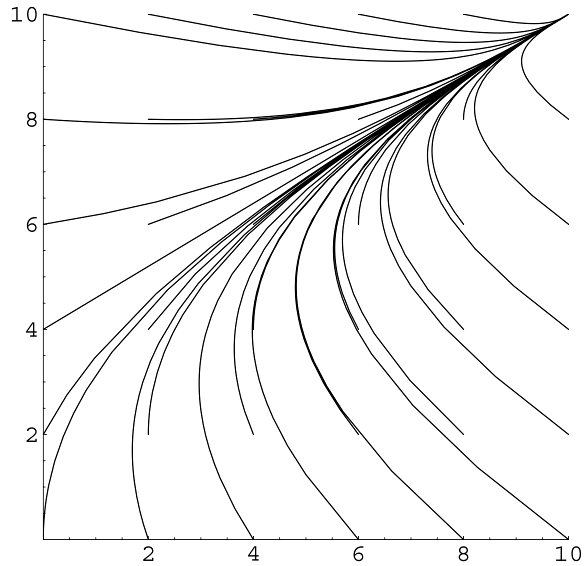


Figure 7-9 Both concentrations approach 10, regardless of the initial conditions

```
Out[1470]= 1/14 e^(-2 t) (-20 - 120 e^(7 t/5)
            +140 e^(2 t) - 3 x0 + 3 e^(7 t/5) x0 + 5 y0 + 9 e^(7 t/5) y0)
```

Then, we use `Table` and `Flatten` to construct a list of (pairs of) functions to be plotted with `ParametricPlot`. `Short` is used to display an abbreviated portion of `toplot`.

```
In[1471]:= toplot = Flatten[
                Table[{sol[[1, 1, 2]], sol[[1, 2, 2]]},
                {x0, 0, 10, 2}, {y0, 0, 10, 2}], 1];

           Short[toplot, 2]
Out[1471]= {<<1>>, <<34>>, {10, 10}}
```

The list of functions in `toplot` is then graphed with `ParametricPlot` for $0 \leq t \leq 7$ in Figure 7-9.

```
In[1472]:= ParametricPlot[Evaluate[toplot], {t, 0, 7},
             PlotRange → {{0, 10}, {0, 10}},
             AspectRatio → 1]
```

■

7.2.3 Population Problems

In Chapter 3, population problems were discussed that were based on the principle that the rate at which a population grows (or decays) is proportional to the number present in the population at any time t. Hence, if $x(t)$ represents the population at time t, $dx/dt = kx$ for some constant k. This idea can be extended to problems involving more than one population and leads to systems of ordinary differential equations. We illustrate several situations through the following examples. Note that in each problem, we determine the rate at which a population of size P changes with the equation

$$\frac{dP}{dt} = \left(\text{rate entering}\right) - \left(\text{rate leaving}\right).$$

We begin by determining the population in two neighboring territories. Suppose that the population x and y of two neighboring territories depends on several factors. The birth rate of x is a_1 while that of y is b_1. The rate at which citizens of x move to y is a_2 while that at which citizens move from y to x is b_2. Finally, the mortality rate of each territory is disregarded. Determine the respective populations of these two territories for any time t.

Using the principles of previous examples, we have that the rate at which population x changes is

$$\frac{dx}{dt} = a_1x - a_2x + b_1y = (a_1 - a_2)x + b_2y$$

while the rate at which population y changes is

$$\frac{dy}{dt} = b_1y - b_2y + a_2x = (b_1 - b_2)y + a_2x.$$

Therefore, the system of equations that must be solved is

$$\begin{cases} dx/dt = (a_1 - a_2)x + b_1y \\ dy/dt = a_2x + (b_1 - b_2)y \end{cases} \tag{7.7}$$

where the initial populations of the two territories $x(0) = x_0$ and $y(0) = y_0$ are given.

EXAMPLE 7.2.3: Determine the populations $x(t)$ and $y(t)$ in each territory if $a_1 = 5$, $a_2 = 4$, $b_1 = -1$, and $b_2 = 1$ given that $x(0) = 60$ and $y(0) = 10$.

SOLUTION: In this example, the initial-value problem that models the situation is

$$\begin{cases} dx/dt = x + y \\ dy/dt = 4x - 2y \\ x(0) = 60, \; y(0) = 10 \end{cases}$$

which we solve with DSolve.

```
In[1473]:= Clear[x, y]

           sol = DSolve[{x'[t] == x[t] + y[t],
                  y'[t] == 4 x[t] - 2 y[t], x[0] == 60,
                  y[0] == 10}, {x[t], y[t]}, t]

Out[1473]= {{x[t] → 10 e^(-3 t) (1 + 5 e^(5 t)),
             y[t] → 10 e^(-3 t) (-4 + 5 e^(5 t))}}
```

We graph these two population functions with Plot in Figure 7-10. Notice that as t increases, the two populations are approximately the same.

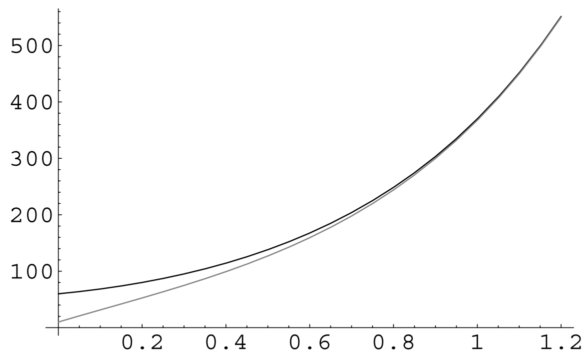


Figure 7-10 As t increases, the two populations are approximately the same

```
In[1474]:= Plot[Evaluate[{x[t],y[t]}/.sol],
              {t,0,1.2},
              PlotStyle → {GrayLevel[0],GrayLevel[0.5]}]
```

■

Population problems that involve more than two neighboring populations can be solved with a system of differential equations as well. Suppose that the population of three neighboring territories x, y, and z depends on several factors. The birth rates of x, y, and z are a_1, b_1, and c_1, respectively. The rate at which citizens of x move to y is a_2 while that at which citizens move from x to z is a_3. Similarly, the rate at which citizens of y move to x is b_2 while that at which citizens move from y to z is b_3. Also, the rate at which citizens of z move to x is c_2 while that at which citizens move from z to y is c_3. Suppose that the mortality rate of each territory is ignored in the model.

The system of equations in this case is similar to that derived in the previous example. The rate at which population x changes is

$$\frac{dx}{dt} = a_1x - a_2x - a_3x + b_2y + c_2z = (a_1 - a_2 - a_3)\,x + b_2y + c_2z,$$

while the rate at which population y changes is

$$\frac{dy}{dt} = b_1y - b_2y - b_3y + a_2x + c_3z = (b_1 - b_2 - b_3)\,y + a_2x + c_3z,$$

and that of z is

$$\frac{dz}{dt} = c_1z - c_2z - c_3z + a_3x + b_3y = (c_1 - c_2 - c_3)\,z + a_3x + b_3y.$$

Hence, we must solve the 3×3 system

$$\begin{cases} dx/dt = (a_1 - a_2 - a_3)\,x + b_2y + c_2z \\ dy/dt = (b_1 - b_2 - b_3)\,y + a_2x + c_3z \\ dz/dt = (c_1 - c_2 - c_3)\,z + a_3x + b_3y \end{cases} \tag{7.8}$$

where the initial populations $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$ are given.

EXAMPLE 7.2.4: Determine the population of the three territories if $a_1 = 3$, $a_2 = 0$, $a_3 = 2$, $b_1 = 4$, $b_2 = 2$, $b_3 = 1$, $c_1 = 5$, $c_2 = 3$, and $c_3 = 0$ if $x(0) = 50$, $y(0) = 60$, and $z(0) = 25$.

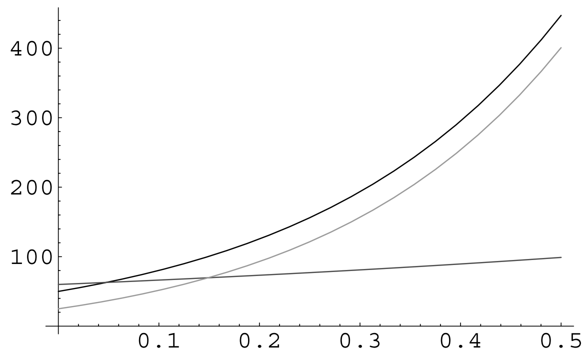


Figure 7-11 Three neighboring territories

SOLUTION: We solve the initial-value problem

$$\begin{cases} dx/dt = x + 2y + 3z \\ dy/dt = y \\ dz/dt = 2x + y + 2z \\ x(0) = 50,\ y(0) = 60,\ z(0) = 25 \end{cases}$$

with DSolve.

```
In[1475]:= Clear[x,y]

           sol = DSolve[{x′[t] == x[t] + 2 y[t]
                 +3 z[t], y′[t] == y[t], z′[t] == 2 x[t] + y[t]
                 +2 z[t], x[0] == 50, y[0] == 60, z[0] == 25},
                   {x[t], y[t], z[t]}, t]
Out[1475]= {{x[t] → e^-t (-3 - 10 e^2t + 63 e^5t),
               y[t] → 60 e^t, z[t] → e^-t (2 - 40 e^2t + 63 e^5t)}}
```

The graphs of these three population functions are generated with Plot in Figure 7-11. We notice that although y was initially greater than populations x and z, these populations increase at a much higher rate than does y.

```
In[1476]:= Plot[Evaluate[{x[t], y[t], z[t]}/.sol],
              {t, 0, 0.5},
              PlotStyle → {GrayLevel[0], GrayLevel[0.3],
              GrayLevel[0.6]}]
```

■

7.3 Applications that Lead to Nonlinear Systems

Several special equations and systems that arise in the study of many areas of applied mathematics can be solved using the techniques of Chapter 6. These include the predator–prey population dynamics problem, the Van der Pol equation that models variable damping in a spring–mass system, and the Bonhoeffer–Van der Pol (BVP) oscillator. We begin by considering the Lotka–Volterra system, which models the interaction between two populations.

7.3.1 Biological Systems: Predator–Prey Interactions, The Lotka–Volterra System, and Food Chains in the Chemostat

The Lotka–Volterra System

Let $x(t)$ and $y(t)$ represent the number of members at time t of the prey and predator populations, respectively. (Examples of such populations include fox/rabbit and shark/seal.) Suppose that the positive constant a is the birth rate of $x(t)$ so that in the absence of the predator $dx/dt = ax$ and that c is the death rate of y which indicates that $dy/dt = -cy$ in the absence of the prey population. In addition to these factors, the number of interactions between predator and prey affects the number of members in the two populations. Note that an interaction increases the growth of the predator population and decreases the growth of the prey population, because an interaction between the two populations indicates that a predator overtakes a member of the prey population. In order to include these interactions in the model, we assume that the number of interactions is directly proportional to the product of $x(t)$ and $y(t)$. Therefore, the rate at which $x(t)$ changes with respect to time is $dx/dt = ax - bxy$. Similarly, the rate at which $y(t)$ changes with respect to time is $dy/dt = -cy + dxy$. Therefore, we must solve the **Lotka–Volterra system**

$$\begin{cases} dx/dt = ax - bxy \\ dy/dt = -cy + dxy \end{cases} \tag{7.9}$$

subject to the initial populations $x(0) = x_0$ and $y(0) = y_0$.

EXAMPLE 7.3.1: Find and classify the equilibrium points of the Lotka–Volterra system.

SOLUTION: We solve $\begin{cases} ax - bxy = 0 \\ -cy + dxy = 0 \end{cases}$ to see that the equilibrium points are (0, 0) and $(c/d, a/b)$.

```
In[1477]:= f[x_, y_] = ax - bxy;
           g[x_, y_] = -c y + dxy;
           Solve[{f[x, y] == 0, g[x, y] == 0}, {x, y}]
Out[1477]= {{x → 0, y → 0}, {x → c/d, y → a/b}}
```

To classify these equilibrium points, we first calculate the Jacobian matrix of the nonlinear system.

The Jacobian matrix is also called the **variational matrix**.

```
In[1478]:= jac = ( D[f[x, y], x]  D[f[x, y], y]
                   D[g[x, y], x]  D[g[x, y], y] );
           MatrixForm[jac]
Out[1478]= ( a - b y   -b x
               d y    -c + d x )
```

At (0, 0), we have $\mathbf{J}(0, 0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ with eigenvalues $\lambda_1 = -c$ and $\lambda_2 = a$.

```
In[1479]:= jac /. {x- > 0, y- > 0}//Eigenvalues
Out[1479]= {a, -c}
```

Because these eigenvalues are real with opposite sign, we classify (0, 0) as a saddle; (0, 0) is unstable. Similarly, at $(c/d, a/b)$, we have $\mathbf{J}(c/d, a/b) = \begin{pmatrix} 0 & -bc/d \\ ad/b & 0 \end{pmatrix}$ with eigenvalues $\lambda_{1,2} = \pm i\sqrt{ac}$.

```
In[1480]:= jac /. {x- > c/d, y- > a/b}//Eigenvalues
Out[1480]= {- i √a √c, i √a √c}
```

Therefore, the point $(c/d, a/b)$ is classified as a center in the linearized system. We show the direction field associated with the system using the values $a = 2$, $b = 1$, $c = 3$, and $d = 1$ in Figure 7-12. The direction field indicates that all solutions oscillate about the center.

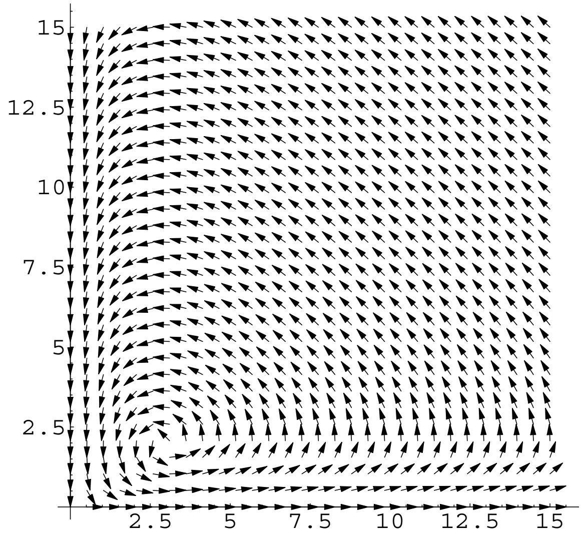


Figure 7-12 Typical direction field associated with the Lotka–Volterra system

```
In[1481]:= << Graphics`PlotField`

           PlotVectorField[{2x - xy, -3y + xy},
             {x, 0, 15}, {y, 0, 15}, Axes- > Automatic,
             AxesOrigin- > {0, 0}, ScaleFunction- > (1&),
             PlotPoints- > 30]
```

This observation is confirmed by graphing several curves in the phase plane of the system for these values of a, b, c, and d. See Figure 7-13.

```
In[1482]:= Clear[x, y, t, s]

           graph[s0_] := Module[{numsol, pp, pxy},
                          GraphicsArray[{pxy, pp}]]
               numsol =
                 NDSolve[{x'[t] == 2 x[t] - x[t] y[t],
                     y'[t] == -3 y[t] + x[t] y[t], x[0] == 3s0,
                     y[0] == 2s0}, {x[t], y[t]}, {t, 0, 15}];
           pp = ParametricPlot[{x[t], y[t]}/.numsol,
                  {t, 0, 4}, Compiled → False,
                  PlotRange → {{0, 15}, {0, 15}},
                  AspectRatio → 1, Ticks → {{3}, {2}},
                  DisplayFunction → Identity];
```

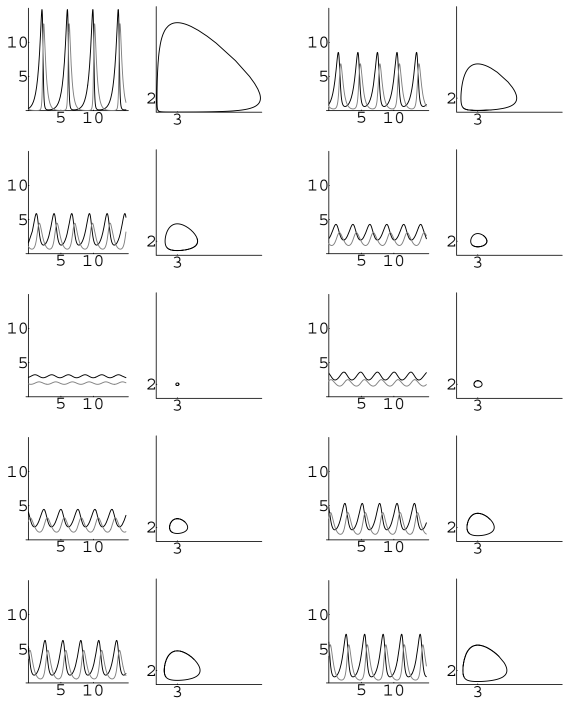


Figure 7-13 Typical solutions of the Lotka–Volterra system

```
In[1482]:= pxy = Plot[Evaluate[{x[t], y[t]}/.numsol],
              {t, 0, 15},
              PlotStyle → {GrayLevel[0],
              GrayLevel[0.5]}, PlotRange → {0, 15},
              AspectRatio → 1,
              Ticks- > {{5, 10}, {5, 10}},
              DisplayFunction → Identity];

In[1483]:= graphs = Table[graph[s], {s, 0.1, 2, 1.9/9}];

In[1484]:= toshow = Partition[graphs, 2];

           Show[GraphicsArray[toshow]]
```

Notice that all of the solutions oscillate about the center. These solutions reveal the relationship between the two populations: prey, $x(t)$, and predator, $y(t)$. As we follow one cycle counterclockwise beginning, for example, near the point (3, 2), we notice that as $x(t)$ increases, then $y(t)$ increases until $y(t)$ becomes overpopulated. Then, because the prey population is too small to supply the predator population, $y(t)$ decreases which leads to an increase in the population of $x(t)$. At this point, because the number of predators becomes too small to control the number in the prey population, $x(t)$ becomes overpopulated and the cycle repeats itself.

■

An interesting variation of the Lotka–Volterra equations is to assume that a depends strongly on environmental factors and might be given by the differential equation

$$\frac{da}{dt} = -ax + \bar{a} + k\sin(\omega t + \phi), \tag{7.10}$$

where the term $-ax$ represents the loss of nutrients due to species x; $\bar{a}$, k, ω, and ϕ are constants. Observe that incorporating equation (7.10) into system (7.9) results in a nonautonomous system.

EXAMPLE 7.3.2: Suppose that $x(0) = y(0) = a(0) = 0.5$, $b = d = 1$, $c = 0.5$, $\bar{a} = 0.25$, $k = 0.125$, and $\phi = 0$. Plot $x(t)$ and $y(t)$ if $\omega = 0.1$, 0.25, 0.5, 0.75, 1, 1.25, 1.5, and 2.5.

SOLUTION: Given the appropriate parameter values and initial conditions, `solgraph` solves

$$\begin{cases} dx/dt = ax - bxy \\ dy/dt = -cy + dxy \\ da/dt = -ax + \bar{a} + k\sin(\omega t + \phi) \\ x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0 \end{cases} \tag{7.11}$$

plots $x(t)$ (in black) and $y(t)$ (in gray), parametrically plots $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, and displays the results side-by-side. Any options included are passed to the `Show` command. If `{t,a,b}` is omitted from the `solgraph` command, the default is `{t,0,40}`.

```
In[1485]:= solgraph[b_, d_, c_, abar_, k_, ω_, ϕ_]
             [{x0_, y0_, a0_}, ts_:{t, 0, 40},
             opts___] := Module[{numsol, p1, p2},
               numsol = NDSolve[{x′[t] == a[t] x[t]
               -b x[t] y[t], y′[t] == -c y[t]
               +d x[t]y[t], a′[t] == -a[t] x[t]
               +abar + k Sin[ω t + ϕ], x[0] == x0,
               y[0] == y0, a[0] == a0},
               {x[t], y[t], a[t]}, ts];
               p1 = ParametricPlot[{x[t], y[t]}/.numsol,
                    ts, Compiled → False,
                   PlotRange → {{0, 1}, {0, 1}},
                   AspectRatio → Automatic,
                   DisplayFunction → Identity,
                   AxesLabel → {"x", "y"}, Ticks → {{0, 1},
                   {0, 1}}];
               p2 = Plot[Evaluate[{x[t],
                    y[t]}/.numsol], ts,
                   PlotStyle → {GrayLevel[0],
                   GrayLevel[0.4]}, PlotRange → {0, 1},
                   DisplayFunction → Identity,
                   AxesLabel → {"t", "x, y"},
                   Ticks → {Automatic, {0, 1}}];
               Show[GraphicsArray[{p2, p1}], opts]
             ]
```

For example, entering

```
In[1486]:= solgraph[1, 1, 0.5, 0.25, 0.125, 0.3, 0]
             [{0.5, 0.5, 0.5}]
```

graphs the solution to the initial-value problem (7.11) for our parameter values and initial conditions if $\omega = 0.3$ shown in Figure 7-14. We then use `Map` to apply `solgraph` to the list of numbers {0.01, 0.1, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 2.5}. The resulting plots are not displayed because we include the option `DisplayFunction->Identity` in the `solgraph` command.

```
In[1487]:= solgraph[1, 1, 0.5, 0.25, 0.125, 0.3, 0]
             [{0.5, 0.5, 0.5}, {t, 0, 50},
             DisplayFunction → Identity]
Out[1487]= -GraphicsArray-
```

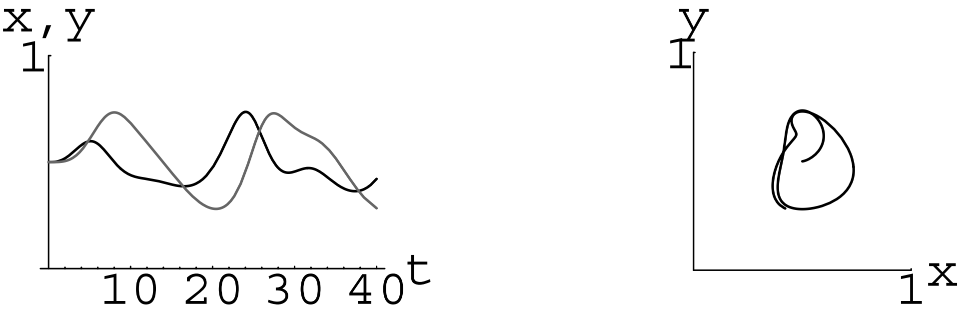


Figure 7-14 $a = a(t)$, $\omega = 0.3$

```
In[1488]:= toshow = Map[solgraph[1, 1, 0.5, 0.25, 0.125,
                #, 0][{0.5, 0.5, 0.5}, {t, 0, 50},
                DisplayFunction → Identity]&,
                {0.1, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 2.5}]

Out[1488]= {-GraphicsArray-, -GraphicsArray-,
             -GraphicsArray-, -GraphicsArray-,
             -GraphicsArray-, -GraphicsArray-,
             -GraphicsArray-, -GraphicsArray-}
```

`Partition` is used to partition `toshow` into two element subsets and the resulting array of graphics is displayed using `Show` and `GraphicsArray` in Figure 7-15.

```
In[1489]:= Show[GraphicsArray[Partition[toshow, 2]]]
```

(Note that if instead you had entered

```
In[1490]:= Show[GraphicsArray[Partition[toshow, 1]]]
```

the plots would have been displayed vertically instead of side-by-side.) From the graphs, we see that larger values of ω appear to stabilize the populations of both species; smaller values of ω appear to cause the size of the populations to oscillate widely.

■

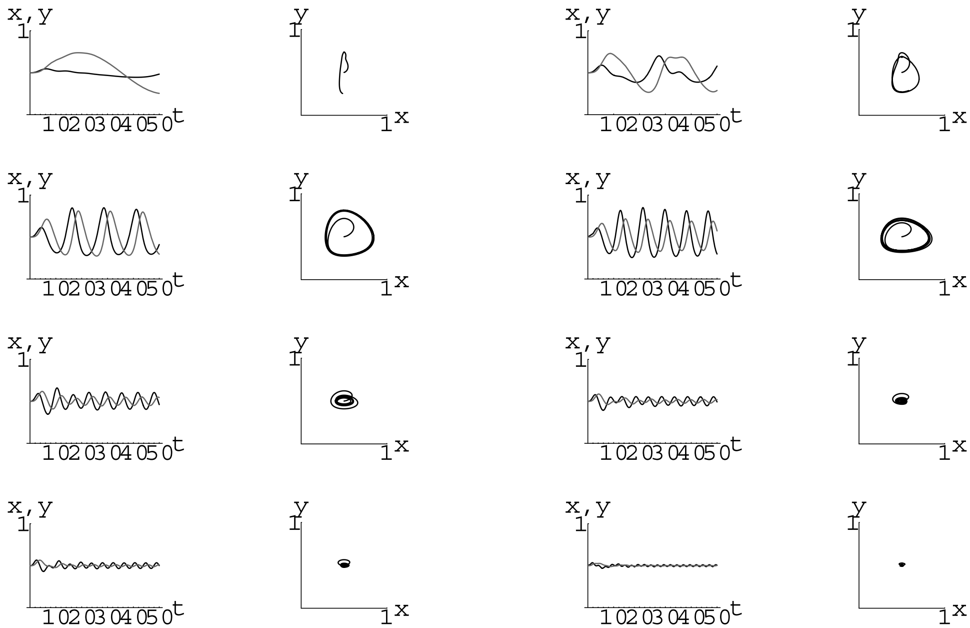


Figure 7-15 Varying ω

7.3.1.1 Simple Food Chain in a Chemostat

See Smith and Waltman's, *The Theory of the Chemostat: Dynamics of Microbial Competition* [24] for a detailed discussion of various chemostat models. Previously, we discussed growth in the chemostat in Example 3.2.6.

The equations that describe a simple food chain in a chemostat are

$$\begin{cases} \dfrac{dS}{dt} = 1 - S - \dfrac{m_1 x S}{a_1 + S} \\ \dfrac{dx}{dt} = \dfrac{m_1 x S}{a_1 + S} - x - \dfrac{m_2 x y}{a_2 + x} \\ \dfrac{dy}{dt} = \dfrac{m_2 x y}{a_2 + x} - y \\ S(0) = S_0,\ x(0) = x_0,\ y(0) = y_0\,. \end{cases} \tag{7.12}$$

In system (7.12), y (the predator) consumes x (the prey) and x consumes the nutrient S.

Now let $\Sigma = 1 - S - x - y$. Then $\Sigma' = -S' - x' - y' = -(1 - S - x - y) = -\Sigma$ so $\Sigma = \Sigma_0 e^{-t}$ and $\lim_{t\to\infty} \Sigma = 0$. In the limit as $t \to \infty$, $\Sigma = 0 = 1 - S - x - y$ so $S = 1 - x - y$ and system (7.12) becomes

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{m_1 x (1 - x - y)}{1 + a_1 - x - y} - x - \dfrac{m_2 x y}{a_2 + x} \\ \dfrac{dy}{dt} = \dfrac{m_2 x y}{a_2 + x} - y \\ x(0) = x_0,\ y(0) = y_0\,. \end{cases} \tag{7.13}$$

The analysis of system (7.13) is quite technical and beyond the scope of this text. We illustrate how Mathematica can assist in carrying out a few of the computations needed when analyzing system (7.13).

See Chapter 3 of Smith and Waltman's *The Theory of the Chemostat: Dynamics of Microbial Competition* [24] for a detailed analysis of system (7.12).

The rest points of system (7.13) are found by solving

$$\begin{cases} \dfrac{m_1 x(1-x-y)}{1+a_1-x-y} - x - \dfrac{m_2 xy}{a_2+x} = 0 \\ \dfrac{m_2 xy}{a_2+x} - y = 0. \end{cases}$$

```
In[1491]:= xeq = x(m1(1 - x - y)/(1 + a1 - x - y) - 1 - m2 y/(a2 + x));
            yeq = y(m2 x/(a2 + x) - 1);

In[1492]:= rps = Solve[{xeq == 0, yeq == 0}, {x, y}]//Simplify
```

$$\text{Out[1492]}= \left\{\{x\to 0, y\to 0\}, \left\{y\to 0, x\to \frac{-1-a1+m1}{-1+m1}\right\},\right.$$
$$\left\{y\to \frac{1}{2(-1+m2)}\left(-1-2\,a2+a2\,m1+a1\,(-1+m2)+m2+\sqrt{((1+a2\,m1-m2)^2+a1^2\,(-1+m2)^2+2\,a1\,(-1+m2)\,(-1+a2\,m1+m2))}\right), x\to \frac{a2}{-1+m2}\right\},$$
$$\left.\left\{y\to \frac{1}{2(-1+m2)}\left(-1-2\,a2+a2\,m1+a1\,(-1+m2)+m2-\sqrt{((1+a2\,m1-m2)^2+a1^2\,(-1+m2)^2+2\,a1\,(-1+m2)\,(-1+a2\,m1+m2))}\right), x\to \frac{a2}{-1+m2}\right\}\right\}$$

From the results, we see that $E_0 = (0, 0)$ is a rest point. If the appropriate quantities are positive, another boundary rest point may exist as well as an interior rest point.

In `jac`, we compute the Jacobian, **J**, of system (7.12).

```
In[1493]:= jac = {{D[xeq, x], D[xeq, y]}, {D[yeq, x], D[yeq, y]}}
```

$$\text{Out[1493]}= \left\{\left\{-1+\frac{m1\,(1-x-y)}{1+a1-x-y}-\frac{m2\,y}{a2+x}+x\left(\frac{m1\,(1-x-y)}{(1+a1-x-y)^2}-\frac{m1}{1+a1-x-y}+\frac{m2\,y}{(a2+x)^2}\right),\right.\right.$$
$$\left. x\left(-\frac{m2}{a2+x}+\frac{m1\,(1-x-y)}{(1+a1-x-y)^2}-\frac{m1}{1+a1-x-y}\right)\right\},$$
$$\left.\left\{\left(-\frac{m2\,x}{(a2+x)^2}+\frac{m2}{a2+x}\right)y, -1+\frac{m2\,x}{a2+x}\right\}\right\}$$

At E_0, $\mathbf{J}(E_0)$ is

```
In[1494] := j0 = jac/.rps[[1]]//FullSimplify
```

$$Out[1494]= \left\{\left\{-1+\frac{m1}{1+a1},0\right\},\{0,-1\}\right\}$$

with eigenvalues

```
In[1495] := Eigenvalues[j0]
```

$$Out[1495]= \left\{-1,-1+\frac{m1}{1+a1}\right\}$$

In the context of the problem, it is desirable for E_0 to be unstable. Thus, we require that $m_1 > 1$ and

$$-1+\frac{m_1}{a_1+1}>0 \qquad \text{or, equivalently,} \qquad 1-\frac{a_1}{m_1-1}<0.$$

We define λ_i to be $\lambda_i = \frac{a_i}{m_i - 1}$.

With this assumption, the boundary point $E_1 = \left(1-\frac{a_1}{m_1-1},0\right) = (1-\lambda_1, 0)$ exists. At E_1, $\mathbf{J}(E_1)$ is given by

```
In[1496] := j1 = jac/.rps[[2]]//FullSimplify
```

$$Out[1496]= \left\{\left\{\frac{(1+a1-m1)(-1+m1)}{a1\,m1},\ (1+a1-m1)\left(\frac{-1+m1}{a1\,m1}-\frac{m2}{1+a1+a2-(1+a2)\,m1}\right)\right\},\ \left\{0,-1+\frac{(1+a1-m1)\,m2}{1+a1+a2-(1+a2)\,m1}\right\}\right\}$$

with eigenvalues

```
In[1497] := Eigenvalues[j1]
```

$$Out[1497]= \left\{\frac{(1+a1-m1)(-1+m1)}{a1\,m1},-1+\frac{(1+a1-m1)\,m2}{1+a1+a2-(1+a2)\,m1}\right\}$$

E_1 may be stable or unstable. It can be shown that E_1 is stable if $\lambda_1 + \lambda_2 > 1$ and a saddle (unstable) if $\lambda_1 + \lambda_2 < 1$. ***If*** an interior rest point exists, Mathematica can compute the Jacobian as well as the eigenvalues. At E_A, the $\mathbf{J}(E_A)$ is

```
In[1498]:= j3 = jac/.rps[[3]]//FullSimplify

Out[1498]= {{ - (2 (a1^2 (-1 + m2)^2
            - (-1 + m1) (1 + a2 m1 - m2)
            × (1 + a2 m1 - m2 + √((1 + a2 m1 - m2)^2
            +a1^2 (-1 + m2)^2 + 2 a1
            × (-1 + m2) (-1 + a2 m1 + m2)))
            -a1 (-1 + m2) (2 - 2 m2
            +m1 (-1 + a2 (m1 - 2 m2) + m2)
            +√((1 + a2 m1 - m2)^2
            +a1^2 (-1 + m2)^2 + 2 a1 (-1 + m2)
            × (-1 + a2 m1 + m2)))))
            /(m2 SuperscriptBox((1 + a1 + a2 m1 - m2
            -a1 m2 + √((1 + a2 m1 - m2)^2+)
            × a1^2 (-1 + m2)^2 + 2 a1 (-1 + m2)
            × (-1 + a2 m1 + m2)))^2),
            - (2 (a1^2 (-1 + m2)^2 + a1 (-1 + m2)
            × ( - 2 + 2 a2 m1 + 2 m2 - √((1 + a2 m1 - m2)^2
            +a1^2 (-1 + m2)^2 + 2 a1 (-1 + m2)
            × (-1 + a2 m1 + m2)))
            + (1 + a2 m1 - m2)
            × (1 + a2 m1 - m2 + √((1 + a2 m1 - m2)^2
            +a1^2 (-1 + m2)^2 + 2 a1
            × (-1 + m2) (-1 + a2 m1 + m2)))))
            /SuperscriptBox((1 + a1 + a2 m1 - m2 - a1 m2
            +√((1 + a2 m1 - m2)^2+) a1^2 (-1 + m2)^2
            +2 a1 (-1 + m2) (-1 + a2 m1 + m2)))^2},
            {    1
             ─────── ((-1 + m2) ( - 1 + a2 (-2 + m1)
             2 a2 m2
            +a1 (-1 + m2) + m2
            +√((1 + a2 m1 - m2)^2 + a1^2 (-1 + m2)^2
            +2 a1 (-1 + m2) (-1 + a2 m1 + m2)))), 0}}
```

The command `Eigenvalues[j3]` returns the eigenvalues of `j3`; however, the result is very lengthy so it is not shown here for length considerations. Refer to Chapter 3 of Smith and Waltman, [24].

Incorporating a second predator, z, of x into system (7.14) results in

$$\begin{cases} \dfrac{dS}{dt} = 1 - S - \dfrac{m_1 x S}{a_1 + S} \\ \dfrac{dx}{dt} = \dfrac{m_1 x S}{a_1 + S} - x - \dfrac{m_2 x y}{a_2 + x} - \dfrac{m_3 x z}{a_3 + x} \\ \dfrac{dy}{dt} = \dfrac{m_2 x y}{a_2 + x} - y \\ \dfrac{dz}{dt} = \dfrac{m_3 x z}{a_3 + x} - z \\ S(0) = S_0,\ x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0 . \end{cases} \tag{7.14}$$

In the same way as with system (7.14), we let $\Sigma = 1 - S - x - y - z$. Then, $\Sigma' = -\Sigma$ so $\lim_{t\to\infty} \Sigma = 0$. Substitution of Σ into system (7.14) and taking the limit $t \to \infty$ results in

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{m_1 x (1 - x - y - z)}{1 + a_1 - x - y - z} - x - \dfrac{m_2 x y}{a_2 + x} - \dfrac{m_3 x z}{a_3 + x} \\ \dfrac{dy}{dt} = \dfrac{m_2 x y}{a_2 + x} - y \\ \dfrac{dz}{dt} = \dfrac{m_3 x z}{a_3 + x} - z \\ S(0) = S_0,\ x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0 . \end{cases} \tag{7.15}$$

System (7.15) can exhibit *very* interesting behavior.

EXAMPLE 7.3.3: Let $a_1 = .3$, $a_2 = .4$, $m_1 = 8$, $m_2 = 4.5$, and $m_3 = 5.0$. If $x(0) = .1$, $y(0) = .1$, and $z(0) = .3$, how does varying a_3 affect the solutions of system (7.15)?

SOLUTION: We define the function `predplot`:

```
predplot[{a1,a2,a3},{m1,m2,m3}][{x0,y0,z0},{t,a,b}
,opts]
```

solves system (7.15) subject to the initial conditions $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$ for $a \le t \le b$, plots $x(t)$ (in black), $y(t)$ (in gray), and $z(t)$ (dashed), parametrically plots x versus y versus z, displays the resulting plots side-by-side, and returns a numerical solution to the initial-value problem. Any options included are passed to the `Show` command. If you do not include any options and omit {t,a,b}, the default is $0 \le t \le 100$.

```
In[1499]:= Clear[predplot];

           predplot[{a1_, a2_, a3_}, {m1_, m2_, m3_}][
              {x0_, y0_, z0_}, ts_ : {t, 0, 100}, opts___] :=
             Module[{numsol, p1, p2, p3},
              numsol = NDSolve[
                 {x'[t] ==
                    x[t]
                      (m1 (1 - x[t] - y[t] - z[t])/
                              (a1 + 1 - x[t]
                              -y[t] - z[t]) - 1-
                          y[t] m2 /(a2 + x[t])
                          -z[t] m3/(a3 + x[t])),
                  y'[t] == y[t] (m2 x[t]/(a2 + x[t]) - 1),
                  z'[t] == z[t] (m3 x[t]/(a3 + x[t]) - 1),
                  x[0] == x0, y[0] == y0, z[0] == z0},
                 {x[t], y[t], z[t]}, ts, MaxSteps → 100000];
              p1 = Plot[Evaluate[{x[t], y[t],
                 z[t]}/.numsol],
                 ts, PlotRange → {0, 1},
                 PlotStyle → {GrayLevel[0],
                 GrayLevel[0.3],
                    Dashing[{0.01}]},
                 DisplayFunction → Identity,
                 Ticks → {{ts[[2]], ts[[3]]}, {0, 1}},
                 AxesLabel → {"t", ""}];
              p2 = ParametricPlot3D[
                 Evaluate[{x[t], y[t], z[t]}/.numsol],
                 ts, PlotRange → {{0, 1}, {0, 1}, {0, 1}},
                 AxesLabel → {"x", "y", "z"},
                 BoxRatios → {1, 1, 1},
                 DisplayFunction → Identity,
                 Ticks → {{0, 1}, {0, 1}, {0, 1}},
                 ViewPoint- > {2.21, 2.211, 1.294},
                 Boxed → False, PlotPoints → 2000];
              Show[GraphicsArray[{p1, p2}], opts];
              numsol
             ]
```

For example, entering

```
In[1500]:= predplot[{0.3, 0.4, 0.455}, {8, 4.5, 5.}]
             [{0.1, 0.1, 0.3}, {t, 50, 60}]
```

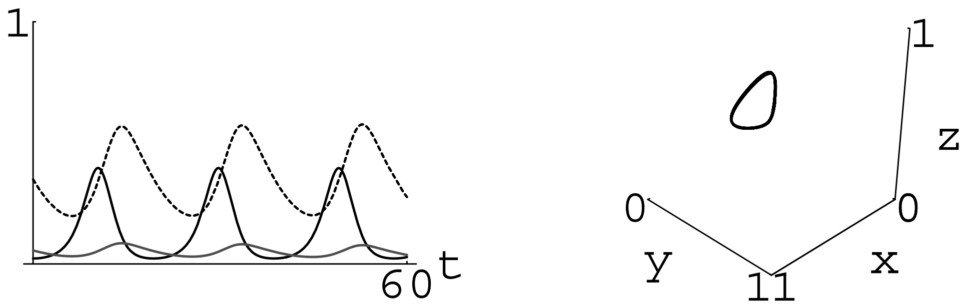


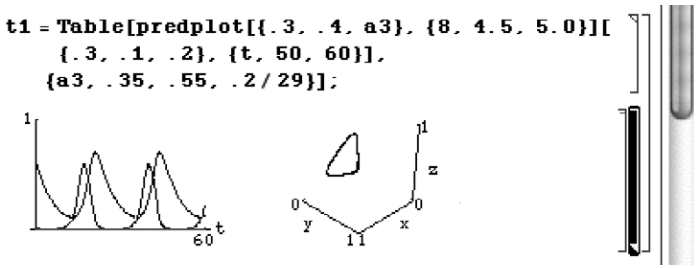
Figure 7-16 If $a_3 = 0.455$, y and z coexist

```
Out[1500]= {{x[t] → InterpolatingFunction[{{50., 60.}},
               <>][t],
             y[t] → InterpolatingFunction[{{50., 60.}},
               <>][t],
             z[t] → InterpolatingFunction[{{50., 60.}},
               <>][t]}}
```

plots the solutions shown in Figure 7-16 using our parameter values and initial conditions for $50 \le t \le 60$ if $a_3 = 0.455$. We vary a_3 in `t1`.

```
In[1501]:= t1 = Table[predplot[{0.3, 0.4, a3}, {8, 4.5, 5.}]
               [{0.3, 0.1, 0.2}, {t, 50, 60}],
               {a3, 0.35, 0.55, 0.2/29}];
```

The resulting graphs result in a striking animation.



You can also visualize the cycles by displaying all the parametric plots together. In `t1`, each result is an approximate solution that we can use. In the following, we use `ParametricPlot3D` to graph each solution.

```
In[1502]:= toshow = Table[ParametricPlot3D[
                Evaluate[{z[t],y[t],x[t]}/.
                t1[[i]][[1]]],{t,50,60},
                PlotRange → {{0,1},{0,1},{0,1}},
                AxesLabel → {"z","y","x"},
                BoxRatios → {1,1,1},
                DisplayFunction → Identity,
                Ticks → {{0,1},{0,1},{0,1}},
                ViewPoint- > {1.47, 2.424, 1.847},
                Boxed → False, PlotPoints → 2000],
                {i,1,30}];
```

The results are displayed together with `Show` in Figure 7-17.

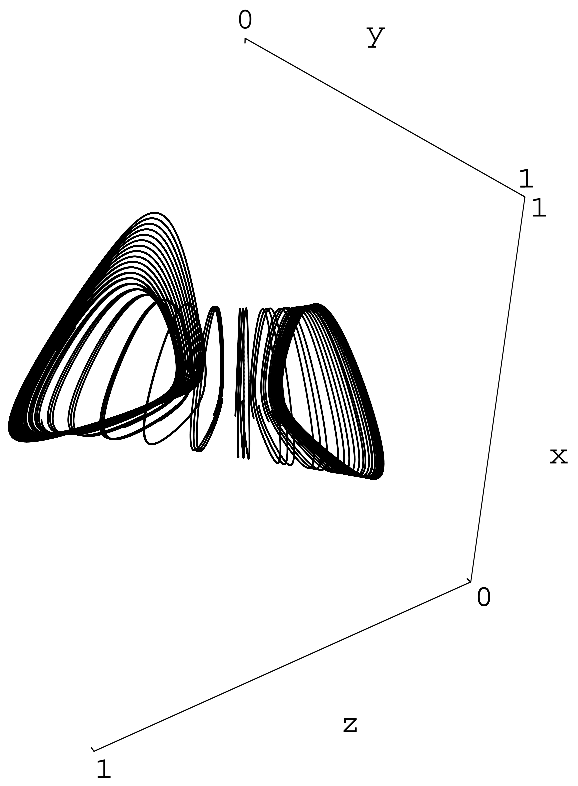


Figure 7-17 In an animation, you can see the limit cycle move from the *xz*-plane to the *xy*-plane as a_3 increases

```
In[1503]:= Show[toshow, DisplayFunction- >
              $DisplayFunction]
```

In the plots we see that if a_3 is small, z dominates the predator population, if a_3 is large, y dominates the predator population. For moderate values, y and z coexist.

■

7.3.1.2 Long Food Chain in a Chemostat

In system (7.12), y predates on x. Incorporating a predator z of y into system (7.12) results in

$$\begin{cases} \dfrac{dS}{dt} = 1 - S - \dfrac{m_1 x S}{a_1 + S} \\ \dfrac{dx}{dt} = \dfrac{m_1 x S}{a_1 + S} - x - \dfrac{m_2 x y}{a_2 + x} \\ \dfrac{dy}{dt} = \dfrac{m_2 x y}{a_2 + x} - y - \dfrac{m_3 y z}{a_3 + y} \\ \dfrac{dz}{dt} = \dfrac{m_3 y z}{a_3 + y} - z. \end{cases} \tag{7.16}$$

As with system (7.12), in the limit as $t \to \infty$, $S = 1 - x - y - z$ so system (7.16) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt} = x\left[f_1(1 - x - y - z) - 1\right] - y f_2(x) \\ \dfrac{dy}{dt} = y\left[f_2(x) - 1\right] - z f_3(y) \\ \dfrac{dz}{dt} = z\left[f_3(y) - 1\right] \end{cases} \tag{7.17}$$

where

$$f_i(u) = \frac{m_i u}{a_i + u}. \tag{7.18}$$

Of course, rigorous analysis of system (7.17) is even more complicated than the analysis of system (7.12).

EXAMPLE 7.3.4: Let $a_1 = .08$, $a_2 = .23$, $m_1 = 10$, $m_2 = 4.0$, and $m_3 = 3.5$. If $x(0) = .3$, $y(0) = .1$, and $z(0) = .2$, how does varying a_3 affect the solutions of system (7.17)?

SOLUTION: We define `longchainplot` in the same way as we defined `predplot` in Example 7.3.3.

```
In[1504]:= longchainplot[{a1_, a2_, a3_}, {m1_, m2_, m3_}][
              {x0_, y0_, z0_}, ts_ : {t, 0, 100}] :=
            Module[{numsol}, numsol = NDSolve[
                 {x'[t] ==
                    x[t]
                         (m1 (1 - x[t] - y[t] - z[t])/
                         (a1 + 1 - x[t] - y[t] - z[t]) - 1) -
                      y[t] m2 x[t]/(a2 + x[t]),
                   y'[t] == y[t] (m2 x[t]/(a2 + x[t]) - 1) -
                      z[t] m3 y[t]/(a3 + y[t]),
                   z'[t] == z[t] (m3 y[t]/(a3 + y[t]) - 1),
                   x[0] == x0, y[0] == y0, z[0] == z0},
                 {x[t], y[t], z[t]}, ts, MaxSteps → 100000];
              p1 =
                Plot[Evaluate[{x[t], y[t], z[t]}/.numsol],
                 ts, PlotRange → {0, 1},
                 PlotStyle → {GrayLevel[0],
                    GrayLevel[0.3], Dashing[{0.01}]},
                 DisplayFunction → Identity,
                 Ticks → {{ts[[2]], ts[[3]]}, {0, 1}},
                 AxesLabel → {"t", ""}];
              p2 = ParametricPlot3D[
                 Evaluate[{x[t], y[t], z[t]}/.numsol],
                 ts, PlotRange → {{0, 1}, {0, 1}, {0, 1}},
                 AxesLabel → {"x", "y", "z"},
                 BoxRatios → {1, 1, 1},
                 DisplayFunction → Identity,
                 Ticks → {{0, 1}, {0, 1}, {0, 1}},
                 ViewPoint- > {2.21, 2.211, 1.294},
                 Boxed → False, PlotPoints → 2000];
              Show[GraphicsArray[{p1, p2}]];
              numsol
            ]
```

For example, entering

```
In[1505]:= longchainplot[{0.08, 0.23, 0.4}, {10, 4, 3.5}]
            [{0.3, 0.1, 0.2}, {t, 50, 60}]

Out[1505]= {{x[t] → InterpolatingFunction
              [{{50., 60.}}, <>][t],
              y[t] → InterpolatingFunction[{{50., 60.}},
              <>][t], z[t] → InterpolatingFunction[{{50.
              , 60.}}, <>][t]}}
```

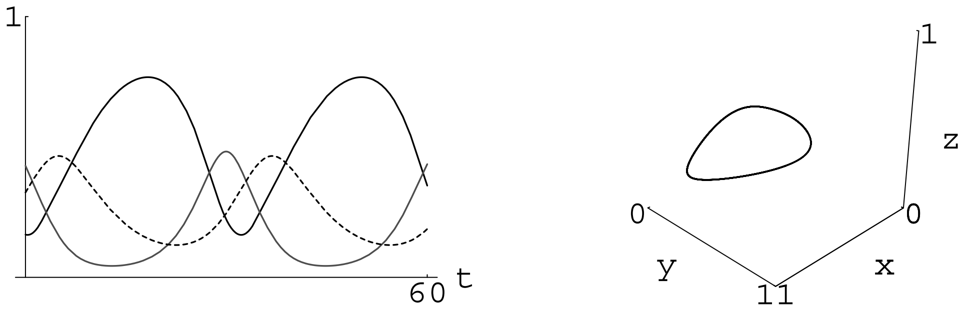


Figure 7-18 $x(t)$ (in black), $y(t)$ (in gray), and $z(t)$ (dashed) if $a_3 = .4$ for $0 \leq t \leq 60$

plots the solution of system (7.17) using our parameter values and initial conditions if $a_3 = .4$ for $50 \leq t \leq 60$ in Figure 7-18. In Figure 7-19, we plot the solutions using the given parameter values and initial conditions using $a_3 = 0.3, 0.26, 0.24, 0.22$, and 0.2 for $975 \leq t \leq 1000$. In the plots, we see that the solution appears chaotic for $a_3 \approx .2$.

```
In[1506]:= r1 = Map[longchainplot[{0.08, 0.23, #},
              {10, 4, 3.5}][{0.3, 0.1, 0.2},
             {t, 975, 1000}]&, {0.3, 0.26, 0.24, 0.22, 0.2}]
```

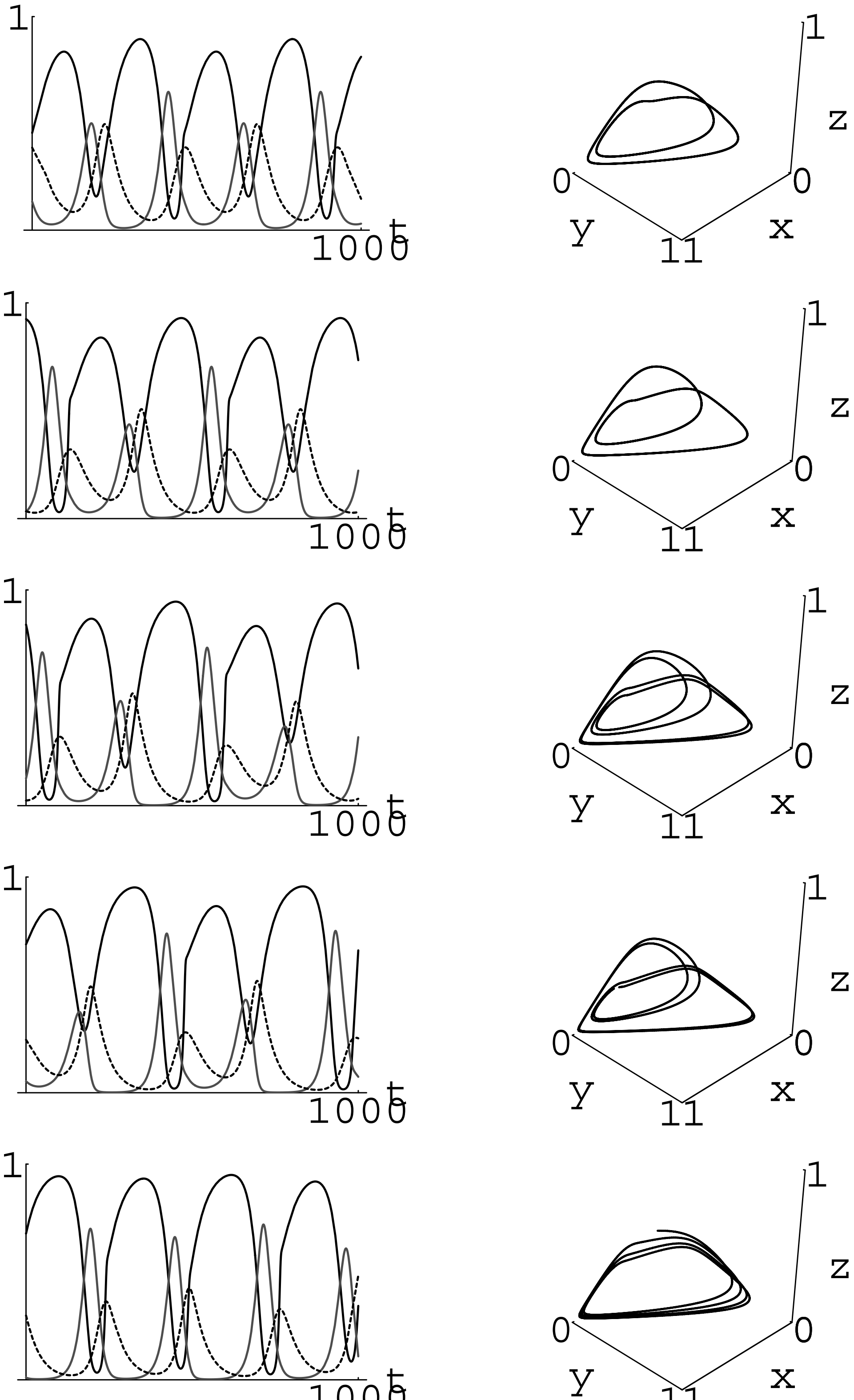
The apparent chaotic behavior for $a_3 = .2$ is more apparent in Figure 7-20, where we graph the solution for $1100 \leq t \leq 1200$.

```
In[1507]:= longchainplot[{0.08, 0.23, 0.2}, {10, 4, 3.5}]
           [{0.3, 0.1, 0.2}, {t, 1100, 1200}]
```

■

7.3.2 Physical Systems: Variable Damping

In some physical systems, energy is fed into the system when there are small oscillations while energy is taken from the system when there are large oscillations. This indicates that the system undergoes "negative damping" for small oscillations and "positive damping" for large oscillations. A differential equation that models this situation is **Van-der-Pol's equation**.


1
t
1000
1
z
0
0
y
x
11

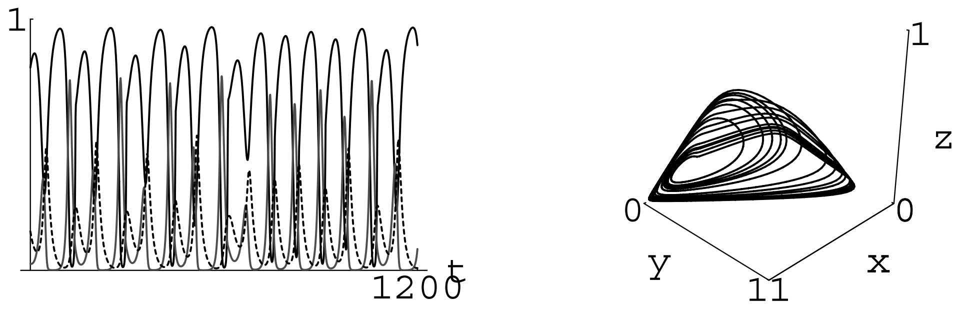


Figure 7-20 The solution appears to be chaotic if $a_3 = .2$

Also see Example 4.1.1.

EXAMPLE 7.3.5 (Van-der-Pol's equation): In the introduction to Chapter 6, we saw that **Van-der-Pol's equation** $x'' + \mu\left(x^2 - 1\right)x' + x = 0$ is equivalent to the system

$$\begin{cases} x' = y \\ y' = \mu\left(1 - x^2\right)y - x. \end{cases}$$

Classify the equilibrium points, use `NDSolve` to approximate the solutions to this nonlinear system, and plot the phase plane.

SOLUTION: We find the equilibrium points by solving $\begin{cases} y = 0 \\ \mu\left(1 - x^2\right)y - x = 0 \end{cases}$. From the first equation, we see that $y = 0$. Then, substitution of $y = 0$ into the second equation yields $x = 0$. Therefore, the only equilibrium point is (0, 0). The Jacobian matrix for this system is

$$\mathbf{J}(x, y) = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & -\mu\left(x^2 - 1\right) \end{pmatrix}.$$

The eigenvalues of $\mathbf{J}(0, 0)$ are $\lambda_{1,2} = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4}\right)$.

```
In[1508]:= Clear[f, g]

           f[x_, y_] = y;

           g[x_, y_] = -x - μ (x^2 - 1) y;

In[1509]:= jac = ( D[f[x, y], x]  D[f[x, y], y]
                   D[g[x, y], x]  D[g[x, y], y] );
```

```
In[1510]:= jac /. {x- > 0, y- > 0}//Eigenvalues
```

$$\text{Out[1510]} = \left\{\frac{1}{2}\left(\mu - \sqrt{-4+\mu^2}\right), \frac{1}{2}\left(\mu + \sqrt{-4+\mu^2}\right)\right\}$$

Notice that if $\mu > 2$, then both eigenvalues are positive and real. Hence, we classify (0, 0) as an **unstable node**. On the other hand, if $0 < \mu < 2$, then the eigenvalues are a complex conjugate pair with a positive real part. Hence, (0, 0) is an **unstable spiral**. (We omit the case $\mu = 2$ because the eigenvalues are repeated.)

We now show several curves in the phase plane that begin at various points for various values of μ. First, we define the function `sol`, which given μ, x_0, and y_0, generates a numerical solution to the initial-value problem

$$\begin{cases} x' = y \\ y' = \mu\left(1 - x^2\right)y - x \\ x(0) = x_0,\ y(0) = y_0 \end{cases}$$

and then parametrically graphs the result for $0 \le t \le 20$.

```
In[1511]:= Clear[sol]

           sol[μ_, {x0_, y0_}, opts___] :=
            Module[{eqone, eqtwo, solt},
            eqone = x′[t] == y[t];
           eqtwo = y′[t] == μ (1 - x[t]^2) y[t] - x[t];
           solt = NDSolve[{eqone, eqtwo, x[0] == x0,
                   y[0] == y0}, {x[t], y[t]}, {t, 0, 20}];
           ParametricPlot[{x[t], y[t]}/.solt,
                {t, 0, 20}, Compiled → False, opts]]
```

We then use `Table` and `Union` to generate a list of ordered pairs `initconds` that will correspond to the initial conditions in the initial-value problem.

```
In[1512]:= initconds1 = Table[{0.1 Cos[t], 0.1 Sin[t]},
                         {t, 0, 2π, 2π/9}];

           initconds2 = Table[{-5, i}, {i, -5, 5, 10/9}];

           initconds3 = Table[{5, i}, {i, -5, 5, 10/9}];

           initconds4 = Table[{i, 5}, {i, -5, 5, 10/9}];

           initconds5 = Table[{i, -5}, {i, -5, 5, 10/9}];
```

```
In[1513]:= initconds = initconds1 ∪ initconds2∪
                initconds3 ∪ initconds4 ∪ initconds5;
```

We then use `Map` to apply `sol` to the list of ordered pairs in `initconds` for $\mu = 1/2$.

```
In[1514]:= somegraphs1 = Map[sol[1/2, #, DisplayFunction- >
                        Identity]&, initconds];

In[1515]:= phase1 = Show[somegraphs1,
                    PlotRange- > {{-5, 5}, {-5, 5}},
                    AspectRatio- > 1, Ticks- > {{-4, 4},
                    {-4, 4}}];
```

Similarly, we use `Map` to apply `sol` to the list of ordered pairs in `initconds` for $\mu = 1, 3/2$, and 3.

```
In[1516]:= somegraphs2 = Map[sol[1, #, DisplayFunction- >
                        Identity]&, initconds];

In[1517]:= phase2 = Show[somegraphs2,
                    PlotRange- > {{-5, 5}, {-5, 5}},
                    AspectRatio- > 1, Ticks- > {{-4, 4},
                    {-4, 4}}];

In[1518]:= somegraphs3 = Map[sol[3/2, #, DisplayFunction- >
                        Identity]&, initconds];

In[1519]:= phase3 = Show[somegraphs3,
                    PlotRange- > {{-5, 5}, {-5, 5}},
                    AspectRatio- > 1, Ticks- > {{-4, 4},
                    {-4, 4}}];

In[1520]:= somegraphs4 = Map[sol[3, #, DisplayFunction- >
                        Identity]&, initconds];

In[1521]:= phase4 = Show[somegraphs3,
                    PlotRange- > {{-5, 5}, {-5, 5}},
                    AspectRatio- > 1, Ticks- > {{-4, 4},
                    {-4, 4}}];
```

We now show all four graphs together in Figure 7-21. In each figure, we see that all of the curves approach a curve called a *limit cycle*. Physically,

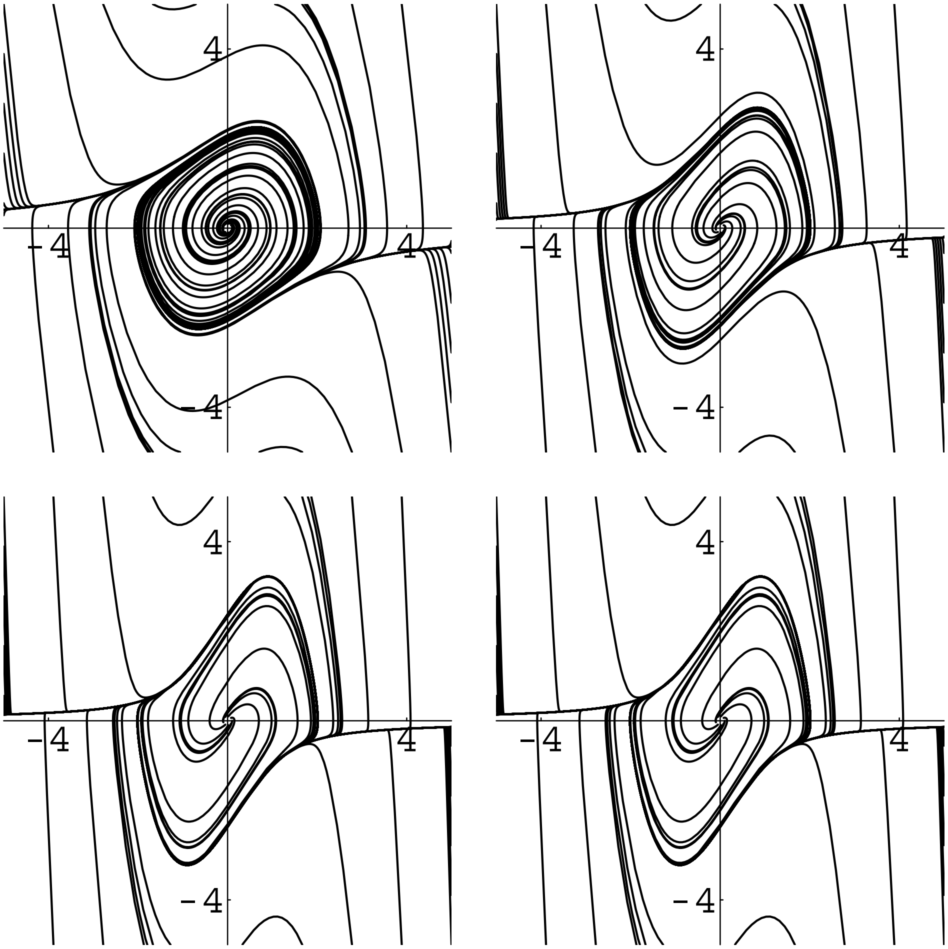


Figure 7-21 Solutions to the Van-der-Pol equation for various values of μ

the fact that the system has a limit cycle indicates that for all oscillations, the motion eventually becomes periodic, which is represented by a closed curve in the phase plane.

```
In[1522]:= Show[GraphicsArray[
              {{phase1,phase2},{phase3,phase4}}]]
```

On the other hand, in Figure 7-22 we graph the solutions that satisfy the initial conditions $x(0) = 1$ and $y(0) = 0$ parametrically and individually for various values of μ. Notice that for small values of μ the system more closely approximates that of the harmonic oscillator because the

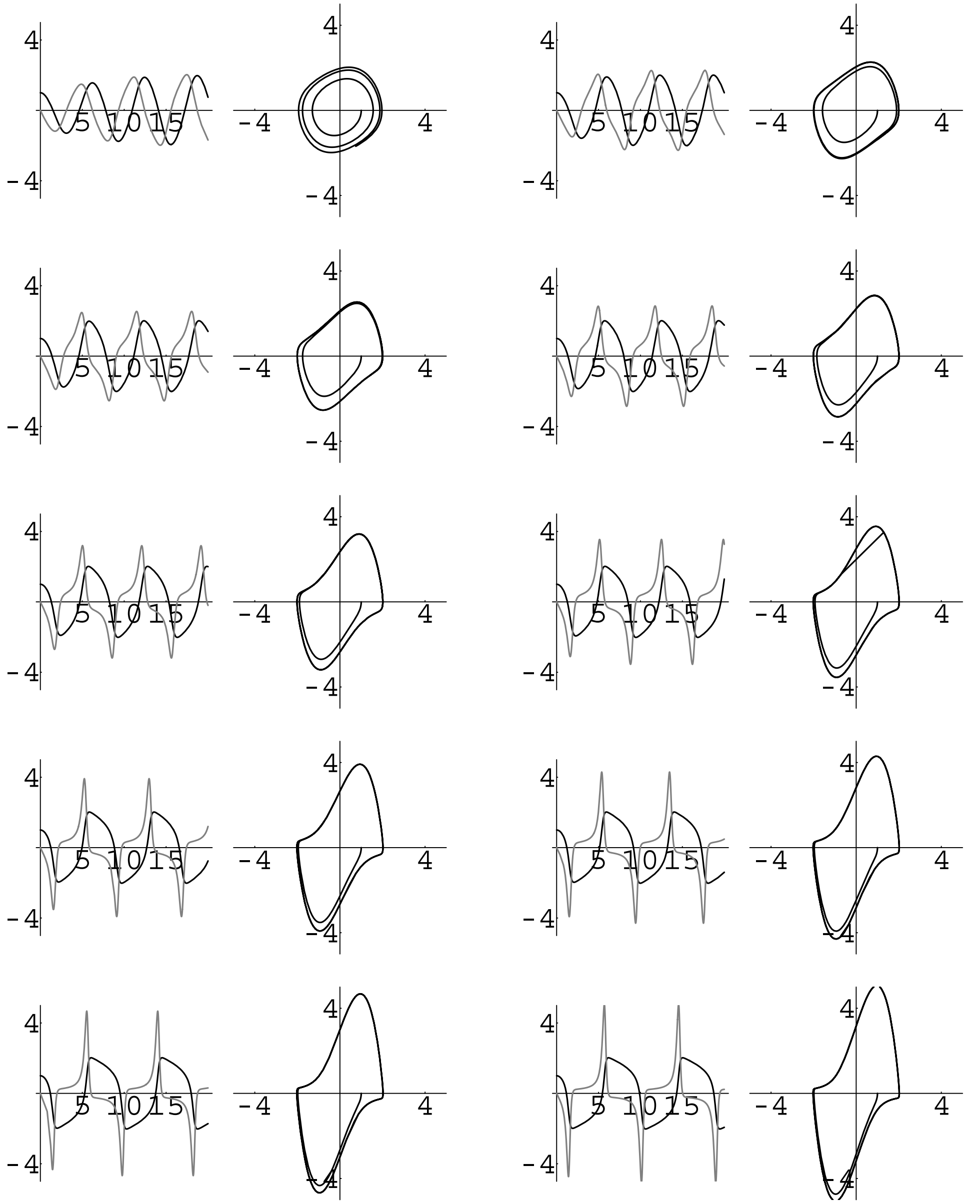


Figure 7-22 The solutions to the Van-der-Pol equation satisfying $x(0) = 1$ and $y(0) = 0$ individually (x in black and y in gray) for various values of μ

damping coefficient is small. The curves are more circular than those for larger values of μ.

```
In[1523]:= Clear[x, y, t, s]

           graph[μ_] := Module[{numsol, pp, pxy},
              numsol = NDSolve[{x'[t] == y[t],
                       y'[t] == μ (1 - x[t]^2) y[t] - x[t],
                       x[0] == 1,
                       y[0] == 0}, {x[t], y[t]}, {t, 0, 20}];
           pp = ParametricPlot[{x[t], y[t]}/.numsol,
                 {t, 0, 20}, Compiled → False,
                 PlotRange → {{-5, 5}, {-5, 5}},
                 AspectRatio → 1, Ticks → {{-4, 4},
                 {-4, 4}}, DisplayFunction → Identity];
           pxy = Plot[Evaluate[{x[t], y[t]}/.numsol],
                 {t, 0, 20},
           PlotStyle → {GrayLevel[0],
                    GrayLevel[0.5]}, PlotRange →
                    {-5, 5}, AspectRatio → 1,
                 Ticks- > {{5, 10, 15}, {-4, 4}},
                 DisplayFunction → Identity];
           GraphicsArray[{pxy, pp}]]

In[1524]:= graphs = Table[graph[i],
                    {i, 0.25, 3, 2.75/9}];

In[1525]:= toshow = Partition[graphs, 2];

           Show[GraphicsArray[toshow]]
```

■

7.3.3 Differential Geometry: Curvature

Let C be a piecewise-smooth curve with parametrization $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$. The **unit tangent vector** to C at t is

Refer to Gray's outstanding text, *Modern Differential Geometry of Curves and Surfaces*[14] which incorporates Mathematica throughout.

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \tag{7.19}$$

The **arc length function**, $s = s(t)$, is defined by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| \, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} \, du. \tag{7.20}$$

Solving equation (7.20) for t, we have $t = t(s)$ and the **parametrization of C with respect to arc length** is $\mathbf{r}(s) = \langle x(t(s)), y(t(s)) \rangle$. When C is parametrized by arc length, $\|\mathbf{r}'(s)\| = 1$ so the unit tangent vector (7.19) is given by $\mathbf{T}(s) = \mathbf{r}'(s)$. The **curvature** of C, $\kappa(s)$, is

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\|. \tag{7.21}$$

Thus, for the curve C parametrized by arc length, $\kappa(s) = \|\mathbf{r}''(s)\|$.

Conversely, a given curvature function determines a plane curve: the curve C parametrized by arc length with curvature $\kappa(s)$ has parametrization $\mathbf{r}(s) = \langle x(s), y(s) \rangle$ where

$$\begin{cases} dx/ds = \cos\theta \\ dy/ds = \sin\theta \\ d\theta/ds = \kappa \\ x(a) = c, \; y(a) = d, \; \theta(0) = \theta_0 . \end{cases} \tag{7.22}$$

You can often use `NDSolve` to solve system (7.22).

EXAMPLE 7.3.6: Plot the curve C for which $\kappa(s) = e^{-s} + e^{s}$ for $-5 \le s \le 5$ if $x(0) = y(0) = \theta(0) = 0$.

SOLUTION: After defining $\kappa(s) = e^{-s} + e^{s}$,

```
In[1526]:= κ[s] = Exp[-s] + Exp[s];
```

we use `NDSolve` to solve system (7.22) using the initial conditions $x(0) = y(0) = \theta(0) = 0$ for $-5 \le s \le 5$.

```
In[1527]:= t1 = NDSolve[{x′[s] == Cos[θ[s]],
                  y′[s] == Sin[θ[s]],
                 θ′[s] == κ[s], x[0] == 0, y[0] == 0,
             θ[0] == 0},
                {x[s], y[s], θ[s]}, {s, -5, 5}]
Out[1527]= {{x[s] → InterpolatingFunction[{{-5., 5.}},
                <>][s],
                y[s] → InterpolatingFunction[{{-5., 5.}},
                <>][s],
                θ[s] → InterpolatingFunction
                <>][s], [{{-5., 5.}}, <>][s]}}
```

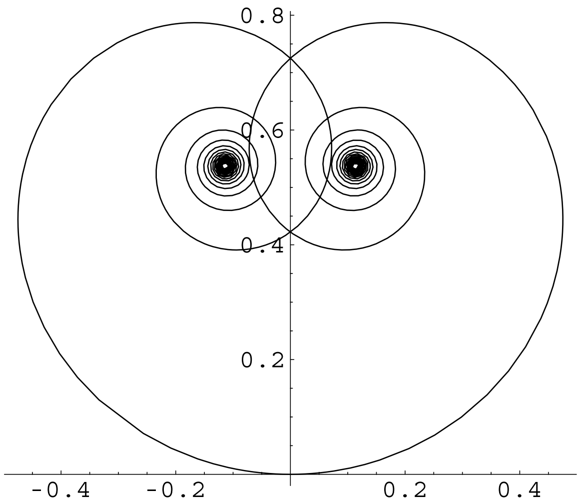


Figure 7-23 For this curve, $\kappa(s) = e^{-s} + e^{s}$

We use `ParametricPlot` to graph the result in Figure 7-23.

```
In[1528]:= ParametricPlot[Evaluate[{x[s],y[s]}/.t1],
              {s,-5,5},AspectRatio → Automatic,
              PlotRange → All]
```

■

Even relatively simple curvature functions can yield remarkable curves. To illustrate, we define the function `curvek`. Given a function $\kappa(s)$, `curvek`[κ[s], {s, a, b}, `opts`] solves system (7.22) using the initial conditions $x(0) = y(0) = \theta(0) = 0$ for $a \le s \le b$, and parametrically plots the result. Any options are passed to the `Parametric Plot` command. If you do not include `{s,a,b}` and do not include any options, the default is $-15 \le s \le 15$.

```
In[1529]:= Clear[curvek,κ];

           curvek[k_,ss_:{s,-15,15},opts___] :=
             Module[{numsol},
               numsol = NDSolve[{x′[s] == Cos[θ[s]],
               y′[s] == Sin[θ[s]],
                    θ′[s] == k,x[0] == 0,y[0] == 0,θ[0] == 0},
                    {x[s],y[s],θ[s]},ss];
               ParametricPlot[Evaluate[{x[s],y[s]}/.numsol],
                 ss,opts,AspectRatio → Automatic]
             ]
```

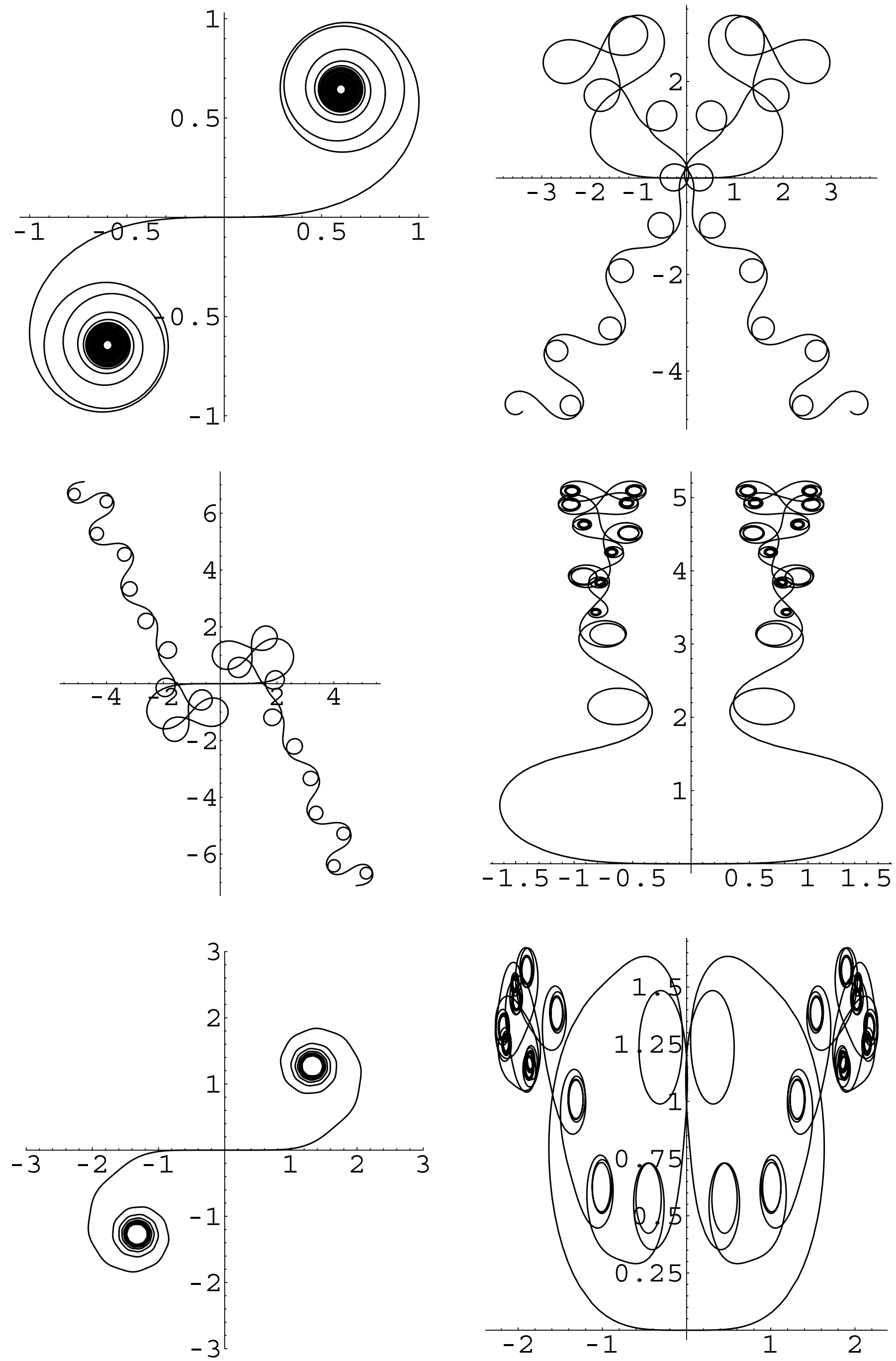


Figure 7-24 You can generate stunning curves by specifying a curvature function

We illustrate the use of curvek using $\kappa(s) = s + \sin s$, $sJ_1(s)$, $sJ_2(s)$, $s\sin(\sin s)$, $s\sin\left(\sin^2 s^2\right)$, and $|s\sin(\sin s)|$. All six plots are shown together as an array in Figure 7-24.

```
In[1530]:= κ[s_] = s + Sin[s];

           p1 = curvek[κ[s], {s, -40, 40}, PlotPoints → 480,
              AspectRatio → 1, DisplayFunction → Identity];

In[1531]:= p2 = curvek[s BesselJ[1, s], {s, -40, 40},
           PlotPoints → 120,
              DisplayFunction → Identity];

In[1532]:= p3 = curvek[s BesselJ[2, s], {s, -40, 40},
           PlotPoints → 120,
              DisplayFunction → Identity];

In[1533]:= κ[s_] = s Sin[Sin[s]];

           p4 = curvek[κ[s], {s, -40, 40},
           PlotPoints → 480,
              AspectRatio → 1, DisplayFunction → Identity];

In[1534]:= κ[s_] = s Sin[Sin[s^2]^2];

           p5 = curvek[κ[s], {s, -15, 15},
           PlotPoints → 480,
              AspectRatio → 1, PlotRange → {{-3, 3}, {-3, 3}},
              DisplayFunction → Identity];

In[1535]:= κ[s_] = Abs[s Sin[Sin[s]]];

           p6 = curvek[κ[s], {s, -40, 40}, PlotPoints → 480,
              AspectRatio → 1, DisplayFunction → Identity];

In[1536]:= Show[GraphicsArray[{{p1, p2}, {p3, p4},
             {p5, p6}}]]
```

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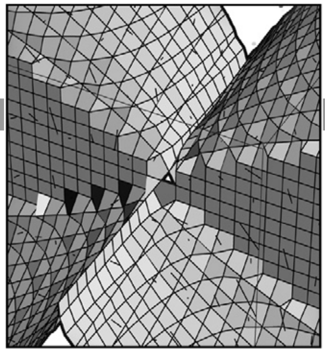
8 Laplace Transform Methods



In previous chapters we have investigated solving the nth-order linear equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{8.1}$$

for y. We have seen that if the coefficients $a_i(x)$ are numbers, we can find a general solution of the equation by first solving the characteristic equation of the corresponding homogeneous equation, forming a general solution of the corresponding homogeneous equation, and then finding a particular solution to the nonhomogeneous equation. If the coefficients $a_i(x)$ are not constants, the situation is more difficult. In particular cases, like when equation (8.1) is a Cauchy–Euler equation, similar techniques can be used. In other cases, we might be able to use a series to find a solution. In each of these situations, however, the function $f(x)$ has typically been a smooth function. If $f(x)$ is not a smooth function, like when $f(x)$ is a piecewise-defined or periodic function, solving equation (8.1) can be substantially more difficult.

In this chapter, we discuss a technique that transforms equation (8.1) into an algebraic equation that can often be solved so that a solution to the differential equation can be obtained.

8.1 The Laplace Transform

8.1.1 Definition of the Laplace Transform

Definition 30 (Laplace Transform). *Let $f(t)$ be a function defined on the interval* $[0, \infty)$. *The **Laplace transform** of $f(t)$ is the function (of s)*

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt. \tag{8.2}$$

The command

```
LaplaceTransform[f[t],t,s]
```

computes the Laplace transform of $f(t)$.

Because the Laplace transform yields a function of s, we often use the notation $\mathcal{L}\{f(t)\} = F(s)$ to denote the Laplace transform of $f(t)$.

EXAMPLE 8.1.1: Compute $\mathcal{L}\{f(t)\}$ if $f(t) = 1$.

SOLUTION: Using the definition, equation (8.2), we have

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, dt = \lim_{M\to\infty} \int_0^M e^{-st}\, dt = \lim_{M\to\infty} \left[-\frac{1}{s} e^{-st}\right]_{t=0}^{t=M}$$
$$= -\frac{1}{s} \lim_{M\to\infty} \left(e^{-sM} - 1\right) = -\frac{1}{s}(0-1) = \frac{1}{s},\ s > 0.$$

Notice that in order for $\lim_{M\to\infty} e^{-sM} = 0$, we must require that $s > 0$. (Otherwise, the limit does not exist.) We can use `Integrate` to evaluate this integral as well.

```
In[1537]:= step1 = ∫_0^capm Exp[-s t] dt
Out[1537]= (1 - e^(-capm s))/s
```

However, Mathematica cannot evaluate $\lim_{M\to\infty} e^{-sM}$ because Mathematica does not assume that $s > 0$.

```
In[1538]:= step2 = Limit[step1, capm → ∞]
Out[1538]= Limit[(1 - e^(-capm s))/s, capm → ∞]
```

Alternatively, we can use `Integrate` to evaluate the improper integral

```
In[1539]:= ∫_0^∞ Exp[-s t] dt
Out[1539]= If[Re[s] > 0, 1/s,
             Integrate[e^(-s t), {t, 0, ∞},
             Assumptions → Re[s] ≤ 0]]
```

or use the command `LaplaceTransform` to compute $\mathcal{L}\{f(t)\}$.

```
In[1540]:= LaplaceTransform[1, t, s]
Out[1540]= 1/s
```

■

EXAMPLE 8.1.2: Compute $\mathcal{L}\{f(t)\}$ if $f(t) = e^{at}$.

SOLUTION: As before, we have

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt \\ &= \lim_{M\to\infty}\left[-\frac{1}{s-a}e^{-(s-a)t}\right]_{t=0}^{t=M} = -\frac{1}{s-a}\lim_{M\to\infty}\left(e^{-(s-a)M}-1\right) \\ &= \frac{1}{s-a},\ s > a.\end{aligned}$$

Notice that we must require $s > a$ so that $\lim_{M\to\infty} e^{-(s-a)M} = 0$. `LaplaceTransform` can be used to compute the Laplace transform of this function as well.

```
In[1541]:= LaplaceTransform[Exp[a t], t, s]
Out[1541]= 1/(-a + s)
```

■

The formula $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$ can now be used to avoid using the definition.

EXAMPLE 8.1.3: Compute: (a) $\mathcal{L}\left\{e^{-3t}\right\}$ and (b) $\mathcal{L}\left\{e^{5t}\right\}$.

SOLUTION: We have that (a) $\mathcal{L}\left\{e^{-3t}\right\} = \dfrac{1}{s-(-3)} = \dfrac{1}{s+3}$, $s > -3$, and (b) $\mathcal{L}\left\{e^{5t}\right\} = \dfrac{1}{s-5}$, $s > 5$.

With Mathematica, we use `Map` to apply the pure function `LaplaceTransform[#,t,s]&` to the list of functions $\left\{e^{-3t},\ e^{5t}\right\}$ to compute both Laplace transforms in a single step.

```
In[1542]:= Map[LaplaceTransform[#, t, s]&,
             {Exp[-3 t], Exp[5 t]}]
```

Out[1542]= $\left\{\dfrac{1}{3+s}, \dfrac{1}{-5+s}\right\}$

■

In most cases, using the definition of the Laplace transform to calculate the Laplace transform of a function is a difficult and time-consuming task.

EXAMPLE 8.1.4: Compute (a) $\mathcal{L}\left\{t^3\right\}$; (b) $\mathcal{L}\{\sin at\}$; and (c) $\mathcal{L}\{\cos at\}$.

SOLUTION: To compute $\mathcal{L}\left\{t^3\right\}$ by hand requires application of integration by parts three times. Instead, we proceed with `Integrate`. First we compute $\int_0^M t^3 e^{-st}\,dt$ and then $\int_0^\infty t^3 e^{-st}\,dt = \lim_{M\to\infty} \int_0^M t^3 e^{-st}\,dt$.

In[1543]:= $\int_0^{\text{capm}} \mathbf{t^3\ Exp[-s\ t]} \,d\mathbf{t}$

Out[1543]= $\dfrac{6 - e^{-\text{capm}\,s}\,(6 + 6\,\text{capm}\,s + 3\,\text{capm}^2\,s^2 + \text{capm}^3\,s^3)}{s^4}$

In[1544]:= $\int_0^{\infty} \mathbf{t^3\ Exp[-s\ t]} \,d\mathbf{t}$

Out[1544]= If$\left[\text{Re}[s] > 0, \dfrac{6}{s^4}, \text{Integrate}[e^{-s\,t}\,t^3,\right.$
$\left.\{t, 0, \infty\}, \text{Assumptions} \to \text{Re}[s] \leq 0]\right]$

The integrals that result when computing $\mathcal{L}\{\sin at\}$ and $\mathcal{L}\{\cos at\}$ using the definition of the Laplace transform each require the use of integration by parts twice. Instead, we use `LaplaceTransform` to compute each Laplace transform.

```
In[1545]:= Clear[t, s]

           Map[LaplaceTransform[#, t, s]&,
             {t^3, Sin[a t], Cos[a t]}]
```

Out[1545]= $\left\{\dfrac{6}{s^4}, \dfrac{a}{a^2+s^2}, \dfrac{s}{a^2+s^2}\right\}$

■

We now discuss the linearity property that enables us to use the transforms that we have found thus far to find the Laplace transform of other functions.

Theorem 18 (Linearity Property). *Let a and b be constants, and suppose that* $\mathcal{L}\{f(t)\}$ *and* $\mathcal{L}\{g(t)\}$ *exist. Then,*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

EXAMPLE 8.1.5: Calculate (a) $\mathcal{L}\{6\}$; (b) $\mathcal{L}\left\{5 - 2e^{-t}\right\}$.

SOLUTION: Using the results obtained in previous examples, we have for (a)

$$\mathcal{L}\{6\} = 6\mathcal{L}\{1\} = 6 \cdot \frac{1}{s} = \frac{6}{s};$$

and for (b)

$$\mathcal{L}\left\{5 - 2e^{-t}\right\} = 5\mathcal{L}\{1\} - 2\mathcal{L}\left\{e^{-t}\right\} = 5 \cdot \frac{1}{s} - 2 \cdot \frac{1}{s-(-1)} = \frac{5}{s} - \frac{2}{s+1}.$$

■

8.1.2 Exponential Order, Jump Discontinuities, and Piecewise-Continuous Functions

In calculus, we learn that some improper integrals diverge, which indicates that the Laplace transform may not exist for some functions. Therefore, we present the following definitions and theorems so that we can better understand the types of functions for which the Laplace transform exists.

Definition 31 (Exponential Order). *A function* $y = f(t)$ *is of **exponential order** b if there are numbers b,* $M > 0$*, and* $T > 0$ *such that*

$$|f(t)| \le Me^{bt}$$

for $t > T$.

In the following sections, we will see that the Laplace transform is particularly useful in solving equations involving piecewise or recursively defined functions.

Definition 32 (Jump Discontinuity). *A function* $y = f(t)$ *has a **jump discontinuity** at* $t = c$ *on the closed interval* $[a, b]$ *if the one-sided limits* $\lim_{t\to c^+} f(t)$ *and* $\lim_{t\to c^-} f(t)$ *are finite, but unequal, values.* $y = f(t)$ *has a **jump discontinuity** at* $t = a$ *if* $\lim_{t\to a^+} f(t)$ *is a finite value different from* $f(a)$. $y = f(t)$ *has a **jump discontinuity** at* $t = b$ *if* $\lim_{t\to b^-} f(t)$ *is a finite value different from* $f(b)$.

Definition 33 (Piecewise Continuous). *A function $y = f(t)$ is* ***piecewise continuous*** *on the finite interval $[a, b]$ if $y = f(t)$ is continuous at every point in $[a, b]$ except at finitely many points at which $y = f(t)$ has a jump discontinuity.*

A function $y = f(t)$ is ***piecewise continuous*** *on $[0, \infty)$ if $y = f(t)$ is piecewise continuous on $[0, N]$ for all N.*

Theorem 19 (Sufficient Condition for Existence of $\mathcal{L}\{f(t)\}$). *Suppose that $y = f(t)$ is a piecewise continuous function on the interval $[0, \infty)$ and that it is of exponential order b for $t > T$. Then, $\mathcal{L}\{f(t)\}$ exists for $s > b$.*

EXAMPLE 8.1.6: Find the Laplace transform of $f(t) = \begin{cases} -1, & 0 \le t < 4 \\ 1, & t \ge 4. \end{cases}$

SOLUTION: Because $y = f(t)$ is a piecewise continuous function on $[0, \infty)$ and of exponential order, $\mathcal{L}\{f(t)\}$ exists. We use the definition and evaluate the integral using the sum of two integrals.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty f(t)e^{-st}\,dt = \int_0^4 -1 \cdot e^{-st}\,dt + \int_4^\infty e^{-st}\,dt \\
&= \left[\frac{1}{s}e^{-st}\right]_{t=0}^{t=4} + \lim_{M\to\infty}\left[-\frac{1}{s}e^{-st}\right]_{t=4}^{t=M} \\
&= \frac{1}{s}\left(e^{-4s} - 1\right) - \frac{1}{s}\lim_{M\to\infty}\left(e^{-Ms} - e^{-4s}\right) = \frac{1}{s}\left(2e^{-4s} - 1\right).
\end{aligned}$$

Using `Cases` (`/;`), we define and graph this piecewise-defined function in Figure 8-1.

```
In[1546]:= Clear[f]

            f[t_] := -1/; 0 ≤ t < 4

            f[t_] := 1/; t ≥ 4
In[1547]:= Plot[f[t], {t, 0, 8}]
```

However, the `LaplaceTransform` command is unable to compute the Laplace transform of $y = f(t)$ when f is defined in this manner. To compute the Laplace transform using Mathematica, we take advantage of the `UnitStep` function, which is defined by

$$\texttt{UnitStep[t]} = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0. \end{cases}$$

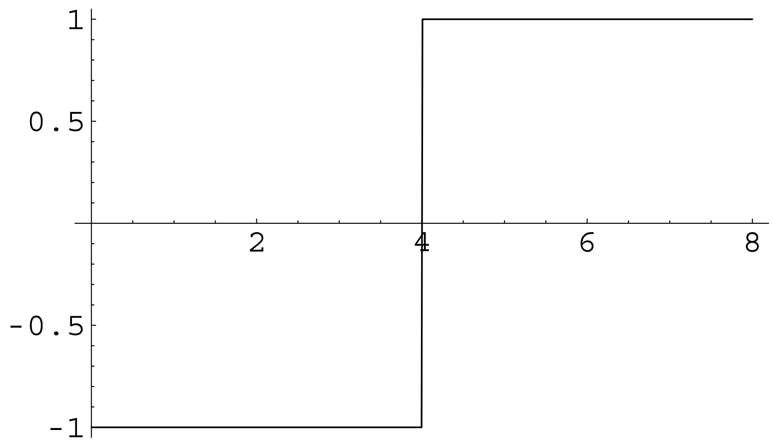


Figure 8-1 Plot of a piecewise-defined function

Thus, $y = f(t)$ is given by UnitStep[t-4]-UnitStep[4-t]. After defining $y = f(t)$ in this manner, we see that LaplaceTransform is then able to compute $\mathcal{L}\{f(t)\}$.

```
In[1548]:= Clear[f]
            f[t_] = UnitStep[t - 4] - UnitStep[4 - t];

In[1549]:= LaplaceTransform[f[t], t, s]
```

Out[1549]= $\frac{e^{-4s}}{s} - \frac{1 - e^{-4s}}{s}$

■

8.1.3 Properties of the Laplace Transform

The definition of the Laplace transform is not easy to apply to most functions. Therefore, we now discuss several properties of the Laplace transform so that numerous transformations can be made without having to use the definition. Most of the properties discussed here follow directly from our knowledge of integrals.

Theorem 20 (Shifting Property). *If* $\mathcal{L}\{f(t)\} = F(s)$ *exists for* $s > b$, *then*

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a). \tag{8.3}$$

EXAMPLE 8.1.7: Find the Laplace transform of (a) $f(t) = e^{-2t}\cos t$ and (b) $f(t) = 4te^{3t}$.

SOLUTION: (a) In this case, $f(t) = \cos t$ and $a = -2$. Using $F(s) = \mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$, we replace each s with $s - a = s + 2$. Therefore,

$$\mathcal{L}\left\{e^{-2t}\cos t\right\} = \frac{s+2}{(s+2)^2+1} = \frac{s+2}{s^2+4s+5}.$$

(b) Using the linearity property, we have $\mathcal{L}\left\{4te^{3t}\right\} = 4\mathcal{L}\left\{te^{3t}\right\}$. To apply the shifting property we have $f(t) = t$ and $a = 3$, so we replace s in $F(s) = \mathcal{L}\{t\} = s^{-2}$ by $s - a = s - 3$.

Therefore,

$$\mathcal{L}\left\{4te^{3t}\right\} = \frac{4}{(s-3)^2}.$$

Identical results are obtained with `LaplaceTransform`.

```
In[1550]:= LaplaceTransform[Exp[-2t] Cos[t], t, s]
```

$$\text{Out[1550]}= \frac{2+s}{5+4s+s^2}$$

```
In[1551]:= LaplaceTransform[4t Exp[3t], t, s]
```

$$\text{Out[1551]}= \frac{4}{(-3+s)^2}$$

■

In order to use the Laplace transform to solve differential equations, we will need to be able to compute the Laplace transform of the derivatives of an arbitrary function, provided the Laplace transform of such a function exists.

Theorem 21 (Laplace Transform of the First Derivative). *Suppose that* $y = f(t)$ *is a piecewise continuous function on the interval and that it is of exponential order* b *for* $t \geq T$. *Then, for* $s > b$

$$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\{f(t)\} - f(0). \tag{8.4}$$

```
In[1552]:= Clear[f]

            LaplaceTransform[f′[t], t, s]
Out[1552]= -f[0] + s LaplaceTransform[f[t], t, s]
```

Using induction, a direct consequence of the theorem is

Theorem 22 (Laplace Transform of the Higher Derivatives). *If* $f^{(i)}(t)$ *is a continuous function on* $[0, \infty)$ *for* $i = 0, 1, \ldots, n-1$ *and* $f^{(n)}(t)$ *is piecewise continuous on* $[0, \infty)$ *and of exponential order b, then for* $s > b$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0). \tag{8.5}$$

We use `LaplaceTransform` to compute the Laplace transform of $f^{(i)}(t)$ for $i = 1, 2, 3, 4, 5$.

```
In[1553]:= derivs = Table[D[f[t], {t, n}], {n, 1, 5}]
Out[1553]= {f′[t], f″[t], f^(3)[t], f^(4)[t], f^(5)[t]}

In[1554]:= Map[LaplaceTransform[#, t, s]&, derivs]
Out[1554]= {-f[0] + s LaplaceTransform[f[t], t, s],
             -s f[0] + s^2 LaplaceTransform[f[t], t, s] - f′[0],
             -s^2 f[0] + s^3 LaplaceTransform[f[t], t, s]
             -s f′[0] - f″[0],
             -s^3 f[0] + s^4 LaplaceTransform[f[t], t, s]
             -s^2 f′[0] - s f″[0]-f^(3)[0],
             -s^4 f[0] + s^5 LaplaceTransform[f[t], t, s]-s^3 f′[0]
             -s^2 f″[0] - s f^(3)[0] - f^(4)[0]}
```

We will use this theorem and corollary in solving initial-value problems. However, we can also use them to find the Laplace transform of a function when we know the Laplace transform of the derivative of the function.

EXAMPLE 8.1.8: Find $\mathcal{L}\left\{\sin^2 kt\right\}$.

SOLUTION: We can use the theorem to find the Laplace transform of $f(t) = \sin^2 kt$. Notice that $f'(t) = 2k \sin kt \cos kt = k \sin 2kt$. Then, because $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ and

$$\mathcal{L}\left\{f'(t)\right\} = \mathcal{L}\{k \sin 2kt\} = k\frac{2k}{s^2 + (2k)^2} = \frac{2k^2}{s^2 + 4k^2},$$

we have $\dfrac{2k^2}{s^2+4k^2} = s\mathcal{L}\{f(t)\} - 0$. Therefore, $\mathcal{L}\{f(t)\} = \dfrac{2k^2}{s\left(s^2+4k^2\right)}$. As in previous examples, we see that the same results are obtained with `LaplaceTransform`.

```
In[1555]:= LaplaceTransform[Sin[kt]^2, t, s]
Out[1555]= 2 k^2/(4 k^2 s + s^3)
```

■

Theorem 23 (Derivatives of the Laplace Transform). *Suppose that $F(s) = \mathcal{L}\{f(t)\}$ where $y = f(t)$ is a piecewise continuous function on $[0, \infty)$ and of exponential order b. Then, for $s > b$,*

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}(s). \tag{8.6}$$

EXAMPLE 8.1.9: Find the Laplace transform of (a) $f(t) = t\cos 2t$ and (b) $f(t) = t^2 e^{-3t}$.

SOLUTION: (a) In this case, $n = 1$ and $F(s) = \mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$. Then

$$\mathcal{L}\{t\cos 2t\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\frac{\left(s^2+4\right) - s\cdot 2s}{\left(s^2+4\right)^2} = \frac{s^2-4}{\left(s^2+4\right)^2}.$$

```
In[1556]:= LaplaceTransform[t Cos[2t], t, s]//
              Simplify
Out[1556]= (-4 + s^2)/(4 + s^2)^2
```

(b) Because $n = 2$ and $F(s) = \mathcal{L}\left\{e^{-3t}\right\} = \dfrac{1}{s+3}$, we have

$$\mathcal{L}\left\{t^2 e^{-3t}\right\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) = \frac{2}{(s+3)^2}.$$

```
In[1557]:= LaplaceTransform[t^2 Exp[-3 t], t, s]
Out[1557]= 2/(3 + s)^3
```

■

EXAMPLE 8.1.10: Find $\mathcal{L}\{t^n\}$.

SOLUTION: Using the theorem with $\mathcal{L}\{t^n\} = \mathcal{L}\{t^n \cdot 1\}$, we have $f(t) = 1$. Then, $F(s) = \mathcal{L}\{1\} = s^{-1}$. Calculating the derivatives of F, we obtain

$$\begin{aligned}\frac{dF}{ds}(s) &= -\frac{1}{s^2}\\ \frac{d^2F}{ds^2}(s) &= \frac{2}{s^3}\\ \frac{d^3F}{ds^3}(s) &= -\frac{3\cdot 2}{s^4}\\ &\vdots\\ \frac{d^nF}{ds^n}(s) &= (-1)^n\frac{n!}{s^{n+1}}.\end{aligned}$$

Therefore,

Recall that for nonnegative integers n, $\Gamma(n+1) = n!$.

$$\mathcal{L}\left\{t^n\right\} = \mathcal{L}\left\{t^n \cdot 1\right\} = (-1)^n(-1)^n\frac{n!}{s^{n+1}} = (-1)^{2n}\frac{n!}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

```
In[1558]:= LaplaceTransform[t^n, t, s]
Out[1558]= s^(-1-n) Gamma[1 + n]
```

■

EXAMPLE 8.1.11: Compute the Laplace transform of $f(t)$, $f'(t)$, and $f''(t)$ if $f(t) = (3t-1)^3$.

SOLUTION: First, $f(t) = (3t-1)^3 = 27t^3 - 27t^2 + 9t - 1$ and $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ so

$$\mathcal{L}\{f(t)\} = 27\frac{3!}{s^4} - 27\frac{2!}{s^3} + 9\frac{1}{s^2} - \frac{1}{s} = \frac{1}{s^4}\left(162 - 54s + 9s^2 - s^3\right).$$

```
In[1559]:= Clear[f]

           f[t_] = (3t - 1)^3;

           lf = LaplaceTransform[f[t], t, s]
Out[1559]= 162/s^4 - 54/s^3 + 9/s^2 - 1/s
```

By the previous theorem, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$. Hence, $\mathcal{L}\left\{f'(t)\right\} = s\cdot\frac{1}{s^4}\left(162 - 54s + 9s^2 - s^3\right) - f(0) = \frac{1}{s^3}\left(162 - 54s + 9s^2 - s^3\right) + 1 = \frac{9}{s^3}\left(18 - 6s + s^2\right)$.

```
In[1560]:= lfprime = LaplaceTransform[f′[t], t, s]//
              Expand
```

Out[1560]= $\frac{162}{s^3} - \frac{54}{s^2} + \frac{9}{s}$

```
In[1561]:= s lf - f[0]//Expand
```

Out[1561]= $\frac{162}{s^3} - \frac{54}{s^2} + \frac{9}{s}$

Similarly $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$:

$$\mathcal{L}\left\{f''(t)\right\} = s^2\frac{1}{s^4}\left(162 - 54s + 9s^2 - s^3\right) - sf(0) - f'(0) = \frac{54}{s^2}(3 - s).$$

```
In[1562]:= lfdoubleprime =
             Expand[LaplaceTransform[f″[t], t, s]]
```

Out[1562]= $\frac{162}{s^2} - \frac{54}{s}$

```
In[1563]:= Expand[s^2 lf - s f[0] - f′[0]]
```

Out[1563]= $\frac{162}{s^2} - \frac{54}{s}$

■

Using the properties of the Laplace transform, we can compute the Laplace transform of a large number of frequently encountered functions. We use `Map`, `LaplaceTransform`, and `TableForm` to compute a table of the Laplace transform of several frequently encountered functions.

```
In[1564]:= r1 = {1, Exp[a t], Sin[k t], Cos[k t], Sinh[k t],
                Cosh[k t], t^n, t^n Exp[a t],
                Exp[a t] Sin[k t],
                Exp[a t] Cos[k t],
                Exp[a t] Sinh[k t],
                Exp[a t] Cosh[k t]};
```

```
In[1565]:= Map[{#,LaplaceTransform[#,t,s]}&,r1]//TableForm
```

Out[1565]=

1	$\frac{1}{s}$
$e^{a t}$	$\frac{1}{-a+s}$
Sin[k t]	$\frac{k}{k^2+s^2}$
Cos[k t]	$\frac{s}{k^2+s^2}$
Sinh[k t]	$\frac{k}{-k^2+s^2}$
Cosh[k t]	$\frac{s}{-k^2+s^2}$
t^n	s^{-1-n} Gamma[1 + n]
$e^{a t} t^n$	$(-a+s)^{-1-n}$ Gamma[1 + n]
$e^{a t}$ Sin[k t]	$\frac{k}{a^2+k^2-2\,a\,s+s^2}$
$e^{a t}$ Cos[k t]	$\frac{-a+s}{a^2+k^2-2\,a\,s+s^2}$
$e^{a t}$ Sinh[k t]	$\frac{k}{a^2-k^2-2\,a\,s+s^2}$
$e^{a t}$ Cosh[k t]	$\frac{-a+s}{a^2-k^2-2\,a\,s+s^2}$

8.2 The Inverse Laplace Transform

8.2.1 Definition of the Inverse Laplace Transform

In the previous section, we were concerned with finding the Laplace transform of a given function either through the use of the definition of the Laplace transform or with one of the numerous properties of the Laplace transform. At that time, we discussed the sufficient conditions for the existence of the Laplace transform. In this section, we will reverse this process: given a function $F(s)$ we want to find a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$.

Definition 34 (Inverse Laplace Transform). *The* ***inverse Laplace transform*** *of the function $F(s)$ is the unique continuous function $f(t)$ on $[0, \infty)$ that satisfies $\mathcal{L}\{f(t)\} = F(s)$. We denote the inverse Laplace transform of $F(s)$ as*

$$f(t) = \mathcal{L}^{-1}\{F(s)\}. \tag{8.7}$$

If the only functions that satisfy this relationship are discontinuous on $[0, \infty)$, we choose a piecewise continuous function on $[0, \infty)$ to be $\mathcal{L}^{-1}\{F(s)\}$.

The table of Laplace transforms listed in the previous section is useful in finding the inverse Laplace transform of a given function. Also, the command

```
InverseLaplaceTransform[F[s],s,t]
```

can often find $\mathcal{L}^{-1}\{F(s)\}$.

EXAMPLE 8.2.1: Find the inverse Laplace transform of (a) $F(s) = \dfrac{1}{s-6}$, (b) $F(s) = \dfrac{2}{s^2+4}$, (c) $F(s) = \dfrac{6}{s^4}$, and (d) $F(s) = \dfrac{6}{(s+2)^4}$.

SOLUTION: (a) Because $\mathcal{L}\left\{e^{6t}\right\} = \dfrac{1}{s-6}$, $\mathcal{L}^{-1}\left\{\dfrac{1}{s-6}\right\} = e^{6t}$. (b) $\mathcal{L}\{\sin 2t\} = \dfrac{2}{s^2+2^2} = \dfrac{2}{s^2+4}$ so $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = \sin 2t$. (c) Note that $\mathcal{L}\left\{t^3\right\} = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$ so $\mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\right\} = t^3$. (c) $F(s) = \dfrac{6}{(s+2)^4}$ is obtained from $F(s) = \dfrac{6}{s^4}$ by substituting $s+2$ for s. Therefore by the shifting property, $\mathcal{L}\left\{e^{-2t}t^3\right\} = \dfrac{6}{(s+2)^4}$, so $\mathcal{L}^{-1}\left\{\dfrac{6}{(s+2)^4}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\right\} = e^{-2t}t^3$. In the same way that we use `LaplaceTransform` to calculate $\mathcal{L}\{f(t)\}$ we use `InverseLaplaceTransform` to calculate $\mathcal{L}^{-1}\{F(s)\}$.

```
In[1566]:= InverseLaplaceTransform[1/(s - 6), s, t]
Out[1566]= e^(6 t)
```

Here, we use `Map` to apply the pure function `InverseLaplaceTransform[#1,s,t] &` to the list of functions $\left\{\dfrac{2}{s^2+4}, \dfrac{6}{s^4}, \dfrac{6}{(s+2)^4}\right\}$.

```
In[1567]:= Map[InverseLaplaceTransform[#1, s, t]&,
              {2/(s^2 + 4), 6/s^4, 6/(s + 2)^4}]
Out[1567]= {2 Cos[t] Sin[t], t^3, e^(-2 t) t^3}
```

■

Theorem 24 (Linearity Property of the Inverse Laplace Transform). *Suppose that* $\mathcal{L}^{-1}\{F(s)\}$ *and* $\mathcal{L}^{-1}\{G(s)\}$ *exist and are continuous on* $[0,\infty)$. *Also, suppose that* a *and* b *are constants. Then,*

$$\mathcal{L}^{-1}\{aF(s)+bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}. \tag{8.8}$$

EXAMPLE 8.2.2: Find the inverse Laplace transform of (a) $F(s) = \frac{1}{s^3}$, (b) $F(s) = -\frac{7}{s^2+16}$, and (c) $F(s) = \frac{5}{s} - \frac{2}{s-10}$.

SOLUTION: (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2.$

```
In[1568]:= InverseLaplaceTransform[1/s^3, s, t]
Out[1568]= t^2/2
```

(b) $\mathcal{L}^{-1}\left\{-\frac{7}{s^2+16}\right\} = -7\mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\} = -7\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{4}{s^2+4^2}\right\} = -\frac{7}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = -\frac{7}{4}\sin 4t.$

```
In[1569]:= InverseLaplaceTransform[-7/(s^2 + 16), s, t]
Out[1569]= -7/4 Sin[4 t]
```

(c) $\mathcal{L}^{-1}\left\{\frac{5}{s} - \frac{2}{s-10}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-10}\right\} = 5 - 2e^{10t}.$

```
In[1570]:= InverseLaplaceTransform[5/s - 2/(s - 10), s, t]
Out[1570]= 5 - 2 e^(10 t)
```

■

Of course, the functions $F(s)$ that are encountered do not have to be of the forms previously discussed. For example, sometimes we must complete the square in the denominator of $F(s)$ before finding $\mathcal{L}^{-1}\{F(s)\}$.

EXAMPLE 8.2.3: Determine $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\}$.

SOLUTION: Notice that all of the forms of $F(s)$ in the table of Laplace transforms involve a term of the form s^2+k^2 in the denominator. However, through shifting, this term is replaced by $(s-a)^2+k^2$. We obtain a term of this form in the denominator by completing the square. This yields

$$\frac{s}{s^2+2s+5} = \frac{s}{\left(s^2+2s+1\right)+4} = \frac{s}{(s+1)^2+4}.$$

Because the variable appears in the numerator, we must write it in the form $s+1$ in order to find the inverse Laplace transform. Doing so, we find that

$$\frac{s}{s^2+2s+5} = \frac{s}{\left(s^2+2s+1\right)+4} = \frac{(s+1)-1}{(s+1)^2+4}.$$

Hence,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\} \\ &= e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t.\end{aligned}$$

As in previous examples, we see that `InverseLaplaceTransform` quickly finds $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\}$.

```
In[1571]:= s1 = InverseLaplaceTransform[s/(s^2 + 2s + 5), s, t]
Out[1571]= 1/4 e^((-1-2 i) t) ((2 - i) + (2 + i) e^(4 i t))
In[1572]:= ExpToTrig[s1]//FullSimplify
Out[1572]= 1/2 e^-t (2 Cos[2 t] - Sin[2 t])
```

■

We assume that $F(s)$ is reduced to lowest terms.

In other cases, partial fractions must be used to obtain terms for which the inverse Laplace transform can be found. Suppose that $F(s) = P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials of degree m and n, respectively. If $n > m$, the method of partial fractions can be used to expand $F(s)$. Recall from calculus, that there are many possible situations that can be solved through partial fractions. We illustrate three cases in the examples that follow.

Linear Factors (Nonrepeated)

In this case, $Q(s)$ can be written as a product of linear factors, so

$$Q(s) = (s-q_1)(s-q_2)\cdots(s-q_n),$$

where $q_1, q_2, \ldots, q_n$ are distinct numbers. Therefore, $F(s)$ can be written as

$$F(s) = \frac{A_1}{s - q_1} + \frac{A_2}{s - q_2} + \cdots + \frac{A_n}{s - q_n},$$

where $A_1, A_2, \ldots, A_n$ are constants that must be determined.

EXAMPLE 8.2.4: Find $\mathcal{L}^{-1}\left\{\frac{3s-4}{s(s-4)}\right\}$.

SOLUTION: In this case, we have distinct linear factors in the denominator. Hence, we write $F(s)$ as

$$\frac{3s-4}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}.$$

Multiplying both sides of this equation by the lowest common denominator $s(s-4)$, we have

$$3s - 4 = A(s-4) + Bs = (A+B)s - 4A.$$

Equating the coefficients of s as well as the constant terms, we see that the system of equations

The set $\{s, 1\}$ is linearly independent.

$$\begin{cases} A + B = 3 \\ -4A = -4 \end{cases}$$

must be satisfied. Mathematica can solve this system of equations with `Solve` or we can solve the equation $3s - 4 = A(s-4) + Bs = (A+B)s - 4A$ for A and B with `SolveAlways`.

```
In[1573] := SolveAlways[3s - 4 == (a + b) s - 4a, s]
Out[1573]= {{a → 1, b → 2}}
```

Hence, $A = 1$ and $B = 2$. Therefore,

$$\frac{3s-4}{s(s-4)} = \frac{1}{s} + \frac{2}{s-4},$$

so

$$\mathcal{L}^{-1}\left\{\frac{3s-4}{s(s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s-4}\right\} = 1 + 2e^{4t}$$

or we can use `InverseLaplaceTransform` as shown in the previous examples.

```
In[1574]:= InverseLaplaceTransform[(3s - 4)/(s (s - 4)), s, t]
Out[1574]= 1 + 2 e^(4 t)
```

Note that we can compute the partial fraction decomposition of $\frac{3s-4}{s(s-4)}$ with `Apart`.

`Apart[f[x]]` computes the partial fraction decomposition of the rational function $f(x)$.

```
In[1575]:= Apart[(3s - 4)/(s (s - 4))]
Out[1575]= 1/(4 - s) + 3/(-4 + s) + 1/s
```

■

Repeated Linear Factors

If $s-q$ is a factor of $Q(s)$ of multiplicity k, the terms in the partial fraction expansion of $F(s)$ that correspond to this factor are

$$\frac{A_1}{s-q} + \frac{A_2}{(s-q)^2} + \cdots + \frac{A_k}{(s-q)^k},$$

where $A_1, A_2, \ldots, A_k$ are constants that must be found.

EXAMPLE 8.2.5: Calculate $\mathcal{L}^{-1}\left\{\frac{5s^2+20s+6}{s^3+2s^2+s}\right\}$.

SOLUTION: After using `Apart`

```
In[1576]:= Apart[(5s^2 + 20s + 6)/(s^3 + 2s^2 + s)]
Out[1576]= 6/s + 9/(1 + s)^2 - 1/(1 + s)
```

we see that

$$\frac{5s^2+20s+6}{s^3+2s^2+s} = \frac{6}{s} - \frac{1}{s+1} + \frac{9}{(s+1)^2}.$$

Therefore,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{5s^2+20s+6}{s^3+2s^2+s}\right\} &= \mathcal{L}^{-1}\left\{\frac{6}{s} - \frac{1}{s+1} + \frac{9}{(s+1)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{6}{s} - \frac{1}{s+1} + 9\frac{1}{(s+1)^2}\right\} \\ &= 6 - e^{-t} + 9te^{-t}.\end{aligned}$$

As expected, we obtain the same results using `InverseLaplace Transform`.

```
In[1577]:= InverseLaplaceTransform[(5s^2 + 20s + 6)/(s^3 + 2s^2 + s), s, t]
Out[1577]= e^-t (-1 + 6 e^t + 9 t)
```

■

Irreducible Quadratic Factors

If $(s-a)^2+b^2$ is a factor of $Q(s)$ of multiplicity k that cannot be reduced to linear factors, the partial fraction expansion of $F(s)$ corresponding to $(s-a)^2+b^2$ is

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{\left[(s-a)^2+b^2\right]^2} + \cdots + \frac{A_k s + B_k}{\left[(s-a)^2+b^2\right]^k}.$$

EXAMPLE 8.2.6: Find $\mathcal{L}^{-1}\left\{\dfrac{2s^3-4s-8}{\left(s^2-s\right)\left(s^2+4\right)}\right\}$.

SOLUTION: As in the previous example, we use `Apart`

```
In[1578]:= Apart[(2s^3 - 4s - 8)/((s^2 - s) (s^2 + 4))]
Out[1578]= -2/(-1 + s) + 2/s + 2 (2 + s)/(4 + s^2)
```

to obtain the partial fraction decomposition. Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s^3-4s-8}{\left(s^2-s\right)\left(s^2+4\right)}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &\quad + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= 2 - 2e^t + 2\cos 2t + 2\sin 2t.\end{aligned}$$

```
In[1579]:= InverseLaplaceTransform[(2s^3 - 4s - 8)/((s^2 - s) (s^2 + 4)), s, t]
Out[1579]= 2 (1 - e^t + Cos[2 t] + Sin[2 t])
```

■

8.2.2 Laplace Transform of an Integral

We have seen that the Laplace transform of the derivatives of a given function can be found from the Laplace transform of the function. Similarly, the Laplace transform of the integral of a given function can also be obtained from the Laplace transform of the original function.

Theorem 25 (Laplace Transform of an Integral). *Suppose that* $F(s) = \mathcal{L}\{f(t)\}$ *where* $y = f(t)$ *is a piecewise continuous function on* $[0, \infty)$ *and of exponential order* b. *Then, for* $s > b$,

$$\mathcal{L}\left\{\int_0^t f(\alpha)\,d\alpha\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}. \tag{8.9}$$

The theorem implies that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{f(t)\}\right\} = \int_0^t f(\alpha)\,d\alpha. \tag{8.10}$$

EXAMPLE 8.2.7: Compute $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+2)}\right\}$.

SOLUTION: In this case, $\dfrac{1}{s(s+2)} = \dfrac{1}{s}\dfrac{1}{s+2}$, so $\mathcal{L}\{f(t)\} = \dfrac{1}{s+2}$. Therefore, $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\} = e^{-2t}$. With the previous theorem, we then have

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \int_0^t e^{-2\alpha}d\alpha = \frac{1}{2}\left(1 - e^{-2t}\right).$$

Note that the same result is obtained with `InverseLaplaceTransform`

```
In[1580]:= InverseLaplaceTransform[1/(s (s + 2)), s, t]

Out[1580]= 1/2 - e^(-2 t)/2
```

or through a partial fraction expansion of $\dfrac{1}{s(s+2)}$: $\dfrac{1}{s(s+2)} = \dfrac{1}{2s} - \dfrac{1}{2(s+2)}$,
$\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+2)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{2s} - \dfrac{1}{2(s+2)}\right\} = \dfrac{1}{2} - \dfrac{1}{2}e^{-2t}$.

■

The following theorem is useful in determining if the inverse Laplace transform of a function $F(s)$ exists.

Theorem 26. *Suppose that* $y = f(t)$ *is a piecewise continuous function on* $[0, \infty)$ *and of exponential order* b. *Then,*

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \mathcal{L}\{f(t)\} = 0.$$

EXAMPLE 8.2.8: Determine if the inverse Laplace transform of the functions exists for (a) $F(s) = \dfrac{2s}{s-6}$; and (b) $F(s) = \dfrac{s^3}{s^2+16}$.

SOLUTION: In both cases, we find $\lim_{s\to\infty} F(s)$. If this value is not zero, then $\mathcal{L}^{-1}\{F(s)\}$ cannot be found. (a) $\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \dfrac{2s}{s-6} = 2 \neq 0$, so $\mathcal{L}^{-1}\left\{\dfrac{2s}{s-6}\right\}$ does not exist. (b) $\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \dfrac{s^3}{s^2+16} = \infty \neq 0$. Thus, $\mathcal{L}^{-1}\left\{\dfrac{s^3}{s^2+16}\right\}$ does not exist.

■

8.3 Solving Initial-Value Problems with the Laplace Transform

Laplace transforms can be used to solve certain initial-value problems. Typically, when we use Laplace transforms to solve an initial-value problem for a function y, we do the following.

1. Compute the Laplace transform of each term in the differential equation.
2. Solve the resulting equation for $\mathcal{L}\{y(t)\}$.
3. Determine y by computing the inverse Laplace transform of $\mathcal{L}\{y(t)\}$.

The advantage of this method is that through the use of the property

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

we transform a linear differential equation to an algebraic equation.

EXAMPLE 8.3.1: Solve the initial-value problem $y' - 4y = e^{4t}$, $y(0) = 0$.

SOLUTION: We begin by taking the Laplace transform of both sides of the differential equation and then solving for $\mathcal{L}\{y(t)\} = Y(s)$. Because $\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$, we have

$$\begin{aligned} \mathcal{L}\left\{y' - 4y\right\} &= \mathcal{L}\left\{e^{4t}\right\} \\ \mathcal{L}\left\{y'\right\} - 4\mathcal{L}\{y\} &= \frac{1}{s-4} \\ sY(s) - 4Y(s) &= \frac{1}{s-4} \\ (s-4)Y(s) &= \frac{1}{s-4} \\ Y(s) &= \frac{1}{(s-4)^2}. \end{aligned}$$

We carry out the same steps with Mathematica. After computing the Laplace transform of each side of the equation,

```
In[1581]:= step1 = LaplaceTransform
              [y'[t] - 4 y[t] == Exp[4t], t, s]

Out[1581]= -4 LaplaceTransform[y[t], t, s]
                                                        1
            +s LaplaceTransform[y[t], t, s] - y[0] == ------
                                                      -4 + s
```

we apply the initial condition

```
In[1582]:= step2 = step1/.y[0] → 0

Out[1582]= -4 LaplaceTransform[y[t], t, s]
                                                1
            +s LaplaceTransform[y[t], t, s] == ------
                                              -4 + s
```

and solve the resulting equation for $\mathcal{L}\{y(t)\} = Y(s)$.

```
In[1583]:= step3 = Solve[step2,
              LaplaceTransform[y[t], t, s]]

                                                   1
Out[1583]= {{LaplaceTransform[y[t], t, s] → ---------}}
                                                    2
                                            (-4 + s)
```

Hence, by using the shifting property with $\mathcal{L}\{t\} = s^{-2}$, we have

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = te^{4t}.$$

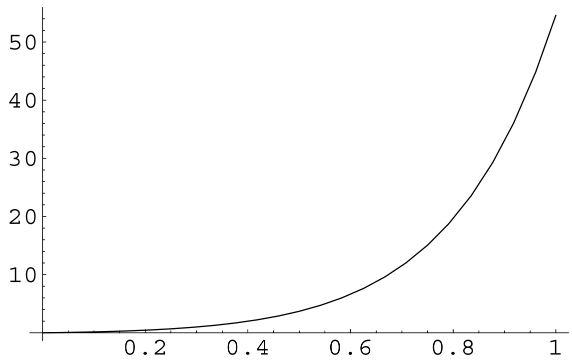


Figure 8-2 In the plot, we see that the initial condition is satisfied

Identical results are obtained using `InverseLaplaceTransform`.

```
In[1584]:= sol = InverseLaplaceTransform[
                  step3[[1, 1, 2]], s, t]
Out[1584]= e^(4 t) t
```

We then graph the solution with `Plot` in Figure 8-2.

```
In[1585]:= Plot[sol, {t, 0, 1}]
```

We can also use `DSolve` to solve the initial-value problem directly.

```
In[1586]:= DSolve[{y'[t] - 4 y[t] == Exp[4t], y[0] == 0},
             y[t], t]
Out[1586]= {{y[t] → e^(4 t) t}}
```

■

As we can see, Laplace transforms are useful in solving nonhomogeneous equations. Hence, problems in Chapter 4 for which the methods of undetermined coefficients or variation of parameters were difficult to apply may be more easily solved through the method of Laplace transforms.

EXAMPLE 8.3.2: Use Laplace transforms to solve $y'' + 4y = e^{-t}\cos 2t$ subject to $y(0) = 0$ and $y'(0) = -1$.

SOLUTION: We proceed by computing the Laplace transform of each side of the equation with `LaplaceTransform`

```
In[1587]:= step1 = LaplaceTransform
                 [y''[t] + 4 y[t] == Exp[-t]
                 Cos[2t], t, s]
```

Out[1587]= $4\,\text{LaplaceTransform}[y[t],t,s] + s^2\,\text{LaplaceTransform}[y[t],t,s] - s\,y[0] - y'[0] == \frac{1+s}{5+2\,s+s^2}$

and then applying the initial conditions $y(0) = 0$ and $y'(0) = -1$ with `ReplaceAll` (`/.`), naming the result `step2`.

```
In[1588]:= step2 = step1/.{y[0] → 0, y'[0] → -1}
```

Out[1588]= $1 + 4\,\text{LaplaceTransform}[y[t],t,s] + s^2\,\text{LaplaceTransform}[y[t],t,s] == \frac{1+s}{5+2\,s+s^2}$

Next, we solve `step2` for the Laplace transform of $y(t)$ and simplify the result, naming the resulting output `step3`

```
In[1589]:= step3 = Solve[step2,
                 LaplaceTransform[y[t], t, s]]
```

Out[1589]= $\left\{\left\{\text{LaplaceTransform}[y[t],t,s] \to \frac{-4-s-s^2}{(4+s^2)\,(5+2\,s+s^2)}\right\}\right\}$

and use `InverseLaplaceTransform` to compute the inverse Laplace transform of `step3`, naming the result `sol`.

In[1590]:= $\text{sol} = \text{Simplify}\left[\text{InverseLaplaceTransform}\left[-\frac{4+s+s^2}{(4+s^2)\,(5+2s+s^2)}, s, t\right]\right] // \text{Expand}$

Out[1590]= $\left(\frac{1}{34} - \frac{2\,i}{17}\right) e^{(-1-2\,i)\,t} + \left(\frac{1}{34} + \frac{2\,i}{17}\right) e^{(-1+2\,i)\,t} - \frac{1}{17}\,\text{Cos}[2\,t] - \frac{4}{17}\,\text{Sin}[2\,t]$

Last, we use `Plot` to graph the solution obtained in `sol` on the interval $[0, 2\pi]$ in Figure 8-3.

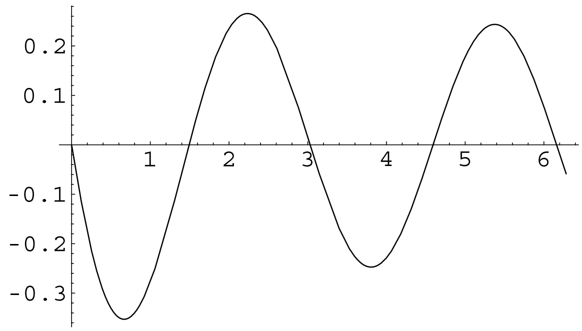


Figure 8-3 In the plot of the solution, we see that the initial conditions are satisfied

```
In[1591]:= Plot[sol, {t, 0, 2π}]
```

As we have seen in many previous examples, DSolve is able to solve the initial-value problem as well.

```
In[1592]:= sol =
             DSolve[{y''[t] + 4y[t] == Exp[-t] Cos[2t],
               y[0] == 0,
               y'[0] == -1}, y[t], t]
```

$$\text{Out[1592]=}\ \left\{\left\{y[t] \to -\frac{1}{68}\, e^{-t}\, (4\, e^{t}\, \text{Cos}[2\,t] - 4\,\text{Cos}[2\,t]\,\text{Cos}[4\,t] + 17\,\text{Sin}[2\,t] + 16\, e^{t}\,\text{Sin}[2\,t] + \text{Cos}[4\,t]\,\text{Sin}[2\,t] - \text{Cos}[2\,t]\,\text{Sin}[4\,t] - 4\,\text{Sin}[2\,t]\,\text{Sin}[4\,t])\right\}\right\}$$

■

Higher-order initial-value problems can be solved with the method of Laplace transforms as well.

EXAMPLE 8.3.3: Solve $y''' + y'' - 6y' = \sin 4t$, $y(0) = 2$, $y'(0) = 0$, $y''(0) = -1$.

SOLUTION: We first note that DSolve is able to quickly find an explicit solution of the initial-value problem.

```
In[1593]:= sol =
             DSolve[{y^(3)[t] + y''[t] - 6 y'[t] == Sin[4t],
               y[0] == 2, y'[0] == 0, y''[0] == -1}, y[t], t]
```

$$\text{Out[1593]= } \left\{\left\{y[t] \to -\frac{1}{1000}\left(e^{-3t}\left(56 - 2125\, e^{3t} + 80\, e^{5t} - 11\, e^{3t}\, \text{Cos}[4t] + 2\, e^{3t}\, \text{Sin}[4t]\right)\right)\right\}\right\}$$

Alternatively, we can use Mathematica to implement the steps encountered when solving the equation using the method of Laplace transforms, as in the previous two examples. Taking the Laplace transform of both sides of the equation, we find

```
In[1594]:= step1 = LaplaceTransform[y^(3)[t]
                 +y''[t] - 6 y'[t] == Sin[4t], t, s]
```

$$\text{Out[1594]= } s^2\, \text{LaplaceTransform}[y[t], t, s] + s^3\, \text{LaplaceTransform}[y[t], t, s] - 6\,(s\, \text{LaplaceTransform}[y[t], t, s] - y[0]) - s\, y[0] - s^2\, y[0] - y'[0] - s\, y'[0] - y''[0] == \frac{4}{16 + s^2}$$

and then we apply the initial conditions, naming the result `step2`.

```
In[1595]:= step2 = step1/.{y[0] → 2, y'[0] → 0,
             y''[0] → -1}
```

$$\text{Out[1595]= } 1 - 2\,s - 2\,s^2 + s^2\, \text{LaplaceTransform}[y[t], t, s] + s^3\, \text{LaplaceTransform}[y[t], t, s] - 6\,(-2 + s\, \text{LaplaceTransform}[y[t], t, s]) == \frac{4}{16 + s^2}$$

Solving for $Y(s)$, we obtain

```
In[1596]:= step3 = Solve[step2,
                   LaplaceTransform[y[t], t, s]]
```

$$\text{Out[1596]= } \left\{\left\{\text{LaplaceTransform}[y[t], t, s] \to \frac{-204 + 32\,s + 19\,s^2 + 2\,s^3 + 2\,s^4}{(16 + s^2)\,(-6\,s + s^2 + s^3)}\right\}\right\}$$

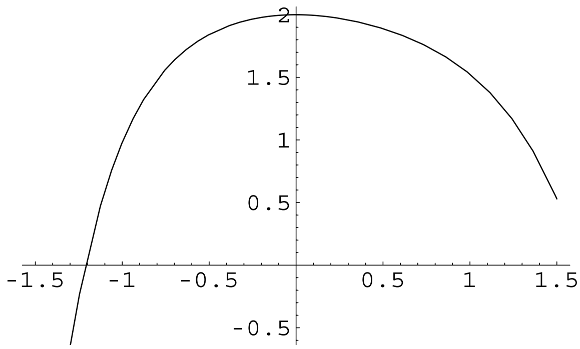


Figure 8-4 Plot of the solution to a third-order initial-value problem

and computing the inverse Laplace transform of `step3` with `InverseLaplaceTransform` yields the solution to the initial-value problem.

```
In[1597]:= sol = InverseLaplaceTransform[step3[[1, 1, 2]],
                  s, t]//Simplify
```

$$\text{Out[1597]}= \frac{2125 - 56\,e^{-3t} - 80\,e^{2t} + 11\,\text{Cos}[4t] - 2\,\text{Sin}[4t]}{1000}$$

Last, a graph of the solution is generated with `Plot` in Figure 8-4.

$$\text{In[1598]:= Plot}\left[\text{sol}, \left\{t, -\frac{3}{2}, \frac{3}{2}\right\}\right]$$

■

Some initial-value problems that involve differential equations with nonconstant coefficients can also be solved with the method of Laplace transforms. However, Laplace transforms do not provide a general method for solving equations with nonconstant coefficients.

EXAMPLE 8.3.4: Solve $\begin{cases} y'' + 2ty' - 4y = 2 \\ y(0) = y'(0) = 0. \end{cases}$

SOLUTION: `DSolve` is able to solve this equation.

```
In[1599]:= sol = DSolve[{y''[t] + 2t y'[t] - 4y[t] == 2,
                  y[0] == 0, y'[0] == 0}, y[t], t]
Out[1599]= {{y[t] → t^2}}
```

Using the method of Laplace transforms, we take the Laplace transform of both sides of the equation.

```
In[1600]:= step1 = LaplaceTransform[
                y''[t] + 2t y'[t] - 4y[t] == 2, t, s]

Out[1600]= -4 LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s]
           +2 LaplaceTransform[t y'[t], t, s]
           -s y[0] - y'[0] == 2/s
```

Next, we apply the initial conditions.

```
In[1601]:= step2 = step1 /. {y[0]- > 0, y'[0]- > 0}

Out[1601]= -4 LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s]
           +2 LaplaceTransform[t y'[t], t, s] == 2/s
```

This is a first-order linear equation that we are able to solve with DSolve. First, in step3, we replace LaplaceTransform[y[t], t, s] with capy[s], which represents $Y(s)$, and $\text{LaplaceTransform}^{(0,0,1)}$[y[t], t, s] with capy′[s], which represents $Y'(s)$. Then in step4 we use DSolve to solve for capy[s].

```
In[1602]:= step3 =
            step2 /.
             {LaplaceTransform[y[t], t, s]- > capy[s],
               LaplaceTransform^(0,0,1) [y[t], t, s]- >
               capy'[s]}

Out[1602]= -4 capy[s] + s^2 capy[s]
           +2 LaplaceTransform[t y'[t], t, s] == 2/s

In[1603]:= Simplify[step3]

Out[1603]= (-4 + s^2) capy[s]
           +2 LaplaceTransform[t y'[t], t, s] == 2/s
```

```
In[1604]:= step4 = DSolve[step3, capy[s], s]
Out[1604]= {{capy[s] →
              2 (-1 + s LaplaceTransform[t y'[t], t, s])
            - ------------------------------------------ }}
                           s (-4 + s^2)
```

These results indicate that $Y(s) = 2s^{-3} + Ce^{\frac{1}{4}s^2 - 3\ln 3}$. Recall that if $\lim_{s\to\infty} Y(s) \neq 0$, $\mathcal{L}^{-1}\{Y(s)\}$ does not exist. Therefore, we must have that $C = 0$. Hence, $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{2s^{-3}\right\} = t^2$.

```
In[1605]:= InverseLaplaceTransform[2/s^3, s, t]
Out[1605]= t^2
```

■

8.4 Laplace Transforms of Step and Periodic Functions

8.4.1 Piecewise-Defined Functions: The Unit Step Function

An important function in modeling many physical situations is the *unit step function*, $\mathcal{U}$.

Definition 35 (Unit Step Function). *The* ***unit step function****, $\mathcal{U}(t-a) = \mathcal{U}_a(t)$, where a is a number defined by*

$$\mathcal{U}(t-a) = \mathcal{U}_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases} \tag{8.11}$$

We can use the function `UnitStep` to define the unit step function:

$$\texttt{UnitStep[t]} = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

so $\mathcal{U}_a(t) = \texttt{UnitStep[t - a]}$.

EXAMPLE 8.4.1: Graph (a) $2\mathcal{U}(t)$, (b) $\frac{1}{2}\mathcal{U}(t-5)$, and (c) $\mathcal{U}(t-2) - \mathcal{U}(t-8)$.

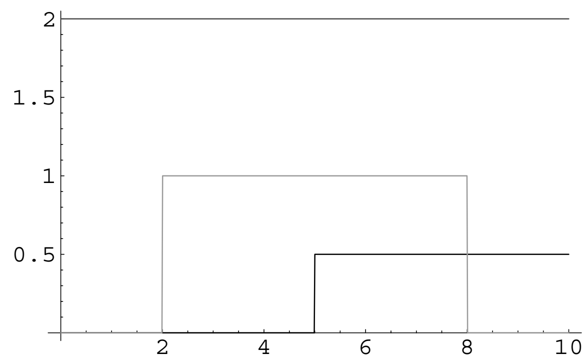


Figure 8-5 Plots of combinations of various step functions

SOLUTION: (a) Here, $2\mathcal{U}(t) = 2\mathcal{U}(t-0)$, so $2\mathcal{U}(t) = 2$ for $t \geq 0$.

(b) In this case, $\frac{1}{2}\mathcal{U}(t-5) = \begin{cases} 0, & t < 5 \\ 1/2, & t \geq 5 \end{cases}$ so the "jump" occurs at $t = 5$.

(c) $\mathcal{U}(t-2) - \mathcal{U}(t-8) = \begin{cases} 0, & t < 2 \text{ or } t \geq 8 \\ 1, & 2 \leq t < 8 \end{cases}$. These functions are graphed using `Plot` and `UnitStep` in Figure 8-5.

```
In[1606]:= Plot[{UnitStep[t - 5]/2, 2 UnitStep[t],
              UnitStep[t - 2] - UnitStep[t - 8]},
              {t, 0, 10},
              PlotStyle → {GrayLevel[0], GrayLevel[0.3],
              GrayLevel[0.6]}]
```

■

The unit step function is useful in defining functions that are piecewise continuous. For example, we can define the function

$$g(t) = \begin{cases} 0, & t < a \\ h(t), & a \leq t < b \\ 0, & t \geq b \end{cases}$$

as

$$g(t) = h(t)\left[\mathcal{U}(t-a) - \mathcal{U}(t-b)\right].$$

Similarly, a function like

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$

can be written as

$$f(t) = g(t)\,[1 - \mathcal{U}(t-a)] + h(t)\mathcal{U}(t-a).$$

The reason for writing piecewise continuous functions in terms of step functions is that we encounter functions of this type in solving initial-value problems. Using our methods in Chapters 4 and 5, we had to solve the problem over each piece of the function. However, the method of Laplace transforms can be used to avoid these complicated calculations.

Theorem 27. *Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > b \ge 0$. If a is a positive constant, then*

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s). \tag{8.12}$$

EXAMPLE 8.4.2: Find $\mathcal{L}\left\{(t-3)^5\mathcal{U}(t-3)\right\}$.

SOLUTION: In this case, $a = 3$ and $f(t) = t^5$. Thus,

$$\mathcal{L}\left\{(t-3)^5\mathcal{U}(t-3)\right\} = e^{-3s}\mathcal{L}\left\{t^5\right\} = e^{-3s}\frac{5!}{s^6} = \frac{120}{s^6}e^{-3s}.$$

Equivalent results are obtained with Mathematica.

```
In[1607]:= LaplaceTransform[(t - 3)^5 UnitStep[t - 3], t, s]
```

$$\text{Out[1607]} = -\frac{243\,e^{-3s}}{s} + \frac{405\,e^{-3s}\,(1+3s)}{s^2} - \frac{270\,e^{-3s}\,(2+6s+9s^2)}{s^3} + \frac{270\,e^{-3s}\,(2+6s+9s^2+9s^3)}{s^4} - \frac{45\,e^{-3s}\,(8+24s+36s^2+36s^3+27s^4)}{s^5} + \frac{3\,e^{-3s}\,(40+120s+180s^2+180s^3+135s^4+81s^5)}{s^6}$$

■

In most cases, we must calculate $\mathcal{L}\{g(t)\mathcal{U}(t-a)\}$ instead of $\mathcal{L}\{g(t-a)\mathcal{U}(t-a)\}$. To solve this problem, we let $g(t) = f(t-a)$, so $f(t) = g(t+a)$. Therefore,

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}. \tag{8.13}$$

EXAMPLE 8.4.3: Calculate $\mathcal{L}\{\sin t\,\mathcal{U}(t-\pi)\}$.

SOLUTION: In this case, $g(t) = \sin t$ and $a = \pi$. Thus,

$$\mathcal{L}\{\sin t\,\mathcal{U}(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} = e^{-\pi s}\mathcal{L}\{-\sin t\}$$
$$= -e^{-\pi s}\frac{1}{s^2+1} = -\frac{e^{-\pi s}}{s^2+1}.$$

The same result is obtained using `LaplaceTransform`.

```
In[1608]:= LaplaceTransform[Sin[t] UnitStep[t - π], t, s]
```

$$\text{Out[1608]=}\; -\frac{e^{-\pi s}}{1+s^2}$$

■

Theorem 28. *Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > b \geq 0$. If a is a positive constant and $y = f(t)$ is continuous on $[0,\infty)$, then*

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)\mathcal{U}(t-a). \tag{8.14}$$

EXAMPLE 8.4.4: Find (a) $\mathcal{L}^{-1}\left\{\dfrac{e^{-4s}}{s^3}\right\}$ and (b) $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s/2}}{s^2+16}\right\}$.

SOLUTION: (a) If we write the expression $\dfrac{e^{-4s}}{s^3}$ in the form $e^{-as}F(s)$, we see that $a = 4$ and $F(s) = s^{-3}$. Hence, $f(t) = \mathcal{L}^{-1}\left\{s^{-3}\right\} = \frac{1}{2}t^2$ and

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^3}\right\} = f(t-4)\mathcal{U}(t-4) = \frac{1}{2}(t-4)^2\mathcal{U}(t-4).$$

(b) In this case, $a = \pi/2$ and $F(s) = \dfrac{1}{s^2+16}$. Then, $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+16}\right\} = \frac{1}{4}\sin 4t$ and

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+16}\right\} = f\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right) = \frac{1}{4}\sin\left[4\left(t-\frac{\pi}{2}\right)\right]\mathcal{U}\left(t-\frac{\pi}{2}\right)$$
$$= \frac{1}{4}\sin 4t\,\mathcal{U}\left(t-\frac{\pi}{2}\right).$$

For each of (a) and (b), the same results are obtained using `InverseLaplaceTransform`, although we must use `Simplify` to simplify the result obtained for (b).

```
In[1609]:= InverseLaplaceTransform[Exp[-4 s]/s^3, s, t]
Out[1609]= 1/2 (-4 + t)^2 UnitStep[-4 + t]
In[1610]:= step1 = InverseLaplaceTransform[
                Exp[-πs/2]/(s^2 + 16), s, t]
Out[1610]= 1/4 Sin[4 (-π/2 + t)] UnitStep[-π/2 + t]
In[1611]:= Simplify[step1]
Out[1611]= 1/4 Sin[4 t] UnitStep[-π/2 + t]
```

■

8.4.2 Solving Initial-Value Problems

With the unit step function, we can solve initial-value problems that involve piecewise continuous functions.

EXAMPLE 8.4.5: Solve $y'' + 9y = \begin{cases} 1, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$ subject to $y(0) = y'(0) = 0$.

SOLUTION: In order to solve this initial-value problem, we must compute $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$. This is a piecewise continuous function so we write it in terms of the unit step function as

$$f(t) = 1\left[\mathcal{U}(t-0) - \mathcal{U}(t-\pi)\right] + 0\left[\mathcal{U}(t-\pi)\right] = \mathcal{U}(t) - \mathcal{U}(t-\pi).$$

Then,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 - \mathcal{U}(t-\pi)\} = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Hence,

$$\begin{aligned} \mathcal{L}\left\{y''\right\} + 9\mathcal{L}\{y\} &= \mathcal{L}\{f(t)\} \\ s^2 Y(s) - sy(0) - y'(0) + 9Y(s) &= \frac{1}{s} - \frac{e^{-\pi s}}{s} \\ \left(s^2+9\right) Y(s) &= \frac{1}{s} - \frac{e^{-\pi s}}{s} \\ Y(s) &= \frac{1}{s\left(s^2+9\right)} - \frac{e^{-\pi s}}{s\left(s^2+9\right)}. \end{aligned}$$

The same steps are performed next with Mathematica. First, we define `eq` to be the equation $y'' + 9y = \begin{cases} 1, & 0 \le t < \pi \\ 0, & t \ge \pi. \end{cases}$

```
In[1612]:= eq = y''[t] + 9 y[t] ==
                 UnitStep[t] - UnitStep[t - π];
```

Next, we use `LaplaceTransform` to compute the Laplace transform of each side of the equation, naming the resulting equation `step1`,

```
In[1613]:= step1 = LaplaceTransform[eq, t, s]

Out[1613]= 9 LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s]
           -s y[0] - y'[0] == 1/s - e^(-π s)/s
```

apply the initial conditions, naming the result `step2`,

```
In[1614]:= step2 = step1 /. {y[0]- > 0, y'[0]- > 0}

Out[1614]= 9 LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s] == 1/s - e^(-π s)/s
```

and solve `step2` for `LaplaceTransform[y[t],t,s]`, naming the result `step3`.

```
In[1615]:= step3 = Solve[step2,
                      LaplaceTransform[y[t], t, s]]

Out[1615]= {{LaplaceTransform[y[t], t, s] →
              e^(-π s) (-1 + e^(π s)) / (s (9 + s^2))}}
```

Then,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+9\right)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s\left(s^2+9\right)}\right\}.$$

Consider $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s\left(s^2+9\right)}\right\}$. In the form of $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}$, $a=\pi$ and $F(s)=\frac{1}{s\left(s^2+9\right)}$. $f(t)=\mathcal{L}^{-1}\left\{F(s)\right\}$ can be found with either a partial fraction expansion or with equation (8.10):

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+9\right)}\right\}=\int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}d\alpha=\int_0^t \frac{1}{3}\sin 3\alpha\, d\alpha$$
$$=-\frac{1}{3}\left[\frac{1}{3}\cos 3\alpha\right]_0^t=\frac{1}{9}-\frac{1}{9}\cos 3t.$$

Then,

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s\left(s^2+9\right)}\right\}=\left[\frac{1}{9}-\frac{1}{9}\cos\left(3(t-\pi)\right)\right]\mathcal{U}(t-\pi)$$
$$=\left[\frac{1}{9}-\frac{1}{9}\cos\left(3t-3\pi\right)\right]\mathcal{U}(t-\pi)=\left[\frac{1}{9}+\frac{1}{9}\cos 3t\right]\mathcal{U}(t-\pi).$$

Combining these results yields the solution

$$y(t)=\mathcal{L}^{-1}\left\{Y(s)\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+9\right)}\right\}-\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s\left(s^2+9\right)}\right\}$$
$$=\frac{1}{9}-\frac{1}{9}\cos 3t-\left[\frac{1}{9}+\frac{1}{9}\cos 3t\right]\mathcal{U}(t-\pi).$$

Equivalent results are obtained with `InverseLaplaceTransform` and `Simplify`.

```
In[1616]:= sol = InverseLaplaceTransform[
                    -1 + e^(-π s)
                  - ------------- , s, t]
                    s (9 + s^2)
```

$$\text{Out[1616]}= \frac{1}{9}\left(1-\text{Cos}[3\,t]-(1+\text{Cos}[3\,t])\,\text{UnitStep}[-\pi+t]\right)$$

```
In[1617]:= Simplify[sol]
```

$$\text{Out[1617]}= \frac{1}{9}\left(1-\text{Cos}[3\,t]-(1+\text{Cos}[3\,t])\,\text{UnitStep}[-\pi+t]\right)$$

We now graph this solution with `Plot` in Figure 8-6.

```
In[1618]:= Plot[sol, {t, 0, 2π}]
```

An equivalent result is obtained using `DSolve` as shown next.

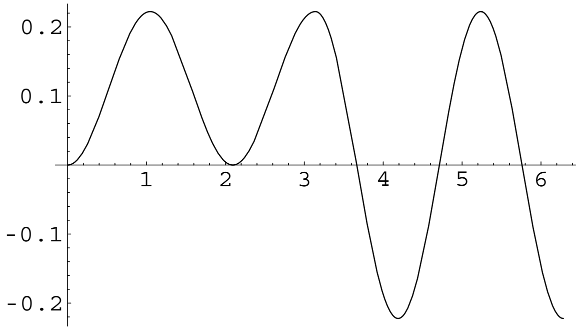


Figure 8-6 Plot of $y(t)$

```
In[1619]:= sol = DSolve[{eq, y[0] == 0, y'[0] == 0}, y[t], t]
Out[1619]= {{y[t] → 1/9 (- Cos[3 t] UnitStep[t]
                +Cos[3 t]^2 UnitStep[t]
                + Sin[3 t]^2 UnitStep[t]
                - Cos[3 t] UnitStep[-π + t]
                -Cos[3 t]^2 UnitStep[-π + t]
                - Sin[3 t]^2 UnitStep[-π + t])}}
```

■

8.4.3 Periodic Functions

Another type of function that is encountered in many areas of applied mathematics is the *periodic function*.

Definition 36 (Periodic Function). *A function $y = f(t)$ is* ***periodic*** *if there is a positive number T such that $f(t + T) = f(t)$ for all $t \geq 0$. The minimum value of T that satisfies this equation is called the* ***period*** *of $y = f(t)$.*

Due to the nature of periodic functions, we can simplify the calculation of the Laplace transform of these functions as indicated in the following theorem.

Theorem 29 (Laplace Transform of Periodic Functions). *Suppose that $y = f(t)$ is a periodic function with period T and that $y = f(t)$ is piecewise continuous on $[0, \infty)$. Then, $\mathcal{L}\{f(t)\}$ exists for $s > 0$ and is given by the definite integral*

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt. \tag{8.15}$$

EXAMPLE 8.4.6: Find the Laplace transform of the periodic function $f(t) = t$, $0 \le t < 1$, and $f(t+1) = f(t)$.

SOLUTION: The period of $y = f(t)$ is $T = 1$. We use `Plot` to generate a graph of $y = f(t)$ on the interval $[0, 4]$ in Figure 8-7.

```
In[1620]:= Clear[f]

            f[t_] := f[t - 1]/; t ≥ 1

            f[t_] := t/; 0 ≤ t < 1
In[1621]:= Plot[f[t], {t, 0, 4}]
```

We use integration by parts,

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-s}}\int_0^1 te^{-st}\,dt \\ &= \frac{1}{1-e^{-s}}\left\{\left[-\frac{te^{-st}}{s}\right]_{t=0}^{t=1} + \int_0^1 \frac{e^{-st}}{s}\,dt\right\} \\ &= \frac{1}{1-e^{-s}}\left\{-\frac{e^{-s}}{s} - \left[\frac{e^{-st}}{s^2}\right]_{t=0}^{t=1}\right\} \\ &= \frac{1}{1-e^{-s}}\left(-\frac{e^{-s}}{s} + \frac{1-e^{-s}}{s^2}\right) = \frac{1-(s+1)e^{-s}}{s^2\left(1-e^{-s}\right)}\end{aligned}$$

or Mathematica

```
In[1622]:= Simplify[(∫_0^1 t Exp[-s t] dt) / (1 - Exp[-s])]
```

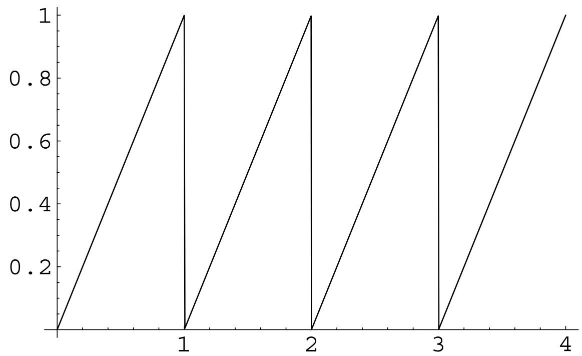


Figure 8-7 Plot of $f(t)$ on the interval $[0, 4]$

```
Out[1622]= (-1 + e^s - s)/((-1 + e^s) s^2)

In[1623]:= term[n_] := LaplaceTransform[UnitStep[t - n],
                t, s]
```

to compute the Laplace transform. Alternatively, note that

$$\begin{aligned} f(t) &= t\,[\mathcal{U}(t) - \mathcal{U}(t-1)] + (t-1)\,[\mathcal{U}(t-1) - \mathcal{U}(t-2)] \\ &\quad + (t-2)\,[\mathcal{U}(t-2) - \mathcal{U}(t-3)] + \cdots \\ &= t - \mathcal{U}(t-1) - \mathcal{U}(t-2) - \mathcal{U}(t-3) - \mathcal{U}(t-4) - \cdots \\ &= t - \sum_{n=1}^{\infty} \mathcal{U}(t-n) \end{aligned}$$

so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\left\{\sum_{n=1}^{\infty} \mathcal{U}(t-n)\right\} = \mathcal{L}\{t\} - \sum_{n=1}^{\infty} \mathcal{L}\{\mathcal{U}(t-n)\}.$$

We use `LaplaceTransform` and `Table`

```
In[1624]:= Table[term[n], {n, 1, 7}]
Out[1624]= {e^-s/s, e^-2 s/s, e^-3 s/s, e^-4 s/s, e^-5 s/s, e^-6 s/s, e^-7 s/s}
```

to see that $\mathcal{L}\{\mathcal{U}(t-n)\} = \frac{1}{s}e^{-ns}$. Next, we use `Sum` and `Together` to calculate

For the geometric series, $\sum_{n=1}^{\infty} r^n$, if $|r| < 1$, $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{t\} - \sum_{n=1}^{\infty} \mathcal{L}\{\mathcal{U}(t-n)\} = \frac{1}{s^2} - \sum_{n=1}^{\infty} \frac{e^{-ns}}{s} \\ &= \frac{1}{s}\left(\frac{1}{s} - \sum_{n=1}^{\infty} \left(e^{-s}\right)^n\right) = \frac{1}{s}\left(\frac{1}{s} - \frac{e^{-s}}{1-e^{-s}}\right). \end{aligned}$$

```
In[1625]:= Together[LaplaceTransform[t, t, s]
              - Sum[1/(Exp[n s] s), {n, 1, ∞}]]
Out[1625]= (-1 + e^s - s)/((-1 + e^s) s^2)
```

■

Laplace transforms can now be used to solve initial-value problems with periodic forcing functions more easily.

EXAMPLE 8.4.7: Solve $y'' + y = f(t)$ subject to $y(0) = y'(0) = 0$ if $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}$ and $f(t + 2\pi) = f(t)$. ($f(t)$ is known as the **half-wave rectification** of $\sin t$.)

SOLUTION: To graph $f(t)$, we begin by defining $g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi. \end{cases}$

```
In[1626] := g[t_] = Sin[t] UnitStep[π - t];
```

Then,

$$\begin{aligned} f(t) &= g(t)\,[\mathcal{U}(t) - \mathcal{U}(t - 2\pi)] + g(t - 2\pi)\,[\mathcal{U}(t - 2\pi) - \mathcal{U}(t - 4\pi)] + \cdots \\ &= \sum_{n=0}^{\infty} g(t - n\pi)\,[\mathcal{U}(t - 2n\pi) - \mathcal{U}(t - 2(n + 1)\pi)] \end{aligned}$$

Thus, the graph of $f(t)$ on the interval $[0, 2k\pi]$, where k represents a positive integer, is obtained by graphing

$$f_k(t) = \sum_{n=0}^{k-1} g(t - n\pi)\,[\mathcal{U}(t - 2n\pi) - \mathcal{U}(t - 2(n + 1)\pi)]$$

on the interval $[0, 2k\pi]$. For convenience, we define `nthterm[n]` to be

$$g(t - n\pi)\,[\mathcal{U}(t - 2n\pi) - \mathcal{U}(t - 2(n + 1)\pi)].$$

```
In[1627] := nthterm[n_] = g[t - 2nπ]
                (UnitStep[t - 2nπ]
                -UnitStep[t - 2 (n + 1) π]);
```

```
                         k-1
In[1628] := f[k_, t_] =  Σ  nthterm[n];
                         n=0
```

Here is $f_2(t)$.

```
In[1629] := f[2, t]
Out[1629]= Sin[t] UnitStep[π - t]
             (UnitStep[t] - UnitStep[-2 π + t])
           + Sin[t] UnitStep[3 π - t]
             (-UnitStep[-4 π + t] + UnitStep[-2 π + t])
```

We graph $f(t)$ on the interval $[0, 10\pi]$ with `Plot` in Figure 8-8.

```
In[1630] := Plot[f[5, t], {t, 0, 10π}]
```

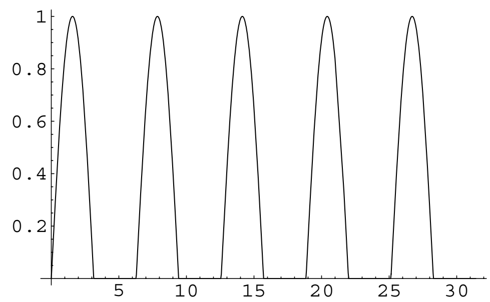


Figure 8-8 The half-wave rectification of $\sin t$ on the interval $[0, 10\pi]$

To solve the initial-value problem we must find $\mathcal{L}\{f(t)\}$. Because the period is $T = 2\pi$, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st}\sin t\,dt + \int_{\pi}^{2\pi} e^{-st}\cdot 0\,dt\right]$$
$$= \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt.$$

We use `Integrate` to evaluate this integral

```
In[1631]:= step1 = Simplify[∫_0^π Exp[-s t] Sin[t] dt / (1 - Exp[-2 π s])]
Out[1631]= e^(π s) / ((-1 + e^(π s)) (1 + s^2))
In[1632]:= lapf = step1//ExpandDenominator
Out[1632]= e^(π s) / (-1 + e^(π s) - s^2 + e^(π s) s^2)
```

and see that

$$\mathcal{L}\{f(t)\} = \frac{e^{\pi s}}{(e^{\pi s}-1)\left(s^2+1\right)} = \frac{1}{(1-e^{-\pi s})\left(s^2+1\right)}.$$

Alternatively, we can use

$$f(t) = \sum_{n=0}^{\infty} g(t-n\pi)\left[\mathcal{U}(t-2n\pi) - \mathcal{U}(t-2(n+1)\pi)\right]$$

to rewrite $f(t)$ as

$$f(t) = \sum_{n=0}^{\infty}(-1)^n \sin t\,\mathcal{U}(t-n\pi).$$

Then,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\sum_{n=0}^{\infty}(-1)^n \sin t\,\mathcal{U}(t-n\pi)\right\} = \sum_{n=0}^{\infty}\mathcal{L}\left\{(-1)^n \sin t\,\mathcal{U}(t-n\pi)\right\}.$$

We use LaplaceTransform and UnitStep to compute $\mathcal{L}\{(-1)^n \sin t\, \mathcal{U}(t-n\pi)\}$, naming the result nthlap,

```
In[1633]:= Clear[nthlap]

In[1634]:= nthlap = LaplaceTransform[
                (-1)^n Sin[t] UnitStep[t - nπ], t, s];

In[1635]:= TableForm[Table[{n, nthlap}, {n, 0, 8}]]
```

Out[1635]=

0	$\frac{2}{1+s^2}$
1	$-\frac{e^{-\pi(i+s)}}{1+s^2}$
2	$\frac{e^{-2\pi(i+s)}}{1+s^2}$
3	$-\frac{e^{-3\pi(i+s)}}{1+s^2}$
4	$\frac{e^{-4\pi(i+s)}}{1+s^2}$
5	$-\frac{e^{-5\pi(i+s)}}{1+s^2}$
6	$\frac{e^{-6\pi(i+s)}}{1+s^2}$
7	$-\frac{e^{-7\pi(i+s)}}{1+s^2}$
8	$\frac{e^{-8\pi(i+s)}}{1+s^2}$

and then use Sum to compute $\sum_{n=0}^{\infty} \mathcal{L}\{(-1)^n \sin t\, \mathcal{U}(t-n\pi)\}$.

In[1636]:= $\frac{\sum_{n=0}^{\infty} \mathrm{Exp}[-n\,\pi s]}{1+s^2}$

Out[1636]= $\frac{e^{\pi s}}{(-1+e^{\pi s})(1+s^2)}$

Taking the Laplace transform of both sides of the differential equation, applying the initial conditions, and solving for $Y(s)$ then gives us

$$\mathcal{L}\left\{y''\right\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{(1-e^{-\pi s})\left(s^2+1\right)}$$

$$Y(s) = \frac{1}{(1-e^{-\pi s})\left(s^2+1\right)^2}.$$

Using lapf, we perform the same steps with Mathematica.

```
In[1637]:= step1 = LaplaceTransform
                [y''[t] + y[t], t, s] == lapf
Out[1637]= LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s]
           -s y[0] - y'[0] == ⅇ^(π s)/(-1 + ⅇ^(π s) - s^2 + ⅇ^(π s) s^2)
In[1638]:= step2 = step1 /. {y[0] -> 0, y'[0] -> 0}
Out[1638]= LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s] ==
             ⅇ^(π s)/(-1 + ⅇ^(π s) - s^2 + ⅇ^(π s) s^2)
In[1639]:= step3 = Solve[step2,
                  LaplaceTransform[y[t], t, s]]
Out[1639]= {{LaplaceTransform[y[t], t, s] →
             ⅇ^(π s)/((-1 + ⅇ^(π s)) (1 + s^2)^2)}}
```

Recall from our work with the geometric series that if $|x| < 1$, then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

Because we do not know the inverse Laplace transform of $\dfrac{1}{(1-e^{-\pi s})\,\frac{1}{(1-e^{-\pi s})\left(s^2+1\right)^2}}$, we must use a geometric series expansion of $\dfrac{1}{1-e^{-\pi s}}$ to obtain terms for which we can calculate the inverse Laplace transform. Using $x = e^{-\pi s}$, this gives us

$$\frac{1}{1-e^{-\pi s}} = 1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + \cdots = \sum_{n=0}^{\infty} e^{-n\pi s},$$

so

$$\begin{aligned} Y(s) &= \left(1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + \cdots\right)\frac{1}{\left(s^2+1\right)^2} \\ &= \frac{1}{\left(s^2+1\right)^2} + \frac{e^{-\pi s}}{\left(s^2+1\right)^2} + \frac{e^{-2\pi s}}{\left(s^2+1\right)^2} + \frac{e^{-3\pi s}}{\left(s^2+1\right)^2} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{e^{-n\pi s}}{\left(s^2+1\right)^2}. \end{aligned}$$

Then,

$$y(t) = \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left\{\frac{e^{-n\pi s}}{\left(s^2+1\right)^2}\right\}.$$

Notice that $\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\}$ is needed to find all of the other terms. Using `InverseLaplaceTransform`,

```
In[1640]:= Expand[InverseLaplaceTransform
              [1/(s^2 + 1)^2, s, t]]
Out[1640]= -1/2 t Cos[t] + Sin[t]/2
```

we have $\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} = \frac{1}{2}(\sin t - t\cos t)$. In fact, we can use `InverseLaplaceTransform` together with `Table` to compute the inverse Laplace transform of the first few terms of the series.

```
In[1641]:= TableForm[
             Table[{n, InverseLaplaceTransform[
                 Exp[-n π s]/(s^2 + 1)^2, s, t]}, {n, 0, 5}]]
            0 1/2 (-t Cos[t] + Sin[t])
            1 1/2 ((-π + t) Cos[t]
               - Sin[t]) UnitStep[-π + t]
            2 1/2 (-(-2 π + t) Cos[t]
               + Sin[t]) UnitStep[-2 π + t]
Out[1641]=  3 1/2 ((-3 π + t) Cos[t]
               - Sin[t]) UnitStep[-3 π + t]
            4 1/2 (-(-4 π + t) Cos[t]
               + Sin[t]) UnitStep[-4 π + t]
            5 1/2 ((-5 π + t) Cos[t]
               - Sin[t]) UnitStep[-5 π + t]
```

Then,

$$
\begin{aligned}
y(t) &= \frac{1}{2}\Big\{(\sin t - t\cos t) + [\sin(t-\pi) - (t-\pi)\cos(t-\pi)]\,\mathcal{U}(t-\pi) \\
&\quad + [\sin(t-2\pi) - (t-2\pi)\cos(t-2\pi)]\,\mathcal{U}(t-2\pi) \\
&\quad + [\sin(t-3\pi) - (t-3\pi)\cos(t-3\pi)]\,\mathcal{U}(t-3\pi)\Big\} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}[\sin(t-n\pi) - (t-n\pi)\cos(t-n\pi)]\,\mathcal{U}(t-n\pi).
\end{aligned}
$$

To graph $y(t)$ on the interval $[0, k\pi]$, where k represents a positive integer, we note that

$$[\sin(t-n\pi)-(t-n\pi)\cos(t-n\pi)]\,\mathcal{U}(t-n\pi)=0$$

for all values of t in $[0, k\pi]$ if $n \geq k$ so we need to graph

$$\frac{1}{2}\sum_{n=0}^{k-1}[\sin(t-n\pi)-(t-n\pi)\cos(t-n\pi)]\,\mathcal{U}(t-n\pi).$$

For convenience, we define `nthterm` to represent

$$\frac{1}{2}[\sin(t-n\pi)-(t-n\pi)\cos(t-n\pi)]\,\mathcal{U}(t-n\pi).$$

```
In[1642]:= nthterm[n_] =
              1/2 (Sin[t - nπ] - (t - nπ) Cos[t - nπ])
               UnitStep[t - nπ];
```

Thus, to graph on the interval , we enter the following commands. See Figure 8-9.

```
In[1643]:= tograph = Sum[nthterm[n], {n, 0, 4}];
```

$$\texttt{tograph} = \sum_{n=0}^{4}\texttt{nthterm[n]};$$

```
In[1644]:= Plot[tograph, {t, 0, 5π}]
```

■

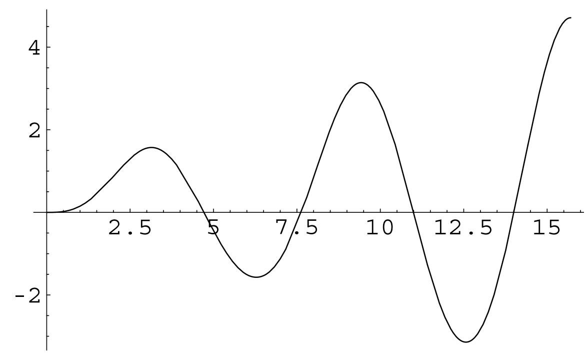


Figure 8-9 Plot of the solution to an initial-value problem with a periodic piecewise continuous forcing function

8.4.4 Impulse Functions: The Delta Function

We now consider differential equations of the form $ay'' + by' + cy = f(t)$ where $f(t)$ is large in magnitude over the short interval centered at t_0, $t_0 - \alpha \le t \le t_0 + \alpha$, and zero otherwise. Hence, we define the **impulse** delivered by the function $f(t)$ as $I(t) = \int_{t_0-\alpha}^{t_0+\alpha} f(t)\,dt$, or because $f(t) = 0$ for t on $(-\infty, t_0 - \alpha) \cup (t_0 + \alpha, \infty)$,

$$I(t) = \int_{-\infty}^{\infty} f(t)\,dt.$$

In order to better understand the impulse function, we let $f(t)$ be defined in the following manner:

$$f(t) = \delta_\alpha (t - t_0) = \begin{cases} \dfrac{1}{2\alpha}, & t_0 - \alpha \le t \le t_0 + \alpha \\ 0, & \text{otherwise.} \end{cases}$$

To graph $\delta_\alpha (t - t_0)$ for several values of α and $t_0 = 0$, we define `del`.

```
In[1645]:= del[t_, t0_, α_] := 1/(2α) /; t0 - α ≤ t ≤ t0 + α

           del[t_, t0_, α_] := 0/; t0 - α > t||t > t0 + α
```

For example, entering

```
In[1646]:= Plot[del[t, 0, 0.25], {t, -1, 1}]
```

graphs $\delta_{1/4}(t)$ on the interval $[-1, 1]$. See Figure 8-10. Similarly, to graph $\delta_i(t)$ for $i = 0.01, 0.02, 0.03, 0.04$, and 0.05, we first define `toplot` using `Table` and then use `Plot` to graph this set of functions on the interval $[-0.1, 0.1]$. See Figure 8-11.

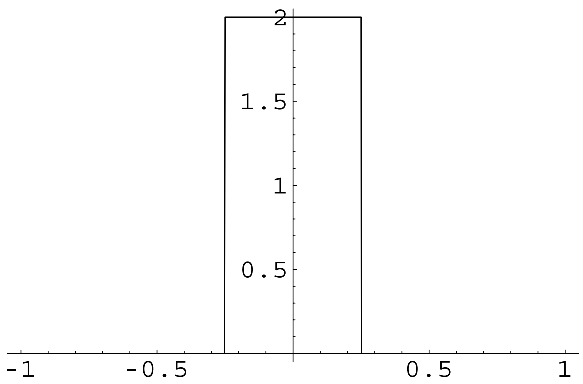


Figure 8-10 Plot of $\delta_{1/4}(t)$ on the interval $[-1, 1]$

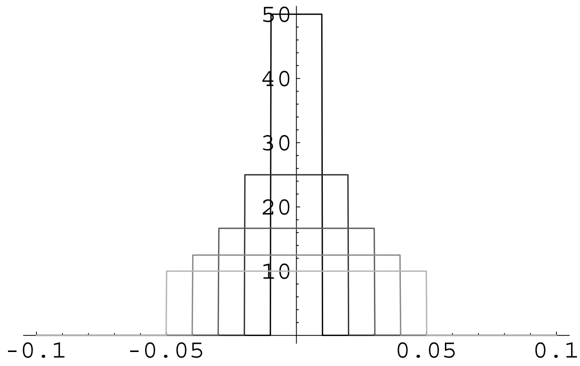


Figure 8-11 Plots of $\delta_i(t)$ for $i = 0.01, 0.02, 0.03, 0.04$, and 0.05

```
In[1647]:= toplot = Table[del[t, 0, α],
                {α, 0.01, 0.05, 0.01}];

In[1648]:= grays = Table[GrayLevel[i], {i, 0, 0.7, 0.7/4}];

           Plot[Evaluate[toplot], {t, -0.1, 0.1},
             PlotStyle- > grays]
```

With this definition, the impulse is given by

$$I(t) = \int_{t_0-\alpha}^{t_0+\alpha} f(t)\,dt = \int_{t_0-\alpha}^{t_0+\alpha} \frac{1}{2\alpha}\,dt = \frac{1}{2\alpha}\left[(t_0+\alpha)-(t_0-\alpha)\right] = \frac{1}{2\alpha}\cdot 2\alpha = 1.$$

Notice that the value of this integral does not depend on α as long as $\alpha \neq 0$. We now try to create the *idealized impulse function* by requiring that $\delta_\alpha(t-t_0)$ act on smaller and smaller intervals. From the integral calculation, we have

$$\lim_{\alpha\to 0} I(t) = 1.$$

We also note that

$$\lim_{\alpha\to 0} \delta_\alpha(t-t_0) = 0,\ t \neq t_0.$$

We use these properties to now define the **idealized unit impulse function**.

The Dirac delta function is not a real-valued function of a single variable. Objects of this type are called **generalized functions**.

Definition 37 (Unit Impulse Function). *The **idealized unit impulse function (Dirac delta function)** δ satisfies*

$$\begin{gathered}\delta(t-t_0) = 0,\ t \neq t_0 \\ \int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1.\end{gathered} \tag{8.16}$$

The Mathematica function `DiracDelta` represents the Dirac delta function.

We now state the following useful theorem involving the unit impulse function.

Theorem 30. *Suppose that $y = g(t)$ is a bounded and continuous function. Then,*

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, g(t)\, dt = g(t_0). \tag{8.17}$$

The Laplace transform of $\delta(t - t_0)$ is found by using the function $\delta_\alpha(t - t_0)$ and L'Hôpital's rule.

Theorem 31. *For $t_0 > 0$,*

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}. \tag{8.18}$$

EXAMPLE 8.4.8: Find (a) $\mathcal{L}\{\delta(t-1)\}$; (b) $\mathcal{L}\{\delta(t-\pi)\}$; and (c) $\mathcal{L}\{\delta(t)\}$.

SOLUTION: (a) In this case, $t_0 = 1$, so $\mathcal{L}\{\delta(t-1)\} = e^{-s}$. (b) With $t_0 = \pi$, $\mathcal{L}\{\delta(t-\pi)\} = e^{-\pi s}$. (c) Because $t_0 = 0$, $\mathcal{L}\{\delta(t)\} = \mathcal{L}\{\delta(t-0)\} = e^{-s\cdot 0} = 1$.

We obtain the same results using `DiracDelta` and `LaplaceTransform` as shown next. We can compute the Laplace transform of each individually.

```
In[1649]:= LaplaceTransform[DiracDelta[t - 1], t, s]
Out[1649]= e^-s
```

Or, we can use `Map` to compute the Laplace transform of all three simultaneously.

```
In[1650]:= Map[LaplaceTransform[#, t, s]&,
              {DiracDelta[t - 1],
              DiracDelta[t - π], DiracDelta[t]}]
Out[1650]= {e^-s, e^-π s, 1}
```

■

EXAMPLE 8.4.9: Solve $y'' + y = \delta(t - \pi) + 1$ subject to $y(0) = y'(0) = 0$.

SOLUTION: As in previous examples, we solve this initial-value problem by taking the Laplace transform of both sides of the differential equation,

```
In[1651]:= step1 = LaplaceTransform[y''[t]
                +y[t] == DiracDelta
                 [t - π] + 1, t, s]
Out[1651]= LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s]
           -s y[0] - y'[0] == e^(-π s) + 1/s
```

applying the initial conditions,

```
In[1652]:= step2 = step1/.{y[0] → 0, y'[0] → 0}
Out[1652]= LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s] == e^(-π s) + 1/s
```

and solving for $Y(s)$.

```
In[1653]:= step3 = Solve[step2,
                  LaplaceTransform[y[t], t, s]]
Out[1653]= {{LaplaceTransform[y[t], t, s] →
             e^(-π s) (e^(π s) + s) / (s (1 + s^2))}}
```

We find $y(t)$ using `InverseLaplaceTransform`.

```
In[1654]:= sol = InverseLaplaceTransform[
                step3[[1, 1, 2]], s, t]
Out[1654]= 1 - Cos[t] - Sin[t] UnitStep[-π + t]
```

We can use `DSolve` to find the solution to the initial-value problem as follows. The result is graphed with `Plot` in Figure 8-12.

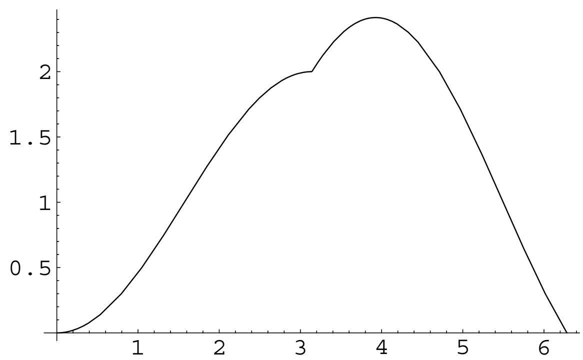


Figure 8-12 At $t = \pi$, an impulse is delivered

```
In[1655]:= sol = DSolve[{y''[t]
                 +y[t] == DiracDelta[t - π] + 1,
                 y[0] == 0, y'[0] == 0}, y[t], t]
Out[1655]= {{y[t] → - Cos[t] + Cos[t]^2
                 +Sin[t]^2 - Sin[t] UnitStep[-π + t]}}

In[1656]:= Plot[y[t]/.sol, {t, 0, 2π}]
```

■

The forcing function may involve a combination of functions as illustrated in the following example.

EXAMPLE 8.4.10: Solve $y'' + 2y' + y = 1 + \delta(t - \pi) + \delta(t - 2\pi)$ subject to $y(0) = y'(0) = 0$.

SOLUTION: After computing the Laplace transform of each side of the equation

```
In[1657]:= step1 = LaplaceTransform[
                 y''[t] + 2 y'[t] + y[t] ==
                 1 + DiracDelta[t - π]
                 +DiracDelta[t - 2π], t, s]
Out[1657]= LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s]
           +2 (s LaplaceTransform[y[t], t, s] - y[0])
           -s y[0] - y'[0] == e^(-2 π s) + e^(-π s) + 1/s
```

and applying the initial conditions,

```
In[1658]:= step2 = step1 /. {y[0]- > 0, y'[0]- > 0}
Out[1658]= LaplaceTransform[y[t], t, s]
           +2 s LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s] == e^(-2 π s)
           +e^(-π s) + 1/s
```

we solve for $Y(s)$

```
In[1659]:= step3 = Solve[step2,
                 LaplaceTransform[y[t], t, s]]
```

```
Out[1659]= {{LaplaceTransform[y[t], t, s] →
              e^(-2 π s) (e^(2 π s) + s + e^(π s) s)
              ──────────────────────────────────────── }}
                          s (1 + s)^2
```

and then compute $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

```
In[1660]:= sol = InverseLaplaceTransform[
                   1 + e^(-2 πs) s + e^(-π s) s
                   ──────────────────────────── , s, t]
                          s (1 + s)^2
Out[1660]= e^-t (-1 + e^t - t - e^(2 π) (2 π - t) UnitStep[-2 π + t]
           -e^π (π - t) UnitStep[-π + t])
```

Equivalent results are obtained with `DSolve` that are then graphed with `Plot` in Figure 8-13.

```
In[1661]:= Clear[y, t, sol]

In[1662]:= sol = DSolve[{y''[t] + 2 y'[t] + y[t] == 1
                 +DiracDelta[t - π] + DiracDelta[t - 2π],
                 y[0] == 0, y'[0] == 0}, y[t], t]
Out[1662]= {{y[t] → e^-t (-1 + e^t - t
                 -2 e^(2 π) π UnitStep[-2 π + t]
                 +e^(2 π) t UnitStep[-2 π + t]
                 -e^π π UnitStep[-π + t]
                 +e^π t UnitStep[-π + t])}}

In[1663]:= Plot[y[t]/.sol, {t, 0, 4π}]
```

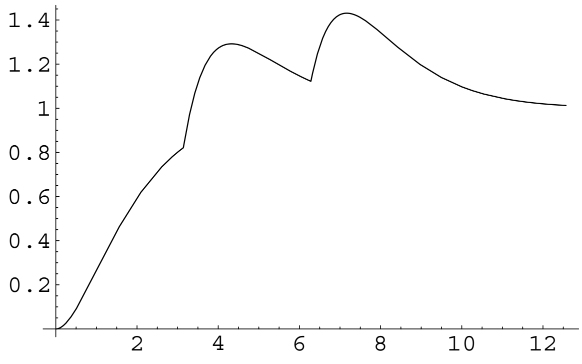


Figure 8-13 Impulses are delivered at $t = \pi$ and $t = 2\pi$

■

8.5 The Convolution Theorem

8.5.1 The Convolution Theorem

In many cases, we are required to determine the inverse Laplace transform of a product of two functions. Just as in integral calculus when the integral of the product of two functions did not produce the product of the integrals, neither does the inverse Laplace transform of the product yield the product of the inverse Laplace transforms. Thus, we state the following theorem.

Theorem 32 (Convolution Theorem). *Suppose that $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and both of exponential order b. Further suppose that $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then,*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{(f * g)(t)\}\} = (f * g)(t) = \int_0^t f(t-v)g(v)\,dv. \qquad (8.19)$$

```
In[1664]:= Clear[f, g]

           LaplaceTransform[∫_0^t f[t - v] g[v] ⅆv, t, s]

Out[1664]= LaplaceTransform[f[t], t, s]
            LaplaceTransform[g[t], t, s]
```

Note that $(f * g)(t) = \int_0^t f(t-v)g(v)\,dv$ is called the **convolution integral**.

EXAMPLE 8.5.1: Compute $(f * g)(t)$ if $f(t) = e^{-t}$ and $g(t) = \sin t$. Verify the Convolution Theorem with these functions.

SOLUTION: We use the definition and integration by parts to obtain

$$\begin{aligned}(f * g)(t) &= \int_0^t f(t-v)g(v)\,dv = \int_0^t e^{-t+v}\sin v\,dv = e^{-t}\int_0^t e^v \sin v\,dv \\ &= e^{-t}\left[\frac{1}{2}e^v(\sin v - \cos v)\right]_0^t = \frac{1}{2}e^{-t}\left[e^t(\sin t - \cos t) - (\sin 0 - \cos 0)\right] \\ &= \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}.\end{aligned}$$

The same results are obtained with Mathematica. After defining `convolution`, which computes $(f * g)(t)$,

```
In[1665]:= Clear[convolution, f, t, g, v];

            convolution[f_, g_] := ∫_0^t f[t - v] g[v] ⅆv
```

we define $f(t)$ and $g(t)$

```
In[1666]:= f[t_] = Exp[-t];

            g[t_] = Sin[t];
```

and then use `convolution` to compute $(f * g)(t)$.

```
In[1667]:= convolution[f, g]
Out[1667]= 1/2 (ⅇ^-t - Cos[t] + Sin[t])
```

Note that $(f * g)(t) = (g * f)(t)$.

```
In[1668]:= convolution[g, f]
Out[1668]= 1/2 (ⅇ^-t - Cos[t] + Sin[t])
```

Now, according to the Convolution Theorem, $\mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = \mathcal{L}\{(f * g)(t)\}$. In this example, we have

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\left\{e^{-t}\right\} = \frac{1}{s+1} \text{ and } G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}.$$

Hence, $\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s^2+1}\right\}$ should equal $(f * g)(t)$. We compute $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s^2+1}\right\}$ with `InverseLaplaceTransform`.

```
In[1669]:= InverseLaplaceTransform[1/((s + 1) (s^2 + 1)), s, t]

Out[1669]= 1/2 (ⅇ^-t - Cos[t] + Sin[t])
```

Hence,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s^2+1}\right\} = \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t,$$

which is the same result as that obtained for $(f * g)(t)$.

■

EXAMPLE 8.5.2: Use the Convolution Theorem to find the Laplace transform of $h(t) = \int_0^t \cos(t - v)\sin v\,dv$.

SOLUTION: Notice that $h(t) = (f * g)(t)$, where $f(t) = \cos t$ and $g(t) = \sin t$. Therefore, by the Convolution Theorem, $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$. Hence,

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = \mathcal{L}\{\cos t\}\,\mathcal{L}\{\sin t\} = \frac{s}{s^2+1}\cdot\frac{1}{s^2+1} = \frac{s}{\left(s^2+1\right)^2}.$$

The same result is obtained with `LaplaceTransform`.

```
In[1670]:= LaplaceTransform[∫_0^t Cos[t - v] Sin[v]ⅆv,
              t, s]//Simplify
Out[1670]=    s
          ─────────
          (1 + s²)²
```

■

8.5.2 Integral and Integrodifferential Equations

The Convolution Theorem is useful in solving numerous problems. In particular, this theorem can be employed to solve **integral equations**, which are equations that involve an integral of the unknown function.

EXAMPLE 8.5.3: Use the Convolution Theorem to solve the integral equation

$$h(t) = 4t + \int_0^t h(t-v)\sin v\,dv.$$

SOLUTION: We first note that the integral in this equation represents $(h * g)(t)$ where $g(t) = \sin t$. Therefore, if we apply the Laplace transform to both sides of the equation, we obtain

$$\begin{aligned}\mathcal{L}\{h(t)\} &= \mathcal{L}\{4t\} + \mathcal{L}\{h(t)\}\,\mathcal{L}\{\sin t\}\\ H(s) &= \frac{4}{s^2} + H(s)\frac{1}{s^2+1},\end{aligned}$$

where $H(s) = \mathcal{L}\{h(t)\}$. The same result is obtained with `Laplace Transform`.

```
In[1671]:= Clear[h]

            step1 = LaplaceTransform[
                h[t] == 4t + ∫_0^t h[t - v] Sin[v]ⅆv, t, s]
Out[1671]= LaplaceTransform[h[t], t, s] ==
              4/s^2 + LaplaceTransform[h[t], t, s]/(1 + s^2)
```

Solving for $H(s)$, we have

$$H(s)\left(1-\frac{1}{s^2+1}\right)=\frac{4}{s^2} \qquad \text{so} \qquad H(s)=\frac{4\left(s^2+1\right)}{s^4}=\frac{4}{s^2}+\frac{4}{s^4}.$$

```
In[1672]:= step2 = Solve[step1,
                      LaplaceTransform[h[t], t, s]]
Out[1672]= {{LaplaceTransform[h[t], t, s] → 4 (1 + s^2)/s^4}}
```

Then by computing the inverse Laplace transform,

```
In[1673]:= sol = InverseLaplaceTransform
                     [step2[[1, 1, 2]], s, t]
Out[1673]= 4 (t + t^3/6)
```

we find that

$$h(t)=\mathcal{L}^{-1}\left\{\frac{4}{s^2}+\frac{4}{s^4}\right\}=4t+\frac{2}{3}t^3.$$

■

Laplace transforms are helpful in solving problems of other types as well. Next, we illustrate how Laplace transforms can be used to solve an **integrodifferential equation**, an equation that involves a derivative as well as an integral of the dependent variable, the unknown function.

EXAMPLE 8.5.4: Solve $\dfrac{dy}{dt}+y+\int_0^t y(u)\,du=1$ subject to $y(0)=0$.

SOLUTION: Because we must take the Laplace transform of both sides of this integrodifferential equation, we first compute

$$\mathcal{L}\left\{\int_0^t y(u)\,du\right\}=\mathcal{L}\{(1*y)(t)\}=\mathcal{L}\{1\}\,\mathcal{L}\{y\}=\frac{Y(s)}{s}.$$

Hence,

$$\begin{aligned}\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{y\} + \mathcal{L}\left\{\int_0^t y(u)\,du\right\} &= \mathcal{L}\{1\}\\ sY(s) - y(0) + Y(s) + \frac{Y(s)}{s} &= \frac{1}{s}\\ s^2Y(s) + sY(s) + Y(s) &= 1\\ Y(s) &= \frac{1}{s^2+s+1}.\end{aligned}$$

The same steps are carried out with Mathematica.

```
In[1674] := step1 = LaplaceTransform[
              y'[t] + y[t] + ∫_0^t y[u] du == 1, t, s]

Out[1674]= LaplaceTransform[y[t], t, s]
           + LaplaceTransform[y[t], t, s] / s
           + s LaplaceTransform[y[t], t, s] - y[0] == 1/s

In[1675] := step2 = step1/.y[0] → 0

Out[1675]= LaplaceTransform[y[t], t, s]
           + LaplaceTransform[y[t], t, s] / s
           + s LaplaceTransform[y[t], t, s] == 1/s

In[1676] := step3 = Solve[step2,
              LaplaceTransform[y[t], t, s]]

Out[1676]= {{LaplaceTransform[y[t], t, s] → 1/(1 + s + s^2)}}
```

Because $Y(s) = \dfrac{1}{s^2+s+1} = \dfrac{1}{(s+1/2)^2 + \left(\sqrt{3}/2\right)^2}$, $y(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t$.

The same solution, which is then graphed on the interval $[0, 3\pi]$ with `Plot` in Figure 8-14, is found with `InverseLaplaceTransform` and named `sol`.

```
In[1677] := sol = InverseLaplaceTransform
              [step3[[1, 1, 2]], s, t]

Out[1677]= 2 e^(-t/2) Sin[√3 t / 2] / √3
```

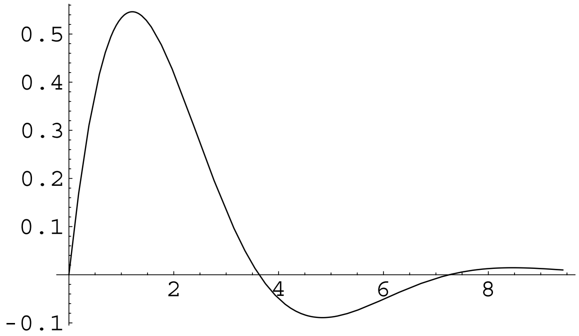


Figure 8-14 Plot of the solution to an integrodifferential equation

```
In[1678]:= Plot[sol,{t,0,3π}]
```

■

8.6 Applications of Laplace Transforms, Part I

8.6.1 Spring–Mass Systems Revisited

Laplace transforms are useful in solving the spring–mass systems that were discussed in earlier sections. Although the method of Laplace transforms can be used to solve all problems discussed in the section on applications of higher-order equations, this method is most useful in alleviating the difficulties associated with problems that involve piecewise-defined forcing functions. Hence, we investigate the use of Laplace transforms to solve the second-order initial-value problem that models the motion of a mass attached to the end of a spring. We found in Chapter 5 that without forcing this situation is modeled by the initial-value problem

$$\begin{cases} mx'' + cx' + kx = 0 \\ x(0) = \alpha, \; x'(0) = \beta, \end{cases} \tag{8.20}$$

where m represents the mass, c the damping coefficient, and k the spring constant determined by Hooke's law. We demonstrate how the method of Laplace transforms is used to solve initial-value problems of this type if the forcing function is discontinuous.

EXAMPLE 8.6.1: Suppose that a mass with $m = 1$ is attached to a spring with spring constant $k = 1$. If there is no resistance due to damping determine the displacement of the mass if it is released from its equilibrium position and is subjected to the force $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2. \end{cases}$

SOLUTION: In this case, the constants are $m = k = 1$ and $c = 0$. The initial position is $x(0) = 0$ and the initial velocity is $x'(0) = 0$. Hence, the initial-value problem that models this situation is

$$x'' + x = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2 \end{cases}, \quad x(0) = 0,\ x'(0) = 0.$$

Because we will take the Laplace transform of both sides of the differential equation, we write $f(t)$ in terms of the unit step function. This gives us

$$f(t) = [\mathcal{U}(t-0) - \mathcal{U}(t-\pi/2)]\sin t = [1 - \mathcal{U}(t-\pi/2)]\sin t,$$

which we graph with `Plot` in Figure 8-15.

```
In[1679]:= Plot[Sin[t] (1 - UnitStep[t - π/2]), {t, 0, π}]
```

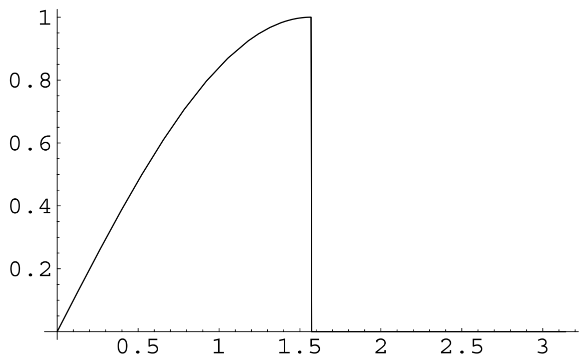


Figure 8-15 Plot of the forcing function

Using the method of Laplace transforms, we compute the Laplace transform of each side of the equation,

```
In[1680]:= step1 = LaplaceTransform[x''[t] + x[t] ==
                Sin[t] (1 - UnitStep[t - π/2]), t, s]
Out[1680]= LaplaceTransform[x[t], t, s]
           +s^2 LaplaceTransform[x[t], t, s]
           -s x[0] - x'[0] == 1/(1 + s^2) - (e^(-π s/2) s)/(1 + s^2)
```

apply the initial conditions,

```
In[1681]:= step2 = step1/.{x[0] → 0, x'[0] → 0}
Out[1681]= LaplaceTransform[x[t], t, s]
           +s^2 LaplaceTransform[x[t], t, s] ==
            1/(1 + s^2) - (e^(-π s/2) s)/(1 + s^2)
```

and solve the resulting equation for $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

```
In[1682]:= step3 = Solve[step2,
                LaplaceTransform[x[t], t, s]]
Out[1682]= {{LaplaceTransform[x[t], t, s] →
            (e^(-π s/2) (e^(π s/2) - s))/(1 + s^2)^2}}
```

The solution is obtained with `InverseLaplaceTransform`.

```
In[1683]:= sol = InverseLaplaceTransform
                [-(-1 + e^(-π s/2) s)/(1 + s^2)^2, s, t]
Out[1683]= 1/4 (-2 t Cos[t] + 2 Sin[t]
           -(π - 2 t) Cos[t] UnitStep[-π/2 + t])
```

The same result is obtained with `DSolve`, which we then graph with `Plot` in Figure 8-16.

```
In[1684]:= Clear[x, t, sol]

           sol = DSolve[{x''[t] + x[t] == Sin[t]
                (1 - UnitStep[t - π/2]), x[0] == 0,
                x'[0] == 0}, x[t], t]//Simplify
```

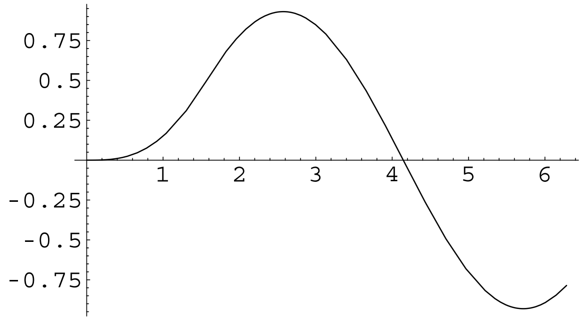


Figure 8-16 For $t \geq \pi/2$, the motion is harmonic

```
Out[1684]= {{x[t] → 1/4 (-2 t Cos[t] + 2 Sin[t]
            -(π - 2 t) Cos[t] UnitStep[-π/2 + t])}}

In[1685]:= Plot[x[t]/.sol, {t, 0, 2π}]
```

Notice that resonance begins on the interval $0 \leq t < \pi/2$. Then, for $t \geq \pi/2$, the motion is harmonic. Hence, although the forcing function is zero for $t \geq \pi/2$, the mass continues to follow the path defined by $x(t)$ indefinitely.

■

EXAMPLE 8.6.2: Suppose that a mass of $m = 1$ is attached to a spring with spring constant $k = 13$. If the mass is subjected to the resistive force due to damping $F_R = 4\,dx/dt$, determine the displacement of the mass if it is released from its equilibrium position and is subjected to the force

$$f(t) = 2t\,[1 - \mathcal{U}(t-1)] + 2\mathcal{U}(t-1) + 10\delta(t-3).$$

SOLUTION: In this case, the initial-value problem is

$$\begin{cases} x'' + 4x' + 13x = 2t\,[1 - \mathcal{U}(t-1)] + 2\mathcal{U}(t-1) + 10\delta(t-3) \\ x(0) = x'(0) = 0. \end{cases}$$

We first graph $2t\,[1 - \mathcal{U}(t-1)] + 2\mathcal{U}(t-1)$ in Figure 8-17.

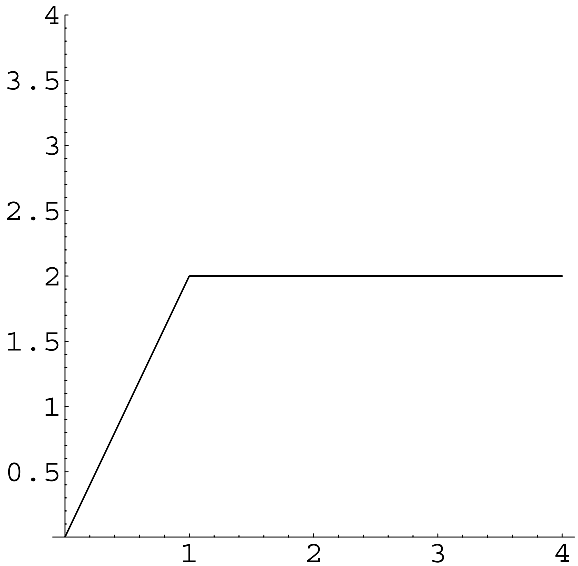


Figure 8-17 Plot of $2t\,[1 - \mathcal{U}(t-1)] + 2\mathcal{U}(t-1)$

```
In[1686]:= Plot[2t (1 - UnitStep[t - 1])
             +2UnitStep[t - 1],
             {t, 0, 4}, PlotRange- > {0, 4},
             AspectRatio- > 1]
```

Using the method of Laplace transforms, we take the Laplace transform of each side of the equation,

```
In[1687]:= step1 = LaplaceTransform[x''[t]
                    +4x'[t] + 13x[t] ==
                     2t (1 - UnitStep[t - 1])
                    +2UnitStep[t - 1]
                    +10DiracDelta[t - 3], t, s]
Out[1687]= 13 LaplaceTransform[x[t], t, s]
           +s^2 LaplaceTransform[x[t], t, s]
           +4 (s LaplaceTransform[x[t], t, s] - x[0])
```

$$-s\,x[0] - x'[0] == 10\,e^{-3s} + \frac{2}{s^2} + \frac{2\,e^{-s}}{s} - \frac{2\,e^{-s}\,(1+s)}{s^2}$$

apply the initial conditions,

```
In[1688]:= step2 = step1 /. {x[0]- > 0, x'[0]- > 0}
```

```
Out[1688]= 13 LaplaceTransform[x[t], t, s]
           +4 s LaplaceTransform[x[t], t, s]
           +s^2 LaplaceTransform[x[t], t, s] ==
```

$$10\,e^{-3s} + \frac{2}{s^2} + \frac{2\,e^{-s}}{s} - \frac{2\,e^{-s}\,(1+s)}{s^2}$$

and solve for $X(s) = \mathcal{L}\{x(t)\}$.

```
In[1689]:= step3 = Solve[step2,
                LaplaceTransform[x[t], t, s]]
Out[1689]= {{LaplaceTransform[x[t], t, s] →
```

$$\frac{2\,e^{-3s}\,(-e^{2s} + e^{3s} + 5\,s^2)}{s^2\,(13 + 4\,s + s^2)}\}\}$$

The solution to the initial-value problem is obtained with `Inverse LaplaceTransform`.

```
In[1690]:= sol =
             InverseLaplaceTransform[
```

$$\frac{2\,(1 - e^{-s} + 5e^{-3s}\,s^2)}{s^2\,(13 + 4s + s^2)}, s, t]//Simplify$$

Out[1690]=

$$2\left(\frac{-24 + (12 - 5\,i)\,e^{(-2-3\,i)\,t}}{1014} + \frac{(12 + 5\,i)\,e^{(-2+3\,i)\,t} + 78\,t}{1014} - \frac{5}{6}\,i\,e^{(-2-3\,i)\,(-3+t)}\left(-1 + e^{6\,i\,(-3+t)}\right) \text{UnitStep}[-3+t] - \frac{1}{1014}\left(\left(-102 + (12 - 5\,i)\,e^{(-2-3\,i)\,(-1+t)} + (12 + 5\,i)\,e^{(-2+3\,i)\,(-1+t)} + 78\,t\right)\text{UnitStep}[-1+t]\right)\right)$$

The same result is obtained with `DSolve`. The solution is graphed with `Plot` in Figure 8-18.

```
In[1691]:= sol = DSolve[{x''[t] + 4x'[t] + 13x[t] ==
                  2t (1 - UnitStep[t - 1])
                  +2UnitStep[t - 1]
                  +10DiracDelta[t - 3], x[0] == 0,
                  x'[0] == 0}, x[t], t]//Simplify
```

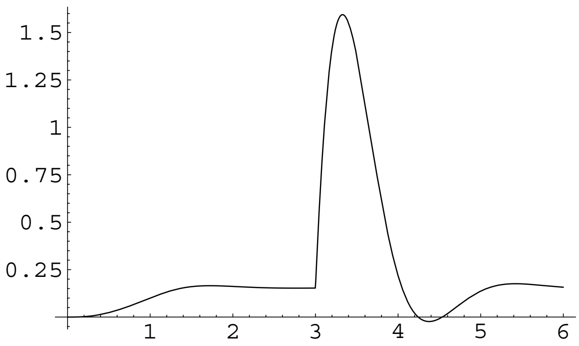


Figure 8-18 Note the effect of the impulse delivered at $t = 3$

```
Out[1691]= {{x[t] → -2/507 e^(-2 t) (12 e^(2 t) - 39 e^(2 t) t - 12 Cos[3 t]
                +5 Sin[3 t] + 845 e^6 Sin[9 - 3 t]
                × UnitStep[-3 + t]
                +(3 e^(2 t) (-17 + 13 t) + 12 e^2 Cos[3 - 3 t]
                +5 e^2 Sin[3 - 3 t]) UnitStep[-1 + t])}}

In[1692]:= Plot[x[t]/.sol, {t, 0, 6}, PlotRange- > All]
```

The graph of the solution shows the effect of the impulse delivered at $t = 3$, which is especially evident when we compare this result to the solution of

$$\begin{cases} x'' + 4x' + 13x = 2t\,[1 - \mathcal{U}(t-1)] + 2\mathcal{U}(t-1) \\ x(0) = x'(0) = 0 \end{cases}$$

shown in Figure 8-19.

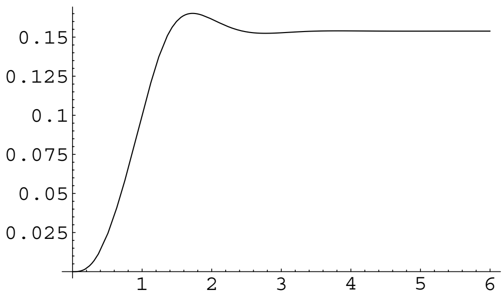


Figure 8-19 No impulse is delivered

```
In[1693]:= sol2 = DSolve[{x''[t] + 4x'[t] + 13x[t] ==
                2t (1 - UnitStep[t - 1])
                +2UnitStep[t - 1], x[0] == 0,
                x'[0] == 0}, x[t], t]//Simplify
```

Out[1693]= $\left\{\left\{x[t] \to -\frac{2}{507} e^{-2t} (12 e^{2t} - 39 e^{2t} t - 12 \text{Cos}[3t] + 5 \text{Sin}[3t] + (3 e^{2t} (-17 + 13t) + 12 e^2 \text{Cos}[3 - 3t] + 5 e^2 \text{Sin}[3 - 3t]) \text{UnitStep}[-1 + t])\right\}\right\}$

```
In[1694]:= Plot[x[t]/.sol2, {t, 0, 6}, PlotRange- > All]
```

■

8.6.2 *L–R–C* Circuits Revisited

Laplace transforms can be used to solve the *L–R–C* circuit problems that were introduced earlier. Recall that the initial-value problem that is used to find the current is

$$\begin{cases} L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t) \\ Q(0) = Q_0,\ I(0) = \frac{dQ}{dt}(0) = I_0 \end{cases} \tag{8.21}$$

where L, R, and C represent the inductance, resistance, and capacitance, respectively. Q is the charge of the capacitor and $dQ/dt = I$, where I is the current. $E(t)$ is the voltage supply. In particular, the method of Laplace transforms is most useful when the supplied voltage, $E(t)$, is piecewise defined.

EXAMPLE 8.6.3: Suppose that we consider a circuit with a capacitor C, a resistor R, and a voltage supply

$$E(t) = \begin{cases} 100,\ 0 \le t < 1 \\ 200 - 100t,\ 1 \le t < 2 \\ 0,\ t \ge 2. \end{cases}$$

If $L = 0$, find $Q(t)$ and $I(t)$ if $Q(0) = 0$, $C = 10^{-2}$ farads, and $R = 100\,\Omega$.

SOLUTION: Because $L = 0$, we can state the first-order initial-value problem as

$$\begin{cases} 100\dfrac{dQ}{dt} + 100Q = \begin{cases} 100, \ 0 \le t < 1 \\ 200 - 100t, \ 1 \le t < 2 \\ 0, \ t \ge 2 \end{cases} \\ Q(0) = 0. \end{cases}$$

First, we rewrite $E(t)$ in terms of the unit step functions as

$$E(t) = 100\,[1 - \mathcal{U}(t-1)] + (200 - 100t)\,[\mathcal{U}(t-1) - \mathcal{U}(t-2)].$$

When we use Mathematica to define $E(t)$, we use a lower-case `e` to avoid ambiguity with `E`, which represents $e \approx 2.71828$. See Figure 8-20.

```
In[1695]:= e[t_] = 100 (1 - UnitStep[t - 1])
                  +(200 - 100t) (UnitStep[t - 1]
                  -UnitStep[t - 2])
Out[1695]= 100 (1 - UnitStep[-1 + t])
           +(200 - 100 t) (-UnitStep[-2 + t]
           +UnitStep[-1 + t])

In[1696]:= Plot[e[t], {t, 0, 4}, PlotRange- > {0, 100}]
```

Now, we take the Laplace transform of both sides of the differential equation,

```
In[1697]:= step1 = LaplaceTransform[
                    100q′[t] + 100q[t] == e[t], t, s]
```

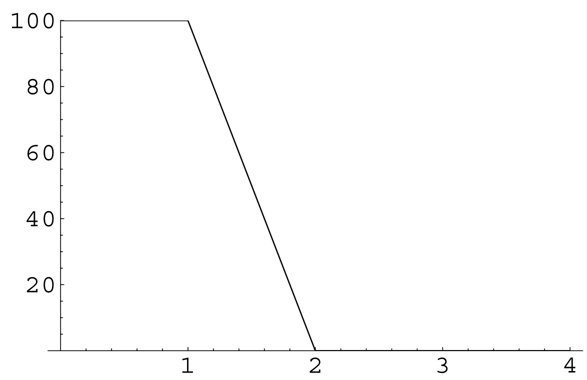


Figure 8-20 Plot of $E(t)$

```
Out[1697]= 100 LaplaceTransform[q[t], t, s]
           +100 (s LaplaceTransform[q[t], t, s]
           -q[0]) == 100/s - 200 e^(-2 s)/s + 100 e^(-s)/s
           - 100 e^(-s) (1 + s)/s^2 + 100 e^(-2 s) (1 + 2 s)/s^2
```

apply the initial condition,

```
In[1698]:= step2 = step1 /. q[0]- > 0
Out[1698]= 100 LaplaceTransform[q[t], t, s]
           +100 s LaplaceTransform[q[t], t, s] == 100/s
           - 200 e^(-2 s)/s + 100 e^(-s)/s
           - 100 e^(-s) (1 + s)/s^2 + 100 e^(-2 s) (1 + 2 s)/s^2
```

and solve for $\mathcal{L}\{Q(t)\}$.

```
In[1699]:= step3 = Solve[step2,
                LaplaceTransform[q[t], t, s]]
Out[1699]= {{LaplaceTransform[q[t], t, s] →
              e^(-2 s) (1 - e^s + e^(2 s) s)/(s^2 (1 + s))}}
```

The solution to the initial-value problem is obtained with `InverseLaplaceTransform`.

```
In[1700]:= sol = InverseLaplaceTransform[
                (e^(-2 s) - e^(-s) + s)/(s^2 (1 + s)), s, t]

Out[1700]= 1 - e^(-t) + (-3 + e^(2-t) + t) UnitStep[-2 + t]
           - (-2 + e^(1-t) + t) UnitStep[-1 + t]
```

The same result is obtained with `DSolve`.

```
In[1701]:= sol = DSolve[{100q'[t]
                +100q[t] == e[t], q[0] == 0},
                q[t], t]//Simplify
Out[1701]= {{q[t] → e^(-t) (-1 + e^t
              + (e^2 + e^t (-3 + t)) UnitStep[-2 + t]
              - (e + e^t (-2 + t)) UnitStep[-1 + t])}}
```

We now compute $I = dQ/dt$ and then graph both $Q(t)$ and $I(t)$ on the interval [0, 4] in Figure 8-21.

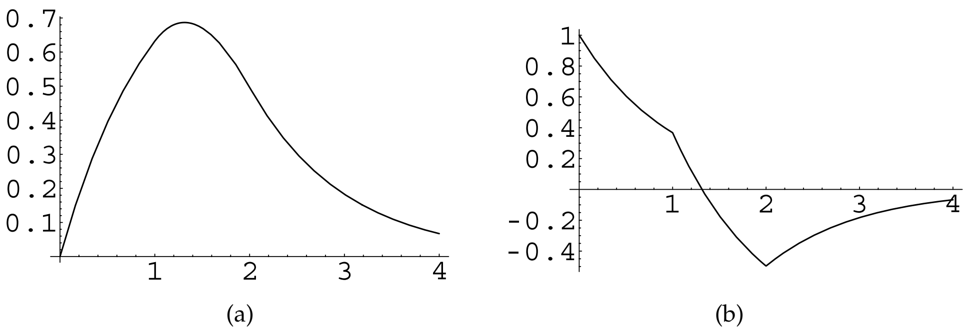


Figure 8-21 (a) $Q(t)$, (b) $I(t)$

```
In[1702]:= i[t_] = D[q[t] /. sol, t];

            pq = Plot[q[t] /. sol, {t, 0, 4},
                   DisplayFunction- > Identity];

            π = Plot[i[t], {t, 0, 4},
                   DisplayFunction- > Identity];

            Show[GraphicsArray[{pq, π}]]
```

From the graph, we see that after the voltage source is turned off at $t = 2$, the charge approaches zero.

■

EXAMPLE 8.6.4: Consider the circuit with no capacitor, $R = 100\,\Omega$, and $L = 100\,H$ if $E(t) = \begin{cases} 100\,V, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$ and $E(t+2) = E(t)$. Find the current $I(t)$ if $I(0) = 0$.

SOLUTION: The differential equation that models the situation is $100Q'' + 100Q' = E(t)$. Now, $Q' = I$, so we can write this equation as $100I' + 100I = E(t)$. Hence, the initial-value problem is

$$\begin{cases} 100I' + 100I = E(t) \\ I(0) = 0. \end{cases}$$

Notice that $E(t)$ is a periodic function, so we first compute $\mathcal{L}\{E(t)\}$

```
In[1703]:= Clear[i, step1, step2]

           lape = Simplify[∫_0^1 100 Exp[-s t] dt / (1 - Exp[-2 s])]

Out[1703]= 100 ⅇ^s / (s + ⅇ^s s)
```

and see $\mathcal{L}\{E(t)\} = \dfrac{100}{s\,(1+e^{-s})}$.

We now compute the Laplace transform of the left side of the equation,

Note that we use `i` to represent I instead of `I` because `I` represents the imaginary number $i=\sqrt{-1}$.

```
In[1704]:= step1 =
             LaplaceTransform[100 i'[t]
             +100 i[t], t, s] == lape
Out[1704]= 100 LaplaceTransform[i[t], t, s]
           +100 (-i[0]
           +s LaplaceTransform[i[t], t, s]) ==
             100 ⅇ^s / (s + ⅇ^s s)
```

apply the initial condition,

```
In[1705]:= step2 = step1 /. i[0]- > 0
Out[1705]= 100 LaplaceTransform[i[t], t, s]
           +100 s LaplaceTransform[i[t], t, s] ==
             100 ⅇ^s / (s + ⅇ^s s)
```

and solve for $\mathcal{L}\{I(t)\}$.

```
In[1706]:= step3 = Solve[step2, LaplaceTransform[
                  i[t], t, s]]
Out[1706]= {{LaplaceTransform[i[t], t, s] →
              ⅇ^s / ((1 + ⅇ^s) s (1 + s))}}
```

As we did before, we write a power series expansion of $\dfrac{1}{1+e^{-s}}$:

We use $\dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = 1 - x + x^2 - x^3 + \cdots$.

$$\frac{1}{1+e^{-s}} = \sum_{n=0}^{\infty}\left(-e^{-s}\right)^n = 1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots.$$

Thus,

$$\mathcal{L}\{I(t)\} = \frac{1}{s(s+1)}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right).$$

Because, $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 - e^{-t}$,

```
In[1707]:= InverseLaplaceTransform[1/(s (s + 1)), s, t]
Out[1707]= 1 - e^-t
```

we have that

$$I(t) = \left(1 - e^{-t}\right) - \left(1 - e^{-(t-1)}\right)\mathcal{U}(t-1) + \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2) - \left(1 - e^{-(t-3)}\right)\mathcal{U}(t-3) + \cdots .$$

We can write this function as

$$I(t) = \begin{cases} 1 - e^{-t}, \ 0 \le t < 1 \\ -e^{-t} + e^{-(t-1)}, \ 1 \le t < 2 \\ 1 - e^{-t} + e^{-(t-1)} - e^{-(t-2)}, \ 2 \le t < 3 \\ -e^{-t} + e^{-(t-1)} - e^{-(t-2)} + e^{-(t-3)}, \ 3 \le t < 4 \\ \vdots \end{cases}$$

To graph $I(t)$ on the interval $[0, n]$, we note that $\mathcal{U}(t-n) = 0$ for $t \le n$ so the graph of $I(t)$ on the interval $[0, n]$ is the same as the graph of

$$\left(1 - e^{-t}\right) - \left(1 - e^{-(t-1)}\right)\mathcal{U}(t-1) + \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2) - \left(1 - e^{-(t-3)}\right)\mathcal{U}(t-3) + \cdots + (-1)^{n-1}\left(1 - e^{-[t-(n-1)]}\right)\mathcal{U}(t-(n-1)).$$

```
In[1708]:= Clear[i]

           i[n_] :=
             i[n] = i[n - 1]
               + (-1)^n (1 - Exp[-(t - n)])
                  UnitStep[t - n]

           i[0] = 1 - Exp[-t];
```

For example, to graph $I(t)$ on the interval $[0, 5]$ we enter

```
In[1709]:= i[4]
Out[1709]= 1 - e^-t + (1 - e^(4-t)) UnitStep[-4 + t]
           - (1 - e^(3-t)) UnitStep[-3 + t]
           + (1 - e^(2-t)) UnitStep[-2 + t]
           - (1 - e^(1-t)) UnitStep[-1 + t]
```

and then use `Plot`. See Figure 8-22.

```
In[1710]:= Plot[i[4], {t, 0, 5}]
```

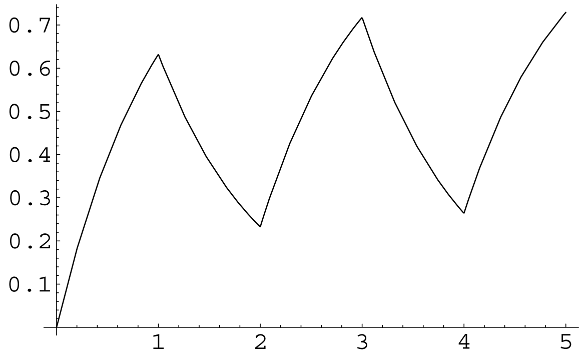


Figure 8-22 Plot of $I(t)$ on the interval [0, 5]

Notice that $I(t)$ increases over the intervals where $E(t) = 100$ and decreases on those where $E(t) = 0$.

■

We can consider the L–R–C circuit in terms of the integrodifferential equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\alpha)\,d\alpha = E(t), \tag{8.22}$$

which is useful when using the method of Laplace transforms to find the current.

EXAMPLE 8.6.5: Find the current $I(t)$ if $L = 1$ Henry, $R = 6\,\Omega$, $C = 1/9$ Farad, $E(t) = 1$ volt, and $I(0) = 0$.

SOLUTION: In this case, we must solve the initial-value problem

$$\begin{cases} \frac{dI}{dt} + 6I + 9\int_0^t I(\alpha)\,d\alpha = 1 \\ I(0) = 0. \end{cases}$$

First, we compute the Laplace transform of each side of the equation,

```
In[1711]:= Clear[i]

           step1 = LaplaceTransform[
               i'[t] + 6i[t]
               +9Integrate[i[α], {α, 0, t}] == 1,
               t, s]
```

```
Out[1711]= -i[0] + 6 LaplaceTransform[i[t], t, s]
             9 LaplaceTransform[i[t], t, s]
           + ------------------------------
                           s
                                                 1
           + s LaplaceTransform[i[t], t, s] == -
                                                 s
```

apply the initial condition,

```
In[1712]:= step2 = step1 /. i[0] -> 0
Out[1712]= 6 LaplaceTransform[i[t], t, s]
             9 LaplaceTransform[i[t], t, s]
           + ------------------------------
                           s
                                                 1
           + s LaplaceTransform[i[t], t, s] == -
                                                 s
```

and solve for $\mathcal{L}\{I(t)\}$.

```
In[1713]:= step3 = Solve[step2,
                   LaplaceTransform[i[t], t, s]]
                                                   1
Out[1713]= {{LaplaceTransform[i[t], t, s] -> --------}}
                                              (3 + s)^2
```

The solution is obtained with `InverseLaplaceTransform`,

```
In[1714]:= sol = InverseLaplaceTransform[
                   step3[[1, 1, 2]], s, t]
Out[1714]= e^(-3 t) t
```

which we graph with `Plot` in Figure 8-23.

```
In[1715]:= Plot[sol, {t, 0, 3}]
```

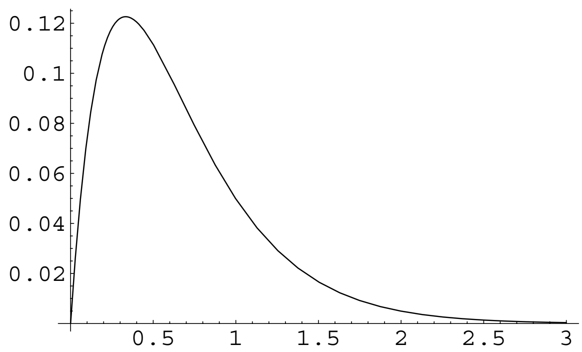


Figure 8-23 $I(t) \to 0$ as $t \to \infty$

■

8.6.3 Population Problems Revisited

Laplace transforms can be used to solve the population problems that were discussed as applications of first-order equations and systems. Laplace transforms are especially useful when dealing with piecewise-defined forcing functions, but they are useful in many other cases as well.

EXAMPLE 8.6.6: Let $x(t)$ represent the population of a certain country. The rate at which the population increases and decreases depends on the growth rate of the country as well as the rate at which people are being added to or subtracted from the population due to immigration or emigration. Hence, we consider the population problem

$$\begin{cases} x' + kx = 1000\,(1 + a\sin t) \\ x(0) = x_0. \end{cases}$$

Solve this problem using Laplace transforms with $k = 3$, $x_0 = 2000$, and $a = 0.2, 0.4, 0.6$, and 0.8. Plot the solution in each case.

SOLUTION: Using the method of Laplace transforms, we begin by computing the Laplace transform of each side of the equation with `LaplaceTransform`,

```
In[1716]:= step1 = LaplaceTransform[
             x'[t] + 3 x[t] == 1000(1 + a Sin[t]), t, s]
```

Out[1716]= $3\,\text{LaplaceTransform}[x[t],t,s] + s\,\text{LaplaceTransform}[x[t],t,s] - x[0] == 1000\left(\frac{1}{s} + \frac{a}{1+s^2}\right)$

apply the initial condition,

```
In[1717]:= step2 = step1 /. x[0]- > 2000
```

Out[1717]= $-2000 + 3\,\text{LaplaceTransform}[x[t],t,s] + s\,\text{LaplaceTransform}[x[t],t,s] == 1000\left(\frac{1}{s} + \frac{a}{1+s^2}\right)$

and then use `Solve` to solve `step2` for $X(s) = \mathcal{L}\{x(t)\}$.

```
In[1718]:= step3 = Solve[step2,
                 LaplaceTransform[x[t], t, s]]
```

Out[1718]= $\left\{\left\{\text{LaplaceTransform}[x[t], t, s] \to \frac{1000\,(1 + 2\,s + a\,s + s^2 + 2\,s^3)}{s\,(3 + s)\,(1 + s^2)}\right\}\right\}$

To find the solution, we use `InverseLaplaceTransform` and name the result `sol`.

```
In[1719]:= sol =
            InverseLaplaceTransform[step3[[1, 1, 2]],
              s, t]//Simplify
```

Out[1719]= $\frac{100}{3}\,(10 + (50 + 3\,a)\,e^{-3\,t} - 3\,a\,(\text{Cos}[t] - 3\,\text{Sin}[t]))$

We use the result to investigate the population for the values of *a* using `Plot`. See Figure 8-24.

```
In[1720]:= toplot = Table[sol, {a, 0.2, 0.8, 0.2}];

In[1721]:= grays = Table[GrayLevel[i],
                 {i, 0, 0.7, 0.7/4}];

           Plot[Evaluate[toplot], {t, 0, 25},
             PlotStyle- > grays, PlotRange- > All,
             AxesOrigin- > {0, 0}]
```

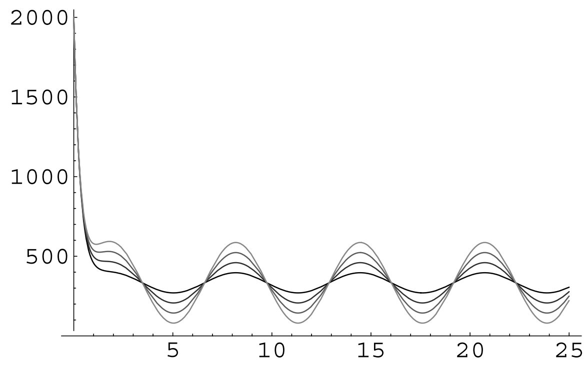


Figure 8-24 Fluctuations in the size of the population are larger for larger values of *a*

■

Application: The Tautochrone

Suppose that from rest, a particle slides down a frictionless curve under the force of gravity. What must the shape of the curve be in order for the time of descent to be independent of the starting position of the particle?

We can determine the shape of the curve using the method of Laplace transforms. Suppose that the particle starts at height y and that its speed is v when it is at a height of z. If m is the mass of the particle and g is the acceleration due to gravity, the speed is found by equating the kinetic and potential energies of the particle with

$$\frac{1}{2}mv^2 = mg(y-z)$$
$$v = \sqrt{2g}\sqrt{y-z}.$$

Let σ denote the arc length along the curve from its lowest point to the particle. Then, the time required for the descent is

$$\text{time} = \int_0^{\sigma(y)} \frac{1}{v}\,d\sigma = \int_0^y \frac{1}{v}\frac{d\sigma}{dz} = \int_0^y \frac{1}{v}\phi(z)dz,$$

where $\phi(y) = d\sigma/dy$. The time is constant and $v = \sqrt{2g}\sqrt{y-z}$ so we have

$$\int_0^y \frac{\phi(z)}{\sqrt{y-z}}dz = c_1,$$

where c_1 is a constant. To use a convolution, we multiply by $e^{-sy}\,dy$ and integrate:

$$\int_0^\infty e^{-sy}\int_0^y \frac{\phi(z)}{\sqrt{y-z}}dz\,dy = \int_0^\infty e^{-sy}c_1\,dy$$
$$\mathcal{L}\left\{\phi * y^{-1/2}\right\} = \mathcal{L}\{c_1\}.$$

Using the Convolution Theorem, we simplify to obtain

$$\mathcal{L}\{\phi\}\,\mathcal{L}\left\{y^{-1/2}\right\} = \frac{c_1}{s}.$$

```
In[1722]:= step1 = LaplaceTransform[∫_0^y ϕ[z]/√(y - z) dz == c1,
              y, s]

Out[1722]= √π LaplaceTransform[ϕ[y], y, s]/√s == c1/s
```

Then, $\mathcal{L}\{\phi\} = \dfrac{c_1}{\sqrt{\pi s}}$.

```
In[1723]:= step2 = Solve[step1,
                      LaplaceTransform[ϕ[y], y, s]]
Out[1723]= {{LaplaceTransform[ϕ[y], y, s] → c1/(√π √s)}}
```

We use `InverseLaplaceTransform` to compute $\phi = \mathcal{L}^{-1}\left\{\dfrac{c_1}{\sqrt{\pi s}}\right\} = \frac{c_1}{\pi}y^{-1/2} = ky^{-1/2}$.

```
In[1724]:= step3 = InverseLaplaceTransform[
                      step2[[1, 1, 2]], s, y]
Out[1724]= c1/(π √y)
```

Recall that $\phi(y) = d\sigma/dy$ represents arc length. Then, $\phi(y) = d\sigma/dy = \sqrt{1 + (dx/dy)^2}$ and substitution of $\phi = ky^{-1/2}$ into this equation gives us

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = ky^{-1/2} \qquad \text{or} \qquad 1+\left(\frac{dx}{dy}\right)^2 = \frac{k^2}{y}.$$

We solve this equation for dx/dy to obtain $\dfrac{dx}{dy} = \sqrt{\dfrac{k^2}{y} - 1}$. With the substitution $y = k^2\sin^2\theta$ we obtain

$$\begin{aligned} dx &= \sqrt{\frac{k^2}{k^2\sin^2\theta} - 1}\cdot 2k^2\sin\theta\cos\theta\,d\theta = \sqrt{\frac{k^2\left(1-\sin^2\theta\right)}{k^2\sin^2\theta}}\cdot 2k^2\sin\theta\cos\theta\,d\theta \\ &= \frac{\cos\theta}{\sin\theta}\cdot 2k^2\sin\theta\cos\theta\,d\theta = 2k^2\cos\theta\,d\theta \end{aligned}$$

and integration results in $x(\theta) = \frac{1}{2}k^2(2\theta + \sin 2\theta) + C_1$. To find C_1, we apply the initial condition $x(0) = 0$ to see that $C_1 = 0$ and $x(\theta) = \frac{1}{2}k^2(2\theta + \sin 2\theta)$.

```
In[1725]:= x[θ_, k_] = ∫ 2k² Cos[θ]² dθ
Out[1725]= 2 k² (θ/2 + 1/4 Sin[2 θ])
```

Using the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ yields $y(\theta) = k^2\sin^2\theta = \frac{1}{2}k^2(1-\cos 2\theta)$.

```
In[1726]:= y[theta_, k_] = k^2 Sin[θ]^2
Out[1726]= k² Sin[θ]²
```

We use `ParametricPlot` to graph $\begin{cases} x = x(\theta) \\ y = y(\theta) \end{cases}$, $-\pi/2 \le \theta \le 0$ for various values

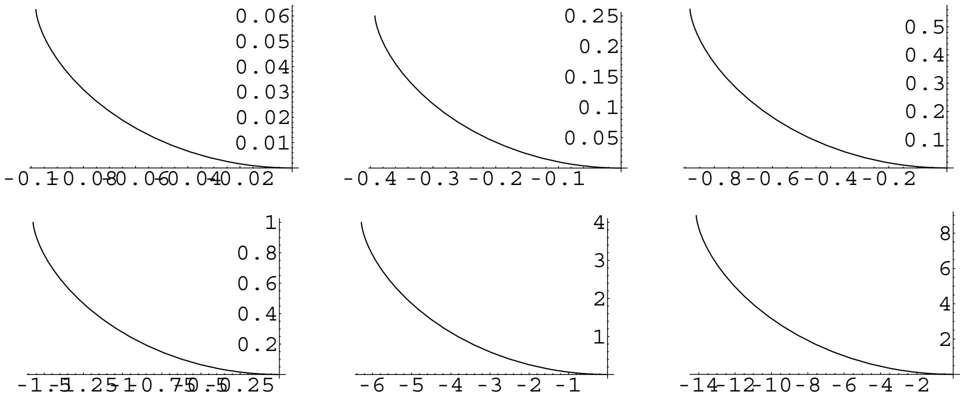


Figure 8-25 Increasing k increases the length of the curve

of k in Figure 8-25.

```
In[1727]:= somegraphs =
              Map[ParametricPlot[
              {x[θ, #], y[θ, #]},
              {θ, -π/2, 0},
              DisplayFunction- > Identity]&,
              {0.25, 0.5, 0.75, 1, 2, 3}];

In[1728]:= toshow = Partition[somegraphs, 3];

In[1729]:= Show[GraphicsArray[toshow]]
```

The graphs illustrate that increasing the value of k increases the length of the curve. The time is independent of the choice of y (that is, the choice of θ). Therefore,

$$\text{time} = \int_0^y \frac{\phi(z)}{\sqrt{y-z}}dz = \int_0^y \frac{ky^{-1/2}}{\sqrt{y-z}}dz = -2k\left[\sqrt{\frac{y-z}{y}}\right]_0^y = -2k \cdot -1 = 2k.$$

8.7 Laplace Transform Methods for Systems

In many cases, Laplace transforms can be used to solve initial-value problems that involve a system of linear differential equations. This method is applied in much the same way that it was in solving initial-value problems involving higher-order differential equations. In the case of systems of differential equations, however, a system of algebraic equations is obtained after taking the Laplace transform of each equation. After solving the algebraic system for the Laplace transform of each of the unknown functions, the inverse Laplace transform is used to find each unknown function in the solution of the system.

EXAMPLE 8.7.1: Solve $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \sin t \\ 2\cos t \end{pmatrix}$ subject to $\mathbf{X}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

SOLUTION: Let $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Then, we can rewrite this initial-value problem as

$$\begin{cases} x' = y + \sin t \\ y' = x + 2\cos t \\ x(0) = 2,\ y(0) = 0. \end{cases}$$

```
In[1730]:= Clear[x, y]

           sys = {x′[t] == y[t] + Sin[t],
                  y′[t] == x[t] + 2 Cos[t]};
```

Taking the Laplace transform of both sides of each equation yields the system

$$\begin{cases} sX(s) - x(0) = Y(s) + \dfrac{1}{s^2+1} \\ sY(s) - y(0) = X(s) + \dfrac{2s}{s^2+1} \end{cases}$$

```
In[1731]:= step1 = LaplaceTransform[sys, t, s]
Out[1731]= {s LaplaceTransform[x[t], t, s] - x[0] ==
              1/(1 + s^2) + LaplaceTransform[y[t], t, s],
            s LaplaceTransform[y[t], t, s] - y[0] ==
              2 s/(1 + s^2) + LaplaceTransform[x[t], t, s]}
```

and applying the initial condition results in

$$\begin{cases} sX(s) - Y(s) = \dfrac{1}{s^2+1} + 2 \\ -X(s) + sY(s) = \dfrac{2s}{s^2+1}. \end{cases}$$

```
In[1732]:= step2 = step1 /. {x[0]- > 2, y[0]- > 1}
Out[1732]= {- 2 + s LaplaceTransform[x[t], t, s] ==
              1/(1 + s^2) + LaplaceTransform[y[t], t, s],
            -1 + s LaplaceTransform[y[t], t, s] ==
              2 s/(1 + s^2) + LaplaceTransform[x[t], t, s]}
```

We now use `Solve` to solve this system of algebraic equations for $X(s)$ and $Y(s)$.

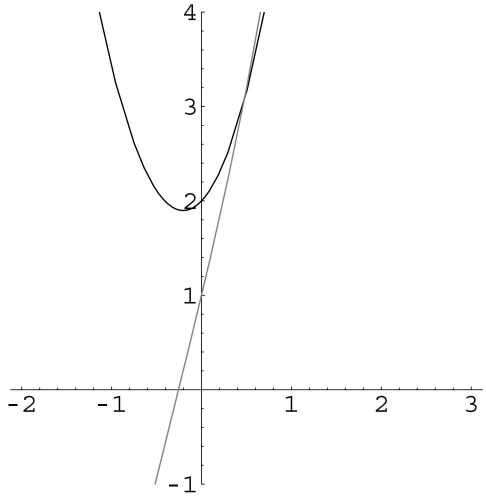
```
In[1733]:= step3 = Solve[step2,
                {LaplaceTransform[x[t], t, s],
                 LaplaceTransform[y[t], t, s]}]
Out[1733]= {{LaplaceTransform[x[t], t, s] ->
               -(-1 - 5 s - s^2 - 2 s^3)/((-1 + s^2) (1 + s^2)),
             LaplaceTransform[y[t], t, s] ->
               -(-3 - s - 4 s^2 - s^3)/((-1 + s^2) (1 + s^2))}}
```

We find $x(t)$ and $y(t)$ with `InverseLaplaceTransform`.

```
In[1734]:= x[t_] =
             InverseLaplaceTransform[
               -(-1 - 5s - s^2 - 2s^3)/((-1 + s^2) (1 + s^2)), s, t]//Simplify

           y[t_] =
             InverseLaplaceTransform[
               -(-3 - s - 4s^2 - s^3)/((-1 + s^2) (1 + s^2)), s, t]//Simplify
Out[1734]= 1/4 (5 e^-t + 9 e^t - 6 Cos[t])
Out[1734]= 1/4 (-5 e^-t + 9 e^t + 2 Sin[t])
```

Last, we graph $x(t)$ and $y(t)$ in Figure 8-26 (a) and $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in Figure 8-26 (b).

```
In[1735]:= Plot[{x[t], y[t]}, {t, -2, 3},
             PlotRange- > {-1, 4}, AspectRatio- > 1,
             PlotStyle- > {GrayLevel[0],
             GrayLevel[0.5]}]
```



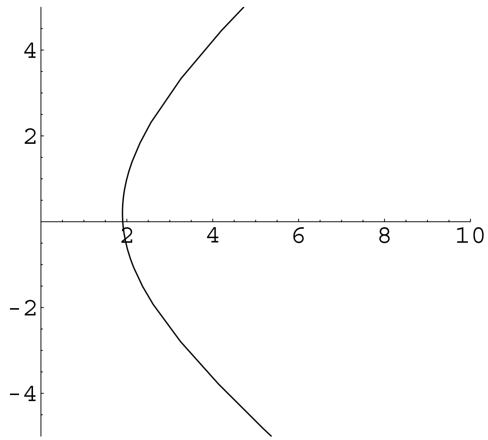


Figure 8-26 (a) $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of x versus y

```
In[1736]:= ParametricPlot[{x[t],y[t]},{t,-2,3},
              PlotRange- > {{0,10},{-5,5}},
              AspectRatio- > 1]
```

■

In some cases, systems that involve higher-order differential equations can be solved with Laplace transforms.

EXAMPLE 8.7.2: Solve $\begin{cases} x'' = -2x - 4y - \cos t \\ y'' = -x - 2y + \sin t \\ x(0) = x'(0) = y(0) = y'(0) = 0. \end{cases}$

SOLUTION: After defining the system of equations in `sys`, we take the Laplace transform of each equation.

```
In[1737]:= Clear[x,y,t]

In[1738]:= sys = {x''[t] == -2x[t] - 4y[t] - Cos[t],
                  y''[t] == -x[t] - 2y[t] + Sin[t]}
Out[1738]= {x''[t] == -Cos[t] - 2 x[t] - 4 y[t],
              y''[t] == Sin[t] - x[t] - 2 y[t]}
```

```
In[1739]:= step1 = LaplaceTransform[sys, t, s]

Out[1739]= {s² LaplaceTransform[x[t], t, s]
              -s x[0] - x′[0] == -   s
                                  ------
                                  1 + s²
              -2 LaplaceTransform[x[t], t, s]
              -4 LaplaceTransform[y[t], t, s],
            s² LaplaceTransform[y[t], t, s]
              -s y[0] - y′[0] ==   1
                                 ------
                                 1 + s²
              -LaplaceTransform[x[t], t, s]
              -2 LaplaceTransform[y[t], t, s]}
```

We then apply the initial conditions and solve the resulting algebraic system of equations for $X(s)$ and $Y(s)$.

```
In[1740]:= step2 =
              step1 /. {x[0]- > 0, x′[0]- > 0,
                 y[0]- > 0, y′[0]- > 0}

Out[1740]= {s² LaplaceTransform[x[t], t, s] == -   s
                                                 ------
                                                 1 + s²
              -2 LaplaceTransform[x[t], t, s]
              -4 LaplaceTransform[y[t], t, s],
            s² LaplaceTransform[y[t], t, s] ==   1
                                               ------
                                               1 + s²
              -LaplaceTransform[x[t], t, s]
              -2 LaplaceTransform[y[t], t, s]}

In[1741]:= step3 =
              Solve[step2,
                 { LaplaceTransform[x[t], t, s],
                  LaplaceTransform[y[t], t, s]}]//
                 Simplify

Out[1741]= {{LaplaceTransform[x[t], t, s] →
              -     4 + 2 s + s³
                --------------------,
                s² (4 + 5 s² + s⁴)
               LaplaceTransform[y[t], t, s] →
                  2 + s + s²
                ------------------}}
                4 s² + 5 s⁴ + s⁶
```

Finally, we use `InverseLaplaceTransform` to compute $x(t) = \mathcal{L}^{-1}\{X(s)\}$ and $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

```
In[1742]:= x[t_] = InverseLaplaceTransform[
                    step3[[1, 1, 2]], s, t]
```

Out[1742]= $-\frac{1}{2} - t + \frac{1}{3}\,(\mathrm{Cos}[t] + 4\,\mathrm{Sin}[t]) + \frac{1}{6}\,(\mathrm{Cos}[2\,t] - \mathrm{Sin}[2\,t])$

```
In[1743]:= y[t_] = InverseLaplaceTransform[
                    step3[[1, 2, 2]], s, t]
```

Out[1743]= $\frac{1}{4} + \frac{t}{2} + \frac{1}{3}\,(-\mathrm{Cos}[t] - \mathrm{Sin}[t]) + \frac{1}{12}\,(\mathrm{Cos}[2\,t] - \mathrm{Sin}[2\,t])$

We see that the initial conditions are satisfied by graphing $x(t)$ and $y(t)$ in Figure 8-27 (a) and $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in Figure 8-27 (b).

```
In[1744]:= Plot[{x[t], y[t]}, {t, -π, 4π},
             PlotRange → {-3 π, 2π}, AspectRatio → 1,
             PlotStyle →
             {GrayLevel[0], GrayLevel[0.5]}]

In[1745]:= ParametricPlot[{x[t], y[t]}, {t, -π, 4π},
             PlotRange → {{-12, 3}, {-3, 12}},
             AspectRatio → 1]
```

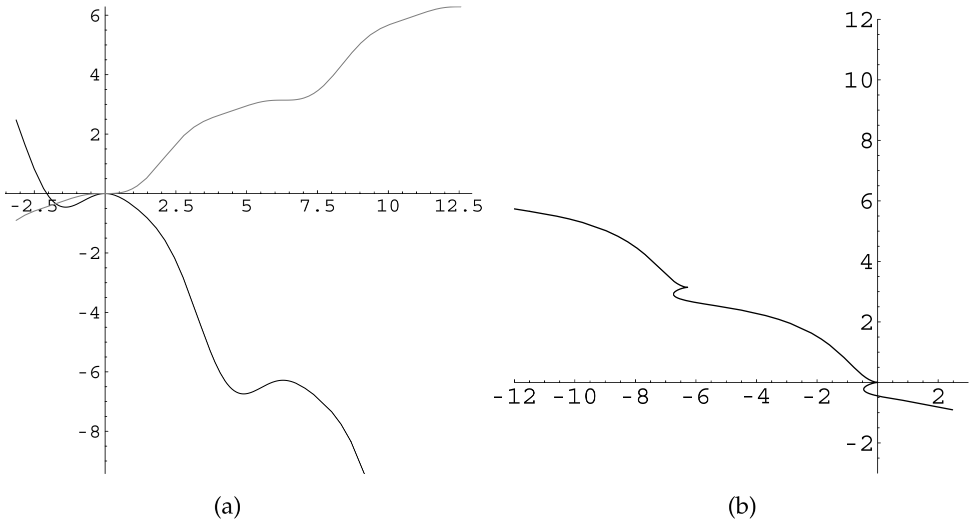


Figure 8-27 (a) $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of x versus y

■

Laplace transform methods are especially useful in solving problems that involve piecewise-defined, periodic, or impulse functions.

EXAMPLE 8.7.3: Solve $\begin{cases} x' = y + 3\delta(t-\pi) \\ y' = -\frac{1}{4}x + 6\delta(t-2\pi) \\ x(0) = 1,\ y(0) = -1. \end{cases}$

SOLUTION: We proceed in the exact same manner as in the previous examples. After defining the system of equations,

```
In[1746]:= Clear[x, y, t]

           sys = {x'[t] == y[t] + 3 DiracDelta[t - π],
                  y'[t] == -x[t] + 6 DiracDelta[t - 2π]}
Out[1746]= {x'[t] == 3 DiracDelta[π - t] + y[t],
             y'[t] == 6 DiracDelta[2 π - t] - x[t]}
```

we use `LaplaceTransform` to compute the Laplace transform of each equation

```
In[1747]:= step1 = LaplaceTransform[sys, t, s]
Out[1747]= {s LaplaceTransform[x[t], t, s] - x[0] ==
             3 e^(-π s) + LaplaceTransform[y[t], t, s],
             s LaplaceTransform[y[t], t, s] - y[0] ==
             6 e^(-2 π s) - LaplaceTransform[x[t], t, s]}
```

and apply the initial conditions.

```
In[1748]:= step2 = step1 /. {x[0]- > 1, y[0]- > -1}
Out[1748]= {-1 + s LaplaceTransform[x[t], t, s] ==
             3 e^(-π s) + LaplaceTransform[y[t], t, s],
             1 + s LaplaceTransform[y[t], t, s] ==
             6 e^(-2 π s) - LaplaceTransform[x[t], t, s]}
```

We then solve the resulting algebraic system of equations for $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$ and use `InverseLaplaceTransform` to compute $x(t)$ and $y(t)$.

```
In[1749]:= step3 = Solve[step2,
                  { LaplaceTransform[x[t], t, s],
                  LaplaceTransform[y[t], t, s]}]
```

```
Out[1749]= {{LaplaceTransform[x[t], t, s] →
              e^(-2πs) (6 - e^(2πs) + 3 e^(πs) s + e^(2πs) s)
              ------------------------------------------------ ,
                              1 + s^2
            LaplaceTransform[y[t], t, s] →
              -1 - 3 e^(-πs) - (1 - 6 e^(-2πs)) s
            - ------------------------------------ }}
                            -1 - s^2

In[1750]:= x[t_] = InverseLaplaceTransform[
                   step3[[1, 1, 2]], s, t]
Out[1750]= Cos[t] - Sin[t] + 6 Sin[t] UnitStep[-2π + t]
           -3 Cos[t] UnitStep[-π + t]

In[1751]:= y[t_] = InverseLaplaceTransform[
                   step3[[1, 2, 2]], s, t]
Out[1751]= -Cos[t] - Sin[t] + 6 Cos[t] UnitStep[-2π + t]
           +3 Sin[t] UnitStep[-π + t]
```

We see that the initial conditions are satisfied by graphing $x(t)$ and $y(t)$ in Figure 8-28 (a) and $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in Figure 8-28 (b).

```
In[1752]:= Plot[{x[t], y[t]}, {t, 0, 4π},
              PlotRange → {-2π, 2π}, AspectRatio → 1,
              PlotStyle → {GrayLevel[0], GrayLevel[0.5]}]
```

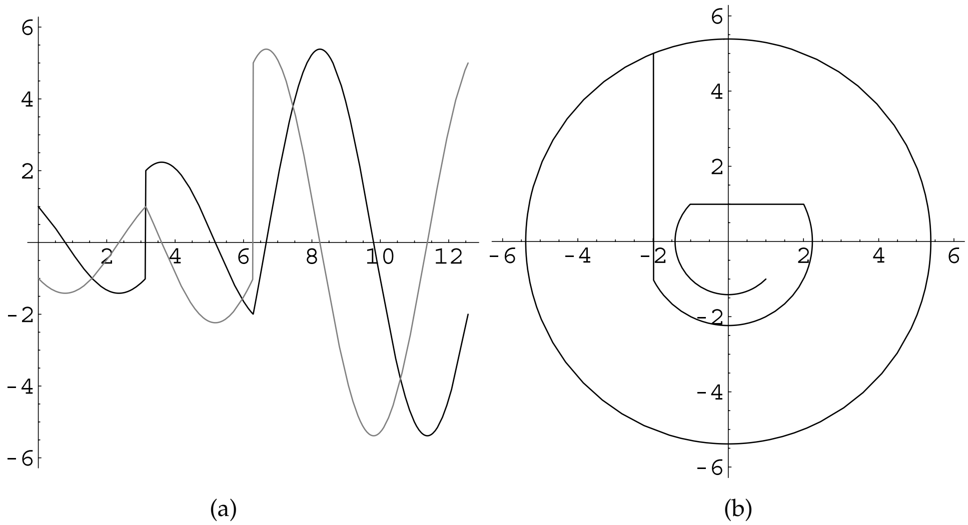


Figure 8-28 (a) $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of x versus y

```
In[1753]:= ParametricPlot[{x[t],y[t]},{t,0,4π},
              PlotRange → {{-2 π, 2π},{-2 π, 2π}},
              AspectRatio → 1]
```

■

EXAMPLE 8.7.4: Solve $\begin{cases} x' = -17x + f(t) \\ y' = \frac{1}{4}x - y - f(t) \\ x(0) = y(0) = 0 \end{cases}$ where $f(t) = \begin{cases} 1+t, & 0 \le t < 1 \\ 3, & t \ge 1. \end{cases}$

SOLUTION: We first rewrite $f(t)$ in terms of the unit step function:

$$f(t) = \begin{cases} 1+t, & 0 \le t < 1 \\ 3, & t \ge 1 \end{cases} = (1+t)\left[1 - \mathcal{U}(t-1)\right] + 3\mathcal{U}(t-1).$$

Then, we define and graph $f(t)$ in Figure 8-29.

```
In[1754]:= Clear[x,y,t,f]

           f[t_] = (1 + t) (1 - UnitStep[t - 1])
                  +3UnitStep[t - 1]

Out[1754]= (1 + t) (1 - UnitStep[-1 + t]) + 3 UnitStep[-1 + t]
```

```
In[1755]:= Plot[f[t],{t,0,4},PlotRange → {0,4},
              AspectRatio → 1]
```

To solve the initial-value problem, we proceed as in the previous examples. First, we define the system of equations.

```
In[1756]:= sys = {x'[t] == -17 y[t] + f[t],
                 y'[t] == x[t]/4 - y[t] - f[t]};
```

Then, we compute the Laplace transform of each equation,

```
In[1757]:= step1 = LaplaceTransform[sys, t, s]
```

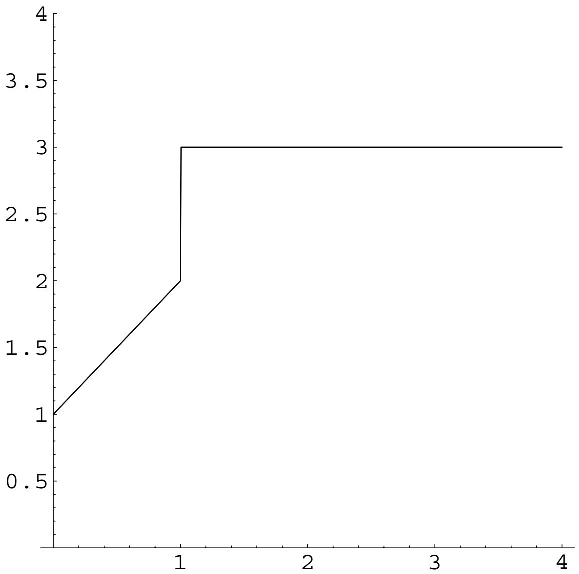


Figure 8-29 A piecewise-defined forcing function

Out[1757]= {s LaplaceTransform[x[t], t, s] − x[0] ==

$$\frac{1}{s^2}+\frac{1}{s}+\frac{2\,e^{-s}}{s}-\frac{e^{-s}\,(1+s)}{s^2}$$

−17 LaplaceTransform[y[t], t, s],

s LaplaceTransform[y[t], t, s] − y[0] ==

$$-\frac{1}{s^2}-\frac{1}{s}-\frac{2\,e^{-s}}{s}+\frac{e^{-s}\,(1+s)}{s^2}$$

$+\frac{1}{4}$ LaplaceTransform[x[t], t, s]

−LaplaceTransform[y[t], t, s]}

apply the initial conditions,

In[1758]:= **step2 = step1 /. {x[0] − > 0, y[0] − > 0}**

Out[1758]= {s LaplaceTransform[x[t], t, s] ==

$$\frac{1}{s^2}+\frac{1}{s}+\frac{2\,e^{-s}}{s}-\frac{e^{-s}\,(1+s)}{s^2}$$

−17 LaplaceTransform[y[t], t, s],

s LaplaceTransform[y[t], t, s] ==

$$-\frac{1}{s^2}-\frac{1}{s}-\frac{2\,e^{-s}}{s}+\frac{e^{-s}\,(1+s)}{s^2}$$

$+\frac{1}{4}$ LaplaceTransform[x[t], t, s]

−LaplaceTransform[y[t], t, s]}

and solve the resulting algebraic system of equations for $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

```
In[1759]:= step3 = Solve[step2,
                {LaplaceTransform[x[t], t, s],
                 LaplaceTransform[y[t], t, s]}]
```

Out[1759]= $\left\{\left\{\text{LaplaceTransform}[x[t],t,s]\to \frac{4\,e^{-s}(18+s)(-1+e^{s}+s+e^{s}s)}{s^{2}(17+4s+4s^{2})},\ \text{LaplaceTransform}[y[t],t,s]\to -\frac{e^{-s}(-1+4s)(-1+e^{s}+s+e^{s}s)}{s^{2}(17+4s+4s^{2})}\right\}\right\}$

The solution is obtained with `InverseLaplaceTransform`.

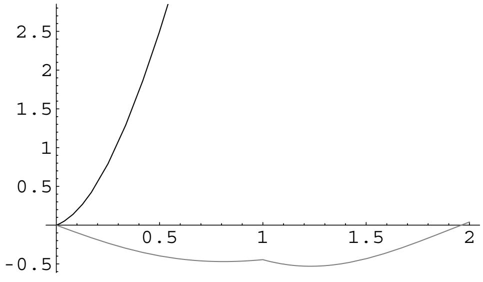
```
In[1760]:= x[t_] = InverseLaplaceTransform[
                step3[[1, 1, 2]], s, t]//Simplify
```

Out[1760]= $\frac{1}{1156}\Big(e^{-2i-\left(\frac{1}{2}+2i\right)t}\Big(e^{2i}\Big((-2008-1437i) -(2008-1437i)\,e^{4it} +16\,e^{\left(\frac{1}{2}+2i\right)t}(251+306t)\Big) -\Big((2888-791i)\,e^{\frac{1}{2}+4i}+(2888+791i) \times e^{\frac{1}{2}+4it}+16\,e^{2i+\left(\frac{1}{2}+2i\right)t}(-667+306t)\Big) \times \text{UnitStep}[-1+t]\Big)\Big)$

```
In[1761]:= y[t_] = InverseLaplaceTransform[
                step3[[1, 2, 2]], s, t]//Simplify
```

Out[1761]= $\frac{1}{2312}\Big(e^{-2i-\left(\frac{1}{2}+2i\right)t}\Big(e^{2i}\Big((220-557i) +(220+557i)\,e^{4it} +8\,e^{\left(\frac{1}{2}+2i\right)t}(-55+17t)\Big) +\Big((-356-633i)\,e^{\frac{1}{2}+4i}-(356-633i) \times e^{\frac{1}{2}+4it}+e^{2i+\left(\frac{1}{2}+2i\right)t}(848-136t)\Big) \times \text{UnitStep}[-1+t]\Big)\Big)$

Last, we confirm that the initial conditions are satisfied by graphing $x(t)$ and $y(t)$ in Figure 8-30 (a) and $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in Figure 8-30 (b).



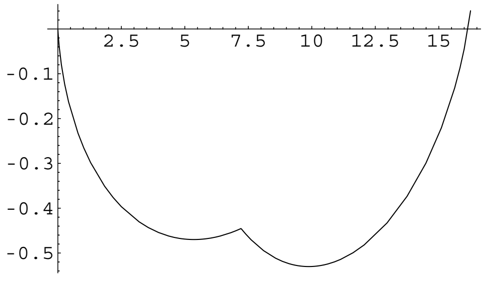


Figure 8-30 (a) $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of x versus y

```
In[1762]:= Plot[{x[t],y[t]},{t,0,2},
              PlotStyle->{GrayLevel[0],GrayLevel[0.5]}]

In[1763]:= ParametricPlot[{x[t],y[t]},{t,0,2}]
```

■

EXAMPLE 8.7.5: Solve $\begin{cases} x' + 2x + 3y = 0 \\ y' - x + 6y = f(t) \\ x(0) = 1,\ y(0) = 0 \end{cases}$ where $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 2, & 2 \le t < 3 \end{cases}$ and $f(t) = f(t-3)$, $t \ge 3$.

SOLUTION: We begin by defining and graphing $f(t)$ in Figure 8-31.

```
In[1764]:= Clear[x,y,t]

In[1765]:= Clear[f]

           f[t_] := 0/; 0 ≤ t < 1;

           f[t_] := 1/; 1 ≤ t < 2;

           f[t_] := 2/; 2 ≤ t < 3

           f[t_] := f[t-3]/; t ≥ 3;

In[1766]:= Plot[f[t],{t,0,9},PlotRange->{-1,8},
              AspectRatio->1]
```

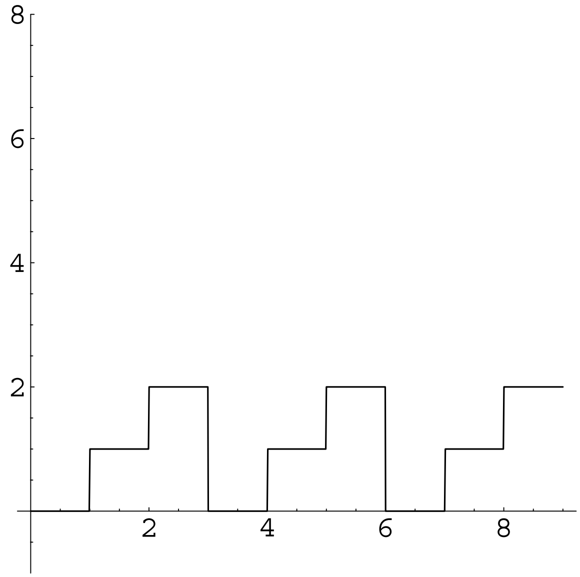


Figure 8-31 A piecewise-defined periodic forcing function

The Laplace transform of the periodic function $f(t)$ is given by equation (8.15). We use `Integrate` and `Simplify` to find $\mathcal{L}\{f(t)\}$, naming the result `lapf`.

```
In[1767]:= lapf = Simplify
              [(∫_1^2 Exp[-s t]ⅆt + ∫_2^3 2 Exp[-s t]ⅆt) / (1 - Exp[-3 s])]
Out[1767]= (2 + ⅇ^s) / (s + ⅇ^s s + ⅇ^(2 s) s)
```

Now we compute the Laplace transform of $x' + 2x + 3y = 0$

```
In[1768]:= leq1 = LaplaceTransform[
               x'[t] + 2x[t] + 3y[t] == 0, t, s]
               /. {x[0]- > 1, y[0]- > 0}
Out[1768]= -1 + 2 LaplaceTransform[x[t], t, s]
           +s LaplaceTransform[x[t], t, s]
           +3 LaplaceTransform[y[t], t, s] == 0
```

and $y' - x + 6y = f(t)$. In each case, we apply the initial conditions as well.

```
In[1769]:= leq2 = LaplaceTransform[y'[t] - x[t] + 6y[t],
               t, s] == lapf /. {x[0]- > 1, y[0]- > 0}
```

```
Out[1769]= -LaplaceTransform[x[t],t,s]
           +6 LaplaceTransform[y[t],t,s]
           +s LaplaceTransform[y[t],t,s] ==
```
$$\frac{2+e^s}{s+e^s s+e^{2s} s}$$

We then use `Solve` to solve this system of equations for $X(s)$ and $Y(s)$.

```
In[1770]:= Solve[{leq1,leq2},
             {LaplaceTransform[x[t],t,s],
             LaplaceTransform[y[t],t,s]}]
Out[1770]= {{LaplaceTransform[x[t],t,s] →
```
$$\frac{1}{2+s}+\frac{3\left(-1-\frac{(2+e^s)(2+s)}{s+e^s s+e^{2s} s}\right)}{(2+s)(15+8s+s^2)},$$

LaplaceTransform[y[t],t,s] →

$$-\frac{-1-\frac{(2+e^s)(2+s)}{s+e^s s+e^{2s} s}}{15+8s+s^2}\}\}$$

Note that `InverseLaplaceTransform` cannot be used to compute $x(t)=\mathcal{L}^{-1}\{X(s)\}$ and $y(t)=\mathcal{L}^{-1}\{Y(s)\}$. Instead, we use `Apart` to rewrite $X(s)$.

In[1771]:= **Apart**$\left[\frac{1}{2+s}+\frac{3\left(-1-\frac{(2e^{-2s}+e^{-s})(2+s)}{(1+e^{-2s}+e^{-s})s}\right)}{(2+s)(15+8s+s^2)}\right]$

Out[1771]= $-\frac{3(2+e^s)}{(1+e^s+e^{2s})s(3+s)(5+s)}+\frac{6+s}{(3+s)(5+s)}$

`InverseLaplaceTransform` quickly calculates $\mathcal{L}^{-1}\left\{\frac{s^2+6s-6}{s(s+3)(s+5)}\right\}$.

In[1772]:= **InverseLaplaceTransform**$\left[\frac{-6+6s+s^2}{s(3+s)(5+s)},\right.$ **s, t**$\left.\right]$

Out[1772]= $-\frac{2}{5}-\frac{11e^{-5t}}{10}+\frac{5e^{-3t}}{2}$

To calculate $\mathcal{L}^{-1}\left\{\frac{3(2+e^{-s})}{\left(1+e^{-s}+e^{-2s}\right)s(s+3)(s+5)}\right\}$, we first rewrite the fraction:

$$\begin{aligned}\frac{3(2+e^{-s})}{\left(1+e^{-s}+e^{-2s}\right)s(s+3)(s+5)} &= \frac{2+e^{-s}}{1+e^{-s}+e^{-2s}}\cdot\frac{3}{s(s+3)(s+5)}\\ &= \frac{2+e^{-s}}{1+e^{-s}+e^{-2s}}\cdot\frac{1-e^{-s}}{1-e^{-s}}\cdot\frac{3}{s(s+3)(s+5)}\\ &= \left(\frac{2}{1-e^{-3s}}+\frac{e^{-s}}{1-e^{-3s}}+\frac{e^{-2s}}{1-e^{-3s}}\right)\cdot\frac{3}{s(s+3)(s+5)}\end{aligned}$$

and then use the geometric series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$:

$$\left(\frac{2}{1-e^{-3s}} + \frac{e^{-s}}{1-e^{-3s}} + \frac{e^{-2s}}{1-e^{-3s}}\right) \cdot \frac{3}{s(s+3)(s+5)} =$$
$$\left(2\sum_{n=0}^{\infty} e^{-3ns} + \sum_{n=0}^{\infty} e^{-(3n+1)s} + \sum_{n=0}^{\infty} e^{-(3n+2)s}\right) \cdot \frac{3}{s(s+3)(s+5)}.$$

Notice that $\mathcal{L}^{-1}\left\{\dfrac{3}{s(s+3)(s+5)}\right\} = \frac{1}{5} + \frac{3}{10}e^{-5t} - \frac{1}{2}e^{-3t}$. We name this function $g(t)$ for later use.

```
In[1773]:= g[t_] = InverseLaplaceTransform[
                        3
                  ----------------- , s, t]
                  s (3 + s) (5 + s)
Out[1773]= 3 ( 1/15 + e^(-5 t)/10 - e^(-3 t)/6 )
```

Previously, we learned that $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$. Thus,

$$\mathcal{L}^{-1}\left\{\left(2\sum_{n=0}^{\infty} e^{-3ns} + \sum_{n=0}^{\infty} e^{-(3n+1)s} + \sum_{n=0}^{\infty} e^{-(3n+2)s}\right) \cdot \frac{3}{s(s+3)(s+5)}\right\} =$$
$$2\sum_{n=0}^{\infty} g(t-3n)\mathcal{U}(t-3n) + \sum_{n=0}^{\infty} g(t-(3n+1))\mathcal{U}(t-(3n+1)) +$$
$$\sum_{n=0}^{\infty} g(t-(3n+2))\mathcal{U}(t-(3n+2))$$

and

$$x(t) = -\frac{2}{5} - \frac{11}{10}e^{-5t} + \frac{5}{2}e^{-3t} + 2\sum_{n=0}^{\infty} g(t-3n)\mathcal{U}(t-3n)$$
$$+ \sum_{n=0}^{\infty} g(t-(3n+1))\mathcal{U}(t-(3n+1)) + \sum_{n=0}^{\infty} g(t-(3n+2))\mathcal{U}(t-(3n+2)).$$

We find $y(t)$ in the same way.

```
                          (2e^(-2 s) + e^(-s)) (2+s)
                     -1 - --------------------------
                           (1+e^(-2 s) + e^(-s)) s
In[1774]:= Apart[ - ------------------------------- ]
                           15 + 8s + s^2
                    1                  (2 + e^s) (2 + s)
Out[1774]= ----------------- + -----------------------------------
           (3 + s) (5 + s)     (1 + e^s + e^(2 s)) s (3 + s) (5 + s)
```

We use `InverseLaplaceTransform` to see that

$$\mathcal{L}^{-1}\left\{\frac{3s+4}{s(s+3)(s+5)}\right\} = \frac{4}{15} - \frac{11}{10}e^{-5t} + \frac{5}{6}e^{-3t}$$

and

$$\mathcal{L}^{-1}\left\{-\frac{s+2}{s(s+3)(s+5)}\right\} = -\frac{2}{15} + \frac{3}{10}e^{-5t} - \frac{1}{6}e^{-3t}.$$

We name the second result $h(t)$ for later use.

```
In[1775]:= InverseLaplaceTransform[(4 + 3s)/(s (3 + s) (5 + s)), s, t]

Out[1775]= 4/15 - (11 e^(-5 t))/10 + (5 e^(-3 t))/6

In[1776]:= Clear[h]

           h[t_] = InverseLaplaceTransform
                     [- (2 + s)/(s (3 + s) (5 + s)), s, t]

Out[1776]= -2/15 + (3 e^(-5 t))/10 - e^(-3 t)/6
```

To calculate $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, we use the results we obtained when calculating $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

$$\mathcal{L}^{-1}\left\{\frac{2+e^{-s}}{1+e^{-s}+e^{-2s}} \cdot \frac{-(s+2)}{s(s+3)(s+5)}\right\} =$$

$$\mathcal{L}^{-1}\left\{\left(2\sum_{n=0}^{\infty} e^{-3ns} - \sum_{n=0}^{\infty} e^{-(3n+1)s} - \sum_{n=0}^{\infty} e^{-(3n+2)s}\right) \cdot \frac{-(s+2)}{s(s+3)(s+5)}\right\} =$$

$$2\sum_{n=0}^{\infty} h(t-3n)\mathcal{U}(t-3n) - \sum_{n=0}^{\infty} h(t-(3n+1))\mathcal{U}(t-(3n+1))$$

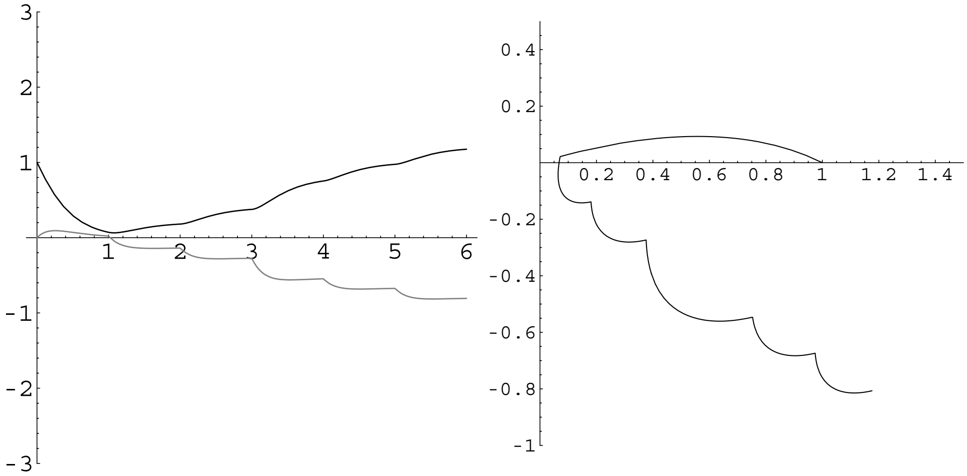
$$-\sum_{n=0}^{\infty} h(t-(3n+2))\mathcal{U}(t-(3n+2))$$

and

$$y(t) = \frac{4}{15} - \frac{11}{10}e^{-5t} + \frac{5}{6}e^{-3t} + 2\sum_{n=0}^{\infty} h(t-3n)\mathcal{U}(t-3n)$$

$$-\sum_{n=0}^{\infty} h(t-(3n+1))\mathcal{U}(t-(3n+1)) - \sum_{n=0}^{\infty} h(t-(3n+2))\mathcal{U}(t-(3n+2)).$$

We then graph the solution on the interval [0, 6] in Figure 8-32.



(a) (b)

Figure 8-32 (a) $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of x versus y

In[1777]:= **xapprox[t_]**

$$= -\frac{2}{5} - \frac{11}{10}e^{-5t} + \frac{5}{2}e^{-3t}$$
$$+2\left(\sum_{n=0}^{6} \mathtt{g[t-3n]\ UnitStep[t-3n]}\right)$$
$$-\sum_{n=0}^{6} \mathtt{g[t-(3n+1)]\ UnitStep[t-(3n+1)]}$$
$$-\sum_{n=0}^{6} \mathtt{g[t-(3n+2)]\ UnitStep[t-(3n+2)];}$$

In[1778]:= **yapprox[t_]**

$$= \frac{4}{15} - \frac{11}{10}e^{-5t} + \frac{5}{6}e^{-3t} +$$
$$+2\left(\sum_{n=0}^{6} \mathtt{h[t-3n]\ UnitStep[t-3n]}\right)$$
$$-\sum_{n=0}^{6} \mathtt{h[t-(3n+1)]\ UnitStep[t-(3n+1)]}$$
$$-\sum_{n=0}^{6} \mathtt{h[t-(3n+2)]\ UnitStep[t-(3n+2)];}$$

```
In[1779]:= Plot[{xapprox[t],yapprox[t]},
             {t,0,6},
             PlotStyle->{GrayLevel[0],GrayLevel[0.5]},
             PlotRange->{-3,3},AspectRatio->1]

In[1780]:= ParametricPlot[{xapprox[t],yapprox[t]},
             {t,0,6},
             PlotRange->{{0,1.5},{-1,0.5}},
             AspectRatio->1]
```

■

8.8 Applications of Laplace Transforms, Part II

8.8.1 Coupled Spring–Mass Systems

The motion of a mass attached to the end of a spring was modeled with a second-order linear differential equation with constant coefficients in Chapter 5. Similarly, if a second spring and mass are attached to the end of the first mass, then the model becomes that of a system of second-order equations. To more precisely state the problem, let masses m_1 and m_2 be attached to the ends of springs S_1 and S_2 having spring constants k_1 and k_2, respectively. Then, spring S_2 is attached to the base of mass m_1.

Suppose that $x(t)$ and $y(t)$ represent the vertical displacement from the equilibrium position of springs S_1 and S_2, respectively. Because spring S_2 undergoes both elongation and compression when the system is in motion (due to the spring S_1 and the mass m_2), then according to Hooke's law, S_2 exerts the force $k_2(y-x)$ on m_2 while S_1 exerts the force $-k_1x$ on m_1. Therefore, the force acting on mass m_1 is the sum $-k_1x+k_2(y-x)$ and that acting on m_2 is $-k_2(y-x)$. Hence, using Newton's second law, $F=ma$, with each mass, we have the system

$$\begin{cases} m_1\dfrac{d^2x}{dt^2} = -k_1x+k_2(y-x) \\ m_2\dfrac{d^2y}{dt^2} = -k_2(y-x). \end{cases} \tag{8.23}$$

The initial position and velocity of the two masses m_1 and m_2 are given by $x(0)$, $x'(0)$, $y(0)$, and $y'(0)$, respectively. If external forces $F_1(t)$ and $F_2(t)$ are applied to the masses, the system (8.23) becomes

$$\begin{cases} m_1 \dfrac{d^2x}{dt^2} = -k_1 x + k_2(y-x) + F_1(t) \\ m_2 \dfrac{d^2y}{dt^2} = -k_2(y-x) + F_2(t). \end{cases} \tag{8.24}$$

Therefore, the method of Laplace transforms can be used to solve problems of this type.

EXAMPLE 8.8.1: Consider the spring–mass system with $m_1 = m_2 = 1$, $k_1 = 3$, and $k_2 = 2$. Find the position functions $x(t)$ and $y(t)$ if $x(0) = 0$, $x'(0) = 1$, $y(0) = 1$, and $y'(0) = 0$. (Assume there are no external forces.)

SOLUTION: In order to find $x(t)$ and $y(t)$, we must solve the initial-value problem

$$\begin{cases} \dfrac{d^2x}{dt^2} = -5x + 2y \\ \dfrac{d^2y}{dt^2} = 2x - 2y \\ x(0) = 0,\ x'(0) = 1,\ y(0) = 1,\ y'(0) = 0. \end{cases}$$

We use `LaplaceTransform` to take the Laplace transform of both sides of each equation.

```
In[1781]:= eqs = {x''[t] == -5x[t] + 2y[t],
                  y''[t] == 2x[t] - 2y[t]};

In[1782]:= step1 = LaplaceTransform[eqs, t, s]

Out[1782]= {s^2 LaplaceTransform[x[t], t, s] - s x[0]
           -x'[0] == -5 LaplaceTransform[x[t], t, s]
           +2 LaplaceTransform[y[t], t, s],
             s^2 LaplaceTransform[y[t], t, s] - s y[0]
           -y'[0] == 2 LaplaceTransform[x[t], t, s]
           -2 LaplaceTransform[y[t], t, s]}
```

We then apply the initial conditions.

```
In[1783]:= step2 =
              step1 /. {x[0]- > 0, x′[0]- > 1,
              y[0]- > 1, y′[0]- > 0}

Out[1783]= {-1 + s^2 LaplaceTransform[x[t], t, s] ==
            -5 LaplaceTransform[x[t], t, s]
            +2 LaplaceTransform[y[t], t, s],
              -s + s^2 LaplaceTransform[y[t], t, s] ==
              2 LaplaceTransform[x[t], t, s]
              -2 LaplaceTransform[y[t], t, s]}
```

We solve this system of algebraic equations for $X(s)$ and $Y(s)$ with `Solve`.

```
In[1784]:= step3 =
              Solve[step2, {LaplaceTransform[x[t], t, s],
                 LaplaceTransform[y[t], t, s]}]
```

Out[1784]= $\left\{\left\{\text{LaplaceTransform[x[t], t, s]} \to -\frac{-2-2s-s^2}{6+7s^2+s^4},\ \text{LaplaceTransform[y[t], t, s]} \to -\frac{-2-5s-s^3}{6+7s^2+s^4}\right\}\right\}$

Taking the inverse Laplace transform with `InverseLaplaceTransform` yields $x(t)$ and $y(t)$.

In[1785]:= `x[t_] =`

$$\text{InverseLaplaceTransform}\left[-\frac{-2-2s-s^2}{6+7s^2+s^4}, s, t\right]$$

Out[1785]= $\frac{1}{5}(2\,\text{Cos}[t]+\text{Sin}[t])-\frac{2}{15}\left(3\,\text{Cos}\left[\sqrt{6}\,t\right]-\sqrt{6}\,\text{Sin}\left[\sqrt{6}\,t\right]\right)$

In[1786]:= `y[t_] =`

$$\text{InverseLaplaceTransform}\left[-\frac{-2-5s-s^3}{6+7s^2+s^4}, s, t\right]$$

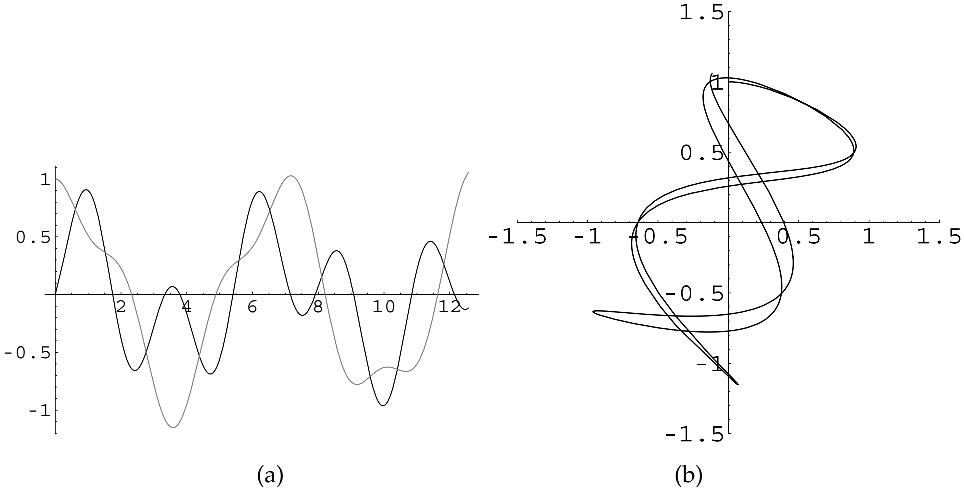


Figure 8-33 (a) $x(t)$ (in black) and $y(t)$ (in gray). (b) Parametric plot of x versus y

```
Out[1786]= 2/5 (2 Cos[t] + Sin[t])
           + 1/15 (3 Cos[√6 t] - √6 Sin[√6 t])
```

We graph $x(t)$ and $y(t)$ together in Figure 8-33 (a) and then parametrically in Figure 8-33 (b). Note that $y(t)$ starts at (0, 1) while $x(t)$ has initial point (0, 0). Also, the phase plane is different from those discussed in previous sections. One of the reasons for this is that the equations in the system of differential equations are second-order instead of first-order.

```
In[1787]:= Plot[{x[t],y[t]},{t,0,4π},
             PlotStyle → {GrayLevel[0],
             GrayLevel[0.5]}]
```

```
In[1788]:= ParametricPlot[{x[t],y[t]},{t,0,4π},
             PlotRange → {{-3/2, 3/2},{-3/2, 3/2}},
             AspectRatio → 1]
```

We can illustrate the motion of the spring in nearly the same way as we did in Chapter 5. First, we define the functions `zigzag` and `spring2`.

```
In[1789]:= Clear[spring, zigzag, length, points, pairs]

           zigzag[{a_, b_}, {c_, d_}, n_, eps_] :=
             Module[{length, points, pairs},

               length = d - b;

               points = Table[b + i length/n,
                 {i, 1, n - 1}];

               pairs = Table[{a + (-1)^i eps,
                 points[[i]]},
                 {i, 1, n - 1}];

               PrependTo[pairs, {a, b}];

               AppendTo[pairs, {c, d}];

               Line[pairs]
               ]

In[1790]:= spring2[t_, len1_, len2_, opts___] :=
             Show[Graphics[
             {zigzag[{0, -x[t]}, {0, len1}, 20, 0.025],
             PointSize[0.025], Point[{0, len1}],
             zigzag[{0, -y[t] - len2}, {0, -x[t]},
               20, 0.025],
             PointSize[0.075], Point[{0, -x[t]}],
             PointSize[0.05],
             Point[{0, -y[t] - len2}]}], opts,
             Axes- > Automatic,
             AxesStyle- > GrayLevel[0.5],
             Ticks- > None,
             AspectRatio- > 1,
             PlotRange- > {{-1/2, 1/2}, {-2.2, 1.2}},
             DisplayFunction- > Identity]
```

Next, we define `tvals` to be a list of 16 evenly spaced numbers between 0 and 4π.

```
In[1791]:= tvals = Table[t, {t, 0, 4π, 4π/15}];
```

`Map` is then used to apply `spring2` to the list of numbers in `tvals`.

```
In[1792]:= graphs = Map[spring2[#, 1, 1]&, tvals];
```

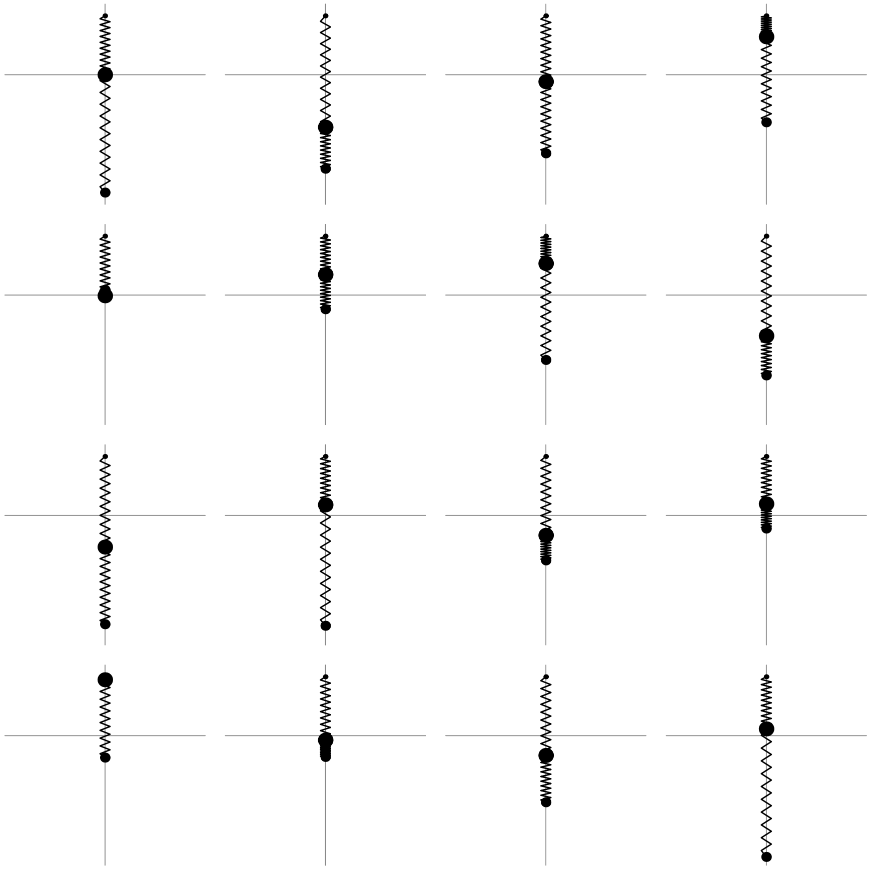


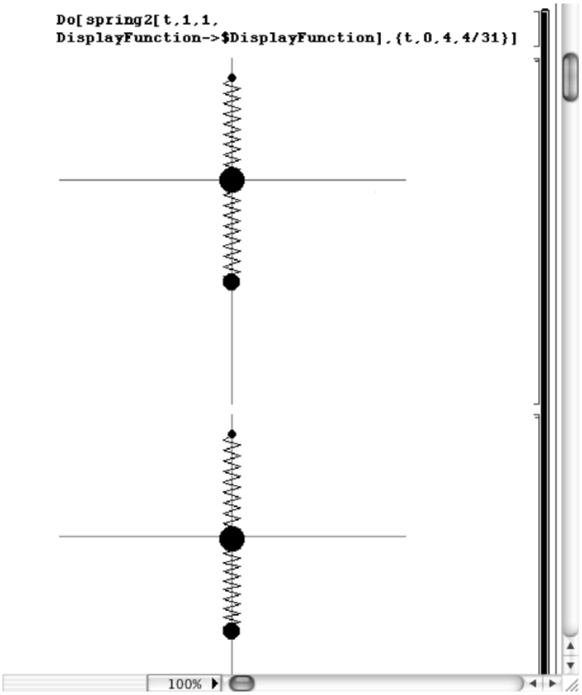
Figure 8-34 Visualizing the motion of a coupled spring–mass system

The resulting list of graphics is partitioned into four element subsets with `Partition` and displayed using `Show` and `GraphicsArray` in Figure 8-34.

```
In[1793] := toshow = Partition[graphs, 4];

            Show[GraphicsArray[toshow]]
```

If the option `DisplayFunction->$DisplayFunction` is included in the `spring2` command, we can use a `Do` loop to generate a set of graphics that can then be animated as follows.

```
In[1794] := Do[spring2[t, 1, 1,
              DisplayFunction- > $DisplayFunction],
              {t, 0, 4, 4/31}]
```



■

8.8.2 The Double Pendulum

In a method similar to that of the simple pendulum in Chapter 5 and that of the coupled spring–mass system, the motion of a double pendulum as shown in Figure 8-35 is modeled by the following system of equations using the approximation $\sin\theta \approx \theta$ for small displacements

$$\begin{cases} (m_1 + m_2)\,{l_1}^2 \dfrac{d^2\theta_1}{dt^2} + m_2 l_1 l_2 \dfrac{d^2\theta_2}{dt^2} + (m_1 + m_2)\, l_1 g \theta_1 = 0 \\ m_2 {l_2}^2 \dfrac{d^2\theta_2}{dt^2} + m_2 l_1 l_2 \dfrac{d^2\theta_1}{dt^2} + m_2 l_2 g \theta_2 = 0 \end{cases} \tag{8.25}$$

where θ_1 represents the displacement of the upper pendulum and θ_2 that of the lower pendulum. Also, m_1 and m_2 represent the mass attached to the upper and lower pendulums, respectively, while the length of each is given by l_1 and l_2.

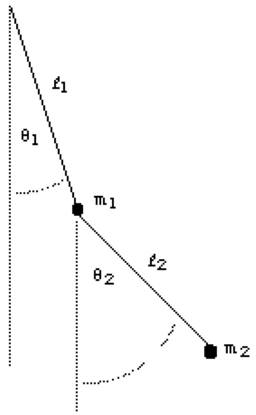


Figure 8-35 A double pendulum

EXAMPLE 8.8.2: Suppose that $m_1 = 3$, $m_2 = 1$, and each pendulum has length 16. If $\theta_1(0) = 1$, $\theta_1'(0) = 1$, $\theta_2(0) = 0$, and $\theta_2'(0) = -1$, solve the double pendulum problem using $g = 32$. Plot the solution.

SOLUTION: In this case, the system to be solved is

$$\begin{cases} 4 \cdot 16^2 \dfrac{d^2\theta_1}{dt^2} + 16^2 \dfrac{d^2\theta_2}{dt^2} + 4 \cdot 16 \cdot 32\theta_1 = 0 \\ 16^2 \dfrac{d^2\theta_2}{dt^2} + 16^2 \dfrac{d^2\theta_1}{dt^2} + 16 \cdot 32\theta_2 = 0 \end{cases}$$

which we simplify to obtain

$$\begin{cases} 4\dfrac{d^2\theta_1}{dt^2} + \dfrac{d^2\theta_2}{dt^2} + 8\theta_1 = 0 \\ \dfrac{d^2\theta_2}{dt^2} + \dfrac{d^2\theta_1}{dt^2} + 2\theta_2 = 0. \end{cases}$$

In the following code, we let $x(t)$ and $y(t)$ represent $\theta_1(t)$ and $\theta_2(t)$, respectively. First, we use `DSolve` to solve the initial-value problem.

```
In[1795]:= sol =
             DSolve[{4 x''[t] + y''[t] + 8 x[t] == 0,
                 x''[t] + y''[t] + 2 y[t] == 0, x[0] == 1,
                 x'[0] == 1, y[0] == 0, y'[0] == -1},
                 {x[t], y[t]}, t]
```

```
Out[1795]= {{x[t] → 1/8 (4 Cos[2 t]
                +4 Cos[2 t/√3] + 3 Sin[2 t]
                +√3 Sin[2 t/√3]),
          y[t] → 1/4 (- 4 Cos[2 t] + 4 Cos[2 t/√3]
                 -3 Sin[2 t] + √3 Sin[2 t/√3])}}
```

We define `sys` to be the system of equations and use `LaplaceTransform` to compute the Laplace transform of each equation.

```
In[1796]:= step1 = LaplaceTransform[sys, t, s]

Out[1796]= {8 LaplaceTransform[x[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s] - s y[0]
           +4 (s^2 LaplaceTransform[x[t], t, s]
           -s x[0] - x'[0]) - y'[0] == 0,
           s^2 LaplaceTransform[x[t], t, s]
           +2 LaplaceTransform[y[t], t, s]
           +s^2 LaplaceTransform[y[t], t, s]
           -s x[0] - s y[0] - x'[0] - y'[0] == 0}
```

Next, we apply the initial conditions and solve the resulting system of equations for $\mathcal{L}\{\theta_1(t)\} = X(s)$ and $\mathcal{L}\{\theta_2(t)\} = Y(s)$.

```
In[1797]:= step2 =
             step1 /. {x[0]- > 1, x'[0]- > 1, y[0]- > 0,
                 y'[0]- > -1}

Out[1797]= {1 + 8 LaplaceTransform[x[t], t, s]
           +4 (-1 - s + s^2 LaplaceTransform[x[t], t, s])
           +s^2 LaplaceTransform[y[t], t, s] == 0,
             -s + s^2 LaplaceTransform[x[t], t, s]
             +2 LaplaceTransform[y[t], t, s]
             +s^2 LaplaceTransform[y[t], t, s] == 0}

In[1798]:= step3 = Solve[step2,
                  { LaplaceTransform[x[t], t, s],
                  LaplaceTransform[y[t], t, s]}]
```

```
Out[1798]= {{LaplaceTransform[x[t],t,s] →
              -(-6 - 8 s - 3 s^2 - 3 s^3)/(16 + 16 s^2 + 3 s^4),
            LaplaceTransform[y[t],t,s] →
              -(-8 s + 3 s^2)/(16 + 16 s^2 + 3 s^4)}}
```

InverseLaplaceTransform is then used to find $\theta_1(t)$ and $\theta_2(t)$.

```
In[1799]:= x[t_] = InverseLaplaceTransform[
              -(-6 - 8s - 3s^2 - 3s^3)/(16 + 16s^2 + 3s^4), s, t]

Out[1799]= 1/8 (4 Cos[2 t] + 4 Cos[2 t/Sqrt[3]]
            +3 Sin[2 t] + Sqrt[3] Sin[2 t/Sqrt[3]])
```

```
In[1800]:= y[t_] = InverseLaplaceTransform[
              -(-8 s + 3s^2)/(16 + 16s^2 + 3s^4), s, t]

Out[1800]= - Cos[2 t] + Cos[2 t/Sqrt[3]]
            -3/2 Cos[t] Sin[t] + 1/4 Sqrt[3] Sin[2 t/Sqrt[3]]
```

These two functions are graphed together in Figure 8-36 (a) and parametrically in Figure 8-36 (b).

```
In[1801]:= Plot[{x[t],y[t]},{t,0,20},
             PlotStyle- > {GrayLevel[0],
             GrayLevel[0.5]}]
```

```
In[1802]:= ParametricPlot[{x[t],y[t]},{t,0,20},
             PlotRange- > {{-5/2,5/2},{-5/2,5/2}},
             AspectRatio- > 1]
```

We can illustrate the motion of the pendulum as follows. First, we define the function pen2.

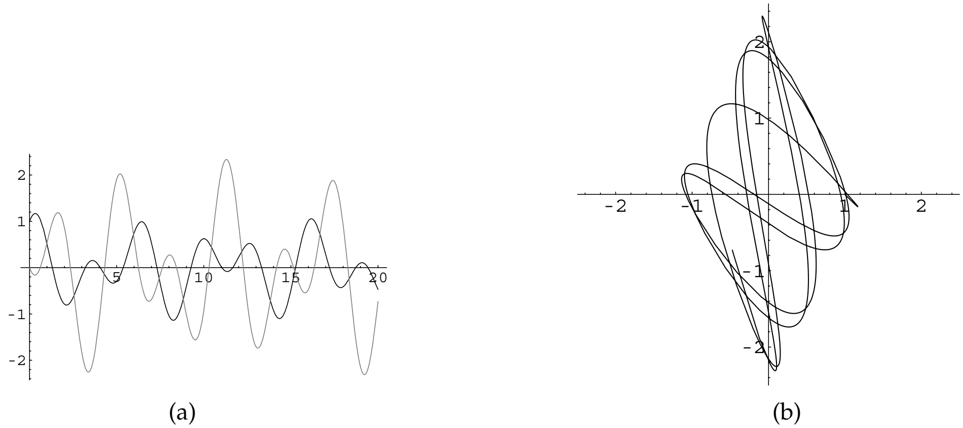


Figure 8-36 (a) $\theta_1(t)$ (in black) and $\theta_2(t)$ (in gray) as functions of t. (b) Parametric plot of $\theta_1(t)$ versus $\theta_2(t)$

```
In[1803]:= Clear[pen2]

           pen2[t_, len1_, len2_] := Module[{pt1, pt2},
             pt1 = {len1 Cos[3π/2 + x[t]],
                 len1 Sin[3π/2 + x[t]]};
             pt2 = {len1 Cos[3π/2 + x[t]]
                 +len2 Cos[3π/2 + y[t]],
                 len1 Sin[3π/2 + x[t]]
                 +len2 Sin[3π/2 + y[t]]};
             Show[
              Graphics[{Line[{{0, 0}, pt1}],
                 PointSize[0.05], Point[pt1],
                 Line[{pt1, pt2}], PointSize[0.05],
                 Point[pt2]}], Axes → Automatic,
                 Ticks → None,
                 AxesStyle → GrayLevel[0.5],
                 PlotRange → {{-32, 32}, {-34, 0}},
                 DisplayFunction → Identity]]
```

Next, we define `tvals` to be a list of 16 evenly spaced numbers between 0 and 10. `Map` is then used to apply `pen2` to the list of numbers

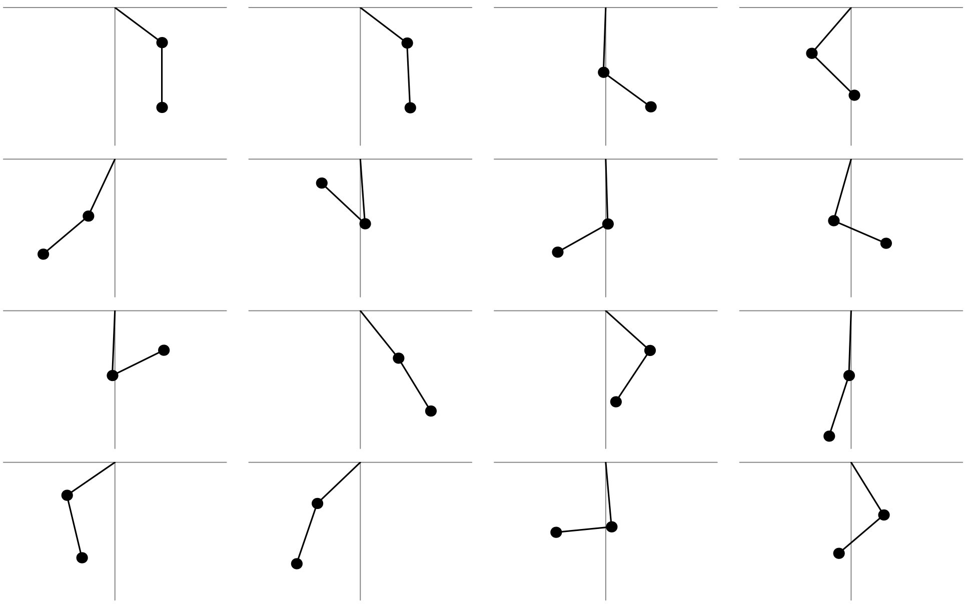


Figure 8-37 The double pendulum for 16 equally spaced values of t between 0 and 10

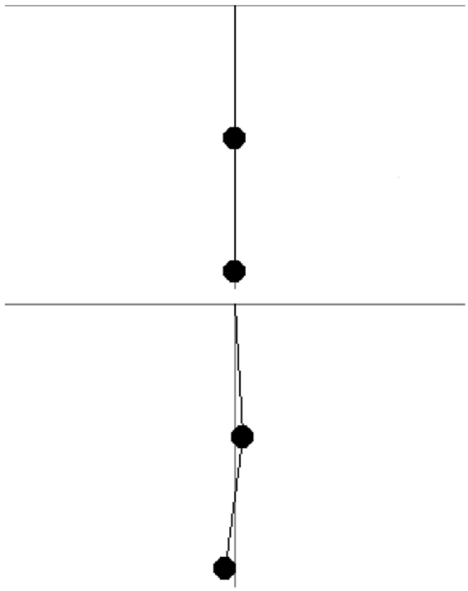
in `tvals`. The resulting set of graphics is partitioned into four element subsets and displayed using `Show` and `GraphicsArray` in Figure 8-37.

```
In[1804]:= tvals = Table[t, {t, 0, 10, 10/15}];

In[1805]:= graphs = Map[pen2[#, 16, 16]&, tvals];

In[1806]:= toshow = Partition[graphs, 4];

In[1807]:= Show[GraphicsArray[toshow]]
```

If the option `DisplayFunction->$DisplayFunction` is included in the `pen2` command, we can use a `Do` loop to generate a set of graphics that can then be animated.

```
In[1808]:= Clear[t]

           Do[pen2[t, 16, 16,
              DisplayFunction- > $DisplayFunction],
              {t, 0, 10, 10/31}]
```

```
Clear[t]
Do[pen2[t,16,16,DisplayFunction-
>$DisplayFunction],{t,0,10,10/31}]
```



■

Sources: M. L. James, G. M. Smith, J. C. Wolford, P. W. Whaley, *Vibration of Mechanical and Structural Systems with Microcomputer Applications*, Harper & Row (1989), pp. 282–286. Robert K. Vierck, *Vibration Analysis*, Second Edition, HarperCollins (1979), pp. 266–290.

Application: Free Vibration of a Three-Story Building

If you have ever gone to the top of a tall building like the Sears Tower or Empire State Building on a windy day you may have been acutely aware of the sway of the building. In fact, all buildings sway, or vibrate, naturally. Usually, we are only aware, if ever, of the sway of a building when we are in a very tall building or in a building during an event like an earthquake. In some tall buildings, like the John Hancock Building in Boston, the sway of the building during high winds is reduced by installing a tuned mass damper at the top of the building which oscillates at the same frequency as the building but out of phase. We will investigate the sway of a three-story building and then try to determine how we would investigate the sway of a tall building.

We make two assumptions to solve this problem. First, we assume that the mass distribution of the building can be represented by the lumped masses at the different levels. Second, we assume that the girders of the structure are infinitely rigid in comparison with the supporting columns. With these assumptions, we can determine the motion of the building by interpreting the columns as springs in parallel.

Assume that the coordinates x_1, x_2, and x_3 as well as the velocities and accelerations are positive to the right. Also assume that $x_3 > x_2 > x_1$.

In applying Newton's second law of motion, recall that we have assumed that acceleration is in the positive direction. Therefore, we sum forces in the same direction as the acceleration positively, and others negatively. With this configuration, Newton's second law on each of the three masses yields the following system of differential equations

$$\begin{aligned} -k_1x_1 + k_2(x_2 - x_1) &= m_1\frac{d^2x_1}{dt^2} \\ -k_2(x_2 - x_1) + k_3(x_3 - x_2) &= m_2\frac{d^2x_2}{dt^2} \\ -k_3(x_3 - x_2) &= m_3\frac{d^2x_3}{dt^2} \end{aligned} \tag{8.26}$$

which we write as

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} + (k_1 + k_2)x_1 - k_2x_2 &= 0 \\ m_2\frac{d^2x_2}{dt^2} - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 &= 0 \\ m_3\frac{d^2x_3}{dt^2} - k_3x_2 + k_3x_3 &= 0 \end{aligned} \tag{8.27}$$

where m_1, m_2, and m_3 represent the mass of the building on the first, second, and third levels, and k_1, k_2, and k_3, corresponding to the spring constants, represent the total stiffness of the columns supporting a given floor.

If we attempt to find an exact solution with the method of Laplace transforms, we find that each denominator of $\mathcal{L}\{x_1(t)\}$, $\mathcal{L}\{x_2(t)\}$, and $\mathcal{L}\{x_3(t)\}$ is a positive function of s. Therefore, the roots are complex and solutions will involve sines and/or cosines. (Here, we use $x(t)$, $y(t)$, and $z(t)$ in the place of $x_1(t)$, $x_2(t)$, and $x_3(t)$.)

```
In[1809]:= Clear[x, y, rule, eq1, eq2]

           eq1 = m₁ x''[t] + (k₁ + k₂) x[t] - k₂ y[t] == 0;

           eq2 = m₂ y''[t] - k₂ x[t] + (k₂ + k₃) y[t] - k₃ z[t] == 0;

           eq3 = m₃ z''[t] - k₃ y[t] + k₃ z[t] == 0;
```

```
In[1810]:= step1 = LaplaceTransform[{eq1, eq2, eq3}, t, s]
Out[1810]= {-LaplaceTransform[y[t], t, s] k2
             +LaplaceTransform[x[t], t, s] (k1 + k2)
             +m1 (s^2 LaplaceTransform[x[t], t, s]
             -s x[0] - x'[0]) == 0,
             -LaplaceTransform[x[t], t, s] k2
             -LaplaceTransform[z[t], t, s] k3
             +LaplaceTransform[y[t], t, s] (k2 + k3)
             +m2 (s^2 LaplaceTransform[y[t], t, s]
             -s y[0] - y'[0]) == 0,
             -LaplaceTransform[y[t], t, s] k3
             +LaplaceTransform[z[t], t, s] k3
             +m3 (s^2 LaplaceTransform[z[t], t, s]
             -s z[0] - z'[0]) == 0}

In[1811]:= step2 = Solve[step1, {LaplaceTransform[x[t], t, s],
                   LaplaceTransform[y[t], t, s],
                   LaplaceTransform[
                   z[t], t, s]}]//Simplify
Out[1811]= {{LaplaceTransform[z[t], t, s] →
               1/(k3 + s^2 m3) (s m3 z[0] + m3 z'[0]
             -(k3 (-k2 m1 (k3 + s^2 m3) (s x[0] + x'[0])
             +(k1 + k2 + s^2 m1) (-m2 (k3 + s^2 m3) (s y[0] + y'[0])
             -k3 m3 (s z[0] + z'[0]))))/(-k2^2 (k3 + s^2 m3)
             +(k1 + k2 + s^2 m1) (-k3^2 + (k2 + k3 + s^2 m2)
              (k3 + s^2 m3)))),
             LaplaceTransform[x[t], t, s] →
               (s^2 m1 (s^2 m2 m3 + k3 (m2 + m3)) (s x[0] + x'[0])
             +k2 (s^2 m3 (m1 (s x[0] + x'[0]) + m2 (s y[0] + y'[0]))
             +k3 (m1 (s x[0] + x'[0]) + m2 (s y[0] + y'[0])
             +m3 (s z[0] + z'[0]))))/
               (k1 (k2 (k3 + s^2 m3)
             +s^2 (s^2 m2 m3 + k3 (m2 + m3)))
             +s^2 (s^2 m1 (s^2 m2 m3 + k3 (m2 + m3))
             +k2 (s^2 (m1 + m2) m3 + k3 (m1 + m2 + m3)))),
             LaplaceTransform[y[t], t, s] →
             -(-k2 m1 (k3 + s^2 m3) (s x[0] + x'[0])
             +(k1 + k2 + s^2 m1) (-m2 (k3 + s^2 m3) (s y[0] + y'[0])
             -k3 m3 (s z[0] + z'[0])))/(-k2^2 (k3 + s^2 m3)
             +(k1 + k2 + s^2 m1) (-k3^2 + (k2 + k3 + s^2 m2)
                (k3 + s^2 m3)))}}
```

Suppose that $k_1 = 3$, $k_2 = 2$, $k_3 = 1$, $m_1 = 1$, $m_2 = 2$, and $m_3 = 3$ and that the initial conditions are $x(0) = 0$, $x'(0) = 1/4$, $y(0) = 0$, $y'(0) = -1/2$, $z(0) = 0$, and $z'(0) = 1$.

```
In[1812]:= step3 = step2 /. {k1- > 3, k2- > 2, x[0]- > 0, k3- > 1,
                m1- > 1, m2- > 2, m3- > 3, x′[0]- > 1/4,
                y[0]- > 0, y′[0]- > -1/2,
                z[0]- > 0, z′[0]- > 1}//Simplify
```

Out[1812]= $\left\{\left\{\text{LaplaceTransform}[z[t], t, s] \to \frac{57 + 76 s^2 + 12 s^4}{12 + 90 s^2 + 82 s^4 + 12 s^6}, \text{LaplaceTransform}[x[t], t, s] \to \frac{18 - 13 s^2 + 6 s^4}{4 (6 + 45 s^2 + 41 s^4 + 6 s^6)}, \text{LaplaceTransform}[y[t], t, s] \to \frac{21 - 23 s^2 - 6 s^4}{12 + 90 s^2 + 82 s^4 + 12 s^6}\right\}\right\}$

For these values, we use InverseLaplaceTransform to compute $x(t) = \mathcal{L}^{-1}\{X(s)\}$, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, and $z(t) = \mathcal{L}^{-1}\{Z(s)\}$. First, we compute $x(t)$. The result is very long so we do not display it here.

Instead, we use Short to view a portion of this result. Note that several terms are given in terms of Root.

Root[p[x],k] represents the kth root of the polynomial equation $p(x) = 0$.

In[1813]:= x[t_] = InverseLaplaceTransform$\left[\frac{18 - 13s^2 + 6s^4}{4 (6 + 45s^2 + 41s^4 + 6s^6)}, s, t\right]$;

```
In[1814]:= Short[x[t], 3]
```

Out[1814]= $\frac{e^{\ll 1 \gg} (\ll 1 \gg)}{48 \ll 8 \gg \sqrt{\text{Root}[6 + 45 \#1 + 41 \#1^2 + 6 \#1^3 \&, 3]}}$

In this case, we cannot find exact solutions of the equation $6s^6 + 41s^4 + 45s^2 + 6 = 0$. Nevertheless, we can use NRoots to approximate the solutions of this equation.

In[1815]:= NRoots$\left[6 + 45s^2 + 41s^4 + 6s^6 == 0, s\right]$

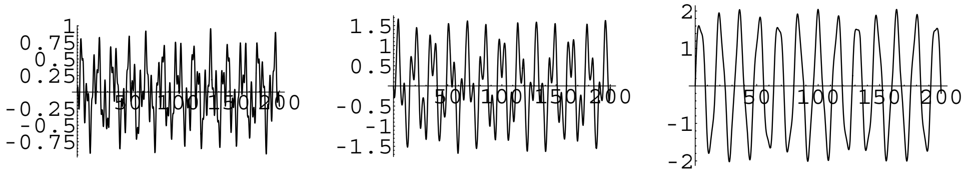


Figure 8-38 The sway of a building is periodic

```
Out[1815]= s == -7.70372 × 10^-34 - 0.393222 ⅈ ||
            s == -7.70372 × 10^-34 + 0.393222 ⅈ || s == 0. - 1.08402 ⅈ ||
            s == 0. + 1.08402 ⅈ || s == 7.70372 × 10^-34 - 2.34598 ⅈ ||
            s == 7.70372 × 10^-34 + 2.34598 ⅈ
```

Now, we use `InverseLaplaceTransform` to compute $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ and $z(t) = \mathcal{L}^{-1}\{Z(s)\}$.

```
In[1816]:= y[t_] = InverseLaplaceTransform[
                  (21 - 23s^2 - 6s^4)/(2 (6 + 45s^2 + 41s^4 + 6s^6)), s, t];

In[1817]:= z[t_] = InverseLaplaceTransform[
                  (57 + 76s^2 + 12s^4)/(2 (6 + 45s^2 + 41s^4 + 6s^6)), s, t];
```

The graphs of $x(t)$, $y(t)$, and $z(t)$ shown in Figure 8-38 indicate that they are indeed periodic functions.

```
In[1818]:= px = Plot[x[t], {t, 0, 200},
                DisplayFunction- > Identity];

           py = Plot[y[t], {t, 0, 200},
                DisplayFunction- > Identity];

           pz = Plot[z[t], {t, 0, 200},
                DisplayFunction- > Identity];

           Show[GraphicsArray[{px, py, pz}]]
```

We can construct an outline of a three-story building and observe its vibration. The width and height of the floors were selected arbitrarily to be 20 and 1, respectively. See Figure 8-39.

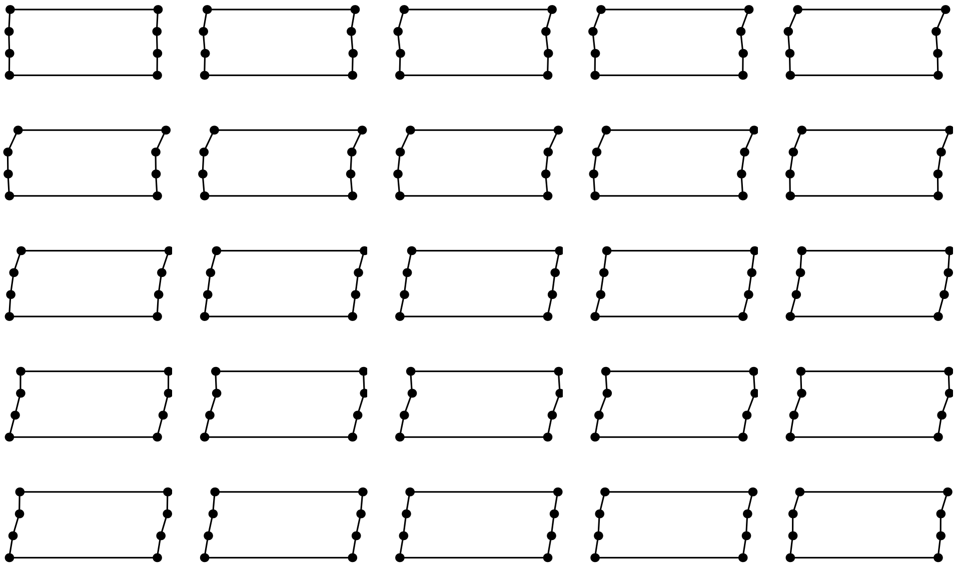


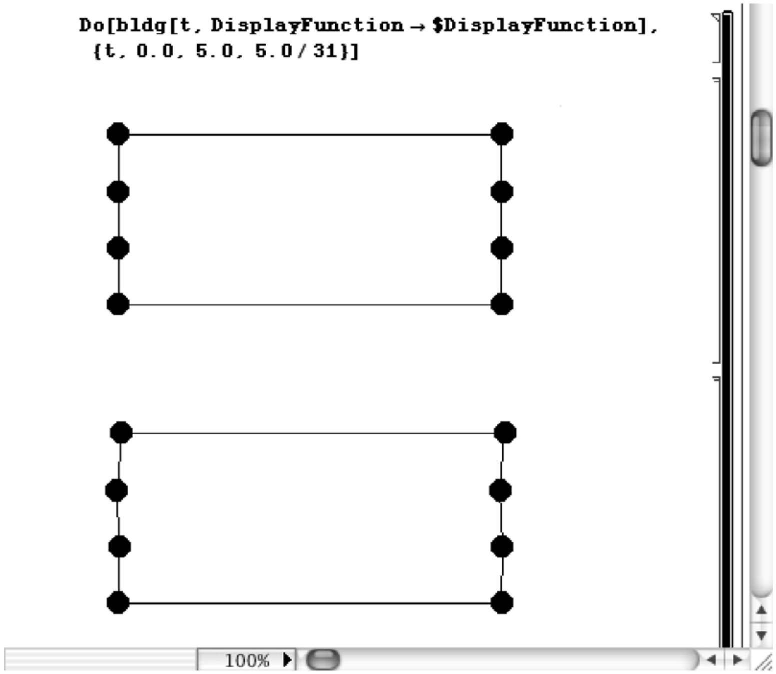
Figure 8-39 Visualizing the sway of a building

```
In[1819]:= Clear[bldg]

           bldg[t_,opts___] :=
                 Show[Graphics[{Line[{{0,0},{20,0}}],
                 PointSize[0.05],Point[{0,0}],
                 Point[{20,0}],Line[{{0,0},{x[t],1}}],
                 Point[{x[t],1}],Line[{{20,0},
                 {20 + x[t],1}}],Point[{20 + x[t],1}],
                 Line[{{x[t],1},{y[t],2}}],
                 Point[{y[t],2}],
                 Line[{{20 + x[t],1},{20 + y[t],2}}],
                 Point[{20 + y[t],2}],
                 Line[{{y[t],2},{z[t],3}}],
                 Point[{z[t],3}],Line[{{20 + y[t],2},
                 {20 + z[t],3}}],Point[{20 + z[t],3}],
                 Line[{{z[t],3},{20 + z[t],3}}]}],opts,
                 Axes → None,Ticks → None,
                 PlotRange → {{-2,22},{-1,4}},
                 DisplayFunction- > Identity]]

In[1820]:= graphs = Table[bldg[t],{t,0.1,6.1,6./24}];
```

```
In[1821]:= toshow = Partition[graphs, 5];

           Show[GraphicsArray[toshow]]
```

If you include the option `DisplayFunction->$DisplayFunction` in the `bldg` command, you can use a `Do` loop and animate the resulting graphics as indicated next.

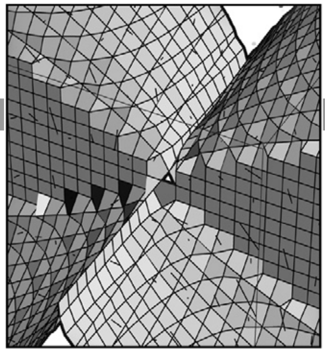
```
In[1822]:= Do[bldg[t, DisplayFunction → $DisplayFunction],
             {t, 0., 5., 5./31}]
```



Increasing the number of stories increases the size of the system of differential equations. A five-story building corresponds to a system of five second-order differential equations; a 50-story building, a system of 50 second-order differential equations, and so on.

Eigenvalue Problems and Fourier Series

9



In previous chapters, we have seen that many physical situations can be modeled by either ordinary differential equations or systems of ordinary differential equations. However, to understand the motion of a string at a particular location and at a particular time, the temperature in a thin wire at a particular location and a particular time, or the electrostatic potential at a point on a plate, we must solve partial differential equations as each of these quantities depends on (at least) two independent variables.

Wave equation	$c^2 u_{xx} = u_{tt}$
Heat equation	$u_t = c^2 u_{xx}$
Laplace's equation	$u_{xx} + u_{yy} = 0$

In Chapter 10, we introduce a particular method for solving these partial differential equations (as well as others). In order to carry out this method, however, we introduce the necessary tools in this chapter. We begin with a discussion of boundary-value problems and their solutions.

9.1 Boundary-Value Problems, Eigenvalue Problems, Sturm–Liouville Problems

9.1.1 Boundary-Value Problems

In previous sections, we have solved initial-value problems. However, at this time we will consider boundary-value problems which are solved in much the same way as initial-value problems except that the value of the function and its derivatives are given at two values of the independent variable instead of one. The general form of a second-order (two-point) boundary-value problem is

$$\begin{cases} a_2(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_0(x)y = f(x),\ a < x < b \\ k_1 y(a) + k_2\dfrac{dy}{dx}(a) = \alpha,\ h_1 y(b) + h_2\dfrac{dy}{dx}(b) = \beta \end{cases} \tag{9.1}$$

where k_1, k_2, α, h_1, h_2, and β are constants and at least one of k_1, k_2 and at least one of h_1, h_2 is not zero.

Note that if $\alpha = \beta = 0$, then we say the problem has **homogeneous boundary conditions**. We also consider boundary-value problems that include a parameter in the differential equation. We solve these problems, called **eigenvalue problems**, in order to investigate several useful properties associated with their solutions.

EXAMPLE 9.1.1: Solve $\begin{cases} y'' + y = 0,\ 0 < x < \pi \\ y'(0) = 0,\ y'(\pi) = 0. \end{cases}$

SOLUTION: Because the characteristic equation is $k^2+1 = 0$ with roots $k_{1,2} = \pm i$, a general solution of $y'' + y = 0$ is $y = c_1 \cos x + c_2 \sin x$ and it follows that $y' = -c_1 \sin x + c_2 \cos x$. Applying the boundary conditions, we have $y'(0) = c_2 = 0$. Then, $y = c_1 \cos x$. With this solution, we have $y'(\pi) = -c_1 \sin \pi = 0$ for any value of c_1. Therefore, there are infinitely many solutions, $y = c_1 \cos x$, of the boundary-value problem, depending on the choice of c_1. In this case, we are able to use `DSolve` to solve the boundary-value problem

```
In[1823]:= sol = DSolve[{y''[x] + y[x] == 0, y'[0] == 0
                    y'[π] == 0}, y[x], x]
Out[1823]= {{y[x] → C[1] Cos[x]}}
```

We confirm that the boundary conditions are satisfied for any value of `C[1]` by graphing several solutions with `Plot` in Figure 9-1.

```
In[1824]:= toplot = Table[y[x] /. sol /. C[1]- > i,
                     {i, -5, 5}];

           grays = Table[GrayLevel[i], {i, 0, 0.5, 0.5/10}];

In[1825]:= Plot[Evaluate[toplot], {x, 0, π},
             PlotStyle → grays]
```

■

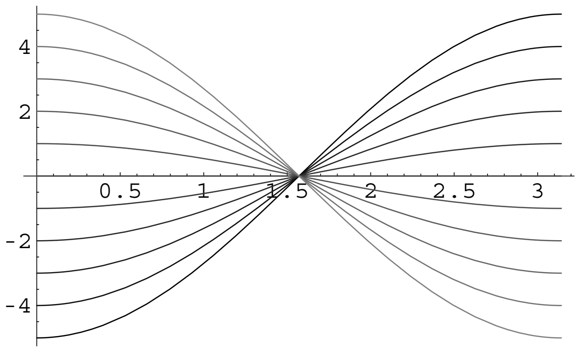


Figure 9-1 The boundary-value problem has infinitely many solutions

From the result in the example, we notice a difference between *initial-value problems* and *boundary-value problems*: an initial-value problem (that meets the hypotheses of the Existence and Uniqueness Theorem) has a unique solution while a boundary-value problem may have one solution, more than one solution, or no solution.

See Chapter 4 and Theorem 2.

EXAMPLE 9.1.2: Solve $\begin{cases} y'' + y = 0, \ 0 < x < \pi \\ y'(0) = 0, \ y'(\pi) = 1. \end{cases}$

SOLUTION: Using the general solution obtained in the previous example, we have $y = c_1 \cos x + c_2 \sin x$. As before, $y'(0) = c_2 = 0$, so $y = c_1 \cos x$. However, because $y'(\pi) = -c_1 \sin \pi = 0 \neq 1$, the boundary conditions cannot be satisfied with any choice of c_1. Therefore, there is no solution to the boundary-value problem.

■

As indicated in the general form of a boundary-value problem, the boundary conditions in these problems can involve the function and its derivative. However, this modification to the problem does not affect the method of solution.

EXAMPLE 9.1.3: Solve $\begin{cases} y'' - y = 0, \ 0 < x < 1 \\ y'(0) + 3y(0) = 0, \ y'(1) + y(1) = 1. \end{cases}$

SOLUTION: The characteristic equation is $k^2 - 1 = 0$ with roots $k_{1,2} = \pm 1$. Hence, a general solution is $y = c_1 e^x + c_2 e^{-x}$ with derivative $y' = c_1 e^x - c_2 e^{-x}$. Applying $y'(0) + 3y(0) = 0$ yields $y'(0) + 3y(0) = c_1 - c_2 + 3(c_1 + c_2) = 4c_1 + 2c_2 = 0$. Because $y'(1) + y(1) = 1$,

$$y'(1) + y(1) = c_1 e^1 - c_2 e^{-1} + c_1 e^1 + c_2 e^{-1} = 2c_1 e = 1,$$

so $c_1 = \frac{1}{2e}$ and $c_2 = -\frac{1}{e}$. Thus, the boundary-value problem has the unique solution $y = \frac{1}{2e} e^x - \frac{1}{e} e^{-x} = \frac{1}{2} e^{x-1} - e^{-x-1}$, which we confirm with Mathematica. See Figure 9-2.

```
In[1826]:= sol = DSolve[{y''[x] - y[x] == 0, y'[0] + 3y[0] == 0,
                y'[1] + y[1] == 1}, y[x], x]
```

$$\text{Out[1826]= } \left\{\left\{y[x] \to \frac{1}{2} e^{-1-x} \left(-2 + e^{2x}\right)\right\}\right\}$$

```
In[1827]:= Plot[y[x]/.sol, {x, 0, 1},
             AspectRatio → Automatic]
```

■

9.1.2 Eigenvalue Problems

We now consider **eigenvalue problems**, boundary-value problems that include a parameter. Values of the parameter for which the boundary-value problem has a nontrivial solution are called **eigenvalues** of the problem. For each eigenvalue, the nontrivial solution that satisfies the problem is called the **corresponding eigenfunction**.

If a value of the parameter leads to the trivial solution, then the value is not considered an eigenvalue of the problem.

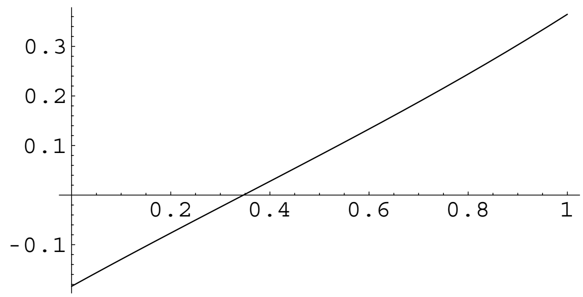


Figure 9-2 The boundary-value problem has a unique solution

EXAMPLE 9.1.4: Solve the eigenvalue problem $y'' + \lambda y = 0$, $0 < x < p$, subject to $y(0) = 0$ and $y(p) = 0$.

In Example 9.1.5 we solve the same differential equation but use the boundary conditions $y'(0) = 0$ and $y'(p) = 0$.

SOLUTION: Notice that the differential equation in this problem differs from those solved earlier because it includes the parameter λ. However, we solve it in a similar manner by solving the characteristic equation $k^2 + \lambda = 0$. Of course, the values of k depend on the value of the parameter λ. Hence, we consider the following three cases.

1. ($\lambda = 0$) In this case, the characteristic equation is $k^2 = 0$ with roots $k_{1,2} = 0$, which indicates that a general solution is $y = c_1 x + c_2$. Application of the boundary condition $y(0) = 0$ yields $y(0) = c_1 \cdot 0 + c_2 = 0$, so $c_2 = 0$. For the second condition, $y(p) = c_1 p = 0$, so $c_1 = 0$ and $y = 0$. Because we obtain the trivial solution, $\lambda = 0$ is *not* an eigenvalue.
2. ($\lambda < 0$) To represent λ as a negative value, we let $\lambda = -\mu^2 < 0$. Then, the characteristic equation is $k^2 - \mu^2 = 0$, so $k_{1,2} = \pm\mu$. Therefore, a general solution is, $y = c_1 e^{\mu x} + c_1 e^{-\mu x}$ (or $y = c_1 \cosh \mu x + c_1 \sinh \mu x$). Substitution of the boundary condition $y(0) = 0$ yields $y(0) = c_1 + c_2 = 0$, so $c_2 = -c_1$. Because $y(p) = 0$ indicates that $y = c_1 e^{\mu p} + c_1 e^{-\mu p} = 0$, substitution gives us the equation $y(p) = c_1 e^{\mu p} - c_1 e^{-\mu p} = c_1 \left(e^{\mu p} - e^{-\mu p}\right)$. Notice that $e^{\mu p} - e^{-\mu p} = 0$ only if $e^{\mu p} = e^{-\mu p}$ which can only occur if $\mu = 0$ or $p = 0$. If $\mu = 0$, then $\lambda = -\mu^2 = -0^2 = 0$ which contradicts the assumption that $\lambda < 0$. We also assumed that $p > 0$, so $e^{\mu p} - e^{-\mu p} > 0$. Hence, $y(p) = c_1 \left(e^{\mu p} - e^{-\mu p}\right)$ implies that $c_1 = 0$, so $c_2 = -c_1 = 0$ as well. Because $\lambda < 0$ leads to the trivial solution $y = 0$, there are no negative eigenvalues.
3. ($\lambda > 0$) To represent λ as a positive value, we let $\lambda = \mu^2 > 0$. Then, we have the characteristic equation $k^2 + \mu^2 = 0$ with complex conjugate roots $k_{1,2} = \pm\mu i$. Thus, a general solution is $y = c_1 \cos \mu x + c_2 \sin \mu x$. Because $y(0) = c_1 \cos \mu \cdot 0 + c_2 \sin \mu \cdot 0 = c_1$, the boundary condition $y(0) = 0$ indicates that $c_1 = 0$. Hence, $y = c_2 \sin \mu x$. Application of $y(p) = 0$ yields $y(p) = c_2 \sin \mu p$, so either $c_2 = 0$ or $\sin \mu p = 0$. Selecting $c_2 = 0$ leads to the trivial solution that we want to avoid, so we determine the values of μ that satisfy $\sin \mu p = 0$. Because $\sin n\pi = 0$ for integer values

of n, $\sin \mu p = 0$ if $\mu p = n\pi$, $n = 1, 2, \ldots$ Solving for μ, we have $\mu = n\pi/p$, so the eigenvalues are

$$\lambda = \lambda_n = \mu^2 = \left(\frac{n\pi}{p}\right)^2, \quad n = 1, 2, \ldots .$$

Notice that the subscript n is used to indicate that the parameter depends on the value of n. (Notice also that we omit $n = 0$, because the value $\mu = 0$ was considered in Case 1.) For each eigenvalue, the corresponding eigenfunction is obtained by substitution into $y = c_2 \sin \mu x$. Because c_2 is arbitrary, we choose $c_2 = 1$. Therefore, the eigenvalue $\lambda_n = (n\pi/p)^2$, $n = 1, 2, \ldots$ has corresponding eigenfunction

$$y(x) = y_n(x) = \sin \frac{n\pi x}{p}, \quad n = 1, 2, \ldots .$$

We did not consider negative values of n because $\sin(-n\pi x/p) = -\sin(n\pi x/p)$; the negative sign can be taken into account in the constant; we do not obtain additional eigenvalues or eigenfunctions by using $n = -1, -2, \ldots$.

■

We will find the eigenvalues and eigenfunctions in Example 9.1.4 quite useful in future sections. The following eigenvalue problem will be useful as well.

In Example 9.1.4 we solve the same differential equation but use the boundary conditions $y(0) = 0$ and $y(p) = 0$.

EXAMPLE 9.1.5: Solve $y'' + \lambda y = 0$, $0 < x < p$, subject to $y'(0) = 0$ and $y'(p) = 0$.

SOLUTION: Notice that the only difference between this problem and that in Example 9.1.4 is in the boundary conditions. Again, the characteristic equation is $k^2 + \lambda = 0$, so we must consider the three cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. Note that a general solution in each case is the same as that obtained in Example 9.1.4. However, the final results may differ due to the boundary conditions.

1. ($\lambda = 0$) Because $y = c_1 x + c_2$, $y' = c_1$. Therefore, $y'(0) = c_1 = 0$, so $y = c_2$. Notice that this constant function satisfies $y'(p) = 0$ for all values of c_2. Hence, if we choose $c_2 = 1$, then $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y = y_0(x) = 1$.
2. ($\lambda < 0$) If $\lambda = -\mu^2 < 0$, then $y = c_1 e^{\mu x} + c_2 e^{-\mu x}$ and $y' = c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x}$. Applying the first condition results in $y'(0) = c_1\mu - c_2\mu = 0$, so $c_1 = c_2$. Therefore, $y'(p) = c_1 \mu e^{\mu p} - c_1 \mu e^{-\mu p}$=0 which is not

possible unless $c_1 = 0$, because $\mu \neq 0$ and $p \neq 0$. Thus, $c_1 = c_2 = 0$, so $y = 0$. Because we have the trivial solution, there are no negative eigenvalues.

3. ($\lambda > 0$) By letting $\lambda = \mu^2$, $y = c_1 \cos \mu x + c_2 \sin \mu x$ and $y' = -c_1\mu \sin \mu x + c_2\mu \cos \mu x$. Hence, $y'(0) = c_2\mu = 0$, so $c_2 = 0$. Consequently, $y'(p) = -c_1\mu \sin \mu p = 0$ which is satisfied if $\mu p = n\pi$, $n = 1, 2, \ldots$ Therefore, the eigenvalues are

$$\lambda = \lambda_n = \left(\frac{n\pi}{p}\right)^2, \; n = 1, \, 2, \, \ldots .$$

Note that we found $c_2 = 0$ in $y = c_1 \cos \mu x + c_2 \sin \mu x$, so the corresponding eigenfunctions are

$$y = y_n = \cos \frac{n\pi x}{p}, \; n = 1, \, 2, \, \ldots$$

if we choose $c_1 = 1$.

■

EXAMPLE 9.1.6: Consider the eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) + y'(1) = 0$. (a) Show that the positive eigenvalues $\lambda = \mu^2$ satisfy the relationship $\mu = -\tan \mu$. (b) Approximate the first eight positive eigenvalues. Notice that for larger values of μ, the eigenvalues are approximately the vertical asymptotes of $y = \tan \mu$, so $\lambda_n \approx [(2n-1)\pi/2]^2$, $n = 1, 2, \ldots$.

SOLUTION: In order to solve the eigenvalue problem, we consider the three cases.

1. ($\lambda = 0$) The problem $y'' = 0$, $y(0) = 0$, $y(1) + y'(1) = 0$ has the solution $y = 0$, so $\lambda = 0$ is not an eigenvalue.

```
In[1828]:= DSolve[{y''[x] == 0, y[0] == 0,
              y[1] + y'[1] == 0}, y[x], x]
Out[1828]= {{y[x] → 0}}
```

2. ($\lambda < 0$) Similarly, $y'' - \mu^2 y = 0$, $y(0) = 0$, $y(1) + y'(1) = 0$ has solution $y = 0$, so there are no negative eigenvalues.

```
In[1829]:= DSolve[{y''[x] - μ^2 y[x] == 0, y[0] == 0,
              y[1] + y'[1] == 0}, y[x], x]
Out[1829]= {{y[x] → 0}}
```

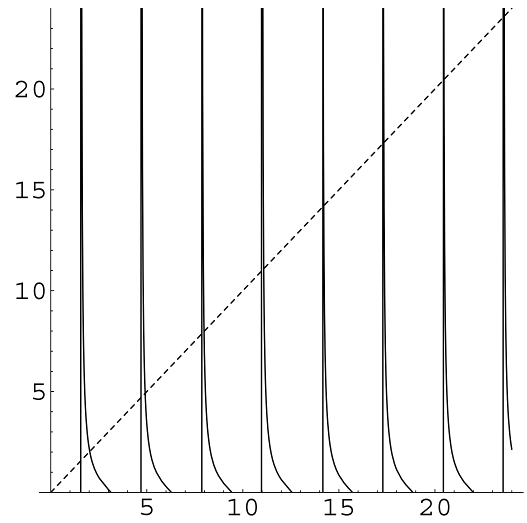


Figure 9-3 The eigenvalues are the x-coordinates of the points of intersection of $y = x$ and $y = -\tan x$

3. ($\lambda > 0$) If $\lambda = \mu^2 > 0$, we solve $y'' + \mu^2 y = 0$, $y(0) = 0$, $y(1) + y'(1) = 0$. In this case, the result returned by `DSolve` is incorrect.

```
In[1830]:= DSolve[{y''[x] + μ^2 y[x] == 0, y[0] == 0,
             y[1] + y'[1] == 0}, y[x], x]
Out[1830]= {{y[x] → 0}}
```

A general solution of $y'' + \mu^2 y = 0$ is $y = A\cos\mu x + B\sin\mu x$. Applying $y(0) = 0$ indicates that $A = 0$, so $y = B\sin\mu x$. Applying $y(1) + y'(1) = 0$ where $y' = \mu B\cos\mu x$ yields $B\sin\mu + \mu B\cos\mu = 0$. Because we want to avoid requiring that $B = 0$, we note that this condition is satisfied if $-\sin\mu = \mu\cos\mu$ or $-\tan\mu = \mu$. To approximate the first eight positive roots of this equation, we graph $y = -\tan x$ and $y = x$ simultaneously in Figure 9-3. (We only look for positive roots because $\tan(-\mu) = -\tan\mu$, meaning that no additional eigenvalues are obtained by considering negative values of μ.) The eigenfunctions of this problem are $y = \sin\mu x$ where μ satisfies $-\tan\mu = \mu$.

```
In[1831]:= Plot[{-Tan[x], x}, {x, 0, 24},
             PlotRange → {0, 24}, PlotStyle → {GrayLevel[0],
             Dashing[{0.01}]}, AspectRatio → 1]
```

In Figure 9-3, notice that roots are to the right of the vertical asymptotes of $y = -\tan x$ which are $x = (2n-1)\pi/2$, n any integer. We use

FindRoot to obtain approximations to the roots using initial guesses near the asymptotes. Here, we guess 0.1 unit to the right of $(2n-1)\pi/2$ for $n = 1, 2, \ldots, 8$.

```
In[1832]:= kvals = Table[FindRoot[-Tan[x] == x,
                {x, (2n - 1) π/2 + 0.1}], {n, 1, 8}]
Out[1832]= {{x → 2.02876}, {x → 4.91318}, {x → 7.97867},
            {x → 11.0855}, {x → 14.2074}, {x → 17.3364},
            {x → 20.4692}, {x → 23.6043}}
```

Therefore, the first eight roots are approximately 2.02876, 4.91318, 7.97867, 11.0855, 14.2074, 17.3364, 20.4692, and 23.6043. As x increases, the roots move closer to the value of x at the vertical asymptotes of $y = -\tan x$. We can compare the two approximations by finding a for the first eight vertical asymptotes, $x = a$.

```
In[1833]:= Table[N[(2n - 1) π/2], {n, 1, 8}]
Out[1833]= {1.5708, 4.71239, 7.85398, 10.9956, 14.1372,
            17.2788, 20.4204, 23.5619}
```

The first eight eigenvalues are approximated by squaring the elements of kvals. We call this list evals.

```
In[1834]:= evals = Table[kvals[[j, 1, 2]]^2, {j, 1, 8}]
Out[1834]= {4.11586, 24.1393, 63.6591, 122.889, 201.851,
            300.55, 418.987, 557.162}
```

■

9.1.3 Sturm–Liouville Problems

Because of the importance of eigenvalue problems, we express these problems in the general form

$$a_2(x)y'' + a_1(x)y' + [a_0(x) + \lambda]\, y = 0, \; a < x < b, \tag{9.2}$$

where $a_2(x) \neq 0$ on $[a, b]$ and the boundary conditions at the endpoints $x = a$ and $x = b$ can be written as

$$k_1 y(a) + k_2 y'(a) = 0 \qquad \text{and} \qquad h_1 y(b) + h_2 y'(b) = 0 \tag{9.3}$$

for the constants k_1, k_2, h_1, and h_2 where at least one of h_1, h_2 and at least one of k_1, k_2 is not zero. Equation (9.2) can be rewritten by letting

$$p(x) = e^{\int a_1(x)/a_2(x)\,dx}, \quad q(x) = \frac{a_0(x)}{a_2(x)}p(x), \quad \text{and} \quad s(x) = \frac{p(x)}{a_2(x)}. \tag{9.4}$$

By making this change, equation (9.2) can be rewritten as the equivalent equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda s(x))\, y = 0, \tag{9.5}$$

which is called a **Sturm–Liouville equation** and along with appropriate boundary conditions is called a **Sturm–Liouville problem**. This particular form of the equation is known as **self-adjoint form**, which is of interest because of the relationship of the function $s(x)$ and the solutions of the problem.

EXAMPLE 9.1.7: Place the equation $x^2y'' + 2xy' + \lambda y = 0$, $x > 0$, in self-adjoint form.

SOLUTION: In this case, $a_2(x) = x^2$, $a_1(x) = 2x$, and $a_0(x) = 0$. Hence, $p(x) = e^{\int a_1(x)/a_2(x)\,dx} = e^{\int 2x/x^2\,dx} = e^{2\ln x} = x^2$, $q(x) = \dfrac{a_0(x)}{a_2(x)}p(x) = 0$, and $s(x) = \dfrac{p(x)}{a_2(x)} = \dfrac{x^2}{x^2} = 1$, so the self-adjoint form of the equation is $\dfrac{d}{dx}\left(x^2\dfrac{dy}{dx}\right) + \lambda y = 0$. We see that our result is correct by differentiating.

■

Solutions of Sturm–Liouville problems have several interesting properties, two of which are included in the following theorem.

Theorem 33 (Linear Independence and Orthogonality of Eigenfunctions). *If $y_m(x)$ and $y_n(x)$ are eigenfunctions of the regular Sturm–Liouville problem*

$$\begin{cases} \dfrac{d}{dx}\left(p(x)\dfrac{dy}{dx}\right) + (q(x) + \lambda s(x))\, y = 0 \\ k_1 y(a) + k_2 y'(a) = 0,\ h_1 y(b) + h_2 y'(b) = 0. \end{cases} \tag{9.6}$$

where $m \neq n$, $y_m(x)$ and $y_n(x)$ are ***linearly independent*** *and the* ***orthogonality condition*** *$\int_a^b s(x)y_m(x)y_n(x)\,dx = 0$ holds.*

Because we integrate the product of the eigenfunctions with the function $s(x)$ in the orthogonality condition, we call $s(x)$ the **weighting function**.

EXAMPLE 9.1.8: Consider the eigenvalue problem $y'' + \lambda y = 0$, $0 < x < p$, subject to $y(0) = 0$ and $y(p) = 0$ that we solved in Example 9.1.4. Verify that the eigenfunctions $y_1 = \sin(\pi x/p)$ and $y_2 = \sin(2\pi x/p)$ are linearly independent. Also, verify the orthogonality condition.

SOLUTION: We can verify that $y_1 = \sin(\pi x/p)$ and $y_2 = \sin(2\pi x/p)$ are linearly independent by computing the Wronskian.

```
In[1835]:= Clear[x, p]

           caps = {Sin[πx/p], Sin[2πx/p]};

           ws = Simplify[Det[{caps, ∂ₓcaps}]]
Out[1835]= -(2 π Sin[πx/p]^3)/p
```

We see that the Wronskian is not the zero function by evaluating it for a particular value of x; we choose $x = p/2$.

```
In[1836]:= ws/.x → p/2
Out[1836]= -(2 π)/p
```

Because $W\{y_1, y_2\}$ is not zero for all values of x, the two functions are linearly independent. In self-adjoint form, the equation is $y'' + \lambda y = 0$, with $s(x) = 1$. Hence, the orthogonality condition is $\int_0^p y_m(x) y_n(x)\,dx = 0$, $m \neq n$, which we verify for y_1 and y_2.

Of course, these two properties hold for any choices of m and n, $m \neq n$.

```
In[1837]:= ∫₀ᵖ Sin[πx/p] Sin[2πx/p] dx
Out[1837]= 0
```

■

9.2 Fourier Sine Series and Cosine Series

9.2.1 Fourier Sine Series

Recall the eigenvalue problem $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0,\ y(p) = 0 \end{cases}$ that was solved in Example 9.1.4. The eigenvalues of this problem are $\lambda = \lambda_n = (n\pi/p)^2$, $n = 1, 2, \ldots,$ with corresponding eigenfunctions $\phi_n(x) = \sin(n\pi x/p)$, $n = 1, 2, \ldots.$

We will see that for some functions $y = f(x)$, we can find coefficients c_n so that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{p}. \tag{9.7}$$

A series of this form is called a **Fourier sine series**. To make use of these series, we must determine the coefficients c_n. We accomplish this by taking advantage of the orthogonality properties of eigenfunctions stated in Theorem 33.

Because the differential equation $y'' + \lambda y = 0$ is in self-adjoint form, we have that $s(x) = 1$. Therefore, the orthogonality condition is $\int_0^p \sin(n\pi x/p)\,\sin(m\pi x/p)\,dx$, $m \neq n$. In order to use this condition, multiply both sides of $f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/p)$ by the eigenfunction $\sin(m\pi x/p)$ and $s(x) = 1$. Then, integrate the result from $x = 0$ to $x = p$ (because the boundary conditions of the corresponding eigenvalue problem are given at these two values of x). This yields

$$\int_0^p f(x)\sin\frac{m\pi x}{p}\,dx = \int_0^p \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx.$$

Assuming that term-by-term integration is allowed on the right-hand side of the equation, we have

$$\int_0^p f(x)\sin\frac{m\pi x}{p}\,dx = \sum_{n=1}^{\infty} \int_0^p c_n \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx.$$

Recall that the eigenfunctions $\phi_n(x)$, $n = 1, 2, \ldots$ are orthogonal, so $\int_0^p \sin(n\pi x/p)$ $\sin(m\pi x/p)\,dx = 0$ if $m \neq 0$. On the other hand if $m = n$,

$$\begin{aligned}\int_0^p \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx &= \int_0^p \sin^2\frac{n\pi x}{p}\,dx \\ &= \frac{1}{2}\int_0^p \left(1 - \cos\frac{2n\pi x}{p}\right)dx \\ &= \frac{1}{2}\left[x - \frac{p}{2n\pi}\sin\frac{2n\pi x}{p}\right]_0^p = \frac{p}{2}.\end{aligned}$$

```
In[1838]:= ∫_0^p Sin[nπx/p]^2 dx
Out[1838]= p/2 - p Sin[2 n π]/(4 n π)
```

Therefore, each term in the sum $\sum_{n=1}^{\infty} c_n \int_0^p \sin(n\pi x/p)\sin(m\pi x/p)\,dx$ equals zero except when $m = n$. Hence, $\int_0^p f(x)\sin(n\pi x/p)\,dx = \frac{1}{2}c_n p$, so the Fourier sine series coefficients are given by

$$c_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx, \tag{9.8}$$

where we assume that $y = f(x)$ is integrable on $[0, p]$.

EXAMPLE 9.2.1: Find the Fourier sine series for $f(x) = x$, $0 \le x \le \pi$.

SOLUTION: In this case, $p = \pi$. Using integration by parts we have,

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \\ &= \frac{2}{\pi}\left[-\frac{1}{n}x\cos nx\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\frac{1}{n}\cos nx\,dx = -\frac{2}{n}\cos n\pi + \frac{2}{\pi}\left[\frac{1}{n^2}\sin nx\right]_0^{\pi} \\ &= -\frac{2}{n}\cos n\pi + \frac{2}{n^2}(\sin n\pi - \sin 0) = -\frac{2}{n}\cos n\pi. \end{aligned}$$

```
In[1839]:= ∫_0^π (2x Sin[nx])/π dx

Out[1839]= 2 (-(π Cos[n π])/n + Sin[n π]/n^2) / π
```

Observe that n is an integer so $\cos n\pi = (-1)^n$. Hence, $c_n = -\frac{2}{n}(-1)^n = (-1)^{n+1}\frac{2}{n}$, and the Fourier sine series is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{\pi} = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin nx \\ &= 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \cdots . \end{aligned}$$

We can use a finite number of terms of the series to obtain a trigonometric polynomial that approximates $f(x) = x$, $0 \le x \le \pi$ as follows. Let $f_k(x) = 2\sum_{n=1}^{k}(-1)^{n+1}\frac{1}{n}\sin nx$. Then, $f_k(x) = f_{k-1}(x) + (-1)^{k+1}\frac{2}{k}\sin kx$. Thus, to calculate the kth partial sum of the Fourier sine series, we need only add $(-1)^{k+1}\frac{2}{k}\sin kx$ to the $(k-1)$st partial sum: we need not recompute all k terms of the kth partial sum if we know the $(k-1)$st partial sum. Using this observation, we define the recursively defined function `f` to return the kth partial sum of the series. We use the form `f[k_]:=f[k] =...` so that Mathematica "remembers" each $f_k(x)$ that is computed. The advantage of doing so is that Mathematica need not recompute $f_k(x)$ to compute $f_{k+1}(x)$.

```
In[1840]:= Clear[f]

           f[k_] := f[k] = f[k - 1] + 2 (-1)^(k+1) Sin[kx] / k

           f[1] = 2 Sin[x];

In[1841]:= Table[{n, f[n]}, {n, 1, 5}]//TableForm
```

$$
\begin{array}{ll}
1 & 2\,\mathrm{Sin}[x] \\
2 & 2\,\mathrm{Sin}[x]-\mathrm{Sin}[2\,x] \\
3 & 2\,\mathrm{Sin}[x]-\mathrm{Sin}[2\,x]+\frac{2}{3}\,\mathrm{Sin}[3\,x] \\
4 & 2\,\mathrm{Sin}[x]-\mathrm{Sin}[2\,x]+\frac{2}{3}\,\mathrm{Sin}[3\,x]-\frac{1}{2}\,\mathrm{Sin}[4\,x] \\
5 & 2\,\mathrm{Sin}[x]-\mathrm{Sin}[2\,x]+\frac{2}{3}\,\mathrm{Sin}[3\,x]-\frac{1}{2}\,\mathrm{Sin}[4\,x] \\
 & +\frac{2}{5}\,\mathrm{Sin}[5\,x]
\end{array}
$$

Out[1841]=

We now graph $f(x)$ on $[0, \pi]$ along with several of the partial sums of the sine series in Figure 9-4. As we increase the number of terms used in approximating $f(x)$, we improve the accuracy. Notice from the graphs that none of the partial sums attain the value of $f(\pi) = \pi$ at $x = \pi$. This is due to the fact that at $x = \pi$, each of the partial sums yield a value of 0. Hence, our approximation can only be reliable on the interval $0 < x < \pi$. In general, however, we are only assured of accuracy at points of continuity of $f(x)$ on the open interval.

```
In[1842]:= somegraphs =
             Table[Plot[{x, f[n]}, {x, 0, π},
               PlotStyle → {GrayLevel[0.5],
               GrayLevel[0]},
               DisplayFunction → Identity, Ticks → {{0, π},
               {1, 2, 3}}], {n, 3, 30, 9}];

           toshow = Partition[somegraphs, 2];

           Show[GraphicsArray[toshow], AspectRatio → 1]
```

■

EXAMPLE 9.2.2: Find the Fourier sine series for $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ -1, & 1 \le x \le 2. \end{cases}$

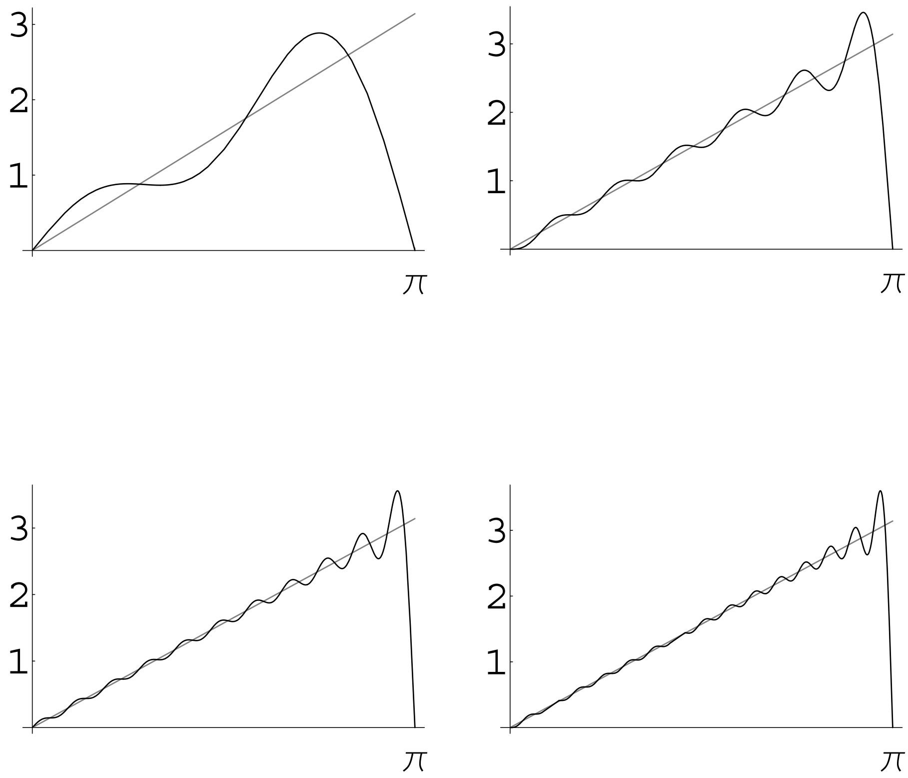


Figure 9-4 $f(x) = x$, $0 \le x \le \pi$, shown with the 3rd, 12th, 21st, and 30th partial sums of its Fourier sine series

SOLUTION: Because $f(x)$ is defined on $0 \le x \le 2$, $p = 2$. Hence,

$$c_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \frac{2}{n\pi}\left(-2\cos\frac{n\pi}{2} + \cos n\pi + 1\right).$$

```
In[1843]:= c[n_] = ∫_0^1 Sin[nπx/2] dx - ∫_1^2 Sin[nπx/2] dx
Out[1843]= 2/(n π) - 4 Cos[n π/2]/(n π) + 2 Cos[n π]/(n π)
```

We use `Table` to calculate a few of the c_n's.

```
In[1844]:= Table[{n, c_n}, {n, 1, 15}]//TableForm
```

Out[1844]=

1	0
2	$\frac{4}{\pi}$
3	0
4	0
5	0
6	$\frac{4}{3\pi}$
7	0
8	0
9	0
10	$\frac{4}{5\pi}$
11	0
12	0
13	0
14	$\frac{4}{7\pi}$
15	0

As we can see, most of the coefficients are zero. In fact, only those c_n's where n is an odd multiple of 2 yield a nonzero value. For example, $c_6 = c_{2\cdot 3} = \frac{2}{6\pi} \cdot 4 = \frac{4}{3\pi}$, $c_{10} = c_{2\cdot 5} = \frac{2}{10\pi} \cdot 4 = \frac{4}{5\pi}, \ldots,$

$$c_{2(2n-1)} = \frac{4}{(2n-1)\pi}, \; n = 1, 2, \ldots$$

so we have the series

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin \frac{2(2n-1)\pi x}{2} = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1)\pi x$$
$$= \frac{4}{\pi} \sin \pi x + \frac{4}{3\pi} \sin 3\pi x + \frac{4}{5\pi} \sin 5\pi x + \cdots .$$

As in Example 9.2.1, we graph $f(x)$ with several partial sums of the Fourier sine series in Figure 9.2.1.

```
In[1845]:= Clear[f, g]

           g[x_] := 1/; 0 ≤ x < 1

           g[x_] := -1/; 1 ≤ x ≤ 2

In[1846]:= f[k_] := f[k] = f[k - 1] + 4 Sin[(2k - 1) πx]/((2k - 1) π)

           f[1] = 4 Sin[πx]/π;

In[1847]:= Table[{n, f[n]}, {n, 1, 5}]//TableForm
```

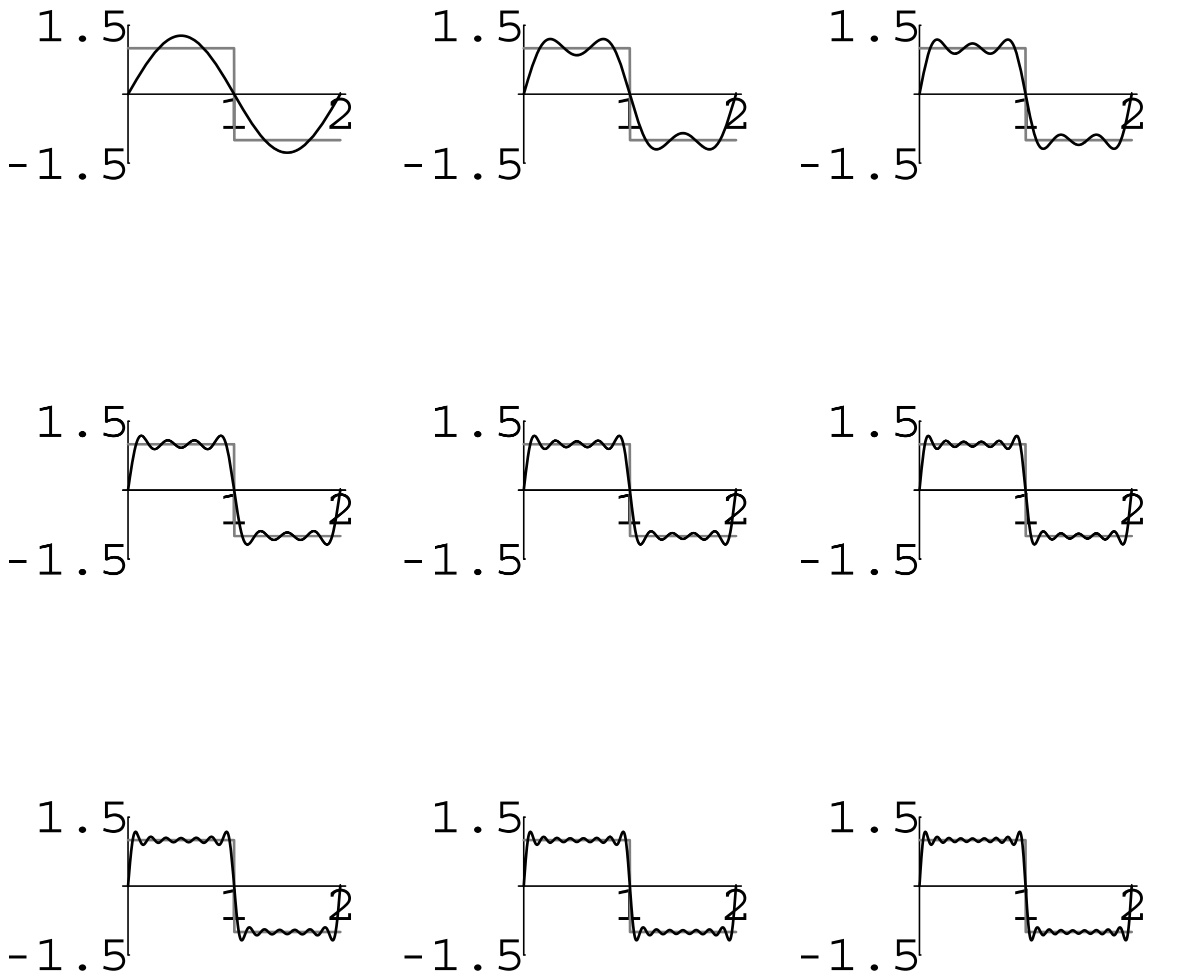


Figure 9-5 At the jump discontinuity at $x = 1$, the Fourier sine series converges to $\frac{1}{2}\left(\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right) = \frac{1}{2}(1-1) = 0$

```
Out[1847]=
1   4 Sin[π x] / π
2   4 Sin[π x] / π + 4 Sin[3 π x] / (3 π)
3   4 Sin[π x] / π + 4 Sin[3 π x] / (3 π) + 4 Sin[5 π x] / (5 π)
4   4 Sin[π x] / π + 4 Sin[3 π x] / (3 π) + 4 Sin[5 π x] / (5 π)
      + 4 Sin[7 π x] / (7 π)
5   4 Sin[π x] / π + 4 Sin[3 π x] / (3 π) + 4 Sin[5 π x] / (5 π)
      + 4 Sin[7 π x] / (7 π) + 4 Sin[9 π x] / (9 π)
```

Notice that with a large number of terms the approximation is quite good at values of x where $f(x)$ is continuous.

```
In[1848]:= somegraphs =
             Table[Plot[{g[x], f[n]}, {x, 0, 2},
               PlotStyle → {GrayLevel[0.5],
               GrayLevel[0]},
               DisplayFunction → Identity,
               PlotRange → {-1.5, 1.5}, Ticks → {{1, 2},
               {-1.5, 1.5}}], {n, 1, 9}];

            toshow = Partition[somegraphs, 3];

            Show[GraphicsArray[toshow], AspectRatio → 1]
```

The behavior of the series near points of discontinuity (in that the approximation overshoots the function) is called the **Gibbs phenomenon**. The approximation continues to "miss" the function even though more and more terms from the series are used!

In `somegraphs`, we observe the graph of the error function `Abs[g[x]-f[n]]` for $n = 1, 2, \ldots, 9$. Notice that the error remains "large" at the points of discontinuity, $x = 0, 1, 2$, even for "large" values of n. See Figure 9.2.1.

```
In[1849]:= somegraphs =
             Table[Plot[Abs[g[x] - f[n]], {x, 0, 2},
               DisplayFunction → Identity,
               PlotRange → {0, 1}, Ticks → {{1, 2},
               {0, 1}}], {n, 1, 9}];

            toshow = Partition[somegraphs, 3];
            Show[GraphicsArray[toshow], AspectRatio → 1]
```

■

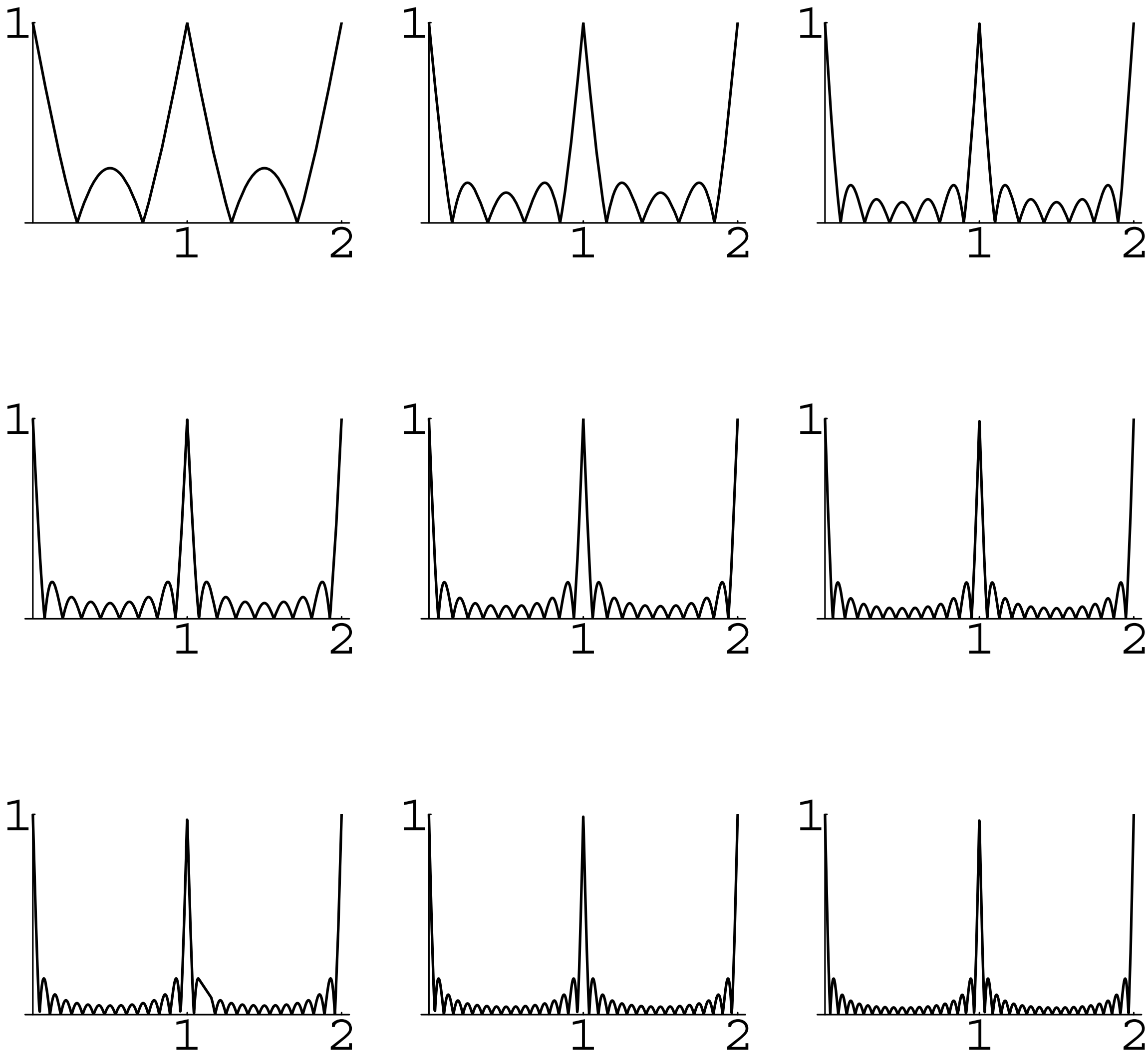


Figure 9-6 The Fourier sine series converges to $f(x)$ on the open intervals where $f(x)$ is continuous

9.2.2 Fourier Cosine Series

We solved this eigenvalue problem in Example 9.1.5.

Another important eigenvalue problem that has useful eigenfunctions is

$$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = y'(p) = 0 \end{cases}$$

which has eigenvalues and eigenfunctions given by

$$\lambda_n = \begin{cases} 0, \ n = 0 \\ (n\pi/p)^2, \ n = 1, \ 2, \ \ldots \end{cases} \quad \text{and} \quad y_n(x) = \begin{cases} 1, \ n = 0 \\ \cos(n\pi x/p), \ n = 1, \ 2, \ \ldots . \end{cases}$$

Therefore, for some functions $f(x)$, we can find a series expansion of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p}. \tag{9.9}$$

We call this expansion a **Fourier cosine series** where in the first term (associated with $\lambda_0 = 0$), the constant $\frac{1}{2}a_0$ is written in this form for convenience in finding the formula for the coefficients a_n, $n = 1, 2, \ldots$. We find these coefficients in a manner similar to that followed to find the coefficients in the Fourier sine series. Notice that in this case, the orthogonality condition is $\int_0^p \cos(n\pi x/p)\cos(m\pi x/p)\,dx = 0$, $m \neq n$. We use this condition by multiplying both sides of the series expansion by $\cos(m\pi x/p)$ and integrating from $x = 0$ to $x = p$. This yields

$$\int_0^p f(x)\cos\frac{m\pi x}{p}\,dx = \int_0^p \frac{1}{2}a_0 \cos\frac{m\pi x}{p}\,dx + \int_0^p \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p}\cos\frac{m\pi x}{p}\,dx.$$

Assuming that term-by-term integration is allowed,

$$\int_0^p f(x)\cos\frac{m\pi x}{p}\,dx = \int_0^p \frac{1}{2}a_0 \cos\frac{m\pi x}{p}\,dx + \sum_{n=1}^{\infty}\int_0^p a_n \cos\frac{n\pi x}{p}\cos\frac{m\pi x}{p}\,dx.$$

If $m = 0$, then this equation reduces to

$$\int_0^p f(x)\,dx = \int_0^p \frac{1}{2}a_0\,dx + \sum_{n=1}^{\infty}\int_0^p a_n \cos\frac{n\pi x}{p}\,dx$$

where $\int_0^p \cos(n\pi x/p)\,dx = 0$ and $\int_0^p \frac{1}{2}a_0\,dx = \frac{1}{2}pa_0$. Therefore, $\int_0^p f(x)\,dx = \frac{1}{2}pa_0$, so

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx. \tag{9.10}$$

If $m > 0$, we note that by the orthogonality property $\int_0^p \cos(n\pi x/p)\cos(m\pi x/p)\,dx = 0$, $m \neq n$. We also note that $\int_0^p \frac{1}{2}a_0 \cos(m\pi x/p)\,dx = 0$ and $\int_0^p \cos^2(n\pi x/p)\,dx = \frac{1}{2}p$. Hence, $\int_0^p f(x)\cos(n\pi x/p)\,dx = 0 + a_n \cdot \frac{1}{2}p$. Solving for a_n, we have

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx,\ n = 1,\ 2,\ \ldots. \tag{9.11}$$

Notice that equation (9.11) is valid for $n = 0$ because $\cos\dfrac{0 \cdot \pi x}{p} = 1$.

EXAMPLE 9.2.3: Find the Fourier cosine series for $f(x) = x$, $0 \le x \le \pi$.

SOLUTION: In this case, $p = \pi$. Hence,

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^\pi = \pi$$

and using integration by parts we find that

If n is an integer, $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$.

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\cos\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx \\ &= \frac{2}{\pi}\left\{\left[\frac{1}{n}x\sin nx\right]_0^\pi - \int_0^\pi \frac{1}{n}\sin nx\,dx\right\} = \frac{2}{\pi}\left[\frac{1}{n^2}\cos nx\right]_0^\pi \\ &= \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]. \end{aligned}$$

Notice that for even values of n, $(-1)^n - 1 = 0$. Therefore, $a_n = 0$ if n is even. On the other hand, if n is odd, $(-1)^n - 1 = -2$. Hence, $a_1 = -\frac{4}{\pi}$, $a_3 = -\frac{4}{9\pi}$, $a_5 = -\frac{4}{25\pi}$, ...,

$$a_{2n-1} = -\frac{4}{(2n-1)^2\pi},$$

so the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos\frac{n\pi x}{\pi} = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x.$$

We plot the function with several terms of the series in Figure 9-7. Compare these results to those obtained when approximating this function with a sine series. Which series yields the better approximation with the fewer number of terms?

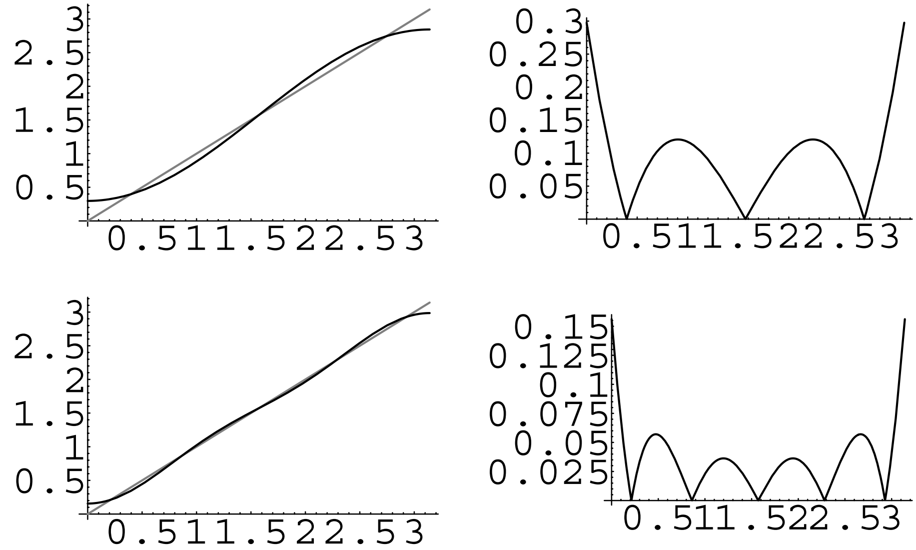


Figure 9-7 Partial sums of the Fourier cosine series shown with $f(x) = x$, $0 \le x \le \pi$ on the left and the absolute value of the difference between the two on the right

```
In[1850]:= Clear[f]

           f[n_] := f[n] = f[n - 1] - (4 Cos[(2n - 1) x])/(π (2n - 1)^2)

           f[0] = π/2;

In[1851]:= p1 = Plot[Evaluate[{x, f[1]}], {x, 0, π},
                PlotStyle → {GrayLevel[0.5], GrayLevel[0]},
                DisplayFunction → Identity];

           p2 = Plot[Evaluate[Abs[x - f[1]]], {x, 0, π},
                DisplayFunction → Identity];

           p3 = Plot[Evaluate[{x, f[2]}], {x, 0, π},
                PlotStyle → {GrayLevel[0.5], GrayLevel[0]},
                DisplayFunction → Identity];

           p4 = Plot[Evaluate[Abs[x - f[2]]], {x, 0, π},
                DisplayFunction → Identity];

           Show[GraphicsArray[{{p1, p2}, {p3, p4}}]]
```

■

9.3 Fourier Series

9.3.1 Fourier Series

The eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0, \ -p \le x \le p \\ y(-p) = y(p), \ y'(-p) = y'(p) \end{cases}$$

has eigenvalues

$$\lambda_n = \begin{cases} 0, \ n = 0 \\ (n\pi/p)^2, \ n = 1, \ 2 \ldots \end{cases}$$

and eigenfunctions

$$y_n(x) = \begin{cases} 1, \ n = 0 \\ a_n \cos \dfrac{n\pi x}{p} + b_n \sin \dfrac{n\pi x}{p}, \ n = 1, \ 2 \ldots \end{cases}$$

so we can consider a series made up of these functions. Hence, we write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right), \tag{9.12}$$

which is called a **Fourier series**. As was the case with Fourier sine and Fourier cosine series, we must determine the coefficients a_0, a_n $(n = 1, 2, \ldots)$, and b_n $(n = 1, 2, \ldots)$. Because we use a method similar to that used to find the coefficients in Section 9.2, we state the value of several integrals next.

$$\begin{gathered} \int_{-p}^{p} \cos \frac{n\pi x}{p}\, dx = 0 \\ \int_{-p}^{p} \sin \frac{n\pi x}{p}\, dx = 0 \\ \int_{-p}^{p} \cos \frac{m\pi x}{p} \sin \frac{n\pi x}{p}\, dx = 0 \\ \int_{-p}^{p} \cos \frac{m\pi x}{p} \cos \frac{n\pi x}{p}\, dx = \begin{cases} 0, \ m \ne n \\ p, \ m = n \end{cases} \\ \int_{-p}^{p} \sin \frac{m\pi x}{p} \sin \frac{n\pi x}{p}\, dx = \begin{cases} 0, \ m \ne n \\ p, \ m = n. \end{cases} \end{gathered} \tag{9.13}$$

We begin by finding a_0 and a_n ($n = 1, 2, \ldots$). Multiplying both sides of equation (9.12) by $\cos(m\pi x/p)$ and integrating from $x = -p$ to $x = p$ (because of the boundary conditions) yields

$$\begin{aligned}\int_{-p}^{p} f(x)\cos\frac{m\pi x}{p}\,dx &= \int_{-p}^{p}\frac{1}{2}a_0\cos\frac{m\pi x}{p}\,dx \\ &\quad+\int_{-p}^{p}\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p}\cos\frac{m\pi x}{p}+b_n\sin\frac{n\pi x}{p}\cos\frac{m\pi x}{p}\right)dx \\ &= \int_{-p}^{p}\frac{1}{2}a_0\cos\frac{m\pi x}{p}\,dx \\ &\quad+\sum_{n=1}^{\infty}\left(\int_{-p}^{p}a_n\cos\frac{n\pi x}{p}\cos\frac{m\pi x}{p}\,dx+\int_{-p}^{p}b_n\sin\frac{n\pi x}{p}\cos\frac{m\pi x}{p}\,dx\right).\end{aligned}$$

If $m = 0$, we notice that all of the integrals that we are summing have the value zero. Thus, this equation simplifies to

$$\begin{aligned}\int_{-p}^{p} f(x)\,dx &= \int_{-p}^{p}\frac{1}{2}a_0\,dx \\ \int_{-p}^{p} f(x)\,dx &= \frac{1}{2}a_0\cdot 2p \\ a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx.\end{aligned} \tag{9.14}$$

If $m \neq 0$, only one of the integrals on the right-hand side of the equation yields a value other than zero and this occurs with

$$\int_{-p}^{p}\cos\frac{m\pi x}{p}\cos\frac{n\pi x}{p}\,dx = \begin{cases}0, & m \neq n\\ p, & m = n\end{cases}$$

if $m = n$. Hence,

$$\begin{aligned}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx &= p\cdot a_n \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx,\; n = 1, 2, \ldots.\end{aligned} \tag{9.15}$$

We find b_n $(n = 1, 2, \dots)$ by multiplying the series by $\sin(m\pi x/p)$ and integrating from $x = -p$ to $x = p$. This yields

$$\begin{aligned}\int_{-p}^{p} f(x)\sin\frac{m\pi x}{p}\,dx &= \int_{-p}^{p}\frac{1}{2}a_0\sin\frac{m\pi x}{p}\,dx \\ &\quad + \int_{-p}^{p}\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p}\sin\frac{m\pi x}{p} + b_n\sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\right)dx \\ &= \int_{-p}^{p}\frac{1}{2}a_0\sin\frac{m\pi x}{p}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left(\int_{-p}^{p}a_n\cos\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx + \int_{-p}^{p}b_n\sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}\,dx\right).\end{aligned}$$

Again, we note that only one of the integrals on the right-hand side of the equation is not zero. In this case, we use

$$\int_{-p}^{p}\sin\frac{m\pi x}{p}\sin\frac{n\pi x}{p}\,dx = \begin{cases}0, & m \neq n \\ p, & m = n\end{cases}$$

to obtain

$$\begin{aligned}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx &= p\cdot b_n \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx,\ n = 1, 2, \dots.\end{aligned} \tag{9.16}$$

Definition 38 (Fourier Series). *Suppose that $y = f(x)$ is defined on $-p \le x \le p$. The **Fourier series** for $f(x)$ is*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right), \tag{9.17}$$

where

$$\begin{aligned}a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx,\ n = 1, 2, \dots, \text{ and} \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx,\ n = 1, 2, \dots.\end{aligned} \tag{9.18}$$

The following theorem tells us that the Fourier series for any function converges to the function except at points of discontinuity.

Theorem 34 (Convergence of Fourier Series). *Suppose that $f(x)$ and $f'(x)$ are piecewise continuous functions on $-p \le x \le p$. Then the Fourier series for $f(x)$ on $-p \le x \le p$ converges to $f(x)$ at every x where $f(x)$ is continuous.*

If $f(x)$ is discontinuous at $x = a$, the Fourier series converges to the average

$$\frac{1}{2}\left(\lim_{x\to a+} f(x) + \lim_{x\to a-} f(x)\right).$$

EXAMPLE 9.3.1: Find the Fourier series for $f(x) = \begin{cases} 1, & -2 \le x < 0 \\ 2, & 0 \le x < 2 \end{cases}$ where $f(x+4) = f(x)$.

SOLUTION: In this case, $p = 2$. First we find

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0} 1\cdot dx + \frac{1}{2}\int_{0}^{2} 2\cdot dx = 3,$$

```
In[1852] := a0 = 1/2 ∫_{-2}^{0} 1 dx + 1/2 ∫_{0}^{2} 2 dx
Out[1852]= 3
```

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{0}\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi x}{2}\,dx = 0,$$

```
In[1853] := a[n_] = 1/2 ∫_{-2}^{0} Cos[nπx/2] dx + 1/2 ∫_{0}^{2} 2 Cos[nπx/2] dx
Out[1853]= 3 Sin[n π]
           ----------
              n π
```

and

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{0}\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} 2\sin\frac{n\pi x}{2}\,dx$$
$$= \frac{1}{n\pi}(1-\cos n\pi) = \frac{1}{n\pi}\left(1-(-1)^n\right).$$

```
In[1854] := b[n_] = 1/2 ∫_{-2}^{0} Sin[nπx/2] dx + 1/2 ∫_{0}^{2} 2 Sin[nπx/2] dx//
            Simplify
Out[1854]= 1 - Cos[n π]
           ------------
               n π
```

Therefore, at the values of x for which $f(x)$ is continuous

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \left[1 - (-1)^n\right] \frac{1}{n\pi} \sin \frac{n\pi x}{2} = \frac{3}{2} + \frac{2}{\pi} \sin \frac{\pi x}{2} + \frac{2}{3\pi} \sin \frac{3\pi x}{2} + \frac{2}{5\pi} \sin \frac{5\pi x}{2} + \cdots .$$

We now graph $f(x)$ with several partial sums of the Fourier series. First, we define $p_k(x)$ to be the kth partial sum of the Fourier series and then $f(x)$.

```
In[1855]:= p_k_[x_] := 3/2 + Sum[(a_n Cos[nπx/2] + b_n Sin[nπx/2]), {n, 1, k}]

In[1856]:= f[x_] := f[x - 4]/; x ≥ 2

           f[x_] := 1/; -2 ≤ x < 0

           f[x_] := 2/; 0 ≤ x < 2
```

Given k, the function `comp` graphs $f(x)$ and $p_k(x)$ on the interval $[-2, 6]$, which corresponds to two periods of $f(x)$, as well as the error, $|f(x) - p_k(x)|$. The resulting two graphics objects are displayed side-by-side.

```
In[1857]:= comp[k_] := Module[{p1, p2},
             p1 = Plot[{f[x], p_k[x]}, {x, -2, 6},
                   PlotRange → {0, 3},
                   PlotStyle → {GrayLevel[0.5],
                   GrayLevel[0]},
                   DisplayFunction → Identity];
             p2 = Plot[Abs[f[x] - p_k[x]], {x, -2, 6},
                   PlotRange → All,
                   DisplayFunction → Identity];
           Show[GraphicsArray[{p1, p2}]]]
```

We then use `Map` to generate these graphs for $k = 3, 5, 11$, and 15. See Figure 9-8.

```
In[1858]:= Map[comp, {3, 5, 11, 15}]
```

The graphs show that if we extend the graph of $f(x)$ over more periods, then the approximation by the Fourier series carries over to those intervals.

■

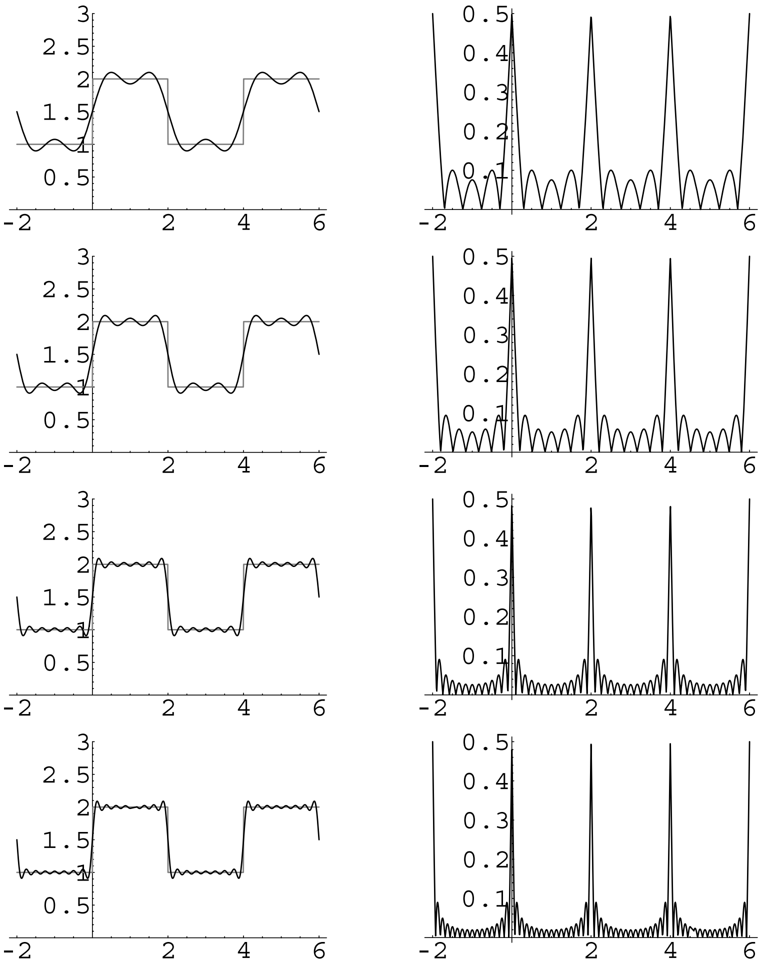


Figure 9-8 The Fourier series converges to $f(x)$ at points of continuity and to the average of the left and right-hand limits at points of discontinuity

---

EXAMPLE 9.3.2: Find the Fourier series for $f(x) = \begin{cases} 0, & -1 \le x < 0 \\ \sin \pi x, & 0 \le x < 1 \end{cases}$ where $f(x+2) = f(x)$.

SOLUTION: In this case, $p = 1$, so $a_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 \sin \pi x\,dx = 2/\pi$ and

```
In[1859]:= a0 = ∫_0^1 Sin[πx]dx
Out[1859]= 2/π
```

$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_0^1 \sin \pi x \cos n\pi x\,dx$. The value of this integral depends on the value of n. If $n = 1$, we have $a_1 = \int_0^1 \sin \pi x \cos \pi x\,dx = \frac{1}{2}\int_0^1 \sin 2\pi x\,dx = 0$, where we use the identity $\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$.

```
In[1860]:= a1 = ∫_0^1 Sin[πx] Cos[πx]dx
Out[1860]= 0
```

If $n \neq 1$, we use the identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$ to obtain

$$\begin{aligned}
a_n &= \frac{1}{2}\int_0^1 [\sin(1-n)\pi x + \sin(1+n)\pi x]\,dx \\
&= -\frac{1}{2}\left[\frac{\cos(1-n)\pi x}{(1-n)\pi} + \frac{\cos(1+n)\pi x}{(1+n)\pi}\right]_0^1 \\
&= -\frac{1}{2}\left\{\left[\frac{\cos(1-n)\pi}{(1-n)\pi} + \frac{\cos(1+n)\pi}{(1+n)\pi}\right] - \left[\frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi}\right]\right\}, \\
&\quad n = 2, 3, \ldots.
\end{aligned}$$

```
In[1861]:= an_ = ∫_0^1 Sin[πx] Cos[nπx]dx
Out[1861]= -1/(2 (-1 + n) π) + 1/(2 (1 + n) π) + Cos[(-1 + n) π]/(2 (-1 + n) π)
            - Cos[(1 + n) π]/(2 (1 + n) π)
```

Notice that if n is odd, both $1-n$ and $1+n$ are even. Hence, $\cos(1-n)\pi x = \cos(1+n)\pi x = 1$, so

$$a_n = -\frac{1}{2}\left\{\left[\frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi}\right] - \left[\frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi}\right]\right\} = 0$$

if n is odd. On the other hand, if n is even, $1-n$ and $1+n$ are odd. Therefore, $\cos(1-n)\pi x = \cos(1+n)\pi x = -1$, so

$$\begin{aligned}
a_n &= -\frac{1}{2}\left\{\left[\frac{-1}{(1-n)\pi} + \frac{-1}{(1+n)\pi}\right] - \left[\frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi}\right]\right\} \\
&= \frac{1}{(1-n)\pi} + \frac{1}{(1+n)\pi} = \frac{2}{(1-n)(1+n)\pi} = -\frac{2}{(n-1)(n+1)\pi}
\end{aligned}$$

if n is even. We confirm this observation by computing several coefficients.

```
In[1862]:= Table[{n, a_n}, {n, 1, 10}]//TableForm
```

Out[1862]=

1	0
2	$-\frac{2}{3\pi}$
3	0
4	$-\frac{2}{15\pi}$
5	0
6	$-\frac{2}{35\pi}$
7	0
8	$-\frac{2}{63\pi}$
9	0
10	$-\frac{2}{99\pi}$

Putting this information together, we can write the coefficients as

$$a_{2n} = -\frac{2}{(2n-1)(2n+1)\pi}, \quad n = 1, 2, \ldots .$$

Similarly, $b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx = \int_0^1 \sin \pi x \sin n\pi x \, dx$, so if $n = 1$,

$$b_1 = \int_0^1 \sin^2 \pi x \, dx = \frac{1}{2} \int_0^1 (1 - \cos 2\pi x) dx = \frac{1}{2} \left[x - \frac{1}{2\pi} \sin 2\pi x \right]_0^1 = \frac{1}{2}.$$

```
In[1863]:= b_1 = ∫_0^1 Sin[πx] Sin[πx] dx
```

Out[1863]= $\frac{1}{2}$

If $n \neq 1$, we use $\sin\alpha \sin\beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$. Hence,

$$b_n = \frac{1}{2} \int_0^1 [\cos(1-n)\pi x - \cos(1+n)\pi x] \, dx = \frac{1}{2} \left[\frac{\sin(1-n)\pi x}{(1-n)\pi} - \frac{\sin(1+n)\pi x}{(1+n)\pi} \right]_0^1$$
$$= 0, \quad n = 2, 3, \ldots .$$

```
In[1864]:= b_n_ = ∫_0^1 Sin[πx] Sin[nπx] dx
```

Out[1864]= $\frac{\text{Sin}[(-1+n)\pi]}{2(-1+n)\pi} - \frac{\text{Sin}[(1+n)\pi]}{2(1+n)\pi}$

Therefore, we write the Fourier series as

$$f(x) = \frac{1}{\pi} + \frac{1}{2} \sin \pi x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos 2n\pi x.$$

We graph $f(x)$ along with several approximations using this series in the same way as in previous examples. Let

$$p_k(x) = \frac{1}{\pi} + \frac{1}{2} \sin \pi x - \frac{2}{\pi} \sum_{n=1}^{k} \frac{1}{(2n-1)(2n+1)} \cos 2n\pi x$$

denote the kth partial sum of the Fourier series. Note that

$$p_k(x) = \frac{1}{\pi} + \frac{1}{2}\sin \pi x - \frac{2}{\pi}\sum_{n=1}^{k-1}\frac{1}{(2n-1)(2n+1)}\cos 2n\pi x$$
$$-\frac{2}{\pi}\frac{1}{(2k-1)(2k+1)}\cos 2k\pi x = p_{k-1}(x) - \frac{2}{\pi}\frac{1}{(2k-1)(2k+1)}\cos 2k\pi x.$$

Thus, to calculate the kth partial sum of the Fourier series, we need only subtract $\frac{2}{\pi}\frac{1}{(2k-1)(2k+1)}\cos 2k\pi x$ from the $(k-1)$st partial sum: we need not recompute all k terms of the kth partial sum if we know the $(k-1)$st partial sum. Using this observation, we define the recursively defined function p to return the kth partial sum of the series.

```
In[1865]:= p[k_] := p[k] = p[k - 1] - 2 Cos[2kπx]/(π ((2k - 1) (1 + 2k)))

            p[0] = 1/π + Sin[πx]/2;
```

We then define $f(x)$.

```
In[1866]:= Clear[f]

            f[x_] := 0/; -1 ≤ x < 0

            f[x_] := Sin[πx]/; 0 ≤ x < 1

            f[x_] := f[x - 2]/; x ≥ 1
```

We graph $f(x)$ along with the second, sixth, and tenth partial sums of the series in Figure 9-9.

```
In[1867]:= Clear[graph]

            graph[k_] := Plot[{f[x],p[k]}, {x, -1, 3},
                           PlotStyle- > {GrayLevel[0.5],
                           GrayLevel[0]},
                           DisplayFunction- > Identity];

In[1868]:= somegraphs = Map[graph, {2, 6, 10}];

            Show[GraphicsArray[somegraphs]]
```

The corresponding errors are graphed in Figure 9-10.

```
In[1869]:= error[k_] := Plot[Abs[f[x] - p[k]], {x, -1, 3},
                           DisplayFunction → Identity,
                           PlotRange → All];
```

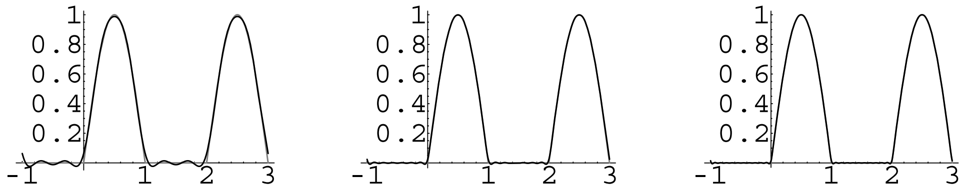


Figure 9-9 The graphs of the 6th and 10th partial sums are virtually indistinguishable from the graph of $f(x)$

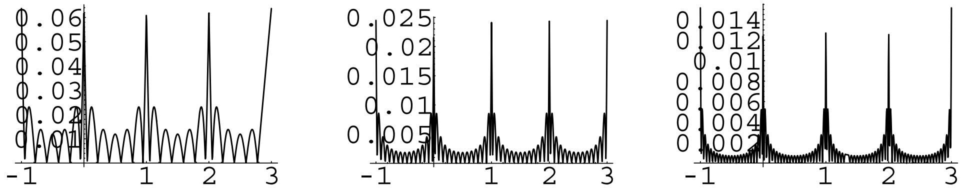


Figure 9-10 The Fourier series converges to $f(x)$ on $(-\infty, \infty)$

```
In[1870]:= somegraphs = Map[error, {2, 6, 10}];

           Show[GraphicsArray[somegraphs]]
```

■

9.3.2 Even, Odd, and Periodic Extensions

In the discussion so far in this section, we have assumed that $f(x)$ was defined on the interval $-p < x < p$. However, this is not always the case. Sometimes, we must take a function that is defined on the interval $0 < x < p$ and represent it in terms of trigonometric functions. Three ways of accomplishing this task is to extend $f(x)$ to obtain (a) an **even** function on $-p < x < p$; (b) an **odd** function on $-p < x < p$; (c) a **periodic** function on $-p < x < p$.

We can notice some interesting properties associated with the Fourier series in each of these three cases by noting the properties of even and odd functions. If $f(x)$ is an even function and $g(x)$ is an odd function, then the product $(f \cdot g)(x) = f(x)g(x)$ is an odd function. Similarly, if $f(x)$ is an even function and $g(x)$ is an even function, then $(f \cdot g)(x)$ is an even function, and if $f(x)$ is an odd function and $g(x)$ is an odd function, then $(f \cdot g)(x)$ is an even function. Recall from integral calculus that if $f(x)$ is odd on $-p \le x \le p$, then $\int_{-p}^{p} f(x)\,dx = 0$ while if $g(x)$ is even on $-p \le x \le p$, then $\int_{-p}^{p} g(x)\,dx = 2\int_{0}^{p} g(x)\,dx$. These properties are useful in determining the coefficients in the Fourier series for the even, odd, and periodic extensions of a function, because $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$ are even and odd periodic functions, respectively, on $-p \le x \le p$.

1. The **even extension** $f_{\text{even}}(x)$ of $f(x)$ is an even function. Therefore,

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f_{\text{even}}(x)\,dx = \frac{2}{p}\int_0^p f(x)\,dx \\ a_n &= \frac{1}{p}\int_{-p}^{p} f_{\text{even}}(x)\cos\frac{n\pi x}{p}\,dx = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx,\ n = 1,\ 2,\ \ldots \\ b_n &= \frac{1}{p}\int_{-p}^{p} f_{\text{even}}(x)\sin\frac{n\pi x}{p}\,dx = 0,\ n = 1,\ 2,\ \ldots. \end{aligned} \tag{9.19}$$

2. The **odd extension** $f_{\text{odd}}(x)$ of $f(x)$ is an odd function, so

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f_{\text{odd}}(x)\,dx = 0 \\ a_n &= \frac{1}{p}\int_{-p}^{p} f_{\text{even}}(x)\cos\frac{n\pi x}{p}\,dx = 0,\ n = 1,\ 2,\ \ldots \\ b_n &= \frac{1}{p}\int_{-p}^{p} f_{\text{even}}(x)\sin\frac{n\pi x}{p}\,dx = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx,\ n = 1,\ 2,\ \ldots. \end{aligned} \tag{9.20}$$

3. The **periodic extension** $f_p(x)$ has period p. Because half of the period is $p/2$,

$$\begin{aligned} a_0 &= \frac{2}{p}\int_0^p f(x)\,dx \\ a_n &= \frac{2}{p}\int_0^p f(x)\cos\frac{2n\pi x}{p}\,dx,\ n = 1,\ 2,\ \ldots \\ b_n &= \frac{2}{p}\int_0^p f(x)\sin\frac{2n\pi x}{p}\,dx,\ n = 1,\ 2,\ \ldots. \end{aligned} \tag{9.21}$$

EXAMPLE 9.3.3: Let $f(x) = x$ on $(0, 1)$. Find the Fourier series for (a) the even extension of $f(x)$; (b) the odd extension of $f(x)$; (c) the periodic extension of $f(x)$.

SOLUTION: (a) Here $p = 1$, so $a_0 = 2\int_0^1 x\,dx = 1$,

$$a_n = 2\int_0^1 x\cos n\pi x\,dx = \frac{2}{n^2\pi^2}(\cos n\pi - 1) = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right],\ n = 1,\ 2,\ \ldots$$

and $b_n = 0, n = 1, 2, \ldots.$

```
In[1871]:= Remove[a]
           a0 = 2 ∫_0^1 x dx
Out[1871]= 1
```

```
In[1872]:= a[n_] = 2 ∫_0^1 x Cos[nπx] dx
Out[1872]= 2 (-1/(n^2 π^2) + Cos[n π]/(n^2 π^2) + Sin[n π]/(n π))
```

Because $a_n = 0$ if n is even, we can represent the coefficients with odd subscripts as $a_{2n-1} = -\dfrac{4}{(2n-1)^2\pi^2}$. Therefore, the Fourier cosine series is

$$f_{\text{even}}(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2} \cos(2n-1)\pi x.$$

We graph the even extension with several terms of the Fourier cosine series by first defining $f(x)$ to be the even extension of $f(x)$ on $(0, 1)$

```
In[1873]:= Clear[f, p]
           f[x_] := -x /; -1 ≤ x < 0
           f[x_] := x /; 0 ≤ x < 1
           f[x_] := f[x - 2] /; x ≥ 1
           f[x_] := f[x + 2] /; x ≤ -1
```

and $p_k(x) = \frac{1}{2} - \sum_{n=1}^{k} \frac{4}{(2n-1)^2\pi^2} \cos(2n-1)\pi x.$

```
In[1874]:= p[k_][x_] := a0/2 + Sum[a[2n-1] Cos[(2n - 1) π x], {n, 1, k}]
```

We then graph $f(x)$ together with $p_1(x)$ and $p_5(x)$ in Figure 9-11.

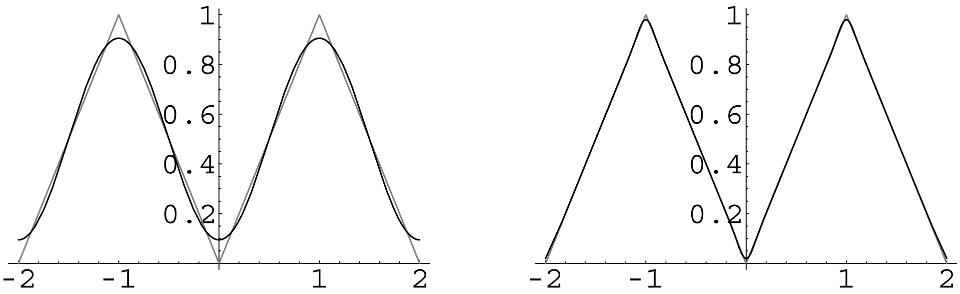


Figure 9-11 Even extension

```
In[1875]:= p1 = Plot[{f[x], p1[x]}, {x, -2, 2},
                PlotStyle → {GrayLevel[0.5],
                GrayLevel[0]},
                DisplayFunction → Identity];

            p2 = Plot[{f[x], p5[x]}, {x, -2, 2},
                PlotStyle → {GrayLevel[0.5],
                GrayLevel[0]},
                DisplayFunction → Identity,
                PlotRange → All];

            Show[GraphicsArray[{p1, p2}]]
```

(b) For the odd extension $f_{\text{odd}}(x)$, we note that $a_0 = 0$, $a_n = 0$, $n = 1, 2, \ldots$, and $b_n = 2\int_0^1 x \sin\frac{n\pi x}{p}, dx = -\frac{2}{n\pi}\cos n\pi = (-1)^{n+1}\frac{2}{n\pi}$, $n = 1, 2, \ldots$.

```
In[1876]:= b_n_ = 2 ∫_0^1 x Sin[n π x] dx

Out[1876]= 2 (- Cos[n π]/(n π) + Sin[n π]/(n^2 π^2))
```

Hence, the Fourier sine series is

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n\pi}\sin n\pi x.$$

We graph the odd extension along with several terms of the Fourier sine series in the same manner as in (a). See Figure 9-12.

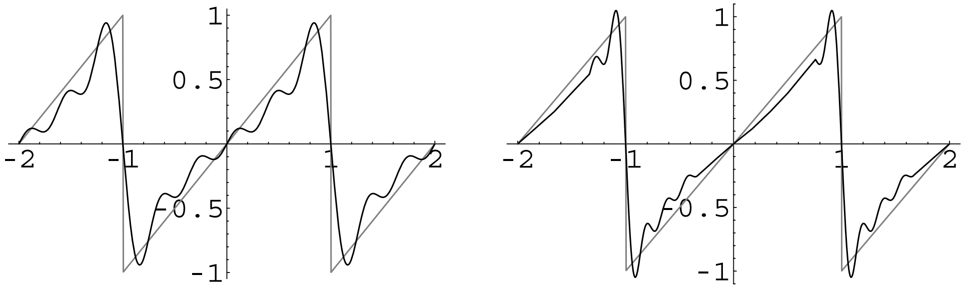


Figure 9-12 Odd extension

```
In[1877]:= Clear[f]

           f[x_] := x/; -1 ≤ x < 1

           f[x_] := f[x - 2]/; x ≥ 1

           f[x_] := f[x + 2]/; x ≤ -1
```

$$In[1878]:= p_{k\_}[x\_] := \sum_{n=1}^{k} b_n \text{ Sin}[n\pi x]$$

```
In[1879]:= p1 = Plot[{f[x], p5[x]}, {x, -2, 2},
                 PlotStyle → {GrayLevel[0.5],
                 GrayLevel[0]},
                 DisplayFunction → Identity];

           p2 = Plot[{f[x], p10[x]}, {x, -2, 2},
                 PlotStyle → {GrayLevel[0.5],
                 GrayLevel[0]},
                 DisplayFunction → Identity];

           Show[GraphicsArray[{p1, p2}]]
```

(c) The periodic extension has period $2p = 1$, so $p = 1/2$. Thus,

$$a_0 = \frac{1}{1/2}\int_0^1 x\,dx = 2\int_0^1 x\,dx = 1$$

$$a_n = 2\int_0^1 x\cos 2n\pi x\,dx = \frac{1}{2}\frac{\cos 2n\pi + 2n\pi\sin 2n\pi}{n^2\pi^2} - \frac{1}{2n^2\pi^2} = 0,\ n = 1,\ 2,\ \ldots \text{ and}$$

$$b_n = 2\int_0^1 x\sin 2n\pi x\,dx = -\frac{1}{2}\frac{-\sin 2n\pi + 2n\pi\cos 2n\pi}{n^2\pi^2} = -\frac{2n\pi}{2n^2\pi^2} = -\frac{1}{n\pi},\ n = 1,\ 2,\ \ldots.$$

```
In[1880]:= Remove[a, b]
```

$$a_0 = 2\int_0^1 x\,dx$$

```
Out[1880]= 1
```

In[1881]:= $a_{n\_} = 2\int_0^1 x\,\mathrm{Cos}[2n\pi x]\,dx$

Out[1881]= $2\left(-\frac{1}{4\,n^2\,\pi^2} + \frac{\mathrm{Cos}[2\,n\,\pi]}{4\,n^2\,\pi^2} + \frac{\mathrm{Sin}[2\,n\,\pi]}{2\,n\,\pi}\right)$

```
In[1882]:= Table[{n, a_n}, {n, 1, 5}]
Out[1882]= {{1, 0}, {2, 0}, {3, 0}, {4, 0}, {5, 0}}
```

In[1883]:= $b_{n\_} = 2\int_0^1 x\,\mathrm{Sin}[2n\pi x]\,dx$

Out[1883]= $2\left(-\frac{\mathrm{Cos}[2\,n\,\pi]}{2\,n\,\pi} + \frac{\mathrm{Sin}[2\,n\,\pi]}{4\,n^2\,\pi^2}\right)$

In[1884]:= Table[{n, b_n}, {n, 1, 5}]

Out[1884]= $\left\{\left\{1, -\frac{1}{\pi}\right\}, \left\{2, -\frac{1}{2\,\pi}\right\}, \left\{3, -\frac{1}{3\,\pi}\right\}, \left\{4, -\frac{1}{4\,\pi}\right\}, \left\{5, -\frac{1}{5\,\pi}\right\}\right\}$

Hence, the Fourier series for the periodic extension is

$$f_p(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin 2n\pi x.$$

We graph the periodic extension with several terms of the Fourier series in the same way as in (a) and (b). See Figure 9-13.

```
In[1885]:= Clear[f]

           f[x_] := x + 1/; -1 ≤ x < 0

           f[x_] := x/; 0 ≤ x < 1

           f[x_] := f[x - 2]/; x ≥ 1

           f[x_] := f[x + 2]/; x ≤ -1
```

In[1886]:= $p_{k\_}[x\_] := \frac{a_0}{2} + \sum_{n=1}^{k} b_n\,\mathrm{Sin}[2n\pi x]$

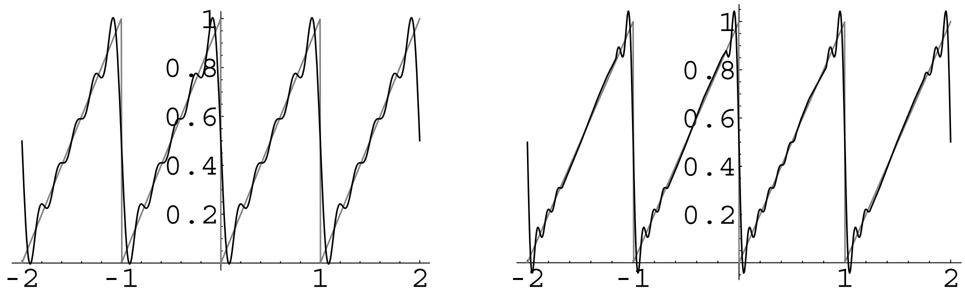


Figure 9-13 Periodic extension

```
In[1887]:= p1 = Plot[{f[x], p5[x]}, {x, -2, 2},
                PlotStyle → {GrayLevel[0.5],
                GrayLevel[0]},
                DisplayFunction → Identity];

            p2 = Plot[{f[x], p10[x]}, {x, -2, 2},
                PlotStyle → {GrayLevel[0.5],
                GrayLevel[0]},
                DisplayFunction → Identity];

            Show[GraphicsArray[{p1, p2}]]
```

■

9.3.3 Differentiation and Integration of Fourier Series

Definition 39 (Piecewise Smooth). *A function $f(x)$, $-p < x < p$ is **piecewise smooth** if $f(x)$ and all of its derivatives are piecewise continuous.*

Theorem 35 (Term-By-Term Differentiation). *Let $f(x)$, $-p < x < p$, be a continuous piecewise smooth function with Fourier series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right).$$

Then, $f'(x)$, $-p < x < p$, has Fourier series

$$\sum_{n=1}^{\infty}\frac{n\pi}{p}\left(-a_n \sin\frac{n\pi x}{p} + b_n \cos\frac{n\pi x}{p}\right).$$

In other words, we differentiate the Fourier series for $f(x)$ term-by-term to obtain the Fourier series for $f'(x)$.

Theorem 36 (Term-By-Term Integration). *Let $f(x)$, $-p < x < p$, be a continuous piecewise smooth function with Fourier series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right).$$

Then, the Fourier series of an antiderivative of $f(x)$ can be found by integrating the Fourier series of $f(x)$ term-by-term.

EXAMPLE 9.3.4: Use the Fourier series for $f(x) = \frac{1}{12}x\left(\pi^2 - x^2\right)$, $-\pi < x < \pi$ to show how term-by-term differentiation and term-by-term integration can be used to find the Fourier series of $g(x) = \frac{1}{12}\pi^2 - \frac{1}{4}x^2$, $-\pi < x < \pi$, and $h(x) = \frac{1}{24}\pi^2x^2\left(1 - \frac{1}{2}x^2\right)$, $-\pi < x < \pi$.

SOLUTION: After defining $f(x)$ and the substitutions in `rule` to simplify our results, we calculate a_0, a_n, and b_n. (Because $f(x)$ is an odd function, $a_n = 0$, $n \geq 0$.)

```
In[1888]:= Clear[f]

            f[x_] = 1/12 x (π^2 - x^2);

In[1889]:= rule = {Sin[nπ] → 0, Cos[nπ] → (-1)^n};

            a0 = ∫_{-π}^{π} f[x] dx / π /.rule
Out[1889]= 0

In[1890]:= a_{n_} = ∫_{-π}^{π} f[x] Cos[nx] dx / π /.rule
Out[1890]= 0

In[1891]:= b_{n_} = ∫_{-π}^{π} f[x] Sin[nx] dx / π /.rule
Out[1891]= -(-1)^n / n^3
```

We define the nth term of $\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right)$ in `fs[n,x]` and the finite sum $\frac{1}{2}a_0 + \sum_{n=1}^{k}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right)$ in `fourier[k]`.

```
In[1892]:= fs[n_] = a_n Cos[nx] + b_n Sin[nx];

In[1893]:= fourier[k_] := fourier[k] = fourier[k - 1]
           +fs[k];
```

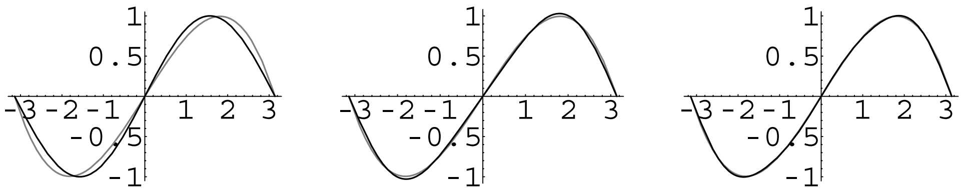


Figure 9-14 The graph of the third partial sum is indistinguishable from the graph of $f(x)$

```
In[1894]:= fourier[0] = a0/2;
```

We see how quickly the Fourier series converges to $f(x)$ by graphing together with `fourier[1]`, `fourier[2]`, and `fourier[3]`. See Figure 9-14.

```
In[1895]:= somegraphs =
             Table[Plot[{f[x], fourier[k]}, {x, -π, π},
               PlotStyle → {GrayLevel[0.5],
               GrayLevel[0]},
               DisplayFunction → Identity], {k, 1, 3}];

           Show[GraphicsArray[somegraphs]]
```

Notice that $g(x) = \frac{1}{12}\pi^2 - \frac{1}{4}x^2$, $-\pi < x < \pi$, is the derivative of $f(x)$, $-\pi < x < \pi$. Of course, we could compute the Fourier series of $f'(x)$, $-\pi < x < \pi$, directly by applying the integral formulas with $g(x)$ to find the Fourier series coefficients. However, the objective here is to illustrate how term-by-term differentiation of the Fourier series for $f(x)$, $-\pi < x < \pi$, gives us the Fourier series for $f'(x) = g(x)$, $-\pi < x < \pi$. We calculate the derivative of $f(x)$ in `df[x]` in order to make graphical comparisons. In `dfs[n]`, we determine the derivative of the nth term of the Fourier series for $f(x)$, $-\pi < x < \pi$, found above, and in `dfourier[k]`, we calculate the kth partial sum of the Fourier series for $f'(x)$, $-\pi < x < \pi$.

Notice that this series does not include a constant term because the derivative of $\frac{1}{2}a_0$ is zero.

```
In[1896]:= df[x_] = D[f[x], x]//Simplify
```

Out[1896]= $\frac{1}{12}(\pi^2 - 3x^2)$

```
In[1897]:= dfs[n_] = D[ fs[n], x]
```

Out[1897]= $-\frac{(-1)^n \text{Cos}[n x]}{n^2}$

```
In[1898]:= dfourier[k_] := dfourier[k] = dfourier[k - 1]
             +dfs[k]; dfourier[0] = 0;
```

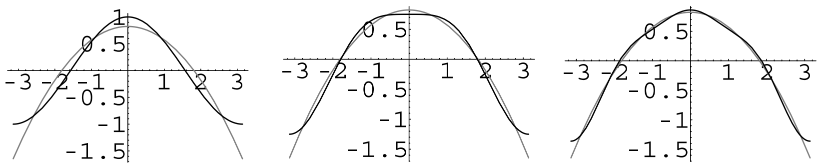


Figure 9-15 Fourier series can be differentiated term-by-term

Next, we graph $f'(x)$, $-\pi < x < \pi$, simultaneously with `dfourier[1]`, `dfourier[2]`, and `dfourier[3]` in Figure 9-15. Again, the convergence of the Fourier series approximations to $f'(x)$, $-\pi < x < \pi$, is quick.

```
In[1899]:= somegraphs =
             Table[Plot[{df[x],dfourier[k]},{x,-π,π},
               PlotStyle→{GrayLevel[0.5],
               GrayLevel[0]},
               DisplayFunction→Identity],{k,1,3}];

           Show[GraphicsArray[somegraphs]]
```

Notice that $h(x)$, $-\pi < x < \pi$, is an antiderivative of $f(x)$, $-\pi < x < \pi$. We calculate this antiderivative in `intf[x]`. Of course, this is the antiderivative of $f(x)$ with zero constant of integration because Mathematica does not include an integration constant. When we integrate the terms of the Fourier series of $f(x)$, $-\pi < x < \pi$, a constant term is not included. However, the Fourier series of the even function $h(x) = \frac{1}{24}\pi^2x^2\left(1-\frac{1}{2}x^2\right)$, $-\pi < x < \pi$ should include the constant term $\frac{1}{2}\tilde{a}_0$. We calculate the value of $\tilde{a}_0$ in `inta[0]` with the integral formula $\frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\,dx$.

In[1900]:= $\textbf{intf[x\_]} = \int \textbf{f[x]}\,d\textbf{x}$

Out[1900]= $\frac{1}{12}\left(\frac{\pi^2 x^2}{2} - \frac{x^4}{4}\right)$

In[1901]:= $\textbf{inta}_0 = \frac{\int_{-\pi}^{\pi}\textbf{intf[x]}\,d\textbf{x}}{\pi}$

Out[1901]= $\frac{7\pi^4}{360}$

In `intfs[n,x]`, we integrate the nth term of the Fourier series of $f(x)$, $-\pi < x < \pi$, found above to determine the coefficients of $\cos nx$ and $\sin nx$ in the Fourier series of $h(x)$, $-\pi < x < \pi$. In `intfourier[k,x]`, we determine the sum of the first k terms of the Fourier series obtained by adding $\frac{1}{2}\tilde{a}_0$ to the expression obtained through term-by-term integration of the Fourier series of $f(x)$, $-\pi < x < \pi$.

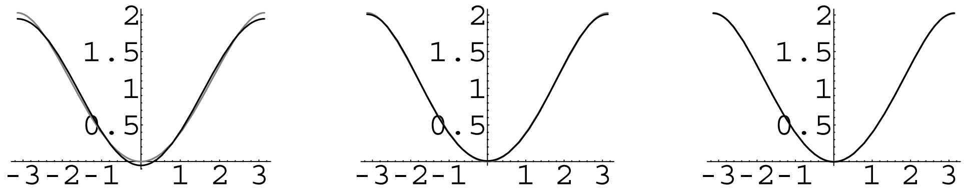


Figure 9-16 Fourier series can be integrated term-by-term

```
In[1902]:= intfs[n_] = ∫ fs[n] dx

            (-1)^n Cos[n x]
Out[1902]= ----------------
                 n^4

In[1903]:= intfourier[k_] := intfourier[k]
                    = intfourier[k - 1] + intfs[k];

           intfourier[0] = inta0/2;
```

By graphing $h(x)$, $-\pi < x < \pi$, simultaneously with the approximation in `intfourier[k,x]` for $k = 1, 2$, and 3, we see how the graphs of Fourier series approximations obtained through term-by-term integration converge to the graph of $h(x)$, $-\pi < x < \pi$, in `intgraph1`, `intgraph2`, and `intgraph3`. See Figure 9-16.

```
In[1904]:= somegraphs =
             Table[Plot[{intf[x], intfourier[k]},
               {x, -π, π}, PlotStyle → {GrayLevel[0.5],
               GrayLevel[0]},
               DisplayFunction → Identity],
               {k, 1, 3}];

           Show[GraphicsArray[somegraphs]]
```

■

9.3.4 Parseval's Equality

Let $f(x)$, $-p < x < p$, be a continuous piecewise smooth function with Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right).$$

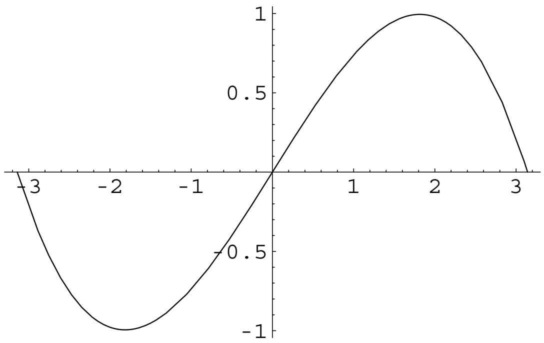


Figure 9-17 $f(x)$ is an odd function

Parseval's Equality states that

$$\frac{1}{p}\int_{-p}^{p}[f(x)]^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = 2A_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{9.22}$$

where $A_0 = \frac{1}{2}a_0$ is the constant term in the Fourier series.

EXAMPLE 9.3.5: Verify Parseval's Equality for $f(x) = \frac{1}{12}x\left(\pi^2 - x^2\right)$, $-\pi < x < \pi$.

SOLUTION: Notice that the function $f(x) = \frac{1}{12}x\left(\pi^2 - x^2\right)$, $-\pi < x < \pi$, is odd as we see from its graph in Figure 9-17.

```
In[1905]:= Clear[f]

            f[x_] = 1/12 x (π^2 - x^2);

            Plot[f[x], {x, -π, π}]
```

Therefore, the only nonzero coefficients in the Fourier series of $f(x)$ are found in b_n. Notice that we simplify the results by using the substitutions defined in `rule`.

```
In[1906]:= rule = {Sin[nπ] → 0, Cos[nπ] → (-1)^n};

            b[n_] = 2 ∫_0^π f[x] Sin[nx] dx / π /.rule

Out[1906]= -(-1)^n / n^3
```

Next, we evaluate $\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx$.

```
In[1907]:= ∫_{-π}^{π} f[x]^2 dx / π
Out[1907]= π^6/945

In[1908]:= N[π^6/945]
Out[1908]= 1.01734
```

We compare this result with the value of $\sum_{n=1}^{\infty} b_n^2$ by calculating $\sum_{n=1}^{k} b_n^2$ for $k = 1, 2, \ldots, 20$. Notice that this sequence of partial sums converges quickly to 1.01734, an approximation of $\frac{1}{945}\pi^6$.

```
In[1909]:= Table[N[Sum_{n=1}^{j} b_n^2], {j, 1, 20}]
Out[1909]= {0.25, 0.265625, 0.266997, 0.267241, 0.267305,
  0.267326, 0.267335, 0.267339, 0.267341,
  0.267342, 0.267342, 0.267342, 0.267343,
  0.267343, 0.267343, 0.267343, 0.267343,
  0.267343, 0.267343, 0.267343}
```

A p-**series** is a series of the form $\sum_{k=1}^{\infty} k^{-p}$. The p-series converges if $p > 1$ and diverges if $0 < p \le 1$.

Thus, for the convergent p-series

$$\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n^3}\right]^2 = \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{1}{945}\pi^6.$$

■

9.4 Generalized Fourier Series

In addition to the trigonometric eigenfunctions that were used to form the Fourier series in Sections 9.2 and 9.3, the eigenfunctions of other eigenvalue problems can be used to form what we call **generalized Fourier series**. We will find that these series will assist in solving problems in applied mathematics that involve physical phenomena that cannot be modeled with trigonometric functions.

Recall Bessel's equation of order zero

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda^2x^2y = 0. \tag{9.23}$$

If we require that the solutions of this differential equation satisfy the boundary conditions $|y(0)| < \infty$ (meaning that the solution is bounded at $x = 0$) and $y(p) = 0$, we can find the eigenvalues of the boundary-value problem

$$\begin{cases} x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + \lambda^2 x^2 y = 0 \\ |y(0)| < \infty, \; y(p) = 0. \end{cases} \tag{9.24}$$

A general solution of Bessel's equation of order zero is $y = c_1 J_0(\lambda x) + c_2 Y_0(\lambda x)$. Because $|y(0)| < \infty$, we must choose $c_2 = 0$ because $\lim_{x\to 0+} Y_0(\lambda x) = -\infty$. Hence, $y(p) = c_1 J_0(\lambda p) = 0$. Just as we did with the eigenvalue problems solved earlier in Section 9.1, we want to avoid choosing $c_1 = 0$, so we must select λ so that $J_0(\lambda p) = 0$.

Let α_n represent the nth zero of the Bessel function of order zero, $J_0(x)$, where $n = 1, 2, \ldots$, which we approximate with `BesselJZeros`. After loading the **Bessel Zeros** package, which is contained in the **NumericalMath** folder (or directory) by entering `<<NumericalMath‘BesselZeros‘`, the command `BesselJZeros[m,n]` returns a list of the first n zeros of $J_m(x)$; `BesselJZeros[m,{p,q}]` returns a list of the pth through qth zeros of $J_m(x)$; `BesselJZeros[0,n]` returns a list of the first n zeros of $J_0(x)$.

The function α_n returns the nth zero of $J_0(x)$.

```
In[1910]:= << NumericalMath‘BesselZeros‘

           αn_ := BesselJZeros[0, {n, n}][[1]]
```

Therefore, in trying to find the eigenvalues, we must solve $J_0(\lambda p) = 0$. From our definition of α_n, this equation is satisfied if $\lambda p = \alpha_n$, $n = 1, 2, \ldots$. Hence, the eigenvalues are $\lambda = \lambda_n = \alpha_n/p$, $n = 1, 2, \ldots$, and the corresponding eigenfunctions are

$$y(x) = y_n(x) = J_0(\lambda_n x) = J_0(\alpha_n x/p), \; n = 1, 2, \ldots.$$

As with the trigonometric eigenfunctions that we found in Sections 9.2 and 9.3, $J_0(\alpha_n x/p)$ can be used to build an eigenfunction series expansion of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\alpha_n x}{p}\right), \tag{9.25}$$

which is called a **Bessel–Fourier series**. We use the orthogonality properties of $J_0(\alpha_n x/p)$ to find the coefficients c_n.

We determine the orthogonality condition by placing Bessel's equation of order zero in the self-adjoint form See Theorem 33.

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \lambda^2 xy = 0.$$

Because the weighting function is $s(x) = x$, the orthogonality condition is

$$\int_0^p xJ_0\left(\frac{\alpha_n x}{p}\right)J_0\left(\frac{\alpha_m x}{p}\right)\,dx = 0,\ n \neq m.$$

Multiplying equation (9.25) by $xJ_0(\alpha_m x/p)$ and integrating from $x = 0$ to $x = p$ yields

$$\begin{aligned}\int_0^p xf(x)J_0\left(\frac{\alpha_m x}{p}\right)\,dx &= \int_0^p \sum_{n=1}^{\infty} c_n xJ_0\left(\frac{\alpha_n x}{p}\right)J_0\left(\frac{\alpha_m x}{p}\right)\,dx\\ &= \sum_{n=1}^{\infty} c_n \int_0^p xJ_0\left(\frac{\alpha_n x}{p}\right)J_0\left(\frac{\alpha_m x}{p}\right)\,dx.\end{aligned}$$

However, by the orthogonality condition, each of the integrals on the right-hand side of the equation equals zero except for $m = n$. Therefore,

$$c_n = \frac{\int_0^p xf(x)J_0\left(\frac{\alpha_n x}{p}\right)\,dx}{\int_0^p x\left[J_0\left(\frac{\alpha_n x}{p}\right)\right]^2\,dx},\ n = 1,\ n = 2,\ \ldots.$$

The value of the integral in the denominator can be found through the use of several of the identities associated with the Bessel functions. Because $\lambda_n = \alpha_n/p$, $n = 1, 2, \ldots$, the function $J_0(\alpha_n x/p) = J_0(\lambda_n x)$ satisfies Bessel's equation of order zero:

$$\frac{d}{dx}\left(x\frac{d}{dx}J_0(\lambda_n x)\right) + \lambda_n^2 xJ_0(\lambda_n x) = 0.$$

Multiplying by the factor $2x\dfrac{d}{dx}J_0(\lambda_n x)$, we can write this equation as

$$\frac{d}{dx}\left(x\frac{d}{dx}J_0(\lambda_n x)\right)^2 + \lambda_n^2 x^2\frac{d}{dx}\left[J_0(\lambda_n x)\right]^2 = 0.$$

Integrating each side of this equation from $x = 0$ to $x = p$ gives us

$$2\lambda_n^2\int_0^p x\left[J_0(\lambda_n x)\right]^2\,dx = \lambda_n^2 p^2\left[J_0'(\lambda_n p)\right]^2 + \lambda_n^2 p^2\left[J_0(\lambda_n p)\right]^2.$$

With the substitution $\lambda_n p = \alpha_n$ the equation becomes

$$2\lambda_n^2\int_0^p x\left[J_0(\lambda_n x)\right]^2\,dx = \lambda_n^2 p^2\left[J_0'(\alpha_n)\right]^2 + \lambda_n^2 p^2\left[J_0(\alpha_n)\right]^2.$$

Now, $J_0(\alpha_n) = 0$, because α_n is the nth zero of $J_0(x)$. Also, with $n = 0$, the identity $\dfrac{d}{dx}\left(x^{-n}J_n(x)\right) = -x^{-n}J_{n+1}(x)$ indicates that $J_0'(\alpha_n) = -J_1(\alpha_n)$. Therefore,

$$\begin{aligned}2\lambda_n^2\int_0^p x\left[J_0(\lambda_n x)\right]^2\,dx &= \lambda_n^2 p^2\left[-J_1(\alpha_n)\right]^2 + \lambda_n^2 p^2\cdot 0\\ \int_0^p x\left[J_0(\lambda_n x)\right]^2\,dx &= \frac{1}{2}p^2\left[J_1(\alpha_n)\right]^2.\end{aligned}$$

Using this expression in the denominator of c_n, the series coefficients are found with

$$c_n = \frac{2}{p^2\left[J_1(\alpha_n)\right]^2}\int_0^p xf(x)J_0\left(\frac{\alpha_n x}{p}\right)dx,\ n = 1,\ 2,\ \ldots. \tag{9.26}$$

EXAMPLE 9.4.1: Find the Bessel–Fourier series for $f(x) = 1 - x^2$ on $0 < x < 1$.

SOLUTION: In this case, $p = 1$, so

$$\begin{aligned} c_n &= \frac{2}{\left[J_1(\alpha_n)\right]^2}\int_0^1 x\left(1-x^2\right)J_0\left(\frac{\alpha_n x}{p}\right)dx \\ &= \frac{2}{\left[J_1(\alpha_n)\right]^2}\left\{\int_0^1 xJ_0\left(\frac{\alpha_n x}{p}\right)dx - \int_0^1 x^3J_0\left(\frac{\alpha_n x}{p}\right)dx\right\}. \end{aligned}$$

Using the formula, $\frac{d}{dx}\left(x^nJ_n(x)\right) = x^nJ_{n-1}(x)$ with $n = 1$ yields

$$\int_0^1 xJ_0\left(\frac{\alpha_n x}{p}\right)dx = \left[\frac{1}{\alpha_n}xJ_1\left(\alpha_n x\right)\right]_0^1 = \frac{1}{\alpha_n}J_1\left(\alpha_n\right).$$

Note that the factor $1/\alpha_n$ is due to the chain rule for differentiating the argument of $J_1\left(\alpha_n x\right)$. We use integration by parts with $u = x^2$ and $dv = xJ_0\left(\alpha_n x\right)$ to evaluate $\int_0^1 x^3J_0\left(\frac{\alpha_n x}{p}\right)dx$. As in the first integral we obtain $v = \frac{1}{\alpha_n}xJ_1\left(\alpha_n x\right)$. Then, because $du = 2x\,dx$, we have

Integration by parts formula: $\int u\,dv = uv - \int v\,du$.

$$\begin{aligned} \int_0^1 x^3J_0\left(\frac{\alpha_n x}{p}\right)dx &= \left[\frac{1}{\alpha_n}x^3J_1\left(\alpha_n x\right)\right]_0^1 - \frac{2}{\alpha_n}\int_0^1 x^2J_1\left(\alpha_n x\right)dx \\ &= \frac{1}{\alpha_n}J_1\left(\alpha_n\right) - \frac{2}{\alpha_n}\left[\frac{1}{\alpha_n}x^2J_2\left(\alpha_n x\right)\right]_0^1 \\ &= \frac{1}{\alpha_n}J_1\left(\alpha_n\right) - \frac{2}{\alpha_n^2}J_2\left(\alpha_n\right). \end{aligned}$$

Thus, the coefficients are

$$\begin{aligned} c_n &= \frac{2}{\left[J_1(\alpha_n)\right]^2}\int_0^1 x\left(1-x^2\right)J_0\left(\frac{\alpha_n x}{p}\right)dx \\ &= \frac{2}{\left[J_1(\alpha_n)\right]^2}\left\{\int_0^1 xJ_0\left(\frac{\alpha_n x}{p}\right)dx - \int_0^1 x^3J_0\left(\frac{\alpha_n x}{p}\right)dx\right\} \\ &= \frac{2}{\left[J_1(\alpha_n)\right]^2}\left[\frac{1}{\alpha_n}J_1\left(\alpha_n\right) - \left(\frac{1}{\alpha_n}J_1\left(\alpha_n\right) - \frac{2}{\alpha_n^2}J_2\left(\alpha_n\right)\right)\right] \\ &= \frac{4J_2\left(\alpha_n\right)}{\alpha_n^2\left[J_1\left(\alpha_n\right)\right]^2},\ n = 1,\ 2,\ \ldots \end{aligned}$$

so that the Bessel–Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4J_2(\alpha_n)}{\alpha_n^2 [J_1(\alpha_n)]^2} J_0(\alpha_n x).$$

We now graph $f(x)$ along with several terms of the series. To do so, we need to compute the values of $J_1(\alpha_n)$ and $J_2(\alpha_n)$ for various values of n. Thus, we first load the **BesselZeros** package and define α_n to be the nth zero of the Bessel function of the first kind of order zero.

Note that you do not need to reload the **BesselZeros** package if you already loaded it during your *current* Mathematica session.

```
In[1911]:= << NumericalMath`BesselZeros`

           αn_ := BesselJZeros[0, {n, n}][[1]]
```

We list the values of $J_1(\alpha_n)$ and $J_2(\alpha_n)$ for various values of n.

```
In[1912]:= Table[{n, BesselJ[1, αn], BesselJ[2, αn]},
             {n, 1, 5}]//TableForm
           1 0.519147  0.431755
           2 -0.340265 -0.123283
Out[1912]= 3 0.271452  0.0627365
           4 -0.23246  -0.0394283
           5 0.206546  0.0276669
```

Next, we define $f(x) = 1 - x^2$ and $p_k(x) = \sum_{n=1}^{k} \frac{4J_2(\alpha_n)}{\alpha_n^2 [J_1(\alpha_n)]^2} J_0(\alpha_n x)$, the kth partial sum of the Bessel–Fourier series. Note that

$$p_k(x) = \sum_{n=1}^{k-1} \frac{4J_2(\alpha_n)}{\alpha_n^2 [J_1(\alpha_n)]^2} J_0(\alpha_n x) + \frac{4J_2(\alpha_k)}{\alpha_k^2 [J_1(\alpha_k)]^2} J_0(\alpha_k x)$$
$$= p_{k-1}(x) + \frac{4J_2(\alpha_k)}{\alpha_k^2 [J_1(\alpha_k)]^2} J_0(\alpha_k x).$$

Thus, to calculate the kth partial sum of the Fourier series, we need only add $\frac{4J_2(\alpha_k)}{\alpha_k^2 [J_1(\alpha_k)]^2} J_0(\alpha_k x)$ to the $(k-1)$st partial sum: we need not recompute all k terms of the kth partial sum if we know the $(k-1)$st partial sum. Using this observation, we define the recursively defined function p to return the kth partial sum of the series.

```
In[1913]:= f[x_] = 1 - x^2;

In[1914]:= p[k_] := p[k] = p[k - 1]
                    + (4 BesselJ[2, αk] BesselJ[0, αk x])/(αk^2 BesselJ[1, αk]^2);

           p[0] = 0;
```

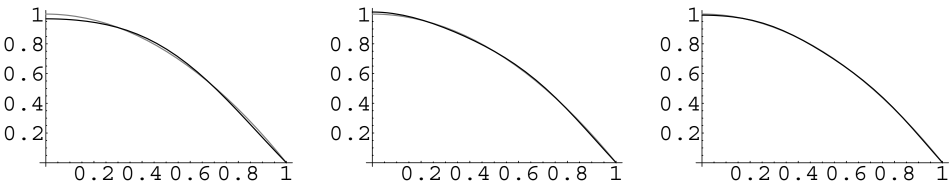


Figure 9-18 The Bessel–Fourier series quickly converges to $f(x)$

The graphs of $f(x)$ and the first three or four partial sums are practically indistinguishable. See Figure 9-18.

```
In[1915]:= p2 = Plot[Evaluate[{f[x],p[2]}],{x,0,1},
                PlotStyle->{GrayLevel[0.5],
                GrayLevel[0]},
                DisplayFunction->Identity];

            p3 = Plot[Evaluate[{f[x],p[3]}],{x,0,1},
                PlotStyle->{GrayLevel[0.5],
                GrayLevel[0]},
                DisplayFunction->Identity];

            p4 = Plot[Evaluate[{f[x],p[4]}],{x,0,1},
                PlotStyle->{GrayLevel[0.5],
                GrayLevel[0]},
                DisplayFunction->Identity];

In[1916]:= Show[GraphicsArray[{p2,p3,p4}]]
```

■

As was the case with Fourier series, we can make a statement about the convergence of the Bessel–Fourier series.

Theorem 37 (Convergence of Bessel–Fourier Series). *Suppose that $f(x)$ and $f'(x)$ are piecewise continuous functions on $0 < x < p$. Then the Bessel–Fourier series for $f(x)$ on $0 < x < p$ converges to $f(x)$ at every x where $f(x)$ is continuous. If $f(x)$ is discontinuous at $x = a$, the Bessel–Fourier series converges to the average*

$$\frac{1}{2}\left(\lim_{x\to a^+} f(x) + \lim_{x\to a^-} f(x)\right) = \frac{1}{2}\left(f(a^+) + f(a^-)\right).$$

Series involving the eigenfunctions of other eigenvalue problems can be formed as well.

EXAMPLE 9.4.2: The eigenvalue problem $\begin{cases} y'' + 2y' - (\lambda - 1)y = 0 \\ y(0) = y(2) = 0 \end{cases}$ has eigenvalues $\lambda_n = -(n\pi/2)^2$ and eigenfunctions $y_n(x) = e^{-x}\sin(n\pi x/2)$. Use these eigenfunctions to approximate $f(x) = e^{-x}$ for $0 < x < 2$.

SOLUTION: In order to approximate $f(x)$, we need the orthogonality condition for these eigenfunctions. We obtain this condition by placing the differential equation in self-adjoint form using the formulas given in equation (9.5). In the general equation, $a_2(x) = 1$, $a_1(x) = 2$, and $a_0(x) = 0$. Therefore, $p(x) = e^{\int 2\,dx} = e^{2x}$ and $s(x) = p(x)/a_2(x) = e^{2x}$, so in self-adjoint form the equation is

$$\frac{d}{dx}\left(e^{2x}\frac{dy}{dx}\right) - (\lambda - 1)e^{2x}y.$$

This means that the orthogonality condition, $\int_a^b s(x)y_n(x)y_m(x)\,dx = 0$ $(m \neq n)$, is

$$\int_0^2 e^{2x}e^{-x}\sin\frac{m\pi x}{2}e^{-x}\sin\frac{n\pi x}{2}\,dx = \int_0^2 \sin\frac{m\pi x}{2}\sin\frac{n\pi x}{2}\,dx = 0,\ m \neq n.$$

We use this condition to determine the coefficients in the eigenfunction expansion

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) = \sum_{n=1}^{\infty} c_n e^{-x}\sin\frac{n\pi x}{2}.$$

Multiplying both sides of this equation by $y_m(x) = e^{-x}\sin(m\pi x/2)$ and $s(x) = e^{2x}$ and then integrating from $x = 0$ to $x = 2$ yields

$$\int_0^2 f(x)e^{2x}e^{-x}\sin\frac{m\pi x}{2}\,dx = \int_0^2 \sum_{n=1}^{\infty} c_n e^{-x}\sin\frac{n\pi x}{2}e^{2x}e^{-x}\sin\frac{m\pi x}{2}\,dx$$
$$\int_0^2 f(x)e^{x}\sin\frac{m\pi x}{2}\,dx = \sum_{n=1}^{\infty}\int_0^2 c_n \sin\frac{n\pi x}{2}\sin\frac{m\pi x}{2}\,dx.$$

Each integral in the sum on the right-hand side of the equation is zero except if $m = n$. In this case, $\int_0^2 \sin^2(n\pi x/2)\,dx = 1$. Therefore,

$$c_n = \int_0^2 f(x)e^{x}\sin\frac{n\pi x}{2}\,dx.$$

For $f(x) = e^{-x}$,

$$c_n = \int_0^2 e^{-x} e^x \sin \frac{n\pi x}{2}\,dx = \int_0^2 \sin \frac{n\pi x}{2}\,dx = -\frac{2}{n\pi}(\cos n\pi - 1).$$

```
In[1917]:= c_n_ = ∫_0^2 Sin[nπx/2] dx
Out[1917]= 2/(n π) - (2 Cos[n π])/(n π)
```

Because $\cos n\pi = (-1)^n$, we can write the eigenfunction expansion of $f(x)$ as

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} -\frac{2}{n\pi}\left((-1)^n - 1\right) e^{-x} \sin \frac{n\pi x}{2} \\ &= e^{-x}\left(\frac{4}{\pi}\sin\frac{\pi x}{2} + \frac{4}{3\pi}\sin\frac{3\pi x}{2} + \frac{4}{5\pi}\sin\frac{5\pi x}{2} + \cdots\right). \end{aligned}$$

We graph $f(x)$ together with

$$p_k(x) = \sum_{n=1}^{k} -\frac{2}{n\pi}\left((-1)^n - 1\right) e^{-x} \sin \frac{n\pi x}{2}$$

for $k = 6$, 10, and 14 in Figure 9-19.

```
In[1918]:= f[x_] = Exp[-x];

In[1919]:= p_k_[x_] := Sum[c_n Exp[-x] Sin[nπx/2], {n, 1, k}];

In[1920]:= p6 = Plot[{f[x], p_6[x]}, {x, 0, 3},
                PlotStyle- > {GrayLevel[0.5],
                GrayLevel[0]},
                DisplayFunction- > Identity];

           p10 = Plot[{f[x], p_10[x]}, {x, 0, 3},
                  PlotStyle- > {GrayLevel[0.5],
                  GrayLevel[0]},
                  DisplayFunction- > Identity];

           p14 = Plot[{f[x], p_14[x]}, {x, 0, 3},
                  PlotStyle- > {GrayLevel[0.5],
                  GrayLevel[0]},
                  DisplayFunction- > Identity];

In[1921]:= Show[GraphicsArray[{p6, p10, p14}]]
```

■

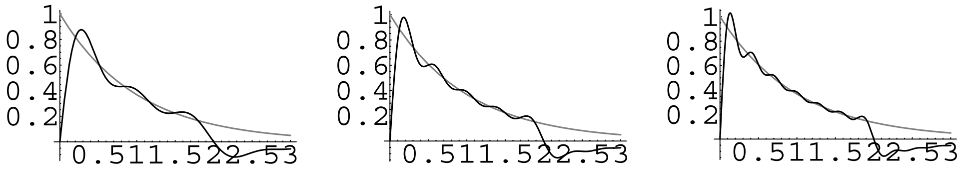


Figure 9-19 Approximating $f(x) = e^{-x}$ with a generalized Fourier series

EXAMPLE 9.4.3: Use the eigenvalues and eigenfunctions of the eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0,\ y(1) + y'(1) = 0 \end{cases}$$

to obtain a generalized Fourier series for $f(x) = x(1-x)$, $0 < x < 1$.

The eigenvalues and corresponding eigenfunctions for this eigenvalue problem are found in Example 9.1.6.

SOLUTION: In Example 9.1.6, the eigenvalues of this problem, $\lambda = k^2$, were shown to satisfy the relationship $k = -\tan k$. In the example, we approximated the first eight roots of this equation to be 2.02876, 4.91318, 7.97867, 11.0855, 14.2074, 17.3364, 20.4692, and 23.6043 entered in `kvals`.

```
In[1922]:= kvals = Table[FindRoot[ - Tan[x] == x,
                {x, (2n - 1) π/2 + 0.1}], {n, 1, 8}]
Out[1922]= {{x → 2.02876}, {x → 4.91318}, {x → 7.97867},
            {x → 11.0855}, {x → 14.2074}, {x → 17.3364},
            {x → 20.4692}, {x → 23.6043}}
```

Let k_n represent the nth positive root of $k = -\tan k$. Therefore, the eigenfunctions of $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0,\ y(1) + y'(1) = 0 \end{cases}$ are $y_n(x) = \sin k_n x$. Because of the orthogonality of the eigenfunctions, we have the orthogonality condition $\int_0^1 \sin k_n x \sin k_m x\, dx = 0$, $m \neq n$. If $m = n$, we have $\int_0^1 \sin^2 k_n x\, dx = \frac{1}{2} - \frac{1}{4k_n} \sin 2k_n$.

```
In[1923]:= ∫_0^1 Sin[kx]^2 dx
Out[1923]= 1/2 - Sin[2 k]/(4 k)
```

Therefore,

$$\int_0^1 \sin^2 k_n x\, dx = \frac{2k_n - \sin 2k_n}{4k_n} = \frac{2k_n - 2\sin k_n \cos k_n}{4k_n}.$$

With the condition $k_n = -\tan k_n$ or $\sin k_n = -k_n \cos k_n$ from the eigenvalue problem, we have

$$\int_0^1 \sin^2 k_n x\, dx = \frac{k_n - \sin k_n \cos k_n}{2k_n} = \frac{k_n - (-k_n \cos k_n)\cos k_n}{2k_n} = \frac{1}{2}\left(1 + \cos^2 k_n\right).$$

To determine the coefficients c_n in the generalized Fourier series $f(x) = \sum_{n=1}^{\infty} c_n \sin k_n x$ using the eigenfunctions $y_n(x) = \sin k_n x$ of the eigenvalue problem, we multiply both sides of $f(x) = \sum_{n=1}^{\infty} c_n \sin k_n x$ by $\sin k_m x$ and integrate from $x = 0$ to $x = 1$. This yields

$$\int_0^1 f(x) \sin k_m x\, dx = \int_0^1 \sum_{n=1}^{\infty} c_n \sin k_n x \sin k_m x\, dx.$$

Assuming uniform convergence of the series, we have

$$\int_0^1 f(x) \sin k_m x\, dx = \sum_{n=1}^{\infty} c_n \int_0^1 \sin k_n x \sin k_m x\, dx.$$

All terms on the right are zero except if $m = n$. In this case, we have

$$\int_0^1 f(x) \sin k_n x\, dx = c_n \int_0^1 \sin^2 k_n x\, dx = \frac{1}{2} c_n \left(1 + \cos^2 k_n\right)$$

so that

$$c_n = \frac{2}{1 + \cos^2 k_n} \int_0^1 f(x) \sin k_n x\, dx.$$

We approximate the value of c_n for $n = 1, 2, \ldots, 8$ in `cvals` using the values of k in `kvals`.

```
In[1924]:= f[x_] := x (1 - x)

cvals = Table[2 NIntegrate[f[x] Sin[(kvals[[j, 1, 2]]) x], {x, 0, 1}] / (1 + Cos[kvals[[j, 1, 2]]]^2),
          {j, 1, 8}]
Out[1924]= {0.213285, 0.104049, -0.0219788, 0.0187303,
           -0.00834994, 0.00734255, -0.00426841,
           0.00387074}
```

We define the sum of the first j terms of $\sum_{n=1}^{\infty} c_n \sin k_n x$ for $f(x) = x(1-x)$, $0 < x < 1$, with `fapprox[x,n]` and then create a table of `fapprox[x,n]` for $n = 1$ to $n = 8$ in `funcs`.

```
In[1925]:= fapprox[x_, j_] := Sum[(cvals[[n]]) Sin[(kvals[[n,
              1, 2]]) x], {n, 1, j}];

In[1926]:= funcs = Table[fapprox[x, j], {j, 1, 8}]
Out[1926]= {0.213285 Sin[2.02876 x],
             0.213285 Sin[2.02876 x]
             +0.104049 Sin[4.91318 x],
             0.213285 Sin[2.02876 x]
             +0.104049 Sin[4.91318 x]
             -0.0219788 Sin[7.97867 x],
             0.213285 Sin[2.02876 x]
             +0.104049 Sin[4.91318 x]
             -0.0219788 Sin[7.97867 x]
             +0.0187303 Sin[11.0855 x],
             0.213285 Sin[2.02876 x]
             +0.104049 Sin[4.91318 x]
             -0.0219788 Sin[7.97867 x]
             +0.0187303 Sin[11.0855 x]
             -0.00834994 Sin[14.2074 x],
             0.213285 Sin[2.02876 x]
             +0.104049 Sin[4.91318 x]
             -0.0219788 Sin[7.97867 x]
             +0.0187303 Sin[11.0855 x]
             -0.00834994 Sin[14.2074 x]
             +0.00734255 Sin[17.3364 x],
             0.213285 Sin[2.02876 x]
             +0.104049 Sin[4.91318 x]
             -0.0219788 Sin[7.97867 x]
             +0.0187303 Sin[11.0855 x]
             -0.00834994 Sin[14.2074 x]
             +0.00734255 Sin[17.3364 x]
             -0.00426841 Sin[20.4692 x],
             0.213285 Sin[2.02876 x]
             +0.104049 Sin[4.91318 x]
             -0.0219788 Sin[7.97867 x]
             +0.0187303 Sin[11.0855 x]
             -0.00834994 Sin[14.2074 x]
             +0.00734255 Sin[17.3364 x]
             -0.00426841 Sin[20.4692 x]
             +0.00387074 Sin[23.6043 x]}
```

We graph $f(x) = x(1-x)$, $0 < x < 1$, simultaneously with the first term of the generalized Fourier series, `funcs[[1]]`, to observe the accuracy in Figure 9-20.

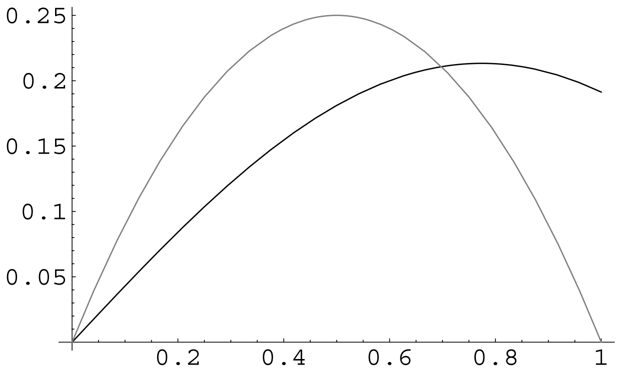


Figure 9-20 Using the first partial sum to approximate $f(x)$ does not result in a very good approximation

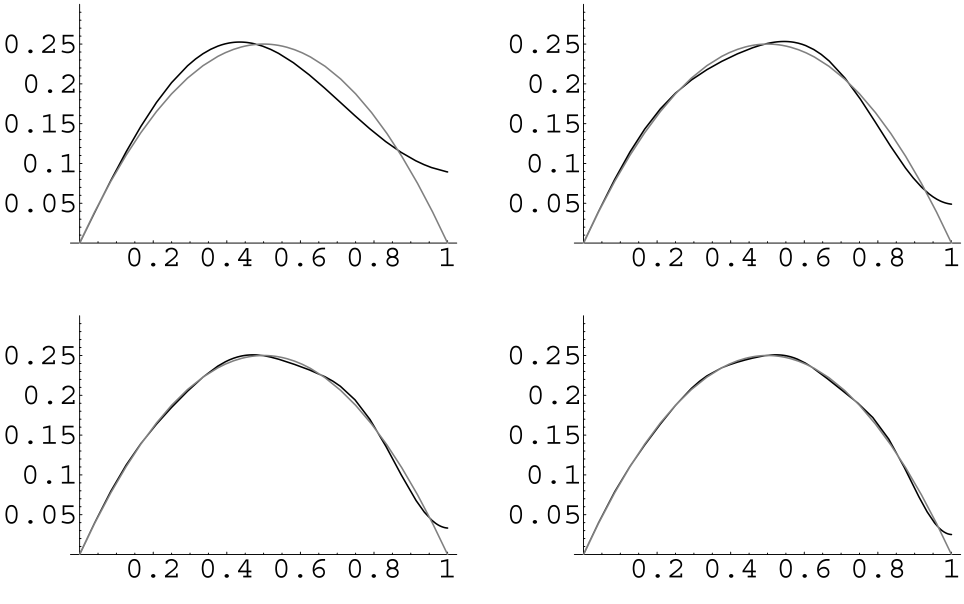


Figure 9-21 The approximation improves when the number of terms in the partial sum is increased

```
In[1927]:= funcs[[1]]
Out[1927]= 0.213285 Sin[2.02876 x]

In[1928]:= Plot[{fapprox[x, 1], f[x]}, {x, 0, 1},
              PlotStyle → {GrayLevel[0],
              GrayLevel[0.5]}]
```

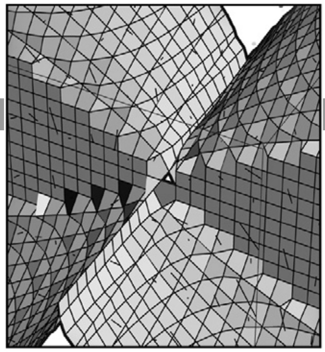
Next, we plot the approximation using the sum of the first 2, 4, 6, and 8 terms of the generalized Fourier series and display the results in Figure 9-21. We see that the approximation improves as the number of terms increases.

```
In[1929]:= graphs =
             Table[Plot[{fapprox[x, j], f[x]}, {x, 0, 1},
               PlotRange → {0, 0.3},
               PlotStyle → {GrayLevel[0],
               GrayLevel[0.5]},
               DisplayFunction → Identity], {j, 2, 8, 2}];

In[1930]:= Show[GraphicsArray[Partition[graphs, 2]]]
```

■

Partial Differential Equations

10



10.1 Introduction to Partial Differential Equations and Separation of Variables

10.1.1 Introduction

We begin our study of partial differential equations with an introduction of some of the terminology associated with the topic. A **linear second-order partial differential equation** (**PDE**) in the two independent variables x and y has the form

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial y \partial x} + C(x, y)\frac{\partial^2 u}{\partial y^2} + D(x, y)\frac{\partial u}{\partial x} + E(x, y)\frac{\partial u}{\partial y} + F(x, y)u = G(x, y), \quad (10.1)$$

where the solution is $u(x, y)$. If $G(x, y) = 0$ for all x and y, we say that the equation is **homogeneous**. Otherwise, the equation is **nonhomogeneous**.

EXAMPLE 10.1.1: Classify the following partial differential equations: (a) $u_{xx} + u_{yy} = u$; (b) $uu_x = x$.

SOLUTION: (a) This equation satisfies the form of the linear second-order partial differential equation (10.1) with $A = C = 1$, $F = -1$, and $B = D = E = 0$. Because $G(x, y) = 0$, the equation is homogeneous.

(b) This equation is nonlinear, because the coefficient of u_x is a function of u. It is also nonhomogeneous because $G(x, y) = x$.

■

Definition 40 (Solution of a Partial Differential Equation). *A* ***solution of a partial differential equation*** *in some region R of the space of the independent variables is a function that possesses all of the partial derivatives that are present in the PDE in some region containing R and satisfies the PDE everywhere in R.*

EXAMPLE 10.1.2: Show that $u(x, y) = y^2 - x^2$ and $u(x, y) = e^y \sin x$ are solutions to Laplace's equation $u_{xx} + u_{yy} = 0$.

SOLUTION: For $u(x, y) = y^2 - x^2$, $u_x(x, y) = -2x$, $u_y(x, y) = 2y$, $u_{xx}(x, y) = -2$, and $u_{yy}(x, y) = 2$, so we have that $u_{xx} + u_{yy} = (-2) + 2 = 0$, which we quickly verify with Mathematica.

```
In[1931]:= Clear[u]

           u[x_, y_] = y^2 - x^2;

In[1932]:= D[u[x, y], {x, 2}] + D[u[x, y], {y, 2}]
Out[1932]= 0
```

Similarly, for $u(x, y) = e^y \sin x$, we have $u_x = e^y \cos x$, $u_y = e^y \sin x$, $u_{xx} = -e^y \sin x$, and $u_{yy} = e^y \sin x$. Therefore, $u_{xx} + u_{yy} = (-e^y \sin x) + e^y \sin x = 0$, so the equation is satisfied for both functions.

```
In[1933]:= Clear[u]

           u[x_, y_] = Exp[y] Sin[x];

In[1934]:= D[u[x, y], {x, 2}] + D[u[x, y], {y, 2}]
Out[1934]= 0
```

We notice that the solutions to Laplace's equation differ in form. This is unlike solutions to homogeneous linear ordinary differential equations. There, we found that solutions were similar in form. (Recall, all solutions could be generated from a general solution.)

■

Some of the techniques used in constructing solutions of homogeneous linear ordinary differential equations can be extended to the study of partial differential equations as we see with the following theorem.

Theorem 38 (Principle of Superposition). *If $u_1, u_2, \ldots, u_m$ are solutions to a linear homogeneous partial differential equation in a region R, then*

$$c_1u_1 + c_2u_2 + \cdots + c_mu_m = \sum_{k=1}^{m} c_ku_k,$$

where $c_1, c_2, \ldots, c_m$ are constants is also a solution in R.

The Principle of Superposition will be used in solving partial differential equations throughout the rest of the chapter. In fact, we will find that equations can have an infinite set of solutions so that we construct another solution in the form of an infinite series.

10.1.2 Separation of Variables

A method that can be used to solve linear partial differential equations is called separation of variables (or the **product method**). Generally, the goal of the method of separation of variables is to transform the partial differential equation into a system of ordinary differential equations each of which depends on only one of the functions in the product form of the solution. Suppose that the function $u(x, y)$ is a solution of a partial differential equation in the independent variables x and y. In separating variables, we assume that u can be written as the product of a function of x and a function of y. Hence,

$$u(x, y) = X(x)Y(y),$$

and we substitute this product into the partial differential equation to determine $X(x)$ and $Y(y)$. Of course, in order to substitute into the differential equation, we must be able to differentiate this product. However, this is accomplished by following the differentiation rules of multivariate calculus:

$$u_x = X'Y, \quad u_{xx} = X''Y, \quad u_{xy} = X'Y', \quad u_y = XY', \quad \text{and} \quad u_{yy} = XY'',$$

where X' represents dX/dx and Y' represents dY/dy. After these substitutions are made and if the equation is separable, we can obtain an ordinary differential equation for X and an ordinary differential equation for Y. These two equations are then solved to find $X(x)$ and $Y(y)$.

EXAMPLE 10.1.3: Use separation of variables to find a solution of $xu_x = u_y$.

SOLUTION: If $u(x, y) = X(x)Y(y)$, then $u_x = X'Y$ and $u_y = XY'$. The equation then becomes

$$xX'Y = XY',$$

which can be written as the separated equation

$$\frac{xX'}{X} = \frac{Y'}{Y}.$$

Notice that the left-hand side of the equation is a function of x while the right-hand side is a function of y. Hence, the only way that this situation can be true is for xX'/X and Y'/Y to both be constant. Therefore,

$$\frac{xX'}{X} = \frac{Y'}{Y} = k,$$

so we obtain the ordinary differential equations $xX' - kX = 0$ and $Y' - ky = 0$. We find X first.

$$\begin{aligned} xX' - kX &= 0 \\ x\frac{dX}{dx} &= kX \\ \frac{1}{X}dX &= \frac{k}{x}dx \\ \ln|X| &= k\ln|x| + c_1 \\ X(x) &= e^{c_1}x^k = C_1x^k. \end{aligned}$$

Similarly, we find

$$\begin{aligned} Y' - kY &= 0 \\ \frac{dY}{dy} &= kY \\ \frac{1}{Y}dY &= k\,dy \\ \ln|Y| &= ky + c_2 \\ Y(y) &= e^{c_2}e^{ky} = C_2e^{ky}. \end{aligned}$$

Therefore, a solution is $u(x, y) = X(x)Y(y) = \left(C_1x^k\right)\left(C_2e^{ky}\right) = C_3x^ke^{ky}$ where k and C_3 are arbitrary constants. DSolve can be used to find a solution of this partial differential equation as well.

```
In[1935]:= Clear[x, y, u]

           DSolve[x D[u[x, y], x] == D[u[x, y], y], u[x, y],
             {x, y}]
Out[1935]= {{u[x, y] → C[1][y + Log[x]]}}
```

In this result, the symbol `C[1]` represents an arbitrary differentiable function. That is, if f is a differentiable function of a single variable, $u(x, y) = f(y + \ln x)$ is a solution to $xu_x = u_y$, which we verify by substituting this result into the partial differential equation.

```
In[1936]:= xD[C[1][y+Log[x]],x] == D[C[1][y+Log[x]],y]
Out[1936]= True
```

■

10.2 The One-Dimensional Heat Equation

One of the more important partial differential equations is the heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{10.2}$$

In one spatial dimension, the solution of the heat equation represents the temperature (at any position x and any time t) in a thin rod or wire of length p. Because the rate at which heat flows through the rod depends on the material that makes up the rod, the constant c^2 which is related to the thermal diffusivity of the material is included in the heat equation. Several different situations can be considered when determining the temperature in the rod. The ends of the wire can be held at a constant temperature, the ends may be insulated, or there can be a combination of these situations.

10.2.1 The Heat Equation with Homogeneous Boundary Conditions

The first problem that we investigate is the situation in which the temperature at the ends of the rod are constantly kept at zero and the initial temperature distribution in the rod is represented as the given function $f(x)$. Hence, the fixed end zero temperature is given in the boundary conditions

$$u(0, t) = u(p, t) = 0$$

while the initial temperature distribution is given by

$$u(x, 0) = f(x).$$

Because the temperature is zero at the endpoints, we say that the problem has **homogeneous boundary conditions**, which are important in finding a solution with separation of variables. We call problems of this type **initial-boundary value problems** (IBVP), because they include initial as well as boundary conditions. Thus, the problem is summarized as

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0,\ u(p,t) = 0,\ t > 0 \\ u(x,0) = f(x),\ 0 < x < p. \end{cases} \tag{10.3}$$

We solve this problem through separation of variables by assuming that

$$u(x,t) = X(x)T(t).$$

Substitution into the heat equation (10.2) yields

$$\frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda$$

where $-\lambda$ is the separation constant. (Note that we selected this constant in order to obtain an eigenvalue problem that was solved in Example 9.1.4.) Separating the variables, we have the two equations

$$T' + c^2\lambda T = 0 \qquad \text{and} \qquad X'' + \lambda X = 0.$$

Now that we have successfully separated the variables, we turn our attention to the homogeneous boundary conditions. In terms of the functions $X(x)$ and $T(t)$, these boundary conditions become

$$u(0,t) = X(0)T(t) = 0 \qquad \text{and} \qquad u(p,t) = X(p)T(t) = 0.$$

In each case, we must avoid setting $T(t) = 0$ for all t, because if this were the case, our solution would be the trivial solution $u(x,t) = X(x)T(t) = 0$. Therefore, we have the boundary conditions

$$X(0) = 0 \qquad \text{and} \qquad X(p) = 0,$$

See Example 9.1.4.

so we solve the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0,\ X(p) = 0. \end{cases}$$

The eigenvalues of this problem are $\lambda_n = (n\pi/p)^2$ with corresponding eigenfunctions $X_n(x) = \sin(n\pi x/p)$. Similarly, a general solution of $T' + c^2\lambda_n T = 0$ is $T_n(t) = Ae^{-c^2\lambda_n t}$, where A is an arbitrary constant and $\lambda_n = (n\pi/p)^2$, $n = 1, 2, \ldots$.

```
In[1937]:= DSolve[capt'[t] + c² λn capt[t] == 0, capt[t], t]
Out[1937]= {{capt[t] → e^(-c² n² π² t) C[1]}}
```

Because $X(x)$ and $T(t)$ both depend on n, the solution $u(x,t) = X(x)T(t)$ does as well. Hence,

$$u_n(x,t) = X_n(t)T_n(t) = c_n \sin\frac{n\pi x}{p}\, e^{-c^2\lambda_n t}$$

where we have replaced the constant A by one that depends on n. In order to find the value of c_n, we apply the initial condition $u(x,0) = f(x)$. Notice that

$$u_n(x,0) = c_n \sin\frac{n\pi x}{p}\, e^{-c^2\lambda_n\cdot 0} = c_n \sin\frac{n\pi x}{p}$$

is satisfied only by functions of the form $\sin(\pi x/p), \sin(2\pi x/p), \ldots$ (which, in general, is not the case). Therefore, we use the principle of superposition to state that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{p}\, e^{-c^2\lambda_n t}$$

is also a solution of the problem, because this solution satisfies the heat equation as well as the boundary conditions. Then, when we apply the initial condition $u(x,0) = f(x)$, we find that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{p}\, e^{-c^2\lambda_n\cdot 0} = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{p} = f(x).$$

Therefore, c_n represents the Fourier sine series coefficients for $f(x)$, which are given by

$$c_n = \frac{2}{p}\int_0^p f(x)\,\sin\frac{n\pi x}{p}\,dx,\ n = 1, 2, \ldots.$$

EXAMPLE 10.2.1: Solve $\begin{cases} u_t = u_{xx},\ 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ u(1,t) = 0,\ t > 0 \\ u(x,0) = 50,\ 0 < x < 1. \end{cases}$

SOLUTION: In this case, $c = 1$, $p = 1$, and $f(x) = 50$. Hence,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin n\pi x\, e^{-\lambda_n t},$$

where

If n is an integer, $\cos n\pi = (-1)^n$.

$$c_n = \frac{2}{1}\int_0^1 50\sin n\pi x\,dx = -\frac{100}{n\pi}(\cos n\pi - 1) = -\frac{100}{n\pi}\left((-1)^n - 1\right),$$

and $\lambda_n = (n\pi)^2$.

```
In[1938]:= c_n_ = 100 ∫_0^1 Sin[nπx] dx
Out[1938]= 100 (1/(n π) - Cos[n π]/(n π))
In[1939]:= λ_n_ = (nπ)^2;
```

Therefore, because $c_n = 0$ if n is even, we write $u(x, t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{(2n-1)\pi} \sin(2n-1)\pi x \, e^{-(2n-1)^2\pi^2 t}.$$

We graph an approximation of $u(x, t)$ at various times by graphing

$$u_k(x,t) = \sum_{n=1}^{k} \frac{200}{(2n-1)\pi} \sin(2n-1)\pi x \, e^{-(2n-1)^2\pi^2 t}$$

if $k = 10$ in Figure 10-1.

```
In[1940]:= uapprox[x_, t_] = Sum[c_n Sin[nπx] Exp[-λ_n t], {n, 1, 41}];
In[1941]:= toplot = Table[uapprox[x, t], {t, 0, 1, 0.05}];
In[1942]:= grays = Table[GrayLevel[i],
               {i, 0, 0.7, 0.7/20}];
           Plot[Evaluate[toplot],
               {x, 0, 1}, PlotStyle- > grays]
```

An alternative approach to visualizing the solution is to generate a density plot of $u(x, t)$ with `DensityPlot` as shown in Figure 10-2.

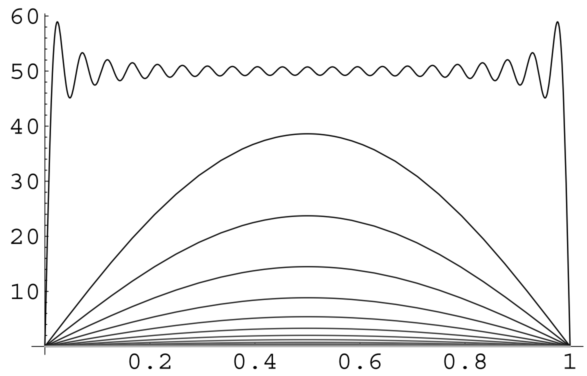


Figure 10-1 $u(x, t)$ for various values of t

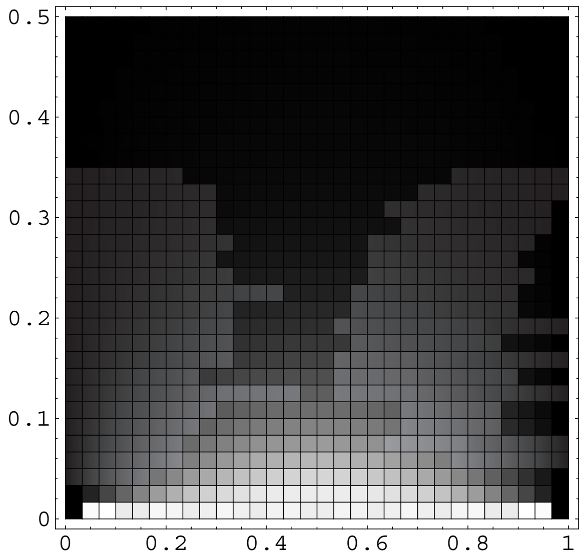


Figure 10-2 A density plot of $u(x, t)$: t corresponds to the vertical axis and x the horizontal axis

```
In[1943]:= DensityPlot[uapprox[x, t], {x, 0, 1}, {t, 0, 0.5},
              PlotPoints- > 30]
```

DensityPlot works in the same way as ContourPlot.

In the density plot, darker shades correspond to smaller values of $u(x, t)$ so we see that as t increases, the temperature throughout the rod approaches zero.

■

10.2.2 Nonhomogeneous Boundary Conditions

The ability to apply the method of separation of variables depends on the presence of homogeneous boundary conditions as we just saw in the previous problem. However, with the heat equation, the temperature at the endpoints may not be held constantly at zero. Instead, consider the case when the temperature at the left-hand endpoint is $T_0 \neq 0$ and at the right-hand endpoint it is $T_1 \neq 0$. Mathematically, we state these **nonhomogeneous boundary conditions** as

$$u(0, t) = T_0 \qquad \text{and} \qquad u(p, t) = T_1,$$

so we are faced with solving the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = T_0,\ u(p,t) = T_1,\ t > 0 \\ u(x,0) = f(x),\ 0 < x < p. \end{cases} \tag{10.4}$$

In this case, we must modify the problem in order to introduce homogeneous boundary conditions to the problem. We do this by using the physical observance that as $t \to \infty$, the temperature in the wire does not depend on t. Hence,

$$\lim_{t\to\infty} u(x,t) = S(x), \tag{10.5}$$

where we call $S(x)$ in equation (10.7) the **steady-state temperature**. Therefore, we let

$$u(x,t) = v(x,t) + S(x), \tag{10.6}$$

where $v(x,t)$ is called the **transient temperature**. We use these two functions to obtain two problems that we can solve. In order to substitute $u(x,t)$ into the heat equation, $u_t = c^2 u_{xx}$, we calculate the derivatives

$$u_t(x,t) = v_t(x,t) + 0 \qquad \text{and} \qquad u_{xx}(x,t) = v_{xx}(x,t) + S''(x).$$

Substitution into the heat equation (10.2) yields

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial^2 v}{\partial x^2} + c^2 S'',$$

so we have the two equations $v_t = c^2 v_{xx}$ and $S'' = 0$. We next consider the boundary conditions. Because

$$u(0,t) = v(0,t) + S(0) = T_0 \qquad \text{and} \qquad u(p,t) = v(p,t) + S(p) = T_1,$$

we can choose the boundary conditions for S to be the nonhomogeneous conditions

$$S(0) = T_0 \qquad \text{and} \qquad S(p) = T_1$$

and the boundary conditions for $v(x,t)$ to be the homogeneous boundary conditions

$$v(0,t) = 0 \qquad \text{and} \qquad v(p,t) = 0.$$

Of course, we have failed to include the initial temperature. Applying this condition, we have $u(x,0) = v(x,0) + S(x) = f(x)$, so the initial condition for v is

$$v(x,0) = f(x) - S(x).$$

Therefore, we have two problems, one for v with homogeneous boundary conditions and one for S that has nonhomogeneous boundary conditions:

$$\begin{cases} S'' = 0, \; 0 < x < p \\ S(0) = T_0, \; S(p) = T_1 \end{cases} \quad \text{and} \quad \begin{cases} v_t = c^2 v_{xx}, \; 0 < x < p, \; t > 0 \\ v(0,t) = 0, \; v(p,t) = 0, \; t > 0 \\ v(x,0) = f(x) - S(x), \; 0 < x < p. \end{cases}$$

Because S is needed in the determination of v, we begin by finding the steady-state temperature and obtain $S(x) = T_0 + \dfrac{T_1 - T_0}{p} x$.

```
In[1944]:= Clear[s, t0]

In[1945]:= DSolve[{s''[x] == 0, s[0] == t0, s[p] == t1}, s[x], x]
Out[1945]= {{s[x] → (p t0 - t0 x + t1 x)/p}}
```

We are now able to find $v(x,t)$ by solving the heat equation with homogeneous boundary conditions for v. Because we solved this problem at the beginning of this section, we do not need to go through the separation of variables procedure. Instead, we use the formula that we derived there using the initial temperature $f(x) - S(x)$. Therefore,

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{p} \, e^{-c^2 \lambda_n t}, \tag{10.7}$$

where $v(x,0) = \sum_{n=1}^{\infty} c_n \sin \dfrac{n\pi x}{p} = f(x) - S(x)$. This means that c_n represents the Fourier sine series coefficients for the function $f(x) - S(x)$ given by

$$c_n = \frac{2}{p} \int_0^p (f(x) - S(x)) \sin \frac{n\pi x}{p} \, dx, \; n = 1, 2, \ldots. \tag{10.8}$$

EXAMPLE 10.2.2: Solve $\begin{cases} u_t = u_{xx}, \; 0 < x < 1, \; t > 0 \\ u(0,t) = 10, \; u(1,t) = 60, \; t > 0 \\ u(x,0) = 10, \; 0 < x < 1. \end{cases}$

SOLUTION: In this case, $c = 1$, $p = 1$, $T_0 = 10$, $T_1 = 60$, and $f(x) = 10$. Therefore, the steady-state solution is

$$S(x) = T_0 + \frac{T_1 - T_0}{p} x = 10 + \frac{60 - 10}{1} x = 10 + 50x.$$

Then, the initial transient temperature is

$$v(x,0) = 10 - (10 + 50x) = -50x$$

so that the series coefficients in the solution (10.7) are given by equation (10.8):

$$c_n = \frac{2}{1}\int_0^1 -50x \sin n\pi x\, dx = -100 \int_0^1 x \sin n\pi x\, dx$$
$$= \frac{100}{n\pi}\cos n\pi = \frac{100}{n\pi}(-1)^n, \ldots$$

```
In[1946]:= c[n_] = -100 ∫_0^1 x Sin[nπx] dx
Out[1946]= -100 (-Cos[n π]/(n π) + Sin[n π]/(n^2 π^2))
In[1947]:= λ[n_] = (nπ)^2;
```

so the transient temperature is

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{p} e^{-c^2\lambda_n t} = \sum_{n=1}^{\infty} (-1)^n \frac{100}{n\pi} \sin n\pi x\, e^{-n^2\pi^2 t}$$

and

$$u(x,t) = v(x,t) + S(x) = 10 + 50x + \sum_{n=1}^{\infty} (-1)^n \frac{100}{n\pi} \sin n\pi x\, e^{-n^2\pi^2 t}.$$

We graph an approximation of $u(x,t)$ for several values of t by graphing

$$10 + 50x + \sum_{n=1}^{30} (-1)^n \frac{100}{n\pi} \sin n\pi x\, e^{-n^2\pi^2 t}.$$

```
In[1948]:= uapprox[x_, t_] = 10+50x+Sum_{n=1}^{30} c[n] Sin[nπx] Exp[-λ[n]t];

In[1949]:= toplot = Table[uapprox[x, t],
                  {t, 0, 0.5, 0.5/20}];

In[1950]:= grays = Table[GrayLevel[i], {i, 0, 0.7, 0.7/20}];

           Plot[Evaluate[toplot], {x, 0, 1},
             PlotStyle- > grays]
```

In Figure 10-3, notice that as $t \to \infty$, $u(x,t) \to S(x)$.

We generate a density plot of this function in Figure 10-4.

```
In[1951]:= DensityPlot[uapprox[x, t], {x, 0, 1}, {t, 0, 0.5},
             PlotPoints- > 30]
```

Notice that the temperature throughout the bar approaches the steady-state temperature as t increases.

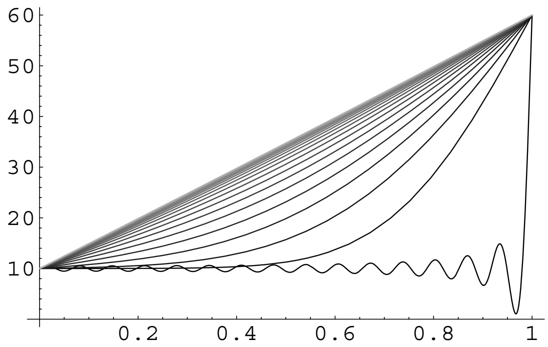


Figure 10-3 An approximation of $u(x, t)$ for 21 equally spaced values of t between 0 and 0.5

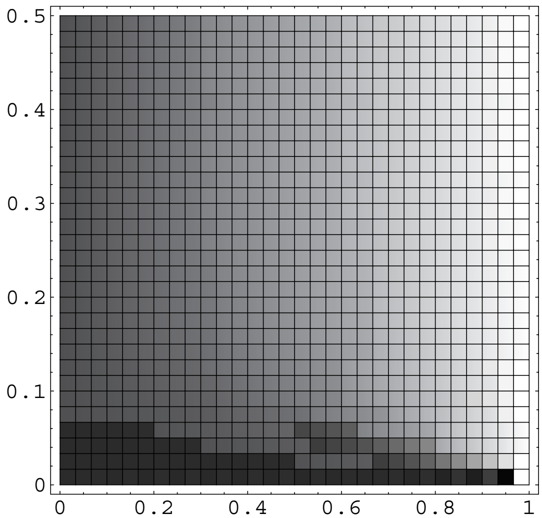


Figure 10-4 Density plot of an approximation of $u(x, t)$

■

10.2.3 Insulated Boundary

Another important situation concerning the flow of heat in a wire involves insulated ends. In this case, heat is not allowed to escape from the ends of the wire. Mathematically, we express these boundary conditions as

$$\frac{\partial u}{\partial x}(0, t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(p, t) = 0,$$

because the rate at which the heat changes along the x-axis at the endpoints $x = 0$ and $x = p$ is zero. Therefore, if we want to determine the temperature in a wire of length p with insulated ends, we solve the initial-boundary-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial u}{\partial x}(0, t) = 0, \ \dfrac{\partial u}{\partial x}(p, t) = 0, \ t > 0 \\ u(x, 0) = f(x), \ 0 < x < p. \end{cases} \tag{10.9}$$

Notice that the boundary conditions are homogeneous, so we can use separation of variables to find $u(x, t) = X(x)T(t)$. By following the steps taken in the solution of the problem with homogeneous boundary conditions, we obtain the ordinary differential equations

$$T' + c^2\lambda T = 0 \qquad \text{and} \qquad X'' + \lambda X = 0.$$

However, when we consider the boundary conditions

$$u_x(0, t) = X'(0)T(0) = 0 \qquad \text{and} \qquad u_x(p, t) = X'(p)T(p) = 0,$$

we wish to avoid letting $T(t) = 0$ for all t (which leads to the trivial solution), so we have the homogeneous boundary conditions

$$X'(0) = 0 \qquad \text{and} \qquad X'(p) = 0.$$

We solve this eigenvalue problem in Example 9.1.5.

Therefore, we solve the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0, \ 0 < x < p \\ X'(0) = 0, \ X'(p) = 0 \end{cases}$$

to find $X(x)$. The eigenvalues and corresponding eigenfunctions of this problem are

$$\lambda_n = \begin{cases} 0, \ n = 0 \\ (n\pi/p)^2, \ n = 1, \ 2, \ \ldots \end{cases} \qquad \text{and} \qquad X_n(x) = \begin{cases} 1, \ n = 0 \\ \cos(n\pi x/p), \ n = 1, \ 2, \ \ldots. \end{cases}$$

Next, we solve the equation $T' + c^2\lambda_n T = 0$. First, for $\lambda_0 = 0$, we have the equation $T' = 0$ which has the solution $T(t) = A_0$, where A_0 is a constant. Therefore, for $\lambda_0 = 0$, the solution is the product

$$u_0(x, t) = X_0(x)T_0(t) = A_0.$$

For $\lambda_n = (n\pi/p)^2$, $T' + c^2\lambda_n T = 0$ has general solution $T_n(t) = a_n e^{-c^2\lambda_n t}$. For these eigenvalues, we have the solution

$$u_n(x, t) = X_n(x)T_n(t) = a_n \cos \frac{n\pi x}{p} \, e^{-c^2\lambda_n t}.$$

Therefore, by the Principle of Superposition, the solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p}\, e^{-c^2\lambda_n t}.$$

Application of the initial temperature yields

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p} = f(x),$$

which is the Fourier cosine series for $f(x)$ where the coefficient A_0 is equivalent to $\frac{1}{2}a_0$ in the original Fourier series given in Section 9.2. Therefore,

$$A_0 = \frac{1}{2}a_0 = \frac{1}{2}\frac{2}{p}\int_0^p f(x)\,dx = \frac{1}{p}\int_0^p f(x)\,dx$$

and (10.10)

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx,\ n = 1,\ 2,\ \ldots.$$

EXAMPLE 10.2.3: Solve $\begin{cases} u_t = u_{xx},\ 0 < x < \pi,\ t > 0 \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0,\ t > 0 \\ u(x,0) = x,\ 0 < x < \pi. \end{cases}$

SOLUTION: In this case, $p = \pi$ and $c = 1$. The Fourier cosine series coefficients for $f(x) = x$ are given by

See Example 9.2.1.

$$A_0 = \frac{1}{2}a_0 = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi n^2}\left[(-1)^n - 1\right],\ n = 1,\ 2,\ \ldots.$$

Therefore, the solution is

$$u(x,t) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi}\cos((2n-1)x)\,e^{-(2n-1)^2 t},$$

where we have used the fact that $a_n = 0$ if n is even. We graph an approximation of $u(x,t)$ by graphing

$$\frac{\pi}{2} - \sum_{n=1}^{40} \frac{4}{(2n-1)^2\pi}\cos((2n-1)x)\,e^{-(2n-1)^2 t}$$

in Figure 10-5 and then a density plot of this function in Figure 10-6.

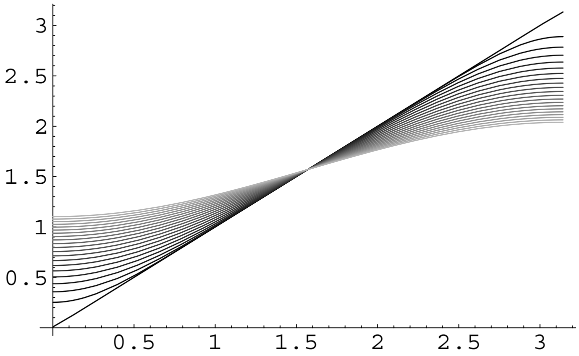


Figure 10-5 An approximation of $u(x, t)$ for 21 equally spaced values of t between 0 and 1

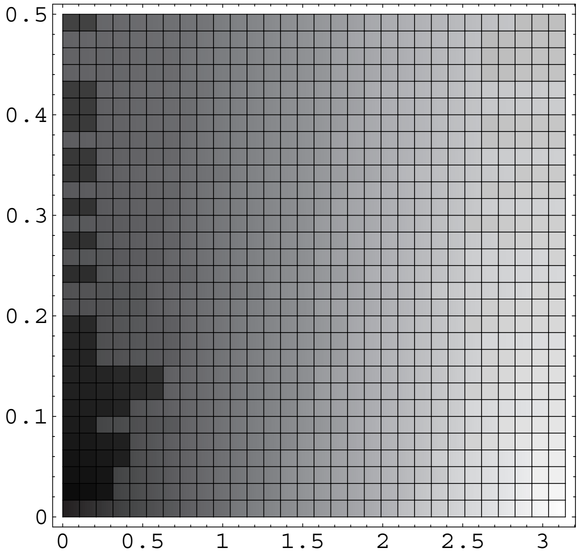


Figure 10-6 A density plot

In[1952]:= **Remove[a]**

$$a_{n\_} = \frac{4}{(2n-1)^2\,\pi};$$

In[1953]:= **uapprox[x_, t_] =** $\frac{\pi}{2} - \sum_{n=1}^{40} a_n \text{Cos}[(2n-1)\,x] \times \text{Exp}\left[-(2n-1)^2\,t\right];$

```
In[1954] := toplot = Table[uapprox[x, t], {t, 0, 1, 1/20}];

In[1955] := grays = Table[GrayLevel[i],
                   {i, 0, 0.7, 0.7/20}];

            Plot[Evaluate[toplot], {x, 0, π},
              PlotStyle- > grays]

In[1956] := DensityPlot[uapprox[x, t], {x, 0, π},
              {t, 0, 0.5}, PlotPoints- > 30]
```

Notice that the temperature eventually becomes $A_0 = \pi/2$ throughout the wire. Temperatures to the left of $x = \pi/2$ increase while those to the right decrease.

■

10.3 The One-Dimensional Wave Equation

The one-dimensional *wave equation* is important in solving an interesting problem.

10.3.1 The Wave Equation

Suppose that we pluck a string (like a guitar or violin string) of length p and constant mass density that is fixed at each end. A question that we might ask is: "What is the position of the string at a particular instance of time?" We answer this question by modeling the physical situation with a partial differential equation, namely the *wave equation* in one spatial variable. We will not go through this derivation as we did with the heat equation, but we point out that it is based on determining the forces that act on a small segment of the string and applying Newton's Second Law of Motion. The partial differential equation that is found is

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \tag{10.11}$$

which is called the (one-dimensional) **wave equation**. In this equation $c^2 = T/\rho$, where T is the tension of the string and ρ is the constant mass of the string per unit length. The solution $u = u(x, t)$ represents the displacement of the string from the x-axis at time t. In order to determine $u = u(x, t)$ we must describe the boundary and initial conditions that model the physical situation. At the ends of the string, the displacement from the x-axis is fixed at zero, so we use the homogeneous boundary conditions

$$u(0,t)=0 \qquad \text{and} \qquad u(p,t)=0$$

for $t > 0$. The motion of the string also depends on the displacement and the velocity at each point of the string at $t = 0$. If the initial displacement is given by $f(x)$ and the initial velocity by $g(x)$, we have the initial conditions

$$u(x,0)=f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0)=g(x)$$

for $0 < x < p$. Therefore, we determine the displacement of the string with the initial-boundary-value problem

$$\begin{cases} c^2\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2},\ 0<x<p,\ t>0 \\ u(0,t)=0,\ u(p,t)=0,\ t>0 \\ u(x,0)=f(x),\ \dfrac{\partial u}{\partial t}(x,0)=g(x),\ 0<x<p. \end{cases} \tag{10.12}$$

Notice that the wave equation requires two initial conditions where the heat equation only needed one. This is due to the fact that there is a second derivative with respect to t while there is only one derivative with respect to t in the heat equation.

This problem is solved through separation of variables by assuming that $u(x,t) = X(x)T(t)$. Substitution into the wave equation yields

$$c^2X''T = XT''$$
$$\frac{X''}{X}=\frac{T''}{c^2T}=-\lambda$$

so we obtain the two second-order ordinary differential equations

$$X''+\lambda X=0 \qquad \text{and} \qquad T''+c^2\lambda T=0.$$

At this point, we solve the equation that involves the homogeneous boundary conditions. As was the case with the heat equation, the boundary conditions in terms of $u(x,t) = X(x)T(t)$ are

$$u(0,t)=X(0)T(t)=0 \qquad \text{and} \qquad u(p,t)=X(p)T(t)=0,$$

so we have

$$X(0)=0 \qquad \text{and} \qquad X(p)=0.$$

Therefore, we determine $X(x)$ by solving the eigenvalue problem

$$\begin{cases} X''+\lambda X=0,\ 0<x<p \\ X(0)=0,\ X(p)=0 \end{cases}$$

which we encountered when solving the heat equation and solved in Section 10.2. The eigenvalues of this problem are

$$\lambda_n = \left(\frac{n\pi}{p}\right)^2, \; n = 1, 2, \ldots$$

with corresponding eigenfunctions

See Example 9.1.4.

$$X_n(x) = \sin\frac{n\pi x}{p}, \; n = 1, 2, \ldots.$$

Next, we solve the equation $T'' + c^2\lambda_n T = 0$. A general solution is

$$T_n(t) = a_n \cos\left(c\sqrt{\lambda_n}t\right) + b_n \sin\left(c\sqrt{\lambda_n}t\right) = a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p},$$

where the coefficients a_n and b_n must be determined. Putting this information together, we obtain

$$u_n(x,t) = \left(a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p},$$

so by the Principle of Superposition, we have

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{cn\pi t}{p} + b_n \sin\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p}.$$

Applying the initial position yields

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{p} = f(x)$$

so a_n is the Fourier sine series coefficient for $f(x)$, which is given by

$$a_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx, \; n = 1, 2, \ldots. \tag{10.13}$$

To determine b_n, we must use the initial velocity. Therefore, we compute

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left(-a_n\frac{cn\pi}{p}\sin\frac{cn\pi t}{p} + b_n\frac{cn\pi}{p}\cos\frac{cn\pi t}{p}\right)\sin\frac{n\pi x}{p}.$$

Then,

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n\frac{cn\pi}{p}\sin\frac{n\pi x}{p} = g(x),$$

so $b_n\dfrac{cn\pi}{p}$ represents the Fourier sine series coefficient for $g(x)$, which means that

$$b_n = \frac{p}{cn\pi}\frac{2}{p}\int_0^p g(x)\sin\frac{n\pi x}{p}\,dx = \frac{2}{cn\pi}\int_0^p g(x)\sin\frac{n\pi x}{p}\,dx, \; n = 1, 2, \ldots. \tag{10.14}$$

EXAMPLE 10.3.1: Solve $\begin{cases} u_{xx} = u_{tt},\ 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ u(1,t) = 0,\ t > 0 \\ u(x,0) = x(1-x),\ u_t(x,0) = 0,\ 0 < x < 1. \end{cases}$

SOLUTION: For this problem, $c = p = 1$, $f(x) = x(1-x)$, and $g(x) = 0$. With this information and equation (10.13) we compute

$$a_n = \frac{2}{1}\int_0^1 x(1-x)\sin n\pi x\,dx = -\frac{4}{n^3\pi^3}\cos n\pi + \frac{4}{n^3\pi^3} = \frac{4}{n^3\pi^3}\left[1-(-1)^n\right],\quad n = 1, 2, \ldots.$$

```
In[1957]:= Remove[a]

           a[n_] = 2 ∫_0^1 x (1 - x) Sin[nπx] dx

Out[1957]= 2 (2/(n^3 π^3) - (2 Cos[n π])/(n^3 π^3) - Sin[n π]/(n^2 π^2))
```

With $g(x) = 0$, we use equation (10.14) to see that the coefficients $b_n = 0$ for all n. Using the fact that $a_n = 0$ for even values of n, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3}\cos((2n-1)\pi t)\sin((2n-1)\pi x).$$

We illustrate the motion of the string by graphing

$$u_k(x,t) = \sum_{n=1}^{k} \frac{8}{(2n-1)^3\pi^3}\cos((2n-1)\pi t)\sin((2n-1)\pi x)$$

using $k = 10$ for 16 equally spaced values of t between 0 and 1 in Figure 10-7.

```
In[1958]:= u[x_, t_] = Sum[8 Cos[(2n - 1) πt] Sin[(2n - 1) πx] / ((2n - 1)^3 π^3), {n, 1, 10}];

In[1959]:= somegraphs =
             Table[Plot[u[x, t], {x, 0, 1},
             DisplayFunction → Identity,
             PlotRange → {-0.3, 0.3},
             Ticks → {{0, 1}, {-0.3, 0.3}}], {t, 0, 1, 1/15}];

           toshow = Partition[somegraphs, 4];

           Show[GraphicsArray[toshow]]
```

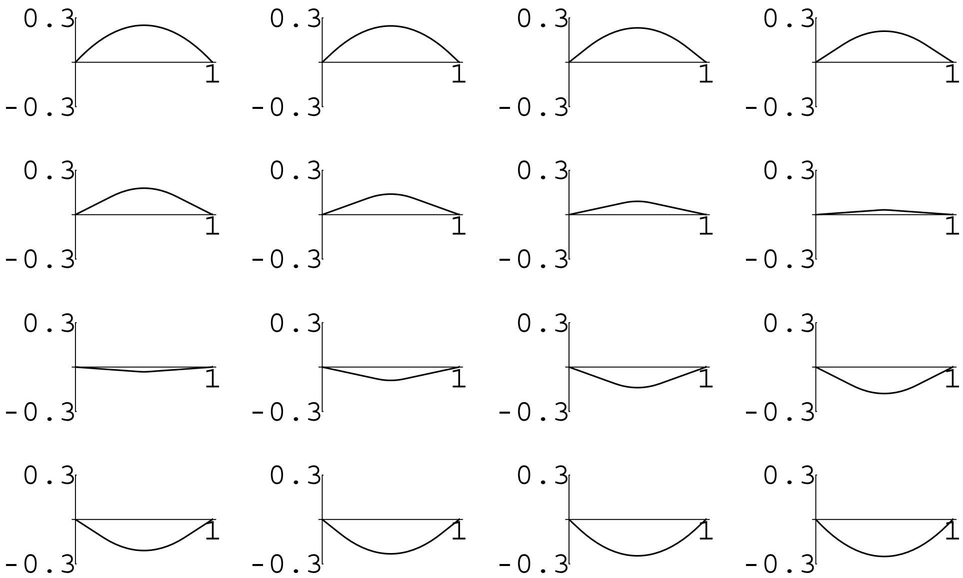


Figure 10-7 Visualizing the motion of a string

To see the motion of the string, you can use a Do loop like

```
Do[Plot[u[x,t],{x,0,1},PlotRange->{-0.3,0.3},
    Ticks->{{0,1},{-0.3,0.3}}],{t,0,2,2/59}]
```

to generate a sequence of graphs and animate the result.

■

EXAMPLE 10.3.2: Solve $\begin{cases} u_{xx} = u_{tt},\ 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ u(1,t) = 0,\ t > 0 \\ u(x,0) = \sin \pi x,\ u_t(x,0) = 3x + 1,\ 0 < x < 1. \end{cases}$

SOLUTION: The appropriate parameters and initial conditions are defined first.

```
In[1960]:= Remove[a,b]

           f[x_] = Sin[πx];

           g[x_] = 3x + 1;
```

Next, the functions to determine the coefficients a_n and b_n in the series approximation of the solution $u = u(x, t)$ are defined.

```
In[1961]:= a_n_ = 2 ∫_0^1 f[x] Sin[nπx] dx
```

$$\text{Out[1961]=}\ 2\left(\frac{\text{Sin}[(-1+n)\pi]}{2(-1+n)\pi} - \frac{\text{Sin}[(1+n)\pi]}{2(1+n)\pi}\right)$$

```
In[1962]:= b_n_ = (2 ∫_0^1 g[x] Sin[nπx] dx)/(nπ)
```

$$\text{Out[1962]=}\ \frac{2\left(\frac{1}{n\pi} - \frac{4\,\text{Cos}[n\pi]}{n\pi} + \frac{3\,\text{Sin}[n\pi]}{n^2\pi^2}\right)}{n\pi}$$

Because n represents an integer, these results indicate that $a_n = 0$ for all $n \geq 1$. We use `Table` to calculate the first 10 values of b_n.

```
In[1963]:= Table[{n, b_n, b_n//N}, {n, 1, 10}]//TableForm
```

Out[1963]=

1	$\frac{10}{\pi^2}$	1.01321
2	$-\frac{3}{2\pi^2}$	-0.151982
3	$\frac{10}{9\pi^2}$	0.112579
4	$-\frac{3}{8\pi^2}$	-0.0379954
5	$\frac{2}{5\pi^2}$	0.0405285
6	$-\frac{1}{6\pi^2}$	-0.0168869
7	$\frac{10}{49\pi^2}$	0.0206778
8	$-\frac{3}{32\pi^2}$	-0.00949886
9	$\frac{10}{81\pi^2}$	0.0125088
10	$-\frac{3}{50\pi^2}$	-0.00607927

The function `u` defined next computes the nth term in the series expansion. Hence, `uapprox` determines the approximation of order k by summing the first k terms of the expansion, as illustrated with `uapprox[10]`.

Notice that we define `uapprox[n]` so that Mathematica "remembers" the terms `uapprox` that are computed. That is, Mathematica need not recompute `uapprox[n-1]` to compute `uapprox[n]` provided that `uapprox[n-1]` has already been computed.

```
In[1964]:= Clear[u, uapprox]

In[1965]:= u[n_] = b_n Sin[nπt] Sin[nπx];

In[1966]:= uapprox[k_] := uapprox[k] = uapprox[k - 1] + u[k];

           uapprox[1] = u[1];
```

```
In[1967]:= uapprox[10]
```

Out[1967]=

$$\frac{10 \text{Sin}[\pi t] \text{Sin}[\pi x]}{\pi^2} - \frac{3 \text{Sin}[2\pi t] \text{Sin}[2\pi x]}{2\pi^2} + \frac{10 \text{Sin}[3\pi t] \text{Sin}[3\pi x]}{9\pi^2} - \frac{3 \text{Sin}[4\pi t] \text{Sin}[4\pi x]}{8\pi^2} + \frac{2 \text{Sin}[5\pi t] \text{Sin}[5\pi x]}{5\pi^2} - \frac{\text{Sin}[6\pi t] \text{Sin}[6\pi x]}{6\pi^2} + \frac{10 \text{Sin}[7\pi t] \text{Sin}[7\pi x]}{49\pi^2} - \frac{3 \text{Sin}[8\pi t] \text{Sin}[8\pi x]}{32\pi^2} + \frac{10 \text{Sin}[9\pi t] \text{Sin}[9\pi x]}{81\pi^2} - \frac{3 \text{Sin}[10\pi t] \text{Sin}[10\pi x]}{50\pi^2}$$

To illustrate the motion of the string, we graph `uapprox[10]`, the tenth partial sum of the series, on the interval [0, 1] for 16 equally spaced values of t between 0 and 2 in Figure 10-8.

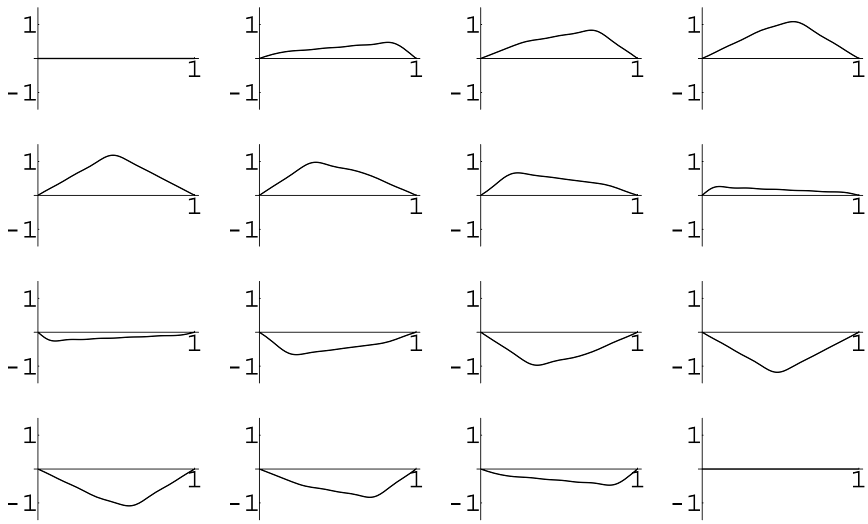
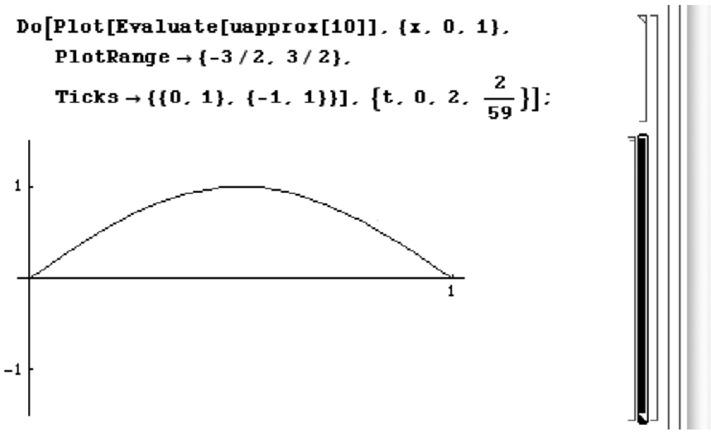


Figure 10-8 Visualizing the motion of a string

```
In[1968]:= somegraphs =
             Table[Plot[Evaluate[uapprox[10]], {x, 0, 1},
               DisplayFunction → Identity,
               PlotRange → {-3/2, 3/2},
               Ticks → {{0, 1}, {-1, 1}}], {t, 0, 2, 2/15}];

           toshow = Partition[somegraphs, 4];

           Show[GraphicsArray[toshow]]
```

If instead we wished to see the motion of the string, we can use a Do loop to generate many graphs and animate the result. We show a frame from the resulting animation.

```
In[1969]:= Do[Plot[Evaluate[uapprox[10]], {x, 0, 1},
               PlotRange → {-3/2, 3/2},
               Ticks → {{0, 1}, {-1, 1}}], {t, 0, 2, 2/59}];
```



■

10.3.2 D'Alembert's Solution

An interesting version of the wave equation is to consider a string of infinite length. Therefore, the boundary conditions are no longer of importance. Instead, we simply work with the wave equation with the initial position and velocity functions. In order to solve the problem

$$\begin{cases} c^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x), & \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{cases} \tag{10.15}$$

we use the change of variables $r = x + ct$ and $s = x - ct$. Using the Chain Rule, we compute the derivatives u_{xx} and u_{tt} in terms of the variables r and s:

$$\begin{aligned} u_x &= u_r r_x + u_s s_x = u_r + u_s, \\ u_{xx} &= (u_r + u_s)_r\, r_x + (u_r + u_s)_s\, s_x = u_{rr} + 2u_{rs} + u_{ss}, \\ u_t &= u_r r_t + u_s s_t = cu_r - cu_s = c\,(u_r - u_s)\,, \end{aligned}$$

and

$$u_{tt} = c\,[(u_r - u_s)_r\, r_t + (u_r - u_s)_s\, s_t] = c^2\,(u_{rr} - 2u_{rs} + u_{ss})\,.$$

Substitution into the wave equation yields

$$\begin{aligned} c^2 u_{xx} &= u_{tt} \\ c^2\,(u_{rr} + 2u_{rs} + u_{ss}) &= c^2\,(u_{rr} - 2u_{rs} + u_{ss}) \\ 4c^2 u_{rs} &= 0 \\ u_{rs} &= 0. \end{aligned}$$

The partial differential equation $u_{rs} = 0$ can be solved by first integrating with respect to s to obtain

$$u_r = f(r),$$

where $f(r)$ is an arbitrary function of r. Then, integrating with respect to r, we have

$$u(r,s) = F(r) + G(s),$$

where F is an antiderivative of f and G is an arbitrary function of s. Returning to our original variables then gives us

$$u(x,t) = F(x + ct) + G(x - ct).$$

We see that this is the solution that `DSolve` returns as well. (Note that `C[1]` and `C[2]` represent the arbitrary functions F and G.)

```
In[1970]:= Clear[u, c, x]

In[1971]:= DSolve[c^2 D[u[x, t], {x, 2}] == D[u[x, t], {t, 2}],
              u[x, t], {x, t}]
```

Out[1971]= $\left\{\left\{u[x,t] \to C[1]\left[t - \frac{\sqrt{c^2}\,x}{c^2}\right] + C[2]\left[t + \frac{\sqrt{c^2}\,x}{c^2}\right]\right\}\right\}$

The functions F and G are determined by the initial conditions which indicate that

$$u(x, 0) = F(x) + G(x) = f(x)$$

and

$$u_t(x, 0) = cF'(x) + cG'(x) = g(x).$$

We can rewrite the second equation by integrating to obtain

$$\begin{aligned} F'(x) - G'(x) &= \frac{1}{c} g(x) \\ F(x) - G(x) &= \frac{1}{c} \int_0^x g(v)\, dv. \end{aligned}$$

Therefore, we solve the system

$$\begin{aligned} F(x) + G(x) &= f(x) \\ F(x) - G(x) &= \frac{1}{c} \int_0^x g(v)\, dv \end{aligned}$$

for $F(x)$ and $G(x)$. Adding these equations yields

$$F(x) = \frac{1}{2} \left(f(x) + \frac{1}{c} \int_0^x g(v)\, dv \right)$$

and subtracting gives us

$$G(x) = \frac{1}{2} \left(f(x) - \frac{1}{c} \int_0^x g(v)\, dv \right).$$

Therefore,

$$F(x + ct) = \frac{1}{2} \left(f(x + ct) + \frac{1}{c} \int_0^{x+ct} g(v)\, dv \right)$$

and

$$G(x - ct) = \frac{1}{2} \left(f(x - ct) - \frac{1}{c} \int_0^{x-ct} g(v)\, dv \right),$$

so the solution is

$$u(x, t) = \frac{1}{2} \left(f(x + ct) + f(x - ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(v)\, dv. \tag{10.16}$$

EXAMPLE 10.3.3: Solve $\begin{cases} u_{xx} = u_{tt}, \ -\infty < x < \infty, \ t > 0 \\ u(x, 0) = 2\left(1 + x^2\right)^{-1}, \ u_t(x, 0) = 0. \end{cases}$

SOLUTION: Using equation (10.16) with $c = 1$, $f(x) = 2\left(1 + x^2\right)^{-1}$, and $g(x) = 0$, we have the solution

$$\begin{aligned} u(x, t) &= \frac{1}{2}\left(f(x + ct) + f(x - ct)\right) = \frac{1}{2}\left[\frac{2}{1 + (x + t)^2} + \frac{2}{1 + (x - t)^2}\right] \\ &= \frac{1}{1 + (x + t)^2} + \frac{1}{1 + (x - t)^2}. \end{aligned}$$

We plot the solution for $t = 0$ to $t = 15$ to illustrate the motion of the string of infinite length in Figure 10-9.

```
In[1972]:= Clear[u, x, t]

            u[x_, t_] = 1/(1 + (x + t)^2) + 1/(1 + (x - t)^2);
```

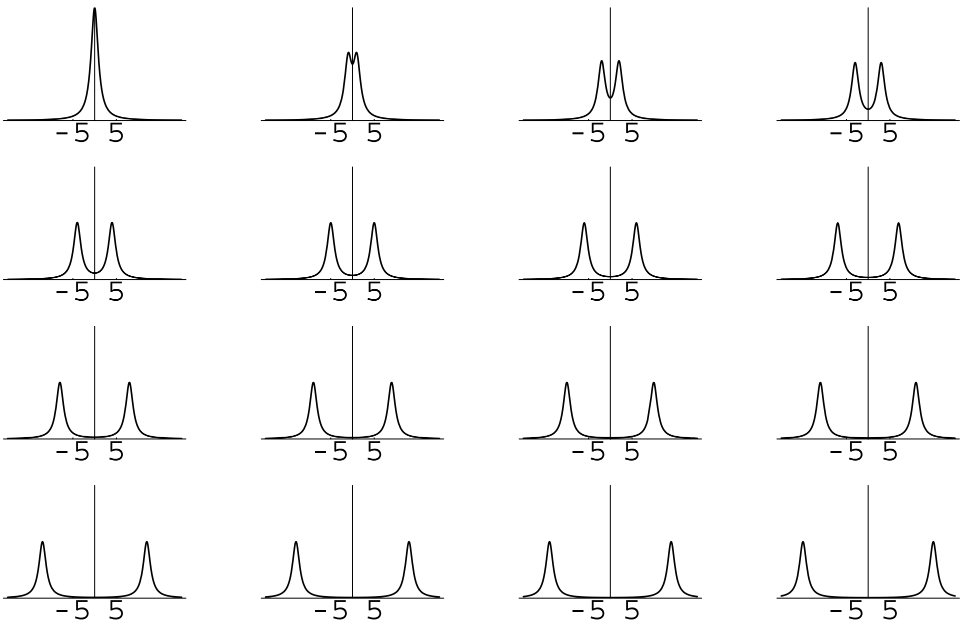


Figure 10-9 A traveling wave solution

```
In[1973]:= somegraphs = Table[
              Plot[u[x, t], {x, -20, 20},
              PlotRange → {0, 2},
              DisplayFunction → Identity,
              Ticks → {{-5, 5}, {0, 5}}],
                {t, 0, 15}];

            toshow = Partition[somegraphs, 4];

            Show[GraphicsArray[toshow]]
```

Alternatively, you can use the following `Do` loop to generate several graphs and animate the results to see the motion of the string.

```
Do[Plot[u[x,t],{x,-20,20},PlotRange->{0,3/2},
     AxesStyle->GrayLevel[.5]],{t,1,15}]
```

D'Alembert's solution is sometimes referred to as the **traveling wave solution** due to the behavior of its graph. The waves appear to move in opposite directions along the x-axis as t increases, as we can see in the graphs.

■

10.4 Problems in Two Dimensions: Laplace's Equation

10.4.1 Laplace's Equation

Laplace's equation, often called the **potential equation**, is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (10.17)$$

in rectangular coordinates and is one of the most useful partial differential equations in that it arises in many fields of study. These include fluid flows as well as electrostatic and gravitational potential. Because the potential $u = u(x, y)$ does not depend on time, no initial condition is required, so we are faced with solving a pure boundary-value problem when working with Laplace's equation. The boundary conditions can be stated in different forms. If the value of the solution is given around the boundary of the region, then the boundary-value problem is called a **Dirichlet problem** whereas if the normal derivative of the solution is given around the boundary, the problem is known as a **Neumann problem**. We now investigate the solutions to Laplace's equation in a rectangular region by, first, stating the general form of the Dirichlet problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b \\ u(x,0) = f_1(x),\ u(x,b) = f_2(x),\ 0 < x < a \\ u(0,y) = g_1(y),\ u(a,y) = g_2(y),\ 0 < y < b. \end{cases} \tag{10.18}$$

This boundary-value problem is solved through separation of variables. We begin by considering the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b \\ u(x,0) = 0,\ u(x,b) = f(x),\ 0 < x < a \\ u(0,y) = 0,\ u(a,y) = 0,\ 0 < y < b. \end{cases} \tag{10.19}$$

In this case, we assume that

$$u(x,y) = X(x)Y(y)$$

so substitution into Laplace's equation (10.17) yields

$$\begin{aligned} X''Y + XY'' &= 0 \\ \frac{X''}{X} = -\frac{Y''}{Y} &= -\lambda, \end{aligned}$$

where $-\lambda$ is the separation constant. Therefore, we have the ordinary differential equations $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$. Notice that the boundary conditions along the lines $x = 0$ and $x = a$ are homogeneous. In fact, because $u(0,y) = X(0)Y(y) = 0$ and $u(a,y) = X(a)Y(y) = 0$, we have $X(0) = 0$ and $X(a) = 0$. Therefore, we first solve the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0,\ 0 < x < a \\ X(0) = 0,\ X(a) = 0 \end{cases}$$

which was solved with $a = p$ in Section 9.1. There, we found the eigenvalues and corresponding eigenfunctions to be $\lambda_n = (n\pi/a)^2$ and $X_n(x) = \sin(n\pi x/a)$, $n = 1, 2, \ldots$. We then solve the equation $Y'' - \lambda Y = 0$. From our experience with second-order equations, we know that $Y_n(y) = a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}$, which can be written in terms of the hyperbolic trigonometric functions as

See Example 9.1.4.

$$Y_n(y) = A_n \cosh \lambda_n y + B_n \sinh \lambda_n y = A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a}.$$

Then, using the homogeneous boundary condition $u(x,0) = X(x)Y(0) = 0$, which indicates that $Y(0) = 0$, we have

$$Y_n(0) = A_n \cosh 0 + B_n \sinh 0 = A_n = 0,$$

so $A_n = 0$ for all n. Therefore, $Y_n(y) = B_n \sinh \lambda_n y$, and a solution of equation (10.19) is

$$u_n(x,y) = B_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a},$$

so by the Principle of Superposition,

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}$$

is also a solution, where the coefficients are determined with the boundary condition $u(x,b) = f(x)$. Substitution into the solution yields

$$u(x,b) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = f(x),$$

where $B_n \sinh(n\pi b/a)$ represents the Fourier sine series coefficients given by

$$\begin{aligned} B_n \sinh \frac{n\pi b}{a} &= \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a}\, dx \\ B_n &= \frac{2}{a \sinh \dfrac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx. \end{aligned} \tag{10.20}$$

EXAMPLE 10.4.1: Solve $\begin{cases} u_{xx} + u_{yy} = 0, \ 0 < x < 1, \ 0 < y < 2 \\ u(x,0) = 0, \ u(x,2) = x(1-x), \ 0 < x < 1 \\ u(0,y) = 0, \ u(1,y) = 0, \ 0 < y < 2. \end{cases}$

SOLUTION: In this case, $a = 1$, $b = 2$, and $f(x) = x(1-x)$. Therefore,

$$\begin{aligned} B_n &= \frac{2}{\sinh 2n\pi}\int_0^1 x(1-x) \sin n\pi x\, dx = \frac{2}{\sinh 2n\pi}\left(-\frac{2}{n^3\pi^3}\cos n\pi + \frac{2}{n^3\pi^3}\right) \\ &= \frac{4}{n^3\pi^3 \sinh 2n\pi}\left[1 - (-1)^n\right], \ n = 1, 2, \ldots \end{aligned}$$

```
In[1974]:= 2 ∫_0^1 x (1 - x) Sin[nπx] dx / Sinh[2nπ]
Out[1974]= 2 Csch[2 n π] (2/(n^3 π^3) - 2 Cos[n π]/(n^3 π^3) - Sin[n π]/(n^2 π^2))
```

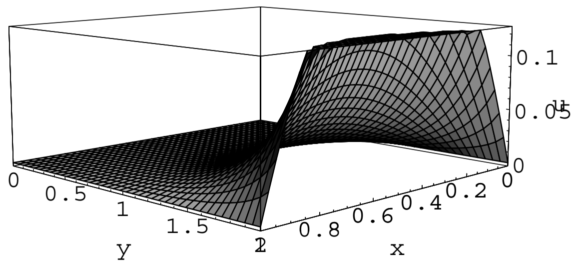


Figure 10-10 Approximating a solution of Laplace's equation

so the solution is

$$u(x,y)=\sum_{n=1}^{\infty} B_n \sinh n\pi y \sin n\pi x$$
$$=\sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3 \sinh 2(2n-1)\pi} \sinh(2n-1)\pi y \sin(2n-1)\pi x.$$

We plot $u(x, y)$ using the first 15 terms of the series solution in Figure 10-10.

```
In[1975]:= Clear[u]
```

$$\texttt{u[x\_, y\_]} = \sum_{n=1}^{15} \frac{\texttt{8 Sinh[(2n - 1) }\pi\texttt{y] Sin[(2n - 1) }\pi\texttt{x]}}{\texttt{(2n - 1)}^3\,\pi^3\,\texttt{Sinh[2 (2n - 1) }\pi\texttt{]}};$$

```
In[1976]:= Plot3D[u[x, y], {x, 0, 1}, {y, 0, 2},
             ViewPoint → {2.365, 2.365, 0.514},
             AxesLabel → {"x", "y", "u"}, PlotPoints → 40]
```

Alternatively, we can generate a contour or density plot of $u(x, y)$ as shown in Figure 10-11.

```
In[1977]:= cplot = ContourPlot[u[x, y], {x, 0, 1}, {y, 0, 2},
                      PlotPoints- > 30,
                      DisplayFunction- > Identity];

           dplot = DensityPlot[u[x, y], {x, 0, 1}, {y, 0, 2},
                      PlotPoints- > 30,
                      DisplayFunction- > Identity];

           Show[GraphicsArray[{cplot, dplot}]]
```

We notice that the value of $u(x, y)$ decreases to zero away from the boundary $y = 2$.

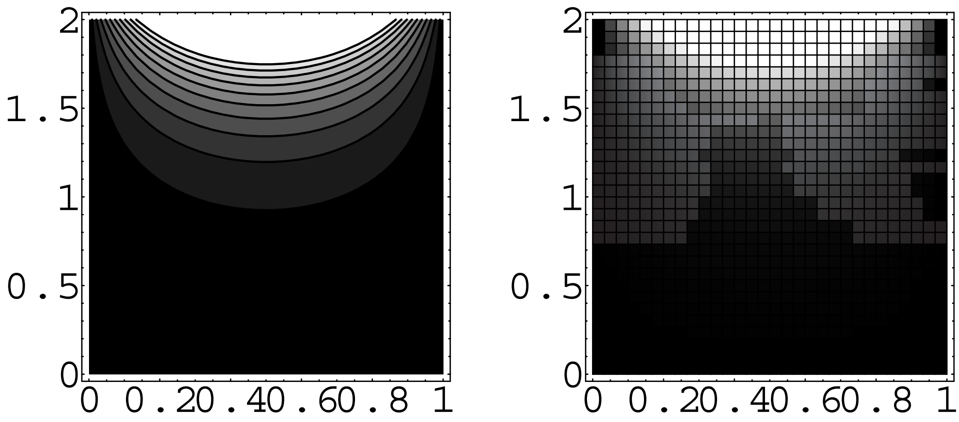


Figure 10-11 On the left, a contour plot and on the right, a density plot

■

Any version of Laplace's equation on a rectangular region can be solved through separation of variables as long as we have a pair of homogeneous boundary conditions in the same variable.

EXAMPLE 10.4.2: Solve $\begin{cases} u_{xx} + u_{yy} = 0, \; 0 < x < \pi, \; 0 < y < 1 \\ u(x, 0) = 0, \; u(x, 1) = 0, \; 0 < x < \pi \\ u(0, y) = \sin 2\pi y, \; u(\pi, y) = 4, \; 0 < y < 1. \end{cases}$

SOLUTION: As we did in the previous problem, we assume that $u(x, y) = X(x)Y(y)$. Notice that this problem differs from the previous problem in that the homogeneous boundary conditions are in terms of the variable y. Hence, when we separate variables, we use a different constant of separation. This yields

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda,$$

so we have the ordinary differential equations $X'' - \lambda X = 0$ and $Y'' + \lambda Y = 0$. Therefore, with the homogeneous boundary conditions $u(x, 0) = X(x)Y(0) = 0$ and $u(x, 1) = X(x)Y(1) = 0$, we have $Y(0) = 0$ and $Y(1) = 0$.

The eigenvalue problem

$$\begin{cases} Y'' + \lambda Y = 0, \ 0 < y < 1 \\ Y(0) = 0, \ Y(1) = 0 \end{cases}$$

has eigenvalues $\lambda_n = (n\pi/1)^2 = n^2\pi^2$, $n = 1, 2, \ldots$ and eigenfunctions $Y_n(y) = \sin n\pi y$, $n = 1, 2, \ldots$. We then solve the equation $X'' - \lambda_n X = 0$ obtaining $X_n(x) = a_n e^{n\pi x} + b_n e^{-n\pi x}$, which can be written in terms of the hyperbolic trigonometric functions as

$$X_n(x) = A_n \cosh n\pi x + B_n \sinh n\pi x.$$

Now, because the boundary conditions on the boundaries $x = 0$ and $x = \pi$ are nonhomogeneous, we use the Principle of Superposition to obtain the solution

$$u(x, y) = \sum_{n=1}^{\infty} (A_n \cosh n\pi x + B_n \sinh n\pi x) \sin n\pi y.$$

Therefore,

$$u(0, y) = \sum_{n=1}^{\infty} A_n \sin n\pi y = \sin 2\pi y,$$

so $A_2 = 1$ and $A_n = 0$ for $n \neq 2$. Similarly,

$$u(\pi, y) = A_2 \cosh 2\pi^2 + \sum_{n=1}^{\infty} B_n \sinh n\pi^2 \sin n\pi y = 4,$$

which indicates that

$$\sum_{n=1}^{\infty} B_n \sinh n\pi^2 \sin n\pi y = 4 - \cosh 2\pi^2.$$

Then, $B_n \sinh n\pi^2$ are the Fourier sine series coefficients for the constant function $4 - \cosh 2\pi^2$ which are given by

$$\begin{aligned} B_n \sinh n\pi^2 &= \frac{2}{1} \int_0^1 \left(4 - \cosh 2\pi^2\right) \sin n\pi y \, dy = -2\left(4 - \cosh 2\pi^2\right) \left[\frac{1}{n\pi} \cos n\pi y\right]_0^1 \\ &= -\frac{2\left(4 - \cosh 2\pi^2\right)}{n\pi} \left[(-1)^n - 1\right], \ n = 1, 2, \ldots . \end{aligned}$$

From this formula, we see that $B_n = 0$ if n is even. Therefore, we express these coefficients as

$$B_{2n-1} = \frac{4\left(4 - \cosh 2\pi^2\right)}{(2n-1)\pi}, \ n = 1, 2, \ldots$$

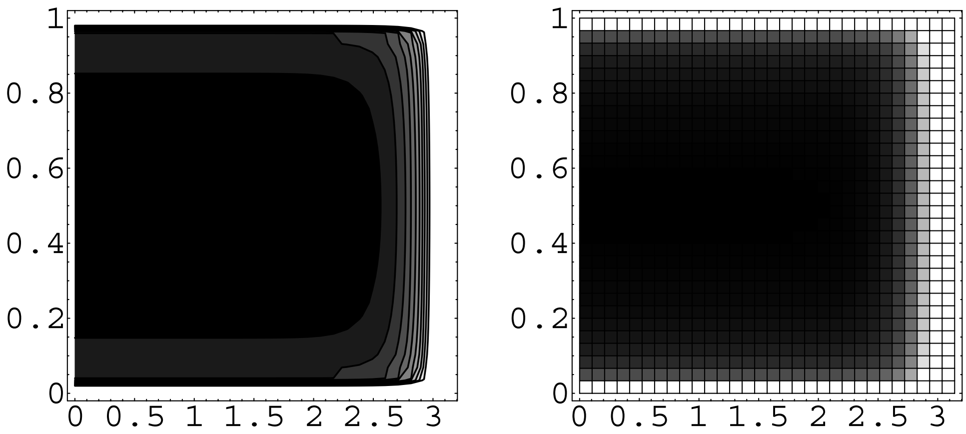


Figure 10-12 On the left, a contour plot and on the right, a density plot

so that the solution is

$$u(x, y) = \cosh 2\pi x + \sum_{n=1}^{\infty} \frac{4\left(4 - \cosh 2\pi^2\right)}{(2n-1)\pi} \sin(2n-1)\pi y.$$

As in the first example, we generate contour and density plots of an approximation of u. See Figure 10-12.

```
In[1978]:= Clear[u]

           u[x_, y_] =
             Cosh[2πx]
                30
             + Σ   4 (4 - Cosh[2π²]) Sin[(2n - 1) πy]
               n=1 ──────────────────────────────────── ;
                              (2n - 1) π

In[1979]:= p1 = ContourPlot[u[x, y], {x, 0, π}, {y, 0, 1},
                 PlotPoints → 30,
                 DisplayFunction- > Identity];

           p2 = DensityPlot[u[x, y], {x, 0, π}, {y, 0, 1},
                 PlotPoints → 30,
                 DisplayFunction- > Identity];

           Show[GraphicsArray[{p1, p2}]]
```

■

10.5 Two-Dimensional Problems in a Circular Region

In some situations, the region on which we solve a boundary-value problem or an initial-boundary-value problem is not rectangular in shape. For example, we usually do not have rectangular shaped drumheads and the heating elements on top of the stove are not square. Instead, these objects are typically circular in shape, so we find the use of polar coordinates convenient. In this section, we discuss problems of this type by presenting two important problems solved on circular regions, Laplace's equation which is related to the steady-state temperature and the wave equation which is used to find the displacement of a drumhead.

10.5.1 Laplace's Equation in a Circular Region

In calculus, we found that polar coordinates are useful in solving many problems. The same can be said for solving boundary-value problems in a circular region. With the change of variables

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

we transform Laplace's equation in rectangular coordinates, $u_{xx} + u_{yy} = 0$, to polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = 0,\ 0 < r < \rho,\ -\pi < \theta < \pi, \qquad (10.21)$$

where ρ is the radius of the drumhead. Recall that for the solution of Laplace's equation in a rectangular region, we had to specify a boundary condition on each of the four boundaries of the rectangle. However, in the case of a circle, there are not four sides, so we must alter the boundary conditions. Because in polar coordinates the points (r, π) and $(r, -\pi)$ are equivalent for the same value of r, we want our solution and its derivative with respect to θ to match at these points (so that the solution is smooth). Therefore, two of the boundary conditions are

$$u(r, -\pi) = u(r, \pi) \qquad \text{and} \qquad \frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi) \qquad (10.22)$$

for $0 < r < \rho$. Also, we want our solution to be bounded at $r = 0$, so another boundary condition is $|u(0, \theta)| < \infty$ for $-\pi < \theta < \pi$. Finally, we specify the value of the solution around the boundary of the circle $r = \rho$ to be $u(\rho, \theta) = f(\theta)$ for

$-\pi < \theta < \pi$. Therefore, we solve the following boundary-value problem to solve Laplace's equation (the Dirichlet problem) in a circular region of radius ρ:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2} = 0,\ 0 < r < \rho,\ -\pi < \theta < \pi \\ u(r, -\pi) = u(r, \pi),\ \dfrac{\partial u}{\partial \theta}(r, -\pi) = \dfrac{\partial u}{\partial \theta}(r, \pi),\ 0 < r < \rho \\ |u(0, \theta)| < \infty,\ u(\rho, \theta) = f(\theta),\ -\pi < \theta < \pi. \end{cases} \tag{10.23}$$

Using separation of variables, we assume that $u(r, \theta) = R(r)H(\theta)$. Substitution into Laplace's equation yields

$$\begin{aligned} R''H + \frac{1}{r}R'H + RH'' &= 0 \\ R''H + \frac{1}{r}R'H &= -RH'' \\ \frac{rR'' + R'}{rR} &= -\frac{H''}{H} = \lambda. \end{aligned}$$

Therefore, we have the ordinary differential equations

$$H'' + \lambda H = 0 \qquad \text{and} \qquad r^2R'' + rR' - \lambda R = 0.$$

Notice that the boundary conditions given by equation (10.22) imply that

$$R(r)H(-\pi) = R(r)H(\pi) \qquad \text{and} \qquad R(r)H'(-\pi) = R(r)H'(\pi),$$

so that

$$H(-\pi) = H(\pi) \qquad \text{and} \qquad H'(-\pi) = H'(\pi).$$

This means that we begin by solving the eigenvalue problem

$$\begin{cases} H'' + \lambda H = 0,\ -\pi < \theta < \pi \\ H(-\pi) = H(\pi),\ H'(-\pi) = H'(\pi). \end{cases}$$

The eigenvalues and corresponding eigenfunctions of this problem are

$$\lambda_n = \begin{cases} 0,\ n = 0 \\ n^2,\ n = 1, 2, \ldots \end{cases} \qquad \text{and} \qquad H_n(\theta) = \begin{cases} 1,\ n = 0 \\ a_n \cos n\theta + b_n \sin n\theta,\ n = 1, 2, \ldots. \end{cases}$$

Because $r^2R'' + rR' - \lambda_n^2 R = 0$ is a Cauchy–Euler equation, we assume that $R(r) = r^m$:

$$\begin{aligned} m(m-1)r^2 r^{m-2} + mrr^{m-1} - \lambda_n r^m &= 0 \\ r^m\left[m(m-1) + m - \lambda_n\right] &= 0. \end{aligned}$$

Therefore,

$$m^2 - \lambda_n^2 = 0 \qquad \text{so} \qquad m = \pm\lambda_n.$$

If $\lambda_0 = 0$, then $R_0(r) = a_0 + b_0 \ln r$. However, because we require that the solution be bounded near $r = 0$ and $\lim_{r\to 0^+} \ln r = -\infty$, we must choose $b_0 = 0$. Therefore, $R_0(r) = a_0$. On the other hand, if $\lambda_n = n^2$, $n = 1, 2, \ldots$, then $R_n(r) = a_n r^n + b_n r^{-n}$. Similarly, because $\lim_{r\to 0^+} r^{-n} = \infty$, we must let $b_n = 0$, so $R_n(r) = a_n r^n$. By the Principle of Superposition, we have the solution

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right).$$

We find the coefficients by applying the boundary condition $u(\rho,\theta) = f(\theta)$. This yields

$$u(\rho,\theta) = a_0 + \sum_{n=1}^{\infty} \rho^n \left(a_n \cos n\theta + b_n \sin n\theta\right) = f(\theta)$$

so a_0, a_n, and b_n are related to the Fourier series coefficients in the following way:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \\ a_n &= \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta,\ n = 1,\ 2,\ \ldots, \text{and} \\ b_n &= \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta,\ n = 1,\ 2,\ \ldots. \end{aligned} \tag{10.24}$$

EXAMPLE 10.5.1: Solve $\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2} = 0,\ 0 < r < 2,\ -\pi < \theta < \pi \\ u(r,-\pi) = u(r,\pi),\ \dfrac{\partial u}{\partial \theta}(r,-\pi) = \dfrac{\partial u}{\partial \theta}(r,\pi),\ 0 < r < 2 \\ |u(0,\theta)| < \infty,\ u(2,\theta) = |\theta|,\ -\pi < \theta < \pi. \end{cases}$

SOLUTION: Notice that $f(\theta) = |\theta|$ is an even function on $-\pi < \theta < \pi$. Therefore, $b_n = 0$ for $n = 1, 2, \ldots$, a_0 is given by

$$a_0 = \frac{1}{2\pi}\int_{\pi}^{\pi} |\theta|\,d\theta = \frac{1}{\pi}\int_0^{\pi} \theta\,d\theta = \frac{\pi}{2},$$

```
In[1980]:= ∫_0^π θ dθ / π
Out[1980]= π/2
```

and a_n is given by

$$\begin{aligned} a_n &= \frac{1}{2^n\pi}\int_{\pi}^{\pi} |\theta|\cos n\theta\,d\theta = \frac{1}{2^{n-1}\pi}\int_0^{\pi} \theta\cos n\theta\,d\theta \\ &= \frac{1}{2^{n-1}n^2\pi}(\cos n\pi - 1) = \frac{1}{2^{n-1}n^2\pi}\left[(-1)^n - 1\right],\ n = 1,\ 2,\ \ldots. \end{aligned}$$

```
In[1981]:= a[n_] = ∫_0^π θ Cos[nθ] dθ / (π 2^(n-1)) //Simplify
```

```
Out[1981]= 2^(1-n) (-1 + Cos[n π] + n π Sin[n π]) / (n^2 π)
```

Notice that $a_{2n} = 0$, $n = 1, 2, \ldots$, while $a_{2n-1} = \dfrac{-2}{2^{2n-1}(2n-1)^2\pi}$, $n = 1, 2, \ldots$, so the solution is

$$u(r, \theta) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{1}{2^{2n-2}(2n-1)^2\pi} r^{2n-1} \cos(2n-1)\theta.$$

In the same manner as in previous examples, we graph an approximation of this solution. See Figure 10-13.

```
In[1982]:= Clear[u, r, θ]

u[r_, θ_] = π/2 - Σ_{n=1}^{20} r^(2n-1) Cos[(2n - 1)θ] / (π 2^(2n-2) (2n - 1)^2);
```

```
In[1983]:= ParametricPlot3D[{r Cos[θ], r Sin[θ], u[r, θ]},
              {r, 0, 2}, {θ, -π, π}, Boxed → False,
              PlotPoints → 35]
```

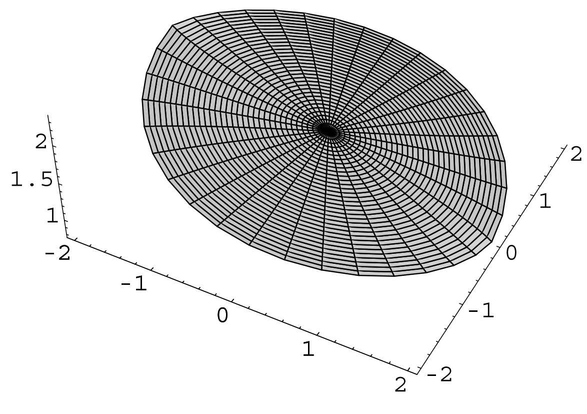


Figure 10-13 Plot of a solution to Laplace's equation in polar coordinates

■

10.5.2 The Wave Equation in a Circular Region

One of the more interesting problems involving two spatial dimensions (x and y) is the wave equation,

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}. \tag{10.25}$$

This is due to the fact that the solution to this problem represents something with which we are all familiar, the displacement of a drumhead. Because drumheads are circular in shape, we investigate the solution of the wave equation in a circular region. Therefore, we transform the wave equation into polar coordinates. Previously, we saw that converting Laplace's equation from rectangular coordinates (x, y) to polar coordinates (r, θ) results in the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

so it follows that the wave equation in polar coordinates becomes

$$c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)=\frac{\partial^2 u}{\partial t^2}. \tag{10.26}$$

If we assume that the displacement of the drumhead from the xy-plane at time t is the same at equal distances from the origin, we say that the solution $u = u(r, \theta)$ is **radially symmetric**. Therefore, $\partial^2 u/\partial\theta^2 = 0$, so the wave equation can be expressed in terms of r and t as

If the solution is radially symmetric, the value of u does not depend on the angle θ.

$$c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2}. \tag{10.27}$$

Of course, to find $u = u(r, t)$ we need the appropriate boundary and initial conditions. Because the circular boundary of the drumhead $r = \rho$ must be fixed so that it does not move we say that $u(\rho, t) = 0$ for $t > 0$. Then, as we had in Laplace's equation on a circular region, we require that the solution $u = u(r, t)$ be bounded near the origin, so we have the condition $|u(0, t)| < \infty$ for $t > 0$. The initial position and initial velocity functions are given as functions of r as

$$u(r, 0) = f(r) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(r, 0) = g(r)$$

for $0 < r < \rho$. Therefore, the initial-boundary-value problem to find the displacement $u = u(r, t)$ of a circular drumhead (of radius ρ) is given by

$$\begin{cases} c^2\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)=\dfrac{\partial^2 u}{\partial t^2}, \; 0<r<\rho, \; t>0 \\ u(\rho, t)=0, \; |u(0,t)|<\infty, \; t>0 \\ u(r,0)=f(r), \; \dfrac{\partial u}{\partial t}(r,0)=g(r), \; 0<r<\rho. \end{cases} \tag{10.28}$$

As with other problems, we are able to use separation of variables to find $u = u(r, t)$ by assuming that $u(r, t) = R(r)T(t)$. Substitution into the wave equation yields

$$c^2\left(R''T + \frac{1}{r}R'T\right) = RT''$$
$$\frac{rR'' + R'}{rR} = \frac{T''}{c^2T} = -k^2,$$

where $-k^2$ is the separation constant. Separating the variables, we have the ordinary differential equations

$$r^2R'' + rR' + k^2r^2R = 0 \qquad \text{and} \qquad T'' + c^2k^2T = 0.$$

We recognize the equation $r^2R'' + rR' + k^2r^2R = 0$ as Bessel's equation of order zero that has solution

$$R(r) = c_1J_0(kr) + c_2Y_0(kr),$$

where J_0 and Y_0 are the Bessel functions of order zero of the first and second kind, respectively. In terms of R, we express the boundary condition $|u(0, t)| < \infty$ as $|R(0)| < \infty$. Therefore, because $\lim_{r\to 0^+} Y_0(kr) = -\infty$, we must choose $c_2 = 0$. Applying the other boundary condition, $R(\rho) = 0$, we have

$$R(\rho) = c_1J_0(k\rho) = 0,$$

so to avoid the trivial solution with $c_1 = 0$, we have $k\rho = \alpha_n$, where α_n is the nth zero of $J_0(x)$. Because k depends on n, we write

$$k_n = \frac{\alpha_n}{\rho}.$$

The solution of $T'' + c^2k_n^2T = 0$ is

$$T_n(t) = A_n \cos ck_nt + B_n \sin ck_nt,$$

so with the Principle of Superposition, we form the solution

$$u(r, t) = \sum_{n=1}^{\infty} (A_n \cos ck_nt + B_n \sin ck_nt)\, J_0\,(k_nr)\,,$$

where the coefficients A_n and B_n are found through application of the initial position and velocity functions. With

$$u(r,0)=\sum_{n=1}^{\infty}A_nJ_0(k_nr)=f(r)$$

and the orthogonality conditions of the Bessel functions, we find that

$$A_n=\frac{\int_0^{\rho} rf(r)J_0(k_nr)\,dr}{\int_0^{\rho} r\left[J_0(k_nr)\right]^2 dr}=\frac{2}{\left[J_1(\alpha_n)\right]^2}\int_0^{\rho} rf(r)J_0(k_nr)\,dr,\ n=1,\ 2,\ \ldots .$$

Similarly, because

$$\frac{\partial u}{\partial t}(r,0)=\sum_{n=1}^{\infty}\left(-ck_nA_n\sin ck_nt+ck_nB_n\cos ck_nt\right)J_0(k_nr)$$

we have

$$u_t(r,0)=\sum_{n=1}^{\infty}ck_nB_nJ_0(k_nr)=g(r).$$

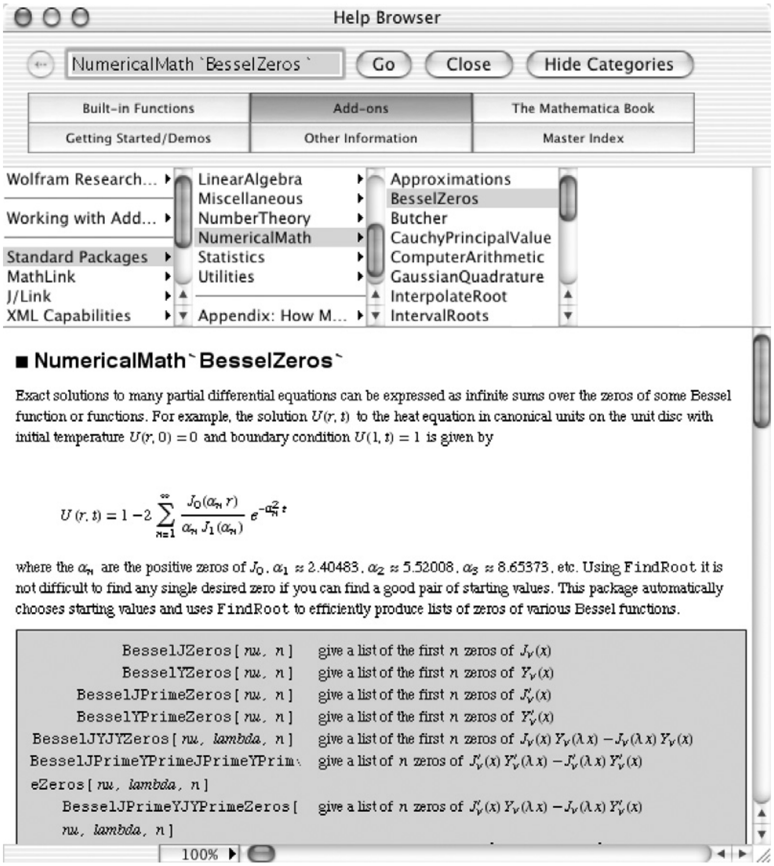
Therefore,

$$B_n=\frac{\int_0^{\rho} rg(r)J_0(k_nr)\,dr}{ck_n\int_0^{\rho} r\left[J_0(k_nr)\right]^2 dr}=\frac{2}{ck_n\left[J_1(\alpha_n)\right]^2}\int_0^{\rho} rg(r)J_0(k_nr)\,dr,\ n=1,\ 2,\ \ldots .$$

As a practical matter, in nearly all cases, these formulas are difficult to evaluate.

EXAMPLE 10.5.2: Solve
$$\begin{cases}\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}=\dfrac{\partial^2 u}{\partial t^2},\ 0<r<1,\ t>0\\ u(1,t)=0,\ |u(0,t)|<\infty,\ t>0\\ u(r,0)=r(r-1),\ \dfrac{\partial u}{\partial t}(r,0)=\sin\pi r,\ 0<r<1.\end{cases}$$

SOLUTION: In this case, $\rho=1$, $f(r)=r(r-1)$, and $g(r)=\sin\pi r$. To calculate the coefficients, we will need to have approximations of the zeros of the Bessel functions, so we load the **BesselZeros** package, which is contained in the **NumericalMath** folder (or directory) and define α_n to be the nth zero of $y=J_0(x)$.




```
In[1984]:= << NumericalMath`BesselZeros`
```

```
In[1985]:= αn_ := αn = BesselJZeros[0, {n, n}][[1]]
```

Next, we define the constants ρ and c and the functions $f(r) = r(r-1)$, $g(r) = \sin \pi r$, and $k_n = \alpha_n/\rho$.

```
In[1986]:= c = 1;
           ρ = 1;
           f[r_] = r (r - 1);
           g[r_] = Sin[π r];
           kn_ := kn = αn/ρ;
```

The formulas for the coefficients A_n and B_n are then defined so that an approximate solution may be determined. (We use lower-case letters to avoid any possible ambiguity with built-in Mathematica functions.) Note that we use `NIntegrate` to approximate the coefficients and avoid the difficulties in integration associated with the presence of the Bessel function of order zero.

```
In[1987]:= a[n_] :=
              a[n] = (2 NIntegrate[r f[r] BesselJ[0, k[n] r],
                   {r, 0, ρ}])/BesselJ[1, α[n]]^2;
```

```
In[1988]:= b[n_] :=
              b[n] = (2 NIntegrate[r g[r] BesselJ[0, k[n] r],
                   {r, 0, ρ}])/(c k[n] BesselJ[1, α[n]]^2)
```

We now compute the first 10 values of A_n and B_n. Because `a` and `b` are defined using the form $a_{n\_} := a_n = \ldots$ and $b_{n\_} := b_n = \ldots$, Mathematica remembers these values for later use.

```
In[1989]:= Table[{n, a[n], b[n]}, {n, 1, 10}]//TableForm
```

Out[1989]=

1	1	0.52118
2	0.208466	-0.145776
3	0.00763767	-0.0134216
4	0.0383536	-0.00832269
5	0.00534454	-0.00250503
6	0.0150378	-0.00208315
7	0.00334937	-0.000882012
8	0.00786698	-0.000814719
9	0.00225748	-0.000410202
10	0.00479521	-0.000399219

The *n*th term of the series solution is defined in `u`. Then, an approximate solution is obtained in `uapprox` by summing the first 10 terms of `u`.

```
In[1990]:= u[n_, r_, t_] := (a[n] Cos[c k[n] t] + b[n] Sin[c k[n] t])
              BesselJ[0, k[n] r];
```

```
In[1991]:= uapprox[r_, t_] = Sum[u[n, r, t], {n, 1, 10}];
```

$$\text{uapprox}[r\_, t\_] = \sum_{n=1}^{10} u[n, r, t];$$

We graph `uapprox` for several values of t in Figure 10-14.

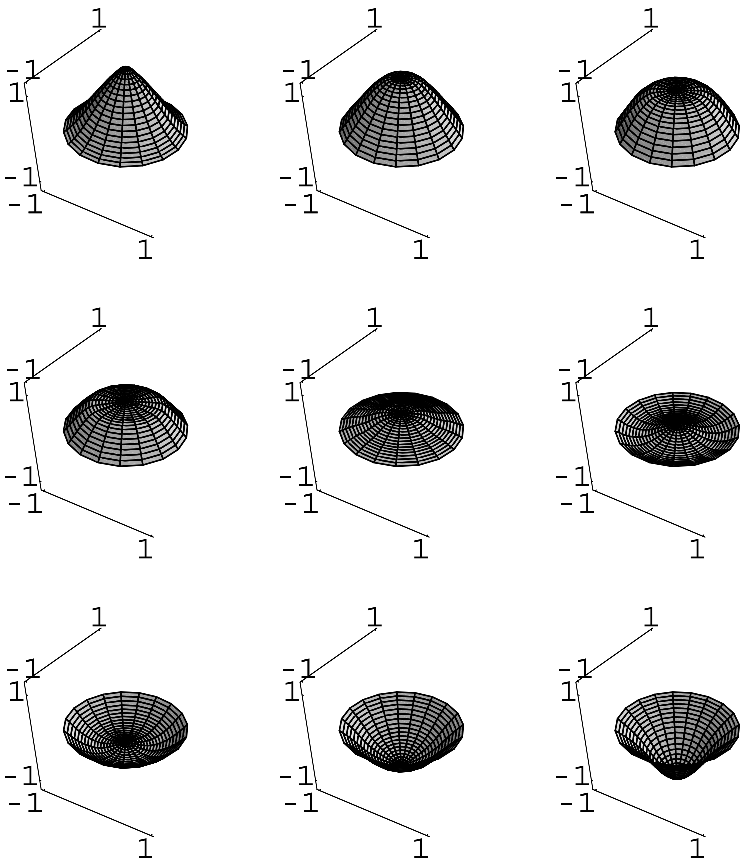
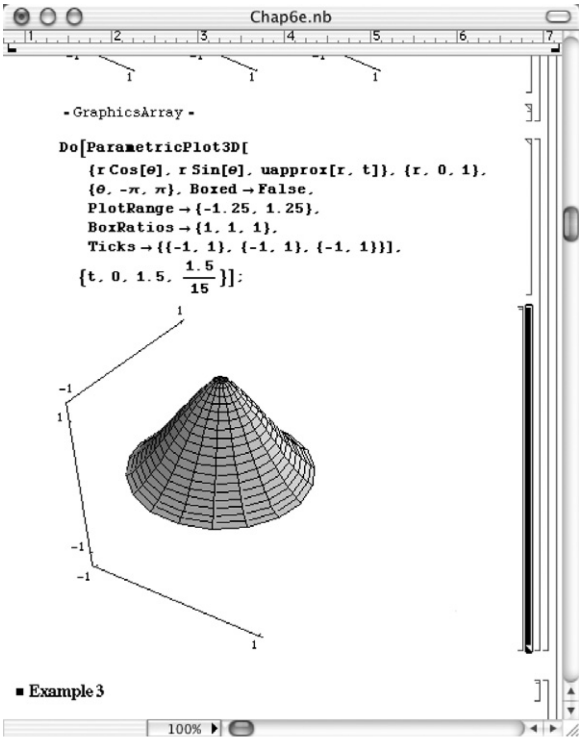


Figure 10-14 The drumhead for nine equally spaced values of *t* between 0 and 1.5

```
In[1992]:= somegraphs =
             Table[ParametricPlot3D[
               {r Cos[θ], r Sin[θ], uapprox[r, t]},
               {r, 0, 1}, {θ, -π, π}, Boxed → False,
               PlotRange → {-1.25, 1.25},
               BoxRatios → {1, 1, 1}, Ticks → {{-1, 1},
               {-1, 1}, {-1, 1}},
               DisplayFunction →
               Identity], {t, 0, 1.5, 1.5/8}];

           toshow = Partition[somegraphs, 3];

           Show[GraphicsArray[toshow]]
```

In order to actually watch the drumhead move, we can use a Do loop to generate several graphs and animate the result. Be aware, however, that generating many three-dimensional graphics and then animating the results uses a great deal of memory and can take considerable time, even on a relatively powerful computer. We show one frame from the animation that results from the following Do loop.

```
In[1993]:= Do[ParametricPlot3D[
             {r Cos[θ], r Sin[θ], uapprox[r, t]},
             {r, 0, 1}, {θ, -π, π}, Boxed → False,
             PlotRange → {-1.25, 1.25},
             BoxRatios → {1, 1, 1}, Ticks → {{-1, 1},
             {-1, 1}, {-1, 1}}], {t, 0, 1.5, 1.5/15}];
```



■

The problem that depends on the angle θ is more complicated to solve. Due to the presence of $\partial^2 u/\partial\theta^2$ we must include two more boundary conditions in order to solve the initial-boundary-value problem. So that the solution is a smooth function, we require the "artificial" boundary conditions

$$u(r,\pi,t)=u(r,-\pi,t) \qquad \text{and} \qquad \frac{\partial u}{\partial\theta}u(r,\pi,t)=\frac{\partial u}{\partial\theta}u(r,-\pi,t)$$

for $0<r<\rho$ and $t>0$. Therefore, we solve the problem

$$\begin{cases} c^2\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}\right)=\dfrac{\partial^2 u}{\partial t^2},\ 0<r<\rho,\ -\pi<\theta<\pi,\ t>0 \\ u(\rho,\theta,t)=0,\ |u(0,\theta,t)|<\infty,\ -\pi\le\theta\le\pi,\ t>0 \\ u(r,\pi,t)=u(r,-\pi,t),\ \dfrac{\partial u}{\partial\theta}(r,\pi,t)=\dfrac{\partial u}{\partial\theta}(r,-\pi,t),\ 0<r<\rho,\ t>0 \\ u(r,\theta,0)=f(r,\theta),\ \dfrac{\partial u}{\partial t}(r,\pi,0)=g(r,\theta),\ 0<r<\rho,\ -\pi<\theta<\pi. \end{cases} \tag{10.29}$$

Using separation of variables and assuming that $u(r,\theta,t)=R(r)H(\theta)T(t)$, we obtain that a general solution is given by

$$\begin{aligned} u(r,\theta,t) = {} & \sum_n a_{0n}J_0(\lambda_{0n}r)\cos(\lambda_{0n}ct)+\sum_{m,n}a_{mn}J_m(\lambda_{mn}r)\cos(m\theta)\cos(\lambda_{mn}ct) \\ & +\sum_{m,n}b_{mn}J_m(\lambda_{mn}r)\sin(m\theta)\cos(\lambda_{mn}ct)+\sum_n A_{0n}J_0(\lambda_{0n}r)\sin(\lambda_{0n}ct) \\ & +\sum_{m,n}A_{mn}J_m(\lambda_{mn}r)\cos(m\theta)\sin(\lambda_{mn}ct) \\ & +\sum_{m,n}B_{mn}J_m(\lambda_{mn}r)\sin(m\theta)\sin(\lambda_{mn}ct) \end{aligned}$$

where J_m represents the mth Bessel function of the first kind, α_{mn} denotes the nth zero of the Bessel function $y=J_m(x)$, and $\lambda_{mn}=\alpha_{mn}/\rho$. The coefficients are given by the following formulas.

$$a_{0n}=\frac{\int_0^{2\pi}\int_0^{\rho}f(r,\theta)J_0(\lambda_{0n}r)\,r\,dr\,d\theta}{2\pi\int_0^{\rho}[J_0(\lambda_{0n}r)]^2\,r\,dr} \qquad a_{mn}=\frac{\int_0^{2\pi}\int_0^{\rho}f(r,\theta)J_m(\lambda_{mn}r)\cos(m\theta)\,r\,dr\,d\theta}{\pi\int_0^{\rho}[J_m(\lambda_{mn}r)]^2\,r\,dr}$$

$$b_{mn}=\frac{\int_0^{2\pi}\int_0^{\rho}f(r,\theta)J_m(\lambda_{mn}r)\sin(m\theta)\,r\,dr\,d\theta}{\pi\int_0^{\rho}[J_m(\lambda_{mn}r)]^2\,r\,dr} \qquad A_{0n}=\frac{\int_0^{2\pi}\int_0^{\rho}g(r,\theta)J_0(\lambda_{0n}r)\,r\,dr\,d\theta}{2\pi\lambda_{0n}c\int_0^{\rho}[J_0(\lambda_{0n}r)]^2\,r\,dr}$$

$$A_{mn} = \frac{\int_0^{2\pi} \int_0^{\rho} g(r,\theta) J_m(\lambda_{mn} r) \cos(m\theta)\, r\, dr\, d\theta}{\pi \lambda_{mn} c \int_0^{\rho} [J_m(\lambda_{mn} r)]^2\, r\, dr} \quad B_{mn} = \frac{\int_0^{2\pi} \int_0^{\rho} g(r,\theta) J_m(\lambda_{mn} r) \sin(m\theta)\, r\, dr\, d\theta}{\pi \lambda_{mn} c \int_0^{\rho} [J_m(\lambda_{mn} r)]^2\, r\, dr}$$

EXAMPLE 10.5.3: Solve

$$\begin{cases} 10^2 \left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} \right) = \dfrac{\partial^2 u}{\partial t^2},\ 0 < r < 1, \\ \qquad -\pi < \theta < \pi,\ t > 0 \\ u(1,\theta,t) = 0,\ |u(0,\theta,t)| < \infty,\ -\pi \le \theta \le \pi,\ t > 0 \\ u(r,\pi,t) = u(r,-\pi,t),\ \dfrac{\partial u}{\partial \theta}(r,\pi,t) \\ \qquad = \dfrac{\partial u}{\partial \theta}(r,-\pi,t),\ 0 < r < 1,\ t > 0 \\ u(r,\theta,0) = \cos(\pi r/2)\sin\theta,\ \dfrac{\partial u}{\partial t}(r,\theta,0) \\ \qquad = (r-1)\cos(\pi\theta/2),\ 0 < r < 1, \\ \qquad -\pi < \theta < \pi. \end{cases}$$

SOLUTION: To calculate the coefficients, we will need to have approximations of the zeros of the Bessel functions, so we load the **BesselZeros** package, which is contained in the **NumericalMath** folder (or directory) and define α_{mn} to be the nth zero of $y = J_m(x)$. We illustrate the use of α_{mn} by using it to compute the first five zeros of $y = J_0(x)$.

```
In[1994]:= << NumericalMath`BesselZeros`
```

```
In[1995]:= α_{m_,n_} := α_{m,n} = BesselJZeros[m, {n, n}][[1]]
```

```
In[1996]:= Table[α_{0,n}, {n, 1, 5}]

Out[1996]= {2.40483, 5.52008, 8.65373, 11.7915, 14.9309}
```

The appropriate parameter values as well as the initial condition functions are defined as follows. Notice that the functions describing the initial displacement and velocity are defined as the product of functions. This enables the subsequent calculations to be carried out using `NIntegrate`.

```
In[1997]:= Clear[a, f, f1, f2, g1, g2, A, c, g, capa, capb, b]

           c = 10;

           ρ = 1;

           f1[r_] = Cos[π r/2];

           f2[θ_] = Sin[θ];

           f[r_, θ_] := f[r, θ] = f1[r] f2[θ];

           g1[r_] = r - 1;

           g2[θ_] = Cos[π θ/2];

           g[r_, θ_] := g[r, θ] = g1[r] g2[θ];
```

The coefficients a_{0n} are determined with the function a.

```
In[1998]:= Clear[a]

In[1999]:= a[n_] :=
             a[n] =
              N[(NIntegrate[f1[r] BesselJ[0, α0,n r] r,
                 {r, 0, ρ}] NIntegrate[f2[t],
                 {t, 0, 2π}])/
                (2π NIntegrate[r BesselJ[0, α0,n r]^2,
                 {r, 0, ρ}])];
```

Hence, as represents a table of the first five values of a_{0n}. Chop is used to round off very small numbers to zero.

```
In[2000]:= as = Table[a[n]//Chop, {n, 1, 5}]
Out[2000]= {0, 0, 0, 0, 0}
```

Because the denominator of each integral formula used to find a_{mn} and b_{mn} is the same, the function bjmn which computes this value is defined next. A table of nine values of this coefficient is then determined.

```
In[2001]:= bjmn[m_, n_] :=
             bjmn[m, n] =
              N[NIntegrate[r BesselJ[m, αm,n r]^2, {r, 0, ρ}]]

           Table[Chop[bjmn[m, n]], {m, 1, 3}, {n, 1, 3}]
```

```
Out[2001]= {{0.0811076, 0.0450347, 0.0311763},
  {0.0576874, 0.0368243, 0.0270149},
  {0.0444835, 0.0311044, 0.0238229}}
```

We also note that in evaluating the numerators of a_{mn} and b_{mn} we must compute $\int_0^{\rho} r f_1(r) J_m(\alpha_{mn} r)\, dr$. This integral is defined in fbjmn and the corresponding values are found for $n = 1, 2, 3$ and $m = 1, 2, 3$.

```
In[2002]:= Clear[fbjmn]

  fbjmn[m_, n_] :=
    fbjmn[m, n] =
      N[NIntegrate[f1[r] BesselJ[m, α_{m,n} r] r,
        {r, 0, ρ}]]

  Table[Chop[fbjmn[m, n]], {m, 1, 3}, {n, 1, 3}]

Out[2002]= {{0.103574, 0.020514, 0.0103984},
  {0.0790948, 0.0275564, 0.0150381},
  {0.0628926, 0.0290764, 0.0171999}}
```

The formula to compute a_{mn} is then defined and uses the information calculated in fbjmn and bjmn. As in the previous calculation, the coefficient values for $n = 1, 2, 3$ and $m = 1, 2, 3$ are determined.

```
In[2003]:= a[m_, n_] :=
    a[m, n] =
      N[(fbjmn[m, n] NIntegrate[f2[t] Cos[m t],
        {t, 0, 2π}])/(π bjmn[m, n])];

  Table[Chop[a[m, n]], {m, 1, 3}, {n, 1, 3}]

Out[2003]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

A similar formula is then defined for the computation of b_{mn}.

```
In[2004]:= b[m_, n_] :=
    b[m, n] =
      N[(fbjmn[m, n] NIntegrate[f2[t] Sin[m t],
        {t, 0, 2π}])/(π bjmn[m, n])];

  Table[Chop[b[m, n]], {m, 1, 3}, {n, 1, 3}]

Out[2004]= {{1.277, 0.455514, 0.333537}, {0, 0, 0}, {0, 0, 0}}
```

Note that defining the coefficients in this manner `a[m_,n_]:=a[m,n]=...+` and `b[m_,n_]:=b[m,n]=...`, so that Mathematica "remembers" previously computed values, reducing computation time. The values of A_{0n} are found similar to those of a_{0n}. After defining the function `capa` to calculate these coefficients, a table of values is then found.

```
In[2005]:= capa[n_] :=
             capa[n] =
              N[(NIntegrate[g1[r] BesselJ[0, α0,n r] r,
                 {r, 0, ρ}]
                NIntegrate[g2[t], {t, 0, 2π}])/
                 (2π c α0,n NIntegrate[
                  r BesselJ[0, α0,n r]^2, {r, 0, ρ}])];

           Table[Chop[capa[n]], {n, 1, 6}]

Out[2005]= {0.00142231, 0.0000542518, 0.0000267596,
             6.41976 × 10^-6, 4.95843 × 10^-6, 1.88585 × 10^-6}
```

The value of the integral of the component of g, `g1`, which depends on r and the appropriate Bessel functions, is defined as `gbjmn`.

```
In[2006]:= gbjmn[m_, n_] := gbjmn[m, n] = NIntegrate[g1[r]*
             BesselJ[m, αm,n r] r, {r, 0, ρ}]//N

           Table[gbjmn[m, n]//Chop, {m, 1, 3}, {n, 1, 3}]

Out[2006]= {{-0.0743906, -0.019491, -0.00989293},
             {-0.0554379, -0.0227976, -0.013039},
             {-0.0433614, -0.0226777, -0.0141684}}
```

Then, A_{mn} is found by taking the product of integrals, `gbjmn` depending on r and one depending on θ. A table of coefficient values is generated in this case as well.

```
In[2007]:= capa[m_, n_] :=
             capa[m, n] =
              N[(gbjmn[m, n] NIntegrate[g2[t] Cos[m t],
                 {t, 0, 2π}])/(π αm,n c bjmn[m, n])];

           Table[Chop[capa[m, n]], {m, 1, 3}, {n, 1, 3}]
```

```
Out[2007]= {{0.0035096, 0.000904517, 0.000457326},
  {-0.00262692, -0.00103252, -0.000583116},
  {-0.000503187, -0.000246002, -0.000150499}}
```

Similarly, the B_{mn} are determined.

```
In[2008]:= capb[m_, n_] :=
  capb[m, n] =
   N[(gbjmn[m, n] NIntegrate[g2[t] Sin[m t],
    {t, 0, 2π}])/(π α_{m,n} c bjmn[m, n])];

 Table[Chop[capb[m, n]], {m, 1, 3}, {n, 1, 3}]
Out[2008]= {{0.00987945, 0.00254619, 0.00128736},
  {-0.0147894, -0.00581305, -0.00328291},
  {-0.00424938, -0.00207747, -0.00127095}}
```

Now that the necessary coefficients have been found, we construct an approximate solution to the wave equation by using our results. In the following, `term1` represents those terms of the expansion involving a_{0n}, `term2` those terms involving a_{mn}, `term3` those involving b_{mn}, `term4` those involving A_{0n}, `term5` those involving A_{mn}, and `term6` those involving B_{mn}.

```
In[2009]:= Clear[term1, term2, term3, term4, term5, term6]

 term1[r_, t_, n_] :=
  a[n] BesselJ[0, α_{0,n} r] Cos[α_{0,n} c t];

 term2[r_, t_, θ_, m_, n_] :=
  a[m, n] BesselJ[m, α_{m,n} r] Cos[m θ] Cos[α_{m,n} c t];

 term3[r_, t_, θ_, m_, n_] :=
  b[m, n] BesselJ[m, α_{m,n} r] Sin[m θ] Cos[α_{m,n} c t];

 term4[r_, t_, n_] :=
  capa[n] BesselJ[0, α_{0,n} r] Sin[α_{0,n} c t];

 term5[r_, t_, θ_, m_, n_] :=
  capa[m, n] BesselJ[m, α_{m,n} r] Cos[m θ]
  Sin[α_{m,n} c t];

 term6[r_, t_, θ_, m_, n_] :=
  capb[m, n] BesselJ[m, α_{m,n} r] Sin[m θ]
   Sin[α_{m,n} c t];
```

Therefore, our approximate solution is given as the sum of these terms as computed in u.

```
In[2010]:= Clear[u]

           u[r_, t_, th_] :=
             Sum[term1[r, t, n], {n, 1, 5}] + Sum[term2[r, t, th, m, n], {m, 1, 3}, {n, 1, 3}]
             + Sum[term3[r, t, th, m, n], {m, 1, 3}, {n, 1, 3}] + Sum[term4[r, t, n], {n, 1, 5}]
             + Sum[term5[r, t, th, m, n], {m, 1, 3}, {n, 1, 3}]
             + Sum[term6[r, t, th, m, n], {m, 1, 3}, {n, 1, 3}];

           uc = Compile[{r, t, th}, u[r, t, th]]
Out[2010]= CompiledFunction[{r, t, th}, u[r, t, th],
             -CompiledCode-]
```

The solution is *compiled* in `uc`. The command `Compile` is used to compile functions. `Compile` returns a `CompiledFunction` which represents the compiled code. Generally, compiled functions take less time to perform computations than uncompiled functions although compiled functions can only be evaluated for numerical arguments.

Next, we define the function `tplot` which uses `ParametricPlot3D` to produce the graph of the solution for a particular value of t. Note that the x and y coordinates are given in terms of polar coordinates.

```
In[2011]:= Clear[tplot]

           tplot[t_] := ParametricPlot3D[{r Cos[θ],
             r Sin[θ], uc[r, t, θ]}, {r, 0, 1}, {θ, -π, π},
             PlotPoints → {20, 20},
             BoxRatios → {1, 1, 1}, Shading → False,
             Axes → False, Boxed → False,
             DisplayFunction → Identity]
```

A table of nine plots for nine equally spaced values of t from $t = 0$ to $t = 1$ using increments of 1/8 is then generated. This table of graphs is displayed as a graphics array in Figure 10-15.

```
In[2012]:= somegraphs = Table[tplot[t], {t, 0, 1, 1/8}];

           toshow = Partition[somegraphs, 3];

           Show[GraphicsArray[toshow]]
```

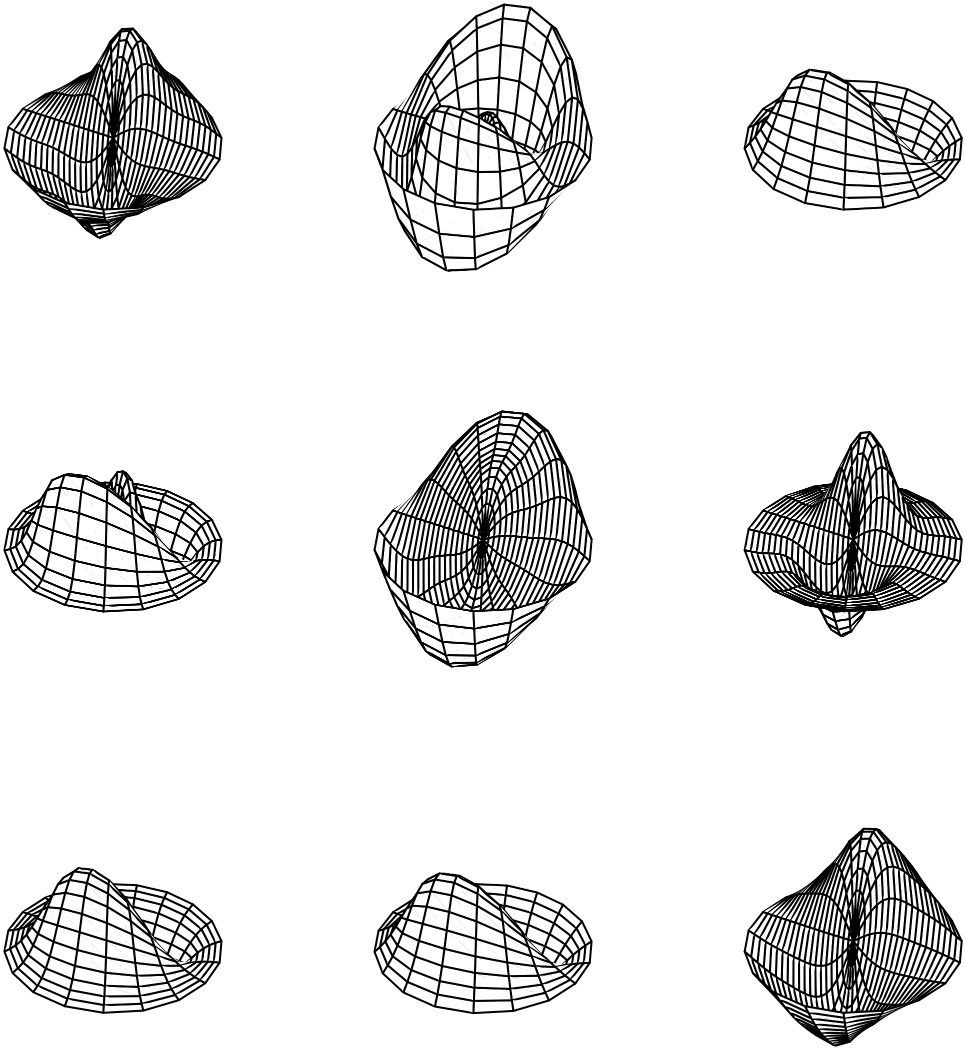


Figure 10-15 The drumhead for nine equally spaced values of t from $t = 0$ to $t = 1$

Of course, we can generate many graphs with a `Do` loop and animate the result as in the previous example. Be aware, however, that generating many three-dimensional graphics and then animating the results uses a great deal of memory and can take considerable time, even on a relatively powerful computer.

■

10.5.3 Other Partial Differential Equations

A partial differential equation of the form

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u) \tag{10.30}$$

is called a **first-order, quasi-linear partial differential equation**. In the case when $c(x, y, u) = 0$, equation (10.30) is **homogeneous**; if a and b are independent of u, equation (10.30) is **almost linear**; and when $c(x, y, u)$ can be written in the form $c(x, y, u) = d(x, y)u + s(x, y)$, equation (10.30) is **linear**. Quasi-linear partial differential equations can frequently be solved using the *method of characteristics*.

EXAMPLE 10.5.4: Use the *method of characteristics* to solve the initial-value problem $\begin{cases} -3xtu_x + u_t = xt \\ u(x, 0) = x. \end{cases}$

SOLUTION: For this problem, the *characteristic system* is

$$\partial x/\partial r = -3xt, \qquad x(0, s) = s$$
$$\partial t/\partial r = 1, \qquad t(0, s) = 0$$
$$\partial u/\partial r = xt, \qquad u(0, s) = s.$$

We begin by using `DSolve` to solve $\partial t/\partial r = 1$, $t(0, s) = 0$

```
In[2013]:= d1 = DSolve[{D[t[r], r] == 1, t[0] == 0}, t[r], r]
Out[2013]= {{t[r] → r}}
```

and obtain $t = r$. Thus, $\partial x/\partial r = -3xr$, $x(0, s) = s$ which we solve next

```
In[2014]:= d2 = DSolve[{D[x[r], r] == -3 x[r] r, x[0] == s},
                x[r], r]
Out[2014]= {{x[r] → e^(-3 r^2/2) s}}
```

and obtain $x = se^{-3r^2/2}$. Substituting $r = t$ and $x = se^{-3r^2/2}$ into $\partial u/\partial r = xt$, $u(0, s) = s$ and using `DSolve` to solve the resulting equation yields the following result, named `d3`.

```
In[2015]:= d3 = DSolve[{D[u[r], r] == e^(-3 r^2/2) s r, u[0] == s},
                u[r], r]
Out[2015]= {{u[r] → 1/3 e^(-3 r^2/2) (-1 + 4 e^(3 r^2/2)) s}}
```

To find $u(x, t)$, we must solve the system of equations

$$\begin{cases} t = r \\ x = se^{-3r^2/2} \end{cases}$$

for r and s. Substituting $r = t$ into $x = se^{-3r^2/2}$ and solving for s yields $s = xe^{3t^2/2}$. Thus, the solution is given by replacing the values obtained above in the solution obtained in `d3`. We do this below by using `ReplaceAll` (`/.`) to replace each occurrence of r and s in `d3[[1,1,2]]`, the solution obtained in `d3`, by the values $r = t$ and $s = xe^{3t^2}/2$. The resulting output represents the solution to the initial-value problem.

```
In[2016]:= d3[[1,1,2]] /. {r->t, s->x Exp[3/2 t^2]}//
             Simplify
```

$$\text{Out[2016]= } \frac{1}{3}\left(-1+4\,e^{\frac{3t^2}{2}}\right)x$$

In this example, `DSolve` can also solve this first-order partial differential equation.

Next, we use `DSolve` to find a general solution of $-3xtu_x + u_t = xt$ and name the resulting output `gensol`.

```
In[2017]:= gensol =
             DSolve[-3x t D[u[x,t],x] + D[u[x,t],t] == x t,
                u[x,t], {x,t}]
```

$$\text{Out[2017]= } \left\{\left\{u[x,t] \to \frac{1}{3}\left(-x+3\,C[1]\left[\frac{1}{6}\left(3t^2+2\,\text{Log}[x]\right)\right]\right)\right\}\right\}$$

The output

$$\text{Out[2017]= } C[1]\left[-\frac{3t^2}{2}-\text{Log}[x]\right]$$

represents an arbitrary function of $-\frac{3}{2}t^2 - \ln x$. The explicit solution is extracted from `gensol` with `gensol[[1,1,2]]`, the same way that results are extracted from the output of `DSolve` commands involving ordinary differential equations.

```
In[2018]:= gensol[[1,1,2]]
```

$$\text{Out[2018]= } \frac{1}{3}\left(-x+3\,C[1]\left[\frac{1}{6}\left(3t^2+2\,\text{Log}[x]\right)\right]\right)$$

To find the solution that satisfies $u(x, 0) = x$ we replace each occurrence of t in the solution by 0.

```
In[2019]:= gensol[[1,1,2]] /. t->0
```

$$\text{Out[2019]= } \frac{1}{3}\left(-x+3\,C[1]\left[\frac{\text{Log}[x]}{3}\right]\right)$$

Thus, we must find a function $f(x)$ so that

$$-\frac{1}{3}x + f(-\ln x) = x$$
$$f(-\ln x) = \frac{4}{3}x.$$

Certainly $f(t) = \frac{4}{3}e^{-t}$ satisfies the above criteria. We define $f(t) = \frac{4}{3}e^{-t}$ and then compute $f(-\ln x)$ to verify that $f(-\ln x) = \frac{4}{3}x$.

```
In[2020]:= Clear[f]

           f[t_] = 4 Exp[-t]/3;

           f[-Log[x]]
```

$$\text{Out[2020]=}\ \frac{4\,x}{3}$$

Thus, the solution to the initial-value problem is given by $-\frac{1}{3}x+f\left(-\frac{3}{2}t^2 - \ln x\right)$ which is computed and named `sol`. Of course, the result returned is the same as that obtained previously.

$$\text{In[2021]:= sol = Simplify}\left[-\frac{x}{3} + f\left[-\frac{3\,t^2}{2} - \text{Log}[x]\right]\right]$$

$$\text{Out[2021]=}\ \frac{1}{3}\left(-1 + 4\,e^{\frac{3\,t^2}{2}}\right)x$$

Last, we use `Plot3D` to graph `sol` on the rectangle $[0, 20] \times [-2, 2]$ in Figure 10-16. The option `ClipFill->None` is used to indicate that por-

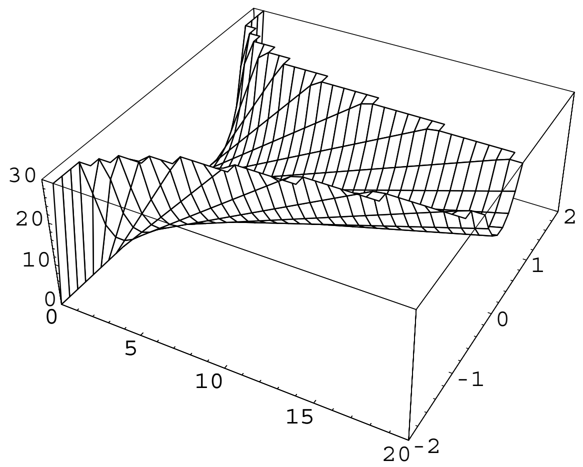


Figure 10-16 Plot of $u(x, t) = \frac{1}{3}x\left(4e^{3t^2/2} - 1\right)$

tions of the resulting surface which extend past the bounding box are not shown: nothing is shown where the surface is clipped.

```
In[2022]:= Plot3D[sol, {x, 0, 20}, {t, -2, 2},
              PlotRange → {0, 30}, PlotPoints → 30,
              ClipFill → None, Shading → False]
```

■

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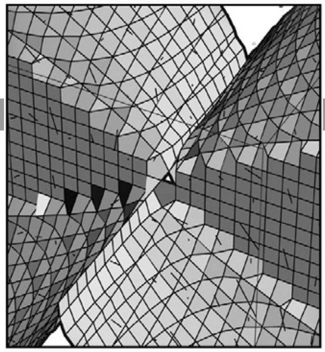
Appendix: Getting Started

A



Introduction to Mathematica

Mathematica, first released in 1988 by Wolfram Research, Inc.,

`http://www.wolfram.com/`,

is a system for doing mathematics on a computer. Mathematica combines symbolic manipulation, numerical mathematics, outstanding graphics, and a sophisticated programming language. Because of its versatility, Mathematica has established itself as the computer algebra system of choice for many computer users. Among the over 1,000,000 users of Mathematica, 28% are engineers, 21% are computer scientists, 20% are physical scientists, 12% are mathematical scientists, and 12% are business, social, and life scientists. Two-thirds of the users are in industry and government with a small (8%) but growing number of student users. However, due to its special nature and sophistication, beginning users need to be aware of the special syntax required to make Mathematica perform in the way intended. You will find that calculations and sequences of calculations most frequently used by beginning users are discussed in detail along with many typical examples. In addition, the comprehensive index not only lists a variety of topics but also cross-references commands with frequently used options. *Mathematica By Example* serves as a valuable tool and reference to the beginning user of Mathematica as well as to the more sophisticated user, with specialized needs.

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web: `http://www.wolfram-media.com`

A Note Regarding Different Versions of Mathematica

With the release of Version 5 of Mathematica, many new functions and features have been added to Mathematica. We encourage users of earlier versions of Mathematica to update to Version 5 as soon as they can. All examples in *Mathematica By Example*, Third Edition, were completed with Version 5. In most cases, the same results will be obtained if you are using Version 4.0 or later, although the appearance of your results will almost certainly differ from that presented here. Occasionally, however, particular features of Version 5 are used and in those cases, of course, these features are not available in earlier versions. If you are using an earlier or later version of Mathematica, your results may not appear in a form identical to those found in this book: some commands found in Version 5 are not available in earlier versions of Mathematica; in later versions some commands will certainly be changed, new commands added, and obsolete commands removed. For details regarding these changes, please see *The Mathematica Book* [28]. You can determine the version of Mathematica you are using during a given Mathematica session by entering either the command `$Version` or the command `$VersionNumber`. In this text, we assume that Mathematica has been correctly installed on the computer you are using. If you need to install Mathematica on your computer, please refer to the documentation that came with the Mathematica software package.

On-line help for upgrading older versions of Mathematica and installing new versions of Mathematica is available at the Wolfram Research, Inc. website:

`http://www.wolfram.com/`.

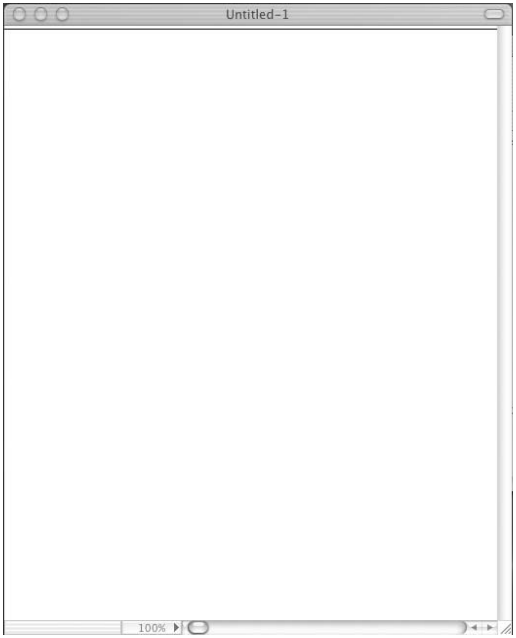
Getting Started with Mathematica

We begin by introducing the essentials of Mathematica. The examples presented are taken from algebra, trigonometry, and calculus topics that you are familiar with to assist you in becoming acquainted with the Mathematica computer algebra system.

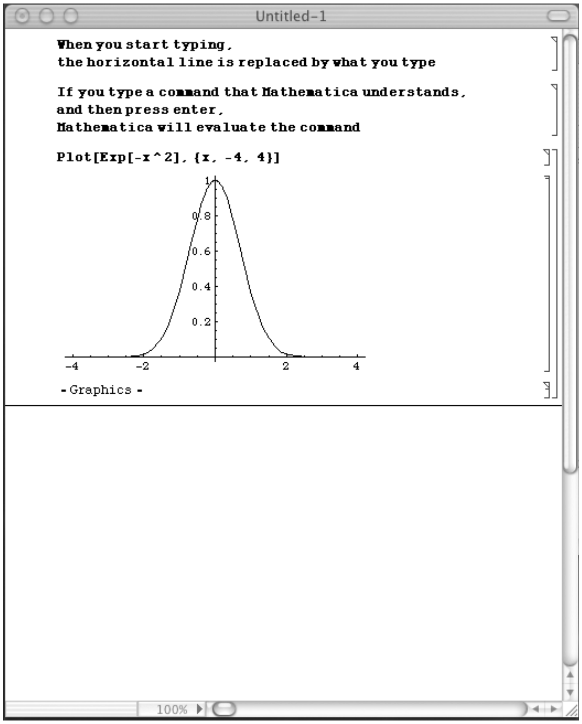
We assume that Mathematica has been correctly installed on the computer you are using. If you need to install Mathematica on your computer, please refer to the documentation that came with the Mathematica software package.

Start Mathematica on your computer system. Using Windows or Macintosh mouse or keyboard commands, activate the Mathematica program by selecting the Mathematica icon or an existing Mathematica document (or notebook), and then clicking or double-clicking on the icon.

If you start Mathematica by selecting the Mathematica icon, a blank untitled notebook is opened, as illustrated in the following screen shot.

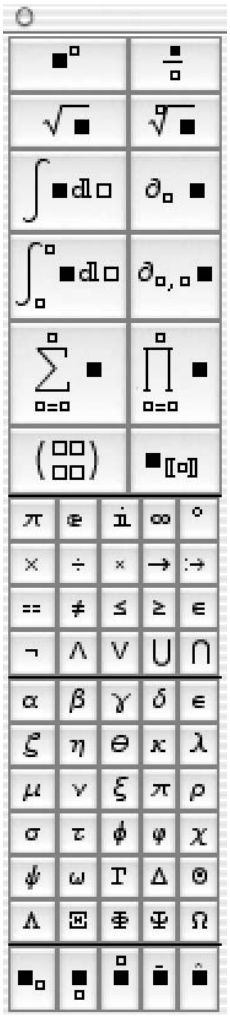


When you start typing, the thin black horizontal line near the top of the window is replaced by what you type.



With some operating systems, **Enter** evaluates commands and **Return** yields a new line

The **Basic Input** palette:



Once Mathematica has been started, computations can be carried out immediately. Mathematica commands are typed and the black horizontal line is replaced by the command, which is then evaluated by pressing **Enter**. Note that pressing **Enter** or **Return** evaluates commands and pressing **Shift-Return** yields a new line. Output is displayed below input. We illustrate some of the typical steps involved in working with Mathematica in the calculations that follow. In each case, we type the command and press **Enter**. Mathematica evaluates the command, displays the result, and inserts a new horizontal line after the result. For example, typing `N[`, then pressing the π key on the **Basic Input** palette, followed by typing `,50]` and pressing the enter key

```
In[2023]:= N[π, 50]
Out[2023]= 3.141592653589793238462643383279502884197169399375106
             2.09749446
```

returns a 50-digit approximation of π. Note that both π and `Pi` represent the mathematical constant π so entering `N[Pi,50]` returns the same result.

The next calculation can then be typed and entered in the same manner as the first. For example, entering

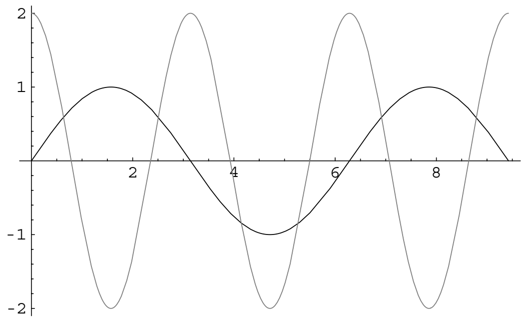


Figure A-1 A two-dimensional plot

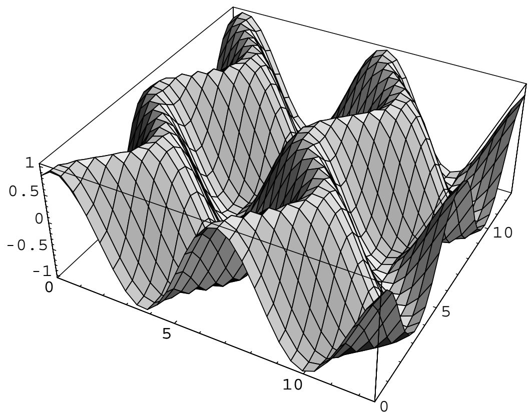


Figure A-2 A three-dimensional plot

Notice that every Mathematica command begins with capital letters and the argument is enclosed by square brackets `[...]`.

```
In[2024]:= Plot[{Sin[x], 2 Cos[2x]}, {x, 0, 3π},
              PlotStyle- > {GrayLevel[0], GrayLevel[0.5]}]
```

graphs the functions $y = \sin x$ and $y = 2\cos 2x$ on the interval $[0, 3\pi]$ shown in Figure A-1. Similarly, entering

```
In[2025]:= Plot3D[Sin[x + Cos[y]], {x, 0, 4π}, {y, 0, 4π},
              PlotPoints- > {30, 30}]
```

graphs the function $z = \sin(x + \cos y)$ for $0 \le x \le 4\pi$ and $0 \le y \le 4\pi$ shown in Figure A-2.

To type x^3 in Mathematica, press the ▪ button on the **Basic Input** palette, type x in the base position, and then click (or tab to) the exponent position and type 3.

Notice that all three of the following commands

In[2026]:= **Solve[x^3 - 2x + 1 == 0]**

Out[2026]= $\left\{\{x \to 1\}, \left\{x \to \frac{1}{2}\left(-1-\sqrt{5}\right)\right\}, \left\{x \to \frac{1}{2}\left(-1+\sqrt{5}\right)\right\}\right\}$

```
In[2027]:= Solve[x^3 - 2*x + 1 == 0]
Out[2027]=
                         1                        1
           {{x -> 1}, {x -> - (-1 - Sqrt[5])}, {x -> - (-1 + Sqrt[5])}}
                         2                        2
```

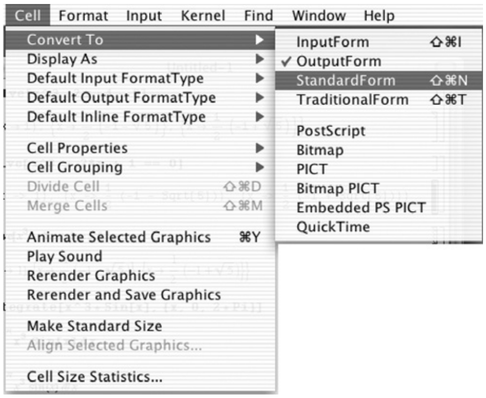
In[2028]:= **Solve[x^3 - 2 x + 1 == 0]**

Out[2028]= $\left\{\{x \to 1\}, \left\{x \to \frac{1}{2}\left(-1-\sqrt{5}\right)\right\}, \left\{x \to \frac{1}{2}\left(-1+\sqrt{5}\right)\right\}\right\}$

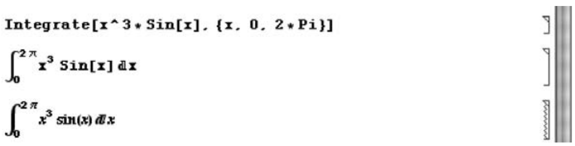
solve the equation $x^3 - 3x + 1 = 0$ for x.

In the first case, the input and output are in **StandardForm**, in the second case, the input and output are in **InputForm**, and in the third case, the input and output are in **TraditionalForm**. Move the cursor to the Mathematica menu,

select **Cell**, and then **ConvertTo**, as illustrated in the following screen shot.

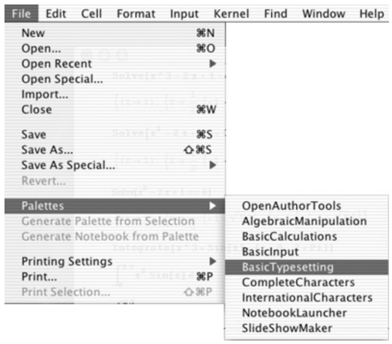


You can change how input and output appear by using **ConvertTo** or by changing the default settings. Moreover, you can determine the form of input/output by looking at the cell bracket that contains the input/output. For example, even though all three of the following commands look different, all three evaluate $\int_0^{2\pi} x^3 \sin x\, dx$.



A cell bracket like this] means the input is in **InputForm**; the output is in **OutputForm**. A cell bracket like this] means the contents of the cell are in **StandardForm**. A cell bracket like this] means the contents of the cell are in **TraditionalForm**. Throughout *Mathematica By Example*, Third Edition, we display input and output using **InputForm** or **StandardForm**, unless otherwise stated.

To enter code in **StandardForm**, we often take advantage of the **BasicTypesetting** palette, which is accessed by going to **File** under the Mathematica menu and then selecting **Palettes**

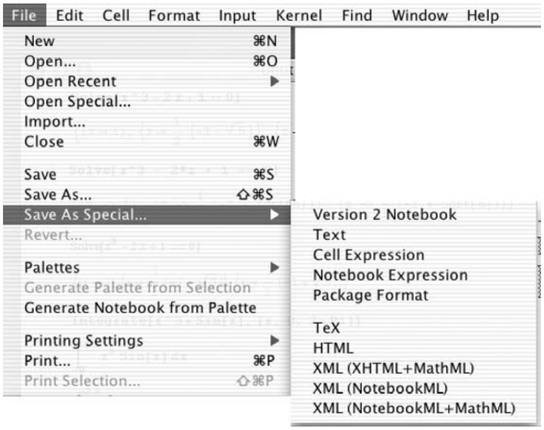


followed by **BasicTypesetting**.

Use the buttons to create templates and enter special characters. Alternatively, you can find a complete list of typesetting shortcuts in *The Mathematica Book*, Appendix 12, Listing of Named Characters [28].

Mathematica sessions are terminated by entering `Quit[]` or by selecting **Quit** from the **File** menu, or by using a keyboard shortcut, like **command-Q**, as with other applications. They can be saved by referring to **Save** from the **File** menu.

Mathematica allows you to save notebooks (as well as combinations of cells) in a variety of formats, in addition to the standard Mathematica format.



Remark. Input and text regions in notebooks can be edited. Editing input can create a notebook in which the mathematical output does not make sense in the sequence it appears. It is also possible to simply go into a notebook and alter input without doing any recalculation. This also creates misleading notebooks. Hence, common sense and caution should be used when editing the input regions of notebooks. Recalculating all commands in the notebook will clarify any confusion.

Five Basic Rules of Mathematica Syntax

In order for the Mathematica user to take full advantage of this powerful software, an understanding of its syntax is imperative. Although all of the rules of Mathematica syntax are far too numerous to list here, knowledge of the following five rules equips the beginner with the necessary tools to start using the Mathematica program with little trouble.

1. The arguments of *all* functions (both built-in ones and ones that you define) are given in brackets `[...]`. Parentheses `(...)` are used for grouping operations; vectors, matrices, and lists are given in braces `{...}`; and double square brackets `[[...]]` are used for indexing lists and tables.
2. Every word of a built-in Mathematica function begins with a capital letter.
3. Multiplication is represented by $*$ or a space between characters. Enter `2*x*y` or `2x y` to evaluate $2xy$ *not* `2xy`.
4. Powers are denoted by ^. Enter `(8*x^3)^(1/3)` to evaluate $(8x^3)^{1/3} = 8^{1/3}(x^3)^{1/3} = 2x$ instead of `8x^1/3`, which returns `8x/3`.
5. Mathematica follows the order of operations *exactly*. Thus, entering `(1+x)^1/x` returns $\frac{(1+x)^1}{x}$ while `(1+x)^(1/x)` returns $(1+x)^{1/x}$. Similarly, entering `x^3x` returns $x^3 \cdot x = x^4$ while entering `x^(3x)` returns x^{3x}.

   Remark. If you get no response or an incorrect response, you may have entered or executed the command incorrectly. In some cases, the amount of memory allocated to Mathematica can cause a crash. Like people, Mathematica is not perfect and errors can occur.

Loading Packages

Although Mathematica contains many built-in functions, some other functions are contained in **packages** that must be loaded separately. A tremendous number of additional commands are available in various packages that are shipped with each version of Mathematica. Experienced users can create their own packages; other packages are available from user groups and MathSource, which electronically distributes Mathematica-related products. For information about MathSource, visit

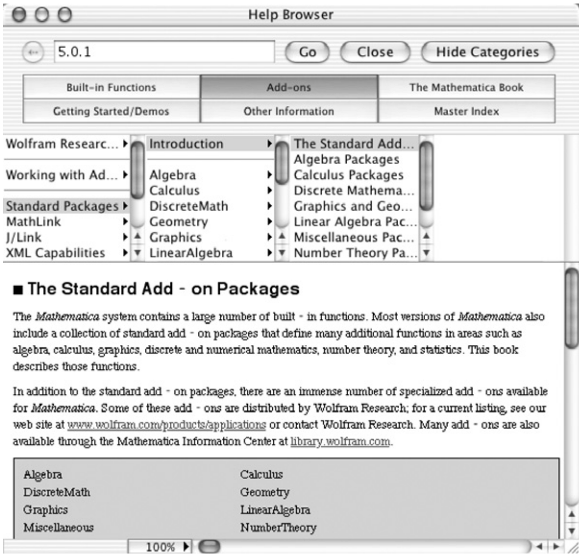
`http://library.wolfram.com/infocenter/MathSource/`

or send the message "help" to `mathsource@wri.com`. If desired, you can purchase MathSource on a CD directly from Wolfram Research, Inc. or you can access MathSource from the Wolfram Research World Wide Web site

`http://www.wri.com` or `http://www.wolfram.com`.

Descriptions of the various packages shipped with Mathematica are found in the **Help Browser**. From the Mathematica menu, select **Help** followed by **Add-Ons...**

to see a list of the standard packages.



Information regarding the packages in each category is obtained by selecting the category from the **Help Browser**'s menu.

Packages are loaded by entering the command

```
<<directory`packagename`
```

where **directory** is the location of the package **packagename**. Entering the command `<<directory`Master`` makes all the functions contained in all the packages in **directory** available. In this case, each package need not be loaded individually. For example, to load the package **Shapes** contained in the **Graphics** folder (or directory), we enter `<<Graphics`Shapes``.

```
In[2029]:= << Graphics`Shapes`
```

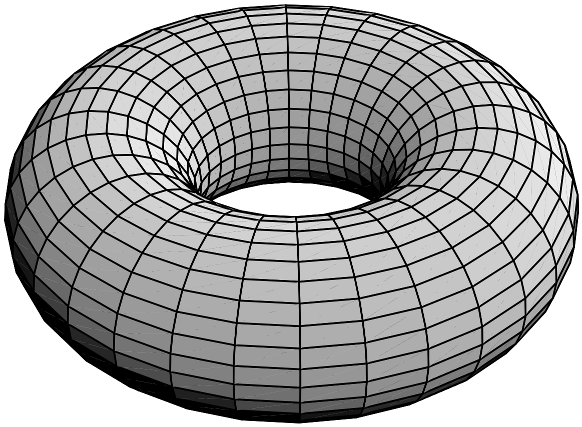


Figure A-3 A torus created with `Torus`

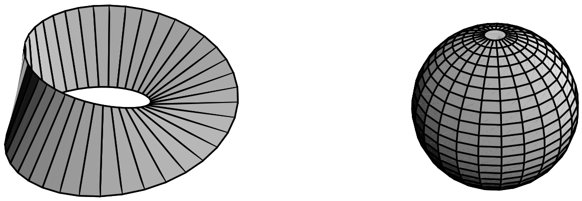


Figure A-4 A Möbius strip and a sphere

After the **Shapes** package has been loaded, entering

```
In[2030]:= Show[Graphics3D[Torus[1, 0.5, 30, 30]], Boxed → False]
```

generates the graph of a torus shown in Figure A-3. Next, we generate a Möbius strip and a sphere and display the two side-by-side using `GraphicsArray` in Figure A-4.

```
In[2031]:= mstrip = Graphics3D[MoebiusStrip[1, 0.5, 40], Boxed → False];

           sph = Graphics3D[Sphere[1, 25, 25], Boxed → False];

           Show[GraphicsArray[{mstrip, sph}]]
```

The **Shapes** package contains definitions of familiar three-dimensional shapes including the cone, cylinder, helix, and double helix. In addition, it allows us to perform transformations like rotations and translations on three-dimensional graphics.

A Word of Caution

When users take advantage of packages frequently, they often encounter error messages. One error message that occurs frequently is when a command is entered before the package is loaded. For example, the command `GramSchmidt[{v1,v2,...,vn}]` returns an orthonormal set of vectors with the same span as the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$. Here, we attempt to use the command `GramSchmidt`, which is contained in the **Orthogonalization** package located in the **LinearAlgebra** folder before the package has been loaded. Mathematica does not yet know the meaning of `GramSchmidt` so our input is returned.

```
In[2032]:= GramSchmidt[{{1,1,0},{0,2,1},{1,0,3}}]
Out[2032]= GramSchmidt[{{1,1,0},{0,2,1},{1,0,3}}]
```

At this point, we load the **Orthogonalization** package, which contains the `GramSchmidt` command, located in the **LinearAlgebra** folder. Several error messages result.

```
In[2033]:= << LinearAlgebra`Orthogonalization`
GramSchmidt::shdw : Symbol GramSchmidt appears in multiple contexts
{LinearAlgebra`Orthogonalization`, Global`};
definitions in context LinearAlgebra`Orthogonalization`
may shadow or be shadowed by other definitions.
```

In fact, when we reenter the command, we obtain the same result as that obtained previously.

```
In[2034]:= GramSchmidt[{{1,1,0},{0,2,1},{1,0,3}}]
Out[2034]= GramSchmidt[{{1,1,0},{0,2,1},{1,0,3}}]
```

However, after using the command `Remove`, the command `GramSchmidt` works as expected. Alternatively, we can quit Mathematica, restart, load the package, and then execute the command.

```
In[2035]:= Remove[GramSchmidt]

In[2036]:= GramSchmidt[{{1,1,0},{0,2,1},{1,0,3}}]
```

Out[2036]= $\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}, \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, \left\{\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right\}\right\}$

Similarly, we can take advantage of other commands contained in the **Orthogonalization** package like `Normalize` which normalizes a given vector.

```
In[2037]:= Normalize[{1,2,3}]
```

Out[2037]= $\left\{\frac{1}{\sqrt{14}}, \sqrt{\frac{2}{7}}, \frac{3}{\sqrt{14}}\right\}$

Getting Help from Mathematica

Becoming competent with Mathematica can take a serious investment of time. Hopefully, messages that result from syntax errors are viewed lightheartedly. Ideally, instead of becoming frustrated, beginning Mathematica users will find it challenging and fun to locate the source of errors. Frequently, Mathematica's error messages indicate where the error(s) has (have) occurred. In this process, it is natural that you will become more proficient with Mathematica. In addition to Mathematica's extensive help facilities, which are described next, a tremendous amount of information is available for all Mathematica users at the Wolfram Research website:

`http://www.wolfram.com/.`

One way to obtain information about commands and functions, including user-defined functions, is the command `?`. `?object` gives a basic description and syntax information of the Mathematica object `object`. `??object` yields detailed information regarding syntax and options for the object `object`.

EXAMPLE A.1: Use `?` and `??` to obtain information about the command `Plot`.

SOLUTION: `?Plot` uses basic information about the `Plot` function

```
?Plot
Plot[f, {x, xmin, xmax}] generates a plot of f as a
   function of x from xmin to xmax. Plot[{f1, f2, ... },
   {x, xmin, xmax}] plots several functions fi. More...
```

while `??Plot` includes basic information as well as a list of options and their default values.

```
??Plot

Plot[f, {x, xmin, xmax}] generates a plot of f as a
   function of x from xmin to xmax. Plot[{f1, f2, ... },
   {x, xmin, xmax}] plots several functions fi. More...

Attributes[Plot] = {HoldAll, Protected}

Options[Plot] = {AspectRatio → 1/GoldenRatio,
  Axes → Automatic, AxesLabel → None, AxesOrigin → Automatic,
  AxesStyle → Automatic, Background → Automatic,
  ColorOutput → Automatic, Compiled → True,
  DefaultColor → Automatic, DefaultFont :→ $DefaultFont,
  DisplayFunction :→ $DisplayFunction, Epilog → {},
  FormatType :→ $FormatType, Frame → False, FrameLabel → None,
  FrameStyle → Automatic, FrameTicks → Automatic,
  GridLines → None, ImageSize → Automatic, MaxBend → 10.,
  PlotDivision → 30., PlotLabel → None, PlotPoints → 25,
  PlotRange → Automatic, PlotRegion → Automatic,
  PlotStyle → Automatic, Prolog → {}, RotateLabel → True,
  TextStyle :→ $TextStyle, Ticks → Automatic}
```

■

`Options[object]` returns a list of the available options associated with `object` along with their current settings. This is quite useful when working with a Mathematica command such as `ParametricPlot` which has many options. Notice that the default value (the value automatically assumed by Mathematica) for each option is given in the output.

EXAMPLE A.2: Use `Options` to obtain a list of the options and their current settings for the command `ParametricPlot`.

SOLUTION: The command `Options[ParametricPlot]` lists all the options and their current settings for the command `ParametricPlot`.

```
Options[ParametricPlot]

{AspectRatio → 1/GoldenRatio, Axes → Automatic,
 AxesLabel → None, AxesOrigin → Automatic,
 AxesStyle → Automatic, Background → Automatic,
 ColorOutput → Automatic, Compiled → True,
 DefaultColor → Automatic, DefaultFont :→ $DefaultFont,
 DisplayFunction :→ $DisplayFunction, Epilog → {},
 FormatType :→ $FormatType, Frame → False, FrameLabel → None,
 FrameStyle → Automatic, FrameTicks → Automatic,
 GridLines → None, ImageSize → Automatic, MaxBend → 10.,
 PlotDivision → 30., PlotLabel → None, PlotPoints → 25,
 PlotRange → Automatic, PlotRegion → Automatic,
 PlotStyle → Automatic, Prolog → {}, RotateLabel → True,
 TextStyle :→ $TextStyle, Ticks → Automatic}
```

■

As indicated above, `??object` or, equivalently, `Information[object]` yields the information on the Mathematica object `object` returned by both `?object` and Options[object] in addition to a list of attributes of object. Note that object may be either a user-defined object or a built-in Mathematica object.

EXAMPLE A.3: Use ?? to obtain information about the commands Solve and Map. Use Information to obtain information about the command PolynomialLCM.

SOLUTION: We use ?? to obtain information about the commands Solve and Map including a list of options and their current settings.

```
?? Solve

Solve[eqns, vars] attempts to solve an equation
   or set of equations for the variables vars. Solve[
   eqns, vars, elims] attempts to solve the equations
   for vars, eliminating the variables elims. More...

Attributes[Solve] = {Protected}

Options[Solve] = {InverseFunctions → Automatic,
  MakeRules → False, Method → 3, Mode → Generic, Sort → True,
  VerifySolutions → Automatic, WorkingPrecision → ∞}

?? Map

Map[f, expr] or f /@ expr applies f to each element on the
   first level in expr. Map[f, expr, levelspec] applies
   f to parts of expr specified by levelspec. More...

Attributes[Map] = {Protected}

Options[Map] = {Heads → False}
```

Similarly, we use Information to obtain information about the command PolynomialLCM including a list of options and their current settings.

```
Information[PolynomialLCM]

PolynomialLCM[poly1, poly2, ... ] gives
  the least common multiple of the polynomials
  polyi. PolynomialLCM[poly1, poly2, ... ,
  Modulus->p] evaluates the LCM modulo the prime p.

Attributes[PolynomialLCM] = {Listable, Protected}

Options[PolynomialLCM] =
 {Extension → None, Modulus → 0, Trig → False}
```

■

The command Names["form"] lists all objects that match the pattern defined in form. For example, Names["Plot"] returns Plot, Names["*Plot"] returns all objects that end with the string Plot, Names["Plot*"] lists all objects that begin with the string `Plot`, and `Names["*Plot*"]` lists all objects that contain the string `Plot`. `Names["form",SpellingCorrection->True]` finds those symbols that match the pattern defined in `form` after a spelling correction.

EXAMPLE A.4: Create a list of all built-in functions beginning with the string `Plot`.

SOLUTION: We use `Names` to find all objects that match the pattern `Plot`.

```
In[2038]:= Names["Plot"]
Out[2038]= {Plot}
```

Next, we use `Names` to create a list of all built-in functions beginning with the string `Plot`.

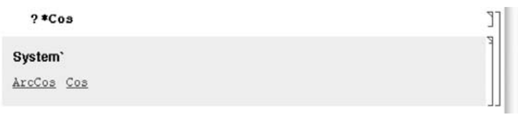
```
In[2039]:= Names["Plot * "]
Out[2039]= {Plot, Plot3D, Plot3Matrix, PlotDivision,
             PlotJoined, PlotLabel, PlotPoints, PlotRange,
             PlotRegion, PlotStyle}
```

■

As indicated above, the `?` function can be used in many ways. Entering `?letters*` gives all Mathematica objects that begin with the string `letters`; `?*letters*` gives all Mathematica objects that contain the string `letters`; and `?*letters` gives all Mathematica commands that end in the string `letters`.

EXAMPLE A.5: What are the Mathematica functions that (a) end in the string `Cos`; (b) contain the string `Sin`; and (c) begin with the string `Polynomial`?

SOLUTION: Entering



returns all functions ending with the string `Cos`, entering

```
?*Sin*

System`
ArcSin                    SingularityDepth
ArcSinh                   SingularValueDecomposition
FourierSinTransform       SingularValueList
IncludeSingularTerm       SingularValues
InverseFourierSinTransform Sinh
Sin                       SinhIntegral
SingleLetterItalics       SinIntegral
```

returns all functions containing the string `Sin`, and entering

```
?Polynomial*

System`
PolynomialForm  PolynomialMod       PolynomialReduce
PolynomialGCD   PolynomialQ         PolynomialRemainder
PolynomialLCM   PolynomialQuotient  Polynomials
```

returns all functions that begin with the string `Polynomial`.

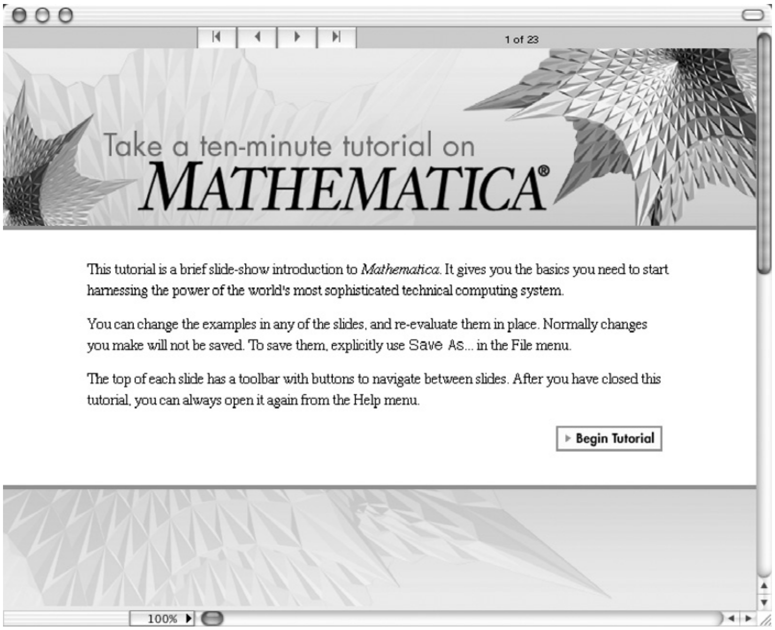
■

Mathematica Help

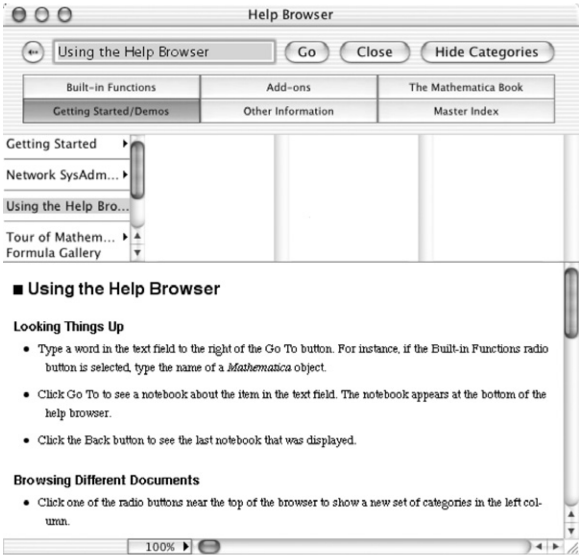
Additional help features are accessed from the Mathematica menu under **Help**. For basic information about Mathematica, go to **Help** and select **Help Browser...**

If you are a beginning Mathematica user, you may choose to select **Welcome Screen...**

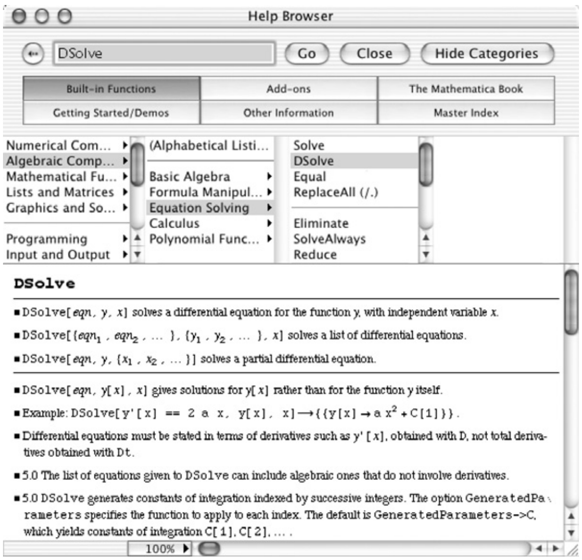
and then select **Ten-Minute Tutorial**



or **Help Browser**.

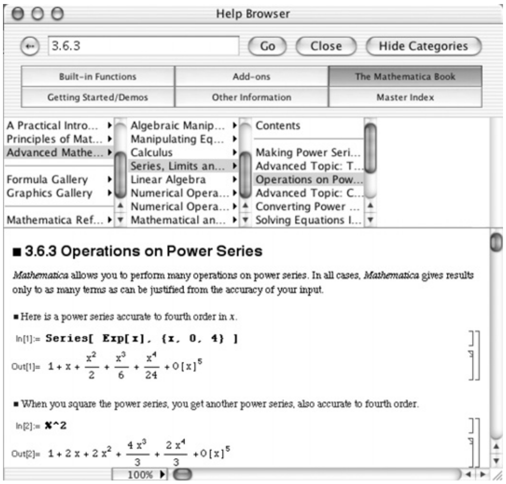


To obtain information about a particular Mathematica object or function, open the **Help Browser**, type the name of the object, function, or topic and press the **Go** button. Alternatively, you can type the name of a function that you wish to obtain help about, select it, go to **Help**, and then select **Find in Help...** as we do here with the `DSolve` function.

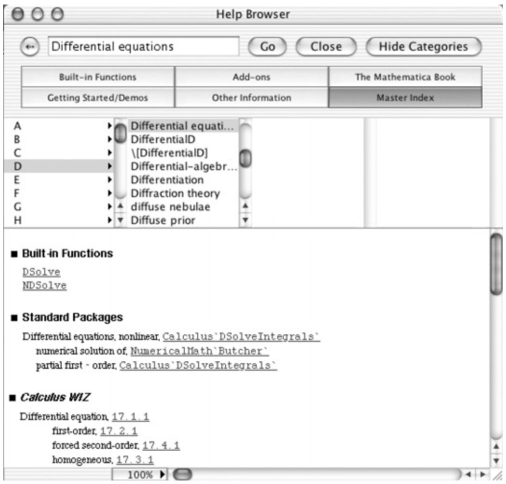


A typical help window not only contains a detailed description of the command and its options but also several examples that illustrate the command as well as hyperlinked cross-references to related commands and *The Mathematica Book* [28], which can be accessed by clicking on the appropriate links.

You can also use the **Help Browser** to access the on-line version of *The Mathematica Book* [28]. Here is a portion of Section 3.6.3, Operations on Power Series.



The **Master Index** contains hyperlinks to all portions of Mathematica help.



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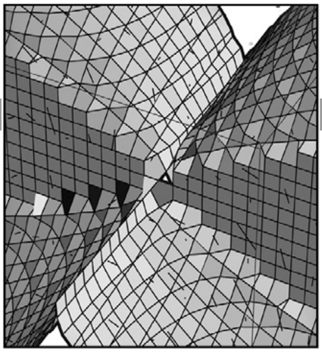
The Mathematica Menu



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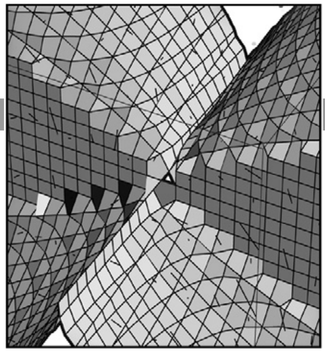
Bibliography



[1] Abell, Martha and Braselton, James, *Mathematica By Example*, Third Edition, Academic Press, 2004.

[2] Abell, Martha and Braselton, James, *Modern Differential Equations*, Second Edition, Harcourt, 2001.

[3] Apostol, Tom, *Mathematical Analysis*, Second Edition, Addison-Wesley, 1974.

[4] Barnsley, Michael, *Fractals Everywhere*, Second Edition, Morgan Kaufmann, 2000.

[5] Boyce, William E. and DiPrima, Richard C., *Elementary Differential Equations and Boundary-Value Problems*, Seventh Edition, John Wiley & Sons, 2000.

[6] Coddington, Earl and Levinson, Norman, *Theory of Ordinary Differential Equations*, Robert E. Krieger Publishing Company/McGraw Hill, 1955/1984.

[7] Corduneanu, C., *Principles of Differential and Integral Equations*, Chelsea Publishing, 1977.

[8] Devaney, Robert L. and Keen, Linda (eds.), *Chaos and Fractals: The Mathematics Behind the Computer Graphics*, Proceedings of Symposia in Applied Mathematics, Volume 39, American Mathematical Society, 1989.

[9] Edwards, C. Henry and Penney, David E. *Calculus with Analytic Geometry*, Fifth Edition, Prentice Hall, 1998.

[10] Edwards, C. Henry and Penney, David E. *Differential Equations and Boundary Value Problems: Computing and Modeling*, Third Edition, Pearson/Prentice Hall, 2004.

[11] Gaylord, Richard J., Kamin, Samuel N., and Wellin, Paul R., *Introduction to Programming with Mathematica*, Second Edition, TELOS/Springer-Verlag, 1996.

[12] Giordano, Frank R., Weir, Maurice D., and Fox, William P., *A First Course in Mathematical Modeling*, Third Edition, Thomson/Brooks/Cole, 2003.
[13] Graff, Karl F., *Wave Motion in Elastic Solids*, Oxford University Press/Dover, 1975/1991.
[14] Gray, Alfred, *Modern Differential Geometry of Curves and Surfaces*, Second Edition, CRC Press, 1997.
[15] Gray, John W., *Mastering Mathematica: Programming Methods and Applications*, Second Edition, Academic Press, 1997.
[16] Herriott, Scott R., *College Algebra through Functions and Models*, Preliminary Edition, Brooks/Cole, 2002.
[17] Jordan, D.W. and Smith, Peter, *Nonlinear Ordinary Differential Equations: An Introduction to Dynamical Systems*, Third Edition, Oxford Applied and Engineering Mathematics, Oxford University Press, 1999.
[18] Kyreszig, Erwin, *Advanced Engineering Mathematics*, Eighth Edition, John Wiley & Sons, 1999.
[19] Larson, Roland E., Hostetler, Robert P., and Edwards, Bruce H., *Calculus with Analytic Geometry*, Sixth Edition, Houghton Mifflin, 1998.
[20] Maeder, Roman E., *The Mathematica Programmer II*, Academic Press, 1996.
[21] Maeder, Roman E., *Programming in Mathematica*, Third Edition, Addison-Wesley, 1996.
[22] Rabenstein, Albert L., *Introduction to Ordinary Differential Equations*, Academic Press, 1966.
[23] Robinson, Clark, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, Second Edition, CRC Press, 1999.
[24] Smith, Hal L. and Waltman, P., *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, 1995.
[25] Stewart, James, *Calculus: Concepts and Contexts*, Second Edition, Brooks/Cole, 2001.
[26] Weisstein, Eric W., *CRC Concise Encyclopedia of Mathematics*, CRC Press, 1999.
[27] Wolfram, Stephen, *A New Kind of Science*, Wolfram Media, 2002.
[28] Wolfram, Stephen, *The Mathematica Book*, Fourth Edition, Wolfram Media, 2004.
[29] Zwillinger, Daniel, *Handbook of Differential Equations*, Second Edition, Academic Press, 1992.

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