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Wolfgang Ebeling
Klaus Hulek
Knut Smoczyk *Editors*

Complex and Differential Geometry

Conference held at Leibniz Universität Hannover, September 14 – 18, 2009

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Wolfgang Ebeling • Klaus Hulek • Knut Smoczyk
Editors

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Foreword

This volume contains the proceedings of the conference "Complex and Differential Geometry 2009", held at Leibniz Universität Hannover from September 14 – 18, 2009. The aim of the conference was to bring specialists from differential geometry and (complex) algebraic geometry together, to discuss new developments in and the interaction between these fields. The articles in this book cover a broad range of subjects from topics in (classical) algebraic geometry and complex geometry, including (holomorphic) symplectic and Poisson geometry, to differential geometry (with an emphasis on curvature flows) and topology.

This volume is based on contributions both by conference speakers and by participants, including in two cases articles from mathematicians who were unable to attend the meeting in Hannover.

The book provides a variety of survey articles giving valuable accounts of important developments in the areas discussed. A. Beauville and E. Markman write about holomorphic symplectic manifolds. Whereas Beauville's contribution concentrates on open problems, Markman's article discusses and develops recent work by Verbitsky on the global Torelli theorem for these manifolds. Bauer, Catanese and Pignatelli report on new results concerning the classification of surfaces of general type with vanishing geometric genus. The paper by S. Rollenske provides the reader with an overview of Dolbeault cohomology of nilmanifolds with left-invariant complex structure. In his contribution F. Leitner gives an exposition of some aspects of the theory of conformal holonomy. Kähler-Einstein manifolds and their classification is the topic of M. Kühnel's survey paper, where he also discusses problems concerning the existence and uniqueness of complete Ricci-flat Kähler metrics. M. Lönne discusses braid monodromy of plane curves and explores the new area of knotted monodromy. Submanifolds in Poisson geometry are the subject of M. Zambon's article, and M. Kureš' contribution provides a survey of some algebraic and differential geometric aspects of Weil algebras.

The (classical) theory of special sets of points in projective geometry and Cremona groups are the topic of I. Dolgachev's contribution, while S. Cynk discusses the Fulton-Johnson class of complete intersections. T. Peternell's paper contains new results concerning generic ampleness of the cotangent bundle of non uni-ruled projective manifolds. It is shown in G. K. Sankaran's article that every smooth projective curve can be embedded into a given toric threefold. J.-M. Hwang and W.-K. To discuss the Buser-Sarnak invariant and use this to prove results on the projective normality of abelian varieties. In his contribution N. Mok discusses established and new results on singularities of holomorphic maps between complex hyperbolic space forms. In his paper on vector bundles on curves, N. Hitchin studies polyvector fields on moduli spaces of such bundles.

Flows played an important role in the talks presented at the conference. This is reflected in a number of papers. T. Behrndt and S. Brendle discuss the generalized Lagrangian mean curvature flow in Kähler manifolds and the Ricci flow respectively. Ricci flows also feature in the article by X. Cao and Z. Zhang, who prove differential Harnack estimates. Y.-I. Lee finally provides detailed computations for constructing translating solutions from self-similar solutions for the Lagrangian mean curvature flow.

It is our pleasure to thank all the organizations and people who made this conference a success. We are grateful to Leibniz Universität Hannover and the DFG funded Graduiertenkolleg GRK 1463 "Analysis, Geometry and String Theory" for financial support. The organization of the conference would not have been possible without X. Bogomolec, S. Heidemann, K. Ludwig and M. Schunert. We are particularly indebted to N. Behrens, A. Frühbis-Krüger, S. Gährs and L. Habermann for their substantial contribution to both the organization of the conference and the editing of this volume. Finally, thanks go to R. Timpe, whose TEXnical expertise was invaluable for producing the final form of these Proceedings.

Hannover, February 2011

Wolfgang Ebeling
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Surfaces of general type with geometric genus zero: a survey

Ingrid Bauer, Fabrizio Catanese and Roberto Pignatelli*

Abstract In the last years there have been several new constructions of surfaces of general type with $p_g = 0$, and important progress on their classification. The present paper presents the status of the art on surfaces of general type with $p_g = 0$, and gives an updated list of the existing surfaces, in the case where $K^2 = 1, \ldots, 7$. It also focuses on certain important aspects of this classification.

Keywords Surfaces of general type with genus 0. Godeaux surfaces. Campedelli surfaces. Burniat surfaces. Bloch conjecture. Actions of finite groups.
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1 Introduction

It is nowadays well known that minimal surfaces of general type with $p_g(S) = 0$ have invariants $p_g(S) = q(S) = 0, 1 \leq K_S^2 \leq 9$, hence they yield a finite number of irreducible components of the moduli space of surfaces of general type.

At first glance this class of surfaces seems rather narrow, but we want to report on recent results showing how varied and rich is the botany of such surfaces, for which a complete classification is still out of reach.

These surfaces represent for algebraic geometers an almost prohibitive test case about the possibility of extending the fine Enriques classification of special surfaces to surfaces of general type.

On the one hand, they are the surfaces of general type which achieve the minimal value 1 for the holomorphic Euler-Poincaré characteristic $\chi(S) := p_g(S) - q(S) + 1$, so a naive (and false) guess is that they should be "easier" to understand than other surfaces with higher invariants; on the other hand, there are pathologies (especially concerning the pluricanonical systems) or problems (cf. the Bloch conjecture ([Blo75]) asserting that for surfaces with $p_g(S) = q(S) = 0$ the group of zero cycles modulo rational equivalence should be isomorphic to $\mathbb{Z}$), which only occur for surfaces with $p_g = 0$.

Surfaces with $p_g(S) = q(S) = 0$ have a very old history, dating back to 1896 ([Enr96], see also [EnrMS], I, page 294, and [Cas96]) when Enriques constructed the so called Enriques surfaces in order to give a counterexample to the conjecture of Max Noether that any such surface should be rational, immediately followed by Castelnuovo who constructed a surface with $p_g(S) = q(S) = 0$ whose bicanonical pencil is elliptic.

The first surfaces of general type with $p_g = q = 0$ were constructed in the 1930' s by Luigi Campedelli and by Lucien Godeaux (cf. [Cam32], [God35]): in their honour minimal surfaces of general type with $K_S^2 = 1$ are called numerical Godeaux surfaces, and those with $K_S^2 = 2$ are called numerical Campedelli surfaces.

In the 1970's there was a big revival of interest in the construction of these surfaces and in a possible attempt to classification.

After rediscoveries of these and other old examples a few new ones were found through the efforts of several authors, in particular Rebecca Barlow ([Bar85a]) found a simply connected numerical Godeaux surface, which played a decisive role in the study of the differential topology of algebraic surfaces and 4-manifolds (and also in the discovery of Kähler Einstein metrics of opposite sign on the same manifold, see [CL97]).

A (relatively short) list of the existing examples appeared in the book [BPV84], (see [BPV84], VII, 11 and references therein, and see also [BHPV04] for an updated slightly longer list).

There has been recently important progress on the topic, and the goal of the present paper is to present the status of the art on surfaces of general type with $p_g = 0$, of course focusing only on certain aspects of the story.

Our article is organized as follows: in the first section we explain the "fine" classification problem for surfaces of general type with $p_g = q = 0$. Since the solution to this problem is far from sight we pose some easier problems which could have a greater chance to be solved in the near future.

Moreover, we try to give an update on the current knowledge concerning surfaces with $p_g = q = 0$.

In the second section, we shortly review several reasons why there has been a lot of attention devoted to surfaces with geometric genus p_g equal to zero: Bloch's conjecture, the exceptional behaviour of the pluricanonical maps and the interesting questions whether there are surfaces of general type homeomorphic to Del Pezzo surfaces. It is not possible that a surface of general type be diffeomorphic to a rational surface. This follows from Seiberg-Witten theory which brought a breakthrough establishing in particular that the Kodaira dimension is a differentiable invariant of the 4-manifold underlying an algebraic surface.

Since the first step towards a classification is always the construction of as many examples as possible, we describe in section three various construction methods for algebraic surfaces, showing how they lead to surfaces of general type with $p_g = 0$. Essentially, there are two different approaches, one is to take quotients, by a finite or infinite group, of known (possibly non-compact) surfaces, and the other is in a certain sense the dual one, namely constructing the surfaces as Galois coverings of known surfaces.

The first approach (i.e., taking quotients) seems at the moment to be far more successful concerning the number of examples that have been constructed by this method. On the other hand, the theory of abelian coverings seems much more useful to study the deformations of the constructed surfaces, i.e., to get hold of the irreducible, resp. connected components of the corresponding moduli spaces.

In the last section we review some recent results which have been obtained by the first two authors, concerning the connected components of the moduli spaces corresponding to Keum-Naie, respectively primary Burniat surfaces.

2 Notation

For typographical reasons, especially lack of space inside the tables, we shall use the following non standard notation for a finite cyclic group of order m:

$$\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m.$$

Furthermore Q_8 will denote the quaternion group of order 8,

$$Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}.$$

As usual, $\mathfrak{S}_n$ is the symmetric group in n letters, $\mathfrak{A}_n$ is the alternating subgroup.

$D_{p,q,r}$ is the generalized dihedral group admitting the following presentation:

$$D_{p,q,r} = \langle x, y | x^p, y^q, xyx^{-1}y^{-r} \rangle,$$

while $D_n = D_{2,n,-1}$ is the usual dihedral group of order $2n$.

$G(n,m)$ denotes the m-th group of order n in the MAGMA database of small groups.

Finally, we have semidirect products $H \rtimes \mathbb{Z}_r$; to specify them, one should indicate the image $\varphi \in Aut(H)$ of the standard generator of $\mathbb{Z}_r$ in $Aut(H)$. There is no space in the tables to indicate φ, hence we explain here which automorphism φ will be in the case of the semidirect products occurring as fundamental groups.

For $H = \mathbb{Z}^2$ either r is even, and then φ is $-Id$, or $r = 3$ and φ is the matrix $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

Else H is finite and $r = 2$; for $H = \mathbb{Z}_3^2$, φ is $-Id$; for $H = \mathbb{Z}_2^4$, φ is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Concerning the case where the group G is a semidirect product, we simply refer to [BCGP08] for more details.

Finally, Π_g is the fundamental group of a compact Riemann surface of genus g.

3 The classification problem and "simpler" sub-problems

The history of surfaces with geometric genus equal to zero starts about 120 years ago with a question posed by Max Noether.

Assume that $S \subset \mathbb{P}^N_{\mathbb{C}}$ is a smooth projective surface. Recall that the *geometric genus* of S:

$$p_g(S) := h^0(S, \Omega^2_S) := \dim H^0(S, \Omega^2_S),$$

and the *irregularity* of S:

$$q(S) := h^0(S, \Omega^1_S) := \dim H^0(S, \Omega^1_S),$$

are *birational invariants* of S.

Trying to generalize the one dimensional situation, Max Noether asked the following:

Question 1 Let S be a smooth projective surface with $p_g(S) = q(S) = 0$. Does this imply that S is rational?

The first negative answer to this question is, as we already wrote, due to Enriques ([Enr96], see also [EnrMS], I, page 294) and Castelnuovo, who constructed counterexamples which are surfaces of special type (this means, with Kodaira dimension ≤ 1. *Enriques surfaces* have Kodaira dimension equal to 0, Castelnuovo surfaces have instead Kodaira dimension 1).

After the already mentioned examples by Luigi Campedelli and by Lucien Godeaux and the new examples found by Pol Burniat ([Bur66]), and by many other authors, the discovery and understanding of surfaces of general type with $p_g = 0$ was considered as a challenging problem (cf. [Dol77]): a complete fine classification however soon seemed to be far out of reach.

Maybe this was the motivation for D. Mumford to ask the following provocative

Question 2 (Montreal 1980) Can a computer classify all surfaces of general type with $p_g = 0$?

Before we comment more on Mumford's question, we shall recall some basic facts concerning surfaces of general type.

Let S be a *minimal* surface of general type, i.e., S does not contain any rational curve of self intersection (-1), or equivalently, the canonical divisor K_S of S is nef and big ($K_S^2 > 0$). Then it is well known that

$$K_S^2 \geq 1, \ \chi(S) := 1 - q(S) + p_g(S) \geq 1.$$

In particular, $p_g(S) = 0 \implies q(S) = 0$. Moreover, we have a coarse moduli space parametrizing minimal surfaces of general type with fixed χ and K^2.

Theorem 1 *For each pair of natural numbers (x,y) we have the Gieseker moduli space $\mathfrak{M}^{can}_{(x,y)}$, whose points correspond to the isomorphism classes of minimal surfaces S of general type with $\chi(S) = x$ and $K_S^2 = y$.*

It is a quasi projective scheme which is a coarse moduli space for the canonical models of minimal surfaces S of general type with $\chi(S) = x$ and $K_S^2 = y$.

An upper bound for K_S^2 is given by the famous Bogomolov-Miyaoka-Yau inequality:

Theorem 2 ([Miy77b], [Yau77], [Yau78], [Miy82]) *Let S be a smooth surface of general type. Then*

$$K_S^2 \leq 9\chi(S),$$

and equality holds if and only if the universal covering of S is the complex ball $\mathbb{B}_2 := \{(z,w) \in \mathbb{C}^2 | |z|^2 + |w|^2 < 1\}$.

As a note for the non experts: Miyaoka proved in the first paper the general inequality, which Yau only proved under the assumption of ampleness of the canonical divisor K_S. But Yau showed that if equality holds, and K_S is ample, then the universal cover is the ball; in the second paper Miyaoka showed that if equality holds, then necessarily K_S is ample.

Remark 1 Classification of surfaces of general type with $p_g = 0$ means therefore to "understand" the nine moduli spaces $\mathfrak{M}^{can}_{(1,n)}$ for $1 \leq n \leq 9$, in particular, the connected components of each $\mathfrak{M}^{can}_{(1,n)}$ corresponding to surfaces with $p_g = 0$. Here, understanding means to describe the connected and irreducible components and their respective dimensions.

Even if this is the "test-case" with the lowest possible value for the invariant $\chi(S)$ for surfaces of general type, still nowadays we are quite far from realistically seeing how this goal can be achieved. It is in particular a quite non trivial question, given two explicit surfaces with the same invariants (χ, K^2), to decide whether they are in the same connected component of the moduli space.

An easy observation, which indeed is quite useful, is the following:

Remark 2 Assume that S, S' are two minimal surfaces of general type which are in the same connected component of the moduli space. Then S and S' are orientedly diffeomorphic through a diffeomorphism preserving the Chern class of the canonical divisor; whence S and S' are homeomorphic, in particular they have the same (topological) fundamental group.

Thus the fundamental group π_1 is the simplest invariant which distinguishes connected components of the moduli space $\mathfrak{M}^{can}_{(x,y)}$.

So, it seems natural to pose the following questions which sound "easier" to solve than the complete classification of surfaces with geometric genus zero.

Question 3 What are the topological fundamental groups of surfaces of general type with $p_g = 0$ and $K_S^2 = y$?

Question 4 Is $\pi_1(S) =: \Gamma$ residually finite, i.e., is the natural homomorphism $\Gamma \to \hat{\Gamma} = \lim_{H \triangleleft_f \Gamma}(\Gamma/H)$ from Γ to its profinite completion $\hat{\Gamma}$ injective?

Remark 3

1) Note that in general fundamental groups of algebraic surfaces are not residually finite, but all known examples have $p_g > 0$ (cf. [Tol93], [CK92]).

2) There are examples of surfaces S, S' with non isomorphic topological fundamental groups, but whose profinite completions are isomorphic (cf. [Serre64], [BCG07]).

Question 5 What are the best possible positive numbers a, b such that

- $K_S^2 \leq a \implies |\pi_1(S)| < \infty$,
- $K_S^2 \geq b \implies |\pi_1(S)| = \infty$?

In fact, by Yau's theorem $K_S^2 = 9 \implies |\pi_1(S)| = \infty$. Moreover by [BCGP08] there exists a surface S with $K_S^2 = 6$ and finite fundamental group, so $b \geq 7$. On the other hand, there are surfaces with $K^2 = 4$ and infinite fundamental group (cf. [Keu88], [Nai99]), whence $a \leq 3$.

Note that all known minimal surfaces of general type S with $p_g = 0$ and $K_S^2 = 8$ are uniformized by the bidisk $\mathbb{B}_1 \times \mathbb{B}_1$.

Question 6 Is the universal covering of S with $K_S^2 = 8$ always $\mathbb{B}_1 \times \mathbb{B}_1$?

An affirmative answer to the above question would give a negative answer to the following question of F. Hirzebruch:

Question 7 (F. Hirzebruch) Does there exist a surface of general type homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$?

Or homeomorphic to the blow up $\mathbb{F}_1$ of $\mathbb{P}^2$ in one point ?

In the other direction, for $K_S^2 \leq 2$ it is known that the profinite completion $\hat{\pi}_1$ is finite. There is the following result:

Theorem 3

1) $K_S^2 = 1 \implies \hat{\pi}_1 \cong \mathbb{Z}_m$ *for* $1 \leq m \leq 5$ *(cf. [Rei78]).*

2) $K_S^2 = 2 \implies |\hat{\pi}_1| \leq 9$ *(cf. [Rei], [Xia85a]).*

The bounds are sharp in both cases, indeed for the case $K_S^2 = 1$ there are examples with $\pi_1(S) \cong \mathbb{Z}_m$ for all $1 \leq m \leq 5$ and there is the following conjecture

Conjecture 1 (M. Reid) $\mathfrak{M}^{can}_{(1,1)}$ has exactly five irreducible components corresponding to each choice $\pi_1(S) \cong \mathbb{Z}_m$ for all $1 \leq m \leq 5$.

This conjecture is known to hold true for $m \geq 3$ (cf. [Rei78]).

One can ask similar questions:

Question 8

2) Does $K_S^2 = 2$, $p_g(S) = 0$ imply that $|\pi_1(S)| \leq 9$?

3) Does $K_S^2 = 3$ (and $p_g(S) = 0$) imply that $|\pi_1(S)| \leq 16$?

3.1 Update on surfaces with $p_g = 0$

There has been recently important progress on surfaces of general type with $p_g = 0$ and the current situation is as follows:

$K_S^2 = 9$: these surfaces have the unit ball in $\mathbb{C}^2$ as universal cover, and their fundamental group is an arithmetic subgroup Γ of $SU(2,1)$.

This case seems to be completely classified through exciting new work of Prasad and Yeung and of Cartright and Steger ([PY07], [PY09], [CS]) asserting that the moduli space consists exactly of 100 points, corresponding to 50 pairs of complex conjugate surfaces (cf. [KK02]).

$K_S^2 = 8$: we posed the question whether in this case the universal cover must be the bidisk in $\mathbb{C}^2$.

Assuming this, a complete classification should be possible.

The classification has already been accomplished in [BCG08] for the reducible case where there is a finite étale cover which is isomorphic to a product of curves. In this case there are exactly 18 irreducible connected components of the moduli space: in fact, 17 such components are listed in [BCG08], and recently Davide Frapporti ([Frap10]), while rerunning the classification program, found one more family whose existence had been excluded by an incomplete analysis.

There are many examples, due to Kuga and Shavel ([Kug75], [Sha78]) for the irreducible case, which yield (as in the case $K_S^2 = 9$) rigid surfaces (by results of Jost and Yau [JT85]); but a complete classification of this second case is still missing.

The constructions of minimal surfaces of general type with $p_g = 0$ and with $K_S^2 \leq 7$ available in the literature (to the best of the authors' knowledge, and excluding the recent results of the authors, which will be described later) are listed in table 1.

We proceed to a description, with the aim of putting the recent developments in proper perspective.

$K_S^2 = 1$, i.e., *numerical Godeaux surfaces*: recall that by conjecture 1 the moduli space should have exactly five irreducible connected components, distinguished by the order of the fundamental group, which should be cyclic of order at most 5 ([Rei78] settled the case where the order of the first homology group is at least 3; [Bar85a], [Bar84] and [Wer94] were the first to show the occurrence of the two other groups).

$K_S^2 = 2$, i.e., *numerical Campedelli surfaces*: here, it is known that the order of the algebraic fundamental group is at most 9, and the cases of order $8, 9$ have been classified by Mendes Lopes, Pardini and Reid ([MP08], [MPR09], [Rei]), who showed in particular that the fundamental group equals the algebraic fundamental group and cannot be the dihedral group D_4 of order 8. Naie ([Nai99]) showed that the group

Table 1 Minimal surfaces of general type with $p_g = 0$ and $K^2 \leq 7$ available in the literature

K^2	π_1	π_1^{alg}	H_1	References
1	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	[God34][Rei78][Miy76]
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	[Rei78][OP81][Bar84][Nai94]
	?	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[Rei78]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Bar84][Ino94][KL10]
	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Wer94][Wer97]
	$\{1\}$	$\{1\}$	$\{0\}$	[Bar85a][LP07]
	?	$\{1\}$	$\{0\}$	[CG94][DW99]
2	$\mathbb{Z}_9$	$\mathbb{Z}_9$	$\mathbb{Z}_9$	[MP08]
	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	[Xia85a][MP08]
	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	[Cam32][Rei][Pet76][Ino94][Nai94]
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	[Rei][Nai94][Keu88]
	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	[Rei]
	Q_8	Q_8	$\mathbb{Z}_2^2$	[Rei] [Bea96]
	$\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_7$	[Rei91]
	?	$\mathbb{Z}_6$	$\mathbb{Z}_6$	[NP09]
	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	[Cat81][Sup98]
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	[Ino94][Keu88]
	?	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[LP09]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[KL10]
	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[LP09]
	$\{1\}$	$\{1\}$	$\{0\}$	[LP07]
3	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	[Nai94] [Keu88] [MP04a]
	$Q_8 \times \mathbb{Z}_2$	$Q_8 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$	[Bur66][Pet77] [Ino94]
	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}$	[CS]
	$\mathbb{Z}_{13}$	$\mathbb{Z}_{13}$	$\mathbb{Z}_{13}$	[CS]
	Q_8	Q_8	$\mathbb{Z}_2^2$	[CS]
	D_4	D_4	$\mathbb{Z}_2^2$	[CS]
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	[CS]
	$\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_7$	[CS]
	$\mathfrak{S}_3$	$\mathfrak{S}_3$	$\mathbb{Z}_2$	[CS]
	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	[CS]
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	[CS]
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	[CS]
	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[CS]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[KL10][CS]
	?	?	$\mathbb{Z}_2$	[PPS08a]
	$\{1\}$	$\{1\}$	$\{0\}$	[PPS09a][CS]
4	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	[Nai94][Keu88]
	$Q_8 \times \mathbb{Z}_2^2$	$Q_8 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	[Bur66][Pet77][Ino94]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Par10]
	$\{1\}$	$\{1\}$	$\{0\}$	[PPS09b]
5	$Q_8 \times \mathbb{Z}_2^3$	$Q_8 \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	[Bur66][Pet77][Ino94]
	?	?	?	[Ino94]
6	$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_2^3 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^6$	[Bur66][Pet77][Ino94]
	$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_3^3 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_3^3 \subset H_1$	[Kul04]
	?	?	?	[Ino94][MP04b]
7	$1 \to \Pi_3 \times \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^3 \to 1$	$\hat{\pi}_1$	?	[Ino94][MP01a] [BCC10]

D_3 of order 6 cannot occur as the fundamental group of a numerical Campedelli surface. By the work of Lee and Park ([LP07]), one knows that there exist simply connected numerical Campedelli surfaces.

Recently, in [BCGP08], [BP10], the construction of eight families of numerical Campedelli surfaces with fundamental group $\mathbb{Z}_3$ was given. Neves and Papadakis ([NP09]) constructed a numerical Campedelli surface with algebraic fundamental group $\mathbb{Z}_6$, while Lee and Park ([LP09]) constructed one with algebraic fundamental group $\mathbb{Z}_2$, and one with algebraic fundamental group $\mathbb{Z}_3$ was added in the second version of the same paper. Finally Keum and Lee ([KL10]) constructed examples with topological fundamental group $\mathbb{Z}_2$.

Open conjectures are:

Conjecture 2 Is the fundamental group $\pi_1(S)$ of a numerical Campedelli surface finite?

Question 9 Does every group of order ≤ 9 except D_4 and D_3 occur as topological fundamental group (not only as algebraic fundamental group)?

The answer to question 9 is completely open for $\mathbb{Z}_4$; for $\mathbb{Z}_6, \mathbb{Z}_2$ one suspects that these fundamental groups are realized by the Neves-Papadakis surfaces, respectively by the Lee-Park surfaces.

Note that the existence of the case where $\pi_1(S) = \mathbb{Z}_7$ is shown in the paper [Rei91] (where the result is not mentioned in the introduction).

$\underline{K_S^2 = 3}$: here there were two examples of non trivial fundamental groups, the first one due to Burniat and Inoue, the second one to Keum and Naie ([Bur66], [Ino94], [Keu88] [Nai94]).

It is conjectured that for $p_g(S) = 0, K_S^2 = 3$ the algebraic fundamental group is finite, and one can ask as in 1) above whether also $\pi_1(S)$ is finite. Park, Park and Shin ([PPS09a]) showed the existence of simply connected surfaces, and of surfaces with torsion $\mathbb{Z}_2$ ([PPS08a]). More recently Keum and Lee ([KL10]) constructed an example with $\pi_1(S) = \mathbb{Z}_2$.

Other constructions were given in [Cat98], together with two more examples with $p_g(S) = 0, K^2 = 4, 5$: these turned out however to be the same as the Burniat surfaces.

In [BP10], the existence of four new fundamental groups is shown. Then new fundamental groups were shown to occur by Cartright and Steger, while considering quotients of a fake projective plane by an automorphism of order 3.

With this method Cartright and Steger produced also other examples with $p_g(S) = 0$, $K_S^2 = 3$, and trivial fundamental group, or with $\pi_1(S) = \mathbb{Z}_2$.

$\underline{K_S^2 = 4}$: there were known up to now three examples of fundamental groups, the trivial one (Park, Park and Shin, [PPS09b]), a finite one, and an infinite one. In [BCGP08], [BP10] the existence of 10 new groups, 6 finite and 4 infinite, is shown: thus minimal surfaces with $K_S^2 = 4$, $p_g(S) = q(S) = 0$ realize at least 13 distinct topological types. Recently, H. Park constructed one more example in [Par10] raising the number of topological types to 14.

$\underline{K_S^2 = 5, 6, 7}$: there was known up to now only one example of a fundamental group for $K_S^2 = 5, 7$.

Instead for $K_S^2 = 6$, there are the Inoue-Burniat surfaces and an example due to V. Kulikov (cf. [Kul04]), which contains $\mathbb{Z}_3^3$ in its torsion group. Like in the case of primary Burniat surfaces one can see that the fundamental group of the Kulikov surface fits into an exact sequence

$$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_3^3 \to 1.$$

$K_S^2 = 5$: in [BP10] the existence of 7 new groups, four of which finite, is shown: thus minimal surfaces with $K_S^2 = 5$, $p_g(S) = q(S) = 0$ realize at least 8 distinct topological types.

$K_S^2 = 6$: in [BCGP08] the existence of 6 new groups, three of which finite, is shown: thus minimal surfaces with $K_S^2 = 6$, $p_g(S) = q(S) = 0$ realize at least 7 distinct topological types.

$K_S^2 = 7$: we shall show elsewhere ([BCC10]) that these surfaces, constructed by Inoue in [Ino94], have a fundamental group fitting into an exact sequence

$$1 \to \Pi_3 \times \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^3 \to 1.$$

This motivates the following further question (cf. question 5).

Question 10 Is it true that fundamental groups of surfaces of general type with $q = p_g = 0$ are finite for $K_S^2 \leq 3$, and infinite for $K_S^2 \geq 7$?

4 Other reasons why surfaces with $p_g = 0$ have been of interest in the last 30 years

4.1 Bloch's conjecture

Another important problem concerning surfaces with $p_g = 0$ is related to the problem of rational equivalence of 0-cycles.

Recall that, for a nonsingular projective variety X, $A_0^i(X)$ is the group of rational equivalence classes of zero cycles of degree i.

Conjecture 3 Let S be a smooth surface with $p_g = 0$. Then the kernel $T(S)$ of the natural morphism (the so-called *Abel-Jacobi map*) $A_0^0(S) \to \mathrm{Alb}(S)$ is trivial.

By a beautiful result of D. Mumford ([Mum68]), the kernel of the Abel-Jacobi map is infinite dimensional for surfaces S with $p_g \neq 0$.

The conjecture has been proven for $\kappa(S) < 2$ by Bloch, Kas and Liebermann (cf. [BKL76]). If instead S is of general type, then $q(S) = 0$, whence Bloch's conjecture asserts for those surfaces that $A_0(S) \cong \mathbb{Z}$.

Inspite of the efforts of many authors, there are only few cases of surfaces of general type for which Bloch's conjecture has been verified (cf. e.g. [IM79], [Bar85b], [Keu88], [Voi92]).

Recently S. Kimura introduced the following notion of *finite dimensionality* of motives ([Kim05]).

Definition 1 Let M be a motive.

Then M is *evenly finite dimensional* if there is a natural number $n \geq 1$ such that $\wedge^n M = 0$.

M is *oddly finite dimensional* if there is a natural number $n \geq 1$ such that $\mathrm{Sym}^n M = 0$.

And, finally, M is *finite dimensional* if $M = M^+ \oplus M^-$, where M^+ is evenly finite dimensional and M^- is oddly finite dimensional.

Using this notation, he proves the following

Theorem 4

1) The motive of a smooth projective curve is finite dimensional ([Kim05], cor. 4.4.).

2) The product of finite dimensional motives is finite dimensional (loc. cit., cor. 5.11.).

3) Let $f: M \to N$ be a surjective morphism of motives, and assume that M is finite dimensional. Then N is finite dimensional (loc. cit., prop. 6.9.).

4) Let S be a surface with $p_g = 0$ and suppose that the Chow motive of X is finite dimensional. Then $T(S) = 0$ (loc.cit., cor. 7.7.).

Using the above results we obtain

Theorem 5 *Let S be the minimal model of a product-quotient surface (i.e., birational to $(C_1 \times C_2)/G$, where G is a finite group acting effectively on a product of two compact Riemann surfaces of respective genera $g_i \geq 2$) with $p_g = 0$.*

Then Bloch's conjecture holds for S, namely, $A_0(S) \cong \mathbb{Z}$.

Proof Let S be the minimal model of $X = (C_1 \times C_2)/G$. Since X has rational singularities $T(X) = T(S)$.

By thm. 4, 2), 3) we have that the motive of X is finite dimensional, whence, by 4), $T(S) = T(X) = 0$.

Since S is of general type we have also $q(S) = 0$, hence $A_0^0(S) = T(S) = 0$.

Corollary 1 *All the surfaces in table 2, 3, and all the surfaces in [BC04], [BCG08] satisfy Bloch's conjecture.*

4.2 Pluricanonical maps

A further motivation for the study of surfaces with $p_g = 0$ comes from the behavior of the pluricanonical maps of surfaces of general type.

Definition 2 The n-th pluricanonical map

$$\varphi_n := \varphi_{|nK_S|} : S \dashrightarrow \mathbb{P}^{P_n - 1}$$

is the rational map associated to $H^0(\mathscr{O}_S(nK_S))$.

We recall that for a curve of general type φ_n is an embedding as soon as $n \geq 3$, and also for $n = 2$, if the curve is not of genus 2. The situation in dimension 2 is much more complicated. We recall:

Definition 3 The canonical model of a surface of general type is the normal surface

$$X := Proj(\bigoplus_{n=0}^{\infty} H^0(\mathscr{O}_S(nK_S))),$$

the projective spectrum of the (finitely generated) canonical ring.

X is obtained from its minimal model S by contracting all the curves C with $K_S \cdot C = 0$, i.e., all the smooth rational curves with self intersection equal to -2.

The n-th pluricanonical map $\varphi_{|nK_S|}$ of a surface of general type is the composition of the projection onto its canonical model X with $\psi_n := \varphi_{|nK_X|}$. So it suffices to study this last map.

This was done by Bombieri, whose results were later improved by the work of several authors. We summarize these efforts in the following theorem.

Theorem 6 *([Bom73], [Miy76], [BC78], [Cat77], [Reider88], [Fran88], [CC88], [CFHR99])*

Let X be the canonical model of a surface of general type. Then

i) $\varphi_{|nK_X|}$ *is an embedding for all $n \geq 5$;*

ii) $\varphi_{|4K_X|}$ *is an embedding if $K_X^2 \geq 2$;*

iii) $\varphi_{|3K_X|}$ *is a morphism if $K_X^2 \geq 2$ and an embedding if $K_X^2 \geq 3$;*

iv) $\varphi_{|nK_X|}$ *is birational for all $n \geq 3$ unless*

a) either $K^2 = 1$, $p_g = 2$, $n = 3$ or 4.

In this case X is a hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1,1,2,5)$, a finite double cover of the quadric cone $Y := \mathbb{P}(1,1,2)$, $\varphi_{|3K_X|}(X)$ is birational to Y and isomorphic to an embedding of the surface $\mathbb{F}_2$ in $\mathbb{P}^5$, while $\varphi_{|4K_X|}(X)$ is an embedding of Y in $\mathbb{P}^8$.

b) Or $K^2 = 2$, $p_g = 3$, $n = 3$ (in this case X is a double cover of $\mathbb{P}^2$ branched on a curve of degree 8, and $\varphi_{|3K_X|}(X)$ is the image of the Veronese embedding $\nu_3 : \mathbb{P}^2 \to \mathbb{P}^9$).

v) $\varphi_{|2K_X|}$ *is a morphism if $K_X^2 \geq 5$ or if $p_g \neq 0$.*

vi) If $K_X^2 \geq 10$ then $\varphi_{|2K_X|}$ is birational if and only if X does not admit a morphism onto a curve with general fibre of genus 2.

The surfaces with $p_g = 0$ arose as the difficult case for the understanding of the tricanonical map, because, in the first version of his theorem, Bombieri could not determine whether the tricanonical and quadricanonical map of the numerical Godeaux and of the numerical Campedelli surfaces had to be birational. This was later proved in [Miy76], in [BC78], and in [Cat77].

It was already known to Kodaira that a morphism onto a smooth curve with general fibre of genus 2 forces the bicanonical map to factor through the hyperelliptic involution of the fibres: this is called the *standard case* for the nonbirationality of the bicanonical map. Part vi) of Theorem 6 shows that there are finitely many families of surfaces of general type with bicanonical map nonbirational which do not present the standard case. These interesting families have been classified under the hypothesis $p_g > 1$ or $p_g = 1$, $q \neq 1$: see [BCP06] for a more precise account on this results.

Again, the surfaces with $p_g = 0$ are the most difficult and hence the most interesting, since there are "pathologies" which can happen only for surfaces with $p_g = 0$.

For example, the bicanonical system of a numerical Godeaux surface is a pencil, and therefore maps the surface onto $\mathbb{P}^1$, while [Xia85b] showed that the bicanonical map of every other surface of general type has a two dimensional image. Moreover, obviously for a numerical Godeaux surface $\varphi_{|2K_X|}$ is not a morphism, thus showing that the condition $p_g \neq 0$ in the point v) of the Theorem 6 is sharp.

Recently, Pardini and Mendes Lopes (cf. [MP08]) showed that there are more examples of surfaces whose bicanonical map is not a morphism, constructing two families of numerical Campedelli surfaces whose bicanonical system has two base points.

What it is known on the degree of the bicanonical map of surfaces with $p_g = 0$ can be summarized in the following

Theorem 7 ([MP07a],[MLP02], [MP08]) *Let S be a surface with $p_g = q = 0$. Then*

- *if* $K_S^2 = 9 \Rightarrow \deg \varphi_{|2K_S|} = 1$,
- *if* $K_S^2 = 7, 8 \Rightarrow \deg \varphi_{|2K_S|} = 1$ *or* 2,
- *if* $K_S^2 = 5, 6 \Rightarrow \deg \varphi_{|2K_S|} = 1$, 2 *or* 4,

- *if* $K_S^2 = 3,4 \Rightarrow \deg\varphi_{|2K_S|} \leq 5$*; if moreover* $\varphi_{|2K_S|}$ *is a morphism, then* $\deg\varphi_{|2K_S|} = 1$, 2 *or* 4*,*
- *if* $K_S^2 = 2$ *(since the image of the bicanonical map is* $\mathbb{P}^2$*, the bicanonical map is non birational), then* $\deg\varphi_{|2K_S|} \leq 8$*. In the known examples it has degree* 6 *(and the bicanonical system has two base points) or* 8 *(and the bicanonical system has no base points).*

4.3 Differential topology

The surfaces with $p_g = 0$ are very interesting also from the point of view of differential topology, in particular in the simply connected case. We recall Freedman's theorem.

Theorem 8 ([Fre82]) *Let M be an oriented, closed, simply connected topological manifold: then M is determined (up to homeomorphism) by its intersection form*

$$q\colon H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}$$

and by the Kirby-Siebenmann invariant $\alpha(M) \in \mathbb{Z}_2$*, which vanishes if and only if* $M \times [0,1]$ *admits a differentiable structure.*

If M is a complex surface, the Kirby-Siebenmann invariant automatically vanishes and therefore the oriented homeomorphism type of M is determined by the intersection form.

Combining it with a basic result of Serre on indefinite unimodular forms, and since by [Yau77] the only simply connected compact complex surface whose intersection form is definite is $\mathbb{P}^2$ one concludes

Corollary 2 *The oriented homeomorphism type of any simply connected complex surface is determined by the rank, the index and the parity of the intersection form.*

This gives a rather easy criterion to decide whether two complex surfaces are orientedly homeomorphic; anyway two orientedly homeomorphic complex surfaces are not necessarily diffeomorphic.

In fact, Dolgachev surfaces ([Dol77], see also [BHPV04, IX.5]) give examples of infinitely many surfaces which are all orientedly homeomorphic, but pairwise not diffeomorphic; these are elliptic surfaces with $p_g = q = 0$.

As mentioned, every compact complex surface homeomorphic to $\mathbb{P}^2$ is diffeomorphic (in fact, algebraically isomorphic) to $\mathbb{P}^2$ (cf. [Yau77]), so one can ask a similar question (cf. e.g. Hirzebruch's question 7): if a surface is homeomorphic to a rational surface, is it also diffeomorphic to it?

Simply connected surfaces of general type with $p_g = 0$ give a negative answer to this question. Indeed, by Freedman's theorem each simply connected minimal surface S of general type with $p_g = 0$ is orientedly homeomorphic to a Del Pezzo surface of degree K_S^2. Still these surfaces are not diffeomorphic to a Del Pezzo surface because of the following

Theorem 9 ([FQ94]) *Let S be a surface of general type. Then S is not diffeomorphic to a rational surface.*

The first simply connected surface of general type with $p_g = 0$ was constructed by R. Barlow in the 80's, and more examples have been constructed recently by Y. Lee, J. Park, H. Park and D. Shin. We summarize their results in the following

Theorem 10 ([Bar85a], [LP07], [PPS09a], [PPS09b]) $\forall 1 \leq y \leq 4$ *there are minimal simply connected surfaces of general type with* $p_g = 0$ *and* $K^2 = y$.

5 Construction techniques

As already mentioned, a first step towards a classification is the construction of examples. Here is a short list of different methods for constructing surfaces of general type with $p_g = 0$.

5.1 Quotients by a finite (resp. : infinite) group

5.1.1 Ball quotients

By the Bogomolov-Miyaoka-Yau theorem, a surface of general type with $p_g = 0$ is uniformized by the two dimensional complex ball $\mathbb{B}_2$ if and only if $K_S^2 = 9$. These surfaces are classically called *fake projective planes*, since they have the same Betti numbers as the projective plane $\mathbb{P}^2$.

The first example of a fake projective plane was constructed by Mumford (cf. [Mum79]), and later very few other examples were given (cf.[IK98], [Keu06]).

Ball quotients $S = \mathbb{B}_2/\Gamma$, where $\Gamma \leq PSU(2,1)$ is a discrete, cocompact, torsionfree subgroup are strongly rigid surfaces in view of Mostow's rigidity theorem ([Mos73]).

In particular the moduli space $\mathfrak{M}_{(1,9)}$ consists of a finite number of isolated points.

The possibility of obtaining a complete list of these fake planes seemed rather unrealistic until a breakthrough came in 2003: a surprising result by Klingler (cf. [Kli03]) showed that the cocompact, discrete, torsionfree subgroups $\Gamma \leq PSU(2,1)$ having minimal Betti numbers, i.e., yielding fake planes, are indeed arithmetic.

This allowed a complete classification of these surfaces carried out by Prasad and Yeung, Steger and Cartright ([PY07], [PY09]): the moduli space contains exactly 100 points, corresponding to 50 pairs of complex conjugate surfaces.

5.1.2 Product quotient surfaces

In a series of papers the following construction was explored systematically by the authors with the help of the computer algebra program MAGMA (cf. [BC04], [BCG08], [BCGP08], [BP10]).

Let C_1, C_2 be two compact curves of respective genera $g_1, g_2 \geq 2$. Assume further that G is a finite group acting effectively on $C_1 \times C_2$.

In the case where the action of G is free, the quotient surface is minimal of general type and is said to be *isogenous to a product* (see [Cat00]).

If the action is not free we consider the minimal resolution of singularities S' of the normal surface $X := (C_1 \times C_2)/G$ and its minimal model S. The aim is to give a complete classification of those S obtained as above which are of general type and have $p_g = 0$.

One observes that, if the tangent action of the stabilizers is contained in $SL(2,\mathbb{C})$, then X has Rational Double Points as singularities and is the canonical model of a surface of general type. In this case S' is minimal.

Recall the definition of an orbifold surface group (here the word 'surface' stands for 'Riemann surface'):

Definition 4 An *orbifold surface group* of genus g' and multiplicities $m_1, \dots m_r \in \mathbb{N}_{\geq 2}$ is the group presented as follows:

$$\mathbb{T}(g'; m_1, \dots, m_r) := \langle a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r | \\ c_1^{m_1}, \dots, c_r^{m_r}, \prod_{i=1}^{g'} [a_i, b_i] \cdot c_1 \cdot \dots \cdot c_r \rangle.$$

The sequence $(g'; m_1, \dots m_r)$ is called the *signature* of the orbifold surface group.

Moreover, recall the following special case of *Riemann's existence theorem*:

Theorem 11 *A finite group G acts as a group of automorphisms on a compact Riemann surface C of genus g if and only if there are natural numbers $g', m_1, \dots, m_r$, and an 'appropriate' orbifold homomorphism*

$$\varphi \colon \mathbb{T}(g'; m_1, \dots, m_r) \to G$$

such that the Riemann - Hurwitz relation holds:

$$2g - 2 = |G| \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right).$$

"Appropriate" means that φ is surjective and moreover that the image $\gamma_i \in G$ of a generator c_i has order exactly equal to m_i (the order of c_i in $\mathbb{T}(g'; m_1, \dots, m_r)$).

In the above situation g' is the genus of $C' := C/G$. The G-cover $C \to C'$ is branched in r points $p_1, \dots, p_r$ with branching indices $m_1, \dots, m_r$, respectively.

Denote as before $\varphi(c_i)$ by $\gamma_i \in G$ the image of c_i under φ: then the set of stabilizers for the action of G on C is the set

$$\Sigma(\gamma_1, \dots, \gamma_r) := \cup_{a \in G} \cup_{i=0}^{max\{m_i\}} \{a\gamma_1^i a^{-1}, \dots a\gamma_r^i a^{-1}\}.$$

Assume now that there are two epimorphisms

$$\varphi_1 \colon \mathbb{T}(g'_1; m_1, \dots, m_r) \to G,$$

$$\varphi_2 \colon \mathbb{T}(g'_2; n_1, \dots, n_s) \to G,$$

determined by two Galois covers $\lambda_i \colon C_i \to C'_i$, $i = 1, 2$.

We will assume in the following that $g(C_1),\ g(C_2) \geq 2$, and we shall consider the diagonal action of G on $C_1 \times C_2$.

We shall say in this situation that the action of G on $C_1 \times C_2$ is of *unmixed* type (indeed, see [Cat00], there is always a subgroup of G of index at most 2 with an action of unmixed type).

Theorem 12 ([BC04], [BCG05] [BCGP08],[BP10])

1) Surfaces S isogenous to a product with $p_g(S) = q(S) = 0$ form 17 irreducible connected components of the moduli space $\mathfrak{M}^{can}_{(1,8)}$.

2) Surfaces with $p_g = 0$, whose canonical model is a singular quotient $X := (C_1 \times C_2)/G$ by an unmixed action of G form 27 further irreducible families.

3) Minimal surfaces with $p_g = 0$ which are the minimal resolution of the singularities of $X := C_1 \times C_2/G$ such that the action is of unmixed type and X does not have canonical singularities form exactly further 32 irreducible families.

Moreover, $K_S^2 = 8$ if and only if S is isogenous to a product.

We summarize the above results in tables 2 and 3.

Remark 4 1) Recall that, if a diagonal action of G on $C_1 \times C_2$ is not free, then G has a finite set of fixed points. The quotient surface $X := (C_1 \times C_2)/G$ has a finite number of singular points. These can be easily found by looking at the given description of the stabilizers for the action of G on each individual curve.

Assume that $x \in X$ is a singular point. Then it is a cyclic quotient singularity of type $\frac{1}{n}(1,a)$ with $g.c.d(a,n) = 1$, i.e., X is, locally around x, biholomorphic to the quotient of $\mathbb{C}^2$ by the action of a diagonal linear automorphism with eigenvalues $\exp(\frac{2\pi i}{n})$, $\exp(\frac{2\pi ia}{n})$. That $g.c.d(a,n) = 1$ follows since the tangent representation is faithful on both factors.

2) We denote by K_X the canonical (Weil) divisor on the normal surface corresponding to $i_*(\Omega^2_{X^0})$, $i\colon X^0 \to X$ being the inclusion of the smooth locus of X. According to Mumford we have an intersection product with values in $\mathbb{Q}$ for Weil divisors on a normal surface, and in particular we consider the selfintersection of the canonical divisor,

$$K_X^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} \in \mathbb{Q}, \tag{1}$$

which is not necessarily an integer.

K_X^2 is however an integer (equal indeed to K_S^2) if X has only RDP's as singularities.

Table 2 Surfaces isogenous to a product and minimal standard isotrivial fibrations with $p_g = 0$, $K^2 \geq 4$

K^2	Sing X	T_1	T_2	G	N	$H_1(S,\mathbb{Z})$	$\pi_1(S)$
8	$\emptyset$	$2,5^2$	3^4	$\mathfrak{A}_5$	1	$\mathbb{Z}_3^2 \times \mathbb{Z}_{15}$	$1 \to \Pi_{21} \times \Pi_4 \to \pi_1 \to G \to 1$
8	$\emptyset$	5^3	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_{10}^2$	$1 \to \Pi_6 \times \Pi_{13} \to \pi_1 \to G \to 1$
8	$\emptyset$	$3^2,5$	2^5	$\mathfrak{A}_5$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_6$	$1 \to \Pi_{16} \times \Pi_5 \to \pi_1 \to G \to 1$
8	$\emptyset$	$2,4,6$	2^6	$\mathfrak{S}_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$	$1 \to \Pi_{25} \times \Pi_3 \to \pi_1 \to G \to 1$
8	$\emptyset$	$2^2,4^2$	$2^3,4$	G(32,27)	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$	$1 \to \Pi_5 \times \Pi_9 \to \pi_1 \to G \to 1$
8	$\emptyset$	5^3	5^3	$\mathbb{Z}_5^2$	2	$\mathbb{Z}_5^2$	$1 \to \Pi_6 \times \Pi_6 \to \pi_1 \to G \to 1$
8	$\emptyset$	$3,4^2$	2^6	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^4 \times \mathbb{Z}_8$	$1 \to \Pi_{13} \times \Pi_3 \to \pi_1 \to G \to 1$
8	$\emptyset$	$2^2,4^2$	$2^2,4^2$	G(16,3)	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$	$1 \to \Pi_5 \times \Pi_5 \to \pi_1 \to G \to 1$
8	$\emptyset$	$2^3,4$	2^6	$D_4 \times \mathbb{Z}_2$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$	$1 \to \Pi_9 \times \Pi_3 \to \pi_1 \to G \to 1$
8	$\emptyset$	2^5	2^5	$\mathbb{Z}_2^4$	1	$\mathbb{Z}_2^4$	$1 \to \Pi_5 \times \Pi_5 \to \pi_1 \to G \to 1$
8	$\emptyset$	3^4	3^4	$\mathbb{Z}_3^2$	1	$\mathbb{Z}_3^4$	$1 \to \Pi_4 \times \Pi_4 \to \pi_1 \to G \to 1$
8	$\emptyset$	2^5	2^6	$\mathbb{Z}_2^3$	1	$\mathbb{Z}_2^6$	$1 \to \Pi_3 \times \Pi_5 \to \pi_1 \to G \to 1$
8	$\emptyset$	mixed		G(256,3678)	3		
8	$\emptyset$	mixed		G(256,3679)	1		
8	$\emptyset$	mixed		G(64,92)	1		
6	$1/2^2$	$2^3,4$	$2^4,4$	$\mathbb{Z}_2 \times D_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	$1 \to \mathbb{Z}^2 \times \Pi_2 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$
6	$1/2^2$	$2^4,4$	$2,4,6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$1 \to \Pi_2 \to \pi_1 \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to 1$
6	$1/2^2$	$2,5^2$	$2,3^3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_3 \times \mathbb{Z}_{15}$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_{15}$
6	$1/2^2$	$2,4,10$	$2,4,6$	$\mathbb{Z}_2 \times \mathfrak{S}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathfrak{S}_3 \times D_{4,5,-1}$
6	$1/2^2$	$2,7^2$	$3^2,4$	PSL(2,7)	2	$\mathbb{Z}_{21}$	$\mathbb{Z}_7 \times \mathfrak{A}_4$
6	$1/2^2$	$2,5^2$	$3^2,4$	$\mathfrak{A}_6$	2	$\mathbb{Z}_{15}$	$\mathbb{Z}_5 \times \mathfrak{A}_4$
5	$1/3,2/3$	$2,4,6$	$2^4,3$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^2 \to \pi_1 \to D_{2,8,3} \to 1$
5	$1/3,2/3$	$2^4,3$	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_8 \to 1$
5	$1/3,2/3$	$4^2,6$	$2^3,3$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_8 \to 1$
5	$1/3,2/3$	$2,5,6$	$3,4^2$	$\mathfrak{S}_5$	1	$\mathbb{Z}_8$	$D_{8,5,-1}$
5	$1/3,2/3$	$3,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\mathbb{Z}_5 \times Q_8$
5	$1/3,2/3$	$2^3,3$	$3,4^2$	$\mathbb{Z}_2^4 \rtimes \mathfrak{S}_3$	1	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$D_{8,4,3}$
5	$1/3,2/3$	$3,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$
4	$1/2^4$	2^5	2^5	$\mathbb{Z}_2^3$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$
4	$1/2^4$	$2^2,4^2$	$2^2,4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$
4	$1/2^4$	2^5	$2^3,4$	$\mathbb{Z}_2 \times D_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to 1$
4	$1/2^4$	$3,6^2$	$2^2,3^2$	$\mathbb{Z}_3 \times \mathfrak{S}_3$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$
4	$1/2^4$	$3,6^2$	$2,4,5$	$\mathfrak{S}_5$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$
4	$1/2^4$	2^5	$2,4,6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$
4	$1/2^4$	$2^2,4^2$	$2,4,6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_4$
4	$1/2^4$	2^5	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_4$
4	$1/2^4$	$2^3,4$	$2^3,4$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	1	$\mathbb{Z}_4^2$	$G(32,2)$
4	$1/2^4$	$2,5^2$	$2^2,3^2$	$\mathfrak{A}_5$	1	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$
4	$1/2^4$	$2^2,3^2$	$2^2,3^2$	$\mathbb{Z}_3^2 \rtimes Z_2$	1	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$
4	$2/5^2$	$2^3,5$	$3^2,5$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$
4	$2/5^2$	$2,4,5$	$4^2,5$	$\mathbb{Z}_2^4 \rtimes D_5$	3	$\mathbb{Z}_8$	$\mathbb{Z}_8$
4	$2/5^2$	$2,4,5$	$3^2,5$	$\mathfrak{A}_6$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$

Table 3 Minimal standard isotrivial fibrations with $p_g = 0$, $K^2 \leq 3$

K^2	Sing X	T_1	T_2	G	N	$H_1(S,\mathbb{Z})$	$\pi_1(S)$
3	$1/5, 4/5$	$2^3, 5$	$3^2, 5$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$
3	$1/5, 4/5$	$2, 4, 5$	$4^2, 5$	$\mathbb{Z}_2^4 \rtimes D_5$	3	$\mathbb{Z}_8$	$\mathbb{Z}_8$
3	$1/3, 1/2^2, 2/3$	$2^2, 3, 4$	$2, 4, 6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
3	$1/5, 4/5$	$2, 4, 5$	$3^2, 5$	$\mathfrak{A}_6$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$
2	$1/3^2, 2/3^2$	$2, 6^2$	$2^2, 3^2$	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_3$	1	$\mathbb{Z}_2^2$	Q_8
2	$1/2^6$	4^3	4^3	$\mathbb{Z}_4^2$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
2	$1/2^6$	$2^3, 4$	$2^3, 4$	$\mathbb{Z}_2 \times D_4$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
2	$1/3^2, 2/3^2$	$2^2, 3^2$	$3, 4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_8$	$\mathbb{Z}_8$
2	$1/3^2, 2/3^2$	$3^2, 5$	$3^2, 5$	$\mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$	2	$\mathbb{Z}_5$	$\mathbb{Z}_5$
2	$1/2^6$	$2, 5^2$	$2^3, 3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_5$	$\mathbb{Z}_5$
2	$1/2^6$	$2^3, 4$	$2, 4, 6$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/3^2, 2/3^2$	$3^2, 5$	$2^3, 3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/2^6$	$2, 3, 7$	4^3	PSL(2,7)	2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/2^6$	$2, 6^2$	$2^3, 3$	$\mathfrak{S}_3 \times \mathfrak{S}_3$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/2^6$	$2, 6^2$	$2, 4, 5$	$\mathfrak{S}_5$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4, 1/2^2, 3/4$	$2, 4, 7$	$3^2, 4$	PSL(2,7)	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4, 1/2^2, 3/4$	$2, 4, 5$	$3^2, 4$	$\mathfrak{A}_6$	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4, 1/2^2, 3/4$	$2, 4, 6$	$2, 4, 5$	$\mathfrak{S}_5$	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
1	$1/3, 1/2^4, 2/3$	$2^3, 3$	$3, 4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$
1	$1/3, 1/2^4, 2/3$	$2, 3, 7$	$3, 4^2$	PSL(2,7)	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2^3, 3$	$\mathbb{Z}_2 \times \mathfrak{S}_4$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$

3) The resolution of a cyclic quotient singularity of type $\frac{1}{n}(1,a)$ with $g.c.d(a,n) = 1$ is well known. These singularities are resolved by the so-called *Hirzebruch-Jung strings*. More precisely, let $\pi \colon S \to X$ be a minimal resolution of the singularities and let $E = \bigcup_{i=1}^m E_i = \pi^{-1}(x)$. Then E_i is a smooth rational curve with $E_i^2 = -b_i$ and $E_i \cdot E_j = 0$ if $|i-j| \geq 2$, while$E_i \cdot E_{i+1} = 1$ for $i \in \{1, \ldots, m-1\}$.

The b_i's are given by the continued fraction

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}}.$$

Since the minimal resolution $S' \to X$ of the singularities of X replaces each singular point by a tree of smooth rational curves, we have, by van Kampen's theorem, that $\pi_1(X) = \pi_1(S') = \pi_1(S)$.

Moreover, we can read off all invariants of S' from the group theoretical data. For details and explicit formulae we refer to [BP10].

Among others, we also prove the following lemma:

Lemma 5.1 *There exist positive numbers D, M, R, B, which depend explicitly (and only) on the singularities of X such that:*

1. $\chi(S') = 1 \implies K_{S'}^2 = 8 - B$;
2. *for the corresponding signatures* $(0; m_1, \ldots, m_r)$ *and* $(0; n_1, \ldots, n_s)$ *of the orbifold surface groups we have* $r, s \leq R$, $\forall\ i\ m_i, n_i \leq M$;
3. $|G| = \frac{K_{S'}+D}{2(-2+\sum_1^r(1-\frac{1}{m_i}))(-2+\sum_1^s(1-\frac{1}{n_i}))}$.

Remark 5 The above lemma 5.1 implies that there is an algorithm which computes all such surfaces S' with $p_g = q = 0$ and fixed $K_{S'}^2$:

a) find all possible configurations (= "baskets") $\mathscr{B}$ of singularities with $B = 8 - K_{S'}^2$;

b) for a fixed basket $\mathscr{B}$ find all signatures $(0; m_1, \ldots, m_r)$ satisfying 2);

c) for each pair of signatures check all groups G of order given by 3), whether there are surjective homomorphisms $\mathbb{T}(0; m_i) \to G$, $\mathbb{T}(0; n_i) \to G$;

d) check whether the surfaces $X = (C_1 \times C_2)/G$ thus obtained have the right singularities.

Still this is not yet the solution of the problem and there are still several difficult problems to be overcome:

- We have to check whether the groups of a given order admit certain systems of generators of prescribed orders, and satisfying moreover certain further conditions (forced by the basket of singularities); we encounter in this way groups of orders 512, 1024, 1536: there are so many groups of these orders that the above investigation is not feasible for naive computer calculations. Moreover, we have to deal with groups of orders > 2000: they are not listed in any database

- If X is singular, we only get subfamilies, not a whole irreducible component of the moduli space. There remains the problem of studying the deformations of the minimal models S obtained with the above construction.
- The algorithm is heavy for K^2 small. In [BP10] we proved and implemented much stronger results on the singularities of X and on the possible signatures, which allowed us to obtain a complete list of surfaces with $K_S^2 \geq 1$.
- We have not yet answered completely the original question. Since, if X does not have canonical singularities, it may happen that $K_{S'}^2 \leq 0$ (recall that S' is the minimal resolution of singularities of X, which is not necessarily minimal!).

Concerning product quotient surfaces, we have proven (in a much more general setting, cf. [BCGP08]) a structure theorem for the fundamental group, which helps us to explicitly identify the fundamental groups of the surfaces we constructed. In fact, it is not difficult to obtain a presentation for these fundamental groups, but as usual having a presentation is not sufficient to determine the group explicitly.

We first need the following

Definition 5 We shall call the fundamental group $\Pi_g := \pi_1(C)$ of a smooth compact complex curve of genus g a *(genus g) surface group*.

Note that we admit also the "degenerate cases" $g = 0, 1$.

Theorem 13 *Let $C_1, \ldots, C_n$ be compact complex curves of respective genera $g_i \geq 2$ and let G be a finite group acting faithfully on each C_i as a group of biholomorphic transformations.*

Let $X = (C_1 \times \ldots \times C_n)/G$, and denote by S a minimal desingularisation of X. Then the fundamental group $\pi_1(X) \cong \pi_1(S)$ has a normal subgroup $\mathscr{N}$ of finite index which is isomorphic to the product of surface groups, i.e., there are natural numbers $h_1, \ldots, h_n \geq 0$ such that $\mathscr{N} \cong \Pi_{h_1} \times \ldots \times \Pi_{h_n}$.

Remark 6 In the case of dimension $n = 2$ there is no loss of generality in assuming that G acts faithfully on each C_i (see [Cat00]). In the general case there will be a group G_i, quotient of G, acting faithfully on C_i, hence the strategy has to be slightly changed in the general case. The generalization of the above theorem, where the assumption that G acts faithfully on each factor is removed, has been proven in [DP10].

We shall now give a short outline of the proof of theorem 13 in the case $n = 2$ (the case of arbitrary n is exactly the same).

We have two appropriate orbifold homomorphisms

$$\varphi_1 \colon \mathbb{T}_1 := \mathbb{T}(g_1'; m_1, \ldots, m_r) \to G,$$

$$\varphi_2 \colon \mathbb{T}_2 := \mathbb{T}(g_2'; n_1, \ldots, n_s) \to G.$$

We define the fibre product $\mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2)$ as

$$\mathbb{H} := \mathbb{H}(G; \varphi_1, \varphi_2) := \{ (x,y) \in \mathbb{T}_1 \times \mathbb{T}_2 \mid \varphi_1(x) = \varphi_2(y) \}. \tag{2}$$

Then the exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \mathbb{T}_1 \times \mathbb{T}_2 \to G \times G \to 1, \tag{3}$$

where $\Pi_{g_i} := \pi_1(C_i)$, induces an exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \mathbb{H}(G; \varphi_1, \varphi_2) \to G \cong \Delta_G \to 1. \tag{4}$$

Here $\Delta_G \subset G \times G$ denotes the diagonal subgroup.

Definition 6 Let H be a group. Then its *torsion subgroup* $\mathrm{Tors}(H)$ is the normal subgroup generated by all elements of finite order in H.

The first observation is that one can calculate our fundamental groups via a simple algebraic recipe:

$$\pi_1((C_1 \times C_2)/G) \cong \mathbb{H}(G; \varphi_1, \varphi_2)/\mathrm{Tors}(\mathbb{H}).$$

The strategy is then the following: using the structure of orbifold surface groups we construct an exact sequence

$$1 \to E \to \mathbb{H}/\mathrm{Tors}(\mathbb{H}) \to \Psi(\hat{\mathbb{H}}) \to 1,$$

where

i) E is finite,

ii) $\Psi(\hat{\mathbb{H}})$ is a subgroup of finite index in a product of orbifold surface groups.

Condition *ii*) implies that $\Psi(\hat{\mathbb{H}})$ is residually finite and "good" according to the following

Definition 7 (J.-P. Serre) Let $\mathbb{G}$ be a group, and let $\tilde{\mathbb{G}}$ be its profinite completion. Then $\mathbb{G}$ is said to be *good* iff the homomorphism of cohomology groups

$$H^k(\tilde{\mathbb{G}}, M) \to H^k(\mathbb{G}, M)$$

is an isomorphism for all $k \in \mathbb{N}$ and for all finite $\mathbb{G}$ - modules M.

Then we use the following result due to F. Grunewald, A. Jaikin-Zapirain, P. Zalesski.

Theorem 14 ([GJZ08]) *Let G be residually finite and good, and let $\varphi\colon H \to G$ be surjective with finite kernel. Then H is residually finite.*

The above theorem implies that $\mathbb{H}/\mathrm{Tors}(\mathbb{H})$ is residually finite, whence there is a subgroup $\Gamma \leq \mathbb{H}/\,\mathrm{Tors}(\mathbb{H})$ of finite index such that

$$\Gamma \cap E = \{1\}.$$

Now, $\Psi(\Gamma)$ is a subgroup of $\Psi(\hat{\mathbb{H}})$ of finite index, whence of finite index in a product of orbifold surface groups, and $\Psi|\Gamma$ is injective. This easily implies our result.

Remark 7 Note that theorem 13 in fact yields a geometric statement in the case where the genera of the surface groups are at least 2. Again, for simplicity, we assume that $n = 2$, and suppose that $\pi_1(S)$ has a normal subgroup $\mathscr{N}$ of finite index isomorphic to $\Pi_g \times \Pi_{g'}$, with $g, g' \geq 2$. Then there is an unramified Galois covering $\hat{S}$ of S such that $\pi_1(\hat{S}) \cong \Pi_g \times \Pi_{g'}$. This implies (see [Cat00]) that there is a finite morphism $\hat{S} \to C \times C'$, where $g(C) = g$, $g(C') = g'$.

Understanding this morphism can lead to the understanding of the irreducible or even of the connected component of the moduli space containing the isomorphism class $[S]$ of S. The method can also work in the case where we only have $g, g' \geq 1$. We shall explain how this method works in section 6.

We summarize the consequences of theorem 12 in terms of "new" fundamental groups of surfaces with $p_g = 0$, respectively "new" connected components of their moduli space.

Theorem 15 *There exist eight families of product-quotient surfaces of unmixed type yielding numerical Campedelli surfaces (i.e., minimal surfaces with $K_S^2 = 2, p_g(S) = 0$) having fundamental group $\mathbb{Z}/3$.*

Our classification also shows the existence of families of product-quotient surfaces yielding numerical Campedelli surfaces with fundamental groups $\mathbb{Z}/5$ (but numerical Campedelli surfaces with fundamental group $\mathbb{Z}/5$ had already been constructed in [Cat81]), respectively with fundamental group $(\mathbb{Z}/2)^2$ (but such fundamental group already appeared in [Ino94]), respectively with fundamental groups $(\mathbb{Z}/2)^3$, Q_8, $\mathbb{Z}/8$ and $\mathbb{Z}/2 \times \mathbb{Z}/4$.

Theorem 16 *There exist six families of product-quotient surfaces yielding minimal surfaces with $K_S^2 = 3, p_g(S) = 0$ realizing four new finite fundamental groups, $\mathbb{Z}/2 \times \mathbb{Z}/6$, $\mathbb{Z}/8$, $\mathbb{Z}/6$ and $\mathbb{Z}/2 \times \mathbb{Z}/4$.*

Theorem 17 *There exist sixteen families of product-quotient surfaces yielding minimal surfaces with $K_S^2 = 4$, $p_g(S) = 0$. Eight of these families realize 6 new finite fundamental groups, $\mathbb{Z}/15$, $G(32,2)$, $(\mathbb{Z}/3)^3$, $\mathbb{Z}/2 \times \mathbb{Z}/6$, $\mathbb{Z}/8$, $\mathbb{Z}/6$. Eight of these families realize 4 new infinite fundamental groups.*

Theorem 18 *There exist seven families of product-quotient surfaces yielding minimal surfaces with $K_S^2 = 5$, $p_g(S) = 0$. Four of these families realize four new finite fundamental groups, $D_{8,5,-1}$, $\mathbb{Z}/5 \times Q_8$, $D_{8,4,3}$, $\mathbb{Z}/2 \times \mathbb{Z}/10$. Three of these families realize three new infinite fundamental groups.*

Theorem 19 *There exist eight families of product-quotient surfaces yielding minimal surfaces with $K_S^2 = 6$, $p_g(S) = 0$ and realizing 6 new fundamental groups, three of them finite and three of them infinite. In particular, there exist minimal surfaces of general type with $p_g = 0$, $K^2 = 6$ and with finite fundamental group.*

5.2 Galois coverings and their deformations

Another standard method for constructing new algebraic surfaces is to consider abelian Galois-coverings of known surfaces.

We shall in the sequel recall the structure theorem on normal finite $\mathbb{Z}_2^r$-coverings, $r \geq 1$, of smooth algebraic surfaces Y. In fact (cf. [Par91], or [BC08] for a more topological approach) this theory holds more generally for any G-covering, with G a finite abelian group.

Since however we do not want here to dwell too much into the general theory and, in most of the applications we consider here only the case $\mathbb{Z}_2^2$ is used, we restrict ourselves to this more special situation.

We shall denote by $G := \mathbb{Z}_2^r$ the Galois group and by $G^* := \mathrm{Hom}(G, \mathbb{C}^*)$ its dual group of characters which we identify to $G^* := \mathrm{Hom}(G, \mathbb{Z}/2)$.

Since Y is smooth any finite abelian covering $f\colon X \to Y$ is flat hence in the eigensheaves splitting

$$f_* \mathscr{O}_X = \bigoplus_{\chi \in G^*} \mathscr{L}_\chi^* = \mathscr{O}_Y \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} \mathscr{O}_Y(-L_\chi).$$

each rank 1 sheaf $\mathscr{L}_\chi^*$ is invertible and corresponds to a Cartier divisor $-L_\chi$.

For each $\sigma \in G$ let $R_\sigma \subset X$ be the divisorial part of the fixed point set of σ. Then one associates to σ a divisor D_σ given by $f(R_\sigma) = D_\sigma$; let x_σ be a section such that $\mathrm{div}(x_\sigma) = D_\sigma$.

Then the algebra structure on $f_* \mathscr{O}_X$ is given by the following (symmetric, bilinear) multiplication maps:

$$\mathscr{O}_Y(-L_\chi) \otimes \mathscr{O}_Y(-L_\eta) \to \mathscr{O}_Y(-L_{\chi+\eta}),$$

given by the section $x_{\chi,\eta} \in H^0(Y, \mathscr{O}_Y(L_\chi + L_\eta - L_{\chi+\eta}))$, defined by

$$x_{\chi,\eta} := \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma.$$

It is now not difficult in this case to show directly the associativity of the multiplication defined above (cf. [Par05] for the general case of an abelian cover).

In particular, the G-covering $f\colon X \to Y$ is embedded in the vector bundle $\mathbb{V} := \bigoplus_{\chi \in G^*} \mathbb{L}_\chi$, where $\mathbb{L}_\chi$ is the geometric line bundle whose sheaf of sections is $\mathscr{O}_Y(L_\chi)$, and is there defined by the equations:

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma.$$

Note the special case where $\chi = \eta$, when $\chi + \eta$ is the trivial character 1, and $z_1 = 1$. In particular, let $\chi_1, \ldots, \chi_r$ be a basis of $G^* = \mathbb{Z}_2^r$, and set $z_i := z_{\chi_i}$. Then we get the following r equations

$$z_i^2 = \prod_{\chi_i(\sigma)=1} x_\sigma. \tag{5}$$

These equations determine the extension of the function fields, hence one gets X as the normalization of the Galois covering given by (5). The main point however is that the previous formulae yield indeed the normalization explicitly under the conditions summarized in the following

Proposition 1 *A normal finite $G \cong \mathbb{Z}_2^r$-covering of a smooth variety Y is completely determined by the datum of*

1. *reduced effective divisors D_σ, $\forall \sigma \in G$, which have no common components,*
2. *divisor classes $L_1, \dots L_r$, for $\chi_1, \dots \chi_r$ a basis of G^*, such that we have the following linear equivalence*

(#) $2L_i \equiv \sum_{\chi_i(\sigma)=1} D_\sigma$.

Conversely, given the datum of 1) and 2) such that #) holds, we obtain a normal scheme X with a finite $G \cong \mathbb{Z}_2^r$-covering $f \colon X \to Y$.

Proof (Idea of the proof.) It suffices to determine the divisor classes L_χ for the remaining elements of G^*. But since any χ is a sum of basis elements, it suffices to exploit the fact that the linear equivalences

$$L_{\chi+\eta} \equiv L_\eta + L_\chi - \sum_{\chi(\sigma)=\eta(\sigma)=1} D_\sigma$$

must hold, and apply induction. Since the covering is well defined as the normalization of the Galois cover given by (5), each L_χ is well defined. Then the above formulae determine explicitly the ring structure of $f_* \mathcal{O}_X$, hence X. Finally, condition 1 implies the normality of the cover.

A natural question is of course: when is the scheme X a variety? I.e., X being normal, when is X connected, or, equivalently, irreducible? The obvious answer is that X is irreducible if and only if the monodromy homomorphism

$$\mu \colon H_1(Y \setminus (\cup_\sigma D_\sigma), \mathbb{Z}) \to G$$

is surjective.

Remark 8 From the extension of Riemann's existence theorem due to Grauert and Remmert ([GR58]) we know that μ determines the covering. It is therefore worthwhile to see how μ is related to the datum of 1) and 2).

Write for this purpose the branch locus $D := \sum_\sigma D_\sigma$ as a sum of irreducible components D_i. To each D_i corresponds a simple geometric loop γ_i around D_i, and we set $\sigma_i := \mu(\gamma_i)$. Then we have that $D_\sigma := \sum_{\sigma_i = \sigma} D_i$. For each character χ, yielding a double covering associated to the composition $\chi \circ \mu$, we must find a divisor class L_χ such that $2L_\chi \equiv \sum_{\chi(\sigma)=1} D_\sigma$.

Consider the exact sequence

$$H^{2n-2}(Y,\mathbb{Z}) \to H^{2n-2}(D,\mathbb{Z}) = \oplus_i \mathbb{Z}[D_i] \to H_1(Y \setminus D,\mathbb{Z}) \to H_1(Y,\mathbb{Z}) \to 0$$

and the similar one with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$. Denote by Δ the subgroup image of $\oplus_i \mathbb{Z}_2[D_i]$. The restriction of μ to Δ is completely determined by the knowledge of the σ_i's, and we have

$$0 \to \Delta \to H_1(Y \setminus D,\mathbb{Z}_2) \to H_1(Y,\mathbb{Z}_2) \to 0.$$

Dualizing, we get

$$0 \to H^1(Y,\mathbb{Z}_2) \to H^1(Y \setminus D,\mathbb{Z}_2) \to \mathrm{Hom}(\Delta,\mathbb{Z}_2) \to 0.$$

The datum of $\chi \circ \mu$, extending $\chi \circ \mu|_\Delta$ is then seen to correspond to an affine space over the vector space $H^1(Y,\mathbb{Z}_2)$: and since $H^1(Y,\mathbb{Z}_2)$ classifies divisor classes of 2-torsion on Y, we infer that the different choices of L_χ such that $2L_\chi \equiv \sum_{\chi(\sigma)=1} D_\sigma$ correspond bijectively to all the possible choices for $\chi \circ \mu$.

Applying this to all characters, we find how μ determines the building data.

Observe on the other hand that if μ is not surjective, then there is a character χ vanishing on the image of μ, hence the corresponding double cover is disconnected.

But the above discussion shows that $\chi \circ \mu$ is trivial iff this covering is disconnected, if and only if the corresponding element in $H^1(Y \setminus D,\mathbb{Z}_2)$ is trivial, or, equivalently, iff the divisor class L_χ is trivial.

We infer then

Corollary 3 *Use the same notation as in prop. 1. Then the scheme X is irreducible if* $\{\sigma | D_\sigma > 0\}$ *generates G.*

Or, more generally, if for each character χ *the class in* $H^1(Y \setminus D,\mathbb{Z}_2)$ *corresponding to* $\chi \circ \mu$ *is nontrivial, or, equivalently, the divisor class* L_χ *is nontrivial.*

Proof We have seen that if $D_\sigma \geq D_i \neq 0$, then $\mu(\gamma_i) = \sigma$, whence we infer that μ is surjective.

An important role plays here once more the concept of *natural deformations*. This concept was introduced for bidouble covers in [Cat84], definition 2.8, and extended to the case of abelian covers in [Par91], definition 5.1. The two definitions do not exactly coincide, because Pardini takes a much larger parameter space: however, the deformations appearing with both definitions are the same. To avoid confusion we call Pardini's case the case of *extended natural deformations*.

Definition 8 Let $f\colon X \to Y$ be a finite $G \cong \mathbb{Z}_2^r$ covering with Y smooth and X normal, so that X is embedded in the vector bundle $\mathbb{V}$ defined above and is defined by equations

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma.$$

Let $\psi_{\sigma,\chi}$ be a section $\psi_{\sigma,\chi} \in H^0(Y, \mathscr{O}_Y(D_\sigma - L_\chi))$, given $\forall \sigma \in G, \chi \in G^*$. To such a collection we associate an *extended natural deformation*, namely, the subscheme of $\mathbb{V}$ defined by equations

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} \left(\sum_\theta \psi_{\sigma,\theta} \cdot z_\theta \right).$$

We have instead a (restricted) *natural deformation* if we restrict ourselves to the θ's such that $\theta(\sigma) = 0$,and we consider only an equation of the form

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} \left(\sum_{\theta(\sigma)=0} \psi_{\sigma,\theta} \cdot z_\theta \right).$$

One can generalize some results, even removing the assumption of smoothness of Y, if one assumes the $G \cong \mathbb{Z}_2^r$-covering to be *locally simple*, i.e., to enjoy the property that for each point $y \in Y$ the σ's such that $y \in D_\sigma$ are a linearly independent set. This is a good notion since (compare [Cat84], proposition 1.1) if also X is smooth the covering is indeed locally simple.

One has for instance the following result (see [Man01], section 3):

Proposition 2 *Let $f : X \to Y$ be a locally simple $G \cong \mathbb{Z}_2^r$ covering with Y smooth and X normal. Then we have the exact sequence*

$$\oplus_{\chi(\sigma)=0}(H^0(\mathscr{O}_{D_\sigma}(D_\sigma - L_\chi))) \to \mathrm{Ext}^1_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X) \to \mathrm{Ext}^1_{\mathscr{O}_X}(f^*\Omega^1_Y, \mathscr{O}_X).$$

In particular, every small deformation of X is a natural deformation if

1. $H^1(\mathscr{O}_Y(-L_\chi)) = 0$,
2. $\mathrm{Ext}^1_{\mathscr{O}_X}(f^*\Omega^1_Y, \mathscr{O}_X) = 0$.

If moreover

3. $H^0(\mathscr{O}_Y(D_\sigma - L_\chi)) = 0 \ \forall \sigma \in G, \chi \in G^*$,

every small deformation of X is again a $G \cong \mathbb{Z}_2^r$*-covering.*

Proof (Comments on the proof.)

In the above proposition condition 1) ensures that

$$H^0(\mathscr{O}_Y(D_\sigma - L_\chi)) \to H^0(\mathscr{O}_{D_\sigma}(D_\sigma - L_\chi))$$

is surjective.

Condition 2 and the above exact sequence imply then that the natural deformations are parametrized by a smooth manifold and have surjective Kodaira-Spencer map, whence they induce all the infinitesimal deformations.

Remark 9 In the following section we shall see examples where surfaces with $p_g = 0$ arise as double covers and as bidouble covers. In fact there are many more surfaces arising this way, see e.g. [Cat98].

6 Keum-Naie surfaces and primary Burniat surfaces

In the nineties J.H. Keum and D. Naie (cf. [Nai94], [Keu88]) constructed a family of surfaces with $K_S^2 = 4$ and $p_g = 0$ as double covers of an Enriques surface with eight nodes and calculated their fundamental group.

We want here to describe explicitly the moduli space of these surfaces.

The motivation for this investigation arose as follows: consider the following two cases of table 2 whose fundamental group has the form

$$\mathbb{Z}^4 \hookrightarrow \pi_1 \twoheadrightarrow \mathbb{Z}_2^2 \to 0.$$

These cases yield 2 families of respective dimensions 2 and 4, which can also be seen as $\mathbb{Z}_4 \times \mathbb{Z}_2$, resp. $\mathbb{Z}_2^3$, coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ branched in a divisor of type $(4,4)$, resp. $(5,5)$, consisting entirely of horizontal and vertical lines. It turns out that their fundamental groups are isomorphic to the fundamental groups of the surfaces constructed by Keum-Naie.

A straightforward computation shows that our family of dimension 4 is equal to the family constructed by Keum, and that both families are subfamilies of the one constructed by Naie.

As a matter of fact each surface of our family of $\mathbb{Z}_2^3$ - coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ has 4 nodes. These nodes can be smoothened simultaneously in a 5 - dimensional family of $\mathbb{Z}_2^3$ - Galois coverings of $\mathbb{P}^1 \times \mathbb{P}^1$.

It suffices to take a smoothing of each D_i, which before the smoothing consisted of a vertical plus a horizontal line.The full six dimensional component is obtained then as the family of natural deformations of these Galois coverings.

It is a standard computation in local deformation theory to show that the six dimensional family of natural deformations of smooth $\mathbb{Z}_2^3$ - Galois coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ is an irreducible component of the moduli space. We will not give the details of this calculation, since we get a stronger result by another method.

In fact, the main result of [BC09a] is the following:

Theorem 20 *Let S be a smooth complex projective surface which is homotopically equivalent to a Keum-Naie surface. Then S is a Keum-Naie surface.*

The moduli space of Keum-Naie surfaces is irreducible, unirational of dimension equal to six. Moreover, the local moduli space of a Keum-Naie surface is smooth.

The proof resorts to a slightly different construction of Keum-Naie surfaces. We study a $\mathbb{Z}_2^2$-action on the product of two elliptic curves $E_1' \times E_2'$. This action has 16 fixed points and the quotient is an 8-nodal Enriques surface. Constructing S as a double cover of the Enriques surface is equivalent to constructing an étale $\mathbb{Z}_2^2$-covering $\hat{S}$ of S, whose existence can be inferred from the structure of the fundamental group, and which is obtained as a double cover of $E_1' \times E_2'$ branched in a $\mathbb{Z}_2^2$-invariant divisor of type $(4,4)$. Because $S = \hat{S}/\mathbb{Z}_2^2$.

The structure of this étale $\mathbb{Z}_2^2$-covering $\hat{S}$ of S is essentially encoded in the fundamental group $\pi_1(S)$, which can be described as an affine group $\Gamma \subset \mathbb{A}(2,\mathbb{C})$. The key point is that the double cover $\hat{\alpha} : \hat{S} \to E_1' \times E_2'$ is the Albanese map of $\hat{S}$.

Assume now that S' is homotopically equivalent to a Keum-Naie surface S. Then the corresponding étale cover $\hat{S}'$ is homotopically equivalent to $\hat{S}$. Since we know that the degree of the Albanese map of $\hat{S}$ is equal to two (by construction), we can conlude the same for the Albanese map of $\hat{S}'$ and this allows to deduce that also $\hat{S}'$ is a double cover of a product of elliptic curves.

A calculation of the invariants of a double cover shows that the branch locus is a $\mathbb{Z}_2^2$-invariant divisor of type $(4,4)$.

We are going to sketch the construction of Keum-Naie surfaces and the proof of theorem 20 in the sequel. For details we refer to the original article [BC09a].

Let (E,o) be any elliptic curve, with a $G = \mathbb{Z}_2^2 = \{0, g_1, g_2, g_1+g_2\}$ action given by

$$g_1(z) := z + \eta, \quad g_2(z) = -z.$$

Remark 10 Let $\eta \in E$ be a 2 - torsion point of E. Then the divisor $[o] + [\eta] \in Div^2(E)$ is invariant under G, hence the invertible sheaf $\mathscr{O}_E([o]+[\eta])$ carries a natural G-linearization.

In particular, G acts on $H^0(E, \mathscr{O}_E([o]+[\eta]))$, and for the character eigenspaces, we have the following:

Lemma 6.2 *Let E be as above, then:*

$$H^0(E, \mathscr{O}_E([o]+[\eta])) = H^0(E, \mathscr{O}_E([o]+[\eta]))^{++} \oplus H^0(E, \mathscr{O}_E([o]+[\eta]))^{--}.$$

I.e., $H^0(E, \mathscr{O}_E([o]+[\eta]))^{+-} = H^0(E, \mathscr{O}_E([o]+[\eta]))^{-+} = 0.$

Remark 6.1 Our notation is self explanatory, e.g. $H^0(E, \mathscr{O}_E([o]+[\eta]))^{+-} = H^0(E, \mathscr{O}_E([o]+[\eta]))^{\chi}$, where χ is the character of G with $\chi(g_1) = 1$, $\chi(g_2) = -1$.

Let now $E_i' := \mathbb{C}/\Lambda_i$, $i = 1,2$, where $\Lambda_i := \mathbb{Z}e_i \oplus \mathbb{Z}e_i'$, be two complex elliptic curves. We consider the affine transformations $\gamma_1,\ \gamma_2 \in \mathbb{A}(2,\mathbb{C})$, defined as follows:

$$\gamma_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + \frac{e_1}{2} \\ -z_2 \end{pmatrix}, \quad \gamma_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_1 \\ z_2 + \frac{e_2}{2} \end{pmatrix},$$

and let $\Gamma \leq \mathbb{A}(2,\mathbb{C})$ be the affine group generated by γ_1, γ_2 and by the translations e_1, e_1', e_2, e_2'.

Remark 11 i) Γ induces a $G := \mathbb{Z}_2^2$-action on $E_1' \times E_2'$.

ii) While $\gamma_1,\ \gamma_2$ have no fixed points on $E_1' \times E_2'$, the involution $\gamma_1\gamma_2$ has 16 fixed points on $E_1' \times E_2'$. It is easy to see that the quotient $Y := (E_1' \times E_2')/G$ is an Enriques surface having 8 nodes, with canonical double cover the Kummer surface $(E_1' \times E_2')/ < \gamma_1\gamma_2 >$.

We lift the G-action on $E_1' \times E_2'$ to an appropriate ramified double cover $\hat{S}$ such that G acts freely on $\hat{S}$.

To do this, consider the following geometric line bundle $\mathbb{L}$ on $E_1' \times E_2'$, whose invertible sheaf of sections is given by:

$$\mathcal{O}_{E_1' \times E_2'}(\mathbb{L}) := p_1^* \mathcal{O}_{E_1'}([o_1] + [\frac{e_1}{2}]) \otimes p_2^* \mathcal{O}_{E_2'}([o_2] + [\frac{e_2}{2}]),$$

where $p_i : E_1' \times E_2' \to E_i'$ is the projection to the i-th factor.

By remark 10, the divisor $[o_i] + [\frac{e_i}{2}] \in \mathrm{Div}^2(E_i')$ is invariant under G. Therefore, we get a natural G-linearization on the two line bundles $\mathcal{O}_{E_i'}([o_i] + [\frac{e_i}{2}])$, whence also on $\mathbb{L}$.

Any two G-linearizations of $\mathbb{L}$ differ by a character $\chi : G \to \mathbb{C}^*$. We twist the above obtained linearization of $\mathbb{L}$ with the character χ such that $\chi(\gamma_1) = 1$, $\chi(\gamma_2) = -1$.

Definition 9 Let

$$f \in H^0(E_1' \times E_2', p_1^* \mathcal{O}_{E_1'}(2[o_1] + 2[\frac{e_1}{2}]) \otimes p_2^* \mathcal{O}_{E_2'}(2[o_2] + 2[\frac{e_2}{2}]))^G$$

be a G - invariant section of $\mathbb{L}^{\otimes 2}$ and denote by w a fibre coordinate of $\mathbb{L}$. Let $\hat{S}$ be the double cover of $E_1' \times E_2'$ branched in f, i.e.,

$$\hat{S} = \{w^2 = f(z_1, z_2)\} \subset \mathbb{L}.$$

Then $\hat{S}$ is a G - invariant hypersurface in $\mathbb{L}$, and we have a G - action on $\hat{S}$.

We call $S := \hat{S}/G$ a *Keum - Naie surface*, if

- G acts freely on $\hat{S}$, and
- $\{f = 0\}$ has only *non-essential singularities*, i.e., $\hat{S}$ has at most rational double points.

Remark 12 If

$$f \in H^0(E_1' \times E_2', p_1^* \mathcal{O}_{E_1'}(2[o_1] + 2[\frac{e_1}{2}]) \otimes p_2^* \mathcal{O}_{E_2'}(2[o_2] + 2[\frac{e_2}{2}]))^G$$

is such that $\{(z_1, z_2) \in E_1' \times E_2' \mid f(z_1, z_2) = 0\} \cap \mathrm{Fix}(\gamma_1 + \gamma_2) = \emptyset$, then G acts freely on $\hat{S}$.

Proposition 3 *Let S be a Keum - Naie surface. Then S is a minimal surface of general type with* *i)* $K_S^2 = 4$,

ii) $p_g(S) = q(S) = 0$,

iii) $\pi_1(S) = \Gamma$.

i) is obvious, since $K_{\hat{S}}^2 = 16$,

ii) is verified via standard arguments of representation theory.

iii) follows since $\pi_1(\hat{S}) = \pi_1(E_1' \times E_2')$.

Let now S be a smooth complex projective surface with $\pi_1(S) = \Gamma$. Recall that $\gamma_i^2 = e_i$ for $i = 1,2$. Therefore $\Gamma = \langle \gamma_1, e_1', \gamma_2, e_2' \rangle$ and we have the exact sequence

$$1 \to \mathbb{Z}^4 \cong \langle e_1, e_1', e_2, e_2' \rangle \to \Gamma \to \mathbb{Z}_2^2 \to 1,$$

where $e_i \mapsto \gamma_i^2$.

We set $\Lambda_i' := \mathbb{Z}e_i \oplus \mathbb{Z}e_i'$, hence $\pi_1(E_1' \times E_2') = \Lambda_1' \oplus \Lambda_2'$. We also have the two lattices $\Lambda_i := \mathbb{Z}\frac{e_i}{2} \oplus \mathbb{Z}e_i'$.

Remark 13 1) Γ is a group of affine transformations on $\Lambda_1 \oplus \Lambda_2$.

2) We have an étale double cover $E_i' = \mathbb{C}/\Lambda_i' \to E_i := \mathbb{C}/\Lambda_i$, which is the quotient by a semiperiod of E_i'.

Γ has two subgroups of index two:

$$\Gamma_1 := \langle \gamma_1, e_1', e_2, e_2' \rangle, \;\; \Gamma_2 := \langle e_1, e_1', \gamma_2, e_2' \rangle,$$

corresponding to two étale covers of S: $S_i \to S$, for $i = 1,2$.

Then one can show:

Lemma 6.3 *The Albanese variety of S_i is E_i. In particular, $q(S_1) = q(S_2) = 1$.*

Let $\hat{S} \to S$ be the étale $\mathbb{Z}_2^2$-covering associated to $\mathbb{Z}^4 \cong \langle e_1, e_1', e_2, e_2' \rangle \triangleleft \Gamma$. Since $\hat{S} \to S_i \to S$, and S_i maps to E_i (via the Albanese map), we get a morphism

$$f : \hat{S} \to E_1 \times E_2 = \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2.$$

Then the covering of $E_1 \times E_2$ associated to $\Lambda_1' \oplus \Lambda_2' \leq \Lambda_1 \oplus \Lambda_2$ is $E_1' \times E_2'$, and since $\pi_1(\hat{S}) = \Lambda_1' \oplus \Lambda_2'$ we see that f factors through $E_1' \times E_2'$ and that the Albanese map of $\hat{S}$ is $\hat{\alpha} : \hat{S} \to E_1' \times E_2'$.

The proof of the main result follows then from

Proposition 4 *Let S be a smooth complex projective surface, which is homotopically equivalent to a Keum - Naie surface. Let $\hat{S} \to S$ be the étale $\mathbb{Z}_2^2$-cover associated to $\langle e_1, e_1', e_2, e_2' \rangle \lhd \Gamma$ and let*

$$\begin{tikzcd} \hat{S} \arrow[r, "\hat{\alpha}"] \arrow[dr] & E_1' \times E_2' \\ & Y \arrow[u, "\varphi"'] \end{tikzcd}$$

be the Stein factorization of the Albanese map of $\hat{S}$.

Then φ has degree 2 and Y is a canonical model of $\hat{S}$.

More precisely, φ is a double cover of $E_1' \times E_2'$ branched on a divisor of type $(4,4)$.

The fact that S is homotopically equivalent to a Keum-Naie surface immediately implies that the degree of $\hat{\alpha}$ is equal to two.

The second assertion, i.e., that Y has only canonical singularities, follows instead from standard formulae on double covers (cf. [Hor75]).

The last assertion follows from $K_{\hat{S}}^2 = 16$ and $(\mathbb{Z}/2\mathbb{Z})^2$- invariance.

In fact, we conjecture a stronger statement to hold true:

Conjecture 4 Let S be a minimal smooth projective surface such that

i) $K_S^2 = 4$,

ii) $\pi_1(S) \cong \Gamma$.

Then S is a Keum-Naie surface.

We can prove

Theorem 21 *Let S be a minimal smooth projective surface such that*

i) $K_S^2 = 4$*,*

ii) $\pi_1(S) \cong \Gamma$*,*

iii)there is a deformation of S with ample canonical bundle.

Then S is a Keum-Naie surface.

We recall the following results:

Theorem 22 (Severi's conjecture, [Par05]) *Let S be a minimal smooth projective surface of maximal Albanese dimension (i.e., the image of the Albanese map is a surface), then $K_S^2 \geq 4\chi(S)$.*

M. Manetti proved Severi's inequality under the assumption that K_S is ample, but he also gave a description of the limit case $K_S^2 = 4\chi(S)$, which will be crucial for the above theorem 21.

Theorem 23 (M. Manetti, [Man03]) *Let S be a minimal smooth projective surface of maximal Albanese dimension with K_S ample then $K_S^2 \geq 4\chi(S)$, and equality holds if and only if $q(S) = 2$, and the Albanese map $\alpha : S \to \mathrm{Alb}(S)$ is a finite double cover.*

Proof (Proof of theorem 21) We know that there is an étale $\mathbb{Z}_2^2$-cover $\hat{S}$ of S with Albanese map $\hat{\alpha} : \hat{S} \to E_1' \times E_2'$. Note that $K_{\hat{S}}^2 = 4K_S^2 = 16$. By Severi's inequality, it follows that $\chi(S) \leq 4$, but since $1 \leq \chi(S) = \frac{1}{4}\chi(\hat{S})$, we have $\chi(S) = 4$. Since S deforms to a surface with K_S ample, we can apply Manetti's result and obtain that $\hat{\alpha} : \hat{S} \to E_1' \times E_2'$ has degree 2, and we conclude as before.

It seems reasonable to conjecture (cf. [Man03]) the following, which would immediately imply our conjecture 4.

Conjecture 5 Let S be a minimal smooth projective surface of maximal Albanese dimension. Then $K_S^2 = 4\chi(S)$ if and only if $q(S) = 2$, and the Albanese map has degree 2.

During the preparation of the article [BC09a] the authors realized that a completely similar argument applies to *primary Burniat surfaces*.

We briefly recall the construction of Burniat surfaces: for more details, and for the proof that Burniat surfaces are exactly certain Inoue surfaces we refer to [BC09b].

Burniat surfaces are minimal surfaces of general type with $K^2 = 6,5,4,3,2$ and $p_g = 0$, which were constructed in [Bur66] as singular bidouble covers (Galois covers with group $\mathbb{Z}_2^2$) of the projective plane branched on 9 lines.

Let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three non collinear points (which we assume to be the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$) and let's denote by $Y := \hat{\mathbb{P}}^2(P_1, P_2, P_3)$ the Del Pezzo surface of degree 6, blow up of $\mathbb{P}^2$ in P_1, P_2, P_3.

Y is 'the' smooth Del Pezzo surface of degree 6, and it is the closure of the graph of the rational map

$$\varepsilon : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

such that

$$\varepsilon(y_1 : y_2 : y_3) = ((y_2 : y_3), (y_3 : y_1), (y_1 : y_2)).$$

One sees immediately that $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the hypersurface of type $(1,1,1)$:

$$Y = \{((x_1' : x_1), (x_2' : x_2), (x_3' : x_3)) \mid x_1 x_2 x_3 = x_1' x_2' x_3'\}.$$

We denote by L the total transform of a general line in $\mathbb{P}^2$, by E_i the exceptional curve lying over P_i, and by $D_{i,1}$ the unique effective divisor in $|L - E_i - E_{i+1}|$, i.e., the proper transform of the line $y_{i-1} = 0$, side of the triangle joining the points P_i, P_{i+1}.

Consider on Y, for each $i \in \mathbb{Z}_3 \cong \{1,2,3\}$, the following divisors

$$D_i = D_{i,1} + D_{i,2} + D_{i,3} + E_{i+2} \in |3L - 3E_i - E_{i+1} + E_{i+2}|,$$

where $D_{i\,j} \in |L - E_i|$, for $\mathrm{j} = 2,3$, $\mathrm{D_{i\,j}} \neq \mathrm{D_{i\,1}}$, is the proper transform of another line through P_i and $D_{i,1} \in |L - E_i - E_{i+1}|$ is as above. Assume also that all the corresponding lines in $\mathbb{P}^2$ are distinct, so that $D := \sum_i D_i$ is a reduced divisor.

Note that, if we define the divisor $\mathscr{L}_i := 3L - 2E_{i-1} - E_{i+1}$, then

$$D_{i-1} + D_{i+1} = 6L - 4E_{i-1} - 2E_{i+1} \equiv 2\mathscr{L}_i,$$

and we can consider (cf. section 4, [Cat84] and [Cat98]) the associated bidouble cover $X' \to Y$ branched on $D := \sum_i D_i$ (but we take a different ordering of the indices of the fibre coordinates u_i, using the same choice as the one made in [BC09b], except that X' was denoted by X).

We recall that this precisely means the following: let $D_i = \operatorname{div}(\delta_i)$, and let u_i be a fibre coordinate of the geometric line bundle $\mathbb{L}_{i+1}$, whose sheaf of holomorphic sections is $\mathscr{O}_Y(\mathscr{L}_{i+1})$.

Then $X \subset \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$ is given by the equations:

$$u_1 u_2 = \delta_1 u_3, \;\; u_1^2 = \delta_3 \delta_1;$$
$$u_2 u_3 = \delta_2 u_1, \;\; u_2^2 = \delta_1 \delta_2;$$
$$u_3 u_1 = \delta_3 u_2, \;\; u_3^2 = \delta_2 \delta_3.$$

From the birational point of view, as done by Burniat, we are simply adjoining to the function field of $\mathbb{P}^2$ two square roots, namely $\sqrt{\frac{\Delta_1}{\Delta_3}}$ and $\sqrt{\frac{\Delta_2}{\Delta_3}}$, where Δ_i is the cubic polynomial in $\mathbb{C}[x_0,x_1,x_2]$ whose zero set has $D_i - E_{i+2}$ as strict transform.

This shows clearly that we have a Galois cover $X' \to Y$ with group $\mathbb{Z}_2^2$.

The equations above give a biregular model X' which is nonsingular exactly if the divisor D does not have points of multiplicity 3 (there cannot be points of higher multiplicities!). These points give then quotient singularities of type $\frac{1}{4}(1,1)$, i.e., isomorphic to the quotient of $\mathbb{C}^2$ by the action of $\mathbb{Z}_4$ sending $(u,v) \mapsto (iu,iv)$ (or, equivalently, the affine cone over the 4-th Veronese embedding of $\mathbb{P}^1$).

Definition 10 A *primary Burniat surface* is a surface constructed as above, and which is moreover smooth. It is then a minimal surface S with K_S ample, and with $K_S^2 = 6$, $p_g(S) = q(S) = 0$.

A *secondary Burniat surface* is the minimal resolution of a surface X' constructed as above, and which moreover has $1 \leq m \leq 2$ singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface S with K_S nef and big, and with $K_S^2 = 6-m$, $p_g(S) = q(S) = 0$.

A *tertiary (respectively, quaternary) Burniat surface* is the minimal resolution of a surface X' constructed as above, and which moreover has $m = 3$ (respectively $m = 4$) singular points (necessarily of the type described above). Its minimal resolution is then a minimal surface S with K_S nef and big, but not ample, and with $K_S^2 = 6-m$, $p_g(S) = q(S) = 0$.

Remark 14 1) We remark that for $K_S^2 = 4$ there are two possible types of configurations. The one where there are three collinear points of multiplicity at least 3 for the plane curve formed by the 9 lines leads to a Burniat surface S which we call of *nodal type*, and with K_S not ample, since the inverse image of the line joining the 3 collinear points is a (-2)-curve (a smooth rational curve of self intersection -2).

In the other cases with $K_S^2 = 4,5,6$, K_S is instead ample.

2) In the nodal case, if we blow up the two $(1,1,1)$ points of D, we obtain a weak Del Pezzo surface $\tilde{Y}$, since it contains a (-2)-curve. Its anticanonical model Y' has a node (an A_1-singularity, corresponding to the contraction of the (-2)-curve). In the non nodal case, we obtain a smooth Del Pezzo surface $\tilde{Y} = Y'$ of degree 4.

With similar methods as in [BC09a] (cf. [BC09b]) the first two authors proved

Theorem 24 *The subset of the Gieseker moduli space corresponding to primary Burniat surfaces is an irreducible connected component, normal, rational and of* *dimension four. More generally, any surface homotopically equivalent to a primary Burniat surface is indeed a primary Burniat surface.*

Remark 15 The assertion that the moduli space corresponding to primary Burniat surfaces is rational needs indeed a further argument, which is carried out in [BC09b].

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Holomorphic symplectic geometry: a problem list

Arnaud Beauville

Abstract The usual structures of symplectic geometry (symplectic, contact, Poisson) make sense for complex manifolds; they turn out to be quite interesting on projective, or compact Kähler, manifolds. In these notes we review some of the recent results on the subject, with emphasis on the open problems and conjectures.

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1 Introduction

Though symplectic geometry is usually done on real manifolds, the main definitions (symplectic or contact structures, Poisson bracket) make perfect sense in the holomorphic setting. What is less obvious is that these structures are indeed quite interesting in this set-up, in particular on *global* objects – meaning compact, or projective, manifolds. The study of these objects has been much developed in the last 30 years – an exhaustive survey would require at least a book. The aim of these notes is much more modest: we would like to give a (very partial) overview of the subject by presenting some of the open problems which are currently investigated.

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Most of the paper is devoted to holomorphic symplectic (= hyperkähler) manifolds, a subject which has been blossoming in recent years. Two short chapters are devoted to contact and Poisson structures : in the former we discuss the conjectural classification of projective contact manifolds, and in the latter an intriguing conjecture of Bondal on the rank of the Poisson tensor.

2 Compact hyperkähler manifolds

2.1 Basic definitions

The interest for holomorphic symplectic manifolds comes from the following result, stated by Bogomolov in [8] :

Theorem 1 (Decomposition theorem) *Let X be a compact, simply-connected Kähler manifold with trivial canonical bundle. Then X is a product of manifolds of the following two types:*

- *projective manifolds Y of dimension ≥ 3, with $H^0(Y, \Omega_Y^*) = \mathbb{C} \oplus \mathbb{C}\omega$, where ω is a generator of K_Y;*
- *compact Kähler manifolds Z with $H^0(Z, \Omega_Z^*) = \mathbb{C}[\sigma]$, where $\sigma \in H^0(Z, \Omega_Z^2)$ is everywhere non-degenerate.*

This theorem has an important interpretation (and a proof) in terms of Riemannian geometry[1]. By the fundamental theorem of Yau [42], an n-dimensional compact Kähler manifold X with trivial canonical bundle admits a Kähler metric with holonomy group contained in $\mathrm{SU}(n)$ (this is equivalent to the vanishing of the Ricci curvature). By the Berger and de Rham theorems, X is a product of manifolds with holonomy $\mathrm{SU}(m)$ or $\mathrm{Sp}(r)$; this corresponds to the first and second case of the decomposition theorem.

We will call the manifolds of the first type *Calabi-Yau* manifolds, and those of the second type *hyperkähler* manifolds (they are also known as *irreducible holomorphic symplectic*).

[1] See [5] for a more detailed exposition.

2.2 Examples

For Calabi-Yau manifolds we know a huge quantity of examples (in dimension 3, the number of known families approaches 10 000), but relatively little general theory. In contrast, we have much information on hyperkähler manifolds, their period map, their cohomology (see below); what is lacking severely is examples. In fact, at this time we know two families in each dimension [2], and two isolated families in dimension 6 and 10 [32], [33] :

a) Let S be a K3 surface. The symmetric product $S^{(r)} := S^r/\mathfrak{S}_r$ parametrizes subsets of r points in S, counted with multiplicities; it is smooth on the open subset $S^{(r)}_{\mathrm{o}}$ consisting of subsets with r distinct points, but singular otherwise. If we replace "subset" by (analytic) "subspace", we obtain a smooth compact manifold, the *Hilbert scheme* $S^{[r]}$; the natural map $S^{[r]} \to S^{(r)}$ is an isomorphism above $S^{(r)}_{\mathrm{o}}$, but it resolves the singularities of $S^{(r)}$.

Let ω be a non-zero holomorphic 2-form on S. The form $\mathrm{pr}_1^*\omega + \ldots + \mathrm{pr}_r^*\omega$ descends to a non-degenerate 2-form on $S^{(r)}_{\mathrm{o}}$; it is easy to check that this 2-form extends to a symplectic structure on $S^{[r]}$.

b) Let T be a 2-dimensional complex torus. The Hilbert scheme $T^{[r]}$ has the same properties as $S^{[r]}$, but it is not simply connected. This is fixed by considering the composite map $\sigma : T^{[r+1]} \to T^{(r+1)} \xrightarrow{s} T$, where $s(t_1,\ldots,t_r) = t_1 + \ldots + t_r$; the fibre $K_r(T) := \sigma^{-1}(0)$ is a hyperkähler manifold of dimension $2r$ ("generalized Kummer manifold").

c) Let again S be a K3 surface, and $\mathscr{M}$ the moduli space of stable rank 2 vector bundles on S, with Chern classes $c_1 = 0$, $c_2 = 4$. According to Mukai [30], this space has a holomorphic symplectic structure. It admits a natural compactification $\overline{\mathscr{M}}$, obtained by adding classes of semi-stable torsion free sheaves; it is singular along the boundary, but O'Grady constructs a desingularization of $\overline{\mathscr{M}}$ which is a new hyperkähler manifold, of dimension 10.

d) The analogous construction can be done starting from rank 2 bundles with $c_1 = 0$, $c_2 = 2$ on a 2-dimensional complex torus, and taking again some fibre to ensure the simple connectedness. The upshot is a new hyperkähler manifold of dimension 6.

In the two last examples it would seem simpler to start with a moduli space $\mathscr{M}$ for which the natural compactification $\overline{\mathscr{M}}$ is smooth; in that case $\overline{\mathscr{M}}$ is a hyperkähler manifold [30], but it turns out that it is a deformation of $S^{[r]}$ or $K_r(T)$ (Göttsche-Huybrechts, O'Grady, Yoshioka ...). On the other hand, when $\overline{\mathscr{M}}$ is singular, it admits a hyperkähler desingularization only in the two cases considered by O'Grady [23].

Thus it seems that a new idea is required to answer our first problem:

Question 1 *Find new examples of hyperkähler manifolds.*

2.3 The period map

In dimension 2 the only hyperkähler manifolds are K3 surfaces; we know them very well thanks to the *period map*, which associates to a K3 surface S the Hodge decomposition
$$H^2(S,\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \ .$$
This is determined by the position of the line $H^{2,0}$ in $H^2(S,\mathbb{C})$: indeed we have $H^{0,2} = \overline{H^{2,0}}$, and $H^{1,1}$ is the orthogonal of $H^{2,0} \oplus H^{0,2}$ with respect to the intersection form. Note that any non-zero element σ of $H^{2,0}$ (that is, the class of a non-zero holomorphic 2-form) satisfies $\sigma^2 = 0$ and $\sigma \cdot \bar{\sigma} > 0$.

To compare the Hodge structures of different K3 surfaces, we consider *marked* surfaces (S,λ), where λ is an isometry of $H^2(S,\mathbb{Z})$ onto a fixed lattice L, the unique even unimodular lattice L of signature $(3,19)$. Then the data of the Hodge structure on $H^2(S,\mathbb{Z})$ is equivalent to that of the *period point* $\wp(S,\lambda) := \lambda_{\mathbb{C}}(H^{2,0}) \in \mathbb{P}(L_{\mathbb{C}})$. By the above remark this point lies in the domain $\Omega \subset \mathbb{P}(L_{\mathbb{C}})$ defined by the conditions $x^2 = 0, x \cdot \bar{x} > 0$. There is a moduli space $\mathscr{M}_L$ for marked K3 surfaces, which is a *non-Hausdorff* complex manifold; the *period map* $\wp : \mathscr{M}_L \to \Omega_L$ is holomorphic. We know a lot about that map, thanks to the work of many people (Piatetski-Shapiro, Shafarevich, Todorov, Siu, ...):

Theorem 2

1) ("local Torelli") *$\wp$ is a local isomorphism.*

2) ("global Torelli") *If $\wp(S,\lambda) = \wp(S',\lambda')$, S and S' are isomorphic;*

3) ("surjectivity") *Every point of Ω is the period of some marked K3 surface.*

Another way of stating 2) is that S and S' are isomorphic if and only if there is a *Hodge isometry* $H^2(S,\mathbb{Z}) \xrightarrow{\sim} H^2(S',\mathbb{Z})$ (that is, an isometry inducing an isomorphism of Hodge structures). There is in fact a more precise statement, see e.g. [1].

There is a very analogous picture for higher-dimensional hyperkähler manifolds. The intersection form is replaced by a canonical quadratic form $q : H^2(X,\mathbb{Z}) \to \mathbb{Z}$, primitive[2], of signature $(3, b_2 - 3)$ [2]. The easiest way to define it is through the Fujiki relation

$$\int_X \alpha^{2r} = f_X\, q(\alpha)^r \quad \text{for each } \alpha \in H^2(X,\mathbb{Z})\ , \ \text{ where } \dim(X) = 2r\ ;$$

this relation determines f_X (the *Fujiki constant* of X) and the form q; they depend only on the topological type of X.

Let X be a hyperkähler manifold, and L a lattice. A *marking* of type L of X is an isometry $\lambda : (H^2(X,\mathbb{Z}),q) \xrightarrow{\sim} L$. The period of (X,λ) is the point $\lambda_{\mathbb{C}}(H^{2,0}) \in \mathbb{P}(L_{\mathbb{C}})$; as above it belongs to the period domain

$$\Omega_L := \{[x] \in \mathbb{P}(L_{\mathbb{C}}) \mid x^2 = 0\ ,\ x \cdot \bar{x} > 0\}\ .$$

Again we have a non-Hausdorff complex manifold $\mathscr{M}_L$ parametrizing hyperkähler manifolds of a given dimension with a marking of type L; the period map $\wp : \mathscr{M}_L \to \Omega_L$ is holomorphic. We have:

Theorem 3

1) *The period map $\wp : \mathscr{M}_L \to \Omega_L$ is a local isomorphism.*

2) *The restriction of $\wp$ to any connected component of $\mathscr{M}_L$ is surjective.*

1) is proved in [2], and 2) in [17]. What is missing is the analogue of the global Torelli theorem. It has long been known that it cannot hold in the form given in Theorem 2; in fact, it follows from the results of [17] that any birational map $X \overset{\sim}{\dashrightarrow} X'$ induces a Hodge isometry $H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X',\mathbb{Z})$. This is not the only obstruction: Namikawa observed in [31] that if T is a 2-dimensional complex torus, and T^* its dual torus, the Kummer manifolds $K_2(T)$ and $K_2(T^*)$ (1.2.*b*) have the same period (with appropriate markings), but are not bimeromorphic in general. Thus we can only ask:

Question 2 *Let X, X' be two hyperkähler manifolds of the same dimension. If there is a Hodge isometry $\lambda : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X',\mathbb{Z})$, what can we say of X and X'? Can we conclude that X and X' are isomorphic by imposing that λ preserves some extra structure?*

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[2] This means that the associated bilinear form is integral and not divisible by an integer > 1.

One can formulate analogous questions for *polarized* hyperkähler manifolds, requiring that λ preserves the polarization classes. Here again Namikawa's construction provides a counter-example. Nevertheless the recent work of Markman [27] and Verbitsky [39] gives a partial answer to question 2 and its "polarized" analogue, in particular for the case of example 1.2.*a*); see also the discussion in [15], Question 2.6, and the paper of E. Markman in these Proceedings.

2.4 Cohomology

Let X be a hyperkähler manifold. Since the quadratic form q plays such an important role, it is natural to expect that it determines most of the cohomology of X. This was indeed shown by Bogomolov [10] :

Proposition 1 *Let X be a hyperkähler manifold, of dimension $2r$, and let $\mathscr{H}$ be the subalgebra of $H^*(X,\mathbb{C})$ spanned by $H^2(X,\mathbb{C})$.*

1) $\mathscr{H}$ is the quotient of $\mathrm{Sym}^* H^2(X,\mathbb{C})$ by the ideal spanned by the classes α^{r+1} for $\alpha \in H^2(X,\mathbb{C})$, $q_{\mathbb{C}}(\alpha) = 0$.

2) $H^*(X,\mathbb{C}) = \mathscr{H} \oplus \mathscr{H}^{\perp}$, where $\mathscr{H}^{\perp}$ is the orthogonal of $\mathscr{H}$ with respect to the cup-product.

Thus the subalgebra $\mathscr{H}$ is completely determined by the form q and the dimension of X. In contrast, not much is known about the $\mathscr{H}$-module $\mathscr{H}^{\perp}$. Note that it is nonzero for the examples *a*) and *b*) of 1.2, with the exception of $S^{[2]}$ for a K3 surface S.

We do not know much about the quadratic form q either. For the two infinite series of (1.2) we have lattice isomorphisms [2]

$$H^2(S^{[r]},\mathbb{Z}) = H^2(S,\mathbb{Z}) \overset{\perp}{\oplus} \langle 2-2r\rangle \qquad H^2(K_r(T),\mathbb{Z}) = H^2(T,\mathbb{Z}) \overset{\perp}{\oplus} \langle -2-2r\rangle\ ;$$

The lattices of O'Grady's two examples are computed in [36]; they are also even.

Question 3 *Is the quadratic form q always even? More generally, what are the possibilities for q? What are the possibilities for the Fujiki index f_X (see 1.3)?*

2.5 Boundedness

Having so few examples leads naturally to the following question:

Conjecture 1 *There are finitely many hyperkähler manifolds (up to deformation) in each dimension.*

Note that the same question can be asked for Calabi-Yau manifolds, but there it seems completely out of reach.

Huybrechts observes that there are finitely many deformation types of hyperkähler manifolds X of dimension $2r$ such that there exists $\alpha \in H^2(X,\mathbb{Z})$ with $q(\alpha) > 0$ and $\int_X \alpha^{2r}$ bounded [18]. As a corollary, given a real number M, there are finitely many deformation types of hyperkähler manifolds with

$$f_X \leq M \quad , \quad \min\{q(\alpha) \mid \alpha \in H^2(X,\mathbb{Z})\ ,\ q(\alpha) > 0\} \leq M\ .$$

A first approximation to finiteness would be to bound the Betti numbers b_i of X, and in particular b_2. Here we have some more information in the case of fourfolds [16] :

Proposition 2 *Let X be a hyperkähler fourfold. Then either $b_2 = 23$, or $3 \leq b_2 \leq 8$.*

Note that b_2 is 23 for $S^{[2]}$ and 7 for $K_2(T)$ (1.2). [16] contains some more information on the other Betti numbers.

Question 4 *Can we exclude some more cases, in particular $b_2 = 3$? If $b_2 = 23$, can we conclude that X is deformation equivalent to $S^{[2]}$?*

2.6 Lagrangian fibrations

Let (X,σ) be a holomorphic symplectic manifold (not necessarily compact), of dimension $2r$. A *Lagrangian fibration* is a proper map $h : X \to B$ onto a manifold B such that the general fibre F of h is Lagrangian, that is, F is connected, of dimension r, and $\sigma_{|F} = 0$. This implies that the smooth fibres of h are complex tori (Arnold-Liouville theorem).

Suppose $B = \mathbb{C}^r$, so that $h = (h_1,\dots,h_r)$. The functions h_i define what is called in classical mechanics an *algebraically completely integrable hamiltonian system* :

the Poisson brackets $\{h_i, h_j\}$ vanish, the hamiltonian vector fields X_{h_i} commute with each other, they are tangent to the fibres of h and their restriction to a smooth fibre is a linear vector field on this complex torus (see for instance [3]).

The analogue of this notion when X is compact (hence hyperkähler) is a Lagrangian fibration $X \to \mathbb{P}^r$. There are many examples of such fibrations (see a sample below); moreover they turn out to be the only non-trivial morphisms from a hyperkähler manifold to a manifold of smaller dimension :

Theorem 4 *Let X be a hyperkähler manifold, of dimension $2r$, B a Kähler manifold with $0 < \dim B < 2r$, and $f : X \to B$ a surjective morphism with connected fibres. Then:*

1) *f is a Lagrangian fibration;*

2) *If X is projective, $B \cong \mathbb{P}^r$.*

1) is due to Matsushita (see [29], Prop. 24.8), and 2) to Hwang [20]. It is expected that 2) holds without the projectivity assumption on X (see the discussion in the introduction of [20]).

How do we detect the existence of a Lagrangian fibration on a given hyperkähler manifold? In dimension 2 there is a simple answer; a Lagrangian fibration on a K3 surface S is an elliptic fibration, and we have :

Proposition 3 a) *Let L be a nontrivial nef line bundle on S with $L^2 = 0$. There exists an elliptic fibration $f : S \to \mathbb{P}^1$ such that $L = f^*\mathscr{O}_{\mathbb{P}^1}(k)$ for some $k \geq 1$.*

b) *S admits an elliptic fibration if and only if it admits a line bundle $L \neq \mathscr{O}_S$ with $L^2 = 0$.*

The proof of a) is straightforward. b) is reduced to a) by proving that some isometry w of $\mathrm{Pic}(S)$ maps L to a nef line bundle; see for instance [1], VIII, Lemma 17.4.

Proposition 3 has a natural (conjectural) generalization to higher-dimensional hyperkähler manifolds[3] :

Conjecture 2 a) *Let L be a nontrivial nef line bundle on X with $q(L) = 0$. There exists a Lagrangian fibration $f : X \to \mathbb{P}^r$ such that $L = f^*\mathscr{O}_{\mathbb{P}^r}(k)$ for some $k \geq 1$.*

[3] The conjecture has been known to experts for a long time; see the introduction of [38] for a discussion of its history.

b) *There exists a hyperkähler manifold X' bimeromorphic to X and a Lagrangian fibration $X' \to \mathbb{P}^r$ if and only if X admits a line bundle $L \neq \mathscr{O}_S$ with $q(L) = 0$.*

Note that it is not clear whether one of the statements implies the other.

There is some evidence in favor of the conjecture. Let S be a "general" K3 surface of genus g – that is, $\mathrm{Pic}(S) = \mathbb{Z}[L]$ with $L^2 = 2g-2$. Then $\mathrm{Pic}(S^{[r]})$ is a rank 2 lattice with an orthonormal basis (h,e) satisfying $q(h) = 2g-2$, $q(e) = -(2r-2)$ [2]. Taking $r = g$ we find $q(h \pm e) = 0$. The corresponding Lagrangian fibration is studied in [3]: $S^{[g]}$ is birational to the relative compactified Jacobian $\mathscr{J}^g \to |L|$, whose fibre above a curve $C \in |L|$ is the compactified Jacobian $\overline{J^g C}$. $\mathscr{J}^g$ is hyperkähler by [30], and the fibration $\mathscr{J}^g \to |L|$ is Lagrangian. The rational map $S^{[g]} \dashrightarrow |L|$ associates to a general set of g points in S the unique curve of $|L|$ passing through these points.

More generally, suppose that $2g-2 = (2r-2)m^2$ for some integer m. Then $q(h \pm me) = 0$, and indeed $S^{[r]}$ admits a birational model with a Lagrangian fibration. This fibration has been constructed independently in [28] and [37]; $\mathscr{J}^g$ is replaced by a moduli space of *twisted sheaves* on S.

Another argument in favor of the conjecture has been given by Matsushita [29], who proved that a) holds "locally", in the following sense. Let X be a hyperkähler manifold, with a Lagrangian fibration $f : X \to \mathbb{P}^r$, and let $\mathrm{Def}(X)$ be the local deformation space of X. Then the Lagrangian fibration deforms along a hypersurface in $\mathrm{Def}(X)$. Thus any small deformation of X such that the cohomology class of $f^*\mathscr{O}_{\mathbb{P}^r}(1)$ remains algebraic carries a Lagrangian fibration.

A related question, which comes from mathematical physics, is :

Question 5 *Does every hyperkähler manifold admit a deformation with a Lagrangian fibration?*

If Conjecture 2 holds, the answer is positive if and only if the quadratic form q is indefinite. I do not know any serious argument either in favor or against this.

Question 6 *Let X be a hyperkähler manifold, and $T \subset X$ a Lagrangian submanifold which is a complex torus. Is it the fibre of a Lagrangian fibration $X \to \mathbb{P}^r$?*

(A less optimistic version would ask only for a bimeromorphic Lagrangian fibration.)

2.7 Projective families

Deformation theory shows that when the K3 surface S varies, the manifolds $S^{[r]}$ form a hypersurface in their deformation space; thus a general deformation of $S^{[r]}$ is not the Hilbert scheme of a K3 – and we do not know how to describe it. This is not particularly surprising: after all, we do not know either how to describe a general K3 surface. On the other hand, if we start from the family of *polarized* K3 surfaces S of genus g, the projective manifolds $S^{[r]}$ are polarized (in various ways)[4], and the same argument tells us that they form again a hypersurface in their (polarized) deformation space; we should be able to describe a (locally) complete family of projective hyperkähler manifolds which specializes to $S^{[r]}$ in codimension 1.

For $r = 2$ there are indeed a few cases where we can describe the general deformation of $S^{[2]}$ with an appropriate polarization :

1. The Fano variety of lines contained in a cubic fourfold ([7]; $g = 8$)
2. The "variety of sums of powers" associated to a cubic fourfold ([21]; $g = 8$)
3. The double cover of certain sextic hypersurfaces in $\mathbb{P}^5$ ([34]; $g = 6$)
4. The subspace of the Grassmannian $\mathbb{G}(6,10)$ consisting of 6-planes L such that $\sigma_{|L} = 0$, where $\sigma : \wedge^3\mathbb{C}^{10} \to \mathbb{C}$ is a sufficiently general 3-form ([12]; $g = 12$).

Note that K3 surfaces of genus 8 appear in both cases 1) and 2); what happens is that the corresponding polarizations on $S^{[2]}$ are different [22][5].

Question 7 *Describe the general projective deformation of $S^{[2]}$, for S a polarized K3 surface of genus* 1, 2, 3, ... *(and for some choice of polarization on $S^{[2]}$); or at least find more examples of locally complete projective families. Same question with $S^{[r]}$ for $r \geq 3$.*

(With the notation of footnote 4, a natural choice of polarization for $g \geq 3$ is $h - e$.)

A different issue concerns the *Chow ring* of a projective hyperkähler manifold. In [6] and [40] the following conjecture is proposed :

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[4] For S general we have $\operatorname{Pic}(S^{[r]}) = \mathbb{Z}h \overset{\perp}{\oplus} \mathbb{Z}e$ (1.6); the polarizations on $S^{[r]}$ are of the form $ah - be$ with $a, b > 0$.

[5] The Corollary in [22] is slightly misleading: the moduli spaces of *polarized* hyperkähler manifolds of type 1) and of type 2) are disjoint.

Conjecture 3 *Let $D_1,\ldots,D_k$ in $\mathrm{Pic}(X)$, and let $z\in CH(X)$ be a class which is a polynomial in $D_1,\ldots,D_k$ and the Chern classes $c_i(X)$. If $z=0$ in $H^*(X,\mathbb{Z})$, then $z=0$.*

This would follow from a much more general (and completely out of reach) conjecture, for which we refer to the introduction of [6]. Conjecture 3 is proved in [40] for the Hilbert scheme $S^{[n]}$ of a K3 surface for $n\leq 8$, and for the Fano variety of lines on a cubic fourfold.

3 Compact Poisson manifolds

Since hyperkähler manifolds are so rare, it is natural to turn to a more flexible notion. Symplectic geometry provides a natural candidate, *Poisson manifolds*. Recall that a (holomorphic) Poisson structure on a complex manifold X is a bivector field $\tau\in H^0(X,\Lambda^2 T_X)$, such that the bracket $\{f,g\}:=\langle\tau,df\wedge dg\rangle$ defines a Lie algebra structure on $\mathscr{O}_X$. A Poisson structure defines a skew-symmetric map $\tau^\sharp:\Omega^1_X\to T_X$; the *rank* of τ at a point $x\in X$ is the rank of $\tau^\sharp(x)$. It is even (because $\tau^\sharp$ is skew-symmetric). The data of a Poisson structure of rank $\dim X$ is equivalent to that of a (holomorphic) symplectic structure. In general, we have a partition

$$X=\coprod_{s\text{ even}} X_s\quad\text{where }\ X_s:=\{x\in X\mid \mathrm{rk}\,\tau(x)=s\}\ .$$

The following conjecture is due to Bondal ([11], see also [35]):

Conjecture 4 *If X is Fano and s even, $X_{\leq s}:=\coprod_{k\leq s} X_k$ contains a component of dimension $>s$.*

This is much larger than one would expect from a naive dimension count. It implies for instance that a Poisson field which vanishes at some point must vanish along a curve.

The condition “X Fano” is probably far too strong. In fact an optimistic modification would be :

Conjecture 5 *If X_s is non-empty, it contains a component of dimension $>s$.*

Here are some arguments in favor of this conjecture:

Proposition 4 *Let (X, τ) be a compact Poisson manifold.*

1) *Every component of X_s has dimension $\geq s$.*

2) *Let r be the generic rank of τ (r even); assume that $c_1(X)^q \neq 0$ in $H^q(X, \Omega_X^q)$, where $q = \dim X - r + 1$. Then the degeneracy locus $X \setminus X_r$ of τ has a component of dimension $> r - 2$.*

3) *Assume that X is a projective threefold. If X_0 is non-empty, it contains a curve.*

Sketch of proof:

1) Let Z be a component of X_s (with its reduced structure). It is not difficult to prove that Z is a Poisson subvariety of X (see [35]); this means that at a smooth point x of Z, the tensor $\tau(x)$ lives in $\wedge^2 T_x(Z) \subset \wedge^2 T_x(X)$. But this implies $s \leq \dim T_x(Z) = \dim Z$.

2) is proved in [35], §9, under the extra hypothesis $\dim X = r + 1$. The proof extends easily to the slightly more general situation considered here.

3) is proved in [14] by a case-by-case analysis (leading to a complete classification of those Poisson threefolds for which $X_0 = \varnothing$). It would be interesting to have a more conceptual proof.

The paper [35] contains many interesting results on Poisson manifolds; in particular, a complete classification of the Poisson structures on $\mathbb{P}^3$ for which the zero locus contains a smooth curve.

4 Compact contact manifolds

Let X be a complex manifold, of odd dimension $2r+1$. A *contact structure* on X is a one-form θ with values in a line bundle L on X, such that $\theta \wedge (d\theta)^r \neq 0$ at each point of X (though θ is a twisted 1-form, it is easy to check that $\theta \wedge (d\theta)^r$ makes sense as a section of $K_X \otimes L^{r+1}$; in particular, the condition on θ implies $K_X = L^{-r-1}$).

There are only two classes of compact holomorphic contact manifolds known so far:

a) The projective cotangent bundle $\mathbb{P}T^*_M$, where M is any compact complex manifold;

b) Let $\mathfrak{g}$ be a simple complex Lie algebra. The action of the adjoint group on $\mathbb{P}(\mathfrak{g})$ has a unique closed orbit $X_{\mathfrak{g}}$: every other orbit contains $X_{\mathfrak{g}}$ in its closure. $X_{\mathfrak{g}}$ is a contact Fano manifold.

The following conjecture is folklore:

Conjecture 6 *Any projective contact manifold is of type* a) *or* b).

Half of this conjecture is now proved, thanks to [24] and [13]: a contact projective manifold is either Fano with $b_2 = 1$, or of type a). It is easily seen that a homogeneous Fano contact manifold is of type b), so we can rephrase Conjecture 6 as :

Conjecture 7 *A contact Fano manifold is homogeneous.*

I refer to [4] for some evidence in favor of this conjecture, and to [5] for its application to differential geometry, more specifically to quaternion-Kähler manifolds. These are Riemannian manifolds with holonomy $\mathrm{Sp}(1)\mathrm{Sp}(r)$; they are Einstein manifolds, and in particular they have constant scalar curvature. Thanks to work of Salamon and LeBrun [25, 26], a positive answer to Conjecture 7 would imply:

Conjecture 8 *The only compact quaternion-Kähler manifolds with positive scalar curvature are homogeneous.*

These positive homogeneous quaternion-Kähler manifolds have been classified by Wolf [41] : there is one, $M_{\mathfrak{g}}$, for each simple complex Lie algebra $\mathfrak{g}$.

The link between Conjectures 7 and 8 is provided by the twistor space construction. To any quaternion-Kähler manifold M is associated a $\mathbb{S}^2$-bundle $X \to M$, the twistor space, which carries a natural complex structure; when M has positive scalar curvature it turns out that X is a contact Fano manifold – for instance the twistor space of $M_{\mathfrak{g}}$ is $X_{\mathfrak{g}}$. Conjecture 7 implies that X is isomorphic to $X_{\mathfrak{g}}$ for some simple Lie algebra $\mathfrak{g}$; this in turn implies that M is isometric to $M_{\mathfrak{g}}$ and therefore homogeneous.

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Generalized Lagrangian mean curvature flow in Kähler manifolds that are almost Einstein

Tapio Behrndt

Abstract We introduce the notion of Kähler manifolds that are almost Einstein and we define a generalized mean curvature vector field along submanifolds in them. We prove that Lagrangian submanifolds remain Lagrangian, when deformed in direction of the generalized mean curvature vector field. For a Kähler manifold that is almost Einstein, and which in addition has a trivial canonical bundle, we show that the generalized mean curvature vector field of a Lagrangian submanifold is the dual vector field associated to the Lagrangian angle.

Keywords Lagrangian mean curvature flow, almost Calabi-Yau manifolds.
Mathematics Subject Classification (2010) 53C44.

1 Introduction

In a Calabi-Yau manifold with parallel holomorphic volume form Ω there is a distinguished class of submanifolds called special Lagrangian submanifolds. These are oriented Lagrangian submanifolds, which are calibrated with respect to $\mathrm{Re}\,\Omega$. Special Lagrangian submanifolds have received a lot of attention since the work by Strominger, Yau and Zaslow [14], where mirror symmetry is related to special Lagrangian torus fibrations.

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The notion of special Lagrangian submanifolds can be generalized to the case when the ambient manifold is almost Calabi-Yau. An almost Calabi-Yau manifold is a Kähler manifold together with a non-vanishing, not necessarily parallel, holomorphic volume form. A nice property of almost Calabi-Yau manifolds is that they appear in infinite dimensional families, i.e. the moduli space of almost Calabi-Yau structures is infinite dimensional, while Calabi-Yau structures only appear in finite dimensional families due to the theorem of Tian and Todorov [16], [17] and Yau's proof of the Calabi conjecture [20]. Choosing a generic almost Calabi-Yau metric is therefore a much more powerful thing to do than choosing a generic Calabi-Yau metric and, as in the study of moduli spaces of J-holomorphic curves, this could be of importance for the study of moduli spaces of special Lagrangian submanifolds as conjectured by Joyce in [4]. Another nice feature of almost Calabi-Yau manifolds is that explicit almost Calabi-Yau metrics on compact manifolds are known, while there are no non-trivial Calabi-Yau metrics on compact manifolds explicitly known. For instance a quintic in $\mathbb{CP}^4$ equipped with the restriction of the Fubini-Study metric is an almost Calabi-Yau manifold.

Special Lagrangian submanifolds in (almost) Calabi-Yau manifolds have been studied extensively by many authors but up to date there is no general method known how to construct examples of special Lagrangian submanifolds. However, since special Lagrangian submanifolds are calibrated submanifolds they are volume minimizing in their homology class and one is tempted to construct special Lagrangian submanifolds by mean curvature flow of Lagrangian submanifolds. The existence of the Lagrangian mean curvature flow in Kähler-Einstein manifolds was first proved by Smoczyk [11]. Smoczyk shows that the mean curvature flow of a given compact Lagrangian submanifold remains Lagrangian as long as the flow exists. Thus the problem is to find conditions such that the Lagrangian mean curvature flow exists for all time and converges to a special Lagrangian submanifold. One attempt to this was done by Thomas and Yau [15], where they conjecture that a Lagrangian submanifold satisfying a certain stability condition converges smoothly by Lagrangian mean curvature flow to a non-singular special Lagrangian submanifold in the same homology class. In general there are two problems occurring. Firstly one expects that the evolving Lagrangian submanifold develops a finite time singularity. There are only a few longtime convergence results known for Lagrangian mean curvature flow, for instance by Smoczyk [10], Smoczyk and Wang [13] and Wang [18]. The second problem which occurs is that there exist Lagrangian submanifolds without regular Lagrangian volume minimizers in their homology classes. Examples of such Lagrangian submanifolds were found by Wolfson in [19].

In this paper we introduce the notion of Kähler manifolds that are almost Einstein (in particular, these contain the class of almost Calabi-Yau manifolds), and we define a generalized mean curvature vector field along submanifolds in them. We show that Lagrangian submanifolds remain Lagrangian under deformation in direction of the generalized mean curvature vector field and we obtain a generalized version of Smoczyk's result. Therefore we call the deformation of Lagrangian submanifolds in direction of the generalized mean curvature vector field a generalized Lagrangian mean curvature flow. We show that the generalized Lagrangian mean curvature flow is the negative gradient flow of the volume functional of some conformally rescaled metric. Moreover, if the ambient manifold is almost Calabi-Yau, then we prove that the one-form associated to the generalized mean curvature vector field of a Lagrangian submanifold is the differential of the Lagrangian angle. As a consequence we show that if the initial Lagrangian has zero Maslov class, then the generalized Lagrangian mean curvature flow can be integrated to a scalar equation.

We remark here that recently, after the first version of the present paper, Smoczyk and Wang showed that in every almost Kähler manifold that admits an Einstein connection there exists a generalized mean curvature vector field with the property that Lagrangian submanifolds remain Lagrangian under the deformation in its direction. The generalized Lagrangian mean curvature flow introduced by them contains ours in Kähler manifolds that are almost Einstein as an example (see [12] for more details).

2 Lagrangian mean curvature flow in Kähler-Einstein manifolds

We first recall the definition of the mean curvature flow. Let M be a Riemannian manifold and let N be a submanifold of M given by an immersion $F_0 : N \longrightarrow M$. Throughout this paper the term submanifold will mean an immersed submanifold. The second fundamental form of N is defined by

$$\mathrm{II}(X,Y) = \pi_{\nu N}\left(\bar{\nabla}_{dF_0(X)} dF_0(Y)\right), \; X,Y \in \Gamma(TN),$$

where $\bar{\nabla}$ denotes the Levi-Civita connection of M and $\pi_{\nu N}$ the orthogonal projection onto the normal bundle νN of N. The mean curvature vector field $H \in \Gamma(\nu N)$ of N is defined as the trace of the second fundamental form with respect to the induced Riemannian metric on N.

Definition 1 A smooth one parameter family $\{F(.,t)\}_{t\in[0,T)}$, $T>0$, of immersions of N into M is evolving by mean curvature flow if

$$\begin{aligned} \frac{\partial F}{\partial t}(x,t) &= H(x,t),\ (x,t)\in N\times(0,T) \\ F(x,0) &= F_0(x),\ x\in N. \end{aligned} \tag{1}$$

The mean curvature flow is a quasilinear parabolic system and hence, if N is compact, short time existence and uniqueness for given initial data is guaranteed by standard theory of quasilinear parabolic PDEs, see for instance Ladyženskaja et al. [6].

From now on and throughout this paper we let $(M,J,\bar{\omega},\bar{g})$ denote a compact Kähler manifold of real dimension $2n$ with complex structure J, Kähler form $\bar{\omega}$, and Kähler metric $\bar{g}$. The Kähler form and Kähler metric are related by $\bar{g}(JX,Y)=\bar{\omega}(X,Y)$, for $X,Y\in\Gamma(TM)$. The Levi-Civita connection of $\bar{g}$ is denoted by $\bar{\nabla}$ and the Riemann curvature tensor $\bar{R}$ of $\bar{g}$ is $\bar{R}(X,Y)Z=\bar{\nabla}_X\bar{\nabla}_Y Z-\bar{\nabla}_Y\bar{\nabla}_X Z-\bar{\nabla}_{[X,Y]}Z$, for $X,Y,Z\in\Gamma(TM)$. Moreover the Ricci tensor $\bar{R}ic$ of $\bar{g}$ is $\bar{R}ic(X,Y)=\text{trace}\,\bar{R}(.,X)Y$, for $X,Y\in\Gamma(TM)$, and the Ricci form $\bar{\rho}$, which is a real $(1,1)$-form, is defined by $\bar{\rho}(X,Y)=\bar{R}ic(JX,Y)$.

Let L be a compact manifold of real dimension n and $F_0: L\longrightarrow M$ an immersion of L into M. The induced Riemannian metric on L is $g=F_0^*\bar{g}$, and we set $\omega=F_0^*\bar{\omega}$. Assume now that F_0 is a Lagrangian immersion, i.e. $\omega=0$. We recall some basic geometric properties of Lagrangian submanifolds. For any normal vector field $\xi\in\Gamma(\nu L)$ there is a corresponding one form α_ξ on L given by $\alpha_\xi=F_0^*(\xi\lrcorner\bar{\omega})$. The one-form $\alpha_H=F_0^*(H\lrcorner\omega)$ is called the mean curvature form and it satisfies the following important relation first proved by Dazord in [1]:

Proposition 1 *The mean curvature form α_H satisfies*

$$d\alpha_H=F_0^*\bar{\rho}.$$

In particular by Cartan's formula we find

$$F_0^*(\mathscr{L}_H\bar{\omega})=F_0^*(d(H\lrcorner\bar{\omega}))+F_0^*(H\lrcorner d\bar{\omega})=F_0^*\bar{\rho}.$$

Hence, if M is Kähler-Einstein, i.e. $\bar{\rho}=\lambda\bar{\omega}$ for some $\lambda\in\mathbb{R}$, then it follows that the deformation of a Lagrangian submanifold in the direction of the mean curvature vector field is an infinitesimal symplectic motion. A natural question that arises now is whether the Lagrangian condition is preserved under the mean curvature flow. This question was answered positively by Smoczyk in [11]:

Theorem 1 *Let L be a compact n-dimensional manifold and let $F_0 : L \longrightarrow M$ be a Lagrangian immersion into a compact Kähler-Einstein manifold M. Then the mean curvature flow admits a unique smooth solution for a short time and this solution consists of Lagrangian submanifolds.*

3 Generalized Lagrangian mean curvature flow in Kähler manifolds that are almost Einstein

Definition 2 An n-dimensional Kähler manifold $(M, J, \bar{\omega}, \bar{g})$ is called almost Einstein if

$$\bar{\rho} = \lambda \bar{\omega} + n dd^c \psi$$

for some constant $\lambda \in \mathbb{R}$ and some smooth function ψ on M.

From now on we additionally assume that our Kähler manifold $(M, J, \bar{\omega}, \bar{g})$ is almost Einstein. Given an immersion $F_0 : N \longrightarrow M$ of a manifold N into M we define a normal vector field $K \in \Gamma(\nu N)$ along N by

$$K = H - n\pi_{\nu N}\left(\bar{\nabla}\psi\right).$$

We call K the generalized mean curvature vector field of N. Now let L be an n-dimensional manifold and $F_0 : L \longrightarrow M$ a Lagrangian immersion. Then the deformation of L in direction of the generalized mean curvature vector field is an infinitesimal symplectic motion. Indeed by Dazord's result we have

$$F_0^*\left(\mathscr{L}_K \bar{\omega}\right) = d\alpha_H + nF_0^*\left(d\left(d\psi \circ J\right)\right) = F_0^*\left(\bar{\rho} - ndd^c\psi\right) = \lambda F_0^* \bar{\omega} = 0.$$

Also observe that if M is Kähler-Einstein, then K is the mean curvature vector field.

In the remainder we study the generalized mean curvature flow

$$\begin{aligned} \frac{\partial F}{\partial t}(x,t) &= K(x,t),\ (x,t) \in L \times (0,T) \\ F(x,0) &= F_0(x),\ x \in L, \end{aligned} \tag{2}$$

for a given Lagrangian immersion $F_0 : L \longrightarrow M$ of a compact n-dimensional manifold L into M and $\{F(.,t)\}_{t\in[0,T)}$ a smooth one-parameter family of immersions of L into M. In order to establish the short time existence and uniqueness of this flow observe that K as a differential operator differs from H just by lower order terms. Hence K and H have the same principal symbol, so short time existence and uniqueness for (2) follows immediately.

Now let $\{F(.,t)\}_{t\in[0,T)}$ be the solution to the generalized mean curvature flow (2). In the remaining part of this chapter we show that $F(.,t): L \longrightarrow M$ is Lagrangian for each $t \in (0,T)$. As before we denote $g = F(.,t)^*\bar{g}$ and $\omega = F(.,t)^*\bar{\omega}$. Furthermore ∇ will denote the Levi-Civita connection of g and R the Riemannian curvature tensor of g. Let $p \in L$ and choose normal coordinates $\{x^i\}$ on L around p at time $t \in (0,T)$ and coordinates $\{y^\alpha\}$ on M around $F(p,t)$. We have to introduce some notation. We denote $e_i = \frac{\partial F}{\partial x^i}(.,t)$ and we define tensors N and η by $N_i = N(e_i) = \pi_{\nu L}(Je_i)$ and $\eta_{ij} = \eta(e_i,e_j) = \bar{g}(Ne_i,Ne_j)$. Moreover we set $h_{ijk} = h(e_i,e_j,e_k) = -\bar{g}\left(Ne_i,\bar{\nabla}_{e_j}e_k\right)$. Observe that h_{ijk} is symmetric in the last two indices and fully symmetric if $F(.,t)$ is Lagrangian. We also denote $\bar{R}_{klj\underline{i}} = \bar{R}(e_k,e_l,e_j,N(e_i))$. The following formula proved by Smoczyk [11, Lem. 1.4] will be of use later:

Lemma 3.1

$$\begin{aligned}\nabla_l h_{kij} - \nabla_k h_{lij} = \bar{R}_{klj\underline{i}} + \nabla_j\nabla_i\omega_{lk} + \omega_i{}^m\bar{R}_{kljm} + \omega_k{}^m R_{ljim}\\ +\omega_l{}^m R_{jkim} + \eta^{mn}\omega_n^{\,s}\left(h_{mlj}h_{ski} - h_{mkj}h_{sli}\right).\end{aligned}$$

We start by computing the evolution equations of g_{ij} and ω_{ij} at $p \in L$ and time t.

Lemma 3.2

$$i)\ \frac{d}{dt}\omega_{ij} = (d\alpha_K)_{ij}$$

$$ii)\ \frac{d}{dt}g_{ij} = -2\eta^{mn}(\alpha_H)_m h_{nij} + 2nd\psi(II_{ij}).$$

Proof

$$\begin{aligned}\frac{d}{dt}\omega_{ij} &= \bar{\omega}_{\alpha\beta}\left\{\frac{\partial}{\partial x^i}\frac{\partial F^\alpha}{\partial t}\frac{\partial F^\beta}{\partial x^j} + \frac{\partial F^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial F^\beta}{\partial t}\right\}\\ &= \bar{\omega}_{\alpha\beta}\left\{\frac{\partial K^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j} + \frac{\partial F^\alpha}{\partial x^i}\frac{\partial K^\beta}{\partial x^j}\right\}\\ &= \bar{\omega}\left(\frac{\partial K}{\partial x^i},\frac{\partial F}{\partial x^j}\right) - \bar{\omega}\left(\frac{\partial K}{\partial x^j},\frac{\partial F}{\partial x^i}\right) = (d\alpha_K)_{ij}.\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} g_{ij} &= \bar{g}\left(\frac{\partial}{\partial x^i}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^j}\right) + \bar{g}\left(\frac{\partial F}{\partial x^i}, \frac{\partial}{\partial x^j}\frac{\partial F}{\partial t}\right) \\
&= \bar{g}\left(\frac{\partial K}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) + \bar{g}\left(\frac{\partial K}{\partial x^j}, \frac{\partial F}{\partial x^i}\right) \\
&= -2\bar{g}\left(H, \frac{\partial^2 F}{\partial x^i \partial x^j}\right) + 2n\bar{g}\left(\bar{\nabla}\psi, \pi_{\nu L}\left(\frac{\partial^2 F}{\partial x^i \partial x^j}\right)\right) \\
&= -2\eta^{mn}(\alpha_H)_m h_{nij} + 2n d\psi(\mathrm{II}_{ij}).
\end{aligned}$$

■

Using Lemma 3.1 and Lemma 3.2 we can now proceed as in [11] to prove the following lemma.

Lemma 3.3 *Let* $0 < \tau < T$, *then there exists a constant* $C > 0$ *such that for all* $t \in [0, \tau]$

$$\frac{d}{dt}|\omega|^2 \leq \Delta|\omega|^2 + n d\psi\left(\nabla|\omega|^2\right) + C|\omega|^2.$$

Proof Denote $Y = \pi_{\nu L}(\bar{\nabla}\psi)$, so that $K = H - nY$. Then

$$\begin{aligned}
\frac{d}{dt}|\omega|^2 &= \frac{d}{dt} g^{mk} g^{jl} \omega_{mj}\omega_{kl} \\
&= -2\omega^{kl}\omega^m{}_l \frac{d}{dt} g_{mk} + 2\omega^{kl}\frac{d}{dt}\omega_{kl} \\
&= -2\omega^{kl}\omega^m{}_l\left(-2\eta^{st}(\alpha_H)_s h_{tmk} + 2n d\psi(\mathrm{II}_{mk})\right) \\
&\qquad + 2\omega^{kl}\left(\nabla_k(\alpha_H)_l - \nabla_l(\alpha_H)_k - n(d\alpha_Y)_{kl}\right) \\
&= 4\omega^{kl}\omega^m{}_l\eta^{st}(\alpha_H)_s h_{tmk} - 4n\omega^{kl}\omega^m{}_l d\psi(\mathrm{II}_{mk}) \\
&\qquad + 2\omega^{kl}\left(\nabla_k(\alpha_H)_l - \nabla_l(\alpha_H)_k\right) - 2n\omega^{kl}(d\alpha_Y)_{kl} \\
&= 4\omega^{kl}\omega^m{}_l\eta^{st}(\alpha_H)_s h_{tmk} - 4n\omega^{kl}\omega^m{}_l d\psi(\mathrm{II}_{mk}) \\
&\qquad + 2\omega^{kl} g^{pq}\left(\nabla_k h_{lpq} - \nabla_l h_{kpq}\right) - 2n\omega^{kl}(d\alpha_Y)_{kl} \\
&= 4\omega^{kl}\omega^m{}_l\eta^{st}(\alpha_H)_s h_{tmk} - 4n\omega^{kl}\omega^m{}_l d\psi(\mathrm{II}_{mk}) \\
&\quad + 2\omega^{kl} g^{pq}\Big(\bar{R}_{lkq\underline{p}} + \nabla_q\nabla_p\omega_{kl} + \omega_p{}^s\bar{R}_{lkqs} + \omega_l{}^s R_{kqps} + \omega_k{}^s R_{qlps} \\
&\qquad + \eta^{mt}\omega_t{}^s\left(h_{mkq}h_{slp} - h_{mlq}h_{skp}\right)\Big) - 2n\omega^{kl}(d\alpha_Y)_{kl} \\
&= 4\omega^{kl}\omega^m{}_l\eta^{st}(\alpha_H)_s h_{tmk} - 4n\omega^{kl}\omega^m{}_l d\psi(\mathrm{II}_{mk}) + 2\omega^{kl}\bar{R}_{lk}{}^p{}_{\underline{p}} + \Delta|\omega|^2 \\
&\quad - |\nabla\omega|^2 + 2\omega^{kl}\omega_p{}^s\bar{R}_{lk}{}^p{}_s + 2\omega^{kl}\omega_l{}^s R_k{}^p{}_{ps} + 2\omega^{kl}\omega_k{}^s R^p_{lps} \\
&\qquad + 2\omega^{kl}\eta^{mt}\omega_t{}^s\left(h_{mk}{}^p h_{slp} - h_{ml}{}^p h_{skp}\right) - 2n\omega^{kl}(d\alpha_Y)_{kl}.
\end{aligned}$$

For terms of the form $\omega^{sl}\omega^m{}_i T^i_{slm}$ we have

$$\begin{aligned}2\omega^{sl}\omega^{m}{}_{i}T^{i}_{slm} &= 2\sum_{s,l}\left(\omega_{sl}\sum_{m,i}\omega_{mi}T_{islm}\right)\\ &\leq \sum_{s,l}(\omega_{sl})^2+\sum_{s,l}\left(\sum_{m,i}\omega_{ml}T_{islm}\right)^2\\ &\leq |\omega|^2+n^2\sum_{s,l,m,i}(\omega_{ml})^2(T_{islm})^2\leq\left(1+n^2|T|^2\right)|\omega|^2.\end{aligned}$$

Since $L\times[0,\tau]$ is compact we can choose a constant $C>0$ such that for all $t\in[0,\tau]$

$$\frac{d}{dt}|\omega|^2\leq\Delta|\omega|^2+C|\omega|^2+2\omega^{kl}\bar{R}_{lk}{}^{p}{}_{\underline{p}}-2n\omega^{kl}(d\alpha_Y)_{lk}.$$

It remains to find an estimate for the last two terms. We have

$$\begin{aligned}(\alpha_Y)_l &= \bar{\omega}\left(\pi_{\nu L}\left(\bar{\nabla}\psi\right),e_l\right)\\ &= -\bar{g}\left(\bar{\nabla}\psi,J(e_l)\right)-g^{mk}\bar{g}\left(\bar{\nabla}\psi,e_k\right)\bar{\omega}\left(e_m,e_l\right)\\ &= d^c\psi(e_l)-g^{mk}d\psi(e_k)\omega_{ml},\end{aligned}$$

and hence

$$\begin{aligned}(d\alpha_Y)_{kl} &= \frac{\partial}{\partial x^k}d^c\psi(e_l)-\frac{\partial}{\partial x^l}d^c\psi(e_k)\\ &\qquad -\frac{\partial}{\partial x^k}\left(g^{mj}d\psi(e_j)\omega_{ml}\right)+\frac{\partial}{\partial x^l}\left(g^{mj}d\psi(e_j)\omega_{mk}\right)\\ &= dd^c\psi(e_k,e_l)-g^{mj}\omega_{ml}\frac{\partial}{\partial x^k}d\psi(e_j)\\ &\qquad +g^{mj}\omega_{mk}\frac{\partial}{\partial x^l}d\psi(e_j)-g^{mj}d\psi(e_j)\frac{\partial\omega_{ml}}{\partial x^k}+g^{mj}d\psi(e_j)\frac{\partial\omega_{mk}}{\partial x^l}.\end{aligned}$$

Multiplying both sides with $-2n\omega^{kl}$, using the Kähler and the almost Einstein condition, and estimating the quadratic terms in ω we get

$$-2n\omega^{kl}(d\alpha_Y)_{kl}\leq C|\omega|^2-2\omega^{kl}\bar{\rho}_{kl}+2ng^{mj}\omega^{kl}d\psi(e_j)\left(\frac{\partial\omega_{ml}}{\partial x^k}-\frac{\partial\omega_{mk}}{\partial x^l}\right).$$

Using that ω is closed we find

$$\begin{aligned}g^{mj}\omega^{kl}\left(\frac{\partial\omega_{ml}}{\partial x^k}-\frac{\partial\omega_{mk}}{\partial x^l}\right)d\psi(e_j) &= d\psi\left(g^{mj}g^{ik}g^{sl}\omega_{is}\frac{\partial\omega_{kl}}{\partial x^m}e_j\right)\\ &= d\psi\left(\frac{1}{2}g^{mj}\frac{\partial|\omega|^2}{\partial x^m}e_j\right)=\frac{1}{2}d\psi\left(\nabla|\omega|^2\right).\end{aligned}$$

Putting all together yields

$$\frac{d}{dt}|\omega|^2 \leq \Delta|\omega|^2 + nd\psi\left(\nabla|\omega|^2\right) + C|\omega|^2 + 2\omega^{kl}\bar{R}_{lk}{}^{p}{}_{\underline{p}} - 2\omega^{kl}\bar{\rho}_{kl}.$$

Now by definition of the tensor N we have $N(e_p) = J(e_p) - \omega_p{}^m e_m$ and so

$$\bar{R}_{lkq\underline{p}} = \bar{R}(e_l, e_k, e_q, J(e_p)) - \omega_p{}^m \bar{R}_{lkqm}.$$

Multiplying both sides with $2\omega^{kl}g^{pq}$ and estimating the quadratic term in ω gives

$$2\omega^{kl}\bar{R}_{lk}{}^{p}{}_{\underline{p}} \leq 2\omega^{kl}g^{pq}\bar{R}(e_l, e_k, e_q, J(e_p)) + C|\omega|^2.$$

Using the following well known identity from Kähler geometry

$$g^{pq}\bar{R}(e_l, e_k, e_q, J(e_p)) = \bar{\rho}_{kl},$$

we finally obtain

$$\frac{d}{dt}|\omega|^2 \leq \Delta|\omega|^2 + nd\psi\left(\nabla|\omega|^2\right) + C|\omega|^2.$$

■

Applying the parabolic maximum principle we conclude that $F(.,t) : L \longrightarrow M$ is Lagrangian for each $t \in [0,T)$. This motivates the following definition:

Definition 3 A family of Lagrangian submanifolds satisfying (2) is said to evolve by generalized Lagrangian mean curvature flow.

And we have proved the following theorem:

Theorem 2 *Let L be a compact n-dimensional manifold and $F_0 : L \longrightarrow M$ a Lagrangian immersion of L into a compact Kähler manifold M that is almost Einstein. Then the generalized Lagrangian mean curvature flow admits a unique smooth solution for a short time and this solution consists of Lagrangian submanifolds.*

4 A variational approach to the generalized mean curvature flow

Let $\mathscr{S}$ be the infinite dimensional manifold consisting of all compact n-dimensional submanifolds of M. In this chapter we show that the generalized mean curvature flow is the gradient flow of a volume functional on $\mathscr{S}$. Let $N \in \mathscr{S}$, then the tangent space of $\mathscr{S}$ at N consists of the normal vector fields along N and for any Riemannian metric g on M there is a natural L^2-metric on $\mathscr{S}$ given by

$$\langle Y,Z\rangle_{g,L^2} = \int_N g(Y,Z)dV_g,$$

for $Y,Z \in \Gamma(\nu N)$.

We define two conformally rescaled Riemannian metrics $\tilde{g}$ and $\hat{g}$ on M by

$$\tilde{g} = e^{2\psi}\bar{g} \quad \text{and} \quad \hat{g} = e^{\frac{2n}{n+2}\psi}\bar{g}.$$

Then we have the following variational characterization of the generalized mean curvature flow:

Proposition 2 *The generalized mean curvature flow is the negative gradient flow of the volume functional $Vol_{\tilde{g}}$ on $\mathscr{S}$ with respect to the L^2-metric $\langle .,.\rangle_{\hat{g},L^2}$.*

Proof Let $N \in \mathscr{S}$ and let Y be a normal vector field along N. Then the first variation of the volume functional gives

$$\delta_Y \mathrm{Vol}_{\tilde{g}}(N) = -\int_N \tilde{g}(Y,\tilde{H})dV_{\tilde{g}},$$

where $\tilde{H}$ is the mean curvature vector field on N with respect to the metric on N which is induces by $\tilde{g}$. It is easy to show that

$$\tilde{H} = e^{-2\psi}\left(H - n\pi_{\nu N}\left(\bar{\nabla}\psi\right)\right).$$

Hence

$$\begin{aligned}\delta_Y \mathrm{Vol}_{\tilde{g}}(N) &= -\int_N e^{n\psi}\bar{g}(H - n\pi_{\nu N}(\bar{\nabla}\psi),Y)dV_{\bar{g}} \\ &= -\int_N e^{\left(n-\frac{2n}{n+2}-\frac{2n}{n+2}\frac{n}{2}\right)\psi}\hat{g}(K,Y)dV_{\hat{g}} \\ &= -\int_N \hat{g}(K,Y)dV_{\hat{g}} = -\langle K,Y\rangle_{\hat{g},L^2}.\end{aligned}$$

∎

5 The case of almost Calabi-Yau manifolds

We introduce almost Calabi-Yau manifolds and special Lagrangian submanifolds as defined by Joyce in [3, §8.4].

Definition 4 An n-dimensional almost Calabi-Yau manifold $(M, J, \bar{\omega}, \bar{g}, \Omega)$ is an n-dimensional Kähler manifold $(M, J, \bar{\omega}, \bar{g})$ together with a non-vanishing holomorphic volume form Ω.

Given an n-dimensional almost Calabi-Yau manifold $(M, J, \bar{\omega}, \bar{g}, \Omega)$ we can define a smooth function ψ on M by

$$e^{2n\psi}\frac{\bar{\omega}^n}{n!} = (-1)^{\frac{n(n-1)}{2}}\left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}.$$

Here $\bar{\Omega}$ denotes the complex conjugate of Ω. Then $(M, J, \bar{\omega}, \bar{g}, \Omega)$ is Calabi-Yau if and only if ψ vanishes identically. Using $|\Omega|_{\bar{g}} = 2^{\frac{n}{2}} e^{n\psi}$ and the following formula for the Ricci form of a Kähler manifold with trivial canonical bundle (see for instance [3, §7.1])

$$\bar{\rho} = dd^c \log |\Omega|_{\bar{g}}$$

we find

$$\bar{\rho} = n dd^c \psi.$$

Hence almost Calabi-Yau manifolds are almost Einstein and Theorem 2 holds in this case. Let $\tilde{g}$ be a conformally rescaled metric on M defined by $\tilde{g} = e^{2\psi}\bar{g}$. One easily proves that $\operatorname{Re}\Omega$ is a calibrating n-form on $(M, \tilde{g})$. This leads to the definition of special Lagrangian submanifolds in almost Calabi-Yau manifolds.

Definition 5 An oriented Lagrangian submanifold L of an almost Calabi-Yau manifold M is called special Lagrangian if it is calibrated with respect to $\operatorname{Re}\Omega$ for the metric $\tilde{g}$. More generally, an oriented Lagrangian submanifold L is special Lagrangian with phase $\theta_0 \in \mathbb{R}$, if L is calibrated with respect to $\operatorname{Re}(e^{-i\theta_0}\Omega)$ for the metric $\tilde{g}$.

Besides the fact that one is able to write down explicit examples of almost Calabi-Yau metrics on compact manifolds there is another reason for studying almost Calabi-Yau manifolds. Recall that by the theorem of Tian and Todorov the moduli space $\mathscr{M}_{CY}$ of Calabi-Yau metrics of a compact Calabi-Yau manifold is of dimension $h^{1,1}(M) + 2h^{n-1,1}(M) + 1$, where $h^{i,j}(M)$ are the Hodge numbers of M. In particular $\mathscr{M}_{CY}$ is finite dimensional. In the study of moduli spaces of J-holomorphic curves in symplectic manifolds it turns out that for a generic almost complex structure J the moduli space $\mathscr{M}_J$ of embedded J-holomorphic curves is a smooth manifold, while for a fixed almost complex structure J the space $\mathscr{M}_J$ can have singularities (see [8] for details). Now the moduli space $\mathscr{M}_{ACY}$ of almost Calabi-Yau structures is of infinite dimension and therefore choosing a generic almost Calabi-Yau metric is a more powerful thing to do than choosing a generic

Calabi-Yau metric. We explain why this is of certain interest. It was proved by McLean [9] that the moduli space of compact special Lagrangian submanifolds $\mathscr{M}_{SL}$ in a Calabi-Yau manifold is a smooth manifold of dimension $b^1(L)$, the first Betti number of L. An important question is whether it is possible to compactify $\mathscr{M}_{SL}$ in order to count invariants of Calabi-Yau manifolds. One approach to this problem, due to Joyce, is to study the moduli space of special Lagrangian submanifolds with conical singularities in almost Calabi-Yau manifolds (see [5] for a survey of his results). In particular Joyce conjectures that for generic almost Calabi-Yau metrics the moduli space of special Lagrangian submanifolds with conical singularities is a smooth finite dimensional manifold.

We come back to the study of the generalized Lagrangian mean curvature flow. First observe that special Lagrangian submanifolds in an almost Calabi-Yau manifold M are minimal with respect to $\tilde{g}$. By Proposition 2 the generalized Lagrangian mean curvature flow decreases volume with respect to $\tilde{g}$. Therefore the generalized Lagrangian mean curvature flow is in this sense the right flow to consider. Harvey and Lawson show in [2] that

$$F_0^* \Omega = e^{i\theta + n\psi} dV_g,$$

for $F_0 : L \longrightarrow M$ a Lagrangian immersion. The map $\theta : L \longrightarrow S^1$ is called the Lagrangian angle of L. From this we obtain an alternative characterization of special Lagrangian submanifolds.

Proposition 3 *An oriented Lagrangian submanifold L is special Lagrangian with phase θ_0 if and only if*

$$(\cos\theta_0 \operatorname{Im}\Omega - \sin\theta_0 \operatorname{Re}\Omega)|_L = 0.$$

In particular, an oriented Lagrangian submanifold is special Lagrangian with phase θ_0 if and only if the Lagrangian angle is constant $\theta \equiv \theta_0$.

The Lagrangian angle is closely related to the generalized Lagrangian mean curvature flow as proved in the next proposition.

Proposition 4 *Let L be a Lagrangian submanifold of M. Then*

$$\alpha_K = -d\theta.$$

Proof The decomposition

$$\Lambda^n T^* M \otimes \mathbb{C} = \bigoplus_{p+q=n} \Lambda^{p,q} T^* M$$

is invariant under the holonomy representation of $\bar{g}$. Hence there exists a complex one form η on M satisfying $\bar{\nabla}\Omega = \eta \otimes \Omega$. Moreover, since Ω is holomorphic, η is in fact a one form of type $(1,0)$. Using $\Omega \wedge \bar{\Omega} = e^{2n\psi} dV_{\bar{g}}$ we find by computing $\bar{\nabla}\left(\Omega \wedge \bar{\Omega}\right)$ the equality

$$(\eta + \bar{\eta}) \otimes \Omega \wedge \bar{\Omega} = 2nd\psi \otimes \Omega \wedge \bar{\Omega}.$$

Hence $\eta = 2n\partial\psi$ and so $\bar{\nabla}\Omega = 2n\partial\psi \otimes \Omega$. Following the computation by Thomas and Yau [15, Lem. 2.1] we obtain

$$\bar{\nabla}\Omega = (id\theta + nd\psi + i\alpha_H) \otimes \Omega$$

and establish the equality

$$\alpha_H - nd^c\psi = -d\theta.$$

But $\alpha_H - nd^c\psi = \alpha_K$ and hence $\alpha_K = -d\theta$. ■

Now let $\{F(.,t)\}_{t\in[0,T)}$ be the solution to the generalized mean curvature flow with initial condition $F_0 : L \longrightarrow M$ a Lagrangian immersion. Then we have the following proposition:

Proposition 5 *Under the generalized Lagrangian mean curvature flow the Lagrangian angle of L satisfies*

$$\frac{d}{dt}\theta = \Delta\theta + nd\psi(\nabla\theta).$$

Proof On the one hand

$$\frac{d}{dt}e^{i\theta+n\psi}dV_g = i\frac{d\theta}{dt}e^{i\theta+n\psi}dV_g + nd\psi(K)e^{i\theta+n\psi}dV_g + e^{i\theta+n\psi}\frac{d}{dt}dV_g$$

and on the other hand, using $F(.,t)^*\Omega = e^{i\theta+n\psi}dV_g$, we have

$$\frac{d}{dt}e^{i\theta+n\psi}dV_g = F(.,t)^*(\mathscr{L}_K\Omega) = F(.,t)^*(d(K \lrcorner \Omega)) + F(.,t)^*(K \lrcorner d\Omega).$$

Since Ω is holomorphic, $d\Omega = 0$. Moreover by Proposition 4 we have $K = J(\nabla\theta)$ and hence

$$\begin{aligned} F(.,t)^*(d(K \lrcorner \Omega)) &= iF(.,t)^*(d(\nabla\theta \lrcorner \Omega)) = id(e^{i\theta+n\psi}\nabla\theta \lrcorner dV_g) \\ &= ie^{i\theta+n\psi}\left(d(\nabla\theta \lrcorner dV_g) + nd\psi \wedge (\nabla\theta \lrcorner dV_g)\right) \\ &\qquad -e^{i\theta+n\psi}d\theta \wedge (\nabla\theta \lrcorner dV_g) \\ &= ie^{i\theta+n\psi}\left(\Delta\theta + nd\psi(\nabla\theta)\right)dV_g - e^{i\theta+n\psi}|\nabla\theta|^2 dV_g. \end{aligned}$$

Comparing imaginary parts yields

$$\frac{d}{dt}\theta = \Delta\theta + nd\psi(\nabla\theta).$$

■

We end this paper by showing how the generalized Lagrangian mean curvature flow in an almost Calabi-Yau manifold can be integrated to a scalar equation. Let $\hat{\omega}$ be the canonical symplectic structure on the cotangent bundle T^*L of L. Then by the Lagrangian neighbourhood theorem [7, Thm. 3.33] there exists an immersion $\Phi : U \longrightarrow V$ from an open neighbourhood U of the zero section in T^*L onto an open neighbourhood V of L in M, such that $\hat{\omega} = \Phi^*\bar{\omega}$ and $\Phi(x,0) = F_0(x)$ for $x \in L$. It is not hard to see that all Lagrangian submanifolds in M which are C^1-close to L correspond to graphs in T^*L of closed one-forms on L.

Theorem 3 *Let $F_0 : L \longrightarrow M$ be a zero Maslov class Lagrangian, i.e. $\theta : L \longrightarrow \mathbb{R}$ is a well defined smooth function on L, let Φ be as above, and let $\{u(.,t)\}_{t\in[0,T)}$ be a smooth one-parameter family of smooth functions on L satisfying*

$$\begin{aligned}&\frac{\partial u}{\partial t}(x,t) = \theta(x,t),\ (x,t) \in L \times (0,T)\\&u(x,0) = 0,\ x \in L.\end{aligned}$$

Here $\theta(.,t)$ denotes the Lagrangian angle of the Lagrangian immersion $\Phi \circ du(.,t)$ of L into M. Choosing $T > 0$ sufficiently small we can assume that the graph of $du(.,t)$ lies in U for $t \in [0,T)$. Then there exists a family of diffeomorphisms $\{\varphi(.,t)\}_{t\in[0,T)}$ of L, such that the immersions $\{F(.,t)\}_{t\in[0,T)}$ of L into M defined by

$$F(x,t) = \Phi(\varphi(x,t), du(\varphi(x,t),t)),\ x \in L,$$

evolve by generalized Lagrangian mean curvature flow.

The proof of this theorem can be found in [10] in the case when the ambient space is $\mathbb{C}^n$. When the ambient space is a general almost Calabi-Yau manifold the proof is analogous.

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Einstein metrics and preserved curvature conditions for the Ricci flow

Simon Brendle

Abstract Let C be a cone in the space of algebraic curvature tensors. Moreover, let (M,g) be a compact Einstein manifold with the property that the curvature tensor of (M,g) lies in the interior of the cone C at each point on M. We show that (M,g) has constant sectional curvature if the cone C satisfies certain structure conditions.

Keywords Ricci flow, Einstein metric.
Mathematics Subject Classification (2010) *Primary* 53C25. *Secondary* 53C24, 53C44.

1 Introduction

In this note, we study Riemannian manifolds (M,g) with the property that $\mathrm{Ric} = \rho\, g$ for some constant ρ. A Riemannian manifold with this property is called an Einstein manifold. Einstein manifolds arise naturally as critical points of the normalized Einstein-Hilbert action, and have been studied intensively (see e.g. [2]). In particular, it is of interest to classify all Einstein manifolds satisfying a suitable curvature condition. This problem was studied by M. Berger [1]. In 1974, S. Tachibana [9] obtained the following important result:

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Theorem 1 (S. Tachibana) *Let (M,g) be a compact Einstein manifold. If (M,g) has positive curvature operator, then (M,g) has constant sectional curvature. Furthermore, if (M,g) has nonnegative curvature operator, then (M,g) is locally symmetric.*

In a recent paper [3], we proved a substantial generalization of Tachibana's theorem. More precisely, it was shown in [3] that the assumption that (M,g) has positive curvature operator can be replaced by the weaker condition that (M,g) has positive isotropic curvature:

Theorem 2 *Let (M,g) be a compact Einstein manifold of dimension $n \geq 4$. If (M,g) has positive isotropic curvature, then (M,g) has constant sectional curvature. Moreover, if (M,g) has nonnegative isotropic curvature, then (M,g) is locally symmetric.*

The proof of Theorem 2 relies on the maximum principle. One of the key ingredients in the proof is the fact that nonnegative isotropic curvature is preserved by the Ricci flow (cf. [5]).

In this note, we show that the first statement in Theorem 2 can be viewed as a special case of a more general principle. To explain this, we fix an integer $n \geq 4$. We shall denote by $\mathcal{C}_B(\mathbb{R}^n)$ the space of algebraic curvature tensors on $\mathbb{R}^n$. Furthermore, for each $R \in \mathcal{C}_B(\mathbb{R}^n)$, we define an algebraic curvature tensor $Q(R) \in \mathcal{C}_B(\mathbb{R}^n)$ by

$$Q(R)_{ijkl} = \sum_{p,q=1}^{n} R_{ijpq} R_{klpq} + 2 \sum_{p,q=1}^{n} (R_{ipkq} R_{jplq} - R_{iplq} R_{jpkq}).$$

The term $Q(R)$ arises naturally in the evolution equation of the curvature tensor under the Ricci flow (cf. [6]). The ordinary differential equation $\frac{d}{dt}R = Q(R)$ on $\mathcal{C}_B(\mathbb{R}^n)$ will be referred to as the Hamilton ODE.

We next consider a cone $C \subset \mathcal{C}_B(\mathbb{R}^n)$ with the following properties:

(i) C is closed, convex, and $O(n)$-invariant.

(ii) C is invariant under the Hamilton ODE $\frac{d}{dt}R = Q(R)$.

(iii) If $R \in C \setminus \{0\}$, then the scalar curvature of R is nonnegative and the Ricci tensor of R is non-zero.

(iv) The curvature tensor $I_{ijkl} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}$ lies in the interior of C.

We now state the main result of this note:

Theorem 3 *Let $C \subset \mathcal{C}_B(\mathbb{R}^n)$ be a cone which satisfies the conditions (i)–(iv) above, and let (M,g) be a compact Einstein manifold of dimension n. Moreover, suppose* *that the curvature tensor of (M,g) lies in the interior of the cone C for all points $p \in M$. Then (M,g) has constant sectional curvature.*

As an example, let us consider the cone

$$C = \{R \in \mathcal{C}_B(\mathbb{R}^n) : R \text{ has nonnegative isotropic curvature}\}.$$

For this choice of C, the conditions (i) and (iv) are trivially satisfied. Moreover, it follows from a result of M. Micallef and M. Wang (see [7], Proposition 2.5) that C satisfies condition (iii) above. Finally, the cone C also satisfies the condition (ii). This was proved independently in [5] and [8]. Therefore, Theorem 2 is a subcase of Theorem 3.

2 Proof of Theorem 3

The proof of Theorem 3 is similar to the proof of Theorem 16 in [3]. Let (M,g) be a compact Einstein manifold of dimension n with the property that the curvature tensor of (M,g) lies in the interior of C for all points $p \in M$. If (M,g) is Ricci flat, then the curvature tensor of (M,g) vanishes identically. Hence, it suffices to consider the case that (M,g) has positive Einstein constant. After rescaling the metric if necessary, we may assume that $\mathrm{Ric} = (n-1)\,g$. As in [3], we define an algebraic curvature tensor S by

$$S_{ijkl} = R_{ijkl} - \kappa\,(g_{ik}\,g_{jl} - g_{il}\,g_{jk}), \tag{1}$$

where κ is a positive constant. Let κ be the largest real number with the property that S lies in the cone C for all points $p \in M$. Since the curvature tensor R lies in the interior of the cone C for all points $p \in M$, we conclude that $\kappa > 0$. On the other hand, the curvature tensor S has nonnegative scalar curvature. From this, we deduce that $\kappa \leq 1$.

Proposition 1 *The tensor S satisfies*

$$\Delta S + Q(S) = 2(n-1)\,S + 2(n-1)\,\kappa\,(\kappa-1)\,I,$$

where $I_{ijkl} = g_{ik}\,g_{jl} - g_{il}\,g_{jk}$.

Proof The curvature tensor of (M,g) satisfies

$$\Delta R + Q(R) = 2(n-1)\,R \tag{2}$$

(see [3], Proposition 3). Using (1), we compute

$$\begin{aligned} Q(S)_{ijkl} &= Q(R)_{ijkl} + 2(n-1)\,\kappa^2\,(g_{ik}\,g_{jl} - g_{il}\,g_{jk}) \\ &\quad - 2\kappa\,(\mathrm{Ric}_{ik}\,g_{jl} - \mathrm{Ric}_{il}\,g_{jk} - \mathrm{Ric}_{jk}\,g_{il} + \mathrm{Ric}_{jl}\,g_{ik}). \end{aligned}$$

Since $\mathrm{Ric} = (n-1)\,g$, it follows that

$$Q(S) = Q(R) + 2(n-1)\,\kappa\,(\kappa - 2)\,I. \tag{3}$$

Combining (2) and (3), we obtain

$$\Delta S + Q(S) = 2(n-1)\,R + 2(n-1)\,\kappa\,(\kappa - 2)\,I.$$

Since $R = S + \kappa I$, the assertion follows. ■

In the following, we denote by $T_S C$ the tangent cone to C at S.

Proposition 2 *At each point $p \in M$, we have $\Delta S \in T_S C$ and $Q(S) \in T_S C$.*

Proof It follows from the definition of κ that S lies in the cone C for all points $p \in M$. Hence, the maximum principle implies that $\Delta S \in T_S C$. Moreover, since the cone C is invariant under the Hamilton ODE, we have $Q(S) \in T_S C$. ■

Proposition 3 *Suppose that $\kappa < 1$. Then S lies in the interior of the cone C for all points $p \in M$.*

Proof Let us fix a point $p \in M$. By Proposition 2, we have $\Delta S \in T_S C$ and $Q(S) \in T_S C$. Furthermore, we have $-S \in T_S C$ since C is a cone. Putting these facts together, we obtain

$$\Delta S + Q(S) - 2(n-1)\,S \in T_S C.$$

Using Proposition 1, we conclude that

$$2(n-1)\,\kappa\,(\kappa - 1)\,I \in T_S C.$$

Since $0 < \kappa < 1$, it follows that $-2I \in T_S C$. On the other hand, I lies in the interior of the tangent cone $T_S C$. Hence, the sum $-2I + I = -I$ lies in the interior of the tangent cone $T_S C$. By Proposition 5.4 in [4], there exists a real number $\varepsilon > 0$ such that $S - \varepsilon I \in C$. Therefore, S lies in the interior of the cone C, as claimed. ■

Proposition 4 *The algebraic curvature tensor S defined in (1) vanishes identically.*

Proof By definition of κ, there exists a point $p_0 \in M$ such that $S \in \partial C$ at p_0. Hence, it follows from Proposition 3 that $\kappa = 1$. Consequently, the Ricci tensor of S vanishes identically. Since $S \in C$ for all points $p \in M$, we conclude that S vanishes identically. ∎

Since S vanishes identically, the manifold (M, g) has constant sectional curvature. This completes the proof of Theorem 3.

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Differential Harnack estimates for parabolic equations

Xiaodong Cao and Zhou Zhang

Abstract Let $(M, g(t))$ be a solution to the Ricci flow on a closed Riemannian manifold. In this paper, we prove differential Harnack inequalities for positive solutions of nonlinear parabolic equations of the type

$$\frac{\partial}{\partial t} f = \Delta f - f \ln f + Rf.$$

We also comment on an earlier result of the first author on positive solutions of the conjugate heat equation under the Ricci flow.

Keywords differential Harnack inequality, nonlinear parabolic equation, Ricci flow, Ricci soliton.
Mathematics Subject Classification (2010) 53C44.

1 Introduction

Let $(M, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold M. In the first part of this paper, we deal with positive solutions of nonlinear parabolic equations on M. We establish Li-Yau type differential Harnack inequalities for such

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positive solutions. More precisely, $g(t)$ evolves under the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2Rc, \tag{1}$$

where Rc denotes the Ricci curvature of $g(t)$. We first assume that the initial metric $g(0)$ has nonnegative curvature operator, which implies that for all time $t \in [0,T)$, $g(t)$ has nonnegative curvature operator (for example, in the case that dimension is 4, see [7]). Consider a positive function $f(x,t)$ defined on $M \times [0,T)$, which solves the following nonlinear parabolic equation,

$$\frac{\partial f}{\partial t} = \triangle f - f \ln f + Rf, \tag{2}$$

where the symbol $\triangle$ stands for the Laplacian of the evolving metric $g(t)$ and R is the scalar curvature of $g(t)$. For simplicity, we omit $g(t)$ in the above notations. All geometry operators are with respect to the evolving metric $g(t)$.

Differential Harnack inequalities were originated by P. Li and S.-T. Yau in [12] for positive solutions of the heat equation (therefore also known as Li-Yau type Harnack estimates). The technique was then brought into the study of geometric evolution equation by R. Hamilton (for example, see [8]) and has ever since been playing an important role in the study of geometric flows. Applications include estimates on the heat kernel; curvature growth control; understanding the ancient solutions for geometric flows; proving noncollapsing result in the Ricci flow ([17]); etc. See [16] for a recent survey on this subject by L. Ni.

Using the maximum principle, one can see that the solution for (2) remains positive along the flow. It exists as long as the solution for (1) exists. The study of the Ricci flow coupled with a heat-type (or backward heat-type) equation started from R. Hamilton's paper [9]. Recently, there has been some interesting study on this topic. In [17], G. Perelman proved a differential Harnack inequality for the fundamental solution of the conjugate heat equation under the Ricci flow. In [2], the first author proved a differential Harnack inequality for general positive solutions of the conjugate heat equation, which was also proved independently by S. Kuang and Q. S. Zhang in [11]. The study has also been pursued in [3, 6, 15, 20]. Various estimates are obtained recently by M. Bailesteanu, A. Pulemotov and the first author in [1], and by S. Liu in [13]. For nonlinear parabolic equations under the Ricci flow, local gradient estimates for positive solutions of equation

$$\frac{\partial}{\partial t} f = \Delta f + af \ln f + bf,$$

where a and b are constants, have been studied by Y. Yang in [19]. For general evolving metrics a similar estimate has been obtained by A. Chau, L.-F. Tam and C. Yu in [4], by S.-Y. Hsu in [10], and by J. Sun in [18]. In [14], L. Ma proved a gradient estimate for the elliptic equation

$$\Delta f + af \ln f + bf = 0.$$

In (2), if one defines

$$u(x,t) = -\ln f(x,t),$$

then the function $u = u(x,t)$ satisfies the following evolution equation

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 - R - u. \tag{3}$$

The computation from (2) to (3) is standard, which also gives the explicit relation between these two equations.

Our motivation to study (2) under the Ricci flow comes from the geometric interpretation of (3), which arises from the study of expanding Ricci solitons. Recall that given a gradient expanding Ricci soliton (M,g) satisfying

$$R_{ij} + \nabla_i \nabla_j w = -\frac{1}{4} g_{ij},$$

where w is called soliton potential function, we have

$$R(g) + \Delta_g w = -\frac{n}{4}.$$

In sight of this, by taking covariant derivative for the soliton equation and applying the second Bianchi identity, one can see that

$$R(g) + |\nabla_g w|_g^2 + \frac{w}{2} = \text{constant}.$$

Also notice that the Ricci soliton potential function w can be differed by a constant in the above equations. So by choosing this constant properly, we have

$$R(g) + |\nabla_g w|_g^2 = -\frac{w}{2} - \frac{n}{8}.$$

One consequence of the above identities is the following

$$|\nabla_g w|_g^2 = \Delta_g w - |\nabla_g w|_g^2 - R(g) - w. \tag{4}$$

Recall that the Ricci flow solution for an expanding soliton is $g(t) = c(t) \cdot \varphi(t)^* g$ (c.f. [5]), where $c(t) = 1 + \frac{t}{2}$ and the family of diffeomorphism $\varphi(t)$ satisfies, for any $x \in M$,

$$\frac{\partial}{\partial t}(\varphi(t)(x)) = \frac{1}{c(t)} \cdot (\nabla_g w)(\varphi(t)(x)).$$

Thus the corresponding Ricci soliton potential $\varphi(t)^* w$ satisfies

$$\frac{\partial \varphi(t)^* w}{\partial t}(x) = \frac{1}{c(t)}(\nabla_g w)(w)(\varphi(t)(x)) = |\nabla \varphi(t)^* w|^2(x).$$

Along the Ricci flow, (4) becomes

$$|\nabla \varphi^* w|^2 = \Delta \varphi^* w - |\nabla \varphi^* w|^2 - R - \frac{\varphi^* w}{c(t)}.$$

Hence the evolution equation for the Ricci soliton potential is

$$\frac{\partial \varphi(t)^* w}{\partial t} = \Delta \varphi^* w - |\nabla \varphi^* w|^2 - R - \frac{\varphi^* w}{c(t)}. \tag{5}$$

The second nonlinear parabolic equation that we investigate in this paper is

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 - R - \frac{u}{1 + \frac{t}{2}}. \tag{6}$$

Notice that (3) and (6) are closely related and only differ by their last terms.

Our first result deals with (2) and (3).

Theorem 1 *Let* $(M, g(t))$*,* $t \in [0, T)$*, be a solution to the Ricci flow on a closed manifold, and suppose that* $g(0)$ (*and so* $g(t)$) *has weakly positive curvature operator. Let* f *be a positive solution to the heat equation* (2)*,* $u = -\ln f$ *and*

$$H = 2\triangle u - |\nabla u|^2 - 3R - \frac{2n}{t}. \tag{7}$$

Then for all time $t \in (0, T)$

$$H \leqslant \frac{n}{4}.$$

Remark 1 The result can be generalized to the context of M being non-compact. In order for the same argument to work, we need to assume that the Ricci flow solution $g(t)$ is complete with the curvature and all the covariant derivatives being uniformly bounded and the solution u and its derivatives up to the second order are uniformly bounded *(*in the space direction*)* .

Our next result deals with (6), which is also a natural evolution equation to consider, by the previous motivation.

Theorem 2 *Let* $(M, g(t))$, $t \in [0, T)$, *be a solution to the Ricci flow on a closed manifold, and suppose that* $g(0)$ (*and so* $g(t)$) *has weakly positive curvature operator. Let* u *be a smooth solution to* (6), *and define*

$$H = 2\triangle u - |\nabla u|^2 - 3R - \frac{2n}{t}. \tag{7}$$

Then for all time $t \in (0, T)$

$$H \leqslant 0.$$

Remark 2 If f is a positive function such that $f = e^{-u}$, then f satisfies the following evolution equation

$$\frac{\partial f}{\partial t} = \triangle f + Rf - \frac{f \ln f}{1 + \frac{t}{2}}.$$

In [2], the first author studied the conjugate heat equation under the Ricci flow. In particular, the following theorem was proved.

Theorem 3 *[2, Theorem 3.6] Let* $(M, g(t))$, $t \in [0, T]$, *be a solution to the Ricci flow, and suppose that* $g(t)$ *has nonnegative scalar curvature. Let* f *be a positive solution of the conjugate heat equation*

$$\frac{\partial}{\partial t} f = -\triangle f + Rf.$$

Set $v = -\ln f - \frac{n}{2}\ln(4\pi\tau)$, $\tau = T - t$ *and*

$$P = 2\triangle v - |\nabla v|^2 + R - \frac{2n}{\tau}.$$

Then we have

$$\frac{\partial}{\partial \tau} P = \triangle P - 2\nabla P \cdot \nabla v - 2|v_{ij} + R_{ij} - \frac{1}{\tau} g_{ij}|^2 - \frac{2}{\tau} P - 2\frac{|\nabla v|^2}{\tau} - 2\frac{R}{\tau}. \tag{8}$$

Moreover, for all time $t \in [0, T)$,

$$P \leqslant 0.$$

In the last section, we apply a similar trick as in the proof of Theorem 1 and obtain a slightly different result, where we no longer need to assume that $g(t)$ has nonnegative scalar curvature.

2 Proof of Theorem 1 and Application

The evolution equation of u is very similar to what is considered in [3]. So the computation for the very general setting there can be applied.

Proof (Theorem 1) In sight of the definition of H from (7) and comparing with [3, Corollary 2.2], we have

$$\frac{\partial}{\partial t}(\triangle u) = \triangle(\triangle u) - \triangle(|\nabla u|^2) - \triangle R + 2R_{ij}u_{ij} - \Delta u,$$

$$\frac{\partial}{\partial t}|\nabla u|^2 = \triangle(|\nabla u|^2) - 2|\nabla\nabla u|^2 - 2\nabla u \cdot \nabla(|\nabla u|^2) - 2\nabla u \cdot \nabla R - 2|\nabla u|^2.$$

In fact, one can directly apply the computation result there with the only modification because of the extra terms coming from time derivative $\frac{\partial}{\partial t}u$, which are put at the end of the right hand side in the above equalities. Then we have

$$\begin{aligned}\frac{\partial}{\partial t}H =& \triangle H - 2\nabla H \cdot \nabla u - 2\left|u_{ij} - R_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}H - \frac{2}{t}|\nabla u|^2 \\ &- 2\left(\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_iu_j\right) - 2\Delta u + 2|\nabla u|^2,\end{aligned} \tag{9}$$

where the last two terms of the right hand side coming from the extra term $-u$ in (3). Plugging in $-2\Delta u + 2|\nabla u|^2 = -H + |\nabla u|^2 - 3R - \frac{2n}{t}$, one arrives at

$$\begin{aligned}\frac{\partial}{\partial t}H =& \triangle H - 2\nabla H \cdot \nabla u - 2\left|u_{ij} - R_{ij} - \frac{1}{t}g_{ij}\right|^2 - \left(\frac{2}{t} + 1\right)H \\ &+ \left(1 - \frac{2}{t}\right)|\nabla u|^2 - 3R - \frac{2n}{t} - \left(\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_iu_j\right).\end{aligned} \tag{10}$$

In sight of the definition of H (7), for t small enough, we have $H < 0$. Since g_{ij} has weakly positive curvature operator, by the trace Harnack inequality for the Ricci flow proved by R. Hamilton in [8], we have

$$\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_iu_j \geqslant 0.$$

Also we have $R \geqslant 0$. Notice that the term $\left(1 - \frac{2}{t}\right)|\nabla u|^2$ prevents us from obtaining an upper bound for H for $t > 2$.

We can deal with this by the following simple manipulation. To begin with, one observes that from the definition of H,

$$|\nabla u|^2 = 2\left(\Delta u - R - \frac{n}{t}\right) - H - R.$$

We also have the following equality from definition,

$$tr\left(u_{ij} - R_{ij} - \frac{1}{t}g_{ij}\right) = \Delta u - R - \frac{n}{t}.$$

Now we can continue the computation for the evolution of H as follows,

$$\begin{aligned}
\frac{\partial}{\partial t}H \leqslant & \triangle H - 2\nabla H\cdot\nabla u - 2|u_{ij} - R_{ij} - \frac{1}{t}g_{ij}|^2 - \left(\frac{2}{t}+1\right)H - \frac{2}{t}|\nabla u|^2 \\
& - 4R + 2\left(\Delta u - R - \frac{n}{t}\right) - H - \frac{2n}{t} \\
\leqslant & \triangle H - 2\nabla H\cdot\nabla u - \frac{2}{n}\left(\Delta u - R - \frac{n}{t}\right)^2 - \left(\frac{2}{t}+1\right)H - \frac{2}{t}|\nabla u|^2 \\
& - 4R + 2\left(\Delta u - R - \frac{n}{t}\right) - H - \frac{2n}{t} \\
= & \triangle H - 2\nabla H\cdot\nabla u - \left(\frac{2}{t}+2\right)H - \frac{2}{t}|\nabla u|^2 - 4R - \frac{2n}{t} \\
& - \frac{2}{n}\left(\Delta u - R - \frac{n}{t} - \frac{n}{2}\right)^2 + \frac{n}{2} \\
\leqslant & \triangle H - 2\nabla H\cdot\nabla u - \left(\frac{2}{t}+2\right)H - \frac{2}{t}|\nabla u|^2 - 4R - \frac{2n}{t} + \frac{n}{2}.
\end{aligned}$$

The essential step is the second inequality where we make use of the elementary inequality

$$|u_{ij} - R_{ij} - \frac{1}{t}g_{ij}|^2 \geqslant \frac{1}{n}\left(\Delta u - R - \frac{n}{t}\right)^2.$$

Now we can apply maximum principle. The value of H for very small positive t is clearly very negative. So we only need to consider the maximum value point is at $t > 0$ for the desired estimate.

For $\forall T_0 < T$, assume that the maximum in $(0, T_0]$ is taken at $t_0 > 0$. At the maximum value point, using the nonnegativity of $|\nabla u|^2$ and R, one has

$$H \leqslant \frac{-4n + nt_0}{4 + 4t_0} = \frac{n}{4}\left(1 - \frac{5}{t_0+1}\right) \leqslant \frac{n}{4}\left(1 - \frac{5}{T+1}\right).$$

So if $T \leqslant 4$, i.e., for time in $[0,4)$, $H \leqslant 0$. In general, we have

$$H \leqslant \frac{n}{4}.$$

Theorem 1 is thus proved. ■

As a consequence of Theorem 1, we have

Corollary 1 *Let $(M,g(t))$, $t\in[0,T)$, be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ (and so $g(t)$) has weakly positive curvature operator. Let f be a positive solution to the heat equation*

$$\frac{\partial}{\partial t}f=\Delta f-f\ln f+Rf.$$

Assume that (x_1,t_1) and (x_2,t_2), $0<t_1<t_2$, are two points in $M\times(0,T)$. Let

$$\Gamma=\inf_{\gamma}\int_{t_1}^{t_2}e^t\left(|\dot{\gamma}|^2+R+\frac{2n}{t}+\frac{n}{4}\right)dt,$$

where γ is any space-time path joining (x_1,t_1) and (x_2,t_2). Then we have

$$e^{t_1}\ln f(x_1,t_1)\leqslant e^{t_2}\ln f(x_2,t_2)+\frac{\Gamma}{2}.$$

This inequality is in the type of classical Harnack inequalities. The proof is quite standard by integrating the differential Harnack inequality. We include it here for completeness.

Proof Pick a space-time curve connecting (x_1,t_1) and (x_2,t_2), $\gamma(t)=(x(t),t)$ for $t\in[t_1,t_2]$. Recall that $u(x,t)=-\ln f(x,t)$. Using the evolution equation for u, we have

$$\begin{aligned}\frac{d}{dt}u(x(t),t)&=\frac{\partial u}{\partial t}+\nabla u\cdot\dot{\gamma}\\&=\Delta u-|\nabla u|^2-R-u+\nabla u\cdot\dot{\gamma}\\&\leqslant\Delta u-\frac{|\nabla u|^2}{2}-R-u+\frac{|\dot{\gamma}|^2}{2}.\end{aligned}\tag{11}$$

Now by Theorem 1, we have

$$\Delta u=\frac{1}{2}\left(H+|\nabla u|^2+3R+\frac{2n}{t}\right)\leqslant\frac{1}{2}\left(\frac{n}{4}+|\nabla u|^2+3R+\frac{2n}{t}\right).$$

So we have the following estimation,

$$\frac{d}{dt}u(x(t),t)\leqslant\frac{1}{2}\left(|\dot{\gamma}|^2+R+\frac{2n}{t}+\frac{n}{4}\right)-u.$$

For any space-time curve γ, we arrives at

$$\frac{d}{dt}(e^t \cdot u) \leqslant \frac{e^t}{2}\left(|\dot{\gamma}|^2 + R + \frac{2n}{t} + \frac{n}{4}\right).$$

Hence the desired Harnack inequality is proved by integrating t from t_1 to t_2. ■

3 Proof of Theorem 2

In this section we study u satisfying the evolution equation (6) originated from gradient expanding Ricci soliton equation. We investigate the same quantity

$$H = 2\triangle u - |\nabla u|^2 - 3R - \frac{2n}{t}$$

as in the last section. The evolution equation of u, is still very similar to what is considered in [3]. We have slightly different terms coming from time derivative $\frac{\partial}{\partial t}u$ when computing the evolution equation satisfied by H. Comparing with [3, Corollary 2.2], we proceed as follows.

Proof (Theorem 2) Direct computation gives the following equation. The modification from the computation of the reference is minor as illustrated in the proof of Theorem 1.

$$\begin{aligned}\frac{\partial}{\partial t}H =& \triangle H - 2\nabla H \cdot \nabla u - 2|u_{ij} - R_{ij} - \frac{1}{t}g_{ij}|^2 - \frac{2}{t}H - \frac{2}{t}|\nabla u|^2 \\ & -2\left(\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_iu_j\right) + \frac{2}{t+2}\left(-2\Delta u + 2|\nabla u|^2\right),\end{aligned} \tag{12}$$

where the last two terms of the right hand side come from the extra term $-\frac{u}{1+\frac{t}{2}}$ in (6). Plugging in $-2\Delta u + 2|\nabla u|^2 = -H + |\nabla u|^2 - 3R - \frac{2n}{t}$, one arrives at

$$\begin{aligned}\frac{\partial}{\partial t}H =& \triangle H - 2\nabla H \cdot \nabla u - 2|u_{ij} - R_{ij} - \frac{1}{t}g_{ij}|^2 - \left(\frac{2}{t} + \frac{2}{t+2}\right)H - \frac{6}{t+2}R \\ & + \left(\frac{2}{t+2} - \frac{2}{t}\right)|\nabla u|^2 - \frac{4n}{t^2+2t} - 2\left(\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_iu_j\right).\end{aligned} \tag{13}$$

By the definition of H, for t small enough, we have $H < 0$. Since $g(t)$ has weakly positive curvature operator, by the trace Harnack inequality for the Ricci flow ([8]), we have

$$\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_iu_j \geqslant 0.$$

Notice that now the coefficient for $|\nabla u|^2$ on the right hand side is $\frac{2}{t+2} - \frac{2}{t} < 0$, and we have $R \geqslant 0$. So one can conclude directly from maximum principle that $H \leqslant 0$. ∎

4 A Remark on the Conjugate Heat Equation

In this section we point out a simple observation for [2, Theorem 3.6]. The assumption on scalar curvature is not needed below. We follow the original set-up in [2].

Over a closed manifold M^n, $g(t)$ for $t \in [0,T]$ is a solution to the Ricci flow (1), and $f(\cdot,t)$ is a positive solution of the conjugate heat equation

$$\frac{\partial f}{\partial t} = -\Delta f + Rf, \tag{14}$$

where Δ and R are Laplacian and scalar curvature with respect to the evolving metric $g(t)$. Notice that $\int_M f(\cdot,t)d\mu_{g(t)}$ is a constant along the flow.

Set $v = -\log f - \frac{n\log(4\pi\tau)}{2}$, where $\tau = T - t$ and define

$$P := 2\Delta v - |\nabla v|^2 + R - \frac{2n}{\tau}.$$

Now we can prove the following result which is closely related to [2, Theorem 3.6].

Theorem 4 *Let $(M, g(t))$, $t \in [0,T]$, be a solution to the Ricci flow on a closed manifold. f is a positive solution to the conjugate heat equation* (14)*, and v is defined as above. Then we have*

$$\max_M \ (2\Delta v - |\nabla v|^2 + R)$$

increases along the Ricci flow.

Proof The exact computation in [2, Theorem 3.6] gives

$$\frac{\partial P}{\partial \tau} = \Delta P - 2\nabla P \cdot \nabla v - 2|\nabla^2 v + Rc - \frac{1}{\tau}g|^2 - \frac{2}{\tau}P - \frac{2}{\tau}|\nabla v|^2 - \frac{2}{\tau}R.$$

Applying the elementary inequality

$$|\nabla^2 v + Rc - \frac{1}{\tau}g|^2 \geqslant \frac{1}{n}\left(\Delta v + R - \frac{n}{\tau}\right)^2,$$

and noticing that

$$P+|\nabla v|^2+R=2\left(\Delta v+R-\frac{n}{\tau}\right),$$

we arrive at

$$\begin{aligned}\frac{\partial P}{\partial \tau} &\leqslant \Delta P-2\nabla P\cdot\nabla v-\frac{1}{2n}(P+|\nabla v|^2+R)^2-\frac{2}{\tau}(P+|\nabla v|^2+R)\\ &=\Delta P-2\nabla P\cdot\nabla v-\frac{1}{2n}\left(P+|\nabla v|^2+R+\frac{2n}{\tau}\right)^2+\frac{2n}{\tau^2}.\end{aligned}$$

Thus if one defines

$$\widetilde{P}:=P+\frac{2n}{\tau}=2\Delta v-|\nabla v|^2+R,$$

we have

$$\frac{\partial\widetilde{P}}{\partial\tau}\leqslant\Delta\widetilde{P}-2\nabla\widetilde{P}\cdot\nabla v.$$

Hence $\max_M(2\Delta v-|\nabla v|^2+R)$ decreases as τ increases, which means that it increases as t increases. This concludes the proof. ■

Remark 3 Notice that we do not need to introduce τ in *Theorem 4*, but we keep the notation here so it is easy to be compared with [2, Theorem 3.6].

Remark 4 Theorem 4 and [2, Theorem 3.6] estimate quantities differ by $\frac{2n}{\tau}$. Here we do not need to assume nonnegative scalar curvature as in [2, Theorem 3.6]. Moreover, one can also prove this result for complete non-compact manifolds with proper boundness assumption.

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Euler characteristic of a complete intersection

Sławomir Cynk*

Abstract In this paper we study the behaviour of the degree of the Fulton–Johnson class of a complete intersection under a blow–up with a smooth center under the assumption that the strict transform is again a complete intersection. Our formula is a generalization of the genus formula for singular curves in smooth surfaces.

Keywords Euler characteristic, complete intersection, singularities
Mathematics Subject Classification (2010) 14B05, 14E15.

1 Introduction

Let $Y = Y_1 \cap \cdots \cap Y_r$ be a complete intersection in a smooth algebraic n–fold X. If Y is smooth then the topological Euler characteristic of Y is uniquely determined by its cohomology class $[Y]$ and can be computed using the adjunction formula

$$\mathrm{e}(Y) = \deg \frac{c(\mathscr{T}_X) \cap [Y]}{c(\mathscr{N}_{Y|X})} = \int_X \sum_{i=0}^{n-r} \sum_{\substack{\alpha \in \mathbb{Z}^r \\ |\alpha| = n-i-r}} (-1)^{|\alpha|} c_i(X) [Y_1]^{\alpha_1+1} \ldots [Y_r]^{\alpha_r+1}.$$

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If Y is singular the situation becomes much more complicated, the above formula does not hold anymore. In fact the topological Euler characteristic is not determined by the cohomological classes $[Y_i]$. The number

$$\tilde{\mathrm{e}}(Y) = \int_X \sum_{i=0}^{n-r} \sum_{\substack{\alpha\in\mathbb{Z}^r\\ |\alpha|=n-i-r}} (-1)^{|\alpha|} c_i(X)[Y_1]^{\alpha_1+1}\dots[Y_r]^{\alpha_r+1}$$

can be considered as the "expected Euler characteristic" of Y and is equal to the degree of the Fulton–Johnson class c^{FJ} (see [10, Examp. 4.2.6] or [11]). The actual Euler characteristic $\mathrm{e}(Y)$ equals the degree of the Schwartz-MacPherson class c^{SM}. The difference (up to a sign convention) of these two numbers is called the Milnor number and is equal to the degree of the Milnor class $\mathscr{M}(Y) := (-1)^{\dim Y}(c^{\mathrm{FJ}}(Y) - c^{\mathrm{SM}}(Y))$. The notion of the Milnor number goes back to Milnor's work ([15]) where the formula for the Milnor number of isolated hypersurface singularities was given. Milnor number and Milnor class were studied by many authors (see f.i. [1, 3, 4, 16, 18]).

The aim of this paper is to give a method for computing the difference between the degree of the Fulton–Johnson class $\tilde{\mathrm{e}}(Y)$ and the Euler characteristic $\mathrm{e}(\tilde{Y})$ of a non–singular model $\tilde{Y}$ of Y. We shall consider a non–singular model satisfying the following property: there is a sequence of blowing-ups with smooth centers $\sigma : \tilde{X} \longrightarrow X$ such that $\tilde{Y} \subset \tilde{X}$ is the strict transform of Y and moreover it is the intersection of strict transforms $\tilde{Y}_i$ of Y_i (then $\tilde{Y}$ is also a complete intersection).

We shall study separately every blow–up. If $\sigma : \tilde{X} \longrightarrow X$ is a blow–up of a smooth (irreducible) subvariety $C \subset X$, then from the formula for Chern classes of a blow–up ([10, Thm. 15.4]) it follows (Prop. 1) that $\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y)$ is a polynomial in $c(\mathscr{N}_{C|X})$, $c(C)$, $[Y_i]\cap C$, $\mathrm{mult}_C Y_i$ (and the polynomial depends only on $\dim X$, $\dim Y$ and $\dim C$).

We do not write explicit formulae for $\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y)$ for arbitrary values of $\dim X$, $\dim Y$ and $\dim C$. The reason is that the general formula is very complicated even for small values of $\dim X$, $\dim Y$ and $\dim C$. We shall show instead that for explicit values of $\dim X$, $\dim Y$, $\dim C$ the formula for $\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y)$ can be written down explicitly. It is quite easy to compute those formulae with a computer.

Our method is based on expression of the top Fulton–Johnson class in terms of cohomologies of certain sheaves of differential forms on X with poles at a snc divisor supported on the sum of hypersurfaces intersecting at Y. This allows us to consider a smooth ambient space and then study the behavior of those sheaves under blow–up at a smooth subvariety. We used a similar method to study the Hodge numbers of certain hypersurfaces (cf. [5, 6, 8, 19])

Under the additional assumption that the projectivized normal cone $\mathbb{P}(\mathscr{N}_{C|Y})$ to Y in C is a complete intersection of the normal cones $\mathbb{P}(\mathscr{N}_{C|Y_i})$ to Y_i in C we can in a similar manner compute its Euler characteristic $e(\mathbb{P}(\mathscr{N}_{C|Y}))$ and so also $e(\tilde{Y}) - e(Y)$. In this case we can compute not only the Euler characteristic of a smooth model of Y but also of Y itself. If we can find a stratification of C such that over every stratum the Euler characteristic of the fiber of the projection $\mathbb{P}(\mathscr{N}_{C|Y}) \longrightarrow C$ is constant, then the computation of $e(\mathbb{P}(\mathscr{N}_{C|Y}))$ is reduced to computation of Euler characteristics of subvarieties of $\mathbb{P}^{n-\dim C-1}$.

In fact we study the homological Euler characteristic $\chi(\Omega^i_{\tilde{Y}})$, so using similar methods we can compute other invariants of $\tilde{Y}$ like f.i. arithmetic genus, signature or more generally the χ_y–genus.

2 Blow–up of the Fulton–Johnson class

Let X be a complete smooth algebraic manifold (over $\mathbb{C}$) of dimension n and let $Y_1, \dots, Y_r \subset X$ be reduced divisors such that $Y = Y_1 \cap \dots \cap Y_r$ is a complete intersection (i.e. Y is reduced and $\dim Y = n - r$). Consider $\sigma : \tilde{X} \longrightarrow X$ a blow–up of a smooth subvariety $C \subset X$ of dimension d, let E be the exceptional divisor of σ. Denote by $\tilde{Y}_i := \sigma^* Y_i - m_i E$ the strict transform of Y_i by σ, where $m_i = \mathrm{mult}_C Y_i$. Assume that $\tilde{Y} = \tilde{Y}_i \cap \dots \cap \tilde{Y}_r$ is also a complete intersection.

Proposition 1 *For any natural numbers n, r, d there exists a polynomial $W_{n,r,d}$ with integer coefficients s.t.*

$$\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y) = \int_C W_{n,r,d}\left(c(\mathscr{N}_{C|X}), c(C), [Y_1] \cap C, \dots, [Y_r] \cap C, m_1, \dots, m_r\right).$$

Proof We have

$$
\begin{aligned}
&\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y) = \\
&= \int_{\tilde{X}} \sum_{i=0}^{n-r} \sum_{\substack{\alpha\in\mathbb{Z}^r \\ |\alpha|=n-i-r}} (-1)^{|\alpha|} c_i(\tilde{X}) [\tilde{Y}_1]^{\alpha_1+1} \dots [\tilde{Y}_r]^{\alpha_r+1} - \\
&- \int_{X} \sum_{i=0}^{n-r} \sum_{\substack{\alpha\in\mathbb{Z}^r \\ |\alpha|=n-i-r}} (-1)^{|\alpha|} c_i(X) [Y_1]^{\alpha_1+1} \dots [Y_r]^{\alpha_r+1} = \\
&= \int_{\tilde{X}} \Bigg(\sum_{i=0}^{n-r} \sum_{\substack{\alpha\in\mathbb{Z}^r \\ |\alpha|=n-i-r}} (-1)^{|\alpha|} c_i(\tilde{X}) [\tilde{Y}_1]^{\alpha_1+1} \dots [\tilde{Y}_r]^{\alpha_r+1} - \\
&- \sigma^* \Big(\sum_{i=0}^{n-r} \sum_{\substack{\alpha\in\mathbb{Z}^r \\ |\alpha|=n-i-r}} (-1)^{|\alpha|} c_i(X) [Y_1]^{\alpha_1+1} \dots [Y_r]^{\alpha_r+1} \Big) \Bigg)
\end{aligned} \tag{1}
$$

By [10, Thm. 15.4]

$$c(\tilde{X}) - \sigma^* c(X) = i_*[\rho^* c(C) \cdot \alpha],$$

where $\rho : E = \mathbb{P}(\mathscr{N}_{C|X}) \longrightarrow C$ is the natural projection (and the restriction of σ), $i : E \longrightarrow \tilde{X}$ is the inclusion and

$$\alpha = \frac{1}{\zeta} \left[\sum_{k=0}^{n-d} \rho^* c_{n-d-k}(\mathscr{N}_{C|X}) - (1-\zeta) \sum_{k=0}^{n-d} (1+\zeta)^k \rho^* c_{n-d-k}(\mathscr{N}_{C|X}) \right],$$

($\zeta = c_1(\mathscr{O}_E(1))$ is the generator of the cohomology ring of $\mathbb{P}(E)$), substituting the above and $\tilde{Y}_i = \sigma^* Y_i - m_i E$ into (1) and using the properties of the cohomology ring $H^*(\mathbb{P}(E))$ we get the required assertion.

Remark 1 The same result holds in fact for the total class $c^{\mathrm{FJ}}(\tilde{Y}) - \sigma^* c^{\mathrm{FJ}}(Y)$.

Remark 2 The polynomial $W_{n,r,d}$ is isobaric of degree d in $c(\mathscr{N}_{C|X})$, $c(C)$, $[Y_i] \cap C$ and of degree less than or equal to n in $m_1, \dots, m_r$.

3 Differential forms

The computations in the proof of Proposition 1 use calculations in the cohomology ring $H^*(\mathbb{P}^3(E))$, and so they are not suitable for deriving explicit formulae. In this section we shall use another approach. The Euler characteristic of a smooth complete variety Y can be expressed in terms of sheaves of differential forms on Y

$$\mathrm{e}(Y) = \sum_i (-1)^i \chi(\Omega_Y^i).$$

This formula follows either from the Hodge decomposition or

$$c_{\dim Y}(Y) = \sum_i (-1)^i \operatorname{ch}(\Omega_Y^i)$$

and the Hirzebruch–Riemann–Roch theorem ([13, Thm. 21.1.1]).

If Y is a smooth hypersurface in a smooth projective variety X then for any $p \geq 1$ we have the following exact sequences ([9, Prop. 2. 3])

$$\begin{aligned}
&0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p(\log Y) \longrightarrow \Omega_Y^{p-1} \longrightarrow 0\\
&0 \longrightarrow \Omega_X^p(\log Y)(-Y) \longrightarrow \Omega_X^p \longrightarrow \Omega_Y^p \longrightarrow 0,
\end{aligned}$$

where $\Omega_X^p(\log Y)$ is the (locally free) sheaf of differential forms with logarithmic poles along Y and $\Omega_X^p(\log Y)(-Y) = \Omega_X^p(\log Y) \otimes \mathscr{O}_X(-Y)$. The map $\Omega_X^p(\log Y) \longrightarrow \Omega_Y^{p-1}$ is given by the Poincare residue, $\Omega_X^p \longrightarrow \Omega_Y^p$ is the restriction whereas $\Omega_X^p \longrightarrow \Omega_X^p(\log Y)$ and $\Omega_X^p(\log Y)(-Y) \longrightarrow \Omega_X^p$ are inclusions.

Playing with the above exact sequences we get

$$\begin{aligned}
&\chi(\mathscr{O}_Y) = \chi(\mathscr{O}_X) - \chi(\mathscr{O}_X(-Y))\\
&\chi(\Omega_Y^1) = \chi(\Omega_X^1) - \chi(\Omega_X^1(-Y)) - \chi(\mathscr{O}_X(-Y)) + \chi(\mathscr{O}_X(-2Y)).
\end{aligned}$$

and more generally, for any locally free sheaf $\mathscr{F}$ on X and any $p = 0, \ldots, n-1$

$$\chi(\Omega_Y^p \otimes \mathscr{F}) = \sum_{q=0}^{p} (-1)^q \left[\chi(\Omega_X^{p-q}(-qY) \otimes \mathscr{F}) - \chi(\Omega_X^{p-q}(-(q+1)Y) \otimes \mathscr{F})\right].$$

If $Y = Y_1 \cap \cdots \cap Y_r$ is a transversal intersection of smooth divisors then we can inductively get a representation of $\mathrm{e}(Y)$ as a linear combination (with integer coefficients) of $\chi(\Omega_X^p(-(q_1Y_1 + \cdots + q_rY_r)))$.

Observe that all the summands in the above formula make sense for any divisors $Y_1, \ldots, Y_r$, not necessarily smooth. If Y is an arbitrary complete intersection the above formula gives the degree of the Fulton–Johnson class $\tilde{\mathrm{e}}(\tilde{Y})$. Our goal is to compute $\tilde{e}(\tilde{Y}) - \tilde{e}(Y)$, in order to do this we have to study

$$\chi(\Omega_{\tilde{X}}^1(-(q_1\tilde{Y}_1 + \cdots + q_r\tilde{Y}_r)) - \chi(\Omega_X^1(-(q_1Y_1 + \cdots + q_rY_r))),$$

so it is enough to compute for an effective line bundle $\mathscr{L}$ on X and non–negative integers $m,\ p$ the numbers

$$D_p(\mathscr{L},m) := \chi(\Omega^p_{\tilde{X}} \otimes \sigma^*\mathscr{L}^{-1} \otimes \mathscr{O}_{\tilde{X}}(mE)) - \chi(\Omega^p_X \otimes \mathscr{L}^{-1}).$$

Let $k = n-d$ be the codimension of C in X. Denote by $\mathscr{N} := \mathscr{N}_{C|Y}$ the normal bundle of C in Y, by $\mathscr{N}^\vee$ its dual and by $\mathrm{S}^l\mathscr{N}$ the l–th symmetric power of $\mathscr{N}$. We have the following obvious relations

Proposition 2

$$\begin{array}{ll}
\sigma_*\mathscr{O}_{\tilde{X}} \cong \mathscr{O}_X, & \\
R^i\sigma_*\mathscr{O}_{\tilde{X}} = 0 & \textit{for } i > 0, \\
\mathscr{O}_{\tilde{X}}(E)\otimes E \cong \mathscr{O}_E(-1). & \\
\sigma_*(\mathscr{O}_E(l)) \cong \mathrm{S}^l\mathscr{N}^\vee, & \textit{for } l \geq 0, \\
\sigma_*(\mathscr{O}_E(l)) = 0, & \textit{for } l < 0, \\
R^i\sigma_*(\mathscr{O}_E(l)) = 0, & \textit{for } i \neq 0, k-1, \\
R^{k-1}\sigma_*(\mathscr{O}_E(l)) = 0, & \textit{for } l > -k, \\
R^{k-1}\sigma_*(\mathscr{O}_E(l)) \cong \mathrm{S}^{-l-k}\mathscr{N}\otimes \bigwedge^k\mathscr{N}, & \textit{for } l \leq -k.
\end{array}$$

Moreover the following "relative Euler sequence"

$$0 \longrightarrow \Omega^p_{E/C} \longrightarrow \sigma^*(\textstyle\bigwedge^p\mathscr{N}^\vee)\otimes\mathscr{O}_E(-p) \longrightarrow \Omega^{p-1}_{E/C} \longrightarrow 0$$

is exact.

From the above Proposition and the projection formula we get the following

Corollary 1 $R^i\sigma_*\Omega^p_{E/C}(l) = 0$ *unless*

(i) $i = p$ *and* $l = 0$*,*

(ii) $i = 0$ *and* $l \geq p+1$*,*

(iii) $i = k-1$ *and* $l \leq -k$*.*

Moreover

$$
\begin{aligned}
&R^i\sigma_*\Omega^i_{E/C} \cong \mathcal{O}_C,\\
&0 \longrightarrow \sigma_*\Omega^p_{E/C}(l) \longrightarrow \textstyle\bigwedge^p \mathcal{N}^\vee \otimes S^{l-p}\mathcal{N}^\vee \longrightarrow\\
&\quad \longrightarrow \textstyle\bigwedge^{p-1} \mathcal{N}^\vee \otimes S^{l-p+1}\mathcal{N}^\vee \longrightarrow \ldots \longrightarrow S^l\mathcal{N}^\vee \longrightarrow 0,\\
&0 \longrightarrow R^{k-1}\sigma_*\Omega^p_{E/C}(l) \longrightarrow \textstyle\bigwedge^p \mathcal{N}^\vee \otimes S^{p-l-k}\mathcal{N}^\vee \otimes \bigwedge^k \mathcal{N} \longrightarrow\\
&\qquad\qquad \longrightarrow \textstyle\bigwedge^{p-1} \mathcal{N}^\vee \otimes S^{p-l-k-1}\mathcal{N}^\vee \otimes \bigwedge^k \mathcal{N} \longrightarrow \ldots\\
&\qquad\qquad\qquad\qquad \ldots \longrightarrow S^{-l-k}\mathcal{N} \otimes \textstyle\bigwedge^k \mathcal{N} \longrightarrow 0.
\end{aligned}
$$

Proposition 3 *For a non–negative integer m we have*

1. $\sigma_*\mathcal{O}_{\tilde{X}}(mE) \cong \mathcal{O}_X$,
2. $R^{k-1}\sigma_*\mathcal{O}_{\tilde{X}}(mE) \cong \bigoplus_{j=0}^{m-k} S^j(\mathcal{N}) \otimes \bigwedge^k \mathcal{N}$,
3. $R^i\sigma_*\mathcal{O}_{\tilde{X}}(mE) = 0$, *for* $i \neq 0, k-1$.

Proof The case $m = 0$ is obvious. The general case follows by induction from the previous Proposition and the direct image functor applied to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}((m-1)E) \longrightarrow \mathcal{O}_{\tilde{X}}(mE) \longrightarrow \mathcal{O}_E(-m) \longrightarrow 0.$$

We shall compute $D_p(\mathcal{L}, m)$ using the Leray spectral sequence, so we need to study the direct images $R^i\sigma_*\Omega^p_{\tilde{X}}$, for which the description of $\sigma^*\Omega^p_X$ is crucial. If σ is a blow–up of a point then

$$\sigma^*\Omega^p_X \cong \Omega^p_{\tilde{X}}(\log E)(-pE).$$

and so we have for $p = 0,\ 1$

$$
\begin{aligned}
&\sigma^*\mathcal{O}_X \cong \mathcal{O}_{\tilde{X}}\\
&0 \longrightarrow \sigma^*\Omega^1_X \longrightarrow \Omega^1_{\tilde{X}} \longrightarrow \Omega^1_{E/C} \longrightarrow 0.
\end{aligned}
$$

The latter follows easily from the following exact sequences

$$
\begin{gathered}
0 \longrightarrow \Omega^1_{\tilde{X}}(\log E)(-E) \longrightarrow \Omega^1_{\tilde{X}} \longrightarrow \Omega^1_E \longrightarrow 0\\
0 \longrightarrow \Omega^1_{\tilde{X}}(\log E)(-E) \longrightarrow \sigma^*\Omega^1_X \longrightarrow \sigma^*\Omega^1_C \longrightarrow 0\\
0 \longrightarrow \sigma^*\Omega^1_C \longrightarrow \Omega^1_E \longrightarrow \Omega^1_{E/C} \longrightarrow 0
\end{gathered}
$$

For higher values of p the formulae become much more complicated. For $p=2$ we have

$$0 \longrightarrow \Omega^2_{\tilde{X}}(\log E)(-2E) \longrightarrow \Omega^2_{\tilde{X}}(-E) \longrightarrow \Omega^2_E(1) \longrightarrow 0$$
$$0 \longrightarrow \Omega^2_{\tilde{X}}(-E) \longrightarrow \Omega^2_{\tilde{X}}(\log E)(-E) \longrightarrow \Omega^1_E(1) \longrightarrow 0$$
$$0 \longrightarrow \Omega^2_{\tilde{X}}(\log E)(-E) \longrightarrow \Omega^2_{\tilde{X}} \longrightarrow \Omega^2_E \longrightarrow 0$$

The kernel of the map $\Omega^2_E \longrightarrow \Omega^2_{E/C}$ contains $\sigma^*\Omega^2_C$, which is a proper submodule with quotient isomorphic to $\Omega^1_{E/C}\otimes\sigma^*\Omega^1_C$. Consequently we can write the following relation in the Grothendieck K_0 group

$$\begin{aligned}[\Omega^2_{\tilde{X}}] = {} & [\Omega^2_{\tilde{X}}(\log E)(-2E)] + [\Omega^2_{E/C}(1)] + [\sigma^*\Omega^2_C(1)] + [\Omega^1_{E/C}(1)\otimes\sigma^*\Omega^1_C] + \\ & + \chi(\Omega^1_{E/C}(1)) + \chi(\sigma^*\Omega^1_C(1)) + \chi(\Omega^2_{E/C}) + \chi(\sigma^*\Omega^2_C) + \chi(\Omega^1_{E/C}\otimes\sigma^*\Omega^1_C)\end{aligned}$$

In a similar way the kernel of the map $\sigma^*\Omega^2_{\tilde{X}} \longrightarrow \Omega^2_{\tilde{X}}(\log E)(-2E)$ contains $\sigma^*\Omega^1_C(1)\oplus\sigma^*\Omega^2_C$ with quotient isomorphic to $\Omega^1_{E/C}(1)\otimes\sigma^*\Omega^1_C\oplus\sigma^*\Omega^2_C(1)$.

Putting the above formulae together we get

$$[\Omega^2_{\tilde{X}}] - [\sigma^*\Omega^2_X] = [\Omega^1_{E/C}(1)] + [\Omega^2_{E/C}] + [\Omega^2_{E/C}(1)] + [\Omega^1_{E/C}\otimes\sigma^*\Omega^1_C]. \qquad (2)$$

The above formula can be also easily verified in local coordinates, if we denote by $x_1,\dots,x_k,y_{k+1},\dots,y_n$ local coordinates in X such that $x_1,\dots,x_k$ is a local equation of C and consider the affine chart on $\tilde{X}$ in which the blow–up σ is given by

$$\sigma : (x_1,x_{2,},\dots,x_k,y_{k+1},\dots,y_n) \mapsto (x_1,x_1x_2,\dots,x_1x_k,y_{k+1},\dots,y_n).$$

Now, $\sigma^*\Omega^2_X$ is a locally free module generated (locally) by $x_1dx_1\wedge dx_i (i>1)$, $x_1^2dx_i\wedge dx_j (i<j)$, $dx_1\wedge dy_i, x_1dx_i\wedge dy_j (i>1)$, $dy_i\wedge dy_j (i<j)$. So the quotient $\Omega^2_{\tilde{X}}/\sigma^*\Omega^2_X$ (which is supported on E) is a locally free sheaf generated (locally) by restrictions to E of $dx_1\wedge dx_i(i>1)$, $dx_i\wedge dx_j(i<j)$, $dx_i\wedge dx_j(i<j)$ twisted by the conormal bundle of E, $dx_i\wedge dy_j(i>1)$. Although we can recognise the above forms as generators of the vector bundles on the right-hand side of the above formula, we cannot write the quotient as the direct sum because in general Ω^1_E does not split as the direct sum of Ω^1_C and $\Omega^1_{E/C}$. Consequently we have to write the formula in the Grothendieck K_o group.

In the same way we can prove the following formula for any $p \geq 0$:

Proposition 4

$$[\Omega^p_{\tilde{X}}] = [\sigma^*\Omega^p_X] + \sum_{k=1}^{p-1}\sum_{i=1}^{k}[\Omega^k_{E/C}(i)\otimes\sigma^*\Omega_C^{p-k-1}] + \sum_{k=1}^{p}\sum_{i=0}^{k-1}[\Omega^k_{E/C}(i)\otimes\sigma^*\Omega_C^{p-k}].$$

Using the above Proposition we can easily get formulae for $D_p(\mathscr{L},0)$. We shall write down only formulae for $p \leq 2$, as for bigger values of p they become more complicated.

Theorem 5 *Let $m > 0$ be a positive integer.*

(1) $D_0(\mathscr{L},0) = 0,$

(2) $D_1(\mathscr{L},0) = -\chi(\mathscr{L}^{-1}\otimes\mathscr{O}_C),$

(3) $D_2(\mathscr{L},0) = \begin{cases} -\chi(\mathscr{L}^{-1}\otimes\Omega^1_C) & \textit{if} \quad k=2 \\ -\chi(\mathscr{L}^{-1}\otimes\Omega^1_C)+\chi(\mathscr{L}^{-1}\otimes\mathscr{O}_C) & \textit{if} \quad k>2 \end{cases}$

(4) $D_0(\mathscr{L},m) = (-1)^{k-1}\sum_{j=0}^{m-k}\chi(\mathscr{L}^{-1}\otimes \mathrm{S}^j\mathscr{N}\otimes\bigwedge^k\mathscr{N}),$

(5) $$D_1(\mathscr{L},m) = (-1)^{k-1}\left[\sum_{j=0}^{m-k}\chi(\Omega^1_C\otimes\mathscr{L}^{-1}\otimes \mathrm{S}^j\mathscr{N}\otimes\bigwedge^k\mathscr{N})+ \right.$$
$$\left. + \sum_{j=0}^{m-k+1}\chi(\mathscr{N}^\vee\otimes\mathscr{L}^{-1}\otimes \mathrm{S}^j\mathscr{N}\otimes\bigwedge^k\mathscr{N}) - \chi(\mathscr{L}^{-1}\otimes \mathrm{S}^{m-k}\mathscr{N}\otimes\bigwedge^k\mathscr{N})\right]$$

(6) $$D_2(\mathscr{L},m) = (-1)^{k-1}\left[\sum_{j=0}^{m-k}\chi(\Omega^2_C\otimes\mathscr{L}^{-1}\otimes \mathrm{S}^j\mathscr{N}\otimes\bigwedge^k\mathscr{N})+ \right.$$
$$+ \sum_{j=0}^{m-k+1}\chi(\mathscr{N}^\vee\otimes\Omega^1_C\otimes\mathscr{L}^{-1}\otimes \mathrm{S}^j\mathscr{N}\otimes\bigwedge^k\mathscr{N})+$$
$$+ \sum_{j=0}^{m-k+2}\chi(\bigwedge^2\mathscr{N}^\vee\otimes\mathscr{L}^{-1}\otimes \mathrm{S}^j\mathscr{N}\otimes\bigwedge^k\mathscr{N})+$$
$$+\chi(\mathscr{L}^{-1}\otimes \mathrm{S}^{m-k}\mathscr{N}\otimes\bigwedge^k\mathscr{N}) - \chi(\mathscr{N}^\vee\otimes S^{m-k+1}\mathscr{N}\otimes\bigwedge^k\mathscr{N}\otimes\mathscr{L}^{-1})$$
$$\left. -\chi(\mathscr{L}^{-1}\otimes \mathrm{S}^{m-k}\mathscr{N}\otimes\bigwedge^k\mathscr{N}\otimes\Omega^1_C)\right]$$

Proof We shall give proofs of the most complicated assertions (3) and (6), proofs of other assertions are similar (but much simpler). Tensoring the equality (2) with the line bundle $\sigma^*\mathscr{L}^{-1}\otimes\mathscr{O}_{\tilde{X}}(mE)$ we get

$$[\sigma^*(\Omega^2_X \otimes \mathscr{L}^{-1}) \otimes \mathscr{O}_{\tilde{X}}(mE)] = [\Omega^2_{\tilde{X}} \otimes \sigma^* \mathscr{L}^{-1} \otimes \mathscr{O}_{\tilde{X}}(mE)] +$$
$$+[\Omega^1_{E/C}(1-m) \otimes \sigma^* \mathscr{L}^{-1}] + [\Omega^2_{E/C}(-m) \otimes \sigma^* \mathscr{L}^{-1}] +$$
$$+[\Omega^2_{E/C}(1-m) \otimes \sigma^* \mathscr{L}^{-1}] + [\Omega^1_{E/C}(-m) \otimes \sigma^*(\Omega^1_C \otimes \mathscr{L}^{-1})].$$

Compute the Euler characteristic, apply the direct image functor and use Corollary 1. Observe, that we have to consider separately the cases $m = 0$ and $m = 1$, because then we have to compute $R^1\sigma_*\Omega^1_{E/C}$ and $R^2\sigma_*\Omega^2_{E/C}$. The assertion follows by simple computations.

Remark 3 The numbers on the righthand sides of the formulae in the above Theorem represent the holomorphic Euler characteristics of certain locally free sheaves on C. They can be computed using the Hirzebruch–Riemann–Roch Theorem, so the numbers can be expressed (for fixed m) in terms of the Chern classes of the center of the blowing–up C, the normal bundle $\mathscr{N}_{C|X}$ and the restriction to C of the line bundle $\mathscr{L}$.

Example 1 If C is a point then the holomorphic Euler characteristic of any locally free sheaf on C equals its rank, the sheaf Ω^p_C of p–forms ($p > 0$) on C is a zero–sheaf. Simple computations yields

$$D_0(\mathscr{L},m) = (-1)^{k-1}\binom{m}{k}$$
$$D_1(\mathscr{L},m) = (-1)^{k-1}\left[k\binom{m+1}{k} - \binom{m-1}{k-1}\right]$$
$$D_2(\mathscr{L},m) = (-1)^{k-1}\left[\binom{k}{2}\binom{m+2}{k} + \binom{m-1}{k-1} - k\binom{m}{k-1}\right]$$

Example 2 If C is a curve then the holomorphic Euler characteristic of any locally free sheaf $\mathscr{F}$ on C equals by the Riemann–Roch theorem $\deg(c_1(\mathscr{F})) + r(1-g)$, where r is the rank of $\mathscr{F}$, g is the genus of C. Moreover we have $c_1(\mathscr{T} \otimes \mathscr{F}) = \mathrm{rank}\mathscr{F} \cdot c_1(\mathscr{T}) + \mathrm{rank}\mathscr{T} \cdot c_1(\mathscr{F})$, $\mathrm{rank}S^j\mathscr{N} = \binom{j+2}{2}$ and $c_1(S^j\mathscr{N}) = \binom{j+2}{3}c_1(\mathscr{N})$.

If X is a threefold (i.e. the codimension k of the curve C in X is 2) simple computations give

$$\begin{aligned}D_0(\mathscr{L},m) = &\ \frac{1}{2}(m^2-m)(c_1(\mathscr{L})\cdot C)+\frac{1}{6}(-m^3+m)c_1(\mathscr{N})+\\ &+\frac{1}{4}(-m^2+m)c_1(C),\\ D_1(\mathscr{L},m) = &\ \frac{1}{2}(3m^2-m+2)(c_1(\mathscr{L})\cdot C)+\frac{1}{2}(-m^3-m)c_1(\mathscr{N})+\\ &+\frac{1}{4}(-m^2-m-2)c_1(C)\end{aligned}$$

Similarly if $k=3$ (i.e. C is a curve in a fourfold X) then

$$\begin{aligned}D_0(\mathscr{L},m) = &\ \frac{1}{6}(-m^3+3m^2-2m)(c_1(\mathscr{L})\cdot C)+\\ &+\frac{1}{24}(m^4-2m^3-m^2+2m)c_1(\mathscr{N})+\frac{1}{12}(m^3-3m^2+2m)c_1(C),\\ D_1(\mathscr{L},m) = &\ \frac{1}{3}(-2m^3+3m^2-4m+3)(c_1(\mathscr{L})\cdot C)+\\ &+\frac{1}{6}(m^4-m^3+2m^2-2m)c_1(\mathscr{N})+\frac{1}{6}(m^3+2m-3)c_1(C)\end{aligned}$$

Already for the case of C being a surface the formulae become very complicated and it is very difficult to write them down directly. On the other hand for m big they are integer valued polynomials in m, the Riemann–Roch theorem allows to represent them as polynomials in Chern roots of $\mathscr{N}$, $\mathscr{T}_C$ and $\mathscr{O}_X(Y)$ so we can use computers to find $D_p(\mathscr{L},m)$ for a few values of m and, then interpolate this to get the formula for arbitrary m.

Example 3 Using this approach we computed with a computer that for $\dim(C)=2$, $\dim(X)=4$

$$\begin{aligned}&D_0(\mathscr{L},m) = \left(\tfrac{1}{24}m^4-\tfrac{1}{12}m^3-\tfrac{1}{24}m^2+\tfrac{1}{12}m\right)c_2(\mathscr{N})+\\ &+\left(-\tfrac{1}{24}m^2+\tfrac{1}{24}m\right)c_2(C)+\left(-\tfrac{1}{24}m^2+\tfrac{1}{24}m\right)c_1^2(C)+\\ &+\left(-\tfrac{1}{4}m^2+\tfrac{1}{4}m\right)c_1^2(\mathscr{L})C+\left(\tfrac{1}{6}m^3-\tfrac{1}{6}m\right)c_1(\mathscr{L})c_1(\mathscr{N})+\\ &+\left(\tfrac{1}{4}m^2-\tfrac{1}{4}m\right)c_1(\mathscr{L})c_1(C)+\left(-\tfrac{1}{24}m^4+\tfrac{1}{24}m^2\right)c_1^2(\mathscr{N})+\\ &+\left(-\tfrac{1}{12}m^3+\tfrac{1}{12}m\right)c_1(C)c_1(\mathscr{N})\\ &D_1(\mathscr{L},m) = \left(\tfrac{1}{6}m^4-\tfrac{1}{6}m^3+\tfrac{5}{6}m^2+\tfrac{1}{6}m\right)c_2(\mathscr{N})+\\ &+\left(\tfrac{1}{3}m^2-\tfrac{5}{12}m-\tfrac{1}{12}\right)c_2(C)+\left(-\tfrac{1}{6}m^2+\tfrac{1}{12}m-\tfrac{1}{12}\right)c_1^2(C)+\\ &+\left(-m^2+\tfrac{1}{2}m-\tfrac{1}{2}\right)c_1^2(\mathscr{L})C+\left(\tfrac{2}{3}m^3+\tfrac{1}{3}m\right)c_1(\mathscr{L})c_1(\mathscr{N})+\\ &+\left(\tfrac{1}{2}m^2+\tfrac{1}{2}\right)c_1(\mathscr{L})c_1(C)+\left(-\tfrac{1}{6}m^4-\tfrac{1}{3}m^2\right)c_1^2(\mathscr{N})+\\ &+\left(-\tfrac{1}{6}m^3-\tfrac{1}{3}m\right)c_1(C)c_1(\mathscr{N})\end{aligned}$$

4 Euler characteristic computation

Using the results from previous sections we can write down the formula for the number $\tilde{e}(\tilde{Y})-\tilde{e}(Y)$. The actual formula depends on the dimensions of X, Y and C. Let us first consider the case of a hypersurface (i.e. $r=1$). We have $e(Y)=\sum_i(-1)^i\chi(\Omega^i_Y)$, moreover by Serre duality $\chi(\Omega^i_Y)=(-1)^{n-1}\chi(\Omega^{n-1-i}_Y)$ for any smooth variety of dimension $n-1$.

4.1 Hypersurface

Theorem 6 *If Y is a surface in a threefold X and m is the multiplicity of Y along C then*

$$\tilde{e}(\tilde{Y})-\tilde{e}(Y)=\begin{cases}-m^3+2m^2 & \textit{if } \dim C=0\\ (3m^2-2m-1)YC+(-m^3+1)c_1(\mathscr{N})+ & \textit{if } \dim C=1\\ +(-m^2+m)c_1(C) & \end{cases}$$

Proof Since Y is a surface we have

$$\tilde{e}(Y)=2\chi(\mathscr{O}_X)-\chi(\mathscr{O}_X(-Y))-\chi(\mathscr{O}_X(-2Y))-\chi(\Omega^1_X)+\chi(\Omega^1_X(-Y))$$

and consequently

$$\tilde{e}(\tilde{Y})-\tilde{e}(Y)=\chi(\mathscr{O}_C)-D_0(\mathscr{L},m)-D_0(\mathscr{L}^{\otimes 2},2m)+D_1(\mathscr{L},m),$$

where $\mathscr{L}=\mathscr{O}_X(Y)$.

Theorem 7 *If Y is a threefold in a fourfold X then*

$$\tilde{e}(\tilde{Y})-\tilde{e}(Y)=\begin{cases}m^4-3m^3+2m^2+2m & \textit{if } \dim C=0\\ (-m^3+2m^2)c_1(C)+(-m^4+m^3+ & \textit{if } \dim C=1\\ +m^2-m)c_1(\mathscr{N})+(4m^3-6m^2+2)YC & \\ \left(-m^4+m^3+2m^2\right)c_2(\mathscr{N})+ & \textit{if } \dim C=2\\ +\left(m^2-m\right)c_2(C)+\left(6m^2-3m-1\right)Y^2C+ & \\ +\left(-4m^3+2m\right)Yc_1(\mathscr{N})+\left(-3m^2+2m+1\right)\,Yc_1(C)+ & \\ +\left(m^4-m^2\right)c_1^2(\mathscr{N})+\left(m^3-m\right)c_1(C)c_1(\mathscr{N}) & \end{cases}$$

Proof As Y is a threefold

$$\tilde{\mathrm{e}}(Y) = 2\chi(\mathscr{O}_X) - 2\chi(\mathscr{O}_X(-2Y)) - 2\chi(\Omega_X^1) + 2\chi(\Omega_X^1(-Y))$$

and

$$\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y) = 2\chi(\mathscr{O}_C) - 2D_0(\mathscr{L}^{\otimes 2}, 2m) + 2D_1(\mathscr{L}, m).$$

Remark 4 In case we are interested in computing the Euler number of Y itself (not of a smooth model of it), we have to subtract from $\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y)$ the difference $\mathrm{e}(\tilde{Y}) - \mathrm{e}(Y)$ (for every blow–up σ). But outside the exceptional divisor $E = Exc(\sigma)$ the mapping σ is an isomorphism, so

$$\mathrm{e}(\tilde{Y}) - \mathrm{e}(Y) = \mathrm{e}(\tilde{Y} \cap E) - \mathrm{e}(C).$$

The exceptional divisor is a projective bundle over C, hence a smooth manifold of dimension $n-1$ and $\tilde{Y} \cap E$ is a hypersurface in E. Iterating the above we are able to compute $\tilde{Y} \cap E$ and hence $\mathrm{e}(Y)$. Moreover studying a resolution of $\tilde{Y} \cap E$ we can find a stratification S_i of C such that over each stratum the Euler characteristic of fibers of projection $\tilde{Y} \cap E \longrightarrow C$ is constant. Since the fibers of $\tilde{Y} \cap E \longrightarrow C$ are hypersurfaces in a projective space $\mathbb{P}^{n-k-1}$, we are reduced to computing the Euler characteristic of hypersurfaces in projective spaces of dimension smaller than $n-1$ and closures $\bar{S}_i$ of the strata.

4.2 Higher codimension complete intersections

When $r \geq 2$ then the formulae for $W_{n,r,d}$ become very complicated even for small value of n, r, d. For instance when $Y = Y_1 \cap Y_2$ then

$$\begin{aligned}
\chi(\mathscr{O}_Y) &= \chi(\mathscr{O}_X) - \chi(\mathscr{O}_X(-Y_1)) - \chi(\mathscr{O}_X(-Y_2)) + \chi(\mathscr{O}_X(-Y_1 - Y_2)),\\
\chi(\Omega_Y^1) &= \chi(\Omega_X^1) - \chi(\Omega_X^1(-Y_1)) - \chi(\Omega_X^1(-Y_2)) + \chi(\Omega_X^1(-Y_1 - Y_2)) +\\
&\quad - \chi(\mathscr{O}_X(-Y_1)) - \chi(\mathscr{O}_X(-Y_2)) + \chi(\mathscr{O}_X(-2Y_1)) + \chi(\mathscr{O}_X(-2Y_2)) +\\
&\quad + 2\chi(\mathscr{O}_X(-Y_1 - Y_2)) - \chi(\mathscr{O}_X(-2Y_1 - Y_2)) - \chi(\mathscr{O}_X(-Y_1 - 2Y_2)).
\end{aligned}$$

So if $n = 3$ (i.e. Y is a curve) and C is a point

$$\mathrm{e}(\tilde{Y}) - \mathrm{e}(Y) = W_{3,r,0}(m_1, m_2) = m_1 m_2 (m_1 + m_2 - 2).$$

The assumption we made on the resolution of Y in fact concerns not only the variety Y itself but also its representation as a complete intersection of divisors $Y := Y_1 \cap \cdots \cap Y_r$ as can be observed in the following trivial example

Example 4 Let $Y_1 = \{x \in \mathbb{P}^5 : x_1x_2 - x_3x_4 + x_5^2 = 0\}, Y_2 = \{x \in \mathbb{P}^5 : x_5 = 0\}$, then blowing–up the point $[1:0:0:0:0]$ we get the strict transform of $Y = Y_1 \cap Y_2$ a (smooth) transversal intersection of strict transforms $\tilde{Y}_1 \cap \tilde{Y}_2$.

On the other hand Y is also the complete intersection of divisors $Y_1' = \{x \in \mathbb{P}^5 : x_1x_2 - x_3x_4 + x_0x_5 = 0\}$ and Y_2. Now, both divisors are smooth and tangent at $[1:0:0:0:0]$, so the intersection of their strict transforms under blow–up at $[1:0:0:0:0]$ equals the sum of the strict transform of Y and the projectivization of the common tangent space.

In our considerations we did not assume that the center of blow–up is contained in the singular locus of Y. On the other hand in the opposite case, the computations did not take into account the singularities of Y lying on C. For instance, if a normal threefold Y contains a smooth surface C, then C is a Cartier divisor on the smooth part of Y, but it may fail to be Cartier at singular points. In this situation the blow–up of C is not an isomorphism and the above formulae give some description of the singularities of Y lying on C.

Example 5 Let Y be a nodal threefold in a smooth, projective fourfold X, assume that C is a smooth surface containing all nodes of Y. Then the blowing up of C is a small resolution of Y. Since the Milnor number of a node is 1 and the small resolution of a node replaces a point by a line, increasing the Euler number by one, Theorem 7 yields the following formula for the number of singular points of Y:

$$c_2(\mathcal{N}) + Y^2C - Yc_1(\mathcal{N}).$$

If $C \subset \mathbb{P}^n$ is any smooth projective surface, then by [14, Thm. 2.1] a general complete intersection of hypersurfaces of sufficiently large degree satisfies the above assumptions. If moreover C is a complete intersection of hypersurfaces of degrees $e_1, \ldots, e_{n-2}$, Y is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n-3}$, then the formula reads

$$\left(\sum_{i<j} e_ie_j - \sum_{i<j} d_id_j + \left(\sum_{i=1}^{n-3} d_i \right)^2 - \left(\sum_{i=1}^{n-3} d_i \right) \left(\sum_{i=1}^{n-2} e_i \right) \right) e_1 \ldots e_{n-2}$$

5 The χ_y–genus

The presented method computes in fact the holomorphic Euler characteristics of the sheaves $\Omega^p_{\tilde{Y}}$. So we can compute with them also the χ_y–genus of $\tilde{Y}$ defined as

$$\chi_y(\tilde{Y}) = \sum_{p=0}^{n} \chi(\Omega^p_{\tilde{Y}})y^p$$

and hence some other numerical invariants like f.i. **arithmetic genus** $p_a(\tilde{Y}) = \chi(\mathscr{O}_{\tilde{Y}}) = \chi_0(\tilde{Y})$, **signature** $\tau(\tilde{Y}) = \chi_1(\tilde{Y})$. More generally the χ_y–genus is defined for any vector bundle $\mathscr{W}$ on $\tilde{Y}$ by a similar formula $\chi_y(\tilde{Y},\mathscr{W}) = \sum_{p=0}^{n} \chi(\Omega^p_{\tilde{Y}} \otimes \mathscr{W})y^p$. We can apply the presented formulae to compute $\chi_y(\mathscr{W})$ for any vector bundle on $\tilde{Y}$ of the form $\mathscr{W} := \sigma^*\mathscr{V} \otimes \mathscr{O}_{\tilde{X}}(mE)$ for $\mathscr{V}$ a vector bundle on X.

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Cremona special sets of points in products of projective spaces

Igor V. Dolgachev

To the memory of V.I. Arnol'd

Abstract A set of points in the projective plane is said to be Cremona special if its orbit with respect to the Cremona group of birational transformations consists of finitely many orbits of the projective group. This notion was extended by A. Coble to sets of points in higher-dimensional projective spaces and by S. Mukai to sets of points in the product of projective spaces. No classification of such sets is known in these cases. In the present article we survey Coble's examples of Cremona special points in projective spaces and initiate a search for new examples in the case of products of projective spaces. We also extend to the new setting the classical notion of associated points sets.

Keywords Coxeter group, Cremona transformations, associated sets of points.
Mathematics Subject Classification (2010) *Primary* 14E07. *Secondary* 14N05, 11H55.

1 Introduction

Let $W_{p,q,r}$ denote the Coxeter group defined by the Coxeter graph of type $T_{p,q,r}$.

We will be interested in the cases when the group $W_{p,q,r}$ is infinite. It follows from the classification of finite reflection groups that $W_{p,q,r}$ is infinite if and only if

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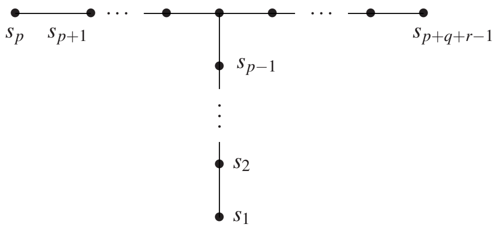

$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\leq 1$. Finite groups $W_{p,q,r}$ are known to be isomorphic to the Weyl groups of root systems of type $A_2\times A_1, A_n, D_5, E_6, E_7, E_8$.

It has been known since the fundamental work of A. Coble [2], [4] that the groups $W_{2,q,r}$ act birationally on the configuration spaces of ordered $q+r$ points in $\mathbb{P}^q$ modulo projective transformations. Roughly speaking, the subgroup generated by the generators $s_p,\ldots,s_{p+q+r-1}$ acts by permutations of points and the additional generator s_{p-1} acts via the standard Cremona transformation of degree q which inverts the coordinates of each point. The stabilizer subgroup of the orbit of a points set can be interpreted as a group of pseudo-automorphisms of the variety obtained by blowing-up this set. Here a pseudo-automorphism means a birational automorphism which is an isomorphism outside of a closed subset of codimension ≥ 2. For example, if $q=2$, the group of automorphisms of the blow-up surface becomes isomorphic to a subgroup of the Coxeter group $W_{2,3,r}$. This result, sometimes attributed to M. Nagata [22], goes back to S. Kantor [18], A. Coble [2], and P. Du Val [12].

It is known that, for a general points set, the automorphism group of the blow-up surface is trivial [15], but in a special case it could be very large in the sense that it is realized as a subgroup of finite index in $W_{2,3,r}$. Coble was the first to initiate a search of points sets in projective spaces which are *special* in the sense that the pseudo-automorphism group of the blow-up variety is realized as a subgroup of finite index in $W_{2,q,r}$. For example, a set of points in the projective plane is special if its orbit with respect to the Cremona group contains only finitely many orbits with respect to the projective group. Coble gave several examples of special sets in the plane and in three-dimensional space. Among these examples is the set of base points of a pencil of plane cubic curves, or the set of nodes of a rational plane sextic, or the set of nodes of a Cayley symmetroid quartic surface. A modern treatment of Coble's theory can be found in [9].

A generalization of the Coble representation of $W_{2,q,r}$ in the group of birational automorphisms of $X_{2,q,r}$ to a representation of an arbitrary Coxeter group $W_{p,q,r}$ on the configuration spaces of ordered points sets in the product of projective spaces was recently given by S. Mukai [21]. He applied this to the construction of new counter-examples to Hilbert's 14th Problem about finite generation of rings of invariants. The purpose of the paper is to initiate a search of special sets of points in this new setting. We also extend the classical notion of association of points sets in projective space to points sets in the product of projective spaces.

2 The Cremona action of $W_{p,q,r}$

We define the configuration space of ordered sets of $q+r$ points in the product of projective spaces $\mathbb{P}_{p,q} := (\mathbb{P}^{q-1})^{q+r}$ to be the GIT-quotient

$$X_{p,q,r} = (\mathbb{P}_{p,q})^{q+r}//\mathrm{SL}(q)^{p-1},$$

where the group $\mathrm{SL}(q)^{p-1}$ acts naturally on the product $\mathbb{P}_{p,q}$ and diagonally on the product $(\mathbb{P}_{p,q})^{q+r}$. We choose a democratic linearization on the product $(\mathbb{P}_{p,q})^{q+r} \cong (\mathbb{P}^{q-1})^{(p-1)(q+r)}$ defined by the invertible sheaf $\mathscr{O}(1)^{\boxtimes(p-1)(q+r)}$. To exclude the trivial cases, we assume that $p,q,r > 1$. The variety $X_{p,q,r}$ is an irreducible rational variety of dimension $D = (p-1)(q-1)(r-1)$.

The group $\mathfrak{S}_{p-1} \times \mathfrak{S}_{q+r}$ acts naturally on $X_{p,q,r}$ by permuting the $p-1$ factors of $\mathbb{P}_{p,q}$ and $q+r$ factors of $(\mathbb{P}_{p,q})^{q+r}$. It is realized as the subgroup of $W_{p,q,r}$ generated by the Coxeter generators $s_1,\ldots,s_{p-2}$ and $s_p,\ldots,s_{p+q+r-1}$. Following Coble, Mukai extends this action to a homomorphism

$$\mathrm{cr}_{p,q,r} : W_{p,q,r} \to \mathrm{Bir}(X_{p,q,r}) \cong \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(z_1,\ldots,z_D)) \tag{1}$$

by defining the action of the remaining generator s_{p-1} and checking that the relations between the generators are preserved. Recall that the standard Cremona transformation T in projective space $\mathbb{P}^{q-1}$ is a birational transformation given by the formula

$$z = [z_0,\ldots,z_q] \mapsto [z_0^{-1},\ldots,z_q^{-1}]. \tag{2}$$

We extend T to the product $(\mathbb{P}^{q-1})^{p-1}$ by the formula

$$(z^{(1)},\ldots,z^{(p-1)}) \mapsto \Big([\frac{1}{z_0^{(1)}},\ldots,\frac{1}{z_{q-1}^{(1)}}],[\frac{z_0^{(2)}}{z_0^{(1)}},\ldots,\frac{z_{q-1}^{(2)}}{z_{q-1}^{(1)}}],\ldots,[\frac{z_0^{(p-1)}}{z_0^{(1)}},\ldots,\frac{z_{q-1}^{(p-1)}}{z_{q-1}^{(1)}}]\Big). \tag{3}$$

A general points set can be represented in $X_{p,q,r}$ by a unique ordered points set $(p_1,\ldots,p_{q+r})$ with the first $q+1$ points equal to the *reference points*

$$[1,0,\ \ldots,0],\ \ldots,\ [0,\ldots,0,1],\ [1,\ldots,1].$$

We use projective transformations in each copy of $\mathbb{P}^{q-1}$ to assume that the projections of the first $q+1$ points $p_1,\dots,p_{q+1}$ are the reference points in $\mathbb{P}^{q-1}$. Then we apply T to the rest of the points. Note that T is not defined at the intersections of the pull-backs of coordinate hyperplanes in the first factor $\mathbb{P}^{q-1}$. One checks that this action of generators preserves the defining relations of the Coxeter group and defines the homomorphism (1). We call this homomorphism the *Cremona action* of $W_{p,q,r}$.

Let $\mathscr{P}$ be an ordered set of $q+r$ distinct points in $\mathbb{P}_{p,q}$ and let

$$\pi_{\mathscr{P}} : V_{\mathscr{P}} \to X^{q+r}$$

be its blow-up. We consider $V_{\mathscr{P}}$ up to a birational isomorphism which is an isomorphism outside a closed subset of codimension ≥ 2 (a *pseudo-isomorphism*). The action of such a birational isomorphism on the group $H^2(V_{\mathscr{P}},\mathbb{Z}) \cong \mathrm{Pic}(V_{\mathscr{P}})$ is well-defined. Denote by $E_i \cong \mathbb{P}^{pq-p-q}$ the exceptional divisor over the point p_i and let $e_i = [E_i]$ be its class in $H^2(V_{\mathscr{P}},\mathbb{Z})$. Let $\mathrm{pr}_i : \mathbb{P}_{p,q} \to \mathbb{P}^{q-1}$ be the projection to the i-th factor and $h_i = [\pi^*(\mathrm{pr}_i^*(H))]$, where H is a hyperplane in $\mathbb{P}^{q-1}$. Then

$$h_1,\dots,h_{p-1},e_1,\dots,e_{q+r}$$

form a basis of $H^2(V_{\mathscr{P}},\mathbb{Z})$. We call it a *geometric basis*. Let

$$h^1,\dots,h^{p-1},e^1,\dots,e^{q+r}$$

be its dual basis in $H_2(V_{\mathscr{P}},\mathbb{Z})$. We can realize it by taking $-e^i$ to be the class of a line in E_i and h^i to be the homology class of a line in $\mathbb{P}^{q-1}$ embedded in $\mathbb{P}_{p,q}$ under the inclusion map in the i-th factor $\iota_i : \mathbb{P}^{q-1} \to (\mathbb{P}^{q-1})^{p-1}$. Let

$$\begin{aligned} \alpha_1 &= h_{p-2} - h_{p-1},\dots,\alpha_{p-2} = h_1 - h_2, \\ \alpha_{p-1} &= h_1 - e_1 - \ldots - e_q, \\ \alpha_p &= e_1 - e_2,\dots,\alpha_{p+q+r-2} = e_{q+r-1} - e_{q+r}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \alpha^1 &= -h^{p-2} + h^{p-1},\dots,\alpha^{p-2} = -h^1 + h^2, \\ \alpha^{p-1} &= (q-2)h^1 + (q-1)h^2 + \ldots + (q-1)h^{p-1} + e^1 + \ldots + e^q, \\ \alpha^p &= e^2 - e^1,\dots,\alpha^{p+q+r-2} = -e_{q+r-1} + e_{q+r}. \end{aligned}$$

We immediately check that the matrix $(\alpha_i,\alpha^j) + 2I_{p+q+r-2}$ is the incidence matrix of the graph $T_{p,q,r}$. The two bases

$$\underline{\alpha} = (\alpha_1, \ldots, \alpha_{p+q+r-2}),\ \underline{\alpha}^\vee = (\alpha^1, \ldots, \alpha^{p+q+r-2})$$

form a root basis. The Weyl group of this root basis acts on $H^2(V_{\mathscr{P}}, \mathbb{Z})$ (resp. on $H_2(V_{\mathscr{P}}, \mathbb{Z})$) as the group generated by the simple reflections

$$s_i : x \to x + (x, \alpha^i)\alpha_i \text{ (resp. } s^i : y \to y + (y, \alpha_i)\alpha^i).$$

Let $e = e_1 + \ldots + e_{q+r}$. It is immediately checked that

$$\begin{aligned}
1 \le i \le p-2 :\ & s_i(h_i) = h_{i+1},\ s(h_j) = h_j, j \neq i, i+1, \\
& s_i(e_j) = e_j,\ j = 1, \ldots, q+r, \\
i = p-1 :\ & s_i(h_1) = (q-1)h_1 - (q-2)e, \\
& s_i(h_j) = h_j + (q-1)(h_1 - e), j \neq 1, \\
& s_i(e_j) = h_1 - e + e_j,\ 1 \le j \le q,\ s_i(e_j) = e_j,\ j > q. \\
i > p :\ & s_i(h_j) = h_j,\ j = 1, \ldots, p-1, \\
& s_i(e_{p-1+i}) = e_{p+i},\ s_i(e_j) = e_j, j \neq p-1+i, p+i.
\end{aligned}$$

It follows that the Weyl group is isomorphic to the Coxeter group $W_{p,q,r}$ with Coxeter generators s_i.

Note that the following vectors are preserved under the action:

$$\begin{aligned}
K_{V_{\mathscr{P}}} &= -q(h_1 + \ldots + h_{p-1}) + (pq - p - q)(e_1 + \ldots + e_{q+r}) \in H^2(V_{\mathscr{P}}, \mathbb{Z}), \\
k_{V_{\mathscr{P}}} &= -q(h^1 + \ldots + h^{p-1}) + e^1 + \ldots + e^{q+r} \in H_2(V_{\mathscr{P}}, \mathbb{Z}).
\end{aligned}$$

One recognizes in $K_{V_{\mathscr{P}}}$ the canonical class of $V_{\mathscr{P}}$. One can check that $\underline{\alpha}$ is a basis of the orthogonal complement of $k_{V_{\mathscr{P}}}$ in $H^2(V_{\mathscr{P}}, \mathbb{Z})$ and $\underline{\alpha}^\vee$ is a basis of the orthogonal complement of $K_{V_{\mathscr{P}}}$ in $H_2(V_{\mathscr{P}}, \mathbb{Z})$.

Another way to look at this is to define a linear map

$$H^2(V_{\mathscr{P}}, \mathbb{Z}) \to H_2(V_{\mathscr{P}}, Z),\ \gamma \to \gamma^\vee,$$

on the basis (h_i, e_j) by

$$h_i \mapsto (q-1)(h^1 + \ldots + h^{p-1}) - h^i,\ i = 1, \ldots, p-1, \quad e_i \mapsto -e^i,\ i = 1, \ldots, q+r.$$

Then $x \cdot y := (x, y^\vee)$ defines a structure of a quadratic lattice on $H^2(V_{\mathscr{P}}, \mathbb{Z})$ with the Gram matrix in the basis $(h_1, \ldots, h_{p-1}, e_1, \ldots, e_{q+r})$ given by block-sum of the square matrix

$$A_{p,q} := \begin{pmatrix} q-2 & q-1 & q-1 & \dots & q-1 \\ q-1 & q-2 & q-1 & \dots & q-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q-1 & q-1 & q-1 & \dots & q-2 \end{pmatrix}$$

and the matrix $-I_{q+r}$. The signature of this lattice is equal to $(1, p+q+r-2)$ and the discriminant is equal to $(-1)^{q+r}(pq-p-q)$. We have

$$K_{V_{\mathscr{P}}}^{\vee} = (pq-p-q)k_{V_{\mathscr{P}}},$$

so $k_{V_{\mathscr{P}}}^{\perp}$ is mapped onto $K_{V_{\mathscr{P}}}^{\perp}$ and

$$K_{V_{\mathscr{P}}}^2 = (pq-p-q)(pqr-pq-pr-qr).$$

This implies that the sublattice $k_{V_{\mathscr{P}}}^{\perp}$ with a basis $\underline{\alpha}$ is an even lattice of signature (t_+, t_-) (or (t_+, t_-, t_0) if $t_0 \neq 0$):

$$\mathrm{sign}(\mathbb{E}_{p,q,r}) = \begin{cases} (0, p+q+r-2) & \text{if } pqr-pq-pr-qr > 0, \\ (0, p+q+r-3, 1) & \text{if } pqr-pq-pr-qr = 0, \\ (1, p+q+r-3) & \text{otherwise.} \end{cases}$$

The group $W_{p,q,r}$ is finite if and only if $pqr-pq-pr-qr > 0$ and it contains an abelian subgroup of finite index if and only if $pqr-pq-pr-qr = 0$.

It is convenient to introduce an abstract quadratic lattice $\mathbf{I}_{p,q,r}$ defined in a basis $(\mathbf{h}_1, \dots, \mathbf{h}_{p-1}, \mathbf{e}_1, \dots, \mathbf{e}_{q+r})$ by the matrix $A_{p,q} \oplus -I_{q+r}$, a vector

$$\mathbf{K}_{p,q,r} = -q(\mathbf{h}_1 + \dots + \mathbf{h}_{p-1}) + (pq-p-q)(\mathbf{e}_1 + \dots + \mathbf{e}_{q+r}),$$

and the sublattice

$$\mathbb{E}_{p,q,r} := (\mathbf{K}_{p,q,r})^{\perp}.$$

with a canonical basis $(\alpha_1, \dots, \alpha_{p+q+r-2})$, where α_i are expressed in terms of $(\mathbf{h}_1, \dots, \mathbf{h}_{p-1}, \mathbf{e}_1, \dots, \mathbf{e}_{q+r})$ by formulas (4). We also define the Weyl group $W(\mathbb{E}_{p,q,r})$ as the group of orthogonal transformations of the lattice $\mathbb{E}_{p,q,r}$ generated by reflections with respect to the vectors α_i:

$$s_i : x \mapsto x + (x \cdot \alpha_i)\alpha_i.$$

This is the Coxeter group corresponding to the $T_{p,q,r}$-graph. For simplicity of notation, we set $W(\mathbb{E}_{p,q,r}) = W_{p,q,r}$.

A choice of a geometric basis in $H^2(V_{\mathscr{P}},\mathbb{Z})$ defines a *geometric marking* of $V_{\mathscr{P}}$, i.e. an isometry of lattices $\varphi : \mathbf{I}_{p,q,r} \to H^2(V_{\mathscr{P}},\mathbb{Z})$ such that $\varphi(\mathbf{K}_{p,q,r}) = K_{V_{\mathscr{P}}}$. Under this isometry the lattice $\mathbb{E}_{p,q,r}$ is mapped onto $K_{V_{\mathscr{P}}}^{\perp}$.

It is easy to see that $W_{p,q,r}$ acts transitively on the root basis $\underline{\alpha}$ of $\mathbb{E}_{p,q,r}$. An element of the $W_{p,q,r}$-orbit of any element from the canonical basis is called a *root*. A root is called *positive* if it can be written as a linear combination of the canonical basis with non-negative coefficients. One can show that any root α is either positive or $-\alpha$ is positive.

Lemma 2.1 *Let* $\alpha = \sum_{i=1}^{p-1} d_i\mathbf{h}_i - \sum_{j=1}^{q+r} m_i\mathbf{e}_j$ *be a positive root. Let* $d = d_1 + \ldots + d_{p-1}$. *Suppose that one of the numbers* d_i *is positive. Then*

(i) $qd = \sum_{j=1}^{q+r} m_j$;

(ii) $(q-1)d^2 - \sum_{i=1}^{p-1} d_i^2 - \sum_{k=1}^{q+r} m_k^2 = -2$;

(iii) $(q-1)d - d_1 < m_1 + \ldots + m_q$, *if* $m_1 \geq m_2 \geq \ldots \geq m_{q+r}$ *and* $d_1 \leq \ldots \leq d_{p-1}$;

(iv) assume $d > 0$, *then* $d_i \geq 0, i = 1,\ldots,p-1$, *and* $m_j \geq 0, j = 1,\ldots,q+r$.

Proof The first equality follows from the condition that $\alpha \cdot \mathbf{K}_{p,q,r} = 0$. The second one follows from the condition that $\alpha^2 = -2$.

(iii) The ordering of the coefficients implies that $\alpha \circ \alpha_i \geq 0$ for $i \neq p-1$. One checks that

$$\alpha \cdot \alpha_{p-1} = (q-1)d - d_1 - \sum_{j=1}^{q} m_j. \tag{5}$$

Assume the inequality does not hold. Then $\alpha \cdot \alpha_i \geq 0$ for all α_i. This means that α belongs to the fundamental chamber of the root system. The proof that it is impossible is the same as in the proof of the corresponding statement for the case $p = 2$ in [9], p. 75.

(iv) It is checked immediately that $d \geq 0$ for a positive root. Let us use induction on d. Assume $d = 1$. Then (i) and (ii) give

$$1 - \sum_{j=1}^{p-1} d_i^2 = \sum_{j=1}^{q} m_j(m_i - 1).$$

The left-hand side is non-positive, the right-hand-side is non-negative. This gives that one of the d_i is equal to 1, all others are zeros. Also, $m_j = 1$ or 0. This checks the assertion.

It is known that s_i transforms the set of positive roots with α_i deleted to a subset of positive roots (see [17], Lemma 1.3). Applying $s_i, i \neq p-1$, we may assume that that d_i's and m'_j are ordered as in (iii). Using (5), we obtain

$$\alpha' = s_{p-1}(\alpha) = \sum_{i=1}^{p-1} d'_i \mathbf{h}_i - \sum_{j=1}^{q+r} m'_i \mathbf{e}_j, \tag{6}$$

where

$$d'_1 = (q-1)d - \sum_{j=1}^{q} m_j,$$
$$m'_j = (q-1)d - d_1 - (m_1 + \ldots + m_q - m_j),\ j = 1, \ldots, q,$$

and all other d_i's and m_j's are unchanged. By (iii), we obtain that $d'_1 < d_1$, and hence $d' = \sum d'_i < d$. If $\sum d'_i > 0$, we are done by induction. Assume that $d' = 0$. Then (i) and (ii) give $2 - \sum d_i^2 = \sum m_j^2$. This gives the following possibilities $\alpha' = h_i - h_j, e_i - e_j, i < j$. But then $\alpha = s_{p-1}(\alpha')$ satisfies (iv) and (v).

Note that a vector in $\mathbf{I}_{p,q,r}$ satisfying (i)-(iv) from the lemma is not necessarily a root. An example in the case $p = 2, q = 3$ is given in [9].

Lemma 2.2 *Let α be a root in $\mathbb{E}_{p,q,r}$ and $\Delta(\alpha)$ be the subset of $\mathbb{P}_{p,q}$ that consists of points sets such that $\varphi_{\mathscr{P}}(\alpha)$ is effective. Then $\Delta(\alpha)$ is a closed proper subset of $\mathbb{P}_{p,q}$.*

Proof If $\alpha = \alpha_i$ or its transform under $s_i, i \neq p-1$, the assertion is obvious. The root $\alpha_i, i < p-1$, is never effective, the roots $\alpha_i, i > p-1$, are effective only if some points coincide, and α_{p-1} is effective only if the first $q+1$ points are in a hyperplane. A points set for which all such roots are not effective will be called a *regular point set*. It follows from Lemma 2.1 that, for roots α with $d = \sum d_i > 0$, the condition that its image is effective reads as the condition of the existence of a hypersurface of multi-degree $(d_1, \ldots, d_{p-1})$ passing through the points $p_i \in \mathscr{P}$ with multiplicity $\geq m_i$. Obviously it is a closed condition. Assume $\Delta(\alpha) = \mathbb{P}_{p,q}$ for some α with $\sum d_i > 0$. Without loss of generality, we may assume that $m_1 \geq \ldots \geq m_{q+r}$ and $d_1 \leq \ldots \leq d_{p-1}$. Also we may assume that $\mathscr{P}$ is a regular set. Now we take $\mathscr{Q}$ from the projective equivalence class of $\mathrm{cr}_{p,q,r}(s_{p-1})([\mathscr{P}])$. Then $\varphi_{\mathscr{Q}}(s_{p-1}(\alpha))$ is an effective root (the transform under T of the hypersurface defining $\varphi_{\mathscr{P}}(\alpha)$). Property (iii) of a positive root together with (6) implies that the hypersurface defined by the root $\varphi_{\mathscr{Q}}(s_{p-1}(\alpha))$ has multi-degree $(d'_1, \ldots, d'_{p-1})$ with $d' = \sum d'_i < d$. Now we can use induction on d. We know that $\Delta(s_{p-1}(\alpha))$ is a proper subset, hence we find a regular $\mathscr{Q}$ such that the root $s_{p-1}(\alpha)$ is not effective.

Definition 1 A points set $\mathscr{P}$ is called *unnodal* if under the geometric marking $\varphi_{\mathscr{P}} : \mathbf{I}_{p,q,r} \to \mathrm{Pic}(V_{\mathscr{Q}})$ defined by $\mathscr{P}$, no root is mapped to an effective divisor class.

It follows from Lemma 2.2 that the set of unnodal points sets is the complement of the union of proper closed subsets $\Delta(\alpha)$, where α is a positive root in $\mathbb{E}_{p,q,r}$. This set is infinitely countable if the lattice $\mathbb{E}_{p,q,r}$ is not negatively definite.

Note that we do not know whether the closed subsets $\Delta(\alpha)$ are hypersurfaces in $\mathbb{P}_{p,q}$ when α is a root different from $\alpha_i, i \neq p-1$. It is true in the cases when $\mathbb{P}_{p,q}$ is a surface.

Proposition 1 *Let $\mathscr{P}$ be an unnodal point set. Then, for any $w \in W_{p,q,r}$, the marking ${}^{w}\varphi := \varphi \circ w^{-1}$ is a geometric marking on $V_{\mathscr{P}}$ defined by the points set $\mathscr{Q}$ such that* $\mathrm{cr}_{p,q,r}([\mathscr{P}]) = [\mathscr{Q}]$.

Proof Let $\mathscr{P}$ be a regular set of points. We use that, for Coxeter generator $s_i \in W_{p,q,r}$, $\mathrm{cr}_{p,q,r}(s_i)$ is defined at $[\mathscr{P}]$. If $s_i \neq s_{p-1}$, then ${}^{s_i}\varphi_{\mathscr{P}} = \varphi_{\mathscr{Q}}$, where $\mathscr{Q}$ is obtained from $\mathscr{P}$ by either permuting the points, or as the image of an automorphism of $\mathbb{P}_{p,q}$ permuting the $(p-1)$ factors. If $i = p-1$, then $[\mathscr{Q}]$ is defined by the points set (3), where T is the standard Cremona transformation of $\mathbb{P}_{p,q}$. Note that the divisor classes E_i' representing $s_{p-1}(e_i), i = 1,\dots,q$, are not contractible on $V_{\mathscr{P}}$ if $\dim \mathbb{P}_{p,q} > 2$. However, they become contractible when we apply to $V_{\mathscr{P}}$ a pseudo-isomorphism. The geometric marking corresponding to a points set $\mathscr{Q}$ representing $\mathrm{cr}_{p,q,r}(s_i)([\mathscr{P}])$ is equal to ${}^{s_i}\varphi$. So, if we show that $\mathscr{Q}$ is again a regular set, we are done.

Let $\mathfrak{l}(w)$ be the minimal length of w as the product of Coxeter generators. This is well-defined in any Coxeter group. Let us prove by induction on $\mathfrak{l}(w)$ that the image of a geometric basis defined by $\mathscr{P}$ under w is a geometric basis. Write w as the product of generators $s_{i_k}\cdots s_{i_1}$, where $k = \mathfrak{l}(w)$. Assume $i_1 \neq p-1$. Then, as we have already observed, s_{i_1} transforms a geometric basis to a geometric basis. It is defined by a regular points set $\mathscr{Q}$ such that $\mathrm{cr}_{p,q,r}(s_{i_1})([\mathscr{P}]) = [\mathscr{Q}]$.

Assume $i_1 = p-1$. Consider the birational transformation $\mathrm{cr}_{p,q,r}(s_{p-1})$ of $X_{p,q,r}$. It transforms the point $[\mathscr{P}]$ corresponding to a normalized points set $\mathscr{P} = (p_1,\dots,p_{q+1},\dots,p_{q+r})$ to the point $[\mathscr{Q}]$, where

$$\mathscr{Q} = (p_1,\dots,p_{q+1},T(p_{q+2}),\dots,T(p_{q+r})) = (p_1',\dots,p_{q+r}').$$

Assume that $\mathscr{Q}$ is not regular. If it does not satisfy the first condition of regularity, then some points in this set coincide. But this could happen only if one of the points $p_i, i > q+1$, in $\mathscr{P}$ lies in an exceptional divisor of T (i.e. the closure of points which are mapped to the set of indeterminacy of $T^{-1}=T$). It is easy to see that this set consists of the union of the pre-images of hyperplanes in the first copy of $\mathbb{P}^{q-1}$ which are spanned by all points $\mathrm{pr}_1(p_1),\ldots,\mathrm{pr}_1(p_{q+1})$ except one, say p_j. Since $h_1-e_1-\ldots-e_q+e_j-e_i$ is not an effective divisor, the images of the points $\{p_1,\ldots,p_q,p_i\}\setminus\{p_j\}$ under the first projection are not contained in a hyperplane. So the new points set $\mathscr{Q}$ consists of distinct points. Next assume that the second condition of regularity is not satisfied. This means that the projections of some points $p'_{j_1},\ldots,p'_{j_{q+1}}$ to some t-factor lie in a hyperplane. Applying $\mathrm{cr}_{p,q,r}(s_i), 1\le i\le p-2$, we may assume that $t=1$. It follows from the definition of the transformation T that $T^*(h_1)=2h_1-e_1-\ldots-e_q$. It agrees with the action of s_{p-1} on the geometric basis. If the points $p'_{j_1},\ldots,p'_{j_{q+1}}$ are projected to a hyperplane in the first factor, then the pre-image of this hyperplane under T is an effective divisor in the class

$$2h_1-e_1-\ldots-e_q-e_{j_1}-\ldots-e_{j_{q+1}}=s_{p-1}(h_1-e_{j_1}-\ldots-e_{j_{q+1}}),$$

where we replace e_{j_k} with e_i if $j_k=i$ for some $i\le q$. Since $h_1-e_{j_1}-\ldots-e_{j_{q+1}}$ is a root, and $s_{p-1}(h_1-e_{j_1}-\ldots-e_{j_{q+1}})$ is again a root, it cannot be effective. This proves that s_{i_1} is well defined on $[\mathscr{P}]$ and transforms it to $[\mathscr{Q}]$, where $\mathscr{Q}$ is a regular point set. It remains to apply induction on $\mathfrak{l}(w)$.

Proposition 2 *Let $\mathscr{P}$ be an unnodal points set and $w\in W_{p,q,r}$ be such that $\mathrm{cr}_{p,q,r}(w)([\mathscr{P}])=[\mathscr{P}]$. Then there exists a pseudo-automorphism $\tau: V_{\mathscr{P}}-\to V_{\mathscr{P}}$ such that $w=\varphi_{\mathscr{P}}^{-1}\circ\tau^*\circ\varphi_{\mathscr{P}}$.*

Proof Let $\sigma: V_{\mathscr{P}}\to\mathbb{P}_{p,q}$ be a birational morphism contracting the divisors $E_1,\ldots,E_{q+r}$ to points $p_1,\ldots,p_{q+r}\in\mathscr{P}$. Then there exists a pseudo-automorphism $\Phi: V_{\mathscr{P}}-\to V'_{\mathscr{P}}$ and a contraction $\sigma': V'_{\mathscr{P}}\to\mathbb{P}_{p,q}$ of divisors E'_i to the same set of points $\mathscr{P}$ such that $\Phi^*(E'_i)=\varphi_{\mathscr{P}}\circ w\circ\varphi_{\mathscr{P}}^{-1}$. Since two blow-ups of the same closed subscheme are isomorphic, there exists an isomorphism $\Psi: V_{\mathscr{P}}-\to V'_{\mathscr{P}}$ which sends E_i to E'_i. The composition $\tau=\Phi\circ\Psi^{-1}: V_{\mathscr{P}}-\to V_{\mathscr{P}}$ is a pseudo-automorphism of $V_{\mathscr{P}}$ whose existence is asserted in the Proposition.

Corollary 1 *Let $\mathscr{P}$ be an unnodal set in $\mathbb{P}_{p,q}$. Then the stabilizer subgroup*

$$(W_{p,q,r})_{\mathscr{P}}:=\{w\in W_{p,q,r}:\mathrm{cr}_{p,q,r}(w)([\mathscr{P}])=[\mathscr{P}]\}$$

is isomorphic to a subgroup of the group $\mathrm{Aut}_{\mathrm{ps}}(V_{\mathscr{P}})$ of the group of pseudo-automorphisms of $V_{\mathscr{P}}$.

Let $\Phi\in\mathrm{Aut}_{\mathrm{ps}}(V_{\mathscr{P}})$ and let Φ^* be its action on $H^2(V_{\mathscr{P}},\mathbb{Z})$. We say that Φ is *Cremona-like* if there exists $w\in W_{p,q,r}$ such that

$$w = \varphi_{\mathscr{P}}^{-1} \circ \Phi^* \circ \varphi_{\mathscr{P}}.$$

Let $\mathrm{Aut}_{\mathrm{cr}}(V_{\mathscr{P}})$ be the subgroup of $\mathrm{Aut}_{\mathrm{ps}}(V_{\mathscr{P}})$ of Cremona-like transformations. We have a natural homomorphism

$$\mathrm{Aut}_{\mathrm{cr}}(V_{\mathscr{P}}) \to W_{p,q,r}, \ \Phi \mapsto \varphi_{\mathscr{P}}^{-1} \circ \Phi^* \circ \varphi_{\mathscr{P}}.$$

It is clear that its kernel is isomorphic to the group of automorphisms of $V_{\mathscr{P}}$ lifted from automorphisms of $\mathbb{P}_{p,q}$.

Remark 1 In the case $p = 2, q = 3$, one can prove, using Noether's Theorem on generators of the planar Cremona group, that $\mathrm{Aut}_{\mathrm{cr}}(V_{\mathscr{P}}) = \mathrm{Aut}(V_{\mathscr{P}})$.

It seems that A. Coble [2] and S. Kantor [19] claimed that $\mathrm{Aut}_{\mathrm{cr}}(V_{\mathscr{P}}) = \mathrm{Aut}_{\mathrm{ps}}(V_{\mathscr{P}})$ in the cases $p = 2$ and q is arbitrary. Their claim was based on their theory of *punctual* or *regular* Cremona transformation of $\mathbb{P}^{q-1}$. A punctual Cremona transformation is a product of projective transformations and the standard Cremona transformations. By Noether's Theorem, any planar Cremona transformation is punctual. Kantor says that a Cremona transformation Φ of $\mathbb{P}^3$ has no fundamental curves of the 1st kind if the graph of Φ transforms any fundamental curve to a fundamental curve of the inverse transformation. He claimed that any such transformation is punctual (citing from [16], p. 318: "the so-called proof is admittedly "gewagt", and merits a stronger adjective"). Another "equivalent definition" ([13], p. 192) of a punctual transformation requires that all base conditions follow from conditions at points. No rigorous proof of equivalence of these definitions is available. What we need is to prove (or disprove) the following assertion:

Let $\sigma_i : V_{\mathscr{P}_i} - \to \mathbb{P}_{p,q}$ be two blow-up varieties and $\Phi : V_{\mathscr{P}_1} - \to V_{\mathscr{P}_2}$ be a pseudo-isomorphism. Then the Cremona transformation $\sigma_2 \circ \Phi \circ \sigma_1^{-1} : \mathbb{P}_{p,q} - \to \mathbb{P}_{p,q}$ is a composition of the standard transformation (3) and automorphisms of $\mathbb{P}_{p,q}$.

3 Examples of Cremona special sets

Definition 2 A points set $\mathscr{P}$ is called *Cremona special* if the image $\mathrm{Aut}_{\mathrm{cr}}(V_{\mathscr{P}})^*$ of $\mathrm{Aut}_{\mathrm{cr}}(V_{\mathscr{P}})$ in $W_{p,q,r}$ is a subgroup of finite index.

Since the condition is vacuous when $W_{p,q,r}$ is a finite group, we will additionally assume that $W_{p,q,r}$ is an infinite group.

All examples of Cremona special sets that I know use the theory of abelian fibrations that I remind below.

Let Y be a smooth projective variety over a field L. There exists an abelian variety $\mathrm{Alb}^0(Y)$ and a torsor $\mathrm{Alb}^1(Y)$ over $\mathrm{Alb}^0(Y)$ satisfying the universal property for morphisms of Y to torsors over an abelian variety [23], [25]. If $Y(K) \neq \emptyset$, then $\mathrm{Alb}^0(Y) = \mathrm{Alb}^1(Y)$ is the Albanese variety $\mathrm{Alb}(Y)$ of Y. The dual abelian variety of $\mathrm{Alb}^0(Y)$ is isomorphic to the (reduced) Picard variety $\mathbf{Pic}^0_{Y/K}$. We say that Y is an *abelian torsor* if the canonical map $Y \to \mathrm{Alb}^1(Y)$ is an isomorphism. In the simplest case when $\dim Y = 1$, this means that Y is a smooth curve of genus 1, and $\mathrm{Alb}^0(Y) = \mathbf{Pic}^0_{Y/K}$ is its Jacobian variety.

Let $f : X \to S$ be a projective morphism of irreducible varieties. We assume that S is smooth and X is $\mathbb{Q}$-factorial with terminal singularities. Assume that a generic fibre X_η is an abelian torsor over the field of rational functions $K = \kappa(\eta)$ of S. Moreover, assume that f is relatively minimal in the sense that $K_X \cdot C \geq 0$ for any curve C contained in a fibre. Let $\mathbf{A} = \mathrm{Alb}^1(X_\eta)$, the group $\mathbf{A}(K)$ acts biregularly on X_η via translations

$$t_a : x \mapsto x + a, \quad x \in X_\eta(\bar{K}), a \in \mathbf{A}(K).$$

This action defines a birational action on X, and, the condition of minimality implies that the action embeds $\mathbf{A}(K)$ into the group $\mathrm{Aut}_{\mathrm{ps}}(X)$ of pseudo-automorphisms of X [20], [24], Lemma 6.2.

Assume that the class group $\mathrm{Cl}(X)$ of Weyl divisors is finitely generated of rank ρ (e.g. X is a unirational variety as it will be in all applications). Let

$$r_\eta : \mathrm{Cl}(X) \to \mathrm{Cl}(X_\eta) \cong \mathrm{Pic}(X_\eta)$$

be the restriction homomorphism. By taking the closure of any point of codimension 1 in X_η, we see that this homomorphism is surjective. Let $\mathrm{Pic}^0(X_\eta) = \mathbf{Pic}^0_{X_\eta/K}(K)$ be the subgroup of Cartier divisor classes of algebraically equivalent to zero. Then we have an exact sequence of finitely generated abelian groups

$$0 \to \mathrm{Cl}_{\mathrm{fib}}(X) \to \mathrm{Cl}(X)^0 \to \mathrm{Pic}(X_\eta)^0 \to 0,$$

where $\mathrm{Cl}_{\mathrm{fib}}(X)$ is the subgroup of $\mathrm{Cl}(X)$ generated by the classes of Weil divisors $\sum n_i D_i$ such that $\mathrm{codim} f(D_i) \geq 1$. Since the dual abelian varieties are isogenous, we obtain that the group $\mathbf{A}(K)$ is finite generated. In the case when $X_\eta(K) \neq \emptyset$, hence $X_\eta \cong \mathbf{A}$, the group $\mathbf{A}(K)$ is called the *Mordell-Weil group* of the fibration. We keep this terminology in the general case and denote the group $\mathbf{A}(K)$ by $\mathrm{MW}(f)$. Counting the ranks, we obtain the *Shioda-Tate* formula

$$\mathrm{rank}\ \mathrm{MW}(f) = \mathrm{rank}\ \mathrm{Cl}(X) - 1 - \mathrm{rank}\ \mathrm{Cl}(S) - \sum_{s \in S^{(1)}} (\#\mathrm{Irr}(X_s) - 1), \tag{7}$$

where $S^{(1)}$ is the set of points of codimension 1 in S and $\mathrm{Irr}(X_s)$ is the set of irreducible components of the fibre X_s.

Example 1 Assume $p = 2, q = 3$, hence $r \geq 6$. The known examples in these cases are due to A. Coble [1], [4]. The first example is an *Halphen set of index* m, the set of 9 base points of an irreducible pencil of curves of degree $3m$ with nine m-multiple base points (an *Halphen pencil* of index m). The proper transform of the pencil in $V_{\mathscr{P}}$ is the linear system $|-mK_{V_{\mathscr{P}}}|$. We assume that all fibres are irreducible which is equivalent to the condition that the set $\mathscr{P}$ is unnodal. When $m = 1$ this can be achieved by assuming that no three points are collinear ([24], Lemma 3.1). Let C be the unique cubic curve passing through the base points $p_1, \ldots, p_9$. We assume that C is nonsingular. Obviously, this is an open condition on $\mathscr{P}$. If we fix a group law on C with an inflection point as the zero, then the condition that $\mathscr{P}$ is an Halphen set of index m is that the points p_i add up in the group law to a point of order m. Let

$$a : \mathbb{Z}^9 \to C, \quad (m_1, \ldots, m_9) \mapsto m_1 p_1 + \ldots + m_9 p_9,$$

where the sum is taken in the group law on C. Assume that $a(\mathbf{m}) \neq 0$ for any vector $\mathbf{m} = (m_1, \ldots, m_9)$ with $(m - m_1, \ldots, m - m_9) \in \mathbb{Z}^9_{\geq 0}$ and $m_1 + \ldots + m_9 < 2m$. This is an open condition on $\mathscr{P}$ that guarantees that there are no curves of degree $d < 2m$ with singular points at the point p_i of multiplicity m_i. Now, if D is a reducible member of the pencil, one of its parts has degree $< 2m$ and passes through the p_i with some multiplicity m_i as above. By assumption, this is impossible, so all members of the pencil are irreducible, and stay the same when we blow-up the nine points. Since all (-2)-curves on $V_{\mathscr{P}}$ are contained in fibres, we obtain that $\mathscr{P}$ is unnodal. Let $f : V_{\mathscr{P}} \to \mathbb{P}^1$ be the elliptic fibration defined by the Halphen pencil. Applying the Shioda-Tate formula, we find that the rank of the Mordell-Weil group is equal to 8. Thus $\mathbb{Z}^8$ embeds into $\mathrm{Aut}_{\mathrm{ps}}(V_{\mathscr{P}}) = \mathrm{Aut}(V_{\mathscr{P}})$. The known structure of $W_{2,3,6}$ shows that the image is a normal subgroup of finite index. More detailed analysis of the action gives that the quotient group is isomorphic to $(\mathbb{Z}/8\mathbb{Z})^8 \rtimes W_{2,3,5}$ (see [15]).

In the second example, $\mathscr{P}$ is a *Coble set*, the ten nodes $p_1, \ldots, p_{10}$ of an irreducible rational plane curve of degree 6. The proper inverse transform C of the sextic is equal to the anti-bicanonical linear system $|-2K_{V_{\mathscr{P}}}|$. The double cover of $V_{\mathscr{P}}$ branched along C is a K3-surface which can be considered as a degeneration of the étale double cover of an Enriques surface. One uses the period theory of Enriques surfaces to show that the locus of Coble sets is irreducible. Also, using some lattice-theoretical methods from [8], one can show that there are only finitely many closed conditions on $\mathscr{P}$ (in fact, exactly 496 conditions) that guarantee that $\mathscr{P}$ is an unnodal set. Let $\mathscr{C}ob \subset P_2^{10}$ be the subset of projective equivalence classes of unnodal Coble sets. This is a locally closed subset invariant with respect to the Cremona action. Let

$$\mathrm{cr}_{2,3,7} : W_{2,3,7} \to \mathrm{Aut}(\mathscr{C}ob)$$

be the action homomorphism. Its kernel is a normal subgroup of $W_{2,3,7}$. Observe that any subset of 9 points in $\mathscr{P}$ is an Halphen set of index 2, The Halphen pencil is generated by the sextic curve and the unique cubic curve through the 9 points taken with multiplicity 2. Conversely, a Coble set is obtained from an Halphen set of index 2 by choosing a singular member of the pencil, its singular point is the tenth point of the set. Fix two points, say p_1 and p_2 and let F_1 and F_2 be the proper transforms in $V_{\mathscr{P}}$ of the cubic curves through $p_3, \ldots, p_{10}$ and $p_i, i = 1, 2$. The linear system $|2F_i|$ is a pencil, the proper transform of the corresponding Halphen pencil. We have $F_i \cdot F_2 = 1$ so that $(2F_1 + 2F_2)^2 = 8$. One can show that the linear system $|2F_1 + 2F_2|$ defines a degree two map from $V_{\mathscr{P}}$ onto a quartic Del Pezzo surface in $\mathbb{P}^4$ with 4 ordinary double points. The image of the curve C is one of these nodes. The deck transformation of the cover defines an isometry of the lattice $K_{V_{\mathscr{P}}}^{\perp} \cong \mathbb{E}_{2,3,7}$ which is conjugate to the isometry $-\mathrm{id}_{\mathbb{E}_{2,3,5}} \oplus \mathrm{id}_U$, where $\mathbb{E}_{2,3,7} \cong U \oplus \mathbb{E}_{2,3,5}$ is an orthogonal decomposition into the sum of two unimodular sublattices. It is known that the normal subgroup of $W_{2,3,7}$ containing this isometry has finite quotient group isomorphic to $\mathrm{O}(10, \mathbb{F}_2)^+$ [7], Theorem 2.10.1.

Recently, in a joint work with S. Cantat, we were able to prove the following.

Theorem 1 *Let $\mathscr{P}$ be a Cremona special set in the plane. Then it is either an Halphen set, or a Coble set, or a set of ≥ 10 points on an irreducible cuspidal curve in characteristic $p > 0$.*

Coble also constructed examples of Cremona special sets in $\mathbb{P}^3$.

Example 2 The first series of examples are analogs of Halphen sets. One considers an elliptic normal curve C of degree 4 in $\mathbb{P}^3$ equipped with the group law defined by a choice of an osculating point. Choose 8 points $\mathscr{P} = \{p_1, \ldots, p_8\}$ on C in general position which add up to a point of order m in the group law. Then one shows that there exists a surface F_{2m} of degree $2m$ with m-multiple points at $p_1, \ldots, p_m$ which does not belong to the family of surfaces of the form $G_m(q_1, q_2) = 0$, where G_m is a homogeneous polynomial of degree m and q_1 and q_2 are quadratic forms in 3 homogeneous coordinates in $\mathbb{P}^3$ such that $C = V(q_1) \cap V(q_2)$. The linear system $|\mathscr{O}_{\mathbb{P}^3}(2m) - m(p_1 + \ldots + p_m)|$ defines a regular map

$$f : V_{\mathscr{P}} \to \mathbb{P}^{m+1}.$$

Its image is the projective cone over a Veronese curve of degree m in $\mathbb{P}^m$. The image of C is the vertex of the cone. If we fix a nonsingular quadric Q in the pencil of quadrics $V(\lambda q_1 + \mu q_2)$, then the restriction of the linear system to Q is a pencil of elliptic curves of degree $4m$ with eight m-multiple points. Thus a general fibre of f is an elliptic curve. If we blow-up the proper transform of C in $V_{\mathscr{P}}$, we obtain an elliptic fibration

$$f' : V'_{\mathscr{P}} \to \mathbf{F}_m;$$

the fibres over the points on the exceptional section of $\mathbf{F}_m$ are m-multiple elliptic curves. The case $m = 1$ corresponds to a well-known set of Cayley octads, the complete intersection of three quadrics. It is discussed in detail in [9]. By choosing the set $\mathscr{P}$ general enough, as in the example of Halphen sets, we find that all fibres over points of codimension 1 are irreducible. Since rank $\mathrm{Pic}(V'_{\mathscr{P}} = 10, \text{rank } \mathrm{Pic}(\mathbf{F}_m) = 2$, applying the Shioda-Tate formula (2), we obtain that rank $\mathrm{MW}(f') = 7$. Thus the kernel of $\mathrm{cr}_{2,4,4}$ contains a normal subgroup isomorphic to $\mathbb{Z}^7$. It is known to be a normal subgroup of $W_{2,4,4}$. If $m = 1$, the quotient group is isomorphic to $W_{2,3,4}$. If $m > 1$ it must be isomorphic to $(\mathbb{Z}/m\mathbb{Z})^7 \rtimes W_{2,3,4}$; however, I confess that I have not checked this.

The second series of examples generalizes a Coble set. It is the set of 10 nodes of a symmetric determinantal quartic surface (*Cayley symmetroid*). This example has been worked out in detail in an unpublished manuscript [7] (see also [8]). The group of pseudo-automorphisms of $V_{\mathscr{P}}$ contains a normal subgroup of $W_{2,4,6}$ with quotient isomorphic to $\mathrm{Sp}(8, \mathbb{F}_2)$.

Here are some new examples in the cases $p > 2$.

Example 3 Take $p = q = r = 3$, so that $W_{3,3,3}$ is the affine Weyl group of type E_6. We are dealing with the set of 6 points in $\mathbb{P}^2 \times \mathbb{P}^2$ modulo $\mathrm{PGL}_3 \times \mathrm{PGL}_3$. Consider the subsets $\mathscr{P} = (x_1,\dots,x_6)$ such that $\dim|h_1+h_2-x_1-\dots-x_6| = 3$ (one more than expected). It is known that this is equivalent to that the two projections of $\mathscr{P}$ to $\mathbb{P}^2$ are associated sets of 6 points in the plane [10]. It is known that the variety of such pairs modulo projective equivalence is isomorphic to an open subset of $\mathbb{P}^4$ [9]. Since $(h_1+h_2)^4 = 6$, for a general set $\mathscr{P}$, the linear system $L = |h_1+h_2-x_1-\dots-x_6|$ has $\mathscr{P}$ as its set of base points, hence defines a morphism

$$f : V_{\mathscr{P}} \to \mathbb{P}^3 \tag{8}$$

whose general fibre is the intersection of three divisors of type $(1,1)$, hence is an elliptic curve. The exceptional divisors $E_1,\dots,E_6$ are disjoint sections of this fibration.

The restriction of f over a general line ℓ in $\mathbb{P}^3$ defines an elliptic fibration $f_\ell : V_\ell \to \mathbb{P}^1$, where V_ℓ is the blow-up of 6 points in a complete intersection of two divisors in $|h_1+h_2|$ in $\mathbb{P}^2\times\mathbb{P}^2$. The latter is known to be isomorphic to a Del Pezzo surface of degree 6, the blow-up of three non-collinear points in the plane embedded into $\mathbb{P}^8$ by the linear system of plane cubics through the three points. Thus f_ℓ is an elliptic fibration obtained from an Halphen pencil of index 1.

Let $\Delta \subset \mathbb{P}^3$ be the locus of points $x\in\mathbb{P}^3$ such that $f^{-1}(x)$ is a singular fibre. Assume that $\mathscr{P}$ is general enough so that Δ is reduced. It follows from the theory of elliptic fibrations that reducible fibres $f^{-1}(x)$ over points of codimension 1 lie over singular points of Δ of codimension 1. Thus, under our assumption all such fibres are irreducible. We may apply the Shioda-Tate formula to obtain that rank $\mathrm{MW}(f) = 6$. Thus the action of the Mordell-Weil group defines a subgroup in the kernel of $\mathrm{cr}_{3,3,3}$ isomorphic to $\mathbb{Z}^6$. This is known to be a normal subgroup of finite index with quotient isomorphic to $W_{2,3,3}$. A subgroup of finite index has only finitely many orbits in the set of roots of $\mathbb{E}_{3,3,3}$. This shows that the infinitely many discriminant subvarieties of $X_{3,3,3}$ restrict to finite many subvarieties on the open set of points $[\mathscr{P}]$ such that the elliptic fibration (8) has reduced discriminant surface Δ. It remains to check that the set of such points is open in the set of points $[\mathscr{P}]$ with $\dim|h_1+h_2-x_1-\dots-x_6| = 3$. For this it is enough to show that this set is non-empty. This can be done by explicit computation by taking a sufficiently general pair of associated sets of 6 points in the plane (see more about the association in the next section).

Example 4 Take $p=q=3, r=4$. We are dealing with the set of 7 points in $\mathbb{P}^2\times\mathbb{P}^2$ modulo $\mathrm{PGL}_3\times\mathrm{PGL}_3$. The linear system of hypersurfaces of bi-degree $(2,2)$ is of dimension 35. If we take 7 general points $p_1,\dots,p_7$, then we expect a unique hypersurface with double points at the p_i's. However, there is a pencil $\mathscr{L}$ of hypersurfaces of bi-degree $(1,1)$ through these points, so there will be a 2-dimensional linear system $\mathscr{N}$ of hypersurfaces of bi-degree $(2,2)$ with double points at points at $\mathscr{L}$. Let us consider a set $\mathscr{P}$ of 7 points such that there exists a hypersurface Z of bi-degree $(2,2)$ with double points at $p_1,\ldots,p_7$ which does not belong to $\mathscr{N}$. Let C be the intersection of Z with the base surface of the pencil $\mathscr{L}$. It is a curve of arithmetic genus 7 with 7 double points, hence it is a rational curve. We have $-K_{V_{\mathscr{P}}} = 3(h_1+h_2-e_1-\ldots-e_6)$, hence $[C] = (-\frac{2}{3}K_{V_{\mathscr{P}}})^3$, so the locus of points $[\mathscr{P}]$ in $X_{3,3,4}$ with the above property is invariant with respect to the Cremona action.

Let $\mathscr{P}_i = \mathscr{P}\setminus\{p_i\}$. The linear system of hypersurfaces of bi-degree $(1,1)$ through $\mathscr{P}_i$ is of dimension 2 and the linear system of hypersurfaces of bi-degree $(2,2)$ is of dimension 6 and contains Z. Then we get a rational map

$$f_i : V_{\mathscr{P}_i} \to \mathbb{P}^6$$

with the image equal to the cone S over the Veronese surface. Its fibres are elliptic curves, the proper transforms of curves of arithmetic genus 7 with 6 double points. The exceptional divisors E_i are 2-sections of the elliptic fibration f_i. This is all similar to the example of a Coble set so that f_i is an analog of the Halphen pencil of index 2 through 9 double points out of ten nodes of the Coble sextic. An elliptic curve F over a field K and a point $x_0\in F$ of degree 2 defines a degree 2 map $F\to\mathbb{P}^1$ over K. It is given by the linear series $|x_0|$. We apply this to our situation to obtain a birational involution σ_i of $V_{\mathscr{P}_i}$ over S. This involution extends to the localization of f_i over any point $s\in S^{(1)}$. This implies that σ_i is a pseudo-automorphism of order 2. This involution is defined at the point p_i and extends to a pseudo-automorphism $\tilde{\sigma}_i$ of $V_{\mathscr{P}}$. Following the analogy with Coble sets, I speculate that any two involutions $\tilde{\sigma}_i,\tilde{\sigma}_j$ commute and the product $\sigma = \tilde{\sigma}_i\circ\tilde{\sigma}_j$ defines an involution of $K_{V_{\mathscr{P}}}^{\perp}\cong\mathbb{E}_{3,3,4}$ conjugate to the involution $-\mathrm{id}_{\mathbb{E}_{2,3,3}}\oplus\mathrm{id}_U$, where $\mathbb{E}_{2,3,3}$ and U are orthogonal summands of $\mathbb{E}_{3,3,4}$. Also I speculate that the normal subgroup of $W_{3,3,4}$ generated by this involution is of finite index in $W_{3,3,4}$ and generates the 2-level congruence subgroup $W_{3,3,4}(2)$. It will follow then that the quotient group is isomorphic to $\mathrm{O}(6,\mathbb{F}_2)^-$ (see [7], Theorem 2.9.1).

4 Association

Examples of Cremona special sets of points in higher-dimensional spaces $\mathbb{P}^n$ can be obtained by the classical construction called the *association* (in modern time known as the *Gale transform*). It is discussed in detail in [9] or [14]. To give an idea, one considers a linear map $\mathbb{C}^{q+r+1} \to \mathbb{C}^{q+1}$ defined by the matrix with columns equal to projective coordinates of the points. The kernel of this linear map is isomorphic to $\mathbb{C}^{r+1}$, and the transpose map defines a map $\mathbb{C}^{q+r} \to \mathbb{C}^{r+1}$ representing $q+r$ points in $\mathbb{P}^r$. This is well-defined on the projective equivalence classes of ordered points sets. The association defines an isomorphism of GIT-quotients

$$\mathrm{as} : X_{2,q,r} \to X_{2,r,q}.$$

The Coble action of $W_{2,p,q} \cong W_{2,q,r}$ commutes with the association so that the image of a Cremona special set in $\mathbb{P}^q$ is a Cremona special set in $\mathbb{P}^r$. However, to see explicitly the action is rather non-trivial geometric problem.

The following nice example is due to B. Totaro [24].

Example 5 Let $p_1,\ldots,p_9$ be the image of an Halphen set of index 1 under a Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$. This set is associated to the set $\mathscr{P}$(see [3], Thm. 20, [11], Proposition 5.4). Consider the linear system of cubic hypersurfaces with double points at $p_1,\ldots,p_9$. Its dimension is equal to 3 and its base locus consists of 45 curves:36 lines through p_i,p_j, and 9 rational normal curves through all points except one. Let $V_{\mathscr{P}} - \to \mathbb{P}^3$ be the corresponding rational map. Its base locus consists of 45 disjoint $\mathbb{P}^1$'s with normal bundle $\mathscr{O}_{\mathbb{P}^1}(-1)^4$. One can perform a flip on these curves giving another smooth 5-fold W in which the 45 curves are replaced with 45 smooth three-dimensional subvarieties S_i isomorphic to $\mathbb{P}^3$ with normal bundle $\mathscr{O}_{\mathbb{P}^1}(-1)^2$. Now we have a morphism $\Phi : W \to \mathbb{P}^3$ with general fibre isomorphic to an abelian surface. The subvarieties S_i are sections and generate the Mordell-Weil group of rank 8. The exceptional divisors E_i cut out on the fibres twice a principal polarization. The abelian fibration Φ is relatively minimal in the sense that $K_W \cdot C = 0$ for any curve contained in a fibre. This implies that the translations by sections act on W by pseudo-automorphism [20]. Thus $\mathrm{Aut}_{\mathrm{ps}}(V_{\mathscr{P}}) \cong \mathrm{Aut}_{\mathrm{ps}}(W)$ contains a subgroup isomorphic to $\mathbb{Z}^8$. It is known that the such a group must be of finite index in $W_{2,3,6}$.

The associated set of the set $\mathscr{P} = \{p_1,\ldots,p_{10}\}$ of 10 nodes of a quartic symmetroid surface is a set of 10 points in $\mathbb{P}^5$ which is equal to the intersection of two Veronese surfaces [3], Theorem 26. The secant variety of a Veronese surface is a cubic hypersurface singular along the surface. This shows that the associated set $\mathscr{Q} = \{q_1,\ldots,q_{10}\}$ is contained in the base locus of a pencil of cubic hypersurfaces with double points at the q_i's. The set of 9 points $\mathscr{Q}_i = \mathscr{Q} \setminus \{q_i\}$ is associated to the set Σ_i of 9 points in the plane equal to the projection of the set $\mathscr{P}_i \setminus \{p_i\}$ from p_i [9], Chapter 3, Prop. 4. It is known that the set Σ_i is equal to the set of base points of a pencil of cubics. In fact, this property distinguishes Cayley symmetroids from other 10-nodal quartic surfaces [6]. Thus any subset of 9 points $\mathscr{Q}_i$ in $\mathscr{Q}$ is a set from Totaro's example. Let $G_i \cong \mathbb{Z}^8$ be the group of pseudo-automorphisms of $V_{\mathscr{Q}_i}$ defined by the Mordell-Weil of the corresponding abelian fibration. The point p_i is a singular point of one of its fibres and is a fixed point of G_i. Thus G_i extends to a group of pseudo-automorphisms of $V_{\mathscr{P}}$. One may ask whether the subgroups G_i generate a subgroup of finite index in $W_{2,4,6}$. I do not know the answer.

Note that the permutation group $\mathfrak{S}_3$ acts on the Weyl group $W_{p,q,r}$ via permuting (p,q,r). The relation between $X_{p,q,r}$ and $X_{p,r,q}$ is the product of $p-1$ copies of the association map. This defines an isomorphism $\mathrm{as}_{23} : X_{p,q,r} \to X_{p,r,q}$. The varieties $X_{q,p,r}$ and $X_{p,q,r}$ are not isomorphic but there exists a natural birational isomorphism. In fact, let us consider $X_{p,q,r}$ as the GIT-quotient of the product of $p-1$ copies of $(\mathbb{P}^{q-1})^{q+r}$ by $\mathrm{PGL}(q)^{p-1}$. Using $\mathrm{PGL}(q)$ in each copy we can fix the first $q+1$ points among $q+r$-points. The quotient becomes birationally isomorphic to $((\mathbb{P}^{q-1})^{r-1})^{p-1}$, which is birationally isomorphic to $\mathbb{C}^{(p-1)(q-1)(r-1)}$. Now if we do the same with $X_{q,p,r}$ we obtain a birational model isomorphic to $((\mathbb{P}^{p-1})^{r-1})^{q-1}$. which is birationally isomorphic to the same space $\mathbb{C}^{(p-1)(q-1)(r-1)}$.

Example 6 Consider the set of Cayley octads in $\mathbb{P}^3$ as a generalized Halphen set of index 1. This is a Cremona special set for $W_{2,4,4}$. It is self-associated, with respect to the symmetry of the Dynkin diagram. Now consider the set of 6 points $q_1,\dots,q_6$ in $(\mathbb{P}^1)^3$. It corresponds to $(p,q,r) = (4,2,4)$. Make the six points Cremona special by requiring that the linear system of divisors of type $(1,1,1)$ containing $\mathscr{P}$ is of dimension one larger than expected. The rational map given by this linear system defines an elliptic fibration $f : V_{\mathscr{P}} \to \mathbb{P}^2$. It has 6 disjoint sections defined by the exceptional divisors $E_1,\dots,E_6$. Applying the Shioda-Tate formula, we obtain that the rank of the Mordell-Weil group is equal to 7. This is a subgroup of finite index in $W_{2,4,4}$.

Conjecture 1 Let $\sigma \in \mathfrak{S}_3$ and $\sigma : X_{p,q,r} \to X_{\sigma(p),\sigma(q),\sigma(r)}$ be the birational map described above. Then

$$\mathrm{cr}_{\sigma(p),\sigma(q),\sigma(r)} = \sigma \circ \mathrm{cr}_{p,q,r} \circ \sigma^{-1}.$$

It follows from this conjecture that the image of a Cremona special set of points under the σ-association is a Cremona special set of points.

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Stable bundles and polyvector fields

Nigel Hitchin

Abstract We introduce an algebra of Schouten-commuting holomorphic polyvector fields on the moduli space of stable G-bundles over a curve by using invariant forms on the Lie algebra. The generators begin in degree three – we prove a vanishing theorem for degree two in the case of $G = GL(n)$.

Keywords polyvector, stable bundle, moduli, Higgs field, Schouten-Nijenhuis.
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1 Introduction

The moduli space of stable bundles on an algebraic curve C is a much-studied object, but there are still new things to learn about it. This paper introduces one more aspect to study, and poses some conjectures about it.

Recall from [7] that, if M is such a moduli space, then its cotangent bundle T^*M defines a completely integrable Hamiltonian system. By this we mean that there are $\dim M$ functionally independent holomorphic functions on T^*M which Poisson-commute and whose common level set is an open set in an abelian variety, on which the Hamiltonian vector fields are linear. These functions are polynomial in the fibre directions and can be understood on the moduli space M itself as holomorphic sections of symmetric powers $\mathrm{Sym}^k T$ of the tangent bundle for various values of k.

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The fact that they Poisson commute is equivalent to the statement that the symmetric tensors commute using the Schouten–Nijenhuis bracket, a natural extension of the Lie bracket on vector fields.

We introduce here a skew-symmetric version of this, identifying holomorphic sections of $\Lambda^k T$ for various values of k (so-called *polyvector fields*) which also Schouten-commute. More precisely we note that at a smooth point of the moduli space of stable holomorphic structures on a principal G-bundle, where G is a complex simple Lie group, the cotangent space is isomorphic to $H^0(C,\mathfrak{g}\otimes K)$ where $\mathfrak{g}$ denotes the adjoint bundle of Lie algebras. Given a bi-invariant differential form ρ on G of degree k, then for $\Phi_i \in H^0(C,\mathfrak{g}\otimes K)$, $\rho(\Phi_1,\ldots,\Phi_k)$ defines a skew form with values in the line bundle K^k. Dually, it defines a homomorphism

$$H^1(C,K^{1-k}) \to H^0(M,\Lambda^k T).$$

By analogy with the symmetric case there are three obvious questions to ask:

- Is this map injective?
- Do these polyvector fields Schouten-commute?
- Is the algebra of all polyvector fields on M generated by these?

In this paper we restrict ourselves mainly to the rank one case where the only invariant form is $B([X,Y],Z)$ where B is the Killing form, but many of our results hold in more generality. We answer in the positive the first question (for genus $g>2$), and show that for general reasons the answer is yes to the second. As to the third question the Verlinde formula shows that the answer is no, though in the final section we discuss some related issues.

What we do show, however, is that for $g>4$, and in the case of vector bundles with coprime degree and rank, there are no polyvector fields of degree two. The vanishing in degree one is well-known [10], [7], so that the first degree where existence holds is precisely where our construction begins. Our proof of the vanishing of $H^0(M,\Lambda^2 T)$ requires another feature of the moduli space, which was the original motivation for this research. In [10], the authors used a holomorphic differential on C to define on the moduli space a nontrivial extension

$$0 \to T^* \to \bigoplus_{i=1}^{2g-2} \mathfrak{g}_{x_i} \to T \to 0$$

where $\mathfrak{g}_{x_i}$ is the restriction of the universal adjoint bundle to $M \times \{x_i\}$ and $x_i \in C$ is a zero of the differential. A considerable part of the paper consists of studying the vector bundles E defined by these extensions in more detail.

The most important point is that the extension class lies in the skew-symmetric part $H^1(M, \Lambda^2 T^*) \subset H^1(M, \mathrm{Hom}(T, T^*))$ and realizes the known isomorphism between the space of differentials $H^0(C, K)$ and $H^1(M, \Lambda^2 T^*)$. This provides an orthogonal structure on E such that the subbundle T^* is maximal isotropic. We also show that these bundles have a natural Courant algebroid structure arising from an infinite-dimensional quotient construction.

One result we need is an isomorphism $H^1(M, T \otimes T^*) \cong H^1(C, \mathscr{O})$, proved in [3]. We shall see this isomorphism being realized as a deformation of the tangent bundle of M by replacing $H^1(C, \mathfrak{g})$ by $H^1(C, \mathfrak{g} \otimes L)$ for a degree zero line bundle L.

2 Polyvector fields

2.1 The construction

We set up the basic framework in the case of a general simple Lie group G. Let C be a compact Riemann surface and M be the moduli space of stable principal G-bundles on C. At a smooth point of M the tangent space T is isomorphic to $H^1(C, \mathfrak{g})$ where $\mathfrak{g}$ denotes the adjoint bundle, and its dual space T^* is, by Serre duality, $H^0(C, \mathfrak{g} \otimes K)$. We shall call sections Φ of $\mathfrak{g} \otimes K$ Higgs fields.

Evaluation of a Higgs field at $x \in C$ defines a homomorphism from T^* to $\mathfrak{g}_x \otimes K_x$ and so a section

$$s_x \in H^0(M, \mathfrak{g}_x \otimes T) \otimes K_x.$$

If $\mathfrak{g}$ now denotes the universal adjoint bundle over the product $M \times C$, then varying x we get a tautological section

$$s \in H^0(M \times C, \mathfrak{g} \otimes (T \boxtimes K)).$$

Consider now the ring of bi-invariant differential forms on G. This is an exterior algebra generated by basic forms whose degrees are given by $k_i = 2m_i + 1$ where m_i are the exponents of the Lie algebra. For each generator σ_i we can evaluate on the section

$$s^{\wedge k_i} \in H^0(M \times C, \Lambda^{k_i}(\mathfrak{g} \otimes (T \boxtimes K)).$$

to obtain

$$s_i \in H^0(M \times C, \Lambda^{k_i} T \boxtimes K^{k_i}) \cong H^0(M, \Lambda^{k_i} T) \otimes H^0(C, K^{k_i})$$

or equivalently by Serre duality a homomorphism

$$A : H^1(C, K^{1-k_i}) \to H^0(M, \Lambda^{k_i} T). \tag{1}$$

Examples:

1. The simplest invariant form for any G is $\sigma(X,Y,Z) = B([X,Y],Z)$ where B is the Killing form. Thus the $(5g-5)$-dimensional space $H^1(C,K^{-2})$ maps to $H^0(M,\Lambda^3 T)$.

2. For each point $x \in C$, evaluation of a section of K^k at x (and a trivialization of K_x^k) defines a linear form on $H^0(C,K^{k_i})$ and hence an element of its dual space $H^1(C,K^{1-k_i})$, so this defines a section $\sigma_x \in H^0(M,\Lambda^{k_i} T)$.

2.2 Injectivity

If the map A in (1) for an invariant form σ of degree k has a non-zero kernel then there is a class $\alpha \in H^1(C,K^{1-k})$ such that for all G-bundles and Higgs fields $\Phi_1,\dots,\Phi_k$

$$\langle \alpha, \sigma(\Phi_1,\dots,\Phi_k) \rangle = 0$$

where $\langle\ ,\ \rangle$ is the Serre duality pairing. Thus for injectivity we need to show that the sections $\sigma(\Phi_1,\dots,\Phi_k)$ generate $H^0(C,K^k)$. Here for simplicity we restrict to the rank one case.

Remark: We should make a remark here about which moduli spaces we are concerned with. The setting for the problem is the space of stable principal G-bundles modulo isomorphism, but the definition of the polyvector fields only depends on the structure of the Lie algebra so it is really the adjoint group which is relevant here. In the case of a linear group the most studied moduli space is that of vector bundles of rank n and degree d with fixed determinant bundle. Especially important is the case where n and d are coprime for the moduli space then is compact and smooth and has a universal vector bundle. But it is the (singular) quotient of this by the operation of tensoring with a line bundle of order n which gives the adjoint bundle moduli space. This point will become relevant in the final section.

Proposition 2.1 *If $g > 2$ the map A is injective for $G = SL(2)$ or $SO(3)$.*

Remark: The map is not injective for $g = 2$. In fact there are two spaces, P^3 is the moduli space of bundles for even degree and the intersection of two quadrics in P^5 for bundles of odd degree. In both cases these are acted on trivially by the hyperelliptic involution τ on C , and in particular the action on sections of $\Lambda^3 T$ is trivial. But $H^0(C,K^3)$ has both invariant and anti-invariant elements under τ.

Proof: There is just one invariant form here – the three-form σ given by $\sigma(X,Y,Z) = B([X,Y],Z)$.

1. We begin with the even degree case, and we may consider a class to be represented by a rank 2 vector bundle E with $\Lambda^2 E$ trivial. Consider first a non-trivial extension of degree zero line bundles

$$0 \to L \to E \to L^* \to 0 \tag{2}$$

defined by $\alpha \in H^1(C,L^2)$. Take a point $x \in C$, and let t denote the tautological section of $\mathscr{O}(x)$. The homomorphism $L^*(-x) \to L^*$ defined by the product with t lifts to E if the class $\alpha t \in H^1(C,L^2(x))$ vanishes. The long exact sequence of

$$0 \to \mathscr{O}_C(L^2) \xrightarrow{t} \mathscr{O}_C(L^2(x)) \to \mathscr{O}_x(L^2(x)) \to 0$$

gives

$$\to H^0(C,L^2(x)) \to L^2(x)|_x \to H^1(C,L^2) \xrightarrow{t} H^1(C,L^2(x)) \to$$

so if $H^0(C,L^2(x)) = 0$ there is a unique α with this property – the image of a vector in $L^2(x)|_x$ under the connecting homomorphism. Moreover from the exact sequence of (2) the lift is then unique.

The lift defines a section of $\mathrm{Hom}(L^*(-x),E) = EL(x)$. This is an inclusion unless it vanishes at x but if that were so, it would come from a section of EL and in the long exact sequence of

$$0 \to L^2 \to EL \to \mathscr{O} \to 0$$

we see that the generator of $H^0(C,\mathscr{O})$ maps to $\alpha \in H^1(C,L^2)$ so if L^2 is non-trivial, $H^0(C,EL) = 0$.

Hence if $H^0(C,L^2(x)) = 0$ (which implies L^2 is non-trivial), the lift of $t : L^*(-x) \to L^*$ to E gives another expression of E as an extension

$$0 \to L^*(-x) \to E \to L(x) \to 0.$$

If $H^0(C, L^2(x)) \neq 0$ then there is a point y such that the divisor class $[L^2] \sim y - x$. This defines a two-dimensional subvariety of the Jacobian and hence for $g > 2$ a generic line bundle L has the property that for all x, $H^0(C, L^2(x)) = 0$.

From [9] a generic element in the $(g+1)$-dimensional space $H^1(C, L^{-2}(-2x))$ defines an extension as above which is a stable bundle. Thus, as we vary L and x and the extension class, E belongs to a family whose generic member is stable. Moreover, although E itself is not stable it is simple, i.e. it has no non-scalar endomorphisms. This means that the rank of $H^0(C, \mathfrak{g} \otimes K) = 3g - 3$ for all bundles in the family.

We shall show that, varying L and x, $\sigma(\Phi_1, \Phi_2, \Phi_3)$ generates $H^0(C, K^3)$ and hence will do so in a generic family of stable bundles.

2. We now need to determine the Higgs fields for E. The adjoint bundle $\mathfrak{g}$ is $\mathrm{End}_0 E$, the bundle of trace zero endomorphisms. We have an exact sequence

$$0 \to E \otimes LK \to \mathrm{End}_0 E \otimes K \to L^{-2}K \to 0$$

and so a section s of $L^{-2}K$ lifts to a Higgs field Φ_1 if it maps in the long exact sequence to zero in $H^1(C, E \otimes LK)$. Now $H^1(C, L^2K)$ is dual to $H^0(C, L^{-2})$ which vanishes if L^2 is non-trivial which means from the long exact sequence of (2) that $H^1(C, E \otimes LK) \cong H^1(C, K)$ and so $s \in H^0(C, L^{-2}K)$ extends if its product with the extension class $\alpha \in H^1(C, L^2)$ vanishes. Choosing the class as above, this means that $s(x) = 0$.

Now let Φ_1 be a lift of s. In the exact sequence

$$0 \to L^2K \to E \otimes LK \to K \to 0$$

since $H^1(C, L^2K) = 0$ the map $H^0(C, E \otimes LK) \to H^0(C, K)$ is surjective, so given a section t of K we can find Φ_2 a section of $E \otimes LK \subset \mathrm{End}_0 E \otimes K$ such that Φ_2 maps to t. Now choose Φ_3 to be any section u of $L^2K \subset E \otimes LK \subset \mathrm{End}_0 E \otimes K$. The $(3g-3)$-dimensional space of Higgs fields can now be seen to be constructed from s, in the $(g-2)$-dimensional subspace of $H^0(C, L^{-2}K)$ consisting of sections that vanish at x, a choice of t in the g-dimensional space of differentials, and an arbitrary section u in the $(g-1)$-dimensional space of sections of L^2K.

It is easy to see then that $\sigma(\Phi_1, \Phi_2, \Phi_3) \in H^0(C, K^3)$ is a multiple of stu. We shall show that we can generate all sections of K^3 vanishing at x this way, for L generic.

3. We use the "base-point free pencil trick" of [1]: let U be a line bundle with sections s_1, s_2 having no common zeros and let F be a vector bundle. Then the kernel of the map

$$\mathbf{C}^2 \otimes H^0(C, F) \to H^0(C, F \otimes U)$$

defined by $(t_1, t_2) \mapsto s_1 t_1 + s_2 t_2$ is isomorphic to $H^0(C, F \otimes U^*)$. Indeed, if $s_1 t_1 = -s_2 t_2$ and s_1, s_2 have no common zeros then $t_1 = us_2, t_2 = -us_1$ for u a section of $F \otimes U^*$.

The bundle L^2K has a basepoint x if $H^1(C, L^2K(-x)) \neq 0$, or by Serre duality if $H^0(C, L^{-2}(x)) \neq 0$. As above, considering $[L^2] \sim y - x$, if $g > 2$ it follows that for generic L, L^2K has no basepoint. Consider the sequence

$$0 \to \mathcal{O}_C L^2 K(-x-y) \to \mathcal{O}_C L^2 K(-x) \to \mathcal{O}_y L^2 K(-x) \to 0.$$

In the long exact sequence we see that if $H^1(C, L^2K(-x-y)) \to H^1(C, L^2K(-x))$ is injective, sections of L^2K vanishing at x do not all vanish at a given point y. This injectivity condition is equivalent to the map $H^0(C, L^{-2}(x)) \to H^0(C, L^{-2}(x+y))$ being surjective. But if $L^{-2}(x+y)$ has a section then $[L^2] \sim x + y - u - v$. Thus if the genus $g > 4$ then for generic L there are no sections. By Riemann-Roch $\dim H^0(C, L^2K) = g - 1$; this system has no basepoint and separates points hence the map $C \to \mathrm{P}^{g-2}$ is injective and we can then use general position arguments.

Take a general divisor $x_1 + \cdots + x_{2g-2}$ of L^2K. By general position the space of sections vanishing on the first $(g-3)$ of these points is a base-point free two-dimensional space: sections of $L^2K(-D)$ where $D = x_1 + \cdots + x_{g-3}$. Taking $F = K$ in the "base-point free pencil trick", we have, by taking products with s_1 and s_2, a subspace of $H^0(C, L^2K^2)$ of dimension $2g - \dim H^0(C, L^{-2}(D))$. But by Riemann-Roch

$$\dim H^0(C, L^{-2}(D)) - \dim H^1(C, L^{-2}(D)) = g - 3 + 1 - g = -2$$

and by Serre duality $\dim H^1(C, L^{-2}(D)) = \dim H^0(C, L^2K(-D)) = 2$. Hence we obtain $H^0(C, L^{-2}(D)) = 0$. This means that s_2, s_2 generate with sections of K a $2g$-dimensional subspace of $H^0(C, L^2K^2)$, which by Riemann-Roch has dimension $(3g-3)$.

However, by general position, for each point x_i there is a section s_i of L^2K which vanishes at all points $x_1,\ldots,x_{g-3}$ except x_i. Multiplying these by sections of K gives a complementary $(g-3)$-dimensional subspace and hence in total $2g+(g-3)=3g-3$ linearly independent sections. Hence sections of L^2K^2 are generated by sections of L^2K and K.

Given $x\in C$ and $2g-3$ general points $x_1,x_2,\ldots,x_{2g-3}$ there is a section q of K^3 vanishing at these points since K^3 defines an embedding $C\subset \mathrm{P}^{5g-6}$. But since $2g-3\geq g$, $2g-3$ general points form the divisor for a generic line bundle $L^{-2}K(-x)$. Hence the divisor of q is of the form $x+D_1+D_2$ where D_1 is the divisor of a section of $L^{-2}K(-x)$ and D_2 of L^2K^2. Using the above result about sections of L^2K^2 we see that sections of $L^{-2}K(-x),K,L^2K$ generate $H^0(C,K^3(-x))$.

Varying x, by genericity we can generate all sections of K^3 from extensions E and hence also from stable bundles.

There remain the cases $g=3,4$. For $g=4$ a generic degree $2g-2=6$ line bundle maps C birationally to a singular sextic curve in P^2 and the genericity theorem holds here. For genus $g=3$, the image of $H^0(C,L^2K)\otimes H^0(C,K)$ has dimension $2\times 3=6=3g-3$ by the base-point trick.

The case of odd degree can be considered as the study of the moduli space of rank 2 vector bundles E with $\Lambda^2E\cong\mathcal{O}(-y)$. Here, from [9] each non-trivial extension

$$0\to L\to E\to L^*(-y)\to 0 \tag{3}$$

is stable (and indeed each stable bundle arises this way). The extension is defined by a class $\alpha\in H^1(C,L^2(y))$ as before and $s\in H^0(C,L^{-2}K(-y))$ lifts to a Higgs field Φ_1 if its product with α vanishes. For some point x we can take as above an extension where, for $g>4$, this condition is that $s(x)=0$. We now need to prove that any section of K^3 vanishing at x is generated by sections of the g-dimensional spaces $H^0(C,L^2K(y)),H^0(C,K)$ and the $(g-3)$-dimensional space $H^0(C,L^{-2}K(-x-y))$ for some y. The argument proceeds as before when L is chosen so that sections of $L^{-2}K$ map C birationally to its image. For genus 3, $L^2K(y)$ defines an embedding for generic L, so sections of $L^2K(y)$ and K generate sections of $L^2K^2(y)$. $\square$

Remark: To consider the case of general G we could at this point use our rank one starting point and take the homomorphism from $SL(2)$ to G given by the principal three-dimensional subgroup [5] to define a G-bundle. This breaks up the Lie algebra $\mathfrak{g}$ into irreducible representations of $SO(3)$ whose dimensions are precisely the degrees of the generators of the algebra of invariant differential forms on G. It seems reasonable to conjecture that the restriction of a generating form of degree $(2m+1)$ to the corresponding subspace of the same dimension is non-zero, but this seems not to have been proved, except for $m=1$ where it is clear. If it were true then a direct generalization of the above would give injectivity in general, though there may well be other means of achieving this. As it stands we can assert that $H^1(C,K^{-2}) \to H^0(M,\Lambda^3 T)$ is injective for all G.

2.3 The Schouten-Nijenhuis bracket

If A,B are sections of $\Lambda^p T$ and $\Lambda^q T$ respectively on any smooth manifold then one can form the *Schouten-Nijenhuis bracket* $[A,B]$ which is a section of $\Lambda^{p+q-1}T$, generalizing the Lie bracket of two vector fields. It has the basic properties:

- For each vector field X, $[X,A] = \mathscr{L}_X A$
- $[A,B] = -(-1)^{(p-1)(q-1)}[B,A]$
- $[A,B\wedge C] = [A,B]\wedge C + (-1)^{(p-1)q} B\wedge [A,C]$

Remark: There is a similar bracket on sections of $\mathrm{Sym}^p T$ and $\mathrm{Sym}^q T$ which corresponds to the Poisson bracket of the corresponding functions on the total space of the cotangent bundle T^* with respect to the canonical symplectic form.

On a complex manifold, the Schouten-Nijenhuis bracket on the sheaf of holomorphic polyvector fields extends to give a Gerstenhaber algebra structure on $H^*(M,\Lambda^* T)$. We shall show here that the global polyvector fields just constructed commute with respect to this bracket.

We adopt an infinite-dimensional viewpoint which can be made rigorous in a standard way by using Banach manifolds and slice theorems. Consider the moduli space M of stable bundles as the quotient of an open set in the space $\mathscr{A}$ of all $\bar{\partial}$-operators $\bar{\partial}_A$ on a fixed C^∞ bundle by the group $\mathscr{G}$ of complex gauge transformations. The space $\mathscr{A}$ is an infinite-dimensional affine space with translation group $\Omega^{0,1}(C,\mathfrak{g})$. The cotangent space at any point is formally $\Omega^0(C,\mathfrak{g}\otimes K) = \Omega^{1,0}(C,\mathfrak{g})$ using the pairing for $a\in\Omega^{0,1}(C,\mathfrak{g})$ and $\Phi\in\Omega^0(C,\mathfrak{g}\otimes K)$,

$$\int_C B(\Phi,a)$$

where B is the Killing form.

Take $\alpha \in \Omega^{01}(C, K^{1-k})$ representing a class in $H^1(C, K^{1-k})$ and an invariant k-form σ on $\mathfrak{g}$. Then define a k-vector field S on $\mathscr{A}$ by evaluating on cotangent vectors $\Phi_i \in \Omega^0(C, \mathfrak{g} \otimes K)$:

$$S(\Phi_1, \ldots, \Phi_k) = \int_C \sigma(\Phi_1, \ldots, \Phi_k)\alpha.$$

Since α is independent of the operator $\bar{\partial}_A$ such a polyvector field on $\mathscr{A}$ is translation invariant (has "constant coefficients"), so any two Schouten-commute.

But S is gauge-invariant because σ is invariant, so under the derivative of the quotient map from the open set of stable points in $\mathscr{A}$ to M,

$$\Lambda^k T_A \mathscr{A} \to \Lambda^k T_{[A]} M$$

the image is independent of the representative point A, and so defines a polyvector field $\bar{S}$ on M. Note that an invariant polyvector field S is not the same as a polyvector field $\bar{S}$ on the quotient but $\bar{S}$ is defined by evaluating on 1-forms which are pulled back. In our case these are *holomorphic* sections Φ_i, and then $\sigma(\Phi_1, \ldots, \Phi_k)$ is a holomorphic section of K^k, and by Stokes' theorem only the Dolbeault cohomology class of α contributes in the definition.

We shall show in Section 4 that there are no holomorphic bivector fields, sections of $\Lambda^2 T$, for $g > 4$ on the moduli space of stable vector bundles when the rank and degree are coprime. Since we have seen above that there always exist non-trivial holomorphic trivector fields, this is some information towards answering the third question in the Introduction. The result most probably extends to other groups but we shall use theorems in the literature which relate to this particularly familiar case.

Our approach revisits a vector bundle on the moduli space first introduced by Narasimhan and Ramanan [10], but where we observe some extra features.

3 Orthogonal bundles on the moduli space

3.1 Courant algebroids

We need the notion of a holomorphic (exact) Courant algebroid. This is a vector bundle E given as an extension

$$0 \to T^* \to E \xrightarrow{\pi} T \to 0$$

with the following properties.

- E has an orthogonal structure – a nondegenerate symmetric form $(\ ,\ )$ such that $T^* \subset E$ is isotropic.
- For local sections u, v there is another local section $[u,v]$, skew-symmetric in u, v, such that :

  (i) if f is a local function $[u, fv] = f[u,v] + (\pi(u)f)v - (u,v)df$

  (ii) $\pi(u)(v,w) = ([u,v] + d(u,v), w) + (v, [u,w] + d(u,w))$

  (iii) $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = d(([u,v],w) + ([w,u],v) + ([v,w],u))/3$

The standard example is $T \oplus T^*$ with symmetric form

$$(X+\xi, X+\xi) = i_X \xi$$

and bracket

$$[X+\xi, Y+\eta] = [X,Y] + \mathscr{L}_X \eta - \mathscr{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi).$$

There is a natural quotient construction for Courant algebroids (see [4]) which we describe next. Suppose M is a manifold with a free proper action of a Lie group G, and suppose that there is a lifted action on E preserving all the structure, in particular being compatible with $\pi : E \to T$ and the natural action on T. The derivative of the group action defines a Lie algebra homomorphism $a \mapsto X_a$ from $\mathfrak{g}$ to vector fields, sections of T, and we ask for an equivariant *extended action* which is a linear map e from $\mathfrak{g}$ to sections of E such that:

- for $a, b \in \mathfrak{g}$ we have $[e(a), e(b)] = e([a,b])$
- $(e(a), e(a)) = 0$
- $\pi(e(a)) = X_a$

Given this data, $e(\mathfrak{g})$ generates a trivial subbundle $F \subset E$ of rank $\dim G$. It is isotropic by the second condition, so $F \subset F^\perp$, and $F^\perp/F$ inherits a nondegenerate symmetric form. The latter is a G-invariant bundle of rank $(2\dim M - 2\dim G) = 2\dim(M/G)$. By G-invariance it descends to a bundle $\bar{E}$ on M/G. The G-invariant sections of $F^\perp/F$ are by definition the sections of $\bar{E}$ on M/G and the bracket on G-invariant sections defines a bracket on sections of $\bar{E}$.

Now $\pi(F)$ is the tangent bundle along the fibres of $M \to M/G$, so π induces a map from $F^{\perp}/F$ to $T(M/G)$ and one can easily deduce that $\bar{E}$ is a Courant algebroid over M/G.

3.2 A family of Courant algebroids

We shall give here an infinite-dimensional example of the above construction to produce (for general G) a family of Courant algebroids over the moduli space of stable G-bundles.

As in Section 2.3 let $\mathscr{A}$ denote the infinite-dimensional space of all holomorphic structures on a fixed principal G-bundle. It is acted on by the group $\mathscr{G}$ of complex gauge transformations, and the quotient of the open set of stable holomorphic structures by $\mathscr{G}$ is the finite-dimensional moduli space of dimension $\dim G(g-1)$. So $\mathscr{A}$ is our manifold with $\mathscr{G}$-action and we are going to define an extended action on the trivial Courant algebroid $T \oplus T^*$. For this we consider as above the cotangent space to be $\Omega^{1,0}(C,\mathfrak{g})$.

To define an extended action we choose a holomorphic 1-form $\theta \in H^0(C,K)$ and define, for ψ in the Lie algebra $\Omega^0(C,\mathfrak{g})$ of $\mathscr{G}$,

$$e(\psi)(a) = (\bar{\partial}_A \psi, \psi\theta) \in \Omega^{0,1}(C,\mathfrak{g}) \oplus \Omega^{1,0}(C,\mathfrak{g}).$$

We check the isotropy condition:

$$(e(\psi), e(\psi)) = \int_C B(\psi\theta, \bar{\partial}_A \psi) = \frac{1}{2}\int_C \bar{\partial}(\theta B(\psi,\psi)) = 0$$

since θ is holomorphic.

To check the bracket condition $[e(\psi), e(\psi')] = e([\psi,\psi'])$ note that $\psi\theta$ is independent of A and is thus a translation-invariant 1-form on $\mathscr{A}$ and hence is closed. Thus using $\mathscr{L}_X = di_X + i_X d$ we have, for $\xi = \psi\theta, \eta = \psi'\theta$, $X = \bar{\partial}_A\psi, Y = \bar{\partial}_A\psi'$

$$\mathscr{L}_X\eta - \mathscr{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) = \frac{1}{2}d(i_X\eta - i_Y\xi).$$

Now

$$i_X\eta - i_Y\xi = \int_C B(\psi'\theta, \bar{\partial}_A\psi) - B(\psi\theta, \bar{\partial}_A\psi')$$

and $d(i_X\eta - i_Y\xi)$ evaluated on $a \in \Omega^{0,1}(C,\mathfrak{g})$ is

$$\int_C B(\psi'\theta,[a,\psi]) - B(\psi\theta,[a,\psi']) = -\int_C \theta B([\psi',\psi],a) - \theta B([\psi,\psi'],a)$$
$$= 2\int_C \theta B([\psi,\psi'],a).$$

This does then define an extended action and we can produce a quotient Courant algebroid as in Section 3.1. In our case the space F generated by the Lie algebra of $\mathscr{G}$ consists of the subspace

$$B^1 = \{(\bar\partial_A\psi,\psi\theta) \in \Omega^{0,1}(C,\mathfrak{g}) \oplus \Omega^{1,0}(C,\mathfrak{g})\}$$

and $F^\perp$ is the space of pairs $(a,\Phi) \in \Omega^{0,1}(C,\mathfrak{g}) \oplus \Omega^{1,0}(C,\mathfrak{g})$ such that

$$\int_C B(\Phi,\bar\partial_A\psi) + B(\psi\theta,a) = 0$$

for all $\psi \in \Omega^0(C,\mathfrak{g})$. By integration by parts this is

$$Z^1 = \{(a,\Phi) \in \Omega^{0,1}(C,\mathfrak{g}) \oplus \Omega^{1,0}(C,\mathfrak{g}) : \bar\partial_A\Phi = a\theta\}.$$

Hence $F^\perp/F = Z^1/B^1$ is the first cohomology group of the complex

$$\Omega^0(C,\mathfrak{g}) \stackrel{\bar\partial+\theta}{\to} \Omega^{0,1}(C,\mathfrak{g}) \oplus \Omega^{1,0}(C,\mathfrak{g}) \stackrel{\bar\partial+\theta}{\to} \Omega^{1,1}(C,\mathfrak{g})$$

or equivalently the hypercohomology $\mathbb{H}^1(C,\mathfrak{g})$ of the short complex of sheaves

$$\mathscr{O}(\mathfrak{g}) \stackrel{\theta}{\to} \mathscr{O}(\mathfrak{g}\otimes K).$$

From the first hypercohomology spectral sequence we have an exact sequence

$$H^0(C,\mathfrak{g}) \to H^0(C,\mathfrak{g}\otimes K) \to \mathbb{H}^1(C,\mathfrak{g}) \to H^1(C,\mathfrak{g}) \to H^1(C,\mathfrak{g}\otimes K)$$

which for stable bundles gives us the expected extension

$$0 \to T^* \to \mathbb{H}^1(C,\mathfrak{g}) \to T \to 0.$$

For the second sequence, if $\mathscr{Q}$ is the quotient sheaf

$$0 \to \mathscr{O}(\mathfrak{g}) \stackrel{\theta}{\to} \mathscr{O}(\mathfrak{g}\otimes K) \to \mathscr{Q} \to 0$$

we have

$$0 \to \mathbb{H}^1(C,\mathfrak{g}) \stackrel{\cong}{\to} H^0(C,\mathscr{Q}) \to 0.$$

But $\mathscr{Q}$ is supported on the zero-set of the differential θ. So for generic θ with simple zeros $x_1,\dots,x_{2g-2}$ we have an isomorphism from $\mathbb{H}^1(C,\mathfrak{g})$ to

$$\bigoplus_{i=1}^{2g-2} (\mathfrak{g} \otimes K)_{x_i}.$$

Denoting by $\mathfrak{g}_x$ the universal adjoint bundle restricted to $M \times \{x\}$ we find that the Courant algebroid E on M produced by our quotient construction is a direct sum of bundles

$$E \cong \bigoplus_{i=1}^{2g-2} \mathfrak{g}_{x_i} \otimes K_{x_i}. \tag{4}$$

Remarks:

1. This vector bundle and its description as an extension appeared in the paper [10]. It is the simplest way to see that the total Pontryagin class of M (the total Chern class of $T \oplus T^*$) is of the form $p(T) = c(\mathfrak{g})^{2g-2}$. Neither the symmetric form nor the Courant bracket played a role in its initial introduction.

2. The extension $0 \to T^* \to E \to T \to 0$ defines a class in $H^1(M, T^* \otimes T^*)$ but the orthogonal structure, and the fact that T^* is isotropic, tells us that the class lies in $H^1(M, \Lambda^2 T^*)$. Each such extension depended on a choice of differential θ so we have a natural homomorphism $H^0(C, K) \to H^1(M, \Lambda^2 T^*)$. For vector bundles this is an isomorphism – in fact from [10] deformations of the point $x \in C$ give non-trivial deformations of $\mathfrak{g}_x$ and hence from (4) non-trivial deformations of E, in particular non-trivial extension classes, so the map is injective; but both spaces are g-dimensional. The more usual description of this isomorphism is the dual one – the intermediate Jacobian of M is isomorphic to the Jacobian of C [10] hence $H^1(C, \mathscr{O}) \cong H^2(M, \Lambda^1 T^*)$.

3.3 *The orthogonal structure*

Let A be a holomorphic structure on the principal bundle and $e \in E_{[A]}$ a vector in the fibre of E over $[A] \in M$. Then e is represented by $(a, \Phi) \in \Omega^{0,1}(C, \mathfrak{g}) \oplus \Omega^{1,0}(C, \mathfrak{g})$ where

$$\bar{\partial}_A \Phi = a\theta.$$

The inner product is defined by

$$(e, e) = \int_C B(\Phi, a).$$

Surround each zero of θ by a small disc and let C' be the complement of these discs, then $a = \theta^{-1}\bar{\partial}_A\varphi$ is smooth on C' and

$$\int_{C'} B(\Phi, a) = \int_{C'} \frac{1}{\theta} B(\Phi, \bar{\partial}_A \Phi) = \frac{1}{2}\int_{C'} \bar{\partial}(\frac{1}{\theta} B(\Phi, \Phi)) = 0.$$

It follows directly on shrinking the discs that, for simple zeros of θ, the orthogonal structure is

$$(e, e) = \pi i \sum_{i=1}^{2g-2} \frac{B(\Phi, \Phi)_{x_i}}{\theta'(x_i)} \tag{5}$$

where $\theta'(x_i) \in K_{x_i}^2$ is the derivative of θ at its zero x_i.

Remarks:

1. Note from this description of the inner product that the decomposition of E in (4) is an orthogonal one.

2. Note also that if Φ is holomorphic then $B(\Phi, \Phi)/\theta$ is a meromorphic differential and the sum of its residues is therefore zero. Hence from (5) $T^* \subset E$ is maximally isotropic.

We can generalize the above by replacing θ by a section s of KL^2 where L is a line bundle of degree zero and considering the hypercohomology of

$$\mathscr{O}(\mathfrak{g} \otimes L^*) \xrightarrow{s} \mathscr{O}(\mathfrak{g} \otimes KL).$$

The quadratic form is defined in the same way as (5), and $H^0(C, \mathfrak{g} \otimes KL)$ is still isotropic but we have lost the Courant bracket.

What we obtain this way is a hypercohomology group

$$0 \to T_L^* \to \mathbb{H}^1(C, \mathfrak{g} \otimes L^*) \to T_L \to 0.$$

where $T_L = H^1(C, \mathfrak{g} \otimes L^*)$. In particular, varying over the moduli space, we see that each line bundle L of degree zero defines a deformation T_L of the tangent bundle.

To summarize, for each effective divisor D of degree $2g-2$ we have produced an orthogonal bundle E_D with the following properties

- E_D has an orthogonal structure
- there is an exact sequence of vector bundles $0 \to T_L^* \to E_D \to T_L \to 0$ where KL^2 is the line bundle defined by D
- T_L^* is a maximal isotropic subbundle

- when L is trivial, E_D has the structure of a holomorphic Courant algebroid

4 A vanishing theorem

We shall use the bundles E_D to prove the following vanishing theorem:

Theorem 4.1 *Let M be the moduli space of rank n, degree d bundles of fixed determinant, with n,d coprime, over a curve of genus $g>4$. Then $H^0(M,\Lambda^2 T)=0$.*

Proof: We return to the situation of a 1-form θ defining an extension

$$0\to T^*\to E\to T\to 0$$

There is an induced sequence of vector bundles

$$0\to A\to \Lambda^2 E\to \Lambda^2 T\to 0 \tag{6}$$

which we shall use to approach $\Lambda^2 T$. Here A is the bundle of Lie subalgebras preserving T^* and is itself an extension

$$0\to \Lambda^2 T^*\to A\to T\otimes T^*\to 0. \tag{7}$$

Consider the bundle $\Lambda^2 E$. From (4) we have

$$\Lambda^2 E\cong \bigoplus_{i<j}(\mathfrak{g}_{x_i}\otimes \mathfrak{g}_{x_j})\oplus \bigoplus_i \Lambda^2\mathfrak{g}_{x_i} \tag{8}$$

The coprime condition means that there is a universal vector bundle. In [8] the authors show that vector bundles U_x on M coming from this universal bundle are stable and isomorphic if and only if $x=y$. If $\mathfrak{g}_x\otimes\mathfrak{g}_y=\mathrm{End}_0 U_x\otimes \mathrm{End}_0 U_y$ has a holomorphic section then by stability this is covariant constant with respect to the connection defined by the Hermitian-Einstein connections on U_x and U_y. This connection has holonomy $U(n)\cdot U(n)$ which means in particular that the section defines an algebra homomorphism from $\mathrm{End}_0 U_x$ to $\mathrm{End}_0 U_y$. By stability this is an isomorphism which means that $U_x\cong L\otimes U_y$ for a line bundle L. But the Picard variety of M is $\mathbf{Z}$ and $c_1(U_x)=c_1(U_y)$ so $U_x\cong U_y$ and $x=y$. It follows that $H^0(M,\mathfrak{g}_{x_i}\otimes\mathfrak{g}_{x_j})=0$ if $i\neq j$. If $x_i=x_j$ then we similarly deduce that the only holomorphic section of $\mathfrak{g}_{x_i}\otimes\mathfrak{g}_{x_i}$ is defined by the Killing form B, which is symmetric and hence $H^0(M,\Lambda^2\mathfrak{g}_{x_i})=0$. From (8) we see that

$$H^0(M,\Lambda^2 E)=0.$$

Since E has an orthogonal structure, $\Lambda^2 E$ is isomorphic to the bundle of skew-adjoint transformations of E and the derivative of any family of deformations of E as an orthogonal bundle defines an element of $H^1(C,\Lambda^2 E)$.

But we saw in the previous section that any holomorphic section with divisor D of a line bundle KL^2 (i.e. any bundle of degree $(2g-2)$) defines an extension

$$0 \to T_L^* \to E_D \to T_L \to 0$$

with an orthogonal structure such that T_L^* is isotropic. We therefore have a family of extensions defined by a 2^{2g}-fold covering (the choice of the line bundle L) of the symmetric product $S^{2g-2}C$ all of which have orthogonal structures. So we have an effectively parametrized $(2g-2)$-dimensional family of bundles deforming E. Each of these bundles has an orthogonal structure so the tangent space to the family is a distinguished $(2g-2)$-dimensional subspace of $H^1(M,\Lambda^2 E)$. But this family also comes with a distinguished maximal isotropic subbundle, so this subspace is the image of a $(2g-2)$-dimensional subspace $V \subseteq H^1(M,A)$.

Now consider the long exact sequence for (7)

$$\to H^0(M,T\otimes T^*) \xrightarrow{h} H^1(M,\Lambda^2 T^*) \to H^1(M,A) \xrightarrow{p} H^1(M,T\otimes T^*) \to H^2(M,\Lambda^2 T^*) \to$$

From [3] for $g>4$ $H^0(M,T\otimes T^*)$ consists of multiples of the identity. The homomorphism h is just the extension class defining E in $H^1(M,\Lambda^2 T^*)$ applied to the identity and so is injective. We know that $H^1(M,\Lambda^2 T^*) \cong H^0(C,K)$, hence from the exact sequence the kernel of p has dimension $(g-1)$.

Now a deformation of E, as a bundle with distinguished subbundle, defines a deformation of the subbundle. The map p in the exact sequence is its derivative. Our $(2g-2)$-dimensional family of deformations of E is parametrized by an effective degree $(2g-2)$ divisor D and defines the deformation T_L of the tangent bundle, where the divisor class of D is $K+2L$. This map factors through the Abel-Jacobi map $u: S^{2g-2}C \to J(C)$ at the divisor of θ, and so p, restricted to the subspace $V \subseteq H^1(M,A)$, factors through the derivative of u.

Writing the map u as

$$u_\alpha = \sum_{i=1}^{2g-2} \int_{x_0}^{x_i} \omega_\alpha$$

for a basis $\{\omega_\alpha\}$ of differentials we see that the image of its derivative is the $(g-1)$-dimensional subspace of $H^1(C,\mathcal{O})$ annihilated by $\theta \in H^0(C,K) = H^1(C,\mathcal{O})^*$. The kernel of p restricted to V is thus $(g-1)$-dimensional and hence coincides with the full kernel of p. Hence $p(V) \subset H^1(M,T\otimes T^*)$ is $(g-1)$-dimensional. From

[3] $H^1(M,T\otimes T^*)\cong H^1(C,\mathscr{O})$ and thus has dimension g. We deduce that either $p: H^1(M,A)\to H^1(M,T\otimes T^*)$ is surjective, and then $\dim H^1(M,A)=(2g-2)+1$ or p maps to a $(g-1)$-dimensional space which means that $V=H^1(M,A)$ and $\dim H^1(M,A)=2g-2$.

Now consider the long exact sequence for (6)

$$\to H^0(M,\Lambda^2E)\to H^0(M,\Lambda^2T)\to H^1(M,A)\to H^1(M,\Lambda^2E)\to .$$

If $V=H^1(M,A)$ then knowing that V maps injectively to $H^1(M,\Lambda^2E)$ and $H^0(M,\Lambda^2E)=0$, we have the required result $H^0(M,\Lambda^2T)=0$. The other alternative is that $\dim H^1(M,A)=2g-1$ in which case $\dim H^0(M,\Lambda^2T)\leq 1$

We now use the exact sequence obtained by tensoring E with T

$$0\to T\otimes T^*\to T\otimes E\to T\otimes T\to 0$$

to yield the exact cohomology sequence

$$0\to H^0(M,T\otimes T^*)\to H^0(M,T\otimes E)\to H^0(M,T\otimes T)\to H^1(M,T\otimes T^*)\to\cdots$$

Now

$$E\cong\bigoplus_{i=1}^{2g-2}\mathfrak{g}_{x_i}$$

so for the term $H^0(M,T\otimes E)$ we need to understand each $H^0(M,T\otimes\mathfrak{g}_x)$. The bundles $\mathfrak{g}_x$ are parametrized by $x\in C$ and so in the complement of a finite set of points in C, $\dim H^0(M,T\otimes\mathfrak{g}_x)$ takes its generic value k, say. For any x we have the section s_x defined in Section 2.1 and so $k\geq 1$. Since the canonical bundle has no base points, a generic canonical differential θ vanishes at points in this complement and so for this bundle E we have

$$\dim H^0(M,T\otimes E)=\sum_{i=1}^{2g-2}\dim H^0(M,T\otimes\mathfrak{g}_{x_i})=(2g-2)k.$$

Now $H^0(M,T\otimes T)=H^0(M,\mathrm{Sym}^2T)\oplus H^0(M,\Lambda^2T)$ and it was proved in [7] that $\dim H^0(M,\mathrm{Sym}^2T^*)=3g-3$. Let n be the dimension of the image of $H^0(M,T\otimes T)$ in $H^1(M,T\otimes T^*)$ in the above sequence then from exactness

$$n+(2g-2)k=1+(3g-3)+\dim H^0(M,\Lambda^2T)$$

using again $\dim H^0(M, T\otimes T^*) = 1$. But $n \geq 0$ and $\dim H^0(M, \Lambda^2 T) \leq 1$ so if $g > 2$ we must have $k = 1$ and $n = g + \dim H^0(M, \Lambda^2 T)$. But $n \leq \dim H^1(M, T\otimes T^*) = g$ and hence $H^0(M, \Lambda^2 T) = 0$. $\square$

Remarks:

1. When $g = 2$, M is the intersection of two quadrics in the 5-dimensional projective space $\mathrm{P}(V)$. A direct calculation shows that $H^0(M, \Lambda^2 T) \cong \Lambda^2 V^*$.

2. From [12] the infinitesimal deformations of the abelian category of coherent sheaves are parametrized by the Hochschild cohomology group $HH^2(M)$ and the vanishing of $H^0(M, \Lambda^2 T)$ and $H^2(M, \mathscr{O})$ means that this is isomorphic to $H^1(M, T)$, the deformations of the complex structure of M which is well-known to be canonically isomorphic to the deformations of the curve C.

3. The evaluation map $H^0(C, \mathfrak{g}\otimes K) \to \mathfrak{g}_x \otimes K_x$ defines as in Section 2.1 a holomorphic section s_x of $\mathrm{Hom}(T^*, \mathfrak{g}_x) = T\otimes \mathfrak{g}_x$ on M. Our calculation above of $k = 1$ shows that for generic x this is the unique section.

5 Generators and relations

5.1 Generators

Suppose now that M is the moduli space of rank 2 bundles of fixed determinant over a curve C of genus g. We have seen from Proposition 2.1 that the $(5g-5)$-dimensional space $H^1(C, K^{-2})$ injects into $H^0(M, \Lambda^3 T)$. This generates maps

$$\Lambda^k H^1(C, K^{-2}) \to H^0(M, \Lambda^{3k} T)$$

and one may ask whether this is surjective, or more generally is it true that any polyvector field is generated by these trivector fields?

Since $\dim M = 3g-3$ we can consider the map from $\Lambda^{g-1} H^1(C, K^{-2})$ to sections of the anticanonical bundle $K_M^{-1} = \Lambda^{3g-3} T$ of M. The Verlinde formula gives this dimension as

$$\dim H^0(M, K_M^{-1}) = 3^{g-1} 2^{2g-1} \pm 2^{2g-1} + 3^{g-1}$$

(where the sign corresponds to even or odd degree), whereas

$$\dim \Lambda^{g-1} H^1(C, K^{-2}) = \binom{5g-5}{g-1}$$

which is smaller.

On the other hand, our polyvector fields are described via the adjoint representation and so are insensitive to the operation of tensoring a rank n stable vector bundle V of fixed determinant by a line bundle of order n. So on the moduli space M of stable vector bundles they are invariant by the action of $H^1(C, \mathbf{Z}_n)$. In the rank 2 case the dimension of the space of invariant sections of K^* is given in [11] as

$$\dim H^0_0(M, K_M^{-1}) = \frac{3^g \pm 1}{2}.$$

Using the inequality

$$\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$$

we have for $g > 2$

$$\dim \Lambda^{g-1} H^1(C, K^{-2}) = \binom{5g-5}{g-1} \geq 5^{g-1} \geq \frac{3^g \pm 1}{2}.$$

It therefore remains a possibility that the invariant trivectors do generate the whole algebra.

5.2 *Some relations*

Recall that for each point $x \in C$ we have (up to a constant) a trivector σ_x defined by evaluation at x:

$$\sigma_x(\Phi_1, \Phi_2, \Phi_3) = B(\Phi_1(x), [\Phi_2(x), \Phi_3(x)]).$$

For $SL(2)$ the three-form $B(X, [Y, Z]))$ is essentially the volume form of the Killing form on the three-dimensional Lie algebra.

Now take $(g-1)$ distinct points $x_1, \ldots, x_{g-1}$ on C and consider evaluating a Higgs field Φ, considered as a cotangent vector to M, at these points. We get a homomorphism

$$\alpha : T^* \to \bigoplus_{i=1}^{g-1} \mathfrak{g}_{x_i}$$

of bundles of the same rank. Taking the top exterior power

$$\Lambda^{3g-3}\alpha : \Lambda^{3g-3}T^* \to \bigotimes_{i=1}^{g-1} \Lambda^3 \mathfrak{g}_{x_i}.$$

The right hand side is just a trivial bundle so this homomorphism defines a section of the anticanonical bundle of M naturally associated to the $(g-1)$ points. In fact it is not hard to see that it is a multiple of

$$\sigma_{x_1} \wedge \sigma_{x_2} \wedge \cdots \wedge \sigma_{x_{g-1}}.$$

This vanishes when α has a non-zero kernel, which is the locus of bundles in M for which there is a Higgs field vanishing at the $(g-1)$ points – a determinant divisor.

If the rank 2 vector bundle has degree zero then by the mod 2 index theorem (as for example in [2]), if $K^{1/2}$ is an odd theta characteristic then

$$\dim H^0(C, \mathfrak{g} \otimes K^{1/2}) > 0.$$

So if $\Psi \in H^0(C, \mathfrak{g} \otimes K^{1/2})$ and a section s of $K^{1/2}$ has divisor $x_1 + x_2 + \cdots + x_{g-1}$ then $\Phi = s\Psi$ is a Higgs field which vanishes at these points. In other words every bundle has a Higgs field vanishing at these points so

$$\sigma_{x_1} \wedge \sigma_{x_2} \wedge \cdots \wedge \sigma_{x_{g-1}} = 0.$$

These are relations in the algebra – one for each of the $2^{g-1}(2^g - 1)$ odd theta characteristics. However we still have for $g > 4$

$$5^{g-1} - 2^{g-1}(2^g - 1) > \frac{3^g + 1}{2}$$

so there must be more.

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Buser-Sarnak invariant and projective normality of abelian varieties

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Abstract We show that a general n-dimensional polarized abelian variety (A,L) of a given polarization type and satisfying $h^0(A,L) \geq \frac{8^n}{2} \cdot \frac{n^n}{n!}$ is projectively normal. In the process, we also obtain a sharp lower bound for the volume of a purely one-dimensional complex analytic subvariety in a geodesic tubular neighborhood of a subtorus of a compact complex torus.

Keywords abelian varieties, projective normality, Buser-Sarnak invariant, Seshadri number.
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1 Introduction and Statement of Results

Let A be an abelian variety of dimension n, and let L be an ample line bundle over A. Such a pair (A,L) is called a *polarized abelian variety*. We are interested in studying the projective normality of (A,L), which plays an important role in the theory of linear series associated to (A,L). For each $r \geq 1$, we consider the multiplication map

$$\rho_r : Sym^r H^0(A,L) \to H^0(A,L^{\otimes r}) \tag{1.1}$$

induced by $(\sigma_1,\cdots,\sigma_r) \to \sigma_1\cdots\sigma_r$ for $\sigma_1,\cdots,\sigma_r \in H^0(A,L)$. Here $Sym^r H^0(A,L)$ denotes the r-fold symmetric tensor power of $H^0(A,L)$. Recall that (A,L) (or simply L) is said to be *projectively normal* if ρ_r is surjective for each $r \geq 1$. The projective normality of a polarized abelian variety (A,L) is well-understood in the case when L is not primitive, i.e., when there exists a line bundle L' such that $L = L'^{\otimes m}$ for some integer $m \geq 2$ (cf. the references in [Iy]). However, not much is known for the case when L is primitive.

In the primitive case, the main interest is to find conditions on the polarization type $d_1|d_2|\cdots|d_n$ of (A,L) or on $h^0(A,L) := \dim_{\mathbb{C}} H^0(A,L)$ (note that $h^0(A,L) = d_1\cdots d_n$) which will guarantee the projective normality of a *general* (A,L) of a given polarization type. Along this line, J. Iyer [Iy] proved the following result:

Theorem 1.1 *([Iy, Theorem 1.2]) Let (A,L) be a polarized simple abelian variety of dimension n. If $h^0(A,L) > 2^n n!$, then L is projectively normal.*

See also [FG] for related results in the lower dimensional cases when $n = 3,4$. These works use the theory of theta functions and theta groups.

Our goal is to relate this problem to the Buser-Sarnak invariant $m(A,L)$ of the polarized abelian variety (cf. [L2, p.291]). Since A is a compact complex torus, one may write $A = \mathbb{C}^n/\Lambda$, where Λ is a lattice in $\mathbb{C}^n$. It is well-known that there exists a unique translation-invariant flat Kähler form ω on A such that $c_1(L) = [\omega] \in H^2(A,\mathbb{Z})$. The real part of ω gives rise to an inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^n$, and we denote by $\|\ \|$ the associated norm on $\mathbb{C}^n$. The Buser-Sarnak invariant is given by

$$m(A,L) := \min_{\lambda\in\Lambda\setminus\{0\}} \|\lambda\|^2. \tag{1.2}$$

In other words, $m(A,L)$ is the square of the minimal length of a non-zero lattice vector in Λ with respect to $\langle\ ,\ \rangle$. The study of this invariant was initiated by Buser and Sarnak in [BS], where they studied it for principally polarized abelian varieties and Jacobians. In particular, they showed the existence of a principally polarized abelian variety (A,L) with

$$m(A,L) \geq \frac{1}{\pi}\sqrt[n]{2L^n}. \tag{1.3}$$

In [Ba], Bauer generalized this to abelian varieties of arbitrary polarization type (cf. [L2, p. 292-293]).

The relevance of the invariant $m(A,L)$ in the study of algebro-geometric questions was first observed by Lazarsfeld [L1], where he obtained a lower bound for the Seshadri number of (A,L) in terms of $m(A,L)$ (cf. [L2, p. 293]). In particular, $m(A,L)$ gives information on generation of jets by $H^0(A,L)$. Furthermore, Bauer used the existence of (A,L) satisfying (1.3) together with Lazarsfeld's above result to obtain the following result:

Theorem 1.2 *([Ba, Corollary 2]) Let (A,L) be a general n-dimensional polarized abelian variety of a given polarization type. If $h^0(A,L) \geq \frac{8^n}{2}\cdot\frac{n^n}{n!}$, then L is very ample.*

Now we state our main result in this paper as follows:

Theorem 1.3 *A general n-dimensional polarized abelian variety (A,L) of a given polarization type and satisfying $h^0(A,L) \geq \frac{8^n}{2}\cdot\frac{n^n}{n!}$ is projectively normal.*

Using Stirling's formula ($n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}}e^{-n}$), one easily sees that our bound in Theorem 1.3 improves Iyer's bound in Theorem 1.1 substantially for large n. Note that our bound in Theorem 1.3 for projective normality is the same as Bauer's bound in Theorem 1.2 for very ampleness. To our knowledge, this is just a coincidence. Although the proofs of both theorems use Bauer's generalization of (1.3), Theorem 1.2 itself is not used in the proof of Theorem 1.3. Finally it is worth comparing Theorem 1.3 with the result in [FG] and [Ru] that there is a polarization type $d_1|d_2|\cdots|d_n$ with $d_1\cdots d_n = h^0(A,L) = \frac{4^n}{2}$ such that no abelian varieties of this polarization type is projectively normal.

We describe briefly our approach as follows. First we obtain an auxiliary result, which is a sharp lower bound for the volume of a purely one-dimensional complex analytic subvariety in a geodesic tubular neighborhood of a subtorus of a compact complex torus (see Proposition 2.3 for the precise statement). As a consequence, we obtain a lower bound of the Seshadri number of the line bundle $p_1^*L\otimes p_2^*L$ along the diagonal of $A\times A$ in terms of $m(A,L)$ (see Proposition 3.2). Here $p_i : A\times A \to A$

denotes the projection onto the i-th factor, $i = 1,2$. We believe that these two auxiliary results are of independent interest beside their application to the projective normality problem. Finally the proof of Theorem 1.3 involves the use of the second auxiliary result and applying Bauer's result mentioned above in (1.3).

2 Volume of subvarieties near a complex subtorus

In this section, we are going to obtain a sharp lower bound for the volume of a purely 1-dimensional complex analytic subvariety in a tubular open neighborhood of a subtorus of a compact complex torus (see Proposition 2.3). This inequality is inspired by an analogous inequality in the hyperbolic setting proved in [HT]. The proof of the current case is much simpler than the one in [HT], using a simple projection argument and Federer's volume inequality for analytic subvarieties in a Euclidean ball in $\mathbb{C}^n$ (cf. e.g. [St] or [L2, p. 300]).

Let $T = \mathbb{C}^n/\Lambda$ be an n-dimensional compact complex torus associated to a lattice $\Lambda \subset \mathbb{C}^n$ and endowed with a flat translation-invariant Kähler form ω. For simplicity, we call (T, ω) a *polarized compact complex torus*. Let $\langle\ ,\ \rangle$ and $\|\ \|$ be the inner product and norm on $\mathbb{C}^n$ associated to ω as in Section 1. Next we let S be a k-dimensional compact complex subtorus of T, where $0 \le k < n$. It is well-known that S is the quotient of a k-dimensional linear subspace $F \cong \mathbb{C}^k$ of $\mathbb{C}^n$ by a sublattice $\Lambda_S \subset \Lambda$ of rank $2k$ and such that $\Lambda_S = \Lambda \cap F$. Let $F^\perp$ be the orthogonal complement of F in $\mathbb{C}^n$ with respect to $\langle\ ,\ \rangle$, and let $q_F : \mathbb{C}^n \to F$ and $q_{F^\perp} : \mathbb{C}^n \to F^\perp$ denote the associated unitary projection maps. Similar to (1.2), we define the *relative Buser-Sarnak invariant* $m(T, S, \omega)$ given by

$$m(T, S, \omega) := \min_{\lambda\in\Lambda\setminus\Lambda_S} \|q_{F^\perp}(\lambda)\|^2. \tag{2.1}$$

In other words, $m(T, S, \omega)$ is the square of the minimal distance of a vector in $\Lambda \setminus \Lambda_S$ from the linear subspace F.

Remark 2.1 *(i) The invariant $m(A,L)$ in (1.2) corresponds to the special case when $S = \{0\}$ and $[\omega] = c_1(L)$, i.e., $m(A,L) = m(A,\{0\},\omega)$.*

(ii) From the discreteness of Λ, the equality $\Lambda_S = \Lambda \cap F$ and the compactness of $S = F/\Lambda_S$, one easily checks that $m(T, S, \omega) > 0$ and its value is attained by some $\lambda \in \Lambda \setminus \Lambda_S$.

With regard to the Riemannian geometry associated to ω, one also easily sees that the geodesic distance function $d_T : T \times T \to \mathbb{R}$ of T with respect to ω can be expressed in terms of $\| \ \|$ given by

$$d_T(x,y) = \inf\{\|z-w\| \,\big|\, p(z)=x,\, p(w)=y\}, \tag{2.2}$$

where $p : \mathbb{C}^n \to T$ denotes the covering projection map. For any given $r > 0$, we consider the open subset of T given by

$$W_r := \{x \in T \,\big|\, d_T(x,S) < r\} \supset S, \tag{2.3}$$

where as usual,

$$d_T(x,S) := \inf_{y\in S} d_T(x,y) = \min\{\|q_{F^\perp}(z)\| \,\big|\, p(z)=x\} \tag{2.4}$$

(note that the second equality in (2.4) follows from standard facts on inner product spaces, and as in Remark 2.1, the minimum value in the last expression in (2.4) is attained by some z). We simply call W_r the *geodesic tubular neighborhood of S in T* of radius r. Next we consider the biholomorphism $\widetilde{\varphi} : F \times F^\perp \to \mathbb{C}^n$ given by

$$\widetilde{\varphi}(z_1,z_2) = z_1 + z_2 \quad \text{for } (z_1,z_2) \in F \times F^\perp. \tag{2.5}$$

It is easy to see that the covering projection map $p \circ \widetilde{\varphi} : F \times F^\perp \to T$ is equivariant under the action of Λ_S on $F \times F^\perp$ given by $(z_1,z_2) \to (z_1+\lambda, z_2)$ for $(z_1,z_2) \in F \times F^\perp$ and $\lambda \in \Lambda_S$. It follows readily that $p \circ \widetilde{\varphi}$ descends to a well-defined covering projection map denoted by $\varphi : S \times F^\perp \to T$ (in particular, φ is a local biholomorphism). Consider the flat translation-invariant Kähler form on $\mathbb{C}^n$ given by

$$\omega_{\mathbb{C}^n} := \frac{\sqrt{-1}}{2}\partial\overline{\partial}\|z\|^2, \quad z \in \mathbb{C}^n, \tag{2.6}$$

which is easily seen to descend to the Kähler form ω on T. Consider also the flat Kähler form on $F^\perp$ given by $\omega_{F^\perp} := \omega_{\mathbb{C}^n}\big|_{F^\perp}$, and for any $r > 0$, let $B_{F^\perp}(r) := \{z \in F^\perp \,\big|\, \|z\| < r\}$ denote the associated open ball of radius r. Let $\omega_S := \omega\big|_S$. Note that $\varphi\big|_{S\times\{0\}}$ is given by the identity map on S. It admits biholomorphic extensions as follows:

Lemma 2.2 *For any real number r satisfying $0 < r \le \frac{\sqrt{m(T,S,\omega)}}{2}$, one has a biholomorphic isometry*

$$\varphi_r : (S,\, \omega_S) \times (B_{F^\perp}(r),\, \omega_{F^\perp}\big|_{B_{F^\perp}(r)}) \to (W_r,\, \omega\big|_{W_r}) \tag{2.7}$$

given by $\varphi_r := \varphi\big|_{S\times B_{F^\perp}(r)}$.

Proof. First we fix a real number r satisfying $0 < r \leq \frac{\sqrt{m(T,S,\omega)}}{2}$. From (2.3), (2.4) and the obvious identity $q_{F^\perp}(\widetilde{\varphi}(z_1,z_2)) = z_2$ for $(z_1,z_2) \in F \times F^\perp$, one easily sees that $\varphi(S \times B_{F^\perp}(r)) \subset W_r$, and thus the map φ_r in (2.7) is well-defined. For each $x \in W_r$, it follows from the second equality in (2.4) that there exists $z \in \mathbb{C}^n$ such that $p(z) = x$ and $\|q_{F^\perp}(z)\| = d_T(x,S) < r$. Now, $q_F(z)$ descends to a point x_S in S, and one easily sees that $\varphi_r(x_S, q_{F^\perp}(z)) = x$ with $(x_S, q_{F^\perp}(z)) \in S \times B_{F^\perp}(r)$. Thus φ_r is surjective. Next we are going to prove by contradiction that φ_r is injective. Suppose φ_r is not injective. Then it implies readily that there exist two points (z_1,z_2), $(z_1',z_2') \in F \times B_{F^\perp}(r)$ such that

(i) either $z_1 - z_1' \notin \Lambda_S$ or $z_2 \neq z_2'$; and

(ii) $z_1 + z_2 - (z_1' + z_2') = \lambda$ for some $\lambda \in \Lambda$

(here (i) means that (z_1,z_2), (z_1',z_2') descend to two different points in $S \times B_{F^\perp}(r)$). In both cases in (i), one easily checks that $\lambda \in \Lambda \setminus \Lambda_S$. On the other hand, one also sees from (ii) that $q_{F^\perp}(\lambda) = z_2 - z_2'$ and thus

$$\|q_{F^\perp}(\lambda)\| \leq \|z_2\| + \|z_2'\| < r + r = 2r \leq \sqrt{m(T,\, S,\, \omega)}, \tag{2.8}$$

which contradicts the definition of $m(T,\, S,\, \omega)$ in (2.1). Thus, φ_r is injective, and we have proved that φ_r is a bihomorphism. Finally from the obvious identity $\|z_1\|^2 + \|z_2\|^2 = \|z_1 + z_2\|^2$ for $(z_1,z_2) \in F \times F^\perp$, and upon taking $\frac{\sqrt{-1}}{2}\partial\overline{\partial}$, one easily sees that $\widetilde{\varphi} : (F,\, \omega_F) \times (F^\perp,\, \omega_{F^\perp}) \to (\mathbb{C}^n,\, \omega_{\mathbb{C}^n})$ is a biholomorphic isometry (cf. (2.6)). It follows readily that the induced covering projection map $\varphi : (S,\, \omega_S) \times (F^\perp,\, \omega_{F^\perp}) \to (T,\, \omega)$ is a local isometry. Upon restricting φ to $S \times B_{F^\perp}(r)$, one sees that the biholomorphism φ_r is an isometry. ∎

For each $x \in S$ and each non-zero holomorphic tangent vector $v \in \mathrm{T}_{x,T}$ orthogonal to $\mathrm{T}_{x,S}$, it is easy to see that there exists a unique 1-dimensional totally geodesic (flat) complex submanifold ℓ of $W_{\frac{\sqrt{m(T,S,\omega)}}{2}}$ passing through x and such that $\mathrm{T}_{x,\ell} = \mathbb{C}v$. We simply call such ℓ an *S-orthogonal line* of $W_{\frac{\sqrt{m(T,S,\omega)}}{2}}$. For a complex analytic subvariety V in an open subset of T, we simply denote by $\mathrm{Vol}\,(V)$ its volume with respect to the Kähler form ω, unless otherwise stated. It is easy to see that for each $0 < r \leq \frac{\sqrt{m(T,S,\omega)}}{2}$, the values of $\mathrm{Vol}\,(\ell \cap W_r)$ are the same for all the S-orthogonal lines ℓ in $W_{\frac{\sqrt{m(T,S,\omega)}}{2}}$. As such, $\mathrm{Vol}\,(\ell \cap W_r)$ is an unambiguously defined number depending on r only (cf. (2.9) below). Next we consider the blow-up $\pi : \widetilde{T} \to T$ of T along S, and denote the associated exceptional divisor by $E := \pi^{-1}(S)$. For a complex analytic subvariety V in an open subset of T such that V has no component lying in S, we denote its strict transform with respect to π by $\widetilde{V} := \overline{\pi^{-1}(V \setminus S)}$. As usual, for an $\mathbb{R}$-divisor Γ and a complex curve C in a complex manifold, we denote by $\Gamma \cdot C$ the intersection number of Γ with C. Our main result in this section is the following

Proposition 2.3 *Let (T, ω) a polarized compact complex torus of dimension n, and let S be a k-dimensional compact complex subtorus of T, where $0 \leq k < n$. Let $\pi : \widetilde{T} \to T$ be the blow-up of T along S with the exceptional divisor $E = \pi^{-1}(S)$ as above. Then for any real number r satisfying $0 < r \leq \frac{\sqrt{m(T,S,\omega)}}{2}$ and any purely 1-dimensional complex analytic subvariety V of the geodesic tubular neighborhood W_r of S such that V has no component lying in S, one has*

$$\begin{aligned} Vol(V) &\geq \pi r^2 \cdot (\widetilde{V} \cdot E) \\ &= Vol(\ell \cap W_r) \cdot (\widetilde{V} \cdot E). \end{aligned} \tag{2.9}$$

In particular, for each $0 < r \leq \frac{\sqrt{m(T,S,\omega)}}{2}$ and each non-negative value s of $\widetilde{V} \cdot E$, the lower bound in (2.9) is attained by the volume of some (and hence any) V consisting of the intersection of W_r with the union of s copies of S-orthogonal lines counting multiplicity.

Proof. Let $V \subset \widetilde{W_r}$ be as above. It is clear that Proposition 2.3 for the general case when V is reducible follows from the special case when V is irreducible, and that (2.9) holds trivially for the case when $V \cap S = \emptyset$. As such, we will assume without loss of generality that

$$V \text{ is irreducible}, \quad V \cap S \neq \emptyset \quad \text{and} \quad V \not\subset S. \tag{2.10}$$

Then $\widetilde{V} \cap E$ consists of a finite number of distinct points $y_1, \cdots, y_\kappa$ with intersection multiplicities $m_1, \cdots, m_\kappa$ respectively, so that

$$\widetilde{V} \cdot E = m_1 + \cdots + m_\kappa. \tag{2.11}$$

By Lemma 2.2, we have a biholomorphic isometry

$$(W_r, \omega\big|_{W_r}) \cong (S \times F^\perp(r),\, \eta_1^* \omega_S + \eta_2^* \omega_{F^\perp}). \tag{2.12}$$

Here $\eta_1 : S \times F^\perp(r) \to S$ and $\eta_2 : S \times F^\perp(r) \to F^\perp(r)$ denote the projections onto the first and second factor respectively. Next we make an identification $F^\perp \cong \mathbb{C}^{n-k}$ with Euclidean coordinates $z_1, z_2, \cdots, z_{n-k}$ associated to an orthonormal basis of $(F^\perp, \langle\ ,\ \rangle\big|_{F^\perp})$. Under this identification, we have

$$F^{\perp}(r) = \{z = (z_1, z_2, \cdots, z_{n-k}) \in \mathbb{C}^{n-k} \,\big|\, |z| < r\}, \quad \text{and} \tag{2.13}$$
$$\omega_{F^{\perp}} = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n-k} dz_i \wedge d\bar{z}_i.$$

Here $|z| = \sqrt{\sum_{i=1}^{n-k} |z_i|^2}$. Note that η_2 (and thus also $\eta_2\big|_V$) is a proper holomorphic mapping, and thus by the proper mapping theorem, $V' := \eta_2(V)$ is a complex analytic subvariety of $F^{\perp}(r)$. From (2.10), one easily sees that V' is irreducible and of pure dimension one, and $\eta_2\big|_V : V \to V'$ is a δ-sheeted branched covering for some $\delta \in \mathbb{N}$. Note that $0 \in V'$ since $V \cap S \neq \emptyset$, and we denote by μ the multiplicity of V' at the origin $0 \in F^{\perp}(r)$. Let $[V]$ (resp. $[V']$) denote the closed positive current defined by integration over V (resp. V') in W_r (resp. $F^{\perp}(r)$). Then via the identifications in (2.13), it follows from Federer's volume inequality for complex analytic subvarieties in a complex Euclidean ball (see e.g. [St] or [L2, p. 300]) that one has

$$\int_{F^{\perp}(r)} [V'] \wedge \omega_{F^{\perp}} \geq \mu \cdot \pi r^2. \tag{2.14}$$

Next we consider a linear projection map $\psi : F^{\perp} \to \mathbb{C}$ from $F^{\perp}$ onto some one-dimensional linear subspace (which we identify with $\mathbb{C}$). It follows readily from the definition of μ that for a generic ψ, $\psi\big|_{V'} : V' \to \psi(V')$ is a μ-sheeted branched covering. Furthermore, by considering the local description of the blow-up map π (cf. e.g. [GH, p. 603]), one easily sees that for each $y_j \in \widetilde{V} \cap E$, $1 \leq j \leq \kappa$, there exists an open neighborhood U_j of y_j in $\widetilde{V}$ such that for a generic ψ, the function $\psi \circ \eta_2 \circ \pi\big|_{U_j} : U_j \to \mathbb{C}$ is a defining function for $E \cap U_j$ in U_j, so that $\psi \circ \eta_2 \circ \pi\big|_{\widetilde{V} \cap U_j}$ is an m_j-sheeted branched covering onto its image, shrinking U_j if necessary. Thus by considering the degree of the map $\psi \circ \eta_2 \circ \pi\big|_{\widetilde{V}}$ for a generic ψ, one gets

$$\delta \cdot \mu = m_1 + \cdots + m_\kappa. \tag{2.15}$$

Under the identification in (2.12), we have

$$
\begin{aligned}
\mathrm{Vol}(V) &= \int_{W_r} [V] \wedge \omega \\
&= \int_{S\times F^{\perp}(r)} [V] \wedge (\eta_1^* \omega_S + \eta_2^* \omega_{F^{\perp}}) \\
&\geq \int_{S\times F^{\perp}(r)} [V] \wedge \eta_2^* \omega_{F^{\perp}} \quad (\text{since } \eta_1^*\omega_S \geq 0) \\
&= \delta \int_{F^{\perp}(r)} [V'] \wedge \omega_{F^{\perp}} \\
&\qquad (\text{upon taking the direct image } \eta_2^*) \\
&\geq \delta \cdot \mu \cdot \pi r^2 \quad (\text{by (2.14)}) \\
&= \pi r^2 \cdot (\widetilde{V} \cdot E) \quad (\text{by (2.11) and (2.15)}),
\end{aligned}
\tag{2.16}
$$

which gives the first line of (2.9). Next we take an S-orthogonal line ℓ of $W_{\frac{\sqrt{m(T,S,\omega)}}{2}}$. Then under the identifications in (2.12), (2.13) and upon making a unitary change of $F^{\perp}$ if necessary, one easily sees that $\ell \cap W_r$ can be given by $\{x\} \times \{(z_1, 0, \cdots, 0) \in \mathbb{C}^{n-k} \big| \, |z_1| < r\}$ for some fixed point $x \in S$, and it follows readily that

$$
\mathrm{Vol}(\ell \cap W_r) = \int_{|z_1|<r} \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 = \pi r^2, \tag{2.17}
$$

which gives the second line of (2.9). Finally we remark that the last statement of Proposition 2.3 is a direct consequence of (2.9), and thus we have finished the proof of Proposition 2.3. ■

3 Seshadri number along the diagonal of $A \times A$

In this section, we let $(A = \mathbb{C}^n/\Lambda, L)$ be a polarized abelian variety of dimension n, and let the associated objects ω, $\langle \ , \ \rangle$, $\| \ \|$ and $m(A,L)$ be as defined in Section 1. Next we consider the Cartesian product $A \times A$, and we denote by $p_i : A \times A \to A$ the projection map onto the i-th factor. It is easy to see that $p_1^* L \otimes p_2^* L$ is an ample line bundle over the $2n$-dimensional (product) abelian variety $A \times A$, and the associated translation-invariant flat Kähler form on $A \times A$ is given by $\omega_{A\times A} := p_1^*\omega + p_2^*\omega$. In particular, one has

$$
[\omega_{A\times A}] = c_1(p_1^* L \otimes p_2^* L) \in H^2(A \times A, \mathbb{Z}). \tag{3.1}
$$

Furthermore, it is easy to see that the diagonal of $A \times A$ given by

$$
D := \{(x,y) \in A \times A \,\big|\, x = y\} \tag{3.2}
$$

is an n-dimensional abelian subvariety of $A\times A$. Let $m(A\times A,D,\omega_{A\times A})$ be the relative Buser-Sarnak invariant as given in (2.1).

Lemma 3.1 *We have*

$$m(A\times A,D,\omega_{A\times A})=\frac{m(A,L)}{2}. \tag{3.3}$$

Proof. First we write $A\times A=(\mathbb{C}^n\times\mathbb{C}^n)/(\Lambda\times\Lambda)$, and we denote by $\langle\ ,\ \rangle_{\mathbb{C}^n\times\mathbb{C}^n}$ and $\|\ \|_{\mathbb{C}^n\times\mathbb{C}^n}$ the inner product and norm on $\mathbb{C}^n\times\mathbb{C}^n$ associated to $\omega_{A\times A}$. It is easy to see that as a compact complex subtorus of $A\times A$, D is isomorphic to the quotient F/Λ_D, where $F:=\{(z,z)\,|\,z\in\mathbb{C}^n\}\subset\mathbb{C}^n\times\mathbb{C}^n$ and $\Lambda_D:=\{(\lambda,\lambda)\,|\,\lambda\in\Lambda\}\subset\Lambda\times\Lambda$. Denote by $F^\perp$ the orthogonal complement of F in $\mathbb{C}^n\times\mathbb{C}^n$ with respect to $\langle\ ,\ \rangle_{\mathbb{C}^n\times\mathbb{C}^n}$, and let $q_{F^\perp}:\mathbb{C}^n\times\mathbb{C}^n\to F^\perp$ be the corresponding unitary projection map. Then for any $(\lambda_1,\lambda_2)\in\Lambda\times\Lambda$, one easily checks that $q_{F^\perp}(\lambda_1,\lambda_2)=(\frac{\lambda_1-\lambda_2}{2},\frac{\lambda_2-\lambda_1}{2})$, and thus

$$\|q_{F^\perp}(\lambda_1,\lambda_2)\|^2_{\mathbb{C}^n\times\mathbb{C}^n}=\|\frac{\lambda_1-\lambda_2}{2}\|^2+\|\frac{\lambda_2-\lambda_1}{2}\|^2=\frac{\|\lambda_1-\lambda_2\|^2}{2}. \tag{3.4}$$

Together with the obvious equality $\{\lambda_1-\lambda_2\,|\,(\lambda_1,\lambda_2)\in(\Lambda\times\Lambda)\setminus\Lambda_D\}=\Lambda\setminus\{0\}$ (and upon writing $\lambda=\lambda_1-\lambda_2$), one gets

$$\inf_{(\lambda_1,\lambda_2)\in(\Lambda\times\Lambda)\setminus\Lambda_D}\|q_{F^\perp}(\lambda_1,\lambda_2)\|^2_{\mathbb{C}^n\times\mathbb{C}^n}=\frac{1}{2}\inf_{\lambda\in\Lambda\setminus\{0\}}\|\lambda\|^2, \tag{3.5}$$

which, upon recalling (1.2) and (2.1), gives (3.3) immediately. ∎

Next we let $\pi:\widetilde{A\times A}\to A\times A$ be the blow-up of $A\times A$ along D with the associated exceptional divisor given by $E:=\pi^{-1}(D)$. We consider the line bundle $p_1^*L\otimes p_2^*L$ over $A\times A$, and denote its pull-back to $\widetilde{A\times A}$ by

$$\mathscr{L}:=\pi^*(p_1^*L\otimes p_2^*L). \tag{3.6}$$

Then the Seshadri number $\varepsilon(p_1^*L\otimes p_2^*L,D)$ of $p_1^*L\otimes p_2^*L$ along D is defined by

$$\varepsilon(p_1^*L\otimes p_2^*L,D):=\sup\{\varepsilon\in\mathbb{R}\,|\,\mathscr{L}-\varepsilon E\text{ is nef on }\widetilde{A\times A}\} \tag{3.7}$$

(see e.g. [L2, Remark 5.4.3] for the general definition and [D] for its origin). Here as usual, an $\mathbb{R}$-divisor Γ on an algebraic manifold M is said to be nef if $\Gamma\cdot C\geq 0$ for any algebraic curve $C\subset M$. Our main result in this section is the following

Proposition 3.2 *Let (A,L) be a polarized abelian variety of dimension n, and let $\mathscr{L}$ be as in (3.6). Then $\mathscr{L}-\alpha E$ is nef on $\widetilde{A\times A}$ for all $0\leq\alpha\leq\frac{\pi}{8}\cdot m(A,L)$. In particular, we have*

$$\varepsilon(p_1^*L \otimes p_2^*L, D) \geq \frac{\pi}{8} \cdot m(A, L). \tag{3.8}$$

Proof. First it is easy to see from (3.6) that $\mathscr{L}$ is nef, and thus the proposition holds for the case when $\alpha = 0$. Now we fix a number α satisfying $0 < \alpha \leq \frac{\pi}{8} \cdot m(A, L)$. Then it is easy to see from Lemma 3.1 that $\alpha = \pi r^2$ for some r satisfying $0 < r \leq \frac{\sqrt{m(A\times A, D, \omega_{A\times A})}}{2}$. For each such r, we let W_r be the geodesic tubular neighborhood of D in $A \times A$ of radius r as defined in (2.3) (with T and S there given by $A \times A$ and D respectively). Let C be an algebraic curve in $\widetilde{A \times A}$. First we consider the case when C is irreducible and $C \not\subset E$, so that $\pi(C) \not\subset D$ and C coincides with the strict transform of $\pi(C)$ with respect to the blow-up map π (i.e., $C = \widetilde{\pi(C)}$ in terms of the notations in Section 2). Then by (3.1), (3.6) and upon taking the direct image π_*, we get

$$\begin{aligned} \mathscr{L} \cdot C &= \int_{\widetilde{A\times A}} [C] \wedge \pi^* \omega_{A\times A} \\ &= \int_{A\times A} [\pi(C)] \wedge \omega_{A\times A} \\ &\geq \int_{W_r} [\pi(C)] \wedge \omega_{A\times A} \\ &\geq \pi r^2 \cdot (E \cdot C) \quad \text{(by Proposition 2.3)}, \\ &= \alpha \cdot (E \cdot C). \end{aligned} \tag{3.9}$$

In other words, we have

$$(\mathscr{L} - \alpha E) \cdot C \geq 0. \tag{3.10}$$

Next we consider the case when C is irreducible and $C \subset E$. By considering translation-invariant vector fields on D and $A \times A$, one easily sees that the normal bundle $N_{D|(A\times A)}$ is holomorphically trivial over D. It follows readily that the line bundle $[E]\big|_E$ is isomorphic to $\sigma^* \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$, where $\sigma : D \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ denotes the projection onto the second factor. Hence $E \cdot C \leq 0$ for any irreducible curve $C \subset E$. Together with the nefness of $\mathscr{L}$, it follows readily that (3.10) also holds for the irreducible case when $C \subset E$. Finally one easily sees that (3.10) for the case when C is reducible follows readily from the case when C is irreducible. Thus we have finished the proof of the nefness of $\mathscr{L} - \alpha D$ for all $0 \leq \alpha \leq \frac{\pi}{8} \cdot m(A, L)$, which also leads to (3.8) readily. ∎

4 Projective normality

In this section, we are going to give the proof of Theorem 1.3, and we follow the notation in Section 3. First we have

Proposition 4.1 *Let (A,L), n, E and $\mathscr{L}$ be as in Proposition 3.2. If $\mathscr{L}\otimes\mathscr{O}(-nE)$ is nef and big, then L is projectively normal.*

Proof By [Iy, Proposition 2.1], one knows that the surjectivity of the multiplication maps ρ_r in (1.1) for all $r \geq 1$ will follow from the surjectivity of ρ_2 (i.e., the case when $r=2$). Thus to prove that L is projectively normal, it suffices to show that the multiplication map

$$\rho : H^0(A,L)\otimes H^0(A,L) \longrightarrow H^0(A,L^{\otimes 2}) \tag{4.1}$$

(as given in (1.1)) is surjective. We are going to reduce this to the question of vanishing of a certain cohomology group on $\widetilde{A\times A}$ following the standard approach in [BEL, Section 3]. Here $\pi : \widetilde{A\times A} \to A\times A$ is the blow-up of $A\times A$ along the diagonal D as in Section 3. Consider the short exact sequence on $A\times A$ given by

$$0 \longrightarrow p_1^*L\otimes p_2^*L\otimes\mathscr{I} \longrightarrow p_1^*L\otimes p_2^*L \longrightarrow p_1^*L\otimes p_2^*L\big|_D \longrightarrow 0, \tag{4.2}$$

where $\mathscr{I}$ denotes the ideal sheaf of D. Note that $p_1^*L\otimes p_2^*L\big|_D \cong L^{\otimes 2}$ under the natural isomorphism $D\cong A$, and one has $H^0(A\times A, p_1^*L\otimes p_2^*L)\cong H^0(A,L)\otimes H^0(A,L)$ by the Künneth formula. Together with the long exact sequence associated to (4.2), one easily sees that ρ is surjective if $H^1(A\times A, p_1^*L\otimes p_2^*L\otimes\mathscr{I})=0$. But one also easily checks that

$$\begin{aligned} H^1(A\times A, p_1^*L\otimes p_2^*L\otimes\mathscr{I}) &= H^1(\widetilde{A\times A}, \mathscr{L}\otimes\mathscr{O}(-E)) \\ &= H^1(\widetilde{A\times A}, K_{\widetilde{A\times A}}\otimes\mathscr{L}\otimes\mathscr{O}(-nE)), \end{aligned} \tag{4.3}$$

where the last line follows from the isomorphism $K_{\widetilde{A\times A}} = \pi^*K_{A\times A} + \mathscr{O}((n-1)E) = \mathscr{O}((n-1)E)$. Finally if $\mathscr{L}\otimes\mathscr{O}(-nE)$ is nef and big, then it follows from Kawamata-Viehweg vanishing theorem that $H^1(\widetilde{A\times A}, K_{\widetilde{A\times A}}\otimes\mathscr{L}\otimes\mathscr{O}(-nE))=0$, which together with (4.3), imply that ρ is surjective. ∎

Lemma 4.2 *Let (A,L), n, E and $\mathscr{L}$ be as in Proposition 3.2. If $\mathscr{L}\otimes\mathscr{O}(-nE)$ is nef and $L^n > (2n)^n$, then $\mathscr{L}\otimes\mathscr{O}(-nE)$ is big.*

Proof Note that

$$\mathscr{L}^{2n} = (p_1^* L \otimes p_2^* L)^{2n} = \frac{(2n)!}{n! \cdot n!} L^n \cdot L^n.$$

Recall that we have the identification $E = D \times \mathbb{P}^{n-1}$ from the proof of Proposition 3.2. Denoting by $\sigma : E \to \mathbb{P}^{n-1}$ and $\eta : E \to D = A$ the projections, we have $\mathscr{O}(E)|_E = \sigma^* \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$ and $\mathscr{L}|_E = \eta^*(L \otimes L)$. From these, a straight-forward calculation gives

$$(\mathscr{L} \otimes \mathscr{O}(-nE))^{2n} = \frac{(2n)!}{n! \cdot n!} \cdot L^n \cdot (L^n - (2n)^n). \tag{4.4}$$

Together with the well-known fact that a nef line bundle is big if and only if its top self-intersection number is positive, one obtains the lemma readily. ■

Finally we complete the proof of our main result as follows:

Proof of Theorem 1.3. Let $\mathscr{A}_{(d_1, \cdots, d_n)}$ denote the moduli space of n-dimensional polarized abelian varieties (A, L) of a given polarization type $d_1 | d_2 | \cdots | d_n$ and satisfying

$$d_1 \cdots d_n \geq \frac{8^n}{2} \cdot \frac{n^n}{n!}, \tag{4.5}$$

and recall that

$$h^0(A, L) = d_1 \cdots d_n = \frac{L^n}{n!} \quad \text{for all } (A, L) \in \mathscr{A}_{(d_1, \cdots, d_n)}. \tag{4.6}$$

By [Ba, Theorem 1], there exists some $(A_o, L_o) \in \mathscr{A}_{(d_1, \cdots, d_n)}$ such that

$$m(A_o, L_o) = \frac{1}{\pi} \sqrt[n]{2L_o^n}. \tag{4.7}$$

Let $\mathscr{L}_o$ be the line bundle over the blow-up $\widetilde{A_o \times A_o}$ of $A_o \times A_o$ along the diagonal (with exceptional divisor E_o) as in Proposition 3.2. From (4.5), (4.6) and (4.7), one easily checks that $n \leq \frac{\pi}{8} \cdot m(A_o, L_o)$. Thus it follows from Proposition 3.2 that $\mathscr{L}_o \otimes \mathscr{O}(-nE_o)$ is nef. One also easily checks from (4.5) and (4.6) that $L_o^n > (2n)^n$, and thus by Lemma 4.2, the nef line bundle $\mathscr{L}_o \otimes \mathscr{O}(-nE_o)$ is also big. Then it follows from Proposition 4.1 that (A_o, L_o) is projectively normal. Finally it is easy to see that the existence of a projective normal (A_o, L_o) implies readily that a general (A, L) in $\mathscr{A}_{(d_1, \cdots, d_n)}$ is projectively normal. ■

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Complete Kähler-Einstein manifolds

Marco Kühnel

Abstract Classifying Kähler-Einstein manifolds has progressed very far for compact manifolds. In the non-compact setting, a lot of encouraging results have been obtained, with the greatest gap of knowledge for the Ricci-flat case. This article wants to present the state of the art of classification and explain current problems and questions with respect to existence and uniqueness of complete Ricci-flat Kähler metrics.

Keywords Kähler-Einstein manifolds, open manifolds, Kähler cones.
Mathematics Subject Classification (2010) 32Q20.

1 The Classification Problem

Let X be a compact complex manifold and let

$$K(X) := \{\text{classes of closed positive } (1,1)\text{-forms}\} \subset H^{1,1}(X)$$

denote the Kähler cone in Dolbeault cohomology. It is the set of fundamental forms ω of Kähler metrics g. The class of the Ricci form

$$\operatorname{Ric} g := \frac{i}{2\pi}\partial\overline{\partial}\log\det\omega$$

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is called the first Chern class $c_1(X) \in H^{1,1}(X)$ and independent of the chosen metric g. We say $\alpha \in H^{1,1}(X)$ is positive (negative) if it can be represented by a positive (negative) $(1,1)$-form.

Definition 1.1 *A Kähler metric g on X is called Kähler-Einstein if there is a constant K such that* Ric $g = K\omega$.

Obviously, existence of a Kähler-Einstein metric implies that $c_1(X)$ has a sign. Roughly, the classification of compact Kähler-Einstein manifolds is the following.

Theorem 1.2 (Aubin [A76], Yau [Y78], Calabi [C54], Futaki [F83], Kobayashi [K84]) *Let X be a compact Kähler manifold.*

1. $c_1(X) > 0 \not\Rightarrow \exists\, g$ *Kähler-Einstein with* $K = 1$.
2. $c_1(X) = 0 \in H^{1,1}(X) \Rightarrow \forall\, \alpha \in K(X)\ \exists^1\, g$ *Kähler-Einstein with* $[g] = \alpha$ *and* $K = 0$.
3. $c_1(X) < 0 \Rightarrow \exists^1 g$ *Kähler-Einstein with* $K = -1$.

Note that the three cases mutually exclude each other.

In the Ricci-positive case, several obstructions are known: The automorphism group of X has to be reductive ([M57]) and the generalized Futaki invariants have to vanish in order to allow for a Kähler-Einstein metric.

Switching to non-compact manifolds we have to add another condition: g should be complete, i.e. all maximal geodesics have infinite length. Otherwise g cannot possibly relate to the topology of X.

This leads to the following questions:

(Q1) Is there a classification of non-compact Kähler-Einstein manifolds analogues to Theorem 1.2?

(Q2) Are the three cases of the sign of Ric mutually exclusive also for non-compact manifolds?

(Q3) What is the correct notion of a Kähler cone for non-compact manifolds?

Let us begin with tackling Q1. Looking at non-compact Kähler manifolds, one finds by the uniformization theorem an interesting classification in one dimension.

Theorem 1.3 *Let X be a non-compact complex curve. Then either* *(i)* *the universal cover of X is the unit disc and there is a complete Ricci-negative Kähler-Einstein metric;*

(ii) *or* $X = \mathbb{C}P^1 \setminus \{p,q\}$ *for points* $p,q \in \mathbb{C}P^1$ *not necessarily distinct; in particular, the universal cover is* $\mathbb{C}$ *and there is a complete Ricci-flat Kähler metric on* X.

In particular, there is no Ricci positive curve. This is a general obstruction for Kähler-Einstein metrics on non-compact manifolds.

Theorem 1.4 (Bonnet, Myers [M41]) *A complete Riemannian manifold with Ricci curvature bounded from below by a positive constant has bounded diameter. In particular, every Kähler-Einstein manifold with* $c_1(X) > 0$ *is compact.*

This leaves us with the Ricci-negative and Ricci-flat cases. Question Q2 can now be answered affirmatively by the Generalized Schwarz Lemma of Yau:

Theorem 1.5 (Yau [Y78b]) *Let* (M,g) *be a complete Kähler manifold with* $Sc(g) \geq -K_1$ *for some* $K_1 \geq 0$ *and* (N,h) *an hermitian manifold with* $\mathrm{Ric}\, h \leq -K_2 h$ *for some* $K_2 > 0$. *Moreover, we assume* $\dim M = \dim N = n$. *If* $f : M \longrightarrow N$ *is a holomorphic map, then*

$$f^* \omega_h^n \leq (\frac{K_1}{nK_2})^n \omega_g^n.$$

Application of Theorem 1.5 to the identity map under the assumption $K_1 = 0$ leads to a contradiction, so there cannot be complete Ricci-flat and Ricci-negative Kähler metrics at the same time.

Corollary 1.1 *A Kähler manifold cannot allow for a complete Ricci-flat Kähler metric and a complete Kähler-Einstein metric with negative scalar curvature simultaneously.*

Theorem 1.3 tells us that, apart from $\mathbb{C}$ and $\mathbb{C}^*$ every domain in $\mathbb{C}$ allows for a Ricci-negative Kähler-Einstein metric. In higher dimensions, natural domains to consider are domains of holomorphy.

Theorem 1.6 (Cheng/Yau [CY80], Mok/Yau [MY83]) *Let* $X \subset \mathbb{C}^n$ *be a bounded domain. X is a domain of holomorphy if and only if there is a complete Kähler-Einstein metric on X with negative scalar curvature.*

So we have rich examples of complete Kähler-Einstein manifolds with negative scalar curvature and no non-compact Kähler-Einstein manifolds with positive scalar curvature; only the non-compact Ricci-flat case remains more elusive.

2 Open manifolds

According to another version of Yau's Schwarz Lemma, the only bounded holomorphic functions on a complete Ricci-flat Kähler manifold are the constants. In particular, no bounded domain in $\mathbb{C}^n$ allows for a complete Ricci-flat Kähler metric. On the other hand, complements of analytic sets in compact complex manifolds share this function theoretic property. The easiest case is the complement of a divisor, being a Stein manifold, if the divisor is ample.

Definition 2.1 *A complex manifold X is called open, if there is a compact complex manifold $\overline{X}$ and an effective divisor D such that $X \cong \overline{X} \setminus D$.*

Indeed, if we restrict to open manifolds, there is some kind of classification like the one in Theorem 1.2. If $D \in |-K_{\overline{X}}|$, then adjunction yields an isomorphism $\Omega^n_X \cong \mathscr{O}_X$, a property sufficient for the existence of a Ricci-flat Kähler metric in the compact case. So we expect the anticanonical linear system to be distinguished compactifying divisors, to say the least.

Theorem 2.2 (Tian, Yau [TY90, TY91, TY86, Y93]) *Let $\overline{X}$ be a compact Kähler manifold, D a normal crossings divisor and $X := \overline{X} \setminus D$.*

(i) If $K_{\overline{X}} + D$ is ample, then there exists on X a unique complete Kähler-Einstein metric with negative scalar curvature.

(ii) If $K_{\overline{X}} + D = 0$, $-K_{\overline{X}}$ is ample and D is smooth, then there is a complete Ricci-flat Kähler metric on X.

The case of negatively curved Kähler-Einstein metrics satisfactorily settled by Theorem 2.2, a lot of questions concerning the flat case are still unanswered, including uniqueness (cf. Question Q3). So now we concentrate on Ricci-flat open manifolds.

3 Complete Ricci-flat open manifolds

3.1 The assumptions of the classification result

The asymmetry of assumptions in Theorem 2.2 is a current field of interest. The following elementary example of Yu [Yu08] sheds some light on it.

Example 1 Let $\overline{X} := \mathbb{C}P^n$ and $D := \{p(z)\prod_{i=0}^{n-1} z_i = 0\}$, where

$$p(z) = z_0^{m-1} z_n + P(z_0, \ldots, z_{n-1})$$

for a homogeneous polynomial P of degree $m \geq 2$. Then

$$X \cong (\mathbb{C}^*)^n$$

and hence carries a complete Ricci-flat Kähler metric, although $K_X + D$ is ample. This shows that the normal crossings condition for the negative case is essential, on the one hand, and that the condition $K_X + D = 0$ is not necessary for the Ricci-flat case, if one forgets about the smoothness of D, on the other hand.

Theorem 2.2 does not address the case $K_X + D < 0$. Recall that there cannot be a complete Ricci-positive Kähler-Einstein metric! Indeed, in cases you might think of as D being non-reduced (in a $\mathbb{Q}$-sense) one obtains a complete Ricci-flat Kähler metric:

Theorem 3.1 (Tian/Yau [TY91], Bando/Kobayashi [BK90]) *Let $\overline{X}$ be a compact Kähler manifold with $c_1(\overline{X}) > 0$ and D a smooth divisor. Assume $K_{\overline{X}} + \alpha D = 0$ for some rational $\alpha > 1$ and that D allows for a Kähler-Einstein metric. Then there is a complete Ricci-flat Kähler metric on X.*

The condition that D allows for a Kähler-Einstein metric can be weakened by using the language of Sasakian geometry (cf. [C09]). Of course, there are also singular reduced D with $K_{\overline{X}} + D \leq 0$ allowing for complete Ricci-flat Kähler metrics on X, e.g. $\overline{X} := \mathbb{C}P^n, D := H_1 + \cdots + H_k$ for hyperplanes $H_1, \ldots, H_k$ and $1 \leq k \leq n+1$, but these examples are only so easy to treat because the situation allows an overwhelming amount of symmetry. Between these two extremes there are singular divisors to consider, but without particular symmetries. For this situation nothing is known.

The positivity assumptions for $-K_{\overline{X}}$ given here for the Ricci-flat case can be weakened (see [TY90], [TY91]) but not beyond nefness.

3.2 Parametrizing complete Ricci-flat Kähler metrics

Now what about uniqueness? In the compact case, Ricci-flat Kähler metrics were parametrized by the Kähler cone. Is there an analogue for open manifolds? The rich use of the $\partial\overline{\partial}$-Lemma for compact manifolds suggests to define the Kähler cone via Bott-Chern-cohomology for open manifolds.

Definition 3.2 *Let X be an open manifold. We call*

$$K^{BC}(X) := \{\omega \in H^{1,1}(X) |\ \omega \text{ positive } \}/\sim$$

with

$$\omega \sim \omega' :\Longleftrightarrow \exists\, \varphi \in C^\infty(X):\ \omega - \omega' = i\partial\overline{\partial}\varphi$$

the Bott-Chern-Kähler cone.

Due to the lack of deeper results we discuss again the two extremal cases.

Example 2 Let $\overline{X} = \mathbb{C}P^n$, $D = H_1 + \ldots H_k$ with hyperplanes $H_1, \ldots H_k$ and $1 \leq k \leq n+1$. Then in every class of $K^{BC}(X)$ there is at least one complete Ricci-flat Kähler metric. Considering only the symmetric ones with respect to the group $G = X = \mathbb{C}^{n+1-k} \times (\mathbb{C}^*)^{k-1}$ there is an affine vector space of dimension $\frac{1}{2}(k-1)(k-2)$ of them for every fixed Bott-Chern-Kähler class. (For details and more general situations cf. [K10].)

So uniqueness in every Bott-Chern-Kähler class cannot be achieved when there is rich symmetry. In the smooth case, however, the question becomes irrelevant, at least in topologically simple (resp. generic) situations.

Theorem 3.3 ([KK06]) *Let $\overline{X}$ be a projective manifold of dimension ≥ 3 satisfying $b_1(X) = b_3(X) = 0, b_2(X) = 1$. For any smooth, ample divisor D the Bott-Chern-Kähler cone of X is trivial, i.e. $K^{BC}(X) = [0]$.*

Both results suggest that the correct notion of a Kähler cone should use a finer equivalence relation. One way would be to fix also the asymptotic behaviour of the metric towards D, but neither existence for nor uniqueness inside an asymptotic class are settled today. We present the known results.

3.3 Asymptotic description of the metrics

For comparison we should discuss the negative case first. The asymptotic expansion of the unique Kähler-Einstein metric has been computed by Schumacher [Sch98] for smooth divisors and by Wu [W05] also for divisors with normal crossings. For simplicity, we state here the result for the smooth case.

Let $K_{\overline{X}} + D > 0$ and D be a smooth divisor. We introduce local coordinates on U around D such that the section defining D is the first coordinate, i.e. $S = z^1$. The local projection onto D is denoted by $\pi : U \longrightarrow D$. Let g be the unique Kähler-Einstein metric with $K = -1$ on X, $\|.\|$ a metric on $\mathscr{L}(D)$, suitably chosen and g_D be the unique Kähler-Einstein metric on D with $K = -1$.

Theorem 3.4 ([Sch98]) *There exist* $0 < \alpha \leq 1$ *such that for every* $k \in \mathbb{N}, 0 < \lambda \leq 1$ *there are* $h, \mu, \nu \in C^{k,\lambda}(X)$

$$\begin{aligned} g = {} & \frac{2h}{\|S\|^2(-\log\|S\|^2)^2} \cdot (1 + \mu(-\log\|S\|^2)^{-\alpha}) dS \otimes d\overline{S} \\ & + \sum_{\gamma} O\left(\frac{1}{\|S\|(-\log\|S\|^2)^{1+\alpha}}\right)(dz^\gamma \otimes d\overline{S} + dS \otimes d\overline{z}^\gamma) \\ & + \nu \pi^* g_D. \end{aligned}$$

In particular, the geodesic distance near D is

$$d(x,p) \sim \log(-\log\|S(x)\|^2),$$

when $p \in X$ *is fixed and x is close enough to D.*

As it is more generally known, the volume growth of geodesic balls with fixed center is exponential in this case.

The Ricci-flat case has been treated only recently by Santoro [S08] and independently in [KK10]. At first we want to point out the general strategy used by [TY90]. Let $D \in |-K_{\overline{X}}|$ be a smooth, ample divisor given as the zero locus of the section $S \in \mathscr{L}(D)$ and $\|.\|$ a suitably chosen metric on $\mathscr{L}(\mathscr{D})$. Then

$$\omega_0 := i\partial\overline{\partial}(-\log\|S\|^2)^{\frac{n+1}{n}}$$

gives a good initial metric, "good" meaning that it is a complete Kähler metric and the volume form is asymptotically flat; moreover, in local coordinates with $z^1 = S$

$$\omega^n = \frac{1}{|z^1|^2} dz^1 \wedge d\overline{z^1} \wedge \tilde{\pi}^* \Omega \wedge \pi^* \overline{\tilde{\Omega}} + \text{l.o.t.}$$

for a volume form $\tilde{\Omega}$ on D and $\pi : U \longrightarrow D$ being the projection as above. This metric looks in local coordinates

$$\begin{aligned} g_0 = & \frac{h}{\|S\|^2(-\log\|S\|^2)^{1-\frac{1}{n}}} dS \otimes d\overline{S} \\ & + \sum_{\gamma} O\left(\frac{1}{\|S\|(-\log\|S\|^2)^{1-\frac{1}{n}}}\right)(dz^\gamma \otimes d\overline{S} + dS \otimes d\overline{z^\gamma}) \\ & + (-\log\|S\|^2)^{\frac{1}{n}} \pi^* g_D \end{aligned}$$

for a suitable Kähler metric g_D on D and function $h \in C^\infty(\overline{X})$. The question is how close this is to a Ricci-flat complete Kähler metric.

Theorem 3.5 ([KK10]) *Let $D \in |-K_{\overline{X}}|$ be a smooth ample divisor. There is a complete, Ricci-flat Kähler metric g on X with asymptotics as above such that for every $N > 0$ there is $C > 0$ such that*

$$(1 - C(-\log\|S\|^2)^{-N})g_0 \leq g \leq (1 + C(-\log\|S\|^2)^{-N})g_0$$

Here, g_0 is constructed in essentially the same way as in [TY90].

By solving Laplace equations on the line bundles $\mathscr{L}(nD) \otimes \overline{\mathscr{L}(kD)}$ Santoro [S08] obtains a sequence g_m of initial metrics constructed explicitely as above with metrics $\|.\|_m$ on $\mathscr{L}(D)$ such that the asymptotics of a Ricci-flat, complete Kähler metric can be described by those of g_m up to order $\|S\|^m$.

Theorem 3.6 ([S08]) *Let $D \in |-K_{\overline{X}}|$ be a smooth ample divisor. There is an explicitely constructable sequence g_m of complete Kähler metrics with asymptotics as above, $C_m > 0$ and a Ricci-flat complete Kähler metric g on X such that*

$$(1 - C_m(\|S\|^2)^m)g_m \leq g \leq (1 + C_m(\|S\|^2)^m)g_m.$$

These results imply that the geodesic distance from a fixed point is of order $(-\log\|S\|^2)^{\frac{n+1}{2n}}$ and the volume growth of geodesic balls of radius r around a fixed point is of order $r^{\frac{2n}{n+1}}$. This is an interesting result when viewed from the symmetric case. In case $X = (\mathbb{C}^*)^k \times \mathbb{C}^l$, the volume growth of the symmetric metrics is of order r^{2l+k}, so the exponent is of integer order and at least $n = \dim X$.

4 Crepant Resolutions

Here the existence of a complete Ricci-flat Kähler metric is encoded in terms of Sasakian geometry. At the basis of this technique lies the fact that the metric cone $C(S) = S \times \mathbb{R}^+$ with $\tilde{g} := dr^2 + r^2 g$ over a Sasaki manifold (S, g) – called "Kähler cone" in this context – is Ricci-flat and Kähler if and only if S is Sasaki-Einstein with positive scalar curvature. Now the aim is to resolve the singularity at the apex in such a way that a Ricci-flat Kähler metric asymptotic to the cone metric can be obtained on the resolution. Necessarily, the resolution $\pi : Y \longrightarrow C(S)$ has to be crepant, i.e. $\pi^* K_{C(S)} = K_Y$. For this class, van Coevering found a beautiful result.

Theorem 4.1 ([C10]) *Let $\pi : Y \longrightarrow C(S)$ be a crepant resolution of the Ricci-flat Kähler cone $C(S)$. Then, for each Kähler class $\alpha \in H^2_c(Y, \mathbb{R})$ there is a complete Ricci-flat Kähler metric g with $[g] = \alpha$ and g is asymptotic to the cone metric for any order of derivatives.*

In particular, the volume growth of geodesic balls is of order r^{2n}. Many examples for this Theorem have been constructed before: Joyce [J01] and Kronheimer [K89] proved the existence of an almost locally euclidean Ricci-flat Kähler metric in every Kähler class for crepant resolutions of $\mathbb{C}^n \setminus \{0\}/\Gamma$ with $\Gamma \subset SL(n, \mathbb{C})$ a finite group.

The conditions of Theorem 4.1 need some discussion. The existence of a crepant resolution in dimension 3 implies that the singularity of X is Gorenstein. For toric varieties the situation clarifies further: Futaki, Ono and Wang [FOW06] proved that a toric Gorenstein metric cone over a Sasaki manifold admits a Ricci-flat Kähler cone metric. In dimension 3, any toric Gorenstein singularity allows for a crepant resolution, so that we obtain

Example 3 Let X be a 3-dimensional toric Gorenstein Kähler cone and $\pi : Y \longrightarrow X$ a crepant resolution. In every class of $H^2_c(Y, \mathbb{R})$ there is a complete, Ricci-flat T^n-invariant Kähler metric on X asymptotic to the cone metric.

There are cases of crepant resolutions not covered by Theorem 4.1.

Example 4 Whenever a crepant resolution is small, i.e. the exceptional locus has codimension > 1, then there cannot be a Kähler class in $H^2_c(Y, \mathbb{R})$. For instance, the total space of the line bundle $Y := \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{C}P^1$ can be obtained by a small resolution from $X = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$, the cone over $S^2 \times S^3$. The latter allows for a Sasaki-Einstein metric and Y is known to support a complete, Ricci-flat Kähler metric [CO90].

This example is also instructive from the viewpoint of open manifolds. Y is an open manifold in $\overline{Y} := \mathbb{P}(\mathscr{O}(-1) \oplus \mathscr{O}(-1) \oplus \mathscr{O}) \longrightarrow \mathbb{P}^1$ with $D = \mathscr{O}(1)$. In particular, $-K_{\overline{Y}}$ is ample and not a multiple of D.

The examples show that as well for open manifolds as for crepant resolutions a classification of the Ricci-flat case is still missing. Apart from the ambitious goal to achieve such a classification, one may also ask for a unifying definition of a class of non-compact manifolds allowing for a classification as systematic as the one for compact manifolds. But there are still too many questions unanswered to raise such an issue.

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Fixed point subalgebras of Weil algebras: from geometric to algebraic questions

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Abstract The paper is a survey of some results about Weil algebras applicable in differential geometry, especially in some classification questions on bundles of generalized velocities and contact elements. Mainly, a number of claims concerning the form of subalgebras of fixed points of various Weil algebras are demonstrated.

Keywords Local algebra, Weil algebra, automorphism, fixed point subalgebra, natural operator.
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1 Introduction

Motivated by algebraic geometry, André Weil suggested the treatment of infinitesimal objects as homomorphisms from algebras of smooth functions into some real finite-dimensional commutative algebra with unit in 1950's. In fact, he follows a certain idea of Sophus Lie: so-called A-near points (defined by Weil in [9]) represent 'parametrized infinitesimal submanifolds'. More precisely, let M be a smooth manifold and let $C^\infty(M,\mathbb{R})$ be its ring of smooth functions into $\mathbb{R}$: A-near points of M are defined as $\mathbb{R}$-algebra homomorphism $C^\infty(M,\mathbb{R}) \to A$, where A is a certain local $\mathbb{R}$-algebra A (precisely defined below) now called the Weil algebra. This can be

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regarded as the first notable occurrence of local $\mathbb{R}$-algebras in differential geometry. New concepts, such as Weil algebras, Weil functors, Weil bundles were introduced and they are widely studied, even to this day, because of their considerable generality. In a modern categorical approach to differential geometry, if we interpret geometric objects as bundle functors, then natural transformations represent a number of geometric constructions. In this context, finding a bijection between natural transformations of two Weil functors T^A, T^B (generalizing well-known functors of higher order velocities and, of course, the tangent functor as the first of them) and corresponding morphisms of Weil algebras A and B, has fundamental importance. The theory of natural bundles and operators, including methods for finding natural operators, is very well presented in the monographical work Natural Operations in Differential Geometry [1] (Ivan Kolář, Peter Michor and Jan Slovák, 1993). This paper has the character of a survey: it provides an introduction to Weil algebras and some selected problems which are geometrically motivated and were studied by the author and his collaborators from the algebraic point of view.

2 Starting points: product preserving functors

Let $F\colon \mathbf{Mf} \to \mathbf{FM}$ be a bundle functor from the category **Mf** of manifolds (having smooth manifolds as objects and smooth maps as morphisms) to the category **FM** of fibered manifolds (and fibered manifold morphisms). For example, such a functor is the tangent functor T. For two manifolds M_1, M_2 we denote the standard projection onto the i–th factor by $p_i\colon M_1 \times M_2 \to M_i$, where $i = 1,2$. F is called **product preserving** if the mapping

$$(F(p_1),F(p_2))\colon F(M_1 \times M_2) \to F(M_1) \times F(M_2)$$

is a diffeomorphism for all manifolds M_1, M_2. For a product preserving bundle functor we shall always identify $F(M_1 \times M_2)$ with $F(M_1) \times F(M_2)$ by the diffeomorphism from the definition. The tangent functor T is product preserving. Another example of a product preserving functor is the functor T_k^r of k-dimensional r-th order velocities with $T_1^1 = T$. Further, we obtain a product preserving functor by arbitrary (finite) iterations of product preserving functors.

If we denote by **WA** the category of Weil algebras (the exact definition of Weil algebra is postponed to the next section) and Weil algebra homomorphisms, then the problem of classification of all product preserving functors was solved in works of Kainz and Michor, Luciano and Eck in the 1980's and reads as follows (see [1]):

Product preserving bundle functors from the category **Mf** *of manifolds into the category* **FM** *of fibered manifolds are in bijection with objects of* **WA** *and natural transformations between two such functors are in bijection with the morphisms of* **WA**.

The correspondence is determined by the following construction of the bundle functor T^A from a given Weil algebra A. Let M be a smooth manifold and let A be a Weil algebra. Two smooth maps $g, h\colon \mathbb{R}^k \to M$ are said to *determine the same A-velocity* $j^A g = j^A h$, if for every smooth function $\varphi\colon M \to \mathbb{R}$

$$\pi_A(j_0^r(\varphi \circ g)) = \pi_A(j_0^r(\varphi \circ h))$$

is satisfied. (As usually, we denote here r-jets with the source in $0 \in \mathbb{R}^k$ by j_0^r and an epimorphism from the algebra $\mathbb{D}_k^r = J_0^r(\mathbb{R}^k, \mathbb{R})$ to the algebra A by π_A.) The space $T^A M$ of all A-velocities on M is fibered over M and is called the *Weil bundle*. The functor T^A from **Mf** into **FM** is called the *Weil functor*.

3 To the definition of the Weil algebra

The *Weil algebra* is a local commutative $\mathbb{R}$-algebra A with identity, the nilradical (nilpotent ideal) $\mathfrak{n}_A$ of which has finite dimension as a vector space and $A/\mathfrak{n}_A = \mathbb{R}$. We call the *order* of A the minimum $\mathrm{ord}(A)$ of the integers r satisfying $\mathfrak{n}_A^{r+1} = 0$ and the *width* $\mathrm{w}(A)$ of A the dimension $\dim_{\mathbb{R}}(\mathfrak{n}_A/\mathfrak{n}_A^2)$.

One can assume A is expressed as a finite dimensional factor $\mathbb{R}$-algebra of the algebra $\mathbb{R}[x_1, \ldots, x_k]$ of real polynomials in several indeterminates. Thus, the main example is

$$\mathbb{D}_k^r = \mathbb{R}[x_1, \ldots, x_k]/\mathfrak{m}^{r+1},$$

$\mathfrak{m} = \langle x_1, \ldots, x_k \rangle$ being the maximal ideal of $\mathbb{R}[x_1, \ldots, x_k]$. Evidently, $\mathrm{ord}(\mathbb{D}_k^r) = r$ and $\mathrm{w}(\mathbb{D}_k^r) = k$. Every other such algebra A of order r can be expressed in a form

$$A = \mathbb{R}[x_1, \ldots, x_k]/\mathfrak{j} = \mathbb{R}[x_1, \ldots, x_k]/\mathfrak{i} + \mathfrak{m}^{r+1},$$

where the ideal $\mathfrak{i}$ satisfies $\mathfrak{m}^{r+1} \subsetneqq \mathfrak{i} \subseteq \mathfrak{m}^2$ and is generated by a finite number of polynomials, i.e. $\mathfrak{i} = \langle P_1, \ldots, P_l \rangle$. The fact $\mathfrak{i} \subseteq \mathfrak{m}^2$ implies that the width of A is k as well. It is evident, that such expressions of algebras in question are not unique after all. Clearly, A can be expressed also as

$$A = \mathbb{D}_k^r/\mathfrak{i},$$

where $\mathfrak{i}$ is an ideal in $\mathbb{D}^r_k$. This last definition will be prefered in the paper; we will also frequently move from $\mathbb{D}^r_k/\mathfrak{i}$ to $\mathbb{R}[x_1,\dots,x_k]/\mathfrak{j}$ and back.

Let $\mathrm{Aut}A$ be the group of automorphisms of the algebra A. By a *fixed point* of A we mean every $a \in A$ satisfying $\varphi(a) = a$ for all $\varphi \in \mathrm{Aut}A$. Let

$$SA = \{a \in A; \varphi(a) = a \text{ for all } \varphi \in \mathrm{Aut}A\}$$

be the set of all fixed points of A. It is clear, that SA is a subalgebra of A containing constants (of couse, every automorphism sends 1 into 1), i.e. $SA \supseteq \mathbb{R}$. If $SA = \mathbb{R}$, we say that SA is *trivial*.

4 Weil contact elements

Now, let the Weil algebra A have width $\mathrm{w}(A) = k < m = \dim M$ and order $\mathrm{ord}(A) = r$. Every A-velocity V determines an underlying $\mathbb{D}^1_k$-velocity $\underline{V}$. We say V is *regular*, if $\underline{V}$ is regular, i.e. having maximal rank k (in its local coordinates). Let us denote $\mathrm{reg}T^AM$ the open subbbundle of T^AM of regular velocities on M. The *contact element of type A* or briefly the *Weil contact element* on M determined by $X \in \mathrm{reg}T^AM$ is the equivalence class

$$\mathrm{Aut}A_M(X) = \{\varphi(X); \varphi \in \mathrm{Aut}A\}.$$

We denote by K^AM the set of all contact elements of type A on M. Then

$$K^AM = \mathrm{reg}T^AM/\mathrm{Aut}A$$

has a differentiable manifold structure and $\mathrm{reg}T^AM \to K^AM$ is a principal fiber bundle with structure group $\mathrm{Aut}A$. Moreover, K^AM is a generalization of the bundle of higher order contact elements $K^r_kM = \mathrm{reg}T^r_kM/G^r_k$ introduced by Claude Ehresmann. We remark that the local description of regular velocities and contact elements is covered by the paper [2].

Let us write

$$\varepsilon_A \colon \mathrm{Aut}A \to \mathrm{GL}(\mathfrak{n}_A/\mathfrak{n}_A^2)$$

for the canonical group morphism. If we write as usual $\mathrm{w}(A) = \dim_{\mathbb{R}}(\mathfrak{n}_A/\mathfrak{n}_A^2)$, then $\mathrm{GL}(\mathfrak{n}_A/\mathfrak{n}_A^2)$ reads as $\mathrm{GL}(\mathrm{w}(A),\mathbb{R})$.

Further, the element $\varphi \in \mathrm{Aut}A$ is called *orientation preserving*, if the determinant of $\varepsilon_A(\varphi)$ is positive.

The subgroup of all orientation preserving elements of Aut$A$ will be denoted by $(\mathrm{Aut}A)^+$.

If we factorize

$$\mathrm{reg}T^AM/(\mathrm{Aut}A)^+,$$

we obtain the bundle $K^{A+}M$ of *Weil oriented contact elements*.

As to orientability, we remark that even the case $\mathrm{Aut}A = (\mathrm{Aut}A)^+$ can occur. So, it is suggestive to study the orientability (with interesting references to classical geometric problems) just from the indicated point of view.

5 Subalgebra of fixed points

We use the fact that a Weil algebra A can also be considered as a factor algebra of the algebra $\mathbb{R}[x_1,\dots,x_k]$ of polynomials, i.e. $A = \mathbb{R}[x_1,\dots,x_k]/\mathfrak{j}$ and then $\mathfrak{j}=\mathfrak{i}+\mathfrak{m}^{r+1}$, where $\mathfrak{m} = \langle x_1,\dots,x_k\rangle$ is the maximal ideal in $\mathbb{R}[x_1,\dots,x_k]$. Let $\tau \in \mathbb{R}$, $\tau \neq 0$, and let $H_\tau\colon \mathbb{R}[x_1,\dots,x_k] \to \mathbb{R}[x_1,\dots,x_k]$ be a (linear diagonal) homomorphism acting by

$$\begin{aligned} x_1 &\mapsto \tau x_1 \\ &\dots \\ x_k &\mapsto \tau x_k. \end{aligned}$$

Then it is necessary to determine whether H_τ induces a homomorphism $\bar{H}_\tau\colon A \to A$ or not.

Definition 1 The Weil algebra $A = \mathbb{D}^r_k/\mathfrak{i}$ is called *monomial*, if $\mathfrak{i}$ is monomial.

Proposition 1 *If A is a monomial Weil algebra, then its subalgebra SA of fixed points is trivial.*

Proof It is clear that $\mathfrak{j}$ can be also generated by monomials. The homomorphism H_τ sends every such monomial from $\mathfrak{j}$ again into $\mathfrak{j}$, i.e. $H_\tau(\mathfrak{j}) \subseteq \mathfrak{j}$ and we have the induced homomorphism $\bar{H}_\tau\colon A \to A$. For $\tau \notin \{-1,0,1\}$, $\bar{H}_\tau(a) \neq a$ for every element of $a \in \mathfrak{n}_A$. Thus, SA is trivial. ■

Definition 2 The Weil algebra $A = \mathbb{D}^r_k/\mathfrak{i}$ is called *homogeneous*, if $\mathfrak{i}$ is homogeneous.

If we have a positive gradation $A = \bigoplus_{i\geq 0} A_i$ on a Weil algebra A such that $\mathfrak{n}_A^n = \bigoplus_{i\geq n} A_i$ for each $n \geq 0$, we say that A is *gradable by the radical*, cf. [8]. We remark that A is gradable by the radical if and only if $\mathfrak{i}$ is homogeneous.

Proposition 2 *If A is a homogeneous Weil algebra, then its subalgebra SA of fixed points is trivial.*

Proof The reason is completely identical to that in the previous proposition (see [5] for the original proof): the homomorphism H_τ sends a homogeneous polynomial from $\mathfrak{j}$ again into $\mathfrak{j}$, i.e. $H_\tau(\mathfrak{j}) \subseteq \mathfrak{j}$ and we have the induced homomorphism $\bar{H}_\tau \colon A \to A$. For $\tau \notin \{-1,0,1\}$, $\bar{H}_\tau(a) \neq a$ for every element of $a \in \mathfrak{n}_A$ and SA is trivial. ∎

The idea of the proofs of the two propositions above lies in the fact that H_τ maps $\mathfrak{j}$ into $\mathfrak{j}$. Thus, it is not difficult to derive the following slight generalization. Let $\tau_1,\dots,\tau_k$ be non-zero real numbers and $H_{\tau_1,\dots,\tau_k} \colon \mathbb{R}[x_1,\dots,x_k] \to \mathbb{R}[x_1,\dots,x_k]$ be a (linear diagonal) homomorphism acting by

$$\begin{aligned} x_1 &\mapsto \tau_1 x_1 \\ &\dots \\ x_k &\mapsto \tau_k x_k. \end{aligned}$$

Proposition 3 *If $A = \mathbb{R}[x_1,\dots,x_k]/\mathfrak{j}$ is a Weil algebra with $\mathrm{w}(A) = k$ and if there exist some $\tau_1,\dots,\tau_k \in \mathbb{R} - [-1,1]$ (or $\tau_1,\dots,\tau_k \in (-1,1) - \{0\}$) such that $H_{\tau_1,\dots,\tau_k}(\mathfrak{j}) \subseteq \mathfrak{j}$, then the subalgebra SA of fixed points of A is trivial.*

Proof The idea of the proof of this generalization is clear: if $H_{\tau_1,\dots,\tau_k}(\mathfrak{j}) \subseteq \mathfrak{j}$, then every non-constant monomial from $\mathfrak{j}$ maps onto a monomial in $\mathfrak{j}$ (with the same multidegree), however, not onto the same monomial, because of the impossibility to obtain 1 as a product of τ's. The induced homomorphism preserves this property. ∎

The assertions of the previous three propositions do not hold in the opposite direction — not even the last one, which has the most general presumptions. For example $A = \mathbb{D}_2^4/\langle x^2 + y^3, x^3 + y^4\rangle$ has trivial SA, but there are no $\tau_1,\tau_2 \in \mathbb{R} - [-1,1]$ (or $\tau_1,\tau_2 \in (-1,1) - \{0\}$) such that $H_{\tau_1,\tau_2}(\langle x^2+y^3, x^3+y^4\rangle + \mathfrak{m}^5) \subseteq \langle x^2+y^3, x^3+y^4\rangle + \mathfrak{m}^5$, see [5].

It is now the right time to show that there exist Weil algebras for which their subalgebras of fixed points are not trivial. Examples of such algebras are

$\mathbb{D}_2^4/\langle x^2y+y^4, x^3+xy^2\rangle$ or $\mathbb{D}_3^3/\langle x^2+y^3, xy+z^3, y^2z+yz^2\rangle$. This can be verified by a direct computation (although it is not evident at first sight: see Appendix!). Moreover, the following "order theorem" holds.

Proposition 4 *There is no algebra A with* $\mathrm{w}(A)=1$ *and with nontrivial fixed point subalgebra. There exist algebras A with* $\mathrm{w}(A)=2$ *with a nontrivial fixed point subalgebra if and only if* $\mathrm{ord}(A)\geq 4$. *For all $k>2$, there exist an algebra with* $\mathrm{w}(A)=k$ *and with a nontrivial fixed point subalgebra if and only if* $\mathrm{ord}(A)\geq 3$.

Proof The proof is based on several technical lemmas and we do not write it here for its length. We refer mainly to [7] and also to [6]. ■

Let us follow through a slightly different but also fairly good approach. For a Weil algebra A, the canonical algebra homomorphism $\kappa_A : A \to \mathbb{R}$ can be viewed as the endomorphism $\kappa_A : A \to A$. The group $\mathrm{Aut}A$ of $\mathbb{R}$-algebra automorphisms of A is a real smooth manifold with the usual Euclidean topology. Then the following definition is correct.

Definition 3 A Weil algebra A is said to be *dwindlable* if there is an infinite sequence $\{\varphi_n\}_{n=1}^{\infty}$ of automorphisms $\varphi_n \in \mathrm{Aut}A$ such that $\varphi_n \to \kappa_A$ for $n\to\infty$.

Proposition 5 *If A is a dwindlable Weil algebra, then its subalgebra SA of fixed elements is trivial. Apart from that, there are non-dwindlable Weil algebras with trivial SA.*

Proof If A is dwindlable and SA is not trivial, then there exists an element $0\neq a\in\mathfrak{n}_A$ belonging to SA. As there is also an infinite sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n\in\mathrm{Aut}A$, $\varphi_n\to\kappa_A$ for $n\to\infty$, we deduce for a that $0\neq a=\varphi_n(a)\to\kappa_A(a)=0$ which is a contradiction. On the other hand, $\mathbb{D}_2^5/\langle xy^2+x^5, x^2y+y^5\rangle$ represents an example of a non-dwindlable Weil algebra with trivial SA. ■

Furthermore, let us remark that for a dwindlable Weil algebra A, the group U_A of unipotent automorphisms (i.e. such automorphisms φ for which $\mathrm{id}_A-\varphi$ is a nilpotent endomorphism of A) is a proper subgroup of the connected identity component G_A of $\mathrm{Aut}A$, see [3]. The index of the subgroup G_A also represents an important object of interest, cf. [4].

Let us return to the geometric motivation. From what we have stated, we have deduced in [5] and [6] the following results:
There is a one-to-one correspondence between all natural operators lifting vector

fields from m-manifolds to the bundle functor K^A of Weil contact elements and the subalgebra of fixed elements SA of A.
There is a one-to-one correspondence between all natural affinors on K^A and the subalgebra of fixed elements SA of A.
All natural operators lifting 1-forms from m-dimensional manifolds to the bundle functor K^A of Weil contact elements are classified for the case of dwindlable Weil algebras: they represent constant multiples of the vertical lifting.

To the open problem

We conclude that the main problem of an exact one-to-one characterization of Weil algebras having non-trivial fixed point subalgebras remains open.

Nevertheless, a number of partial (sub-)problems can be mentioned. For example, elements $a \in A$ annihilated by any element of the nilradical $\mathfrak{n}_A$, i.e. having the property $au = 0$ for all $u \in \mathfrak{n}_A$, constitute an ideal which is called the *socle* of A and denoted by $\mathrm{soc}(A)$. Then elements of A in the form $r_1 + r_2 a$, $r_1, r_2 \in \mathbb{R}$, $a \in \mathrm{soc}(A)$ form a subalgebra MA of A. The problem of a relation between SA and MA is also open (with the conjecture: $SA \subseteq MA$).

Appendix: The computation method and two examples

We present a computation method for the description of automorphisms and detecting whether the fixed point subalgebra is trivial or not.

Example 1 The first example is of theoretical importance, see Proposition 4. Let

$$A = \mathbb{D}_2^4 / \langle x^2 y + y^4, x^3 + xy^2 \rangle.$$

The elements of A have the form

$$k_1 + k_2 x + k_3 y + k_4 x^2 + k_5 xy + k_6 y^2 + k_7 x^3 + k_8 x^2 y + k_9 y^3$$

with the simultaneous vanishing of all monomials of the fifth or higher order in common with x^4, $x^3 y$, $x^2 y^2$, xy^3, $x^2 y + y^4$ and $x^3 + xy^2$. We shall describe automorphisms of A. Automorphisms preserve the unit; so, we determine them by saying what is mapped to x and y, for clarity, denoted rather by $\bar{x}$ and $\bar{y}$. Thus, the starting point is a form

$$\begin{aligned}\bar{x} &= Ax+By+Cx^2+Dxy+Ey^2+Fx^3+Gx^2y+Hy^3\\ \bar{y} &= Ix+Jy+Kx^2+Lxy+My^2+Nx^3+Ox^2y+Py^3.\end{aligned}$$

The matrix $\begin{pmatrix} A & B \\ I & J \end{pmatrix}$ must be non-singular and we consider the conditions $\bar{x}^4=0$, $\bar{x}^3\bar{y}=0$, $\bar{x}^2\bar{y}^2=0$, $\bar{x}\bar{y}^3=0$, $\bar{x}^2\bar{y}+\bar{y}^4=0$ and $\bar{x}^3+\bar{x}\bar{y}^2=0$ now. The condition $\bar{x}^4=0$ gives $B=0$. The conditions $\bar{x}^3\bar{y}=0$, $\bar{x}^2\bar{y}^2=0$, $\bar{x}\bar{y}^3=0$ give no new nontrivial relation. The condition $\bar{x}^2\bar{y}+\bar{y}^4=0$ gives $I=0$, $A^2=J^3$. The condition $\bar{x}^3+\bar{x}\bar{y}^2=0$ gives $E=0$, $A^2=J^2$. So, we obtain $J=1$ and $A=-1$ or $A=1$. Hence the automorphisms have the following form

$$\begin{aligned}\bar{x} &= \varepsilon x+Cx^2+Dxy+Fx^3+Gx^2y+Hy^3\\ \bar{y} &= y+Kx^2+Lxy+My^2+Nx^3+Ox^2y+Py^3,\end{aligned}$$

where $\varepsilon\in\{-1,1\}$. (We observe that the group $\mathrm{Aut}A$ of automorphisms has two connected components.) Finally, we solve the equation

$$\begin{aligned}&k_1+k_2\bar{x}+k_3\bar{y}+k_4\bar{x}^2+k_5\bar{x}\bar{y}+k_6\bar{y}^2+k_7\bar{x}^3+k_8\bar{x}^2\bar{y}+k_9\bar{y}^3 =\\ &k_1+k_2x+k_3y+k_4x^2+k_5xy+k_6y^2+k_7x^3+k_8x^2y+k_9y^3\end{aligned}$$

for k_i, $i=1,\dots,9$, by using the described automorphisms. By comparing coefficients at powers of x and y, we find that $k_2=k_3=k_4=k_5=k_6=k_7=k_9=0$ and k_1,k_8 are arbitrary real coefficients. This means

$$SA=\{k_1+k_8x^2y;\ k_1,k_8\in\mathbb{R}\}$$

and $SA\supsetneq\mathbb{R}$.

Example 2 The second example is new. Let

$$A=\mathbb{D}_3^4/\langle x^2+y^3+z^3, x^3+y^3+z^4, xyz\rangle.$$

We start by expressing of elements of A in the form

$$\begin{aligned}&k_1+k_2x+k_3y+k_4z+k_5x^2+k_6xy+k_7y^2+k_8xz+k_9yz+k_{10}z^2+\\ &k_{11}x^2y+k_{12}xy^2+k_{13}x^2z+k_{14}y^2z+k_{15}xz^2+k_{16}yz^2+k_{17}z^3+k_{18}y^2z^2\end{aligned}$$

with the simultaneous vanishing of all monomials of the fifth or higher in common with xyz, x^2y^2, x^2z^2, $x^2+y^3+z^3$, $x^2-x^3+x^2z+z^3$, x^2z+z^4, x^2y+yz^3,

$x^2+x^2z+z^3+xz^3$. The algorithm given above yields after a "bit of calculation" a connected group of automorphisms (we leave its exact expression as an exercise to the reader) and

$$SA=\{k_1+k_5x^2+k_{12}xy^2+k_{13}x^2z+k_{18}y^2z^2;\ k_1,k_5,k_{12},k_{13},k_{18}\in\mathbb{R}\}.$$

Hence, we find that the dimension of the subalgebra SA of fixed points is remarkably high.

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Self-similar solutions and translating solutions

Yng-Ing Lee

Abstract In this note, I provide some detailed computation of constructing translating solutions from self-similar solutions for Lagrangian mean curvature flow discussed in [6] and explore the related geometric meanings. This method works for all mean curvature flows and has great potential to find other new translating solutions.

Keywords Mean curvature flow, self-similar solution, translating solution, Lagrangian.
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1 Introduction

By the first variation formula of area, the *mean curvature vector* points in the direction in which the area decreases most rapidly. *Mean curvature flow* deforms the submanifold in the direction of the mean curvature vector, and thus is a canonical way to construct minimal submanifolds. However, finite-time singularities may occur along the flow. In geometric flows, singularities are often locally modelled on soliton solutions. In the case of mean curvature flows, two types of soliton solutions of particular interest are those moved by scaling or translation in Euclidean space. We recall that solitons moved by scaling must be of the form:

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Definition 1.1 *A submanifold L in Euclidean space $\mathbb{R}^n$ is called a self-similar solution if $H \equiv \alpha F^{\perp}$ on L for some constant α in $\mathbb{R}$, where $F^{\perp}$ is the projection of the position vector F in $\mathbb{R}^n$ to the normal bundle of L, and H is the mean curvature vector of L in $\mathbb{R}^n$. It is called a self-shrinker if $\alpha < 0$ and a self-expander if $\alpha > 0$. It is a minimal submanifold when $\alpha = 0$, which is a static solution of the flow.*

It is not hard to see that if F is a self-similar solution, then F_t defined by $F_t = \sqrt{2\alpha t}\,F$ is a solution to the mean curvature flow. By Huisken's monotonicity formula [3], any central blow up of a finite-time singularity of the mean curvature flow must be a self-shrinker (the generalization to type II singularities is due to Ilmanen [4] and White). The submanifolds which are moved by translation along mean curvature flow must be of the form:

Definition 1.2 *A submanifold L in Euclidean space $\mathbb{R}^n$ is called a translating solution if there exists a constant vector T in $\mathbb{R}^n$ such that $H + V \equiv T$ on L, where V is the component of T tangent to L, and H is the mean curvature vector of L in $\mathbb{R}^n$. An equivalent equation is $H \equiv T^{\perp}$. The 1-parameter family of submanifolds L_t defined by $L_t = L + tT$ for $t \in \mathbb{R}$ is then a solution to mean curvature flow, and we call T a translating vector.*

D. Joyce, M. P. Tsui and the author constructed in [6] many self-similar solutions and translating solutions with different properties for Lagrangian mean curvature flow which requires the solution to be Lagrangian at each time slice. A Lagrangian in $\mathbb{R}^{2n}$ is an n-dimensional submanifold on which the restriction of $\omega = \sum_{j=1}^{n} \mathrm{dx_j} \wedge \mathrm{dy_j}$ vanishes. Our construction of translating solutions in [6] is in fact derived from taking a limit of self-similar solutions. The method works for solutions of all mean curvature flow. The construction in [6] for $n = 1$ gives *Grim reapers* from self-shrinkers (or self-expanders) for curve shortening flow. Other translating solutions for mean curvature flow can also be obtained from the same procedure [8]. Most people are not aware of this interesting approach and when I discuss with some, they appear to be quite interested. I thus feel that it might be desirable of explaining this principle in details. I will first state the method, which is pinned down to the convergent condition; then use the examples in [6] to demonstrate how the condition can be justified. Although it requires efforts to verify everything rigorously, it is relatively easy to find the right family heuristically. I believe that this method will help us to find more translating solutions for mean curvature flow as [8] being the first attempt in this direction. The geometric picture for $n = 2$, i.e. Lagrangian surfaces in $\mathbb{R}^4$, is discussed in details in the last section.

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2 Self-similar solutions to translating solutions

Here we state the principle of constructing translating solutions from self-similar solutions. The theorem is very simple and the most essential part is to arrange a sequence of self-similar solutions such that the condition is satisfied. Many discussions presented here have appeared implicitly and briefly in [6].

Theorem 2.1 *Suppose we have a sequence of self-similar solutions satisfying $F_i^\perp \equiv R_i H_i$ (or $F_i^\perp \equiv -R_i H_i$) with $R_i \to \infty$. Define*

$$\bar{F}_i \equiv F_i - (0,\ldots,0,R_i). \tag{1}$$

Assume that $\bar{F}_i$ converges to F and the corresponding mean curvature vector H_i also converges to the mean curvature vector H of F. Then F is a translating solution with translating vector $(0,\ldots,0,1)$.

Proof Because $\bar{F}_i$ is just a translation of F_i, it has the same mean curvature vector H_i. The self-similar equation can be rewritten as

$$\bar{F}_i^\perp + (0,\ldots,0,R_i)^\perp \equiv R_i H_i.$$

Dividing both sides by R_i, it gives

$$R_i^{-1}\bar{F}_i^\perp + (0,\ldots,0,1)^\perp \equiv H_i.$$

As $\bar{F}_i$ converges to F, the related convergent sequence of points are bounded and hence $R_i^{-1}\bar{F}_i^\perp$ converges to the zero vector for this sequence. We then have $H \equiv (0,\ldots,0,1)^\perp$ and show that F is a translating solution.

One may try to scale a single self-similar solution by R, and apply the above theorem. But this does not work. The sphere is a self-shrinker, and is a good example to check the idea. We need genuine parameters to make the procedure work. The case in [6] has many nontrivial parameters, and becomes an excellent example to illustrate this method. Note that if we write the self-similar equation in the form $H_i \equiv \alpha_i F_i^{\perp}$ as in Definition 1.1, then we require $\alpha_i \to 0$ instead.

Recall that the self-similar solutions obtained in [6] are of the following form:

Theorem 2.2 [6] *Let $\lambda_1, \dots, \lambda_n, C \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$ be constants, I be an open interval in $\mathbb{R}$, and $\theta : I \to \mathbb{R}$ or $\theta : I \to \mathbb{R}/2\pi\mathbb{Z}$ and $w_1, \dots, w_n : I \to \mathbb{C} \setminus \{0\}$ be smooth functions. Suppose that*

$$\begin{aligned} \frac{\mathrm{dw_j}}{\mathrm{ds}} &= \lambda_j e^{i\theta(s)} \overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_n}, \qquad j = 1, \dots, n, \\ \frac{\mathrm{d}\theta}{\mathrm{ds}} &= \alpha \,\mathrm{Im}(e^{-i\theta(s)} w_1 \cdots w_n), \end{aligned} \tag{2}$$

hold in I. Then the submanifold L in $\mathbb{C}^n$ given by

$$L = \left\{ \left(x_1 w_1(s), \dots, x_n w_n(s)\right) : s \in I, \, x_j \in \mathbb{R}, \, \textstyle\sum_{j=1}^n \lambda_j x_j^2 = C \right\}, \tag{3}$$

is Lagrangian, with Lagrangian angle $\theta(s)$ at $(x_1 w_1(s), \dots, x_n w_n(s))$, and its position vector F and mean curvature vector H satisfy $\alpha F^{\perp} = CH$.

Example 2.3 Give $\lambda_1, \dots, \lambda_{n-1}$ and choose $\lambda_n = R$, $C = R$ in Theorem 2.2. For any initial data which will be specified later, we can solve (2) and find a self-similar solution F_R in the form (3) satisfying $F_R^{\perp} = \frac{R}{\alpha} H$. Note that we now use complex coordinates in $\mathbb{C}^n$. Because we want

$$\bar{F}_R = (x_1 w_1(s), \dots, x_n w_n(s)) - (0, \dots, 0, R)$$

converge as $R \to \infty$, we first rewrite

$$w_n = R + \beta^R \quad \text{and} \quad x_n = 1 + R^{-1} \bar{x}_n$$

to absorb the unbounded vector. In new variables, we have

$$\bar{F}_R = (x_1 w_1(s), \dots, \bar{x}_n + \beta^R(s) + R^{-1} \bar{x}_n) \tag{4}$$

and

$$\lambda_1 x_1^2 + \ldots + \lambda_{n-1} x_{n-1}^2 + 2\bar{x}_n + R^{-1} \bar{x}_n^2 = 0 \tag{5}$$

We want to study the limit of (4) as $R \to \infty$ and let us first start with (5). Given $x_1, \dots, x_{n-1}$, we have $A = \sum_{j=1}^{n-1} \lambda_j x_j^2$ fixed and $A + 2\bar{x}_n + R^{-1}\bar{x}_n^2 = 0$. When λ_j are all positive, we have $A \geq 0$, while we have $A \leqslant 0$ when λ_j are all negative, and A can be any number for the other cases. Note that we always have real value solutions for $\bar{x}_n$ because the equation is from (3). Simple algebra gives $\bar{x}_n = -R + \sqrt{R^2 - RA}$ or $\bar{x}_n = -R - \sqrt{R^2 - RA}$. As the second root diverges to $-\infty$ as $R \to \infty$, we investigate the first one only. When $A \geq 0$, we have

$$0 \geq \bar{x}_n = -R + \sqrt{R^2 - RA} = R\big(-1 + \sqrt{1 - R^{-1}A}\big) \geq R\big(-1 + 1 - R^{-1}A\big) \geq -A \quad (6)$$

for R large enough. It follows that $R^{-1}\bar{x}_n^2 \to 0$ as $R \to \infty$ and the limit of (5) becomes

$$\lambda_1 x_1^2 + \ldots + \lambda_{n-1} x_{n-1}^2 + 2\bar{x}_n = 0. \quad (7)$$

When $A < 0$, we have $0 < \bar{x}_n < -A$ instead and still have the same conclusion. Now consider the curve part. We denote the curve satisfying (2) by Γ_R and define $\bar{\Gamma}_R = \Gamma_R - (0, \ldots, 0, R)$ which satisfies

$$\begin{aligned}
\frac{\mathrm{dw_j^R}}{\mathrm{ds}} &= \lambda_j e^{i\theta^R}\, \overline{w_1^R \cdots w_{j-1}^R w_{j+1}^R \cdots w_{n-1}^R}\,(R + \overline{\beta^R}), \qquad j = 1, \ldots, n-1, \\
\frac{\mathrm{d\beta^R}}{\mathrm{ds}} &= R e^{i\theta^R}\, \overline{w_1^R \cdots w_{n-1}^R} \\
\frac{\mathrm{d\theta^R}}{\mathrm{ds}} &= \alpha \,\mathrm{Im}\big(e^{-i\theta^R} w_1^R \cdots w_{n-1}^R (R + \beta^R)\big).
\end{aligned} \quad (8)$$

What we concern is the limit of the image, not the parametrization itself. So we will choose a different parametrization $\tilde{s} = Rs$ to absorb R factor and make the limiting process more transparent. In this new parameter (8) becomes

$$\begin{aligned}
\frac{\mathrm{dw_j^R}}{\mathrm{d\tilde{s}}} &= \lambda_j e^{i\theta^R}\, \overline{w_1^R \cdots w_{j-1}^R w_{j+1}^R \cdots w_{n-1}^R}\,(1 + R^{-1}\overline{\beta^R}), \qquad j = 1, \ldots, n-1, \\
\frac{\mathrm{d\beta^R}}{\mathrm{d\tilde{s}}} &= e^{i\theta^R}\, \overline{w_1^R \cdots w_{n-1}^R} \\
\frac{\mathrm{d\theta^R}}{\mathrm{d\tilde{s}}} &= \alpha \,\mathrm{Im}\big(e^{-i\theta^R} w_1^R \cdots w_{n-1}^R (1 + R^{-1}\beta^R)\big).
\end{aligned} \quad (9)$$

Choose the initial data in (2) to be $\big(w_1(0), \ldots, w_{n-1}(0), R + \beta(0)\big)$, i.e. the same initial data $\big(w_1(0), \ldots, w_{n-1}(0), \beta(0)\big)$ for all R in (9). We need to control $R^{-1}\beta^R$ to take a limit in (9).

To see this, we first recall Theorem B in [6] where we write $w_j \equiv r_j e^{i\varphi_j}$ and $\varphi = \sum_{j=1}^n \varphi_j$, for functions $r_j : I \to (0, \infty)$ and $\varphi_1, \ldots, \varphi_n, \theta : I \to \mathbb{R}$ or $\mathbb{R}/2\pi\mathbb{Z}$. Fix $s_0 \in I$ and define $u : I \to \mathbb{R}$ by

$$u(s) = 2\int_{s_0}^{s} r_1(t) \cdots r_n(t) \cos\big(\varphi(t) - \theta(t)\big)\mathrm{dt}.$$

Then one has $r_j^2(s) \equiv \alpha_j + \lambda_j u(s)$ for $j = 1, \dots, n$ and $s \in I$, where $\alpha_j = r_j^2(s_0)$. Applying the result to the case discussed here, we have

$$\frac{\mathrm{du^R}}{\mathrm{ds}} = 2\sqrt{(\alpha_1 + \lambda_1 u^R)\dots(\alpha_{n-1} + \lambda_{n-1} u^R)(|R + \beta(0)|^2 + Ru^R)}\cos(\varphi^R - \theta^R) \tag{10}$$

or equivalently,

$$\frac{\mathrm{du^R}}{\mathrm{d\tilde{s}}} = 2\sqrt{(\alpha_1 + \lambda_1 u^R)\dots(\alpha_{n-1} + \lambda_{n-1} u^R)(|1 + R^{-1}\beta(0)|^2 + R^{-1}u^R)}\cos(\varphi^R - \theta^R), \tag{11}$$

where $\alpha_j = |w_j(0)|^2$. For R large enough we have

$$\frac{\mathrm{du^R}}{\mathrm{d\tilde{s}}} \leqslant 2\sqrt{(\alpha_1 + \lambda_1 u^R)\dots(\alpha_{n-1} + \lambda_{n-1} u^R)(2 + u^R)}. \tag{12}$$

Hence u^R is uniformly bounded for fixed $\tilde{s}$. On the other hand, we have

$$|R + \beta(0)|^2 + Ru^R = |R + \beta^R|^2.$$

Dividing both sides by R^2, we obtain

$$|1 + R^{-1}\beta(0)|^2 + R^{-1}u^R = |1 + R^{-1}\beta^R|^2.$$

For fixed $\tilde{s}$ the left-hand side tends to 1 as $R \to \infty$, because $\beta(0)$ is fixed and u^R is uniformly bounded. This shows that $R^{-1}\beta^R$ tends to zero as $R \to \infty$, and we can take a limit in (9), which becomes

$$\begin{aligned} \frac{\mathrm{dw_j}}{\mathrm{d\tilde{s}}} &= \lambda_j e^{i\theta}\,\overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_{n-1}}, \qquad j = 1, \dots, n-1, \\ \frac{\mathrm{d}\beta}{\mathrm{d\tilde{s}}} &= e^{i\theta}\,\overline{w_1 \cdots w_{n-1}} \\ \frac{\mathrm{d}\theta}{\mathrm{d\tilde{s}}} &= \alpha\,\mathrm{Im}\big(e^{-i\theta} w_1 \cdots w_{n-1}\big). \end{aligned} \tag{13}$$

The solution for (13) is the limit curve of $\bar{\Gamma}_R$ and θ being the Lagrangian angle of F. This shows that $\bar{F}_R$ in (4) converges to

$$\big(x_1 w_1(\tilde{s}), \dots, x_{n-1} w_{n-1}(\tilde{s}), -\tfrac{1}{2}\textstyle\sum_{j=1}^{n-1} \lambda_j x_j^2 + \beta(\tilde{s})\big) \tag{14}$$

with $x_j \in \mathbb{R}$, $w_1, \dots, w_{n-1}, \beta$ satisfying (13) and the mean curvature vector of $\bar{F}_R$ also converges to that of F as we have $H = J\nabla\theta$ for Lagrangians. Thus (14) is a Lagrangian translating solution with translating vector $(0, \dots, 0, \alpha)$.

Example 2.4 When $n = 1$, the system (2) reduce to that for self-similar solutions of the curve shortening flow. Thus the construction of Example 2.3 gives a way to obtain grim reapers from the self-shrinkers constructed in [2]. For completeness, we repeat the argument below briefly.

We take $\alpha = 1$ (or -1), $\lambda = R$ and $C = R$. The solution has two connected components corresponding to $x = 1$ and $x = -1$ respectively. We look at the branch $w_R(s) \in \mathbb{C}$, and have $w_R^{\perp} = RH_R$ (or $w_R^{\perp} = -RH_R$ for $\alpha = -1$). Define $\beta^R = w_R - R$ and $\tilde{s} = Rs$. Then

$$\frac{\mathrm{d}\beta^{\mathrm{R}}}{\mathrm{d}\tilde{\mathrm{s}}} = e^{i\theta^R} \quad \text{and} \quad \frac{\mathrm{d}\theta^{\mathrm{R}}}{\mathrm{d}\tilde{\mathrm{s}}} = \mathrm{Im}\big(e^{-i\theta^R}(1 + R^{-1}\beta^R)\big). \tag{15}$$

Choose the initial data to be $R + \beta(0)$ in (2). To take a limit in (15), we need to control $R^{-1}\beta^R$. Define

$$u^R(s) = 2\int_0^s r(t)\cos\big(\varphi^R(t) - \theta^R(t)\big)\mathrm{dt}.$$

Proceeding as in Example 2.3, we have

$$\frac{\mathrm{du}^{\mathrm{R}}(\tilde{\mathrm{s}})}{\mathrm{d}\tilde{\mathrm{s}}} = 2\sqrt{|1 + R^{-1}\beta(0)|^2 + R^{-1}u^R}\cos(\varphi^R - \theta^R) \leqslant 2\sqrt{2 + u^R} \tag{16}$$

for R large enough. Hence $u^R(\tilde{s})$ is uniformly bounded for fixed $\tilde{s}$, and

$$|1 + R^{-1}\beta(0)|^2 + R^{-1}u^R(\tilde{s}) = |1 + R^{-1}\beta^R(\tilde{s})|^2.$$

It implies $R^{-1}\beta^R(\tilde{s}) \to 0$ as $R \to \infty$ for fixed $\tilde{s}$ and the limit of (15) becomes

$$\frac{\mathrm{d}\beta}{\mathrm{d}\tilde{\mathrm{s}}} = e^{i\theta} \quad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\mathrm{s}}} = \mathrm{Im}\, e^{-i\theta} = -\sin\theta = N\cdot(1,0), \tag{17}$$

which is exactly the equation for a grim reaper. When $\alpha = -1$, we have negative sign in the second equation of (15) and obtain a grim reaper open to the direction $(-1,0)$.

Example 2.5 We can choose different families in Example 2.3 to take the limit. For instance if $\lambda_{n-k} = \ldots = \lambda_n = R$, $C = (k+1)R$, the translating solution obtained will consist of a k–dimensional plane.

3 The geometric picture for $n = 2$

There are different types of soliton solutions constructed in [6]. In this section we use the 2-dimensional case, i.e. Lagrangian surfaces in $\mathbb{C}^2$, to highlight the geometry of these examples. The properties summarized below have their counterparts (and even more) for higher dimensional cases.

For $n = 2$, four types of self-similar solutions are constructed in [6]. They are

(i) Lagrangian self-expanders diffeomorphic to $S^1 \times \mathbb{R}$ that are asymptotic to a pair of Lagrangian planes intersecting at the origin, the *Lagrangian angle* can be arbitrarily small, and the *Maslov class* is zero;

(ii) Compact Lagrangian self-shrinkers diffeomorphic to $S^1 \times S^1$ whose Maslov class is nonzero;

(iii) Lagrangian self-expanders diffeomorphic to $S^1 \times \mathbb{R}$ that are asymptotic to a pair of Lagrangian cones and the Maslov class is nonzero;

(iv) Lagrangian self-shrinkers diffeomorphic to $S^1 \times \mathbb{R}$ that are asymptotic to a pair of Lagrangian cones and the Maslov class is nonzero.

There are a few parameters for these examples that are related. We can choose different normalizations according to our need. We first count the dimension of freedom. There are two independent families in (i) (up to scaling and $\mathrm{U}(2)$ action). They are determined by the radii $\sqrt{\frac{1}{a_1}}$ and $\sqrt{\frac{1}{a_2}}$ in z_1, z_2 planes of the nearest point to the origin, or equivalently by the angles $0 < \bar{\varphi}_1$ and $0 < \bar{\varphi}_2$ with $\bar{\varphi}_1 + \bar{\varphi}_2 < \frac{\pi}{2}$ of the asymptotic planes

$$L_1 = \left\{(e^{i\bar{\varphi}_1}t_1, e^{i\bar{\varphi}_2}t_2) : t_1, t_2 \in \mathbb{R}\right\}, \quad \text{and} \quad L_2 = \left\{(e^{-i\bar{\varphi}_1}t_1, e^{-i\bar{\varphi}_2}t_2) : t_1, t_2 \in \mathbb{R}\right\}.$$

The examples converge to Lawlor's *special Lagrangians* [7] as one of $a_i \to \infty$ or equivalently as $\bar{\varphi}_1 + \bar{\varphi}_2 \to \frac{\pi}{2}$, when α is fixed; or fix a_1, a_2 and let $\alpha \to 0$. See Theorem C and D in [6] for more details.

Up to scaling and $\mathrm{U}(2)$ action, examples in (ii), (iii), and (iv), are respectively from a dense set of 2-dimensional families (parameterized by the initial data). The other examples in these 2-dimensional families are respectively non-closed Lagrangian self-shrinkers diffeomorphic to $S^1 \times R$, non-closed Lagrangian self-expanders diffeomorphic to R^2, and non-closed Lagrangian self-shrinkers diffeomorphic to R^2. The examples in (iii) and (iv) converge to special Lagrangians constructed by Joyce in [5] as $\alpha \to 0$. Self-shrinkers and self-expanders in (iv) and

(iii) with their same asymptotic cones can be glued together to form an *eternal solution* of *Brakke flow*, which is a weak solution of mean curvature flow, without mass loss as proved in [10]. We remark that Anciaux in [1] constructed special self-expanders in (i) satisfying $\bar{\varphi}_1 = \bar{\varphi}_2$ and obtained some other self-similar solutions; Smoczyk proved that every compact (without boundary) self-similar solution must have nonzero Maslov class [12, Theorem 2.3.5]; and Hamiltonian stationary self-similar Lagrangians as (ii), (iii) and (iv) are constructed by Lee and Wang in [9, 10].

From all our self-similar solutions (including the non-closed ones), we can construct translating solutions.

Example 3.1 For examples in (i), we take $\lambda_1 > 0$, $\lambda_2 = R$, $C = R$, $\alpha > 0$ and obtain translating solutions $\big(xw(s), -\frac{1}{2}\lambda_1 x^2 + \beta(s)\big)$ as discussed in Example 2.3, where $x \in \mathbb{R}$ and

$$\frac{\mathrm{dw}}{\mathrm{ds}} = \lambda_1 e^{i\theta}, \quad \frac{\mathrm{d}\beta}{\mathrm{ds}} = e^{i\theta}\,\bar{w}, \quad \frac{\mathrm{d}\theta}{\mathrm{ds}} = \alpha\,\mathrm{Im}\big(e^{-i\theta}w\big). \tag{18}$$

We can assume $\lambda_1 = 1$ in (18) as redefining $\hat{x} = \sqrt{\lambda_1}x$, $\hat{w} = \frac{w}{\sqrt{\lambda_1}}$, $\hat{s} = \sqrt{\lambda_1}s$ will do the job. Simple calculation gives $\beta(s) = \frac{1}{2}|w(s)|^2 - \frac{i}{\alpha}\theta(s) + K$. Since a translating solution after translation is still a translating solution, we now take the solution to be

$$\big(xw(s), -\frac{1}{2}x^2 + \frac{1}{2}|w(s)|^2 - \frac{i}{\alpha}\theta(s)\big), \tag{19}$$

where $x \in \mathbb{R}$ and $w(s)$ is a self-expander for the curve shortening flow satisfying

$$\frac{\mathrm{dw}}{\mathrm{ds}} = e^{i\theta}, \quad \frac{\mathrm{d}\theta}{\mathrm{ds}} = \alpha\,\mathrm{Im}\big(e^{-i\theta}w\big), \tag{20}$$

with $\alpha > 0$. The solution of (20) can be solved explicitly as in Theorem C of [6]. Namely, if we write $w = re^{i\varphi}$ and reparameterize by a new variable $-\infty < y < \infty$, then

$$r = \sqrt{\frac{1}{a} + y^2}, \quad \varphi(y) = \int_0^y \frac{\mathrm{dt}}{(\frac{1}{a} + t^2)\sqrt{P(t)}}, \quad \theta(y) = \varphi(y) + \arg\big(y + i\sqrt{P(y)}\big), \tag{21}$$

where $P(t) = \frac{1}{t^2}\big((1 + at^2)e^{\alpha t^2} - 1\big)$. If we take $\alpha = 1$, after translation we have (19) as

$$\big(x\sqrt{\frac{1}{a} + y^2}\,e^{i\varphi(y)}, -\frac{x^2 - y^2}{2} - i\theta(y)\big). \tag{22}$$

Note that $\varphi(-y) = -\varphi(y)$ and $\lim_{y\to\infty}\varphi(y) = \bar{\varphi}$ with $0 < \bar{\varphi} < \frac{\pi}{2}$. There is an 1–1 correspondence between $0 < a < \infty$ and $0 < \bar{\varphi} < \frac{\pi}{2}$; when $\bar{\varphi} = \frac{\pi}{2} - \varepsilon$, the corresponding Lagrangian angle $\theta(y)$ lies in $\big(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\big)$. That is, the surface (22) can be arbitrarily close to special Lagrangians which have constant Lagrangian angle.

Up to scaling, translation, and $U(2)$ action, we have one parameter family of such translating solutions.

The blow-down of the solution in (22) is the union of two planes $(X_1e^{i\bar{\varphi}}, X_2)$ and $(X_1e^{-i\bar{\varphi}}, X_2)$ in $\mathbb{C}^2$ with the negative real line of the z_2-plane deleted, where $X_1, X_2 \in \mathbb{R}$. As $a \to \infty$, (22) converges to the special Lagrangian plane $(X_1 i, X_2)$ with multiplicity 2; and as $a \to 0$, the surfaces converge to a grim reaper in the z_2-plane times the real line in the z_1-plane. We can fix a and change α instead, or choose other families. The limits will be different. The exploration of these and further properties will be investigated in other places. I would like to thank Fernando Marques for observing that one may get a grim reaper times $\mathbb{R}$ by letting $a \to 0$. Our examples are rather surprising and of great interest as *almost calibrated Lagrangians*, i.e. when the range of Lagrangian angle is less than π, have better behavior along mean curvature flow and people had been trying to show that such a translating solution must be planes. Our examples show that this is not true in general. In [11] Neves and Tian discuss the relation and importance of understanding translating solutions to the regularity theory of mean curvature flow, and show that under some additional conditions the translating solutions must be planes. Our examples are similar to *cigar solutions* in *Ricci flow*, and thus it is very important to rule them out as blow-ups of Lagrangian mean curvature flow.

Question 3.2 *Can the translating solitons with small Lagrangian angle oscillation constructed above arise as blow-ups of finite time singularities for Lagrangian mean curvature flow?*

Example 3.3 For examples from (ii), we take $\lambda_1 > 0$, $\lambda_2 = R$, $C = R$, $\alpha < 0$ and obtain a translating solution satisfying (19) and (20) with $\alpha < 0$ as discussed in Example 2.3. These translating solutions have infinite oscillation of the Lagrangian angle and thus cannot be the blow-ups of Lagrangian mean curvature flow. For examples from (iii), we take $\lambda_1 < 0$, $\lambda_2 = R$, $C = R$, $\alpha > 0$ and obtain a translating solution with related functions satisfying (18). We can change λ_1 to $-\lambda_1$, w to $-w$, β to $-\beta$, and α to $-\alpha$. They still satisfy (18) and will give the translating solutions constructed from (ii). That is, translating solutions constructed from (iii) are the image of those constructed from (ii) under the negative identity map in $\mathbb{C}^2$. Similarly, translating solutions from (iv) are the image of those from (i) under the negative identity map in $\mathbb{C}^2$.

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Aspects of conformal holonomy

Felipe Leitner

Abstract This is an expository article, which gives an overview about aspects of the theory of conformal holonomy. In particular, we announce a complete geometric description of compact Riemannian conformal manifolds with decomposable conformal holonomy representation. Furthermore, we discuss the relation to almost Einstein structures and generalised Fefferman constructions. Generically, the latter conformal geometries have irreducible conformal holonomy. Reduced conformal holonomy is related to the existence of solutions of certain overdetermined conformally covariant PDE systems. We explain this relation in a unified approach using BGG-sequences.

Keywords Conformal geometry, Cartan and tractor calculus; Conformal holonomy, almost Einstein structures; Overdetermined covariant PDE's.
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1 Introduction

A *conformal structure* on a smooth manifold M is an equivalence class of (pseudo-) Riemannian metric tensors, in which two metrics g and $\tilde{g}$ are equivalent if and only if $\tilde{g} = e^{2f} g$ for some smooth function $f \in C^\infty(M)$. Conformal structures are equivalently defined by *Cartan* and *tractor calculus* (cf. [53, 6]). In fact, conformal geometry is a special case of *parabolic geometry* (cf. [16]). Conformal *invariant theory* studies geometric properties of manifolds, which characterise and distinguish the underlying conformal structures. Well known are local curvature invariants like the *Weyl* tensor, the *Bach* tensor in dimension 4, and more generally, the *Fefferman-Graham obstruction* tensor (cf. [22]). The Fefferman-Graham obstruction occurs as the total metric variation of the Q-curvature action (cf. [27]).

Branson's Q-curvature in turn is closely related to the celebrated *GJMS-operators*, which are conformally covariant differential operators whose principal part is a power of the Laplacian (cf. [28]). In general, the kernel and spectral properties of conformally covariant differential operators are important invariants as well (cf. e.g. [11, 31]). For example, solutions of the *Dirac equation* and *Penrose's twistor equation* [49] are such invariants. The latter PDE system is overdetermined, hence its solvability forces curvature conditions on the underlying conformal structure (cf. [8]). Solutions of such PDE systems (like e.g. conformal Killing vector fields and conformal Killing l-forms) reflect symmetry properties of a conformal manifold (cf. [51, 20]). Also the existence of Einstein metrics in a given conformal class is described by the kernel of a conformally covariant, overdetermined differential operator (cf. [37, 29]). Furthermore, there is an invariant notion of *conformal circles* (resp. *geodesics*; cf. e.g. [16]). Another basic quantity is the famous *Yamabe invariant* (cf. e.g. [35]).

In the current article we are mainly interested in the *conformal holonomy* (cf. [3, 36, 40, 42, 9]) and its standard representation. This conformal invariant is naturally introduced in the framework of Cartan and tractor calculus as the holonomy group of the *tractor connection*. Generically, the conformal holonomy is a subgroup of the *Möbius group*. Reduced conformal holonomy is directly linked to the existence of solutions of certain conformally covariant PDE systems (cf. [36, 37]). Typically, these PDE systems are overdetermined and describe the kernel of the first differential operator in a BGG-sequence (cf. [17]).

The methods that we use in this article are based on parabolic Cartan and tractor calculus. There is also a close link to the *Fefferman-Graham ambient* and *Poincaré-Einstein* models (cf. [22]). The latter is the geometric model, which underlies the

AdS/CFT-correspondence in quantum gravity (cf. [46, 54]). In Section 5 and 6 other parabolic geometries like *CR-geometry* will play an important role. Since the article has an expository motivation, we will only sketch proofs. The presented results about conformal holonomy are collected from various sources of recent times (cf. [3, 36, 37, 40, 42, 43, 44]). The result on the complete geometric description of decomposable holonomy is yet not published. A proof of this result will be given in a forthcoming paper [5]. The current article is based on a lecture, which I gave at the *University of Regensburg* in January 2010.

The course of the article is as follows. In Section 2 we introduce the standard tractor bundle with connection, which then gives rise to the notion of conformal holonomy. In Section 3 we explain the basic relation of conformal holonomy and Einstein metrics. This is a special case of decomposable conformal holonomy. In Section 4 we discuss the general case of decomposable conformal holonomy. In the compact Riemannian signature case exactly two geometric situations occur, the *special Einstein products*, and a degenerate version of these, the *collapsing products with the sphere*. In Section 5 we discuss the first known case of irreducible conformal holonomy, which is the case of unitary conformal holonomy corresponding to the classical Fefferman construction in CR-geometry. In Section 6 we report on a generalised Fefferman construction due to A. Čap, which gives rise to irreducible conformal holonomy as well. During the course of the article we will meet several conformally covariant, overdetermined PDE systems. In the final Section 7 we explain the emergence of these by the unified approach of *BGG-sequences*.

2 Conformal tractor holonomy

Let M^n be a smooth connected manifold of dimension $n \geq 3$. Recall that a *conformal structure* on M is given by a smooth $\mathbb{R}_+$-ray subbundle $\mathcal{Q} \subset S^2T^*M$, whose fibre over $p \in M$ consists of conformally related scalar products on T_pM of a fixed signature (p,q) with $n = p+q$. Smooth sections of $\mathcal{Q}$ are metrics on M, and we denote the set of all such sections by c. Any two sections $g, \tilde{g} \in c$ are related by $\tilde{g} = e^{2f} g$ for some function $f \in C^\infty(M)$, i.e., g and $\tilde{g}$ are conformally equivalent metrics on M. (If $g \in c$ then we also write $c = [g]$.)

2.1 Standard tractors and connection.

The principal $\mathbb{R}_+$-bundle $\pi : \mathscr{Q} \to M$ induces for any representation $t \in \mathbb{R}_+ \mapsto t^{-w/2} \in \mathrm{End}(\mathbb{R})$, $w \in \mathbb{R}$, a natural real line bundle $\mathscr{E}[w]$ over M, which is called the *conformal density bundle of weight* w. The *(conformal) standard tractor bundle* $\mathscr{T}$ of (M,c) is naturally defined as a quotient bundle of rank $n+2$ of the 2-jet prolongation $J^2(\mathscr{E}[1])$ of the weighted bundle $\mathscr{E}[1]$ and admits a composition structure

$$\mathscr{T} \;=\; \mathscr{E}[1] \mathbin{\ooalign{$+$\cr$\supset$}} TM[-1] \mathbin{\ooalign{$+$\cr$\supset$}} \mathscr{E}[-1]\,; \tag{1}$$

$\mathscr{E}[-1]$ may be naturally identified with a subbundle of $\mathscr{T}$ and $TM[-1] = TM \otimes \mathscr{E}[-1]$ is a subbundle of the quotient bundle $\mathscr{T}/\mathscr{E}[-1]$ (cf. [6]). The projection of $\mathscr{T}$ onto $\mathscr{E}[1]$ will be denoted by Π. The standard tractor bundle $\mathscr{T}$ is naturally equipped with a scalar product $\langle\cdot,\cdot\rangle_{\mathscr{T}}$ of signature $(p+1,q+1)$ and a covariant derivative ∇, the so-called *tractor connection*, which preserves the tractor metric $\langle\cdot,\cdot\rangle_{\mathscr{T}}$.

With respect to the choice of a metric g in the given conformal class c on M the weighted bundles $\mathscr{E}[w]$, $w \in \mathbb{R}$, are trivialised and the composition structure (1) splits into the direct sum

$$\mathscr{T} \;\cong_g\; \mathbb{R} \oplus TM \oplus \mathbb{R}\,.$$

Accordingly, any smooth section $T \in \Gamma(\mathscr{T})$ splits into a triple (a,ψ,b), where a,b are smooth functions and ψ is a vector field on M. (By convention, the component a corresponds via g to the projection $\Pi(T)$. Thus we also write $a = \Pi_g(T)$.) With respect to this splitting (induced by $g \in c$) the tractor metric applied to standard tractors $T,\hat{T}$ is expressed by

$$\langle T,\hat{T}\rangle_{\mathscr{T}} \;=\; a\hat{b} + \hat{a}b \;+\; g(\psi,\hat{\psi})\,.$$

And the tractor connection ∇ acts by

$$\nabla_X \begin{pmatrix} a \\ \psi \\ b \end{pmatrix} = \begin{pmatrix} X(a) \\ \nabla^g_X \psi \\ X(b) \end{pmatrix} + \begin{pmatrix} -g(X,\psi) \\ b\cdot X - a\cdot \mathsf{P}^g(X) \\ \mathsf{P}^g(X,\psi) \end{pmatrix}$$

for any $X \in TM$, where ∇^g denotes the Levi-Civita connection of g, and

$$\mathsf{P}^g = \frac{1}{n-2}\left(\frac{scal^g}{2(n-1)} - Ric^g\right)$$

is the *Schouten tensor* in terms of the Ricci tensor Ric^g and the scalar curvature $scal^g$ of g. With $\mathsf{P}^g(X)$ we denote the vector in TM, which is dual to $\mathsf{P}^g(X,\cdot)$ via g. The conformal curvature Ω^∇ of the tractor connection ∇ consists of the *Weyl tensor* W^g and the *Cotton tensor* C^g (cf. e.g. [16, 37]).

Note that under a conformal rescaling of g to $\tilde{g} = e^{2f} \cdot g$ with respect to a smooth function f the triple (a, ψ, b) transforms by

$$(\tilde{a}, \tilde{\psi}, \tilde{b}) \;=\; \Big(\, e^f a \; , \; e^{-f} \cdot (\psi + a \cdot grad^g f) \; , \; e^{-f} \cdot (b - df(\psi) - \frac{a}{2} \| grad^g f \|_g^2) \Big) \; , \tag{2}$$

i.e., the metric $\tilde{g}$ gives rise to a different isomorphism for the tractor bundle $\mathscr{T}$ with the direct sum $\mathbb{R} \oplus TM \oplus \mathbb{R}$. Also note that there is an alternative definition of the standard tractor bundle $\mathscr{T}$ with connection ∇ in the framework of conformal Cartan geometry (cf. Section 7 and [16]).

2.2 *Tractor holonomy.*

Any connection (or covariant derivative) on a vector bundle over a manifold admits a *holonomy group* (with corresponding standard representation) (cf. e.g. [32]). We briefly recall here the definition of holonomy group for the standard tractor bundle $(\mathscr{T}, \nabla)$ over a connected conformal manifold (M, c). For this purpose, let x_o be a base point in M and let Γ_{x_o} be the set of smooth curves $\gamma : [0,1] \to M$, starting and ending in x_o. For any tractor $T_o \in \mathscr{T}_{x_0}$ and any $\gamma \in \Gamma_{x_o}$, there exists a unique covariantly constant tractor field T along γ, i.e., a smooth map $T : [0,1] \to \mathscr{T}$ with $T(t) \in \mathscr{T}_{\gamma(t)}$, $T(0) = T_o$ and $\nabla_{\dot{\gamma}(t)} T = 0$ for all $t \in [0,1]$. Then the map $\mathscr{P}^\gamma_{x_o} : \mathscr{T}_{x_o} \to \mathscr{T}_{x_o}$, $T_o \mapsto T(1)$, is a linear automorphism, which preserves the tractor metric $\langle \cdot, \cdot \rangle_{\mathscr{T}}$. The collection of all these automorphisms $\mathscr{P}^\gamma_{x_o}$, $\gamma \in \Gamma_{x_o}$, forms a Lie subgroup of the orthogonal group $O(\mathscr{T}_{x_o})$. To be precise, we define

$$Hol_{x_o}(\mathscr{T}, \nabla) \; := \; \{ \; \mathscr{P}^\gamma_{x_o} \mid \gamma \in \Gamma_{x_o} \; \} \; \subset O(\mathscr{T}_{x_o}) \; .$$

Note that with respect to any choice of orthonormal basis in $\mathscr{T}_{x_o}$ we can understand $Hol_{x_o}(\mathscr{T}, \nabla)$ as a Lie subgroup of the orthogonal group $O(p+1, q+1)$. Since, for any choice of base point x_o in M and any orthonormal basis in $\mathscr{T}_{x_o}$, the corresponding images of $Hol_{x_o}(\mathscr{T}, \nabla)$ are conjugated in $O(p+1, q+1)$, the isomorphism class of $Hol_{x_o}(\mathscr{T}, \nabla)$ is uniquely determined for $(\mathscr{T}, \nabla)$. Moreover, since $(\mathscr{T}, \nabla)$ is a conformally invariant construction for (M, c), we denote this isomorphism class simply by $Hol(M,c)$, and call this the *conformal holonomy group* of (M,c). The conformal holonomy algebra is denoted by $\mathfrak{hol}(M,c)$.

EXAMPLE. Let us consider the *Möbius sphere* S^n of dimension $n \geq 3$, which is the conformal compactification of the Euclidean space $\mathbb{R}^n$ by adding one point at infinity. The sphere S^n is simply connected. Moreover, the Möbius sphere is conformally flat, i.e., the Weyl and Cotton tensors vanish identically. This implies that the tractor connection ∇ on $\mathscr{T}$ is flat. Hence, $\mathscr{T}$ is parallelisable and the holonomy group $Hol(M,c)$ is trivial. ◇

3 Almost Einstein structures and holonomy

A (pseudo-)Riemannian metric g on a manifold M is called *Einstein* if the Ricci-tensor Ric^g is a constant multiple of g. This is a PDE for the metric tensor g. Einstein metrics are of central interest in geometry and physics. It is interesting to see that Einstein's equations have a natural formulation in the framework of conformal tractor calculus. Also the conformal holonomy $Hol(M,c)$ is suitable to detect Einstein metrics in a given conformal class. We briefly explain these features here. In particular, we introduce the notion of *almost Einstein structures*, which is a slight extension of the notion of *conformally Einstein* metrics. Almost Einstein structures are also related to asymptotically flat and hyperbolic metrics.

Let (M,c) be a connected conformal manifold of dimension $n \geq 3$ and signature (p,q) and let $(\mathscr{T},\nabla)$ be the standard tractor bundle with connection. The 2-jet of a section in $\mathscr{E}[1]$ gives rise in a natural way to a section of $\mathscr{T}$. The corresponding map $\mathbb{S}_0 : \Gamma(\mathscr{E}[1]) \to \Gamma(\mathscr{T})$ is a conformally covariant second order differential operator (cf. Section 7). With respect to the choice of a metric $g \in c$ we have the splitting $\mathscr{T} \cong_g \mathbb{R} \oplus TM \oplus \mathbb{R}$, and the differential operator $\mathbb{S}_0$ is then given by

$$\mathbb{S}_0^g \sigma \;=\; (\,\sigma\,,\, grad^g(\sigma)\,,\, \Box^g\,\sigma), \tag{3}$$

where $\Box^g := -\frac{1}{n}(\Delta^g - tr_g \mathsf{P}^g)$ and $\Delta^g = tr_g Hess^g = tr_g(\nabla^g \circ d)$ is the Laplacian (cf. [37, 29]).

It is a matter of fact that for densities $\omega \in \Gamma(\mathscr{E}[1])$ the equation

$$\nabla \mathbb{S}_0 \omega \;=\; 0 \tag{4}$$

is equivalent to

$$\text{trace-free part of}\,(Hess^g\sigma \;-\; \mathsf{P}^g\cdot\sigma) \;=\; 0\;, \tag{5}$$

where σ is the function which corresponds via $g \in c$ to the density ω. In turn, it is also true that if a tractor $T \in \Gamma(\mathscr{T})$ satisfies $\nabla T = 0$ (i.e. T is ∇-parallel on M), then the component $\sigma = \Pi_g(T) \in C^\infty(M)$ of T with respect to $g \in c$ satisfies (5) and $\mathbb{S}_0^g\sigma = T$.

Note that, since a solution σ of (5) on (M,g) corresponds to a conformal density ω of weight 1 via $g \in c$, the solution σ conformally rescales to a solution $\tilde\sigma = e^f\cdot\sigma$ of (5) with respect to $\tilde g = e^{2f}g$. Hence, if σ is a solution of (5) without zeros, then $\tilde\sigma \equiv 1$ is a solution for $\tilde g = \sigma^{-2}g$, i.e., $\tilde g$ is an Einstein metric in the conformal class $c = [g]$. On the other hand, if $\tilde g = \sigma^{-2}g$ is an Einstein metric, then σ is a solution of (5) with respect to g on M. However, in general, the zero set $\Sigma(\sigma)$ of a non-trivial solution σ of (5) is non-empty (with a dense complement in M). For this reason, we call a generic solution σ of (5) an *almost Einstein structure* of (M,g) (cf. [24]). (The corresponding density ω is also called an *almost Einstein structure* of (M,c). In [37] we named a generic solution of (5) a *nc-Killing function*.) In case $\Sigma(\sigma)$ is empty on (M,g), we also say that g is *conformally Einstein* and $(M,[g])$ is a *conformal Einstein space*.

In the Riemannian signature case the following result on the shape of the zero set of an almost Einstein structure is known. Note that if σ is a solution of (5) then $S(\sigma) := -\langle \mathbb{S}_0^g\sigma, \mathbb{S}_0^g\sigma\rangle_{\mathscr{T}} = \frac{scal^g}{n(n-1)}$ is a well defined real number.

Theorem 1 *[24] Let (M,g,σ) be a Riemannian manifold of dimension $n \geq 3$ with almost Einstein structure σ. If $S(\sigma) > 0$ then $\Sigma(\sigma)$ is empty and $(M,\sigma^{-2}g)$ is Einstein with positive scalar curvature; if $S(\sigma) = 0$ then $\Sigma(\sigma)$ is either empty or consists of isolated points and $(M \setminus \Sigma(\sigma), \sigma^{-2}g)$ is Ricci-flat; if $S(\sigma) < 0$ then the scale singularity set $\Sigma(\sigma)$ is either empty or else is a smooth umbilic hypersurface, and $(M \setminus \Sigma(\sigma), \sigma^{-2}g)$ is Einstein of negative scalar curvature.*

Note that, if $\Sigma(\sigma) \neq \emptyset$ and $S(\sigma) < 0$, then the Einstein metric $\sigma^{-2}g$ is *asymptotically hyperbolic* at the boundary $\Sigma(\sigma)$ of $M \setminus \Sigma(\sigma)$, i.e., the sectional curvature of $\sigma^{-2}g$ goes to -1 at the boundary. If $S(\sigma) = 0$ then $\sigma^{-2}g$ is *asymptotically flat* at the isolated zeros of σ.

As we have seen, the existence of an almost Einstein structure on a manifold (M,g) is equivalent to a ∇-parallel tractor on $(M,[g])$. This in turn implies the existence of a $Hol(M,[g])$-fixed (non-trivial) vector in $\mathbb{R}^{p+1,q+1}$. In fact, there is a natural one-to-one correspondence of almost Einstein structures and $Hol(M,[g])$-fixed (non-trivial) vectors in $\mathbb{R}^{p+1,q+1}$ (with respect to the choice of a basis of $\mathscr{T}$ at some $x_o \in M$).

Theorem 2 *Let (M,c) be a conformal manifold of dimension $n \geq 3$. Then (M,c) is an almost Einstein space if and only if the standard representation of $Hol(M,c)$ fixes a non-trivial vector.*

Remark 1 If there exists an almost Einstein structure σ with $S(\sigma) \neq 0$, then the holonomy representation $\mathbb{R}^{p+1,q+1}$ of $Hol(M,c)$ is *decomposable* (cf. Section 4). For $S(\sigma) = 0$ the holonomy representation is reducible, but generically not decomposable. The latter property of the holonomy representation is sometimes called *weak irreducibility*. $\diamond$

EXAMPLE: (1) Again, let us consider the Möbius sphere S^n of conformal Riemannian geometry. Since $\mathscr{T}$ on S^n is parallelisable, there exist $n+2$ linearly independent ∇-parallel standard tractors in $\Gamma(\mathscr{T})$. Every ∇-parallel standard tractor I with $S(I) := -\langle I, I\rangle_{\mathscr{T}} > 0$ corresponds to a round metric in the conformal class of the Möbius sphere S^n.

Now let us think of the Möbius sphere S^n as the unit sphere in $\mathbb{R}^{n+1}$ with the conformal class of the round standard metric g^o_{rd} (induced by restriction of the Euclidean metric on $\mathbb{R}^{n+1}$). One can easily check that the restriction of any affine linear function on $\mathbb{R}^{n+1}$ is an almost Einstein structure for (S^n, g^o_{rd}).

For example, let x be the first coordinate function of $\mathbb{R}^{n+1}$. Then the restriction of x to the unit sphere S^n is an almost Einstein structure, whose singularity set $\Sigma(x)$ is the intersection of S^n with the hyperplane $\{x = 0\}$, i.e., $\Sigma(x)$ is an equator on S^n. The rescaled metric $x^{-2}g^o_{rd}$ is the hyperbolic metric on the two caps of S^n minus this equator $\{x = 0\}$. Now let σ be the restriction of the function $x-1$ to S^n. Then $p := \Sigma(x)$ is a single pole on S^n, and $(x-1)^{-2}g^o_{rd}$ is the Euclidean metric on $S^n \setminus \{p\}$. Alternatively, the metric $(x-1)^{-2}g^o_{rd}$ can be understood as the pullback of the Euclidean metric on $\mathbb{R}^n$ via stereographic projection from the pole p.

(2) Let us also consider the real projective space $\mathbb{R}P^n$ equipped with the standard metric of constant curvature 1. The only almost Einstein structures on $\mathbb{R}P^n$ are the constant functions $\sigma \neq 0$, whereas e.g. x and $x-1$ on S^n only project locally to almost Einstein structures on $\mathbb{R}P^n$. $\diamond$

Finally, we mention the following characterisation of the Möbius sphere (in Riemannian signature). Let $\mathscr{K}_c$ denote the vector space of ∇-parallel sections in $\mathscr{T}$ on a conformal space (M^n, c). Note that the tractor metric $\langle \cdot, \cdot \rangle_{\mathscr{T}}$ induces a symmetric bilinear form on $\mathscr{K}_c$.

Theorem 3 *[25] Let (M^n, c) be a closed (= compact without boundary) Riemannian conformal manifold of dimension $n \geq 3$ with* $\dim(\mathscr{K}_c) \geq 2$. *Then either* *(i)* (M,c) *is the Möbius sphere* $(S^n,[g^o_{rd}])$*, or*

(ii) for any $K \in \mathscr{K}_c \setminus \{0\}$*, it is necessarily the case that* $S(\Pi(K)) < 0$ *and* $\Sigma(\Pi(K))$ *is non-empty (and hence is a totally umbilic hypersurface in* (M,c)*).*

In particular, the result states that the Möbius sphere is the only closed Riemannian conformal manifold, which is conformally Einstein and admits an almost Einstein structure σ with non-trivial singularity set $\Sigma(\sigma)$. The proof of this uses results about *essential conformal transformation groups* in Riemannian geometry (cf. e.g. [1]).

4 Decomposable conformal holonomy

In the preceding section we have seen that the conformal holonomy representation of an almost Einstein manifold (M,g,σ) with $S(\sigma) \neq 0$ decomposes $\mathbb{R}^{p+1,q+1}$ into a (timelike or spacelike) one-dimensional subspace and a non-degenerate orthogonal complement. Recall that the deRham-Wu Theorem [50, 55] states that a (pseudo-) Riemannian manifold (M,g) with decomposable Riemannian holonomy group is locally isometric to a product of two (pseudo-)Riemannian manifolds. In this section we discuss a similar result for the general case of decomposable conformal holonomy $Hol(M,c)$. In the Riemannian signature case we will see that there occur two types of conformal geometries with decomposable holonomy, the *special Einstein products* and the *collapsing sphere products*. The results of this section are based on the works [36, 43, 44].

4.1 The special Einstein product.

Let (M^n,c) be a connected conformal manifold of dimension $n \geq 3$ and signature (p,q) with tractor bundle $(\mathscr{T},\langle\cdot,\cdot\rangle_{\mathscr{T}})$. Let $\Lambda^{l+1}\mathscr{T}^*$ denote the bundle of $(l+1)$-forms on $\mathscr{T}$ with induced tractor metric $\langle\cdot,\cdot\rangle_{\Lambda^{l+1}\mathscr{T}^*}$ and tractor connection ∇. The tractor bundle of $(l+1)$-forms admits (for $1 \leq l \leq n-1$) a composition structure

$$\Lambda^{l+1}\mathscr{T}^* \;=\; \Lambda^l M[l+1] \,\oplus\!\!\!\!+\, \big(\Lambda^{l+1}M[l+1] \oplus \Lambda^{l-1}M[l-1]\big) \,\oplus\!\!\!\!+\, \Lambda^l M[l-1]$$

(similar to (1)) with natural projection $\Pi : \Lambda^{l+1}\mathscr{T}^* \to \Lambda^l M[l+1]$ onto the bundle of l-forms with conformal weight $l+1$. With respect to any metric $g \in c$ this composition structure splits into $\Lambda^l M \oplus \Lambda^{l+1} M \oplus \Lambda^{l-1} M \oplus \Lambda^l M$, i.e., any *tractor* $(l+1)$*-form* α $(1 \leq l \leq n-1)$ decomposes into a quadruple $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of differential forms on M, where the component α_0 corresponds via g to $\Pi(\alpha)$.

Proposition 1 *(cf. [36, 40]) Let (M,c) be a conformal manifold of dimension $n \geq 3$ and let $\alpha \not\equiv 0$ be a ∇-parallel tractor $(l+1)$-form $(1 \leq l \leq n-1)$. Then with respect to any metric $g \in c$*

(i) *the complement of the zero set $\Sigma(\alpha_0)$ of α_0 is dense in M.*

(ii)

$$\alpha_1 = \frac{1}{l+1} d\alpha_0 \qquad \textit{and} \qquad \alpha_2 = \frac{1}{n-l+1} d^* \alpha_0 \,.$$

(iii) *The component α_0 is a conformal Killing l-form, i.e.,*

$$\nabla^g_X \alpha_0 - \frac{1}{l+1} \iota_X d\alpha_0 + \frac{1}{n-l+1} X^\flat \wedge d^* \alpha_0 \; = \; 0 \qquad \textit{for all } X \in TM \tag{6}$$

(with ι_X the insertion, $X^\flat = g(X,\cdot)$ and d^* the codifferential).

Note that, if α_0 is a conformal Killing 1-form, then the dual vector field X with respect to $g \in c$ is *conformal Killing* on (M,g), i.e., the Lie derivative $L_X g$ is some multiple $\lambda \cdot g$, $\lambda \in C^\infty(M)$, of the metric tensor g. In general, not every conformal Killing l-form stems from a ∇-parallel tractor $(l+1)$-form. In fact, the tractor equation $\nabla\alpha = 0$ implies further equations on the component α_0. If these additional equations are satisfied, we call α_0 a *normal conformal Killing l-form* (cf. Section 7 and [37]).

We call a differential form $\alpha_0 \neq 0$ *simple* at $x_o \in M$ if α_0 is a simple wedge product $a_1 \wedge \ldots \wedge a_l$ of 1-forms. Accordingly, we call a non-trivial tractor $(l+1)$-form $\alpha \in \Lambda^{l+1} \mathscr{T}^*_{x_o}$ *simple* if α is a simple wedge product of tractor 1-forms.

Lemma 4.1 *[36, 40] Let (M,c) be a conformal manifold of dimension $n \geq 3$ and let α be a simple tractor $(l+1)$-form $(1 \leq l \leq n-1)$ with $\langle \alpha, \alpha \rangle_{\Lambda^{l+1}\mathscr{T}^*} \neq 0$. Then with respect to any metric $g \in c$*

(i) all non-trivial components of $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ at any $x_o \in M$ are simple. Either $\|\alpha_0\|_g := \sqrt{|\langle \alpha_0, \alpha_0 \rangle_g|} \neq 0$ or $\alpha_0 = 0$ at x_o.

(ii) In addition, if α is ∇-parallel, then there exist smooth vector fields A, B on $M \setminus \Sigma(\alpha_0)$ such that

$$d\alpha_0 = A^\flat \wedge \alpha_0 \qquad \textit{and} \qquad d^* \alpha_0 = \iota_B \alpha_0 \,.$$

Let $X \in \mathfrak{X}(M)$ be a conformal Killing vector field with non-vanishing length function $\|X\|_g \neq 0$ on a (pseudo-)Riemannian manifold (M,g). It is a matter of fact that, if X is in addition *hypersurface orthogonal*, then the vector field X is parallel with respect to the Levi-Civita connection of the conformally rescaled metric $\tilde{g} = \|X\|_g^{-2} \cdot g$. The following result is a generalisation of this fact to conformal Killing l-forms.

Lemma 4.2 *[36, 40] Let α_0 be a simple conformal Killing l-form with $\|\alpha_0\|_g \neq 0$ such that there are vector fields $A,B \in \mathfrak{X}(M)$ with*

$$d\alpha_0 = A^\flat \wedge \alpha_0 \qquad \text{and} \qquad d^*\alpha_0 = \iota_B \alpha_0 \; .$$

Then the l-form $\tilde{\alpha}_0 = \|\alpha_0\|_g^{-(p+1)} \cdot \alpha_0$ is $\nabla^{\tilde{g}}$-parallel for $\tilde{g} = \|\alpha_0\|_g^{-2} \cdot g$.

We call the holonomy group $Hol(M,c)$ of a conformal manifold (M,c) *decomposable* if there exists a decomposition $\mathbb{R}^{p+1,q+1} = V \oplus W$ into two $Hol(M,c)$-invariant, non-degenerate subspaces V and W. We assume here $r := dim(V) \geq 2$ and $s := dim(W) \geq 2$. (Otherwise, we are in the almost Einstein case of the preceding section.) Since V,W are non-degenerate, (if M is simply connected) there exist volume forms $\alpha \in \Gamma(\Lambda^r \mathscr{T}^*)$ to V and $\beta \in \Gamma(\Lambda^s \mathscr{T}^*)$ to W on M, which are ∇-parallel with constant norm ± 1. It follows immediately from Lemma 4.1 and Lemma 4.2 that there is a unique metric $g \in c$ such that the $(r-1)$-form α_0 (which corresponds to $\Pi(\alpha)$ via g) is simple and ∇^g-parallel on $\tilde{M} := M \setminus \Sigma(\alpha_0)$. For this metric g the $(s-1)$-form β_0 (which corresponds to $\Pi(\beta)$) is ∇^g-parallel on $\tilde{M}$ as well. This shows that the metric g on $\tilde{M}$ is locally isometric to a Riemannian product $g_1 \times g_2$ (even if M is not assumed to be simply connected).

Moreover, since α_0 is normal conformal, the additional equations on α_0 imply that both factors g_1 and g_2 are Einstein metrics with $s(s-1)scal^{g_1} = -r(r-1)scal^{g_2} \neq 0$. (This is exactly the case when the Schouten tensor P has exactly two distinct constant eigenvalues on $\tilde{M}$; cf. [36, 40].) We call a (pseudo-)Riemannian metric $g_1 \times g_2$ of this form a *special Einstein product*. Note that, on the other hand, the volume forms of the (oriented) factors of a special Einstein product give rise to simple ∇-parallel tractor forms (via the splitting operator; cf. Section 7).

Proposition 2 *[3, 36, 40] Let (M,c) be a conformal manifold of dimension $n \geq 3$ with arbitrary signature. Then the conformal holonomy group $Hol(M,c)$ is decomposable if and only if there exists an open dense submanifold $\tilde{M}$ in M with metric $g \in c$, which is either* *(i) Einstein with* $scal^g \neq 0$ *or*

(ii) locally isometric to a special Einstein product $g_1 \times g_2$.

4.2 The collapsing sphere product.

In [43] we have invented the *collapsing sphere product* alias S^l*-doubling* of an *even asymptotically hyperbolic space* $(\overline{M}, g_+)$ with boundary N. This construction works in the realm of conformal Riemannian geometry and produces manifolds with decomposable conformal holonomy. We briefly recall the construction for an arbitrary integer $l \geq 0$.

Let $\overline{M}^{m+1}$ be a smooth manifold of dimension $m+1 \geq 3$ with boundary N, and let g_+ be an *asymptotically hyperbolic (=AH) metric* on the interior $M = \overline{M} \setminus N$. Such a metric can always be written in the form $g_+ = \frac{1}{r^2}(dr^2 + g(r))$ for any *special* defining function r of the boundary N. If the Taylor expansion of the symmetric tensor $g(r)$ at the boundary $r = 0$ only has development terms of even degree with respect to any such r, then we call g_+ an *even* AH metric (cf. [23, 43]).

Let us assume now that g_+ is an even AH metric on $\overline{M}^{m+1}$. Obviously, the product $S^l \times \overline{M}$ of the l-dimensional standard sphere S^l with $\overline{M}$ has boundary $S^l \times N$. Let

$$\Lambda : S^l \times \overline{M} \to D_l\overline{M}$$

be the map, which identifies the sphere S^l at (each point of) the boundary N to a single point. The resulting quotient space $D_l\overline{M}$ with final topology is a manifold without boundary. In fact, since the evenness of g_+ induces an *even structure* on the boundary N (cf. [43]), the space $D_l\overline{M}$ is in a naturally way a smooth manifold of dimension $n := m + l + 1$.

Furthermore, we denote the image $\Lambda(S^l \times N)$ of identified points in $D_l\overline{M}$ by N_p. The set N_p is a smooth submanifold of codimension $l+1$ in $D_l\overline{M}$. We call N_p the *pole* of $D_l\overline{M}$, and $D_l\overline{M} \setminus N_p$ is the *bulk*, which is by construction diffeomorphic to the product space $S^l \times M$. The product $S^l \times M$ admits the conformal structure $[g^o_{rd} \times g_+]$, which is the conformal class of the product metric $g^o_{rd} \times g_+$. It is straightforward to show that this conformal structure on the bulk $S^l \times M$ extends smoothly to $D_l\overline{M}$. We denote the resulting conformal structure on $D_l\overline{M}$ by $c_l[g_+]$, and we call $(D_l\overline{M}, c_l[g_+])$ the *collapsing l-sphere product* (or S^l*-doubling*) of $(\overline{M}, g_+)$. (Note that for $l = 0$ the construction just glues two copies of $\overline{M}$ via the identity map at their boundaries. Since g_+ is even, the conformal structure $[g_+]$ of $\overline{M}$ extends smoothly over N to the *other side* of the doubling space.)

By the results of the previous section, we know that if g_+ is an even AH Einstein metric on the interior of $\overline{M}$, then the bulk of $(D_l\overline{M}, c_l[g_+])$ has decomposable conformal holonomy. Furthermore, since the pole N_p is a singular set in $D_l\overline{M}$, we can easily conclude that $(D_l\overline{M}, c_l[g_+])$ itself has decomposable conformal holonomy. This holonomy group $Hol(D_l\overline{M})$ is equal to that of $(M,[g_+])$. However, the standard representation of $Hol(D_l\overline{M})$ on $\mathbb{R}^{1,n+1}$ decomposes into a trivial representation on the Euclidean space $\mathbb{R}^{l+1}$ of dimension $l+1$ and the holonomy representation of $Hol(M,[g_+])$ on the Minkowski space $\mathbb{R}^{1,m+1}$. (Recall that, since g_+ is Einstein, the conformal holonomy $Hol(M,[g_+])$ acts trivially on a positive definite line in $\mathbb{R}^{1,m+2}$.)

If $\mathscr{S}$ is a subset of the space $\mathscr{K}_c$ of ∇-parallel standard tractors, then we denote by $\Sigma(\mathscr{S})$ the intersection of all singularities sets $\Sigma(\Pi(I))$ for $I \in \mathscr{S}$. In summary, we have the following result.

Proposition 3 *[43] Let $(\overline{M}^{m+1}, g_+)$ be an even AH Einstein space of dimension $m+1 \geq 3$ with Riemannian signature and let $(D_l\overline{M}, c_l[g_+])$ be the corresponding S^l-doubling space with $l \geq 0$. Then*

(i) $(D_l\overline{M}, c_l[g_+])$ has decomposable conformal holonomy.

(ii) The space $\mathscr{K}_c$ of ∇-parallel tractors on $D_l\overline{M}$ decomposes naturally into a direct sum $\mathscr{K}_l \oplus \mathscr{K}_+$ with $dim(\mathscr{K}_l) = l+1$. The non-trivial $I \in \mathscr{K}_l$ satisfy $S(\Pi(I)) < 0$, and the intersection $\Sigma(\mathscr{K}_l)$ of singularities coincides with the pole N_p of $D_l\overline{M}$. The elements I of $\mathscr{K}_+ \setminus \{0\}$ correspond to additional almost Einstein structures on M.

(iii) If the conformal holonomy $Hol(M,[g_+])$ of the AHE space is non-trivial, then $(D_l\overline{M}, c_l[g_+])$ is not conformally equivalent to a special Einstein product in any neighbourhood of the pole N_p.

EXAMPLE: The S^l-doubling $D_l\overline{\mathbb{H}^{m+1}}$ of the hyperbolic $(m+1)$-ball $\mathbb{H}^{m+1}$ is for any $l \geq 0$ conformally equivalent to the Möbius sphere S^n of dimension $n = m+l+1$. In other words, the flat model of conformal Riemannian geometry arises as the collapsing sphere product of the flat hyperbolic model (cf. [33, 43]).

Note that, even so S^n has trivial conformal holonomy group, it is globally not a special Einstein product. Any special Einstein product on S^n degenerates at some pole set. ◇

In [44] we have shown the following reconstruction result for S^l-doublings.

Proposition 4 *[44] Let (F^n, c), $n \geq 3$, be a closed Riemannian conformal space admitting an Euclidean subspace $\mathcal{K}_l \subset \mathcal{K}_c$ of dimension $l+1 \geq 2$ with $\Sigma(\mathcal{K}_l) \neq \emptyset$. Then there exists for any $q \in \Sigma(\mathcal{K}_l)$ a neighbourhood U_q of q in F such that (U_q, c) is conformally equivalent to the S^l-doubling $(D_l\overline{M}, c_l[g_+])$ of some Poincaré-Einstein space $(\overline{M}, g_+)$ of dimension $n-l$.*

Notice the assumption of closedness on F in Proposition 4, even so the result is of local nature. The reason for this assumption is that the proof of Proposition 4 in [44] uses general results for conformal transformation groups on closed Riemannian manifolds. We remark that, if F is simply connected, then Proposition 4 is globally true, i.e., in this case (F^n, c) is the S^l-doubling of some compact Poincaré-Einstein space $(\overline{M}, g_+)$ with boundary. (Also recall that Theorem 3 states that if $dim(\mathcal{K}_c) \geq 2$ then there have to exist almost Einstein structures with hypersurface singularity. However, there are known cases with $dim(\mathcal{K}_c) \geq 2$ and $\Sigma(\mathcal{K}_c) = \emptyset$.)

4.3 The classification in Riemannian signature.

We claim now that the special Einstein products and the collapsing sphere products (which can be understood for obvious reasons as degenerate special Einstein products) are the only possible conformal Riemannian geometries with decomposable holonomy on closed manifolds! To understand this point we need to show that, if the conformal holonomy of a closed Riemannian conformal manifold (F^n, c) decomposes, but the corresponding special Einstein product degenerates on some singularity set, then the conformal holonomy representation must be trivial on a subspace of $\mathbb{R}^{1,n+1}$. This implies (locally) the existence of linearly independent ∇-parallel standard tractors with intersecting scale singularity; then Proposition 4 applies. In other words, the conformal holonomy representation is a product of two non-trivial, irreducible representations only if the corresponding special Einstein product does not *collapse*. The details of this argument are worked out in [5]. Here is a key lemma for the argument, which is just a variation of a well known result, which can be found e.g. in [48] p. 316.

Lemma 4.3 *Let (N, h) be a Riemannian manifold, whose Weyl tensor W^h and Cotton tensor C^h have bounded norm with respect to h on N, and let X be a nowhere vanishing smooth conformal Killing vector field on (N, h) with some maximal integral curve $\gamma^X : I \subset \mathbb{R} \to N$.*

Then, if $\inf_{t\in I}\|X(\gamma^X(t))\|_h = 0$*, the Weyl tensor* W^h *and the Cotton tensor* C^h *vanish at* $\gamma^X(t)\in N$ *for all* $t\in I$*.*

Let us consider a (simply connected) conformal Riemannian space (M^n,c), $n\geq 3$, such that the holonomy representation $\mathbb{R}^{1,n+1}$ decomposes into $V\oplus W$, where V is an Euclidean subspace of dimension $l:=dimV$ with $2\leq l\leq n-1$. Let α be the volume form of the ∇-parallel subbundle of the standard tractor bundle $\mathscr{T}$, which corresponds to the $Hol(M,c)$-invariant subspace V. The corresponding normal conformal Killing $(l-1)$-form (with respect to some $g\in c$) is denoted by α_0.

We assume now that (M,c) is not everywhere locally a special Einstein product, i.e., the zero set $\Sigma(\alpha_0)$ of α_0 is non-empty. Let $q\in\Sigma(\alpha_0)$. A straightforward conclusion of our discussion so far implies: there exists an open subset $Q\cong Q_1\times(0,R)\times Q_2$, $R\in\mathbb{R}_+$, in M with $q\in\partial Q$ and $dimQ_1=l-1$ such that the conformal class c restricted to Q is given by a product metric of the form $h+\frac{dr^2}{r^2}+\frac{g(r)}{r^2}$, $r\in(0,R)$, where h is an Einstein metric with constant sectional curvature 1 on Q_1 for $l>2$ (for $l=2$ we have $dimQ_1=1$), and $\frac{1}{r^2}(dr^2+g(r))$ is an AHE metric for $r\to 0$ on $(0,R)\times Q_2$. Any integral curve of the *Euler vector field* $X:=r\partial r$ on $N:=Q_1\times(0,R)$ satisfies the assumptions of Lemma 4.3. We conclude that the metric $h+\frac{1}{r^2}dr^2$ on N is conformally flat, which implies that h is a round metric for $l>2$ (since h is Einstein). In particular, the holonomy representation on V has to be trivial (also in the case $l=2$). Hence, when M is closed and simply connected we can apply (the global version of) Proposition 4.

Theorem 4 *[5] Let* (F^n,c) *be a closed, simply connected conformal Riemannian manifold of dimension* $n\geq 3$ *with decomposable holonomy* $Hol(F,c)$*. Then one of the following three cases holds true.*

(i) (F,c) *is an almost Einstein space, or*

(ii) (F,c) *is a special Einstein product, or*

(iii) (F,c) *is the collapsing l-sphere product of some AHE manifold with* $l\geq 0$*.*

Note that not every almost Einstein space with hypersurface singularity is the doubling of some AHE space. And, if F is not simply connected, but the universal covering is still closed, then the statement of Theorem 4 remains true locally.

5 The case of unitary conformal holonomy

CR-geometry of hypersurface type is closely related to conformal geometry via the *Fefferman construction*. This construction was invented by C. Fefferman in [21] for boundaries of pseudoconvex domains in $\mathbb{C}^{m+1}$ in order to study their geometric properties and invariant theory. He showed that a trivial circle bundle over a pseudoconvex boundary admits a Lorentzian metric, whose conformal class is invariant under biholomorphisms of the domain. The construction was extended to abstract CR-structures by an intrinsic approach due to J.M. Lee [34], which assigns to any pseudo-Hermitian structure a so-called *Fefferman metric* on the canonical circle bundle. We recall this intrinsic construction. The second part of this section will briefly explain how the Fefferman construction for CR-structures is characterised by *unitary conformal holonomy* (cf. [41]). Thus, in the generic situation here, the conformal holonomy representation is reduced, but irreducible.

5.1 Fefferman construction reviewed.

Let (N^n,H,J) be an oriented, integrable CR-manifold of dimension $n=2m+1\geq 3$ with signature (p,q) (for the Levi form), i.e., H is a contact distribution in TN (of dimension $2m$) and $J:H\to H$ is a complex structure such that $[JX,Y]+[X,JY]$ is a section of H and the *Nijenhuis tensor*

$$\mathscr{N}_J(X,Y):=[X,Y]-[JX,JY]+J([JX,Y]+[X,JY])$$

vanishes for all $X,Y\in\Gamma(H)$. The canonical (complex) line bundle of (N,H,J) is defined by

$$\Lambda^{m+1,0}N\ :=\ \{\ \rho\in\Lambda^{m+1}N\otimes\mathbb{C}:\ i\cdot\iota_X\rho=\iota_{(JX)}\rho\ \text{ for all } X\in H\ \}\ .$$

The positive reals $\mathbb{R}_+$ act by multiplication on $K^*:=\Lambda^{m+1,0}N\setminus\{0\}$ (with deleted zero section), and we set $F_c:=K^*/_{\mathbb{R}_+}$. The triple

$$(\ F_c\ ,\ \pi\ ,\ N\ ) \tag{7}$$

is the canonical S^1-principal bundle of the CR-manifold (N,H,J).

Furthermore, let θ be a pseudo-Hermitian structure on (N,H,J), i.e., the kernel of θ is H. Then the symmetric bilinear form $d\theta(\cdot,J\cdot)$ on H has signature $(2p,2q)$.

The *Tanaka-Webster connection* ∇^W of θ gives rise to a connection 1-form on the principal S^1-fibre bundle F_c, which we denote by $A^W : TF_c \to i\mathbb{R}$. We set

$$A_\theta := A^W - \frac{i}{2(m+1)} scal^W \theta ,$$

which is a connection 1-form on F_c as well. Then the Fefferman metric to θ on F_c is defined by

$$f_\theta := \pi^* d\theta(\cdot, J\cdot) - i\frac{4}{m+2}\pi^*\theta \circ A_\theta$$

(or simply $f_\theta = d\theta(\cdot, J\cdot) - i\frac{4}{m+2}\theta \circ A_\theta$). This is a non-degenerate symmetric 2-tensor on the real tangent bundle of F_c of signature $(2p+1, 2q+1)$. (If the underlying space is strictly pseudoconvex the signature of f_θ is Lorentzian.)

The crucial point of the construction is that the Fefferman conformal class $[f_\theta]$ does not depend on the choice of pseudo-Hermitian form. In fact, rescaling the pseudo-Hermitian form by $\tilde{\theta} := e^{2l}\theta$ with some real function l on N produces the conformally changed Fefferman metric $f_{\tilde{\theta}} = e^{2l} f_\theta$ on F_c. Thus the Fefferman construction assigns to any integrable CR-manifold in a natural and invariant way a conformal manifold of one higher dimension.

There exists a famous characterisation result for Fefferman metrics of integrable CR-manifolds due to G. Sparling in [52] (cf. also [26]) through the existence of a certain Killing vector, i.e., through a solution of an overdetermined, conformally covariant system of PDE's.

Theorem 5 *(Sparling's characterisation) Let (M^{n+1}, g) be a pseudo-Riemannian space of dimension $n+1 \geq 4$ and signature $(2p+1, 2q+1)$. Suppose that g admits a Killing vector V (i.e. $L_V g = 0$) such that*

(i) $g(V,V) = 0$, i.e., V is lightlike,

(ii) $\iota_V W^g = 0$ and $\iota_V C^g = 0$,

(iii) $Ric^g(V,V) > 0$ on M.

Then g is locally isometric to the Fefferman metric of some integrable CR-space (N,H,J) of hypersurface type with signature (p,q) and dimension n.

On the other hand, any Fefferman metric of an integrable CR-space (N,H,J) of hypersurface type admits a Killing vector field V satisfying (1) to (3).

Note that, on the principal S^1-bundle F_c of the Fefferman construction over N, the Killing vector field V of Sparling's characterisation is a fundamental vector field of the S^1-action, i.e., the Killing vector V is vertical with respect to the projection $\pi : F_c \to N$.

Moreover, it was shown in [45] for the 4-dimensional Lorentzian case that locally there exists always a (pair of) *conformal Killing spinor* $\varphi \in \Gamma(\mathscr{S})$ on F_c. This result was extended in [7] to a global result for arbitrary even dimensions $n+1$. Recall that a conformal Killing spinor φ is a solution to the overdetermined spinor field equation

$$\nabla^{\mathscr{S}}_X \varphi + \frac{1}{n+1} X \cdot \not{D} \varphi \;=\; 0 \qquad \text{for all } X \in TF_c \ , \tag{8}$$

where $\nabla^{\mathscr{S}}$ is the spinor connection and $X \cdot \not{D}$ denotes Clifford multiplication with the Dirac operator. Then the *Dirac current* (or *spinor square*) $V_\varphi \in \mathfrak{X}(F_c)$ of φ is the Killing vector of Sparling's characterisation.

5.2. Holonomy characterisation

CR geometry and conformal geometry are both *parabolic geometries* (cf. [16]). It is a matter of fact that parabolic geometries are equipped with a standard tractor bundle and canonical connection. In particular, there is a standard CR-tractor bundle $\mathscr{T}^{cr}$ with connection ∇^{cr} (induced by a canonical Cartan connection) over any CR-manifold (N^n, H, J) of signature (p,q) (in case the canonical line bundle $\Lambda^{m+1,0}N$ admits an $(m+2)$nd root). The structure group of this bundle $\mathscr{T}^{cr}$ with connection is $G = SU(p+1, q+1)$.

There are nowadays several works which investigate the Fefferman construction using the framework of Cartan and tractor calculus (cf. e.g. [14, 2]). The essence is that the lift of the CR-tractor bundle $\mathscr{T}^{cr}$ on a CR-manifold (N^n, H, J) via π to the corresponding Fefferman space F_c is naturally identified with the conformal tractor bundle $\mathscr{T}$ on F_c. Moreover, since (N^n, H, J) is assumed to be integrable, the lift of the canonical Cartan connection (of CR-geometry) to F_c induces the conformal tractor connection on $\mathscr{T}$, and all vertical tangent vectors of $\pi : F_c \to N$ insert trivially into the conformal curvature Ω^∇. Hence, *movement* on F_c in vertical direction does not contribute to the conformal holonomy algebra and, in fact, the restricted holonomy groups of ∇^{cr} on $\mathscr{T}^{cr}$ over (N^n, H, J) and of ∇ on $\mathscr{T}$ over F_c are identical. In particular, the conformal holonomy algebra is reduced (at least) to $\mathfrak{su}(p+1, q+1) \subset \mathfrak{so}(2p+2, 2q+2)$.

Theorem 6 *[38, 40] Let F_c be the Fefferman space of an integrable CR-manifold (N^n, H, J) of signature (p,q). Then*

(i) the conformal holonomy algebra $\mathfrak{hol}(F_c)$ is contained in $\mathfrak{su}(p+1, q+1)$.

(ii) There exists a ∇-parallel, orthogonal complex structure $\mathscr{J}$ on $\mathscr{T}$, which corresponds via $\langle\cdot,\cdot\rangle_{\mathscr{T}}$ and the projection $\Pi : \Lambda^2\mathscr{T}^* \to \mathfrak{X}(F_c)$ to a Killing vector V as in Sparling's characterisation (with respect to any Fefferman metric g on F_c).

On the other hand, the reduction to $U(p+1, q+1)$ of the conformal holonomy group of some conformal manifold F (of even dimension with signature (p,q)) implies the existence of a ∇-parallel orthogonal complex structure $\mathscr{J}$ on $\mathscr{T}$ over F. Via Sparling's characterisation we conclude that the conformal manifold F is (at least locally) conformally equivalent to the Fefferman space of some integrable CR-manifold (N^n, H, J). An alternative proof, which relies on the normality of the canonical Cartan connection, is given in [41].

Theorem 7 *[41] Let (F^{n+1}, c) be a conformal manifold of even dimension $n+1 \geq 4$ and signature $(2p+1, 2q+1)$. If the conformal holonomy group $Hol(F)$ is contained in $U(p+1, q+1)$, then (F^{n+1}, c) is locally conformally equivalent to the Fefferman space of some integrable CR-manifold (N^n, H, J).*

In particular, $Hol(F) \neq U(p+1, q+1)$ for any conformal manifold F of dimension $n+1 = 2(p+q+1)$.

5.3 Fefferman-Einstein metrics.

It was pointed out in [34] that a Fefferman metric f_θ on F_c is never Einstein! However, if the conformal holonomy $Hol(F_c)$ of a Fefferman space F_c preserves a standard tractor, then F_c must be almost Einstein. In fact, any conformally flat space (which is almost Einstein) is locally the Fefferman space of some flat CR-manifold.

In [39] we describe the situation of Fefferman spaces, which are almost Einstein, in general. The key to this description is the fact that any ∇^{cr}-parallel standard CR-tractor I^{cr} on an integrable CR-manifold (N, H, J) corresponds via π naturally to a ∇-parallel standard tractor I on the Fefferman space F_c. And, (N, H, J) admits a ∇^{cr}-parallel standard CR-tractor I^{cr} if and only if the CR-structure on N admits a *TSPE structure* θ; that is a pseudo-Hermitian form θ, whose Webster-Ricci curvature is a multiple of the Levi form $d\theta(\cdot, J\cdot)$ on N, and whose Reeb vector T_θ is a *transverse symmetry* of the CR-structure, i.e., T_θ is a CR-vector field, which is transverse to the distribution H. Any such CR-manifold with TSPE structure stems from a Kähler-Einstein spaces in one dimension lower. Here is the statement of our result (in Lorentzian signature). For a detailed explanation we refer to [39].

Theorem 8 *[39] Let (Q,h,J) be a Riemannian Kähler-Einstein space of dimension $2m$ with scalar curvature $scal^h$.*

(i) If $scal^h = 0$ and the Kähler form is $\omega = d\alpha$ for some 1-form α on Q, then the metric

$$\tilde{f}_h = cos^{-2}(t) \cdot \left(\pi^* h + 4dt \circ (\pi^* \alpha + ds) \right)$$

on $Q \times \{ (s,t) \, : \, -\frac{\pi}{2} < t < \frac{\pi}{2} \} \subset Q \times \mathbb{R}^2$ (with natural projection π onto Q) is Ricci-flat and (locally) conformally related to a Fefferman metric.

(ii) If $scal^h \neq 0$ then the metric

$$\tilde{f}_h = cos^{-2}(t) \cdot \left(\pi^* h - \frac{4m(m+1)}{scal^h} \cdot (\, dt^2 + \frac{\rho_{ac}^2}{(m+1)^2}) \right)$$

on $\mathscr{S}_{ac}(Q) \times (-\frac{\pi}{2}, \frac{\pi}{2})$, where $(\mathscr{S}_{ac}(Q), \pi, Q)$ is the anti-canonical S^1-bundle over Q with Levi-Civita connection $\rho_{ac} : T\mathscr{S}_{ac}(Q) \to i\mathbb{R}$, is Einstein with $scal^{\tilde{f}_h} = \frac{2m+1}{2m} \cdot scal^h$ and (locally) conformally related to a Fefferman metric.

On the other hand, if a Fefferman metric f_h of Lorentzian signature over an integrable CR-space is locally conformally Einstein, then any Einstein metric $\tilde{f} \in [f_h]$ can be brought locally into the form (1) or (2).

Note that any almost Einstein structure on a Fefferman space F_c has singularities. However, since the complex structure $\mathscr{J}$ on $\mathscr{T}$ induces an action on the space of ∇-parallel standard tractors $\mathscr{K}_c$, the dimension of $\mathscr{K}_c$ is even. And for any point $x \in F_c$ there exists an almost Einstein structure, which has no singularity at x.

6 The generalised Fefferman construction

Let G be a semisimple Lie group and $P \subset G$ a parabolic subgroup. We call the pair (G,P) a parabolic *Klein geometry*. A parabolic geometry $(\mathscr{P}, \omega)$ on a smooth manifold M of type (G,P) is a principal P-fibre bundle $\mathscr{P}$ equipped with a Cartan connection $\omega : T\mathscr{P} \to \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G (cf. [16]).

EXAMPLE. The structure group of n-dimensional (oriented) Riemannian conformal geometry is $G = SO(1,n+1)$, and the corresponding parabolic subgroup P is the stabiliser of a null line in $(n+2)$-dimensional Minkowski space. To any conformal manifold (M^n,c) of dimension $n \geq 3$ belongs a canonical Cartan geometry $(\mathscr{P}, \omega_{nor})$ of this type (G,P) (cf. e.g. [16]). $\diamond$

Now let (G,P) be a parabolic Klein geometry and let $\iota : G' \hookrightarrow G$ be an inclusion of semisimple Lie groups such that the G'-orbit of eP in G/P is open. (This is equivalent to surjectivity of the map $\mathfrak{g}' \to \mathfrak{g}/\mathfrak{p}$ induced by $\iota_* : \mathfrak{g}' \to \mathfrak{g}$.) The subgroup $Q := \iota^{-1}(P) = G' \cap P$ is closed in G', but in general not a parabolic subgroup. For that reason, we introduce a parabolic subgroup P' of G', which contains Q. Thus, we have the data of a pair of parabolic Klein geometries (G',P') and (G,P) with inclusion $\iota : G' \hookrightarrow G$, $P' \supset G' \cap P$ and $\mathfrak{g}'/(\mathfrak{g}' \cap \mathfrak{p}) \cong \mathfrak{g}/\mathfrak{p}$.

Having these ingredients, we can explain the generalised Fefferman construction for parabolic geometries due to A. Čap (cf. [13]). Let M' be a smooth manifold with parabolic geometry $(\mathscr{P}', \omega')$ of type (G',P'). The quotient space $M := \mathscr{P}'/Q$ is in a natural way a fibre bundle over M' of type P'/Q with canonical projection π, and $\mathscr{P}'$ is a principal Q-fibre bundle over M. We can extend the principal bundle $\mathscr{P}'$ over M by the group P in order to obtain a principal P-fibre bundle $\mathscr{P}$ over M. Furthermore, the Cartan connection ω' on $\mathscr{P}'$ can naturally be lifted and extended to a Cartan connection $\omega : T\mathscr{P} \to \mathfrak{g}$. This is the generalised Fefferman construction, which generates from a parabolic geometry $(\mathscr{P}', \omega')$ on M' of type (G',P') the parabolic geometry $(\mathscr{P}, \omega)$ on $M := \mathscr{P}'/Q$ of type (G,P). The codimension of the Fefferman construction $\pi : M \to M'$ is the dimension of the homogeneous space P'/Q.

EXAMPLE. **(1) The classical Fefferman construction.** Let (H,J) be a strictly pseudoconvex CR-structure on M' of dimension $n = 2m+1$. In case the canonical line bundle of (M',H,J) admits an $(m+2)$nd root, the CR-structure (H,J) is equivalently given by a parabolic geometry $(\mathscr{P}', \omega'_{nor})$ (with normal Cartan connection) of type (G',P'), where $G' = SU(1,m+1)$ and P' is the stabiliser of a complex null line in $\mathbb{C}^{1,m+1}$ (equipped with $SU(1,m+1)$-invariant Hermitian form). Then we use the canonical inclusion $SU(1,m+1) \subset SO(2,n+1)$. In this case the subgroup Q of G' is the stabiliser of a real null line in $\mathbb{C}^{1,m+1}$ (with respect to the real part of the Hermitian form). Obviously, Q is contained in the parabolic subgroup P', and P'/Q is the homogeneous space of real lines in a complex line, i.e., P'/Q is a circle. In particular, we see that $M := \mathscr{P}'/Q \xrightarrow{\pi} M'$ is a circle bundle of total dimension

$n+1$. The extended lift of $(\mathscr{P}', \omega'_{nor})$ to this circle bundle $M := \mathscr{P}'/Q$ gives rise to a conformal structure on M of Lorentzian signature.

Note that the extended lift of the canonical Cartan connection ω'_{nor} on $\mathscr{P}'$ to $\mathscr{P}$ is the canonical Cartan connection ω_{nor} of conformal geometry on M if and only if (H,J) is integrable on M' (cf. [14]). Moreover, note that the S^1-bundle $M := \mathscr{P}'/Q \xrightarrow{\pi} M'$ corresponds to the choice of an $(m+2)$nd root of the canonical line bundle $\Lambda^{m+1,0}M'$, i.e., the S^1-bundle M is only locally isomorphic to the S^1-bundle F_c over M' of the classical Fefferman construction, defined in Section 5 (7). $\diamond$

For generalised Fefferman constructions, corresponding to pairs (G',P') and (G,P), it is a crucial step to investigate how the canonical Cartan connections on the underlying manifolds M' and $M := \mathscr{P}'/Q$ compare to each other! In general, one can not expect that the extended lift of ω'_{nor} to $\mathscr{P}$ coincides with the canonical Cartan connection on M. This comparison of the canonical connections has to be studied in a case by case consideration.

Finally, we mention some generalised Fefferman constructions, which are nowadays discussed in the literature, and which generate a conformal structure on the corresponding Fefferman space (with a certain conformal holonomy reduction). We only mention the involved structure groups and basic features of these construction without going into further details.

FURTHER EXAMPLES. **(2) Quaternionic contact structures.** A quaternionic contact structure on a smooth manifold M'' of dimension $4n+3$ and signature (p,q) with $n = p+q \geq 1$ is given by a subbundle $H \subset TM''$ of rank $4n$ equipped with an almost quaternionic structure. The graded vector bundle $H \oplus TM''/H$ equipped with the algebraic bracket $\{\cdot,\cdot\} : H \times H \to TM''/H$, which is induced by the Lie bracket of vector fields, is pointwise (for any $p \in M''$) isomorphic to the quaternionic Heisenberg algebra of signature (p,q) (cf. [10, 16, 2]).

The structure group of quaternionic contact geometry is the symplectic group $G'' = Sp(p+1,q+1)$ (assuming the existence of a square root of the canonical complex line bundle over M''), and the corresponding parabolic subgroup P'' is the stabiliser of an isotropic quaternionic line in $\mathbb{H}^{p+q+2}$ (equipped with quaternionic Hermitian form of signature $(p+1,q+1)$).

The *twistor space* M' of a quaternionic contact manifold (M'',H) inherits a natural CR structure. This is the Fefferman construction, which corresponds to the standard inclusion of $G'' = Sp(p+1,q+1)$ in $SU(2p+2,2q+2)$. For $p+q \geq 2$ the induced CR structure on the twistor space M' is integrable of signature $(2p,2q)$ (cf. [16]).

Now we can apply the classical Fefferman construction (as in Example (1)) to the twistor space M', which gives rise to a conformal structure of signature $(4p+3, 4q+3)$ on a smooth manifold M of dimension $4n+6$. This Fefferman space M is also the Fefferman space of the quaternionic contact manifold (M'', H), which corresponds to the inclusion of $G'' = Sp(p+1, q+1)$ in $G = SO(4p+4, 4q+4)$. For $p+q \geq 2$ the conformal holonomy of M is contained in $G'' = Sp(p+1, q+1)$. If $p+q = 1$ and the harmonic torsion of (M'', H) vanishes the same result is true (cf. [2]).

(3) Generic rank two distributions. Let M be a 5-dimensional manifold. A *generic distribution* H of rank two in TM has the property that for any (local) basis X_1, X_2 of H the Lie brackets $T := [X_1, X_2]$, $[T, X_1]$ and $[T, X_2]$ span (pointwise) the whole tangent space TM. Such a generic distribution H defines a parabolic geometry of type (G', P') on M^5. Here the structure group G' is the exceptional simply Lie group G_2 of split type, which is naturally contained in $G = SO(3,4)$. The parabolic subgroup P' is the intersection of P with G', where $P \subset SO(3,4)$ is the parabolic subgroup of conformal geometry. (The parabolic subalgebra $\mathfrak{p}'$ corresponds to the cross in the Satake diagram $\circ\!\Rrightarrow\!\times$ of $\mathfrak{g}' = \mathfrak{g}_2$.)

In particular, it follows that on M^5 with generic distribution H there exists a natural construction of a conformal structure of signature $(2,3)$. This is the Fefferman construction, which corresponds to the inclusion of G_2 in $G = SO(3,4)$, and which is explicitly explained in [15]. Note that here the Fefferman space coincides with the base space M^5. In any case the conformal holonomy of M^5 is contained in G_2. The generic case has irreducible conformal holonomy G_2. The almost Einstein structures on M^5 correspond to those conformal Killing vector fields, which are not infinitesimal automorphisms of the underlying rank two distribution H (cf. [30]).

Note that the geometry of generic rank two distributions was first discovered in [18]. Recently, P. Nurowski noticed in [47] the relation to conformal geometry.

(4) Generic rank three distributions. Now let M^6 be a 6-dimensional smooth manifold. A generic rank three distribution H in TM has the property that the Lie brackets of vector fields in H span (pointwise) the whole tangent space TM. Again, this defines a parabolic geometry on M^6. And the Fefferman construction assigns to H a natural conformal structure of signature $(3,3)$ on M^6. This was first observed by R. Bryant in [12] (cf. also [19]).

The involved structure groups for this Fefferman construction are $G' = Spin(3,4)$, which is an irreducible subgroup of $G = SO(4,4)$. The intersection of the parabolic subgroup P in $SO(4,4)$ (of conformal geometry) gives rise to a parabolic subgroup P' of $G' = Spin(3,4)$. The corresponding parabolic subalgebra $\mathfrak{p}'$ is given by the cross in the Satake diagram ○–○⇛× of $\mathfrak{g}' = \mathfrak{so}(3,4)$. Generically, the Fefferman space M^6 has exceptional irreducible conformal holonomy $Spin(3,4)$ (cf. [4]).

7 Overdetermined PDE and BGG-sequences

So far our discussion was mainly motivated by considerations of the conformal holonomy group. However, during the course of the article we have also noticed that, if the conformal holonomy is reduced, then there exist (locally) ∇-parallel tractors, which in turn correspond to solutions of certain overdetermined PDE systems. In this final section we want to clarify the emergence of these PDE systems. This is explained in terms of BGG-sequences.

Let (M,c) be an (oriented) conformal manifold of dimension $n \geq 3$ and signature (p,q). The conformal structure c on M gives rise in a natural way to a parabolic Cartan geometry $(\mathscr{P},\omega_{nor})$ of type (G,P) (with Lie algebras $(\mathfrak{g},\mathfrak{p})$), where $G = SO(p+1,q+1)$. Then, to any finite representation V of G we have the associated tractor bundle $\mathscr{V} = \mathscr{P} \times_P V$, and the Cartan connection ω_{nor} induces a covariant derivative ∇ on $\mathscr{V}$. More generally, we have the exterior covariant derivatives d^∇ for any $k \in \{0,1,\dots,n\}$, mapping k-forms $\Gamma(\Lambda^k T^*M \otimes \mathscr{V})$ to $(k+1)$-forms $\Gamma(\Lambda^{k+1} T^*M \otimes \mathscr{V})$ with values in $\mathscr{V}$.

Furthermore, for any G-representation V we have *Kostant's codifferential*

$$\partial^* : \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes V \;\to\; \Lambda^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes V \;,$$

which computes the Lie algebra cohomology spaces $H^k(\mathfrak{g}_-,V) = Ker\partial^*/Im\partial^*$. (Note that $\mathfrak{p}$ induces a natural $|1|$-grading $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ on the Lie algebra $\mathfrak{g}$, where $\mathfrak{g}_0 = \mathfrak{co}(p,q)$, $\mathfrak{g}_+ \cong (\mathbb{R}^n)^*$, $\mathfrak{g}_- \cong \mathbb{R}^n$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$ and $\mathfrak{g}_- \cong \mathfrak{g}/\mathfrak{p}$.) Since ∂^* is P-invariant, $\mathfrak{g}_-^* \cong \mathfrak{g}_+$ via the Killing-form and $T^*M \cong \mathscr{P} \times_P \mathfrak{g}_+$, the codifferential induces bundle homomorphisms

$$\partial^* : \Lambda^k T^*M \otimes \mathscr{V} \;\to \Lambda^{k-1} T^*M \otimes \mathscr{V}$$

as well. In particular, for any $k \in \{0,1,\dots,n\}$ we obtain a bundle of cohomology spaces $Ker\partial^*/Im\partial^*$ on M, which we denote by $\mathscr{H}^k\mathscr{V}$. The corresponding canonical projections are denoted by $\Pi : Ker\partial^* \to \mathscr{H}^k\mathscr{V}$. Note that the subgroup P_+ in P, which corresponds to $\mathfrak{p}_+ := \mathfrak{g}_+$ in $\mathfrak{p}$, maps $Ker\partial^*$ to $Im\partial^*$. Thus, the cohomology bundles $\mathscr{H}^k\mathscr{V}$ are associated to the principal $CO(p,q)$-fibre bundle of orthogonal frames on (M,c), i.e., the $\mathscr{H}^k\mathscr{V}$ are tensor bundles on (M,c) of a certain conformal weight (cf. [17]).

Next we observe that for any G-representation V and any $k \in \{0,1,\dots,n\}$ there exists a so-called *splitting operator*

$$\mathbb{S}_k : \Gamma(\mathscr{H}^k\mathscr{V}) \;\to\; \Gamma(\Lambda^k T^*M \otimes \mathscr{V}) \,.$$

This $\mathbb{S}_k$ is a conformally covariant differential operator (of a certain order), which is uniquely determined by the properties that $\partial^* \circ \mathbb{S}_k = 0$, $\partial^* \circ d^\nabla \circ \mathbb{S}_k = 0$ and $\Pi \circ \mathbb{S}_k$ is the identity on $\Gamma(\mathscr{H}^k\mathscr{V})$. Having these operators at hand, we define

$$\mathscr{D}_k := \Pi \circ d^\nabla \circ \mathbb{S}_k : \Gamma(\mathscr{H}^k\mathscr{V}) \to \Gamma(\mathscr{H}^{k+1}\mathscr{V}) \,,$$

which is by construction a conformally covariant differential operator, mapping tensor fields to tensor fields on M (for any $k \in \{0,1,\dots,n\}$). In particular, we obtain the so-called *BGG-sequence*

$$0 \longrightarrow \Gamma(\mathscr{H}^0\mathscr{V}) \xrightarrow{\mathscr{D}_0} \Gamma(\mathscr{H}^1\mathscr{V}) \xrightarrow{\mathscr{D}_1} \dots \xrightarrow{\mathscr{D}_n} \Gamma(\mathscr{H}^n\mathscr{V}) \longrightarrow 0$$

for any given G-representation V. Note that this sequence is a complex if (M,c) is conformally flat. If M is the conformally flat model G/P (= Möbius (pseudo)-sphere), then this is just the tensor product of the deRham complex with V (cf. [17]).

The first differential operator $\mathscr{D}_0 := \Pi \circ \nabla \circ \mathbb{S}_0$ of a BGG-sequence we have met already at several occasions in this text. In fact, in Section 3 (3) we have given the first splitting operator $\mathbb{S}_0$ for the standard representation $V = \mathbb{R}^{p+1,q+1}$ with respect to a metric g. The resulting first BGG-operator $\mathscr{D}_0$ maps densities $\Gamma(\mathscr{E}[1])$ of conformal weight 1 to trace-free symmetric bilinear forms $\Gamma(S^2_0(T^*M))$ of weight 0. The operator $\mathscr{D}_0$ is explicitly given with respect to a metric $g \in c$ by

$$\mathscr{D}_0^g \sigma \;=\; \text{trace-free part of}\,(Hess^g\sigma \;-\; \mathsf{P}^g \cdot \sigma) \;.$$

The kernel of $\mathscr{D}_0$ consists of almost Einstein structures on (M,c) (cf. (5)).

More generally, in Section 4 (6) we have introduced the conformal Killing equation for l-forms. This equation describes the kernel of $\mathscr{D}_0$ for the G-representation $\Lambda^{l+1}(\mathbb{R}^{p+1,q+1})^*$ of tractor $(l+1)$-forms. If α is a ∇-parallel tractor $(l+1)$-form and $n \neq 2l$, then application of the splitting operator $\mathbb{S}_0$ to the l-form $\Pi(\alpha)$ of weight $l+1$ is given with respect to a metric $g \in c$ by

$$\mathbb{S}_0^g \alpha_0 \;=\; \left(\; \alpha_0 \,,\, \frac{1}{l+1} d\alpha_0 \,,\, \frac{1}{n-l+1} d^*\alpha_0 \,,\, \frac{1}{n-2p}(\Delta_l + \frac{scal^g}{2(n-1)})\alpha_0 \right) \,,$$

where Δ_l is minus the Bochner-Laplacian (cf. (3) and [37]). For $l = 1$ the operator $\mathscr{D}_0$ is explicitly given with respect to a metric $g \in c$ by

$$\begin{aligned} \mathscr{D}_0^g : \ \mathfrak{X}(M) \cong \Omega^1(M) &\to \qquad \Gamma(S_0^2(T^*M)), \\ X \qquad\qquad &\mapsto \text{trace-free part of } L_X g \ . \end{aligned}$$

Formulae for $\mathbb{S}_0^g$ and $\mathscr{D}_0^g$ acting on arbitrary l-forms of weight $l+1$ can be found in [29].

Recall that ∇-parallel tractor $(l+1)$-forms correspond to so-called *normal conformal Killing l-forms* (cf. Section 4). These are conformal Killing l-forms, which satisfy certain additional equations (cf. [36, 37, 29]). In general, the kernel of $\mathscr{D}_0$ is bigger than the set of normal conformal Killing l-forms for $0 < l < n$. Obviously, normal conformal Killing l-forms are directly linked to reduced conformal holonomy, and thus occur in several situations of our discussion. In particular, normal conformal Killing l-forms occur in the case of decomposable conformal holonomy (as volume forms; cf. Section 4), in Sparling's characterisation of Fefferman spaces (as lightlike Killing vectors; cf. Section 5) and, of course, in the generalised Fefferman constructions of Section 6 as well.

We also mentioned in Section 5 the existence of conformal Killing spinors on spin Fefferman spaces. This is also explained by BGG-sequences. Namely, if (M,c) is a conformal spin manifold of signature (p,q) (with a given spin structure), then there exists a canonical Cartan geometry $(\tilde{\mathscr{P}}, \tilde{\omega}_{nor})$ of type $(\tilde{G}, \tilde{P})$ on (M,c), where $\tilde{G} = Spin(p+1, q+1)$ is the spinorial Möbius group, and the standard (complex) spinor representation W gives rise to a spinorial BGG-sequence. The first operator $\mathscr{D}_0 : \Gamma(\mathscr{S}) \to \Gamma(\mathscr{H}^1\mathscr{W})$ in this BGG-sequence is *Penrose's twistor operator*, which maps spinors to sections of the kernel of the Clifford multiplication. The spinors in the kernel of $\mathscr{D}_0$ satisfy (with respect to a metric g) the *twistor equation* (8).

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Bifurcation braid monodromy of plane curves

Michael Lönne

Abstract We consider spaces of plane curves in the setting of algebraic geometry and of singularity theory. On one hand there are the complete linear systems, on the other we consider unfolding spaces of bivariate polynomials of Brieskorn-Pham type.
For suitable open subspaces we can define the *bifurcation braid monodromy* taking values in the Zariski resp. Artin braid group. In both cases we give the generators of the image.
These results are compared with the corresponding geometric monodromy. It takes values in the mapping class group of braided surfaces. Our final result gives a precise statement about the interdependence of the two monodromy maps.
Our study concludes with some implication with regard to the unfaithfulness of the geometric monodromy ([W]) and the - yet unexploited - *knotted geometric monodromy*, which takes the ambient space into account.

Keywords braid monodromy, plane algebraic curves.
Mathematics Subject Classification (2010) *Primary* 32S50. *Secondary* 14D05, 32S55, 57Q45.

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1 Introduction

Algebraic geometry and Singularity theory share their interest in discriminant complements. Though they look at different parameter spaces, there is an apparent common feature: A closed subset, the *discriminant*, parametrizes objects which are special (singular) in some sense, and thus distinguished from generic objects corresponding to points in the discriminant complement.

Familiar examples are provided by linear systems resp. versal unfoldings, which contain non-singular and singular elements.

The appropriate tool to study the topology of divisor complements seem to be braid monodromy maps. They were successfully exploited to determine fundamental groups. But their domain is naturally the set of regular values of a projection of the divisor complement.

That is different in the situation considered in [Lö1] where the divisor complement itself is the domain of an interesting braid monodromy map. Therefore the natural question was raised, whether spaces of hypersurfaces in singularity theory and projective geometry support braid monodromy maps and can be better understood by their study.

To get *bifurcation braid monodromy* we discard the *degeneracy locus* consisting of those polynomials which are special with respect to a chosen projection of the domain. The remaining *generic* polynomials are those on which the induced map is a Morse function. In particular its *bifurcation set* consists of only finitely many critical values, their constant cardinality is determined by the topology. Of course paths of such points in the target will precisely be the braids associated to paths of generic polynomials, cf. section 3.

We focus our study to the case of plane projective curves and their close cousins, unfoldings of singular bivariate polynomials of Brieskorn-Pham type $f = x^k + y^{n+1}$.

Our results are expressed in terms of the band generators σ_{ij} which are natural conjugates of the Artin generators $\sigma_i = \sigma_{i\,i+1}$ and which can be identified with the following braid diagrams:

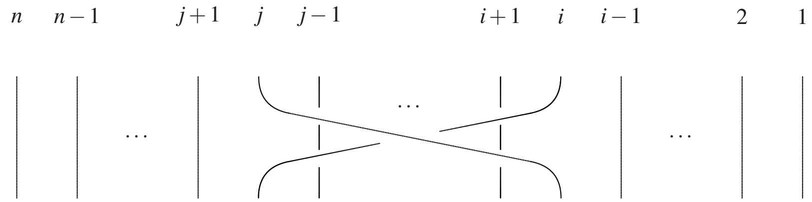


Fig. 1 A generator σ_{ij} of the 'dual' or BKL presentation of the braid group.

Theorem 1 *Suppose the singular polynomial $f = y^{n+1} + x^k$ of Brieskorn-Pham type is considered with respect to the projection along the y-coordinate. Then in* Br_{nk} *the conjugation class of the bifurcation braid monodromy group is represented by*

$$\left\langle \sigma_{ij}^{m_{ij}} \,\middle|\, m_{ij} = \begin{cases} 1 \textit{ if } i \equiv j \mod n, \\ 3 \textit{ if } i \equiv j \pm 1, \{i,j\} \not\equiv \{0,1\} \mod n, \\ 2 \textit{ else} \end{cases} \right\rangle.$$

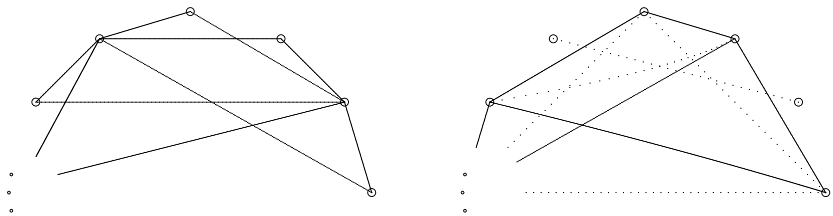


Fig. 2 Both figures depict part of the branch points of $y^4 - 4y + 3x^4$ at the zeroes of $x^{12} = 1$, so $k = 4, l = 3$. The arcs shown on the left hand side correspond to the twists σ_{ij} with $m_{ij} = 3$, those on the right hand side correspond to twists σ_{ij} with $m_{ij} = 2$ and $m_{ij} = 1$ (dotted).

The corresponding result for linear systems of plane curves can be obtained for the open set of plane curves which are transversal to the line at infinity and which do not contain the center of projection $(0:1:0)$:

Theorem 2 *The bifurcation braid monodromy group of plane projective curves of degree d is in the conjugation class of the subgroup of* $\mathrm{Br}_{d(d-1)}$ *generated by the following elements:*

(i) σ_{ij}, *if* $i \equiv j \mod d-1$,

(ii) σ_{ij}^3, *if* $i \equiv j \pm 1, \{i,j\} \not\equiv \{0,1\} \mod d-1$,

(iii) σ_{ij}^2, *if* i, j *not as above,* i) *or* ii),

(iv) $\sigma_1 \sigma_2 \cdots \sigma_{d^2-d-1} \sigma_{d^2-d-1} \cdots \sigma_2 \sigma_1$ *and its conjugates by powers of*
$$\sigma_{d^2-d-1} \sigma_{d^2-d-2} \cdots \sigma_2 \sigma_1.$$

Both bifurcation braid monodromy groups are in fact isomorphic to a group of mapping classes, see Prop. 6 for a proof in case of bivariate polynomials. These mapping classes are obtained as the natural images of a *braided geometric monodromy* to be defined in Section 6.

This isomorphism is another instance of the close connection between algebraic geometry and low dimensional topology, which is witnessed also by

(i) the isomorphism induced by geometric monodromy between the (orbifold) fundamental group of moduli spaces of curves and the mapping class group of the corresponding topological surface,

(ii) the isomorphism between the fundamental group of the space of simple polynomials and the braid group, see Section 3,

(iii) geometric monodromy of plane curves, which induces an injection of the fundamental group of the discriminant complement of polynomials of type A_n and D_n into the mapping class group [PV].

Our ongoing projects aim at a corresponding result in the absence of a projection map. Then there is a kind of *knotted geometric monodromy* with range in the mapping classes of pairs consisting of an ambient space and the embedded hypersurface.

In the basis case that the family of ambient spaces is trivial there is another natural candidate for the range of the geometric monodromy, the fundamental group of higher dimensional configuration spaces. Their study was proposed by Dolgachev and Libgober [DL] as the topological counterpart of spaces of algebraic submanifolds, e.g. smooth projective plane curves in $\mathbf{P}^2$.

The appropriate topological space should contain all topological submanifolds isotopic to a smooth curve C_d of degree d. It can be identified as a coset space for the group $\mathrm{Diff}^o(\mathbf{P}^2)$ of diffeomorphisms of $\mathbf{P}^2$ isotopic to the identity with respect to the subgroup $\mathrm{Diff}^o(\mathbf{P}^2, C_d)$ of diffeomorphisms which induce a diffeomorphism of C_d to itself.

This coset space is the natural topological 'configuration space' in higher dimensions

$$F_{C_d}[\mathbf{P}^2] \quad = \quad \mathrm{Diff}^o(\mathbf{P}^2)/\mathrm{Diff}^o(\mathbf{P}^2, C_d)$$

in analogy to $F_d[S^2] = \mathrm{Diff}^o(S^2)/\mathrm{Diff}^o(S^2, \{p_1, \dots, p_d\})$.

The corresponding quotient map is a fibration which gives rise to a homotopy exact sequence

$$\pi_1 \mathrm{Diff}^o(\mathbf{P}^2) \longrightarrow \pi_1(F_{C_d}[\mathbf{P}^2]) \longrightarrow \pi_0 \mathrm{Diff}^o(\mathbf{P}^2, C_d) \longrightarrow 1$$

where of course the middle group should be called the 'generalised' braid group of algebraic curves in $\mathbf{P}^2$.

This raises a lot of new questions, about the relation between the knotted mapping class group and the fundamental group of higher dimensional configuration spaces, and the respective geometric monodromy maps.

But with the results of this paper it may be conceivable to get hold on injectivity and surjectivity properties of these monodromy maps.

2 Singularity theory

Let us first briefly review some basic notions of singularity theory. We restrict our attention to the case of bivariate polynomials from the beginning. Note that a rigorous treatment would demand the language of germs, but for the sake of clarity we will naively speak of polynomials, (plane) curves and affine spaces.

Definition 1 A holomorphic function f defined in a neighbourhood of $0 \in \mathbf{C}^2$ defines a singular curve, if $0 \in \mathbf{C}^2$ is a critical point of f with critical value $0 \in \mathbf{C}$,

$$f(0) \quad = \quad \partial_x f(0) \quad = \quad \partial_y f(0) \quad = \quad 0.$$

Two singular functions are called *equivalent*, if they differ by a change of coordinates only.

We are also interested in a more restricted equivalence with respect to a linear projection

$$q_x: \quad \mathbf{C}^2 \to \mathbf{C}, \quad (x,y) \mapsto x.$$

Definition 2 Two singular functions are called *equivalent rel* q_x, if they differ by a holomorphic change φ of coordinates only, which fits into a commutative diagram with a suitable biholomorphic ψ:

$$\begin{array}{ccc} \mathbf{C}^2 & \xrightarrow{\varphi} & \mathbf{C}^2 \\ q_x \downarrow & & \downarrow q_x \\ \mathbf{C} & \xrightarrow{\psi} & \mathbf{C} \end{array}$$

The concept of semi-universal unfolding gets hold of all local perturbations of f, at least up to equivalence, resp. equivalence rel q_x.

Suppose now that f and $f\big|_{x=0}$ are isolated singularities. In that case, the semi-universal unfolding rel. q_x associated to f is given by a function F. It is determined by the respective equivalence class of f up to non-canonical isomorphism.

$$F: \mathbf{C}^2 \times \mathbf{C}^{\mu+\mu'} \longrightarrow \mathbf{C},$$

where μ is the Milnor number of f and μ' the Milnor number of $f\big|_{x=0}$.

The following *bifurcation diagram* displays the essential objects and maps for our set-up:

$$\begin{array}{ccl} x,y,u & \mathbf{C}^{\mu+\mu'+2} & \supset \mathscr{C} \\ \downarrow & \downarrow & \\ x,u & \mathbf{C}^{\mu+\mu'+1} & \supset \mathscr{B} \\ \downarrow & \downarrow & \\ u & \mathbf{C}^{\mu+\mu'} & \supset \mathscr{D} \end{array}$$

In this diagram we placed some emphasis on the family of plane curves $\mathscr{C}$, the zero set of F, on the *branch divisor*

$$\begin{aligned} \mathscr{B} &:= \{(x,u) \,|\, F_{x,u}: y \mapsto F(x,y,u) \text{ has singular zero levelset}\} \\ &= \{(x,u) \,|\, F_{x,u}: y \mapsto F(x,y,u) \text{ has multiple roots}\} \end{aligned}$$

and on the *degeneracy locus*

$$\mathscr{D} := \{u \,|\, \mathscr{C}_u \text{ is singular or } q_x|_{\mathscr{C}_u} \text{ is not a Morse function}\}$$

We note the following features:

- $\mathscr{C} \to \mathbf{C}^{\mu+\mu'+1}$ is a finite map with branch locus $\mathscr{B}$,
- $\mathscr{B} \to \mathbf{C}^{\mu+\mu'}$ is a finite map with branch locus $\mathscr{D}$,
- $\mathscr{B}$ is the zero set of a monic polynomial p of degree $\mu+\mu'$ in x with coefficients in $\mathbf{C}[u]$.
- $\mathscr{D}$ is the locus of parameters such that the corresponding monic polynomial p_u has a multiple root.

In particular there is a well-defined Lyashko-Looijenga map on the complement of the degeneracy locus

$$U_{(f)} \quad := \quad \mathbf{C}^{\mu+\mu'} - \mathscr{D} \quad \longrightarrow \quad \mathbf{C}[x], \quad u \quad \mapsto \quad p_u,$$

which maps to monic univariate polynomials of degree $\mu+\mu'$ with simple roots only.

3 Braid monodromy maps and groups

A braid monodromy in general is the map on fundamental groups induced from a topological map on a suitable space to a space which has a braid group as its fundamental group. Here we are only interested in the braid group of the plane and the sphere, i.e. the fundamental groups of the associated configuration spaces.

The configuration space U_d of d points in $\mathbf{C}$ is naturally an open algebraic subset of the affine space $\mathbf{A}_d$ of monic univariate polynomials of degree d. Polynomials in U_d are characterized by the property that they have simple roots only.

Proposition 1 ([Ar]) *The fundamental group of the open subset U_d is isomorphic to the (planar)* braid group Br_d. *It is finitely presented by generators σ_i, $1 \le i < d$ and by relations*

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i, \quad$ *if* $|i-j| > 1,\ 1 \le i, j < d,$

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad$ *if* $1 \le i < d-1,$

The configuration space of d points on $\mathbf{P}^1$, the topological two-sphere, is naturally an open algebraic subset of the projective space associated to the vector space $V_d = \mathrm{Sym}^d \mathbf{C}^2$ of homogenous polynomials of degree d in two variables.

Proposition 2 ([Za]) *The fundamental group of the open set in $\mathbf{P}H^0(\mathscr{O}_{\mathbf{P}^1}(d)) \cong \mathbf{P}V_d$, which consists of homogeneous polynomials with simple roots only, is isomorphic to the* spherical braid group $\mathrm{Br}_d^{\mathrm{s}}$. *It is finitely presented by generators σ_i, $1 \le i < d$ and by relations*

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{if } |i-j| > 1,\ 1 \le i,j < d,$

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{if } 1 \le i < d-1,$

(iii) $\sigma_1 \cdots \sigma_{d-2}\sigma_{d-1}^2\sigma_{d-2}\cdots\sigma_1 = 1.$

Given now any map to $\mathbf{A}_d$ the restriction to the preimage of U_d induces a map from the fundamental group of the preimage to Br_d, the fundamental group of the image. (There is a close analog in the spherical case, of course.)

In our set-up we may just look at a family of bivariate polynomials $f_u(x,y)$ which maps to a family of monic univariate polynomials $p_u(x)$ obtained as the resultant with respect to y of f and its derivative $\partial_y f$ with respect to y,

$$p_u(x) \quad = \quad \mathrm{res}_y(f, \partial_y f).$$

The induced map on fundamental groups is called the *bifurcation braid monodromy*.

In particular, we can apply these considerations to the Lyashko-Looijenga map of the last section. Note again the point stressed there, that the complement $U_{k;n}$ of the degeneracy locus $\mathscr{D}$ is mapped to the discriminant complement U_{nk}.

Definition 3 The *bifurcation braid monodromy group* of a bivariate polynomial f is the image of the bifurcation braid monodromy for the versal unfolding of f relative q_x.

Example 1 The bifurcation braid monodromy of any generic polynomial deformation of a function equivalent to $y^2 - x^k$ rel. q_x is the full braid group Br_k.

To see this it suffices to note that a versal family is parametrized by $\mathbf{A}_k$. More precisely the complement of $\mathscr{D}$ is U_k and the Lyashko-Looijenga map the identity in this case.

Example 2 The bifurcation braid monodromy of the function $f = y^{n+1} + x$ rel. q_x is in the conjugacy class of the subgroup of Br_n generated by

$$\sigma_i^3, \sigma_{i,j}^2, |i-j| \geq 2.$$

In fact the bifurcation diagram for this function is a smooth pull-back of the discriminant diagram of the function t^{n+1}. In terms of an unfolding F and a truncated unfolding F' of t^{n+1}

$$F(t, u_{n-1}, \ldots, u_1, u_0) = F'(t, u_{n-1}, \ldots, u_1) + u_0 = t^{n+1} + \sum_{i=0}^{n-1} u_i t^i.$$

the discriminant diagram is given by

$$\begin{array}{llcl} u = (u_{n-1}, \ldots, u_1, u_0) & \mathbf{C}^n & \supset \mathscr{D} = & \{u \,|\, F_u \text{ has singular zero-level }\} \\ \quad\quad\downarrow & \downarrow & & \\ u' = (u_{n-1}, \ldots, u_1) & \mathbf{C}^{n-1} & \supset \mathscr{B} = & \{u' \,|\, F'_{u'} \text{ is not a Morse function}\}. \end{array}$$

Then we can refer to the corresponding claim for the braid monodromy group of the discriminant diagram, which is shown in [Lö4, section 4] and relies on [L] and [CW].

4 Computation of bifurcation braid monodromy

In this section we want to give the outline of a proof of Theorem 1. The following proposition has been proved with more detail in [Lö5] in the special case of $l = 2$.

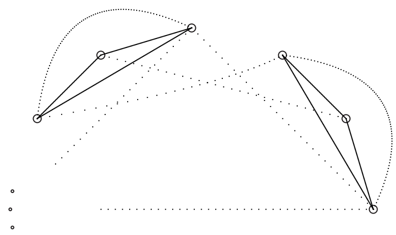
Proposition 3 *The bifurcation braid monodromy group of a plane curve germ $y^{n+1} + x^k$ projected by the x-coordinate is the subgroup of* Br_{nk} *generated by*

(i) σ_{ij}, *if* $i \equiv j \mod n$,

(ii) $\sigma^3_{i,i+1}$, *if there is* $s \equiv 0 \mod n$ *such that* $s < i \leq n+s-1$,

(iii) $(\sigma'_{ij})^2$, *if there is* $s \equiv 0 \mod n$ *such that* $s < i, j \leq n+s, |i-j| > 1$,

where $\sigma'_{ij} = \sigma^{-1}_{i,i+1} \cdots \sigma^{-1}_{j-2,j-1} \sigma_{j-1,j} \sigma_{j-2,j-1} \cdots \sigma_{i,i+1}$.



The figure shows part of the branch locus $x^{12} = 1$ of $y^4 - 4y + 3x^4$, so $k = 4, n = 3$.
The straight arcs correspond to twists σ_{ij} with $m_{ij} = 3$ and $m_{ij} = 1$ (dotted),
the curved arcs to twists ${\sigma'_{ij}}^2$ of the bifurcation braid monodromy group.

Proof To smooth the argument we note one computational detail in advance. The branch locus of a polynomial $f = y^{n+1} - (n+1)p(x)y + nq(x)$ with respect to q_x is obtained by an elementary elimination:

$$\begin{aligned}
\partial_y f = 0 &\implies y^n = p(x) \\
&\overset{f=0}{\implies} yp(x) = q(x) \\
&\implies y^n p^n(x) = q^n(x) \\
&\overset{\partial_y f=0}{\implies} p^{n+1}(x) = q^n(x) \qquad (*)
\end{aligned}$$

In the first step we consider the family $f = y^{n+1} - (n+1)uy + n(x^k + v)$ parametrized by u, v. By definition the bifurcation braid monodromy is induced by the map

$$u, v \quad \mapsto \quad p_{u,v}(x) = \mathrm{discr}_y(f) := \mathrm{res}_y(f, \partial_y f) \overset{(*)}{=} (x^k + v)^n - u^{n+1}.$$

(Let us remark that we feel free to rescale the discriminant without further notice.)

To find the intersection of the uv-parameter plane with the open set of admissible polynomials we have to find the u, v such that $p_{u,v}$ has a multiple root. Again the elimination of x from $p_{u,v}$ and $\partial_x p$ is quite elementary:

$$\begin{aligned}
\partial_x p = 0 &\implies x^{k-1}(x^k + v)^{n-1} = 0 \\
&\implies x^k(x^k + v)^{n-1} = (v - v)(x^k + v)^{n-1} \\
&\overset{p=0}{\implies} u^{n+1} = v(x^k + v)^{n-1} \\
&\overset{p=0}{\implies} u^{n(n+1)} = v^n u^{(n-1)(n+1)} \\
&\implies u^{(n-1)(n+1)}\left(u^{n+1} - v^n\right) = 0 \qquad (**)
\end{aligned}$$

To determine now the braid monodromy we fix a base point at $(u,v) = (1,0)$. The corresponding branch set is given by $p_{(1,0)} = x^{nk} - 1 = 0$, see $(*)$.

For further use we number these branch points according to increasing *arg* ending with $\xi_{nk} = 1$.

The bifurcation braid monodromy is now defined on the fundamental group of the complement to $(**)$. Natural generators are given by a geometric basis on the line $(1,v)$ punctured where $v^n = 1$ and a simple closed path around $u = 0$, see below for details.

On the line $(1,v)$ it suffices to consider paths where v moves along radial rays from 0 to a unit root ξ_{ik}, $1 \leq i \leq n$. Again from $(*)$ we have for $\lambda \in [0,1]$

$$(x^k + \lambda\xi_{ik})^n = 1 \quad \Leftrightarrow \quad x^k = \xi_{jk} - \lambda\xi_{ik} \quad \text{for some } j.$$

Accordingly the k branch points with indices congruent to $i \mod n$ converge along radial rays to 0 while the remaining branch points stay away from these rays.

The local monodromy at the degeneration points $(1,\xi_{ik})$ can be obtained from the family $g_t = y^2 + x^k - t$. Its bifurcation divisor $x^k - t = 0$ is the local model of the singular branch of the bifurcation divisor of f over $(1,\xi_{ik})$.

Accordingly the monodromy of g_t is mapped to the local monodromy of f by a transfer map, which identifies a disc containing the solutions of $x^k = t$ with the disc containing the k branch points converging to the origin.

We now get to the final path from $(1,v)$ around the line $u = 0$. Let ρ be the solution of $\rho^2 = \xi_k$ with positive imaginary part, then we can consider the degeneration along $(u,v) = \big((1-\lambda)^{\frac{n}{n+1}}, \lambda\rho\big)$ with $\lambda \in [0,1]$ and bifurcation according to $(*)$:

$$\begin{aligned} & (x^k + \lambda\rho)^n = (1-\lambda)^n \\ \Longrightarrow \quad & x^k = (1-\lambda)\xi_{jk} - \lambda\rho \quad \text{for some } j \in \{1,\dots,n\}. \end{aligned}$$

Let $x_{i'}$ denote the solution which moves to $\xi_{i'}$ for $\lambda \to 0$. Then one may check for $\lambda \to 1$ that the argument $arg(x_{i'})$ is strictly increasing (resp. decreasing or constant) for $i' \equiv i \mod n$, $0 < i < (n+1)/2$ (resp. $(n+1)/2 < i \leq n$ or $i = (n+1)/2$).

Hence the family degenerates at $\lambda = 1$ only and all branch points are on distinct rays for $\lambda \in [0,1[$. Moreover we observe that for $\lambda \to 1$ the following n-tuples of branch points merge at k distinct points,

$$T_s := \{x_s, x_{s+1}, ..., x_{s+n-1}\} \quad \text{with} \quad s \equiv 0 \mod n.$$

The local monodromy at the degeneration point $(0,\rho)$ can be obtained from the family $f_t = y^{n+1} - (n+1)ty + nx$. Its bifurcation divisor $x^n = t^{n+1}$ is the local model of each singular branch of the bifurcation divisor of f over $(0,\rho)$.

Accordingly each local monodromy of f is obtained by a transfer map from the monodromy of f_t. The transfer map identifies a disc containing the solutions of $x^n = t^{n+1}$ with a disc containing an n-tupel T_s of branch points converging to a singularity over $(0,\rho)$.

From the monodromy of the special family we may get the monodromy of the versal family by the principles of versal braid monodromy [Lö2]. They tell us how to replace the generating braids associated to the special family by groups of braids, which then generate the full braid monodromy group of the versal family.

In fact one has to find first the braid monodromy groups of the local models. Then the transfer maps mentioned above map these group to subgroups of Br_{nk} which generate the bifurcation braid monodromy group.

The local model for the degenerations in the line $(1,v)$ is given by example 1, so for each point $(1,\xi_{ik})$ we have to transfer a full braid group Br_k into Br_{nk}. This is done by a topological disc which contains the ξ_j with $j \equiv i \mod k$. Thus the contribution to the bifurcation braid monodromy group is given by the half-twists σ_{ij} with $i \equiv j \mod k$.

The local model for the degenerations at $(0,\rho)$ is given by example 2, which models the degeneration of each of the k tuples T_i of n branch points. Here we have to transfer the corresponding braids from example 2 to topological discs around the T_i. In this way we get the remaining braids of the claim. ∎

Remark 1 The last argument admittedly is incomplete, since the transfer map in the last case is sensitive to the identification of the disc. Since this ambiguity does not matter in the proof of our theorem, we do not stress the point here.

Proof (of Theorem 1) We have to compare the two subgroups generated by the set of braids given in the assertion of Prop. 3, respectively the set of braids given in the theorem.

Let us focus first on generating braids supported on a topological disc around an n-tupel T_s. In the proposition these elements are listed in 2) and 3) with $n' = s$. As we remarked above, we did not actually prove that these elements generate the monodromy group. Rather, we showed that under some identification of discs generators coincide with the generators of example 2.

In the claim of the theorem the elements concerned are $\sigma_{ij}^2, \sigma_{ij}^3$ listed in 2) and 3) with $s < i < j \leq s+n$. They are readily seen to coincide with the generators of example 2 under a suitable map.

Hence by composition we get a local homeomorphism under which the elements from the proposition are identified with the elements from the theorem.

Since discs around the T_s may be chosen disjoint we deduce the existence of a conjugating braid β which induces the above indentifications simultaneously.

In the second step we consider the generators listed under 1) of the proposition and the theorem. They are the same and generate the same subgroup H of Br_{nk}, but helas we have now to take the conjugation by β into consideration.

We note that β is symmetric in the sense that it commutes with any rigid rotations, which permutes the n-tuples T_s. Therefore β belongs to the subgroup generated by

$$\delta_1 = \sigma_1\sigma_{n+1}\ldots\sigma_{(k-1)n+1}, \ldots, \delta_{n-1} = \sigma_{n-1}\sigma_{2n-1}\ldots\sigma_{nk-1}.$$

The crucial observation is, that H is invariant under conjugation by the δ: In fact $\delta_{i'}$ acts on $\sigma_{i,j}$ with $i \equiv j \mod n$ as

$$\delta_{i'}\sigma_{i,j}\delta_{i'}^{-1} \quad = \quad \begin{cases} \sigma_{i,j}\sigma_{i+1,j+1}\sigma_{i,j}^{-1} & \text{if } i' \equiv i \mod n \\ \sigma_{i-1,j-1} & \text{if } i' \equiv i-1 \mod n \\ \sigma_{i,j} & \text{else} \end{cases}$$

So at this stage we have proved that the monodromy group of the proposition conjugated by β is contained in the group generated by the elements listed in the theorem.

To finish, it suffices to show that the all elements of the theorem not considered till now are in fact redundant. But this can be shown inductively using for $i < j$ that $\sigma_{i,j+n}\sigma_{j,j+n} = \sigma_{j,j+n}\sigma_{i,j}$. ∎

Remark 2 The bifurcation braid monodromy is precisely the subgroup of Br_{nk} generated by those powers of the band generators σ_{ij} which stabilise the periodic sequence of transpositions

$$(12),(23),\cdots,(kk+1),\,(12),(23),\cdots,(kk+1),\cdots,(12),\cdots,(kk+1)$$

of length nk under the Hurwitz action. This sequence encodes of course the finite branched covering of $\mathbf{C}$ by the curve $C := \{f_{(1,0)} = 0\}$ via q_x.

5 Monodromy for spaces of plane projective curves

The space of plane projective curves of degree d is given by $\mathbf{P}H^0_d = \mathbf{P}H^0(\mathscr{O}_{\mathbf{P}^2}(d))$. In analogy to the situation in singularity theory we consider open subsets of curves which have a generic branching property. Of course they are open subsets in the discriminant complement $\mathscr{U}_d$ corresponding to the set of smooth curves.

Notation 1 For a given point $P_0 \in \mathbf{P}^2$ we have the subset of smooth curves disjoint to P_0 which are generic with respect to the projection $q : \mathbf{P}^2 - \{P_0\} \to \mathbf{P}^1$ from P_0:

$$\mathscr{F}_d = \{C \in \mathscr{U}_d \,|\, P_0 \notin C, q|_C \text{ is Morse }\}$$

An open subset is obtained imposing the condition that a line L_0 containing P_0 (say at infinity) has d simple points of intersection with C.

$$\mathscr{F}'_d = \{C \in \mathscr{U}_d \,|\, C \in \mathscr{F}_d, \#C \cap L_0 = d\}$$

Remark 3 Thanks to the homogeneity of $\mathbf{P}^2$ these spaces do not depend on the choice of a projection center and/or line at infinity.

If we introduce homogeneous coordinates $(x : y : z)$ such that $P_0 = (0 : 1 : 0)$ and $L_0 = \{z = 0\}$ we can make the following identifications (with $\mu_d = d(d+3)/2$):

$$\begin{aligned}
\mathbf{A}^{\mu_d} &= \{f \in \mathbf{C}[x,y,z]_d \,\big|\, f(0,1,0) = 1\} \\
\mathscr{F}_d &= \{f \in \mathbf{A}^{\mu_d} \,\big|\, \mathrm{discr}_y(f) \text{ has simple roots only}\} \\
\mathscr{F}'_d &= \{f \in \mathbf{A}^{\mu_d} \,\big|\, \mathrm{discr}_y(f)(x,1) \in U_{d(d-1)}\}
\end{aligned}$$

The complement of $\mathscr{F}_d$ in $\mathbf{A}^{\mu_d}$ is the *weak degeneracy locus* $\mathscr{D}$, that of $\mathscr{F}'_d$ the *degeneracy locus* $\mathscr{D}'$.

There is a pull-back diagram

$$\begin{array}{ccccc} F'_d & \hookrightarrow & \mathscr{F}'_d & \ni & f(x,y,z) \\ \downarrow & & \downarrow & & \downarrow \\ \{y^d+1\} & \hookrightarrow & U_d & \ni & f(1,y,0) \end{array}$$

which defines F'_d as the fibre of the map on the right hand side over the element y^d+1.

Of course F'_d consists of the polynomials, which can be written as a sum of y^d+x^d with a polynomial of degree d that has z as a factor.

Proposition 4 *There is an natural exact sequence of groups*

$$\pi_1(U_{d;d}) \longrightarrow \pi_1(\mathscr{F}'_d) \longrightarrow \mathrm{Br}_d \to 1,$$

where $U_{d;d}$ is $U_{(f)}$ on page 241 of section 2 with $f = y^d + x^d$.

Proof First we want to apply the Zariski theorem on fundamental groups of divisor complements as proved by Bessis [B, section 2]. Consider the map

$$\begin{array}{ccc} \mathbf{A}^{\mu_d} & \longrightarrow & \mathbf{A}^{\mu_d-1} \\ f & \mapsto & f(x,y,1)-f(0,0,1) \end{array}$$

which is the projection along the coefficient of z^d. With a generic choice of parameters $\alpha_1,\alpha_2,\alpha_3$ the fibre F over

$$\{y^d+(\alpha_1 x+\alpha_2)y+x^d+\alpha_3 x\} \quad \in \quad \mathbf{A}^{\mu_d-1}$$

intersects the weak degeneracy locus $\mathscr{D}$ transversally. The second contribution to the degeneracy locus is the pull-back $\mathscr{P}$ of the discriminant in V_d, the complement of U_d.

Hence the Zariski theorem asserts that the following sequence is exact:

$$\pi_1(F - F\cap\mathscr{D}) \longrightarrow \pi_1(\mathscr{F}'_d) \longrightarrow \pi_1(\mathbf{A}^{\mu_d-1}-\mathscr{P}') \longrightarrow 1.$$

where $\mathscr{P}'$ is the pullback of the discriminant in V_d to $\mathbf{A}^{\mu_d-1}$.

Since $F - F\cap\mathscr{D}$ is a subset of F'_d, the first map factors through $\pi_1(F'_d)$. With $\pi_1(\mathbf{A}^{\mu_d-1}-\mathscr{P}') = \pi_1(U_d) = \mathrm{Br}_d$ we then get the exact sequence

$$\pi_1(F'_d) \longrightarrow \pi_1(\mathscr{F}'_d) \longrightarrow \mathrm{Br}_d \longrightarrow 1.$$

It remains to identify the fundamental group on the left with $\pi_1(U_{d;d})$. To do so it is natural to consider F'_d as a subspace of a trivial unfolding of $U_{d;d}$. We claim that the induced map on fundamental groups is then an isomorphism. The proof for that claim has to be copied from the arguments in [Lö3, section 4]. ■

In fact with some more care, it could be proved that the sequence is even short exact.

Definition 4 The *bifurcation braid monodromy* of smooth plane curves of degree d is given by the map

$$\begin{array}{rcl} \pi_1(\mathscr{F}'_d) & \longrightarrow & \mathrm{Br}_{d(d-1)} \\ \big(\text{resp. } \pi_1(\mathscr{F}_d) & \longrightarrow & \mathrm{Br}^{\mathsf{s}}_{d(d-1)} \text{ in the spherical case.}\big) \end{array}$$

which is - in both cases - induced by the map on polynomials

$$f = f(x,y,z) \quad \mapsto \quad \mathrm{res}_y(f, \partial_y f).$$

Of course the spherical braid monodromy group can be given with the same generators, since $\pi_1(\mathscr{F}') \to \pi_1(\mathscr{F})$ is surjective.

Proof (of Theorem 2) We can now outline the proof of the second theorem.
By the exact sequence of Prop. 4, the braid monodromy group of plane curves of degree d contains all braids which belong to the braid monodromy group of the bivariate Fermat polynomial with exponent d. To get sufficiently many additional braids, we have to find lifts of generators of $\pi_1(U_d)$ and add their images in $\mathrm{Br}_{d(d-1)}$.

Again we investigate a special family of curves (here we work with the homogeneous form)

$$f_u \quad = \quad y^d - d(ux^{d-1} + z^{d-1})y + (d-1)x^d.$$

By $(*)$ the branch locus for the family is given by

$$p_u(x,z) \quad = \quad (ux^{d-1} + z^{d-1})^d - x^{d(d-1)}$$

and we can compute again the degeneracy locus

$$
\begin{aligned}
\partial_z p = 0 &\implies z^{d-2}(ux^{d-1}+z^{d-1})^{d-1} = 0 \\
&\implies z^{d-1}(ux^{d-1}+z^{d-1})^{d-1} = (ux^{d-1}-ux^{d-1})(ux^{d-1}+z^{d-1})^{d-1} \\
&\implies (ux^{d-1}+z^{d-1})^{d} = ux^{d-1}(ux^{d-1}+z^{d-1})^{d-1} \\
&\overset{p=0}{\implies} x^{d(d-1)} = ux^{d-1}(ux^{d-1}+z^{d-1})^{d-1} \\
&\implies x^{d^2(d-1)} = u^d x^{d(d-1)}(ux^{d-1}+z^{d-1})^{d(d-1)} \\
&\overset{p=0}{\implies} x^{d^2(d-1)} = u^d x^{d(d-1)} x^{d(d-1)^2} \\
&\implies x^{d^2(d-1)}(u^d-1) = 0
\end{aligned}
$$

This implies $u^d = 1$ since $x = 0$ is only a solution together with $z = 0$, which has no geometric meaning.

We consider now the image of our family under the restriction map:

$$
f_u \quad \mapsto \quad f_u(1,y,0) \quad = \quad y^d - duy + (d-1).
$$

The induced mapping on parameter spaces $\mathbf{C} \to \mathbf{A}_d$ is transversal to the discriminant, hence surjective on fundamental groups. Thus it suffices to find braids associated to a geometric basis of paths in the u-parameter plane punctured at $u^d = 1$.

For the radial path from $u = 0$ to a root $\xi_{j(d-1)}$ of $u^d = 0$ the degeneration is characterized by the following properties:

(i) the order of the punctures according to argument is preserved, which follows from the provable fact, that two punctures never have the same argument, ie. belong never to the same radial ray,

(ii) the punctures with index congruent to j mod d converge to infinity,

(iii) the trace of all other punctures remains bounded.

When u turns in a small circle around $\xi_{j(d-1)}$ most punctures move but very little. In contrast the punctures close to infinity turn on a large circle by the $(d-1)^{\text{th}}$ part of the full circle. With the radial contraction of u they retrace the movement of their d^{th} neighbour.

Any braid thus obtained is

$$
\sigma_1 \cdots \sigma_{(d-1)d-1}\, \sigma_{(d-1)d-1} \cdots \sigma_{(d-2)d+1}\, \sigma_{(d-2)d-1} \cdots \sigma_{(d-3)d+1} \cdots \sigma_{d-1} \cdots \sigma_1,
$$

or a conjugate of it by a power of $\sigma_{(d-1)d-1}\cdots\sigma_1$.

We are free to modify these braids by an element from Prop. 3, and we use this freedom to multiply with braids of the form $\sigma_j^{-1}\cdots\sigma_{j+d-2}^{-1}\sigma_{j+d-1}\sigma_{j+d-2}\cdots\sigma_j$ to get

$$\sigma_1\cdots\sigma_{(d-1)d-1}\,\sigma_{(d-1)d-1}\cdots\sigma_1,$$

and its conjugates by $\sigma_{(d-1)d-1}\cdots\sigma_1$.

These braids have to be conjugated as in the proof of Thm. 1 to get braids which fit with the braids of Thm. 1. To get to our claim we have thus to modify again. ■

6 Braid monodromies versus geometric monodromies

The topological analogue of a plane curve with simple branching along a preferred projection is a simply braided surface:

Definition 5 A *simply braided surface* is a submanifold of dimension two with boundary

$$(S,\partial S) \quad\subset\quad (D^2\times\mathbf{C},S^1\times\mathbf{C})$$

such that

(i) The induced projection $S\to D^2$ is a simple branched covering.

(ii) The induced projection $\partial S\to S^1$ is an unbranched covering.

Remark 4 By the Riemann-Hurwitz formula, the *branch set* $\mathbf{b}$ of branch points of $(S,\partial S)$ has cardinality $|\mathbf{b}|=d-e(S)$, where d is the degree of the covering.

Accordingly the range of our geometric monodromy will consist of mapping classes preserving the braided surface structure.

Definition 6 The *braided mapping class group* $\mathscr{M}(S)$ is the group of isotopy classes of orientation preserving diffeomorphisms of $D^2\times\mathbf{C}$ which

(i) preserve $(S,\partial S)$,

(ii) permute the fibres of $D^2\times\mathbf{C}\to D^2$,

(iii) preserve the fibres of $S^1\times\mathbf{C}\to S^1$,

(iv) are compactly supported.

In the case of versal unfoldings rel q_x of bivariate polynomials the zero set of a generic polynomial is naturally a simply braided surface.

In the case of projective plane curves we look at curves not in the degeneracy locus. The intersection of such a curve with the complement of a small tubular neighbourhood of the line at infinity determines a simply braided surface up to isomorphism. The same is true for families of such curves over a loop. The family of boundaries need not be trivial.

In any case we may define:

Definition 7 The *braided geometric monodromy* is defined on the fundamental group of the complement of the degeneracy locus and takes values in the braided mapping class group of

(i) zero set of a generic polynomial in the case of bivariate polynomials,

(ii) complement of a tubular neighbourhood of the line at infinity in the case of generic projective plane curves.

Since the magnitude of D^2 can be chosen large in comparison with the deformation parameters, we may deduce that the family of boundaries is trivialisable in the case of generic bivariate polynomials.

This observation serves well in the proof of the following comparison result.

Proposition 5 *The braided geometric monodromy group of versal unfolding rel q_x of a bivariate polynomial with isolated singularity is isomorphic to the bifurcation braid monodromy group.*

Proposition 6 *Given a versal unfolding rel q_x of a plane curve singularity. Then the following two monodromy groups are isomorphic:*

(i) the bifurcation braid monodromy group,

(ii) the braided geometric monodromy group.

Proof A representative of a braided mapping class induces a diffeomorphism of the base D^2 preserving the singular values $\mathbf{b}$. The braided mapping class thus determines a mapping class of the punctured base. Hence the bifurcation braid monodromy map factors through braided geometric monodromy map. In fact the bifurcation braid is naturally identified with the induced mapping class of $(D^2, \mathbf{b})$.

Conversely we note that a braid in the bifurcation braid monodromy determines a unique braided mapping class. On one hand it fixes an induced mapping class on $(D^2, \mathbf{b})$. On the other hand the map on the boundary is trivial. Hence there is a unique lift to the braided mapping class group. ∎

Remark 5 The same result is true in the case of plane projective curves, but the proof is more involved, since one has to take into account that the map on the boundary may vary. The crucial step is in fact to determine the map on the boundary from the braid.

There are obvious maps from the braided geometric monodromy of projection germs to the knotted geometric monodromy of plane curve singularities and further to the geometric monodromy.

Proposition 7 *The knotted geometric monodromy is injective for plane curve singularities of type A_n and D_n.*

Proof This follows immediately from [PV], since geometric monodromy factors through knotted geometric monodromy. ∎

We know of the failure of the geometric monodromy to be injective in general by the result of Wajnryb [W]. We also know of its failure to be surjective, see the result of Hirose [H] in the case of projective plane curves.

But there is hope that knotted geometric monodromy is better in the sense that injectivity and surjectivity hold true or fail at least to a lesser extend.

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A survey of Torelli and monodromy results for holomorphic-symplectic varieties

Eyal Markman

Abstract We survey recent results about the Torelli question for holomorphic-symplectic varieties. Following are the main topics. A Hodge theoretic Torelli theorem. A study of the subgroup W_{Exc}, of the isometry group of the weight 2 Hodge structure, generated by reflection with respect to exceptional divisors. A description of the birational Kähler cone as a fundamental domain for the W_{Exc} action on the positive cone. A proof of a weak version of Morrison's movable cone conjecture. A description of the moduli spaces of polarized holomorphic symplectic varieties as monodromy quotients of period domains of type IV.

Keywords Torelli Theorem, Holomorphic symplectic varieties, Moduli spaces, Movable cone
Mathematics Subject Classification (2010) 53C26, 14D20, 14J28, 32G20.

Contents

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1 Introduction

An *irreducible holomorphic symplectic manifold* is a simply connected compact Kähler manifold such that $H^0(X,\Omega^2_X)$ is one-dimensional, spanned by an everywhere non-degenerate holomorphic 2-form [Be1]. There exists a unique non-degenerate symmetric integral and primitive bilinear pairing $(\bullet,\bullet)$ on $H^2(X,\mathbb{Z})$ of signature $(3,b_2(X)-3)$, with the following property. There exists a positive rational number λ_X, such that the equality

$$(\alpha,\alpha)^n = \lambda_X \int_X \alpha^{2n}$$

holds for all $\alpha \in H^2(X,\mathbb{Z})$, where $2n = \dim_{\mathbb{C}}(X)$ [Be1]. If $b_2(X) = 6$, then we require[1] further that $(\alpha,\alpha) > 0$, for every Kähler class α. The pairing is called the *Beauville-Bogomolov pairing* and (α,α) is called the *Beauville-Bogomolov degree* of the class α.

Let S be a $K3$ surface. Then the Hilbert scheme (or Douady space, in the Kähler case) $S^{[n]}$, of length n zero-dimensional subschemes of S, is an irreducible holomorphic symplectic manifold. If $n \geq 2$, then $b_2(S^{[n]}) = 23$ [Be1]. If X is deformation equivalent to $S^{[n]}$, we will say that X is of $K3^{[n]}$*-type.*

Let T be a complex torus with an origin $0 \in T$. Denote by $T^{(n)}$ the n-th symmetric product. Let $T^{(n)} \to T$ be the addition morphism. The composite morphism

$$T^{[n+1]} \longrightarrow T^{(n+1)} \longrightarrow T$$

is an isotrivial fibration. Each fiber is a $2n$-dimensional irreducible holomorphic symplectic manifold, called a *generalized Kummer variety*, and denoted by $K^{[n]}(T)$ [Be1]. If $n \geq 2$, then $b_2\left(K^{[n]}(T)\right) = 7$.

O'Grady constructed two additional irreducible holomorphic symplectic manifolds, a 10-dimensional example X with $b_2(X) = 24$, and a 6-dimensional example Y with $b_2(Y) = 8$ [O'G2, O'G3, R].

We recommend Huybrechts' excellent survey of the subject of irreducible holomorphic symplectic manifolds [Hu3]. The aim of this note is to survey developments related to the Torelli problem, obtained by various authors since Huybrechts' survey was written. The most important, undoubtedly, is Verbitsky's proof of his version of the Global Torelli Theorem [Ver2, Hu6].

1.1 Torelli Theorems

We hope to convince the reader that the concepts of monodromy and parallel-transport operators are essential for any discussion of the Torelli problem.

Definition 1.1 Let X, X_1, and X_2 be irreducible holomorphic symplectic manifolds.

[1] The condition is satisfied automatically by the assumption that the signature is $(3, b_2(X) - 3)$, if $b_2 \neq 6$.

(1) An isomorphism $f : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ is said to be a *parallel-transport operator*, if there exist a smooth and proper family[2] $\pi : \mathscr{X} \to B$ of irreducible holomorphic symplectic manifolds, over an analytic base B, points $b_i \in B$, isomorphisms $\psi_i : X_i \to \mathscr{X}_{b_i}$, $i = 1,2$, and a continuous path $\gamma : [0,1] \to B$, satisfying $\gamma(0) = b_1$, $\gamma(1) = b_2$, such that the parallel transport in the local system $R\pi_*\mathbb{Z}$ along γ induces the homomorphism $\psi_{2_*} \circ f \circ \psi_1^* : H^*(\mathscr{X}_{b_1},\mathbb{Z}) \to H^*(\mathscr{X}_{b_2},\mathbb{Z})$. An isomorphism $g : H^k(X_1,\mathbb{Z}) \to H^k(X_2,\mathbb{Z})$ is said to be a *parallel-transport operator*, if it is the k-th graded summand of a parallel-transport operator f as above.

(2) An automorphism $f : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ is said to be a *monodromy operator*, if it is a parallel transport operator.

(3) The *monodromy group* $Mon(X)$ is the subgroup[3] of $GL[H^*(X,\mathbb{Z})]$ consisting of all monodromy operators. We denote by $Mon^2(X)$ the image of $Mon(X)$ in $O[H^2(X,\mathbb{Z})]$.

(4) Let H_i be an ample line bundle on X_i, $i = 1,2$. An isomorphism $f : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is said to be a *polarized parallel-transport operator* from (X_1,H_1) to (X_2,H_2), if there exists a family $\pi : \mathscr{X} \to B$, satisfying all the properties of part (1), as well as a flat section h of $R^2\pi_*\mathbb{Z}$, such that $h(b_i) = \psi_{i_*}(c_1(H_i))$, $i = 1,2$, and $h(b)$ is an ample class in $H^{1,1}(\mathscr{X}_b,\mathbb{Z})$, for all $b \in B$.

(5) Given an ample line bundle H on X, we denote by $Mon(X,H)$ the subgroup of $Mon(X)$, consisting of polarized parallel transport operators from (X,H) to itself. Elements of $Mon(X,H)$ will be called *polarized monodromy operators* of (X,H).

Following is a necessary condition for an isometry $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ to be a parallel transport operator. Denote by $\widetilde{\mathscr{C}}_X \subset H^2(X,\mathbb{R})$ the cone

$$\{\alpha \in H^2(X,\mathbb{R}) \,:\, (\alpha,\alpha) > 0\}.$$

Then $H^2(\widetilde{\mathscr{C}}_X,\mathbb{Z}) \cong \mathbb{Z}$ and it comes with a canonical generator, which we call the *orientation class* of $\widetilde{\mathscr{C}}_X$ (section 4). Any isometry $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ induces an isomorphism $\bar{g} : \widetilde{\mathscr{C}}_X \to \widetilde{\mathscr{C}}_Y$. The isometry g is said to be *orientation preserving* if

[2] Note that the family may depend on the isomorphism f.

[3] If $f \in Mon(X)$ is associated to a family $\pi' : \mathscr{X}' \to B'$ via an isomorphism $X \cong \mathscr{X}'_{b'}$, and $g \in Mon(X)$ is associated to a family $\pi'' : \mathscr{X}'' \to B''$ via an isomorphism $X \cong \mathscr{X}''_{b''}$, then fg is easily seen to be associated to the family $\pi : \mathscr{X} \to B$, obtained by "gluing" $\mathscr{X}'$ and $\mathscr{X}''$ via the isomorphism $\mathscr{X}'_{b'} \cong X \cong \mathscr{X}''_{b''}$ and connecting B' and B'' at the points b' and b'' to form the (reducible) base B.

$\bar{g}$ is. A parallel transport operator $g: H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ is orientation preserving. When X and Y are $K3$ surfaces, every orientation preserving isometry is a parallel transport operator. This is no longer the case for higher dimensional irreducible holomorphic symplectic varieties [Ma5, Nam2]. A necessary and sufficient criterion for an isometry to be a parallel transport operator is provided in the $K3^{[n]}$-type case, for all $n \geq 1$ (Theorem 9.8).

A *marked pair* (X,η) consists of an irreducible holomorphic symplectic manifold X and an isometry $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ onto a fixed lattice Λ. Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of isomorphism classes of marked pairs (see section 2). There exists a surjective period map $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ onto a period domain ([Hu1], Theorem 8.1). Each point $p \in \Omega_\Lambda$ determines a weight 2 Hodge structure on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, such that the marking η is an isomorphism of Hodge structures. The positive cone $\mathscr{C}_X$ of X is the connected component of the cone $\{\alpha \in H^{1,1}(X,\mathbb{R}) \ : \ (\alpha,\alpha) > 0\}$, containing the Kähler cone $\mathscr{K}_X$. Following is a concise version of the *Global Torelli Theorem* ([Ver2], or Theorem 2.2 below).

Theorem 1.2 *If $P_0(X,\eta) = P_0(\widetilde{X},\tilde{\eta})$, then X and $\widetilde{X}$ are bimeromorphic. A pair (X,η) is the unique point in a fiber of P_0, if and only if $\mathscr{K}_X = \mathscr{C}_X$. This is the case, for example, if the sublattice $H^{1,1}(X,\mathbb{Z})$ is trivial, or of rank 1, generated by an element λ, with $(\lambda,\lambda) \geq 0$.*

The following theorem combines the Global Torelli Theorem with results on the Kähler cone of irreducible holomorphic symplectic manifolds [Hu2, Bou1].

Theorem 1.3 *(A Hodge theoretic Torelli theorem) Let X and Y be irreducible holomorphic symplectic manifolds, which are deformation equivalent.*

(1) X and Y are bimeromorphic, if and only if there exists a parallel transport operator $f: H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, which is an isomorphism of integral Hodge structures.

(2) Let $f: H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ be a parallel transport operator, which is an isomorphism of integral Hodge structures. There exists an isomorphism $\tilde{f}: X \to Y$, such that $f = \tilde{f}_*$, if and only if f maps some Kähler class on X to a Kähler class on Y.

The theorem is proven in section 3.2. It generalizes the Strong Torelli Theorem of Burns and Rapoport [BR] or ([LP], Theorem 9.1).

Given a bimeromorphic map $f: X \to Y$, of irreducible holomorphic symplectic manifolds, denote by $f_*: H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ the homomorphism induced by the closure in $X \times Y$ of the graph of f. The homomorphism f_* is known to be an isometry ([O'G1], Proposition 1.6.2). Set $f^* := (f^{-1})_*$.

The *birational Kähler cone* $\mathscr{BK}_X$ of X is the union of the cones $f^*\mathscr{K}_Y$, as f ranges through all bimeromorphic maps from X to irreducible holomorphic symplectic manifolds Y. Let $Mon^2_{Hdg}(X)$ be the subgroup of $Mon^2(X)$ preserving the Hodge structure. Results of Boucksom and Huybrechts, on the Kähler and birational Kähler cones, are surveyed in section 5. We use them to define a chamber decomposition of the positive cone $\mathscr{C}_X$, via $Mon^2_{Hdg}(X)$-translates of cones of the form $f^*\mathscr{K}_Y$ (Lemma 5.11). These chambers are said to be of *Kähler type*.

Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of marked pairs. A detailed form of the Torelli theorem provides a description of $\mathfrak{M}^0_\Lambda$ as a moduli space of Hodge theoretic data as follows. A point $p \in \Omega_\Lambda$ determines a Hodge structure on Λ, and so a real subspace $\Lambda^{1,1}(p,\mathbb{R})$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, such that a marking η restricts to an isometry $H^{1,1}(X,\mathbb{R}) \to \Lambda^{1,1}(p,\mathbb{R})$, for every pair (X,η) in the fiber $P_0^{-1}(p)$.

Theorem 1.4 *(Theorem 5.16) The map $(X,\eta) \mapsto \eta(\mathscr{K}_X)$ establishes a one-to-one correspondence between points (X,η) in the fiber $P_0^{-1}(p)$ and chambers in the Kähler type chamber decomposition of the positive cone in $\Lambda^{1,1}(p,\mathbb{R})$.*

1.2 The fundamental exceptional chamber

The next few results are easier to understand when compared to the following basic fact about $K3$ surfaces. Let S be a $K3$ surface and κ_0 a Kähler class on S. The effective cone in $H^{1,1}(S,\mathbb{Z})$ is spanned by classes α, such that $(\alpha,\alpha) \geq -2$, and $(\alpha,\kappa_0) > 0$ ([BHPV], Ch. VIII Proposition 3.6). Set[4]

$$\mathcal{S}pe := \{e \in H^{1,1}(S,\mathbb{Z}) \ : \ (\kappa_0,e) > 0, \text{ and } (e,e) = -2\},$$
$$\mathcal{P}ex := \{[C] \in H^{1,1}(S,\mathbb{Z}) \ : \ C \subset S \text{ is a smooth connected rational curve}\}.$$

Clearly, $\mathcal{P}ex$ is contained in $\mathcal{S}pe$. Then the Kähler cone admits the following two characterizations ([BHPV], Ch. VIII Proposition 3.7 and Corollary 3.8).

$$\mathscr{K}_S = \{\kappa \in \mathscr{C}_S \ : \ (\kappa,e) > 0, \text{ for all } e \in \mathcal{S}pe\}. \tag{1.1}$$
$$\mathscr{K}_S = \{\kappa \in \mathscr{C}_S \ : \ (\kappa,e) > 0, \text{ for all } e \in \mathcal{P}ex\}. \tag{1.2}$$

[4] $\mathcal{P}ex$ stands for prime exceptional classes, and $\mathcal{S}pe$ stands for stably prime exceptional classes, as will be explained below.

Equality (1.1) is the simpler one, depending only on the Hodge structure and the intersection pairing. Equality (1.2) expresses the fact that a class $e \in \mathcal{S}pe$ represents a smooth rational curve, if and only if $\overline{\mathscr{K}}_S \cap e^{\perp}$ is a co-dimension one face of the closure of $\mathscr{K}_S$ in $\mathscr{C}_S$.

Let X be a projective irreducible holomorphic symplectic manifold. A *prime exceptional divisor* on X is a reduced and irreducible effective divisor E of negative Beauville-Bogomolov degree. The *fundamental exceptional chamber* of the positive cone is the set

$$\mathscr{F}\mathscr{E}_X \;:=\; \{\alpha \in \mathscr{C}_X \;:\; (\alpha, [E]) > 0, \text{ for every prime exceptional divisor } E\}. \quad (1.3)$$

When X is a $K3$ surface, a prime exceptional divisor is simply a smooth rational curve. Furthermore, the cones $\mathscr{K}_X$, $\mathscr{B}\mathscr{K}_X$, and $\mathscr{F}\mathscr{E}_X$ are equal. If $\dim(X) > 2$, the cone $\mathscr{B}\mathscr{K}_X$ need not be convex. The following is thus a generalization of equality (1.2) in the $K3$ surface case.

Theorem 1.5 *(Theorem 6.17 and Proposition 5.6) $\mathscr{F}\mathscr{E}_X$ is an open cone, which is the interior of a closed generalized convex polyhedron in $\mathscr{C}_X$ (Definition 6.13). The birational Kähler cone $\mathscr{B}\mathscr{K}_X$ is a dense open subset of $\mathscr{F}\mathscr{E}_X$.*

Let E be a prime exceptional divisor on a projective irreducible holomorphic symplectic manifold X. In section 6 we recall that the reflection

$$R_E \;:\; H^2(X,\mathbb{Z}) \;\longrightarrow\; H^2(X,\mathbb{Z}),$$

given by $R_E(\alpha) := \alpha - \frac{2(\alpha,[E])}{([E],[E])}[E]$, is an element of $Mon^2_{Hdg}(X)$ ([Ma7], Corollary 3.6, or Proposition 6.2 below). Let $W_{Exc}(X) \subset Mon^2_{Hdg}(X)$ be the subgroup generated[5] by the reflections R_E, of all prime exceptional divisors in X. In section 6.4 we prove the following analogue of a well known result for $K3$ surfaces ([BHPV], Ch. VIII, Proposition 3.9).

Theorem 1.6 *$W_{Exc}(X)$ is a normal subgroup of $Mon^2_{Hdg}(X)$. Let X_1 and X_2 be projective irreducible holomorphic symplectic manifolds and $f : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ a parallel-transport operator, which preserves the weight 2 Hodge structure. Then there exists a unique element $w \in W_{Exc}(X_2)$ and a birational map $g : X_1 \to X_2$, such that $f = w \circ g_*$. The map g is determined uniquely, up to composition with an automorphism of X_1, which acts trivially on $H^2(X_1,\mathbb{Z})$.*

[5] Definition 6.8 of W_{Exc} is different. The two definitions will be shown to be equivalent in Theorem 6.18.

Let us emphasis the special case $X_1 = X_2 = X$ of the theorem. Denote by $Mon^2_{Bir}(X) \subset O[H^2(X,\mathbb{Z})]$ the subgroup of isometries induced by birational maps from X to itself. Then $Mon^2_{Hdg}(X)$ is the semi-direct product of $W_{Exc}(X)$ and $Mon^2_{Bir}(X)$, by Theorem 6.18 part 5. Theorem 1.6 is proven in section 6.4. The proof relies on a second $Mon^2_{Hdg}(X)$-equivariant chamber decomposition of the positive cone $\mathscr{C}_X$. We call these the *exceptional chambers* (Definition 5.10). $W_{Exc}(X)$ acts simply-transitively on the set of exceptional chambers, one of which is the fundamental exceptional chamber. The walls of a general exceptional chamber are hyperplanes orthogonal to classes of *stably prime-exceptional* line bundles. The latter are higher-dimensional analogues of effective line bundles of degree -2 on a $K3$ surface. Roughly, a line bundle L on X is stably prime-exceptional, if a generic small deformation (X',L') of (X,L) satisfies $L' \cong \mathscr{O}_{X'}(E')$, for a prime exceptional divisor E' on X' (Definition 6.4).

Let X be a projective irreducible holomorphic symplectic manifold. Denote by $\mathrm{Bir}(X)$ the group of birational self-maps of X. The intersection of $\mathscr{F}\mathscr{E}_X$ with the subspace $H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is equal to the interior of the movable cone of X (Definition 6.21 and Lemma 6.22). We prove a weak version of Morrison's movable cone conjecture, about the existence of a rational convex polyhedron, which is a fundamental domain for the action of $\mathrm{Bir}(X)$ on the movable cone (Theorem 6.25). We use it to prove the following result. When X is a $K3$ surface, $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. Hence the following is an analogue of a result of Looijenga and Sterk ([St], Proposition 2.6).

Theorem 1.7 *For every integer $d \neq 0$, the number of $Bir(X)$-orbits of complete linear systems, which contain an irreducible divisor of Beauville-Bogomolov degree d, is finite. For every positive integer k there is only a finite number of $Bir(X)$-orbits of complete linear systems, which contain some irreducible divisor D of Beauville-Bogomolov degree zero, such that the class $[D]$ is k times a primitive class in $H^2(X,\mathbb{Z})$.*

Theorem 1.7 is proven in section 6.5. The proof follows an argument of Looijenga and Sterk, adapted via an analogy between results on the ample cone of a projective $K3$ surface and results on the movable cone of a projective irreducible holomorphic symplectic manifold.

The following is an analogue of the characterization of the Kähler cone of a $K3$ surface given in equation (1.1).

Proposition 1.8 *(Proposition 6.10) The fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, defined in equation (1.3), is equal to the set*

$$\{\alpha \in \mathscr{C}_X \;:\; (\alpha, \ell) > 0, \textit{ for every stably prime exceptional class } \ell\}.$$

The significance of Proposition 1.8 stems from the fact that one has an explicit characterization of the set of stably prime-exceptional classes, in terms of the weight 2 Hodge structure and a certain discrete monodromy invariant, at least in the $K3^{[n]}$-type case (Theorem 9.17). Theorem 1.5 and Proposition 1.8 thus yield an explicit description of the closure of the birational Kähler cone and of the movable cone.

1.3 Torelli and monodromy in the polarized case

In sections 7 and 8 we consider Torelli-type results for polarized irreducible holomorphic symplectic manifolds. Another corollary of the Global Torelli Theorem is the following.

Proposition 1.9 *$Mon^2(X,H)$ is equal to the stabilizer of $c_1(H)$ in $Mon^2(X)$.*

The above proposition is proven in section 7 (see Corollary 7.4).

Coarse moduli spaces of polarized projective irreducible holomorphic symplectic manifolds were constructed by Viehweg as quasi-projective varieties [Vieh]. Given a polarized pair (X,H) representing a point in such a coarse moduli space $\mathscr{V}$, the monodromy group $\Gamma := Mon^2(X,H)$ is an arithmetic group, which acts on a period domain $\mathscr{D}$ associated to $\mathscr{V}$. The quotient $\mathscr{D}/\Gamma$ is a quasi-projective variety [BB]. The following Theorem is a slight sharpening of Corollary 1.24 in [Ver2].

Theorem 1.10 *(Theorem 8.4) The period map $\mathscr{V} \to \mathscr{D}/\Gamma$ embeds each irreducible component $\mathscr{V}$, of the coarse moduli space of polarized irreducible holomorphic symplectic manifolds, as a Zariski open subset of the quasi-projective monodromy-quotient of the corresponding period domain.*

The above theorem provides a bridge between the powerful theory of modular forms, used to study the quotient spaces $\mathscr{D}/\Gamma$, and the theory of projective holomorphic symplectic varieties. The interested reader is referred to the excellent recent survey [GHS2] for further reading on this topic.

1.4 The $K3^{[n]}$-type

In section 9 we specialize to the case of varieties X of $K3^{[n]}$-type and review the results of [Ma2, Ma5, Ma7]. We introduce a Hodge theoretic *Torelli data*, consisting of the weight 2 Hodge structure of X and a certain discrete monodromy invariant (Corollary 9.5). We provide explicit computations, for many of the concepts introduced above, in terms of this Torelli data. We enumerate the connected components of the moduli space of marked pairs of $K3^{[n]}$-deformation type (Corollary 9.10). We determine the monodromy group $Mon^2(X)$, as well as a necessary and sufficient condition for an isometry $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ to be a parallel transport operator (Theorems 9.1 and 9.8). We provide a numerical characterization of the set of stably prime-exceptional line bundles on X (Theorem 9.17). The latter, combined with the general Theorem 1.5 and Proposition 1.8, determines the closure of the birational Kähler cone of X in terms of its Torelli data.

In section 10 we list a few open problems.

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2 The Global Torelli Theorem

Fix a positive integer $b_2 > 3$ and an even lattice Λ of signature $(3, b_2 - 3)$. Let X be an irreducible holomorphic symplectic manifold, such that $H^2(X,\mathbb{Z})$, endowed with its Beauville-Bogomolov pairing, is isometric to Λ. A *marking* for X is a choice of an isometry $\eta : H^2(X,\mathbb{Z}) \to \Lambda$. Two marked pairs (X_1,η_1), (X_2,η_2) are isomorphic, if there exists an isomorphism $f : X_1 \to X_2$, such that $\eta_1 \circ f^* = \eta_2$. There exists a coarse moduli space $\mathfrak{M}_\Lambda$ parametrizing isomorphism classes of marked pairs [Hu1]. $\mathfrak{M}_\Lambda$ is a smooth complex manifold of dimension $b_2 - 2$, but it is non-Hausdorff.

The *period*, of the marked pair (X,η), is the line $\eta[H^{2,0}(X)]$ considered as a point in the projective space $\mathbb{P}[\Lambda\otimes_{\mathbb{Z}}\mathbb{C}]$. The period lies in the period domain

$$\Omega_\Lambda := \{p : (p,p)=0 \quad \text{and} \quad (p,\bar{p})>0\}. \tag{2.1}$$

Ω_Λ is an open subset, in the classical topology, of the quadric in $\mathbb{P}[\Lambda\otimes\mathbb{C}]$ of isotropic lines [Be1]. The period map

$$\begin{aligned} P : \mathfrak{M}_\Lambda &\longrightarrow \Omega_\Lambda, \\ (X,\eta) &\mapsto \eta[H^{2,0}(X)] \end{aligned} \tag{2.2}$$

is a local isomorphism, by the Local Torelli Theorem [Be1].

Given a point $p\in\Omega_\Lambda$, set $\Lambda^{1,1}(p) := \{\lambda\in\Lambda : (\lambda,p)=0\}$. Note that $\Lambda^{1,1}(p)$ is a sublattice of Λ and $\Lambda^{1,1}(p)=(0)$, if p does not belong to the countable union of hyperplane sections $\cup_{\lambda\in\Lambda\setminus\{0\}}[\lambda^\perp\cap\Omega_\Lambda]$. Given a marked pair (X,η), we get the isomorphism $H^{1,1}(X,\mathbb{Z})\cong\Lambda^{1,1}(P(X,\eta))$, via the restriction of η.

Definition 2.1 Let X be an irreducible holomorphic symplectic manifold. The cone $\{\alpha\in H^{1,1}(X,\mathbb{R}) : (\alpha,\alpha)>0\}$ has two connected components. The *positive cone* $\mathscr{C}_X$ is the connected component containing the Kähler cone $\mathscr{K}_X$.

Two points x and y of a topological space M are *inseparable*, if every pair of open subsets U, V, with $x\in U$ and $y\in V$, have a non-empty intersection $U\cap V$. A point $x\in M$ is a *Hausdorff point*, if there does not exist any point $y\in[M\setminus\{x\}]$, such that x and y are inseparable.

Theorem 2.2 *(The Global Torelli Theorem) Fix a connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$.*

(1) ([Hu1], Theorem 8.1) The period map P restricts to a surjective holomorphic map $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$.

(2) ([Ver2], Theorem 1.16) The fiber $P_0^{-1}(p)$ consists of pairwise inseparable points, for all $p\in\Omega_\Lambda$.

(3) ([Hu1], Theorem 4.3) Let (X_1,η_1) and (X_2,η_2) be two inseparable points of $\mathfrak{M}_\Lambda$. Then X_1 and X_2 are bimeromorphic.

(4) The marked pair (X,η) is a Hausdorff point of $\mathfrak{M}_\Lambda$, if and only if $\mathscr{C}_X=\mathscr{K}_X$.

(5) The fiber $P_0^{-1}(p)$, $p\in\Omega_\Lambda$, consists of a single Hausdorff point, if $\Lambda^{1,1}(p)$ is trivial, or if $\Lambda^{1,1}(p)$ is of rank 1, generated by a class α satisfying $(\alpha,\alpha)\geq 0$.

Proof Part (4) of the theorem is due to Huybrechts and Verbitsky. See Proposition 5.14 for a more general description of the fiber $P_0^{-1}[P_0(X,\eta)]$ in terms of the Kähler-type chamber decomposition of the positive cone $\mathscr{C}_X$, and for further details about part (4).

Part (5): $\mathscr{C}_X = \mathscr{K}_X$, if $H^{1,1}(X,\mathbb{Z})$ is trivial, or if $H^{1,1}(X,\mathbb{Z})$ is of rank 1, generated by a class α of non-negative Beauville-Bogomolov degree, by ([Hu1], Corollaries 5.7 and 7.2). The statement of part (5) now follows from part (4). □

Remark 2.3 Verbitsky states part (2) of Theorem 2.2 for a connected component of the Teichmüller space, but Theorem 1.16 in [Ver2] is a consequence of the two more general Theorems 4.22 and 6.14 in [Ver2], and both the Teichmüller space and the moduli space of marked pairs $\mathfrak{M}_\Lambda$ satisfy the hypothesis of these theorems. A complete proof of part (2) of Theorem 2.2 can be found in Huybrechts excellent Bourbaki seminar paper [Hu6].

3 The Hodge theoretic Torelli Theorem

In section 3.1 we review two theorems of Huybrechts, which relate bimeromorphic maps and parallel-transport operators. The Hodge theoretic Torelli Theorem 1.3 is proven in section 3.2.

3.1 Parallel transport operators between inseparable marked pairs

Let X_1 and X_2 be two irreducible holomorphic symplectic manifolds of dimension $2n$. Denote by π_i the projection from $X_1 \times X_2$ onto X_i, $i = 1,2$. Given a correspondence Z in $X_1 \times X_2$, of pure complex co-dimension $2n+d$, denote by $[Z]$ the cohomology class Poincaré dual to Z and by $[Z]_* : H^*(X_1) \to H^{*+2d}(X_2)$ the homomorphism defined by $[Z]_*\alpha := \pi_{2_*}(\pi_1^*(\alpha) \cup [Z])$. The following are two fundamental results of Huybrechts.

Assume that X_1 and X_2 are bimeromorphic. Denote the graph of a bimeromorphic map by $Z \subset X_1 \times X_2$.

Theorem 3.1 *([Hu2], Corollary 2.7) There exists an effective cycle $\Gamma := Z + \sum Y_j$ in $X_1 \times X_2$, of pure dimension $2n$, with the following properties.*

(1) The correspondence $[\Gamma]_* : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ *is a parallel-transport operator.*

(2) The image $\pi_i(Y_j)$ *has codimension* ≥ 2 *in* X_i*, for all* j*. In particular, the correspondences* $[\Gamma]_*$ *and* $[Z]_*$ *coincide on* $H^2(X_1,\mathbb{Z})$*.*

Let (X_1,η_1), (X_2,η_2) be two marked pairs corresponding to inseparable points of $\mathfrak{M}_\Lambda$.

Theorem 3.2 *([Hu1], Theorem 4.3 and its proof) There exists an effective cycle* $\Gamma := Z + \sum_j Y_j$ *in* $X_1 \times X_2$*, of pure dimension* $2n$*, satisfying the following conditions.*

(1) Z *is the graph of a bimeromorphic map from* X_1 *to* X_2*.*

(2) The correspondence $[\Gamma]_* : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ *is a parallel-transport operator. Furthermore, the composition*

$$\eta_2^{-1} \circ \eta_1 : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$$

is equal to the restriction of $[\Gamma]_*$*.*

(3) ([Hu2], Theorem 2.5 and its proof) The codimensions of $\pi_1(Y_j)$ *in* X_1 *and of* $\pi_2(Y_j)$ *in* X_2 *are equal and positive.*

(4) If $\pi_i(Y_j)$ *has codimension* 1*, then it is supported by a uniruled divisor.*

The statement that the isomorphisms $[\Gamma]_*$ in Theorems 3.1 and 3.2 are parallel transport operators is implicit in Huybrechts proofs, so we clarify that point next. In each of the proofs Huybrechts shows that there exist two smooth and proper families $\mathscr{X} \to B$ and $\mathscr{X}' \to B$, over the same one-dimensional disk B, a point b_0 in B, isomorphisms $X_1 \cong \mathscr{X}_{b_0}$ and $X_2 \cong \mathscr{X}'_{b_0}$, and an isomorphism $\tilde{f} : \mathscr{X}_{|_{B\setminus\{b_0\}}} \to \mathscr{X}'_{|_{B\setminus\{b_0\}}}$, compatible with projections to B. The cycle $\Gamma \subset X_1 \times X_2$ is the fiber over b_0 of the closure in $\mathscr{X} \times_B \mathscr{X}'$ of the graph of $\tilde{f}$. Choose a point b_1 in $B \setminus \{b_0\}$ and let γ be a continuous path in B from b_0 to b_1. Let $g_1 : H^*(\mathscr{X}_{b_0},\mathbb{Z}) \to H^*(\mathscr{X}_{b_1},\mathbb{Z})$ and $g_2 : H^*(\mathscr{X}'_{b_0},\mathbb{Z}) \to H^*(\mathscr{X}'_{b_1},\mathbb{Z})$ be the two parallel transport operators along γ. Then the isomorphism $g_2^{-1} \circ g_1 : H^*(\mathscr{X}_{b_0},\mathbb{Z}) \to H^*(\mathscr{X}'_{b_0},\mathbb{Z})$ is induced by the correspondence $[\Gamma]_*$. Furthermore, $g_2^{-1} \circ g_1$ is a parallel transport operator, being a composition of such operators (parallet transport operators form a groupoid, by an argument similar to that used in footnote [3]).

The reader may wonder why the image in X_i of a component Y_j of Γ has codimension ≥ 2 in Theorem 3.1, while the codimension is only ≥ 1 in Theorem 3.2. The reason is that in the proof of Theorem 3.2 one does not have control on the choice of the above mentioned families $\mathscr{X}$ and $\mathscr{X}'$, beyond the condition that $\eta_2 \circ [\Gamma]_* = \eta_1$. In the proof of Theorem 3.1, given a bimeromorphic map $f : X_1 \to X_2$, Huybrechts constructs the above two families $\mathscr{X}$ and $\mathscr{X}'$ in such a way that the following two properties hold. (1) The bimeromorphic map $\tilde{f}$ from $\mathscr{X}$ to $\mathscr{X}'$ restricts to the bimeromorphic map f between the fibers X_1 and X_2 over b_0. (2) $[\Gamma]_*$ restricts to the isometry $f_* : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ (see Theorem 2.5 in [Hu2] and its proof).

3.2 *Proof of the Hodge theoretic Torelli Theorem 1.3*

Proof of part 1: If X and Y are bimeromorphic, then there exists a parallel-transport operator $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, which is an isomorphism of Hodge structures, by Theorem 3.1. Conversely, assume that such f is given. Let $\eta_Y : H^2(Y,\mathbb{Z}) \to \Lambda$ be a marking. Set $\eta_X := \eta_Y \circ f$. The assumption that f is a parallel transport operator implies that (X,η_X) and (Y,η_Y) belong to the same connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$. Both have the same period

$$P(X,\eta_X) = \eta_X(H^{2,0}(X)) = \eta_Y(f(H^{2,0}(X))) = \eta_Y(H^{2,0}(Y)) = P(Y,\eta_Y),$$

where the third equality follows from the assumption that f is an isomorphism of Hodge structures. Hence, (X,η_X) and (Y,η_Y) are inseparable points of $\mathfrak{M}^0_\Lambda$, by Theorem 2.2 part 2. X and Y are thus bimeromorphic, by Theorem 2.2 part 3.

Proof of part 2: Let η_X and η_Y be the markings constructed in the proof of part 1. Note that $f = \eta_Y^{-1} \circ \eta_X$. There exists an *effective* correspondence $\Gamma = Z + \sum_{i=1}^N W_i$ of pure dimension $2n$ in $X \times Y$, such that Z is the graph of a bimeromorphic map, W_i is irreducible, but not necessarily reduced, the images of the projections $W_i \to X$, $W_i \to Y$ have positive co-dimensions, and $[\Gamma]_* : H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ is a parallel transport operator, which is equal to f in degree 2, by Theorem 3.2 and the assumption that the two points (X,η_X) and (Y,η_Y) are inseparable.

Assume that $\alpha \in \mathscr{K}_X$ is a Kähler class, such that $f(\alpha)$ is a Kähler class. The relationship between f and Γ yields:

$$f(\alpha) \; = \; [\Gamma]_*(\alpha) \; = \; [Z]_*(\alpha) + \sum_{i=1}^N [W_i]_*(\alpha).$$

Each class $[W_i]_*(\alpha)$ is either zero or a multiple $c_i[D_i]$ of the class of a prime divisor D_i, where c_i is a positive[6] real number.

We prove next that $[W_i]_*(\alpha) = 0$, for $1 \leq i \leq N$. Write $f(\alpha) = [Z]_*(\alpha) + \sum_{i=1}^N c_i[D_i]$, where c_i are all positive real numbers, and D_i is either a prime divisor, or zero. Set $D := \sum_{i=1}^N c_i D_i$. We need to show that all D_i are equal to zero. The Beauville-Bogomolov degree of α satisfies

$$(\alpha,\alpha) = (f(\alpha), f(\alpha)) = ([Z]_*\alpha, [Z]_*\alpha) + 2\sum_{i=1}^N c_i([Z]_*\alpha, [D_i]) + ([D],[D]).$$

The homomorphism $[Z]_*$, induced by the graph of the bimeromorphic map, is an isometry, by [O'G1], Proposition 1.6.2 (also by the stronger Theorem 3.1). Furthermore, if D_i is non-zero, then D_i is the strict transform of a prime divisor D'_i on X, such that $[Z]_*([D'_i]) = [D_i]$. Set $D' := \sum_{i=1}^N c_i D'_i$. We get the equalities

$$([D],[D]) = -2(\alpha, [D']), \tag{3.1}$$

$$[D] = [Z]_*[D'], \tag{3.2}$$

and

$$([D], f(\alpha)) = ([D], [Z]_*\alpha) + ([D],[D]) \overset{(3.2)}{=} ([D'], \alpha) + ([D],[D]) \overset{(3.1)}{=} -(\alpha, [D']).$$

Now $(\alpha, [D'_i])$ is zero, if $D_i = 0$, and positive, if $D_i \neq 0$, since α is a Kähler class. Hence, the right hand side above is ≤ 0. The left hand side is ≥ 0, due to the assumption that the class $f(\alpha)$ is a Kähler class. Hence, $D'_i = 0$, for all i. We conclude that $[W_i]_*(\alpha) = 0$, for $1 \leq i \leq N$, as claimed.

The equality $[Z]_*(\alpha) = f(\alpha)$ was proven above. Consequently, Z is the graph of a bimeromorphic map, which maps a Kähler class to a Kähler class. Hence, Z is the graph of an isomorphism, by a theorem of Fujiki [F]. □

4 Orientation

Let Ω_Λ be the period domain (2.1). Following are two examples, in which spaces arise with two connected components.

(1) Fix a primitive class $h \in \Lambda$, with $(h,h) > 0$. The hyperplane section

[6] The coefficient c_i is positive since Γ is effective and α is a Kähler class.

$$\Omega_{h^\perp} \ := \ \Omega_\Lambda \cap h^\perp$$

has two connected components.

(2) Let $p \in \Omega_\Lambda$. Set $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Lambda^{1,1}(p,\mathbb{R}) := \{\lambda \in \Lambda_{\mathbb{R}} : (\lambda, p) = 0\}$. Then the cone $\mathscr{C}'_p := \{\lambda \in \Lambda^{1,1}(p,\mathbb{R}) \ : \ (\lambda,\lambda) > 0\}$ has two connected components.

We recall in this section that a connected component $\mathfrak{M}^0_\Lambda$, of the moduli space of marked pairs, determines a choice of a component of $\Omega_{h^\perp}$ and of $\mathscr{C}'_p$, for all $h \in \Lambda$, with $(h,h) > 0$, and for all $p \in \Omega_\Lambda$. Let us first relate the choice of one of the two components in the two examples above. The relation can be explained in terms of the following larger cone. Set

$$\widetilde{\mathscr{C}_\Lambda} := \{\lambda \in \Lambda_{\mathbb{R}} \ : \ (\lambda,\lambda) > 0\}.$$

A subspace $W \subset \Lambda_{\mathbb{R}}$ is said to be *positive*, if the pairing of $\Lambda_{\mathbb{R}}$ restricts to W as a positive definite pairing.

Lemma 4.1

(1) $H^2(\widetilde{\mathscr{C}_\Lambda},\mathbb{Z})$ *is a free abelian group of rank* 1.

(2) Let $e \in \Lambda$ *be an element with* $(e,e) \neq 0$ *and* $R_e : \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}$ *the reflection given by* $R_e(\lambda) = \lambda - \frac{2(e,\lambda)}{(e,e)}e$. *Then* R_e *acts on* $H^2(\widetilde{\mathscr{C}_\Lambda},\mathbb{Z})$ *by* -1, *if* $(e,e) > 0$, *and trivially if* $(e,e) < 0$.

(3) Let W *be a positive three dimensional subspace of* $\Lambda_{\mathbb{R}}$. *Then* $W \setminus \{0\}$ *is a deformation retract of* $\widetilde{\mathscr{C}_\Lambda}$.

Proof (3) Set $I := [0,1]$. We need to construct a continuous map $F : \widetilde{\mathscr{C}_\Lambda} \times I \to \widetilde{\mathscr{C}_\Lambda}$ satisfying

$$\begin{aligned} F(\lambda,0) &= \lambda, && \text{for all } \lambda \in \widetilde{\mathscr{C}_\Lambda}, \\ F(\lambda,1) &\in W \setminus \{0\}, && \text{for all } \lambda \in \widetilde{\mathscr{C}_\Lambda}, \\ F(w,t) &= w, && \text{for all } w \in W \setminus \{0\}. \end{aligned}$$

Choose a basis $\{e_1, e_2, e_3, \dots, e_{b_2}\}$ of $\Lambda_{\mathbb{R}}$, so that $\{e_1,e_2,e_3\}$ is a basis of W, and for $\lambda = \sum_{i=1}^{b_2} x_i e_i$, we have $(\lambda,\lambda) = x_1^2 + x_2^2 + x_3^2 - \sum_{i=4}^{b_2} x_i^2$. Then $\widetilde{\mathscr{C}_\Lambda}$ consists of λ satisfying $x_1^2 + x_2^2 + x_3^2 > \sum_{i=4}^{b_2} x_i^2$. Set $F\left(\sum_{i=1}^{b_2} x_i e_i, t\right) = \sum_{i=1}^{3} x_i e_i + (1-t)\sum_{i=4}^{b_2} x_i e_i$. Then F has the above properties of a deformation retract of $\widetilde{\mathscr{C}_\Lambda}$ onto $W \setminus \{0\}$.

Part (1) follows immediately from part (3).

(2) If $(e,e) > 0$, we can choose a positive 3 dimensional subspace W containing e, and if $(e,e) < 0$ we can choose W to be orthogonal to e. Then $W \setminus \{0\}$ is R_e invariant and R_e acts as stated on $H^2(W \setminus \{0\}, \mathbb{Z})$, hence also on $H^2(\widetilde{\mathscr{C}}_\Lambda, \mathbb{Z})$, by part (3). □

The character $H^2(\widetilde{\mathscr{C}}_\Lambda, \mathbb{Z})$ of $O(\Lambda)$ is known as the *spinor norm*.

A point $p \in \Omega_{h^\perp}$ determines the three dimensional positive definite subspace $W_p := \mathrm{Re}(p) \oplus \mathrm{Im}(p) \oplus \mathrm{span}_{\mathbb{R}}\{h\}$, which comes with an orientation associated to the basis $\{\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), h\}$, for some choice of a non-zero element $\sigma \in p \subset \Lambda_{\mathbb{C}}$. The orientation of the basis is independent of the choice of σ. Consequently, an element $p \in \Omega_{h^\perp}$ determines a generator of $H^2(\widetilde{\mathscr{C}}_\Lambda, \mathbb{Z})$. The two components of $\Omega_{h^\perp}$ are distinguished by the two generators of the rank 1 free abelian group $H^2(\widetilde{\mathscr{C}}_\Lambda, \mathbb{Z})$. We refer to each of the two generators as an *orientation class* of the cone $\widetilde{\mathscr{C}}_\Lambda$.

A point $\lambda \in \mathscr{C}'_p$ determines an orientation of $\widetilde{\mathscr{C}}_\Lambda$ as follows. Choose a class $\sigma \in p$. Again we get the three dimensional positive definite subspace $W_\lambda := \mathrm{Re}(p) \oplus \mathrm{Im}(p) \oplus \mathrm{span}_{\mathbb{R}}\{\lambda\}$, which comes with an orientation associated to the basis $\{\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), \lambda\}$. Consequently, λ determines an orientation of $\widetilde{\mathscr{C}}_\Lambda$. The orientation remains the same as λ varies in a connected component of $\mathscr{C}'_p$. Hence, a connected component of $\mathscr{C}'_p$ determines an orientation of $\widetilde{\mathscr{C}}_\Lambda$.

Let X be an irreducible holomorphic symplectic manifold. Recall that the positive cone $\mathscr{C}_X \subset H^{1,1}(X, \mathbb{R})$ is the distinguished connected component of the cone $\mathscr{C}'_X := \{\lambda \in H^{1,1}(X, \mathbb{R}) \ : \ (\lambda, \lambda) > 0\}$, which contains the Kähler cone (Definition 2.1). Denote by $\widetilde{\mathscr{C}}_X$ the positive cone in $H^2(X, \mathbb{R})$. We conclude that $\widetilde{\mathscr{C}}_X$ comes with a distinguished orientation.

Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of marked pairs and $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ the period map. A marked pair (X, η) in $\mathfrak{M}^0_\Lambda$ determines an orientation of $\widetilde{\mathscr{C}}_\Lambda$, via the isomorphism $\widetilde{C}_X \cong \widetilde{\mathscr{C}}_\Lambda$ induced by the marking η. This orientation of $\widetilde{\mathscr{C}}_\Lambda$ is constant throughout the connected component $\mathfrak{M}^0_\Lambda$. In particular, for each class $h \in \Lambda$, with $(h,h) > 0$, we get a choice of a connected component

$$\Omega^+_{h^\perp} \tag{4.1}$$

of $\Omega_{h^\perp}$, compatible with the orientation of $\widetilde{\mathscr{C}}_\Lambda$ induced by $\mathfrak{M}^0_\Lambda$.

Let $\mathrm{Orient}(\Lambda)$ be the set of two orientations of the positive cone $\widetilde{\mathscr{C}}_\Lambda$. Let

$$\mathrm{orient} \ : \ \mathfrak{M}_\Lambda \quad \to \quad \mathrm{Orient}(\Lambda) \tag{4.2}$$

be the natural map constructed above.

5 A modular description of each fiber of the period map

We provide a modular description of the fiber of the period map $\mathfrak{M}_\Lambda^0 \to \Omega_\Lambda$ from a connected component $\mathfrak{M}_\Lambda^0$ of the moduli space of marked pairs (Theorem 5.16). Throughout this section X is an irreducible holomorphic symplectic manifold, which need not be projective.

5.1 Exceptional divisors

A reduced and irreducible effective divisor $D \subset X$ will be called a *prime* divisor.

Definition 5.1

(1) A set $\{E_1, \ldots, E_r\}$ of prime divisors is *exceptional*, if and only if its Gram matrix $(([E_i],[E_j]))_{ij}$ is negative definite.

(2) An effective divisor E is *exceptional*, if the support of E is an exceptional set of prime divisors.

Definition 5.2 The *fundamental exceptional chamber* $\mathscr{F}\mathscr{E}_X$ is the cone of classes α, such that $\alpha \in \mathscr{C}_X$, and $(\alpha, [E]) > 0$, for every prime exceptional divisor E.

5.1.1 The fundamental exceptional chamber versus the birational Kähler cone

Huybrechts and Boucksom stated an important result (Theorem 5.4 below) in terms of another chamber, which we introduce next.

Definition 5.3 ([Bou2], Section 4.2.2)

(1) A *rational effective* 1-*cycle* C is a linear combination, with positive integral coefficients, of irreducible rational curves on X.

(2) A *uniruled divisor* D is an effective divisor each of which irreducible components D_i is covered by rational curves.

(3) The *fundamental uniruled chamber* $\mathscr{F}\mathscr{U}_X$ is the subset of $\mathscr{C}_X$ consisting of classes $\alpha \in \mathscr{C}_X$, such that $(\alpha, D) > 0$, for every non-zero uniruled divisor D.

(4) The *birational Kähler cone* $\mathscr{BK}_X$ of X is the union of $f^*\mathscr{K}_Y$, as f ranges over all bimeromorphic maps $f : X \to Y$ to an irreducible holomorphic symplectic manifold Y.

Note that the birational Kähler cone is not convex in general.

Theorem 5.4 *([Hu2] and [Bou2], Theorem 4.3)*

(1) The Kähler cone $\mathscr{K}_X$ is equal to the subset of $\mathscr{C}_X$ consisting of classes $\alpha \in \mathscr{C}_X$, such that $\int_C \alpha > 0$, for every non-zero rational effective 1-cycle C.

(2) Let $\alpha \in \mathscr{C}_X$ be a class, such that $\int_C \alpha \neq 0$, for every rational 1-cycle. Then α belongs to $\mathscr{FU}_X$, if and only if α belongs to the birational Kähler cone $\mathscr{BK}_X$.

(3) ([Bou2], Theorem 4.3 part ii, and [Hu1], Corollary 5.2) Let $\alpha \in \mathscr{C}_X$ be a class, which does not belong to $\mathscr{FU}_X$. Assume that $\int_C \alpha \neq 0$, for every rational 1-cycle. Then there exists an irreducible holomorphic symplectic manifold Y, and a bimeromorphic map $f : X \to Y$, such that $f_*(\alpha) = \beta + D'$, where β is a Kähler class on Y and D' is a non-zero linear combination of finitely many uniruled reduced and irreducible divisors with positive real coefficients.

Remark 5.5 Let X be an irreducible holomorphic symplectic manifold. Part (2) of the theorem asserts that if a class α satisfies the assumptions stated, then α is contained in $\mathscr{FU}_X$, if and only if it is contained in $\mathscr{BK}_X$. The 'only if' direction of part (2) is stated in ([Bou2], Theorem 4.3). The 'if' part is the obvious direction. Indeed, let $f : X \to Y$ be a birational map, such that $f_*(\alpha)$ is a Kähler class on Y. Let D be an effective uniruled reduced and irreducible divisor in X, and D' its strict transform in Y. We have $([D], \alpha) = ([D'], f_*(\alpha)) > 0$. Hence, α is in the fundamental uniruled chamber.

Let $\overline{\mathscr{BK}}_X$ be the closure of the birational Kähler cone $\mathscr{BK}_X$ in $\mathscr{C}_X$.

Proposition 5.6 *The following inclusions and equality hold:*

$$\mathscr{BK}_X \subset \mathscr{FU}_X = \mathscr{FE}_X \subset \overline{\mathscr{BK}}_X.$$

Proof An exceptional divisor is uniruled, by ([Bou2], Proposition 4.7). The inclusion $\mathscr{FU}_X \subset \mathscr{FE}_X$ follows. We prove next the inclusion $\mathscr{FE}_X \subset \mathscr{FU}_X$. Let α be a class in $\mathscr{FE}_X$ and D a prime uniruled divisor. If $[D]$ belongs to the closure $\overline{\mathscr{C}}_X$ of the positive cone, then $(\alpha, [D]) > 0$, since α belongs to $\mathscr{C}_X$. Otherwise, $[D]$ is a prime exceptional divisor, and so $(\alpha, [D]) > 0$. The inclusion $\mathscr{FE}_X \subset \mathscr{FU}_X$ follows.

The inclusion $\mathscr{BK}_X \subset \mathscr{FU}_X$ follows from the 'if' direction of Theorem 5.4 part 2, and the inclusion $\mathscr{FE}_X \subset \overline{\mathscr{BK}}_X$ follows from the 'only if' direction. □

The notation $\mathscr{FE}_X$ will replace $\mathscr{FU}_X$ from now on, in view of Proposition 5.6. A class $\alpha \in \mathscr{C}_X$ is said to be *very general*, if $\alpha^\perp \cap H^{1,1}(X,\mathbb{Z}) = 0$.

Corollary 5.7 *Let X_1 and X_2 be irreducible holomorphic symplectic manifolds, $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ a parallel transport operator, which is an isomorphism of Hodge structures, and $\alpha_1 \in \mathscr{FE}_{X_1}$ a very general class. Then $g(\alpha_1)$ belongs to $\mathscr{FE}_{X_2}$, if and only if there exists a bimeromorphic map $f : X_1 \to X_2$, such that $g = f_*$.*

Proof The 'if' part is clear, since f_* induces a bijection between the sets of exceptional divisors on X_i, $i = 1,2$. Set $\alpha_2 := g(\alpha_1)$. There exist irreducible holomorphic symplectic manifolds Y_i and bimeromorphic maps $f_i : X_i \to Y_i$, such that $f_{i_*}(\alpha_i)$ is a Kähler class on Y_i, by part (2) of Theorem 5.4. The homomorphisms $f_{i_*} : H^2(X_i,\mathbb{Z}) \to H^2(Y_i,\mathbb{Z})$ are parallel transport operators, by Theorem 3.1. Thus $(f_2^{-1})^* \circ g \circ f_1^* : H^2(Y_1,\mathbb{Z}) \to H^2(Y_2,\mathbb{Z})$ is a parallel transport operator and a Hodge-isometry, mapping the Kähler class $f_{1_*}(\alpha_1)$ to the the Kähler class $f_{2_*}(\alpha_2)$. Hence, there exists an isomorphism $h : Y_1 \to Y_2$, such that $h_* = (f_2^{-1})^* \circ g \circ f_1^*$, by Theorem 1.3. Thus, $g = [(f_2)^{-1} h f_1]_*$. □

5.1.2 The divisorial Zariski decomposition

The following fundamental result of Bouksom will be needed in section 6.2. The *effective cone* of X is the cone in $H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by the classes of effective divisors. The *algebraic pseudo-effective cone* $\mathscr{P}\mathrm{eff}_X$ is the closure of the effective cone. Boucksom defines a larger transcendental analogue, a cone in $H^{1,1}(X,\mathbb{R})$, which he calls the *pseudo-effective cone* ([Bou2], section 2.3). We will not need the precise definition, but only the fact that the pseudo-effective cone contains $\mathscr{C}_X$ ([Bou2], Theorem 4.3 part (i)). The sum $\mathscr{C}_X + \mathscr{P}\mathrm{eff}_X$ is thus a sub-cone of Boucksom's pseudo-effective cone in $H^{1,1}(X,\mathbb{R})$. Denote by $\overline{\mathscr{FE}_X}$ the closure of the fundamental exceptional chamber in $H^{1,1}(X,\mathbb{R})$.

Theorem 5.8

(1) ([Bou2], Theorem 4.3 part (i), Proposition 4.4, and Theorem 4.8). Let X be an irreducible holomorphic symplectic manifold and α a class in $\mathscr{C}_X + \mathscr{P}\mathrm{eff}_X$. Then there exists a unique decomposition

$$\alpha = P(\alpha) + N(\alpha),$$

where $(P(\alpha), N(\alpha)) = 0$, $P(\alpha)$ *belongs to* $\overline{\mathscr{F}\mathscr{E}}_X$, *and* $N(\alpha)$ *is an exceptional* $\mathbb{R}$-*divisor.*

(2) *([Bou2], Corollary 4.11). Let* L *be a line bundle with* $c_1(L) \in \mathscr{C}_X + \mathscr{P}\mathrm{eff}_X$. *Set* $\alpha := c_1(L)$. *Then the classes* $P(\alpha)$ *and* $N(\alpha)$ *correspond to* $\mathbb{Q}$-*divisors classes, which we denote by* $P(\alpha)$ *and* $N(\alpha)$ *as well. Furthermore, the homomorphism*

$$H^0(X, \mathscr{O}_X(kP(\alpha))) \to H^0(X, L^k)$$

is surjective, for every non-negative integer k, *such that* $kP(\alpha)$ *is an integral class.*

Remark 5.9 The class $P(\alpha)$ is stated as a class in the *modified nef cone* in ([Bou2], Theorem 4.8), but the modified nef cone is equal to the closure of the birational Kähler cone, by ([Bou2], Proposition 4.4), and hence also to $\overline{\mathscr{F}\mathscr{E}}_X$.

Part (2) of the above Theorem implies that the exceptional divisor $N(kc_1(L))$ is the fixed part of the linear system $|L^k|$. In particular, if $c_1(L) = N(c_1(L))$, then the linear system $|L^k|$ is either empty, or consists of a single exceptional divisor. Exceptional divisors are thus rigid.

5.2 A Kähler-type chamber decomposition of the positive cone

Let X be an irreducible holomorphic symplectic manifold. Denote the subgroup of $Mon^2(X)$ preserving the weight 2 Hodge structure by $Mon^2_{Hdg}(X)$. Note that the positive cone $\mathscr{C}_X$ is invariant under $Mon^2_{Hdg}(X)$, since the orientation class of $\widetilde{\mathscr{C}}_X$ is invariant under the whole monodromy group $Mon^2(X)$ (see section 4).

Definition 5.10

(1) An *exceptional chamber* of the positive cone $\mathscr{C}_X$ is a subset of the form $g[\mathscr{F}\mathscr{E}_X]$, $g \in Mon^2_{Hdg}(X)$.

(2) A *Kähler-type chamber* of the positive cone $\mathscr{C}_X$ is a subset of the form $g[f^*(\mathscr{K}_Y)]$, where $g \in Mon^2_{Hdg}(X)$, and $f : X \to Y$ is a bimeromorphic map to an irreducible holomorphic symplectic manifold Y.

Let $Mon^2_{Bir}(X) \subset Mon^2_{Hdg}(X)$ be the subgroup of monodromy operators induced by bimeromorphic maps from X to itself (see Theorem 3.1).

Lemma 5.11

(1) Every very general class $\alpha \in \mathscr{C}_X$ belongs to some Kähler-type chamber.

(2) Every Kähler-type chamber is contained in some exceptional chamber.

(3) If two Kähler-type chambers intersect, then they are equal.

(4) If two exceptional chambers $g_1[\mathscr{F}\mathscr{E}_X]$ and $g_2[\mathscr{F}\mathscr{E}_X]$ contain a common very general class α, then they are equal.

(5) $Mon^2_{Hdg}(X)$ acts transitively on the set of exceptional chambers.

(6) The subgroup of $Mon^2_{Hdg}(X)$ stabilizing $\mathscr{F}\mathscr{E}_X$ is equal to $Mon^2_{Bir}(X)$.

Proof Part (1): There exists an irreducible holomorphic symplectic manifold $\widetilde{X}$ and a correspondence $\Gamma := Z + \sum_i Y_i$ in $X \times \widetilde{X}$, such that Z is the graph of a bimeromorphic map $f : X \to \widetilde{X}$, the restriction $g : H^2(X,\mathbb{Z}) \to H^2(\widetilde{X},\mathbb{Z})$ of $[\Gamma]_*$ is a parallel transport operator, and $g(\alpha)$ is a Kähler class of $\widetilde{X}$, by ([Hu1], Corollary 5.2). Set $h := f^* \circ g$. Then h belongs to $Mon^2_{Hdg}(X)$, by Theorem 3.1, $h(\alpha) = (f^* \circ g)(\alpha)$ belongs to $f^*\mathscr{K}_{\widetilde{X}}$, and $f^*\mathscr{K}_{\widetilde{X}}$ is a Kähler-type chamber, by Definition 5.3. Consequently, $h^{-1}(f^*\mathscr{K}_{\widetilde{X}})$ is a Kähler-type chamber containing α.

Part (2): Let Ch be the Kähler-type chamber $g[f^*(\mathscr{K}_Y)]$, where f, g, and Y are as in Definition 5.10. Then $f^*(\mathscr{F}\mathscr{E}_Y) = \mathscr{F}\mathscr{E}_X$, by Corollary 5.7, and so Ch is contained in the exceptional chamber $g[\mathscr{F}\mathscr{E}_X]$.

Part (3): Let Y_i be irreducible holomorphic symplectic manifolds, $f_i : X \to Y_i$ bimeromorphic maps, $g_i \in Mon^2_{Hdg}(X)$, $i = 1,2$, and α a class in $g_1[f_1^*(\mathscr{K}_{Y_1})] \cap g_2[f_2^*(\mathscr{K}_{Y_2})]$. The composition $\varphi := f_{2_*} \circ g_2^{-1} \circ g_1 \circ f_1^* : H^2(Y_1,\mathbb{Z}) \to H^2(Y_2,\mathbb{Z})$ is a parallel-transport operator, which maps the Kähler class $f_{1_*}(g_1^{-1}(\alpha))$ to the Kähler class $f_{2_*}(g_2^{-1}(\alpha))$. Hence, φ is induced by an isomorphism $\tilde{\varphi} : Y_1 \to Y_2$, by Theorem 1.3. We get the equality $g_1^{-1} g_2 f_2^*(\mathscr{K}_{Y_2}) = f_1^* \tilde{\varphi}^*(\mathscr{K}_{Y_2}) = f_1^*(\mathscr{K}_{Y_1})$. Consequently, $g_1[f_1^*(\mathscr{K}_{Y_1})] = g_2[f_2^*(\mathscr{K}_{Y_2})]$.

Part (4): Set $g := g_2^{-1} g_1$ and $\beta := g_2^{-1}(\alpha)$. Then β belongs to the intersection $g[\mathscr{F}\mathscr{E}_X] \cap \mathscr{F}\mathscr{E}_X$. So $g^{-1}(\beta)$ and β both belong to $\mathscr{F}\mathscr{E}_X$ and g maps the former to the latter. Hence, g is induced by a birational map from X to itself, by Corollary 5.7. Thus, $g[\mathscr{F}\mathscr{E}_X] = \mathscr{F}\mathscr{E}_X$ and so $g_1[\mathscr{F}\mathscr{E}_X] = g_2[\mathscr{F}\mathscr{E}_X]$.

Part (5): The action is transitive, by definition.

Part (6) is an immediate consequence of Corollary 5.7. $\square$

Lemma 5.12 *Let X_1 and X_2 be irreducible holomorphic symplectic manifolds and $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ a parallel transport operator, which is an isomorphism of Hodge structures.*

(1) g maps each exceptional chamber in $\mathscr{C}_{X_1}$ onto an exceptional chamber in $\mathscr{C}_{X_2}$.

(2) g maps each Kähler-type chamber in $\mathscr{C}_{X_1}$ onto a Kähler-type chamber in $\mathscr{C}_{X_2}$.

Proof There exists a bimeromorphic map $h : X_1 \to X_2$, by Theorem 1.3. The homomorphism $h_* : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is a parallel transport operator, and an isomorphism of Hodge structures, by Theorem 3.1.

Part (1): Let f be an element of $Mon^2_{Hdg}(X_1)$. We need to show that $g(f[\mathscr{F}\mathscr{E}_{X_1}])$ is an exceptional chamber in $\mathscr{C}_{X_2}$. Indeed, we have the equalities

$$g(f[\mathscr{F}\mathscr{E}_{X_1}]) = (gfh^*)\{h_*[\mathscr{F}\mathscr{E}_{X_1}]\} = (gfh^*)[\mathscr{F}\mathscr{E}_{X_2}],$$

and gfh^* belongs to $Mon^2_{Hdg}(X_2)$.

Part (2): Any Kähler-type chamber of $\mathscr{C}_{X_1}$ is of the form $f[\tilde{h}^*(\mathscr{K}_{Y_1})]$, where $\tilde{h} : X_1 \to Y_1$ is a bimeromorphic map to an irreducible holomorphic symplectic manifold Y_1, and f is an element of $Mon^2_{Hdg}(X_1)$. We have the equality

$$gf[\tilde{h}^*(\mathscr{K}_{Y_1})] \;=\; (gfh^*)\left\{(h\tilde{h}^{-1})_*(\mathscr{K}_{Y_1})\right\},$$

$(h\tilde{h}^{-1})_*(\mathscr{K}_{Y_1})$ is a Kähler-type chamber of X_2 and gfh^* belongs to $Mon^2_{Hdg}(X_2)$, by Theorem 3.1. Thus $gf[\tilde{h}^*(\mathscr{K}_{Y_1})]$ is a Kähler-type chamber of X_2. $\square$

Corollary 5.13 *Let (X_1,η_1), (X_2,η_2) be two inseparable points in $\mathfrak{M}^0_\Lambda$.*

(1) The composition $\eta_2^{-1}\circ\eta_1$ maps each Kähler-type chamber in $\mathscr{C}_{X_1}$ onto a Kähler-type chamber in $\mathscr{C}_{X_2}$. Similarly, $\eta_2^{-1}\circ\eta_1$ maps each exceptional chamber in $\mathscr{C}_{X_1}$ onto an exceptional chamber in $\mathscr{C}_{X_2}$.

(2) $(\eta_2^{-1}\circ\eta_1)(\mathscr{F}\mathscr{E}_{X_1}) = \mathscr{F}\mathscr{E}_{X_2}$, if and only if there exists a bimeromorphic map f from X_1 to X_2, such that $\eta_2^{-1}\circ\eta_1 = f_*$.

Proof The composition $\eta_2^{-1}\circ\eta_1$ is a parallel-transport operator, and a Hodge-isometry, by Theorem 3.2 part 2. Part (1) follows from Lemma 5.12. Part (2) follows from Corollary 5.7. $\square$

5.3 $\mathfrak{M}_\Lambda$ as the moduli space of Kähler-type chambers

Consider the period map $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ from the connected component $\mathfrak{M}^0_\Lambda$ containing the isomorphism class of the marked pair (X,η). Denote by $\mathscr{K}\mathscr{T}(X)$ the set of Kähler-type chambers in $\mathscr{C}_X$. Let

$$\rho \ : \ P_0^{-1}[P_0(X,\eta)] \ \longrightarrow \ \mathscr{K}\mathscr{T}(X) \tag{5.1}$$

be the map given by $\rho(\widetilde{X},\tilde{\eta}) = (\eta^{-1}\tilde{\eta})(\mathscr{K}_{\widetilde{X}})$. The map ρ is well defined, by Corollary 5.13. $Mon^2_{Hdg}(X)$ acts on $\mathscr{K}\mathscr{T}(X)$, by Lemma 5.12.

Note that each period $P(X,\eta) \in \Omega_\Lambda$ is invariant under the subgroup

$$Mon^2_{Hdg}(X)^\eta \ := \ \{\eta g \eta^{-1} \ : \ g \in Mon^2_{Hdg}(X)\} \tag{5.2}$$

of $O(\Lambda)$. Consequently, $Mon^2_{Hdg}(X)$ acts on the fiber $P_0^{-1}[P_0(X,\eta)]$ of the period map by

$$g(\widetilde{X},\tilde{\eta}) \ := \ (\widetilde{X},\eta g \eta^{-1}\tilde{\eta}).$$

Proposition 5.14

(1) The map ρ is a $Mon^2_{Hdg}(X)$-equivariant bijection.

(2) The marked pair (X,η) is a Hausdorff point of $\mathfrak{M}_\Lambda$, if and only if $\mathscr{C}_X = \mathscr{K}_X$.

(3) ([Hu1], Corollaries 5.7 and 7.2) $\mathscr{C}_X = \mathscr{K}_X$, if $H^{1,1}(X,\mathbb{Z})$ is trivial, or if $H^{1,1}(X,\mathbb{Z})$ is of rank 1, generated by a class α of non-negative Beauville-Bogomolov degree.

Proof Part (1): Assume that $\rho(X_1,\eta_1) = \rho(X_2,\eta_2)$. Then $\eta_2^{-1}\eta_1(\mathscr{K}_{X_1}) = \mathscr{K}_{X_2}$. Hence, $\eta_2^{-1}\eta_1 = f_*$, for an isomorphism $f : X_1 \to X_2$, by Theorem 1.3. Thus, (X_1,η_1) and (X_2,η_2) are isomorphic, and ρ is injective.

Given a Kähler-type chamber Ch in $\mathscr{C}_X$ and a very general class α in Ch, there exists an element $g \in Mon^2_{Hdg}(X)$, such that $g(\alpha)$ belongs to $\mathscr{F}\mathscr{E}_X$, by Lemma 5.11 part 5. There exists an irreducible holomorphic symplectic manifold Y and a bimeromorphic map $h : X \to Y$, such that $h_*(g(\alpha))$ belongs to $\mathscr{K}_Y$, by Theorem 5.4 part 2. Thus, $(h_* \circ g)(Ch) = \mathscr{K}_Y$, by Lemma 5.12. We conclude that $\rho(Y,\eta \circ g^{-1} \circ h^*) = g^{-1}h^*(\mathscr{K}_Y) = Ch$ and ρ is surjective.

Part (2) follows from part (1). □

Fix a connected component $\mathfrak{M}^0_\Lambda$ of the moduli space of marked pairs. We get the following modular description of the fiber $P_0^{-1}(p)$ in terms of the period p. Set

$\Lambda^{1,1}(p,\mathbb{R}) := \{\lambda \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \ : \ (\lambda, p) = 0\}$. Let $\mathscr{C}_p$ be the connected component, of the cone $\mathscr{C}'_p$ in $\Lambda^{1,1}(p,\mathbb{R})$, which is compatible with the orientation of the positive cone $\widetilde{\mathscr{C}}_\Lambda$ determined by $\mathfrak{M}^0_\Lambda$ (see section 4).

Definition 5.15 A *Kähler-type chamber* of $\mathscr{C}_p$ is a subset of the form $\eta(Ch) \subset \mathscr{C}_p$, where (X,η) is a marked pair $\mathfrak{M}^0_\Lambda$ and $Ch \subset \mathscr{C}_X$ is a Kähler-type chamber of X.

Denote by $\mathscr{K}\mathscr{T}(p)$ the set of Kähler-type chambers in $\mathscr{C}_p$. The map

$$\eta \ : \ \mathscr{K}\mathscr{T}(X) \ \longrightarrow \ \mathscr{K}\mathscr{T}(p),$$

sending a Kähler-type chamber $Ch \in \mathscr{K}\mathscr{T}(X)$ to $\eta(Ch)$, is a bijection, for every marked pair (X,η) in the fiber $P_0^{-1}(p)$, by Corollary 5.13 and Proposition 5.14. $Mon^2_{Hdg}(X)^\eta$, given in equation (5.2), is the same subgroup of $O(\Lambda)$, for all $(X,\eta) \in P_0^{-1}(p)$, and we denote it by $Mon^2_{Hdg}(p)$. The following statement is an immediate corollary of Proposition 5.14.

Theorem 5.16 *The map*

$$\rho \ : \ P_0^{-1}(p) \ \longrightarrow \ \mathscr{K}\mathscr{T}(p),$$

given by $\rho(X,\eta) := \eta(\mathscr{K}_X)$*, is a* $Mon^2_{Hdg}(p)$*-equivariant bijection.*

Remark 5.17 Compare Theorem 5.16 with the more detailed analogue for $K3$ surfaces, which is provided in ([LP], Theorem 10.5). Ideally, one would like to have a description of the set $\mathscr{K}\mathscr{T}(p)$, depending only on the period p, the deformation type of X, and possibly some additional discrete monodromy invariant of X (see the invariant ι_X introduced in Corollary 9.5). Such a description would depend on the determination of the Kähler-type chambers in $\mathscr{C}_X$. In particular, it requires a determination of the Kähler cone of an irreducible holomorphic symplectic variety, in terms of the Hodge structure of $H^2(X,\mathbb{Z})$, the Beauville-Bogomolov pairing, and the discrete monodromy invariants of X. The determination of the Kähler cone $\mathscr{K}_X$ in terms of such data is a very difficult problem addressed in a sequence of papers of Hassett and Tschinkel [HT1, HT2, HT3, HT4]. Precise conjectures for the determination of the Kähler cones in the $K3^{[n]}$-type, for all n, and for generalized Kummer fourfolds, are provided in [HT4], Conjectures 1.2 and 1.4. The determination of the birational Kähler cone, in terms of such data, is the subject of section 6.

6 $Mon^2_{Hdg}(X)$ is generated by reflections and $Mon^2_{Bir}(X)$

Throughout this section X denotes a *projective* irreducible holomorphic symplectic manifold. Under the projectivity assumption, we can define a subgroup W_{Exc} of the Hodge-monodromy group $Mon^2_{Hdg}(X)$, which is generated by reflections with respect to classes of prime exceptional divisors (Definition 6.8 and Theorem 6.18 part 3). The fundemental exceptional chamber $\mathscr{F}\mathscr{E}_X$, introduced in Definition 5.2, is the interior of a fundamental domain for the action of the reflection group W_{Exc} on the positive cone $\mathscr{C}_X$. Significant regularity properties follow from this description of $\mathscr{F}\mathscr{E}_X$ (Theorem 6.17). We prove also that W_{Exc} is a normal subgroup of $Mon^2_{Hdg}(X)$ and the latter is a semi-direct-product of W_{Exc} and $Mon^2_{Bir}(X)$ (Theorem 6.18). A weak version of Morrison's movable cone conjecture follows from the above results in the special case of irreducible holomorphic symplectic manifolds (Theorems 1.7 and 6.25).

6.1 Reflections

Let X be a projective irreducible holomorphic symplectic manifold of dimension $2n$ and $E \subset X$ a prime exceptional divisor (Definition 5.1).

Proposition 6.1 *([Dr], Proposition 1.4) There exists a sequence of flops of X, resulting in a smooth birational model X' of X, such that the strict transform E' of E in X' is contractible via a projective birational morphism $\pi : X' \to Y$ onto a normal projective variety Y. The exceptional locus of π is equal to the support of E'.*

Identify $H^2(X,\mathbb{Q})^*$ with $H_2(X,\mathbb{Q})$. Set

$$[E]^\vee \quad := \quad \frac{-2([E],\bullet)}{([E],[E])} \;\in\; H_2(X,\mathbb{Q}).$$

Proposition 6.2 *([Ma7], Corollary 3.6 part 1).*

(1) There exists a Zariski dense open subset $E^0 \subset E$ and a proper holomorphic fibration $\pi : E^0 \to B$, onto a smooth holomorphic symplectic variety of dimension $2n-2$, with the following property. The class $[E]^\vee$ is the class of a generic fiber of π. The generic fiber is either a smooth rational curve, or the union of two homologous smooth rational curves meeting at one point non-tangentially. In par-

ticular, the class $[E]^{\vee}$ *is integral, as is the reflection* $R_E : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$, *given by* $R_E(x) = x + (x,[E]^{\vee})[E]$.

(2) The reflection R_E *belongs to* $Mon^2_{Hdg}(X)$.

Remark 6.3

(1) The proof of Proposition 6.2 relies heavily on Druel's result stated above in Proposition 6.1. The fact that $R_{E'}$ belongs to $Mon^2_{Hdg}(X')$ was proven earlier in ([Ma6], Theorem 1.4), using fundamental work of Namikawa [Nam1] (see [Nam3] for an alternative proof). The author does not know if the analogue of Proposition 6.1 holds for a non-projective irreducible holomorphic symplectic manifold X as well. This is the reason for the projectivity assumption throughout section 6.

(2) The variety B in part (1) of the proposition is an étale cover of a Zariski open subset of the image of E' in Y ([Nam1], section 1.8).

6.2 Stably prime-exceptional line bundles

Let X be an irreducible holomorphic symplectic manifold. Denote by $Def(X)$ the local Kuranishi deformation space of X and let $0 \in Def(X)$ be the special point corresponding to X. Let L be a line bundle on X. Set $\Lambda := H^2(X,\mathbb{Z})$. The period map $P : Def(X) \to \Omega_\Lambda$ embeds $Def(X)$ as an open analytic subset of the period domain Ω_Λ and the intersection $Def(X,L) := Def(X) \cap c_1(L)^{\perp}$ is the Kuranishi deformation space of the pair (X,L), i.e., it consists of deformations of the complex structure of X along which $c_1(L)$ remains of type $(1,1)$. We assume that both $Def(X)$ and the intersection $Def(X,L)$ are simply connected, possibly after replacing $Def(X)$ by a smaller open neighborhood of 0 in the Kuranishi deformation space, which we denote again by $Def(X)$.

Let $\pi : \mathscr{X} \to Def(X)$ be the universal family and denote by X_t the fiber of π over $t \in Def(X)$. Denote by ℓ the flat section of the local system $R^2\pi_*\mathbb{Z}$ through $c_1(L)$ and let $\ell_t \in H^{1,1}(X_t,\mathbb{Z})$ be its value at $t \in Def(X,L)$. Let L_t be the line bundle on X_t with $c_1(L_t) = \ell_t$.

Definition 6.4 A line bundle $L \in \mathrm{Pic}(X)$ is called *stably prime-exceptional*, if there exists a closed analytic subset $Z \subset Def(X,L)$, of positive codimension, such that the linear system $|L_t|$ consists of a prime exceptional divisor E_t, for all $t \in [Def(X,L) \setminus Z]$.

Note that a stably prime-exceptional line bundle L is effective, by the semi-continuity theorem. Furthermore, if we set $\ell := c_1(L)$ and define the reflection $R_\ell(\alpha) := \alpha - 2\frac{(\alpha,\ell)}{(\ell,\ell)}\ell$, then R_ℓ belongs to $Mon^2_{Hdg}(X)$.

Remark 6.5 Note that the linear system $|L|$, of a stably prime-exceptional line bundle L, may have positive dimension, if the Zariski decomposition of Theorem 5.8 is non-trivial. Even if $|L|$ consists of a single exceptional divisor, it may be reducible or non-reduced, i.e., the special point 0 may belong to the closed analytic subset Z in Definition 6.4.

Proposition 6.6 *Let E be a prime exceptional divisor on a projective irreducible holomorphic symplectic manifold X.*

(1) ([Ma7], Proposition 5.2) The line bundle $\mathcal{O}_X(E)$ is stably prime-exceptional.

(2) ([Ma7], Proposition 5.14) Let Y be an irreducible holomorphic symplectic manifold and $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ a parallel-transport operator, which is an isomorphism of Hodge structures. Set $\alpha := g([E]) \in H^{1,1}(Y,\mathbb{Z})$. Then either α or $-\alpha$ is the class of a stably prime-exceptional line bundle.

Example 6.7 Let X be a $K3$ surface. A line bundle L is stably prime-exceptional, if and only if $\deg(L) = -2$, and $(c_1(L), \kappa) > 0$, for some Kähler class κ on X.

Denote by $\mathcal{S}pe \subset H^{1,1}(X,\mathbb{Z})$ the subset of classes of stably prime-exceptional divisors.

Definition 6.8 Let $W_{Exc} \subset Mon^2_{Hdg}(X)$ be the reflection subgroup generated by $\{R_\ell \ : \ \ell \in \mathcal{S}pe\}$.

Note that $R_\ell = R_{-\ell}$.

Corollary 6.9 *The union $\mathcal{S}pe \cup -\mathcal{S}pe$ is a $Mon^2_{Hdg}(X)$-invariant subset of $H^{1,1}(X,\mathbb{Z})$. In particular, W_{Exc} is a normal subgroup of $Mon^2_{Hdg}(X)$*

Proposition 6.10 *The fundamental exceptional chamber $\mathcal{FE}_X$, introduced in Definition 5.2, is equal to the subset*

$$\{\alpha \in \mathcal{C}_X \ : \ (\alpha,\ell) > 0, \text{ for every } \ell \in \mathcal{S}pe\}. \tag{6.1}$$

Proof Denote the exceptional chamber (6.1) by Ch_0. Then $Ch_0 \subset \mathscr{F}\mathscr{E}_X$, since a prime exceptional divisor is stably prime-exceptional, by Proposition 6.6. Let α be a class in $\mathscr{F}\mathscr{E}_X$, $\ell \in \mathcal{S}pe$, and $\ell = P(\ell) + N(\ell)$ its Zariski decomposition of Theorem 5.8. Then $N(\ell)$ is a non-zero exceptional divisor, since $(\ell,\ell) < 0$ and $(P(\ell),P(\ell)) \geq 0$. Furthermore, $(\alpha,P(\ell)) \geq 0$, since α and $P(\ell)$ belong to the closure of the positive cone. Thus, $(\alpha,\ell) \geq (\alpha,N(\ell)) > 0$. We conclude that α belongs to Ch_0 and so $\mathscr{F}\mathscr{E}_X \subset Ch_0$. □

In section 9.2 we will provide a numerical determination of the set $\mathcal{S}pe$, and hence of $\mathscr{F}\mathscr{E}_X$, for X of $K3^{[n]}$-type.

6.3 Hyperbolic reflection groups

Consider the vector space $\mathbb{R}^{n+1}$, endowed with the quadratic form $q(x_0,\ldots,x_n) = x_0^2 - \sum_{i=1}^n x_i^2$. We will denote the inner product space $(\mathbb{R}^{n+1},q)$ by V and denote by (v,w), $v,w \in V$, the inner product, such that $q(v) = (v,v)$. Let $v := (v_0,\ldots,v_n)$ be the coordinates of a vector v in V. The *hyperbolic* (or *Lobachevsky*) *space* is

$$\mathbb{H}^n \ := \ \{v \in V \ : \ q(v) = 1 \text{ and } v_0 > 0\}.$$

$\mathbb{H}^n$ has two additional descriptions. It is the set of $\mathbb{R}_{>0}$ orbits (half lines) in one of the two connected component of the cone $\mathscr{C}'_V := \{v \in V \ : \ q(v) > 0\}$. We will denote by $\mathscr{C}_V$ the chosen connected component of $\mathscr{C}'_V$ and refer to $\mathscr{C}_V$ as the *positive cone*. $\mathbb{H}^n$ also naturally embeds in $\mathbb{P}^n(\mathbb{R})$ as the image of $\mathscr{C}_V$. A *hyperplane* in $\mathbb{H}^n$ is a non-empty intersection of $\mathbb{H}^n$ with a hyperplane in $\mathbb{P}^n(\mathbb{R})$.

The first description of $\mathbb{H}^n$ above depended on the diagonal form of the quadratic form q. The last two descriptions of $\mathbb{H}^n$ produce a copy of $\mathbb{H}^n$ associated more generally to any quadratic form $q(x_0,\ldots,x_n) = \sum_{i,j=0}^n a_{ij}x_ix_j$, $a_{ij} \in \mathbb{Q}$, of signature $(1,n)$. We will consider from now on this more general set-up.

$\mathbb{H}^n$ admits a metric of constant curvature [VS]. Let $O^+(V)$ be the subgroup of the isometry group of V mapping $\mathscr{C}_V$ to itself. Then $O^+(V)$ acts transitively on $\mathbb{H}^n$ via isometries. The stabilizer $Stab_{O^+(V)}(t)$, of every point $t \in \mathbb{H}^n$, is compact, since the hyperplane $t^\perp \subset V$ is negative definite.

A subgroup $\Gamma \subset O^+(V)$ is said to be a *discrete group of motions of* $\mathbb{H}^n$, if for each point $t \in \mathbb{H}^n$, the stabilizer $Stab_\Gamma(t)$ is finite and the orbit $\Gamma \cdot t$ is a discrete subset of $\mathbb{H}^n$. The arithmetic group $O^+(V,\mathbb{Z})$ is a discrete group of motions ([VS], Ch. 1, section 2.2). Furthermore, if a subgroup $\Gamma \subset O^+(V)$ is commensurable to a discrete group of motions, then Γ is a discrete group of motions as well ([VS], Ch. 1, Proposition 1.13). Given a group homomorphism $\widetilde{\Gamma} \to O^+(V)$, we say that $\widetilde{\Gamma}$ *acts on* $\mathbb{H}^n$ *via a discrete group of motions,* if its image $\Gamma \subset O^+(V)$ is a discrete group of motions.

Lemma 6.11 *Let X be a projective irreducible holomorphic symplectic manifold. Then* $Mon^2_{Hdg}(X)$ *acts via a discrete group of motions on the hyperbolic space* $\mathbb{H}_X$ *associated to* $V := H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ *as well as on the hyperbolic space* $\widetilde{\mathbb{H}}_X$ *associated to* $H^{1,1}(X,\mathbb{R})$.

Proof Let ρ be the rank of $\mathrm{Pic}(X)$. The Beauville-Bogomolov pairing restricts to $H^{1,1}(X,\mathbb{Z})$ as a non-degenerate pairing of signature $(1,\rho-1)$. The action of $Mon^2_{Hdg}(X)$ on $\mathbb{H}_X$ factors through the action of $O^+[H^{1,1}(X,\mathbb{Z})]$. The latter acts as a discrete group of motions on $\mathbb{H}_X$ (see [VS], Ch. 1, section 2.2). The statement of the lemma follows for $\mathbb{H}_X$.

Let G be the kernel of the restriction homomorphism $Mon^2_{Hdg}(X) \to O^+[H^{1,1}(X,\mathbb{Z})]$. We prove next that G is a finite group. Let $T(X)$ be the subspace of $H^2(X,\mathbb{R})$ orthogonal to $H^{1,1}(X,\mathbb{Z})$. Set $T^{1,1}(X) := T(X) \cap H^{1,1}(X,\mathbb{R})$. The Beauville-Bogomolov pairing restricts to $T^{1,1}(X)$ as a negative definite pairing. Let $T^+(X) \subset T(X)$ be the orthogonal complement of $T^{1,1}(X)$ in $T(X)$. Then $T^+(X)$ is the two-dimensional positive definite subspace of $T(X)$, spanned by the real and imaginary parts of a holomorphic 2-form on X. G acts faithfully on $T(X)$ and it embeds as a discrete subgroup of the compact group $O(T^+(X)) \times O\left(T^{1,1}(X)\right)$. We conclude that G is finite.

The linear subspace $\mathbb{P}\left(T^{1,1}(X)\right)$ of $\mathbb{P}\left(H^{1,1}(X,\mathbb{R})\right)$ is disjoint from $\widetilde{\mathbb{H}}_X$ and so the orthogonal projection $H^{1,1}(X,\mathbb{R}) \to V$ induces a well defined $Mon^2_{Hdg}(X)$-equivariant map $\pi : \widetilde{\mathbb{H}}_X \to \mathbb{H}_X$. Explicitly, a point $\tilde{v}$ in the positive cone of $H^{1,1}(X,\mathbb{R})$ can be uniquely decomposed as a sum $\tilde{v} = v + t$, with $v \in V$ and $t \in T^{1,1}(X)$, and π takes the image of $\tilde{v}$ in $\widetilde{\mathbb{H}}_X$ to the image of v in $\mathbb{H}_X$.

We show next that $Mon^2_{Hdg}(X)$ acts on $\widetilde{\mathbb{H}}_X$ via a discrete group of motions. Set $\Gamma := Mon^2_{Hdg}(X)/G$. Let $\tilde{x}$ be a point of $\widetilde{\mathbb{H}}_X$ and set $x := \pi(\tilde{x})$. The stabilizing subgroup $Stab_\Gamma(x)$ is finite, since Γ acts on $\mathbb{H}_X$ as a discrete group of motions. The preimage of $Stab_\Gamma(x)$ in $Mon^2_{Hdg}(X)$ is finite and contains the stabilizer of $\tilde{x}$ in $Mon^2_{Hdg}(X)$. Hence, the latter stabilizer is finite. Let y be a point in the orbit $\Gamma \cdot x$ in $\mathbb{H}_X$. Then $\pi^{-1}(y)$ intersects the orbit $Mon^2_{Hdg}(X) \cdot \tilde{x}$ in an orbit of a finite subgroup, namely, an orbit of the preimage of $Stab_\Gamma(y)$ in $Mon^2_{Hdg}(X)$. The orbit

$Mon^2_{Hdg}(X) \cdot \tilde{x}$ is a discrete subset of $\widetilde{\mathbb{H}}_X$, since π restricts to it as a finite map onto the discrete orbit of x in $\mathbb{H}_X$. $\square$

Given an element $e \in V$, with $q(e) < 0$, we get the reflection $R_e \in O^+(V)$, given by $R_e(w) = w - 2\frac{(e,w)}{(e,e)}e$.

Definition 6.12 A *hyperbolic reflection group* is a discrete group of motions of $\mathbb{H}^n$ generated by reflections.

Given a vector $e \in V$, with $q(e) < 0$, set

$$H_e^+ := \{v \in \mathscr{C}_V \ : \ (v,e) > 0\}/\mathbb{R}_{>0}.$$

Define H_e^- similarly using the inequality $(v,e) < 0$. Set $H_e := e^\perp \cap \mathbb{H}^n$, where $e^\perp$ is the hyperplane of $\mathbb{P}(V)$ orthogonal to e. Then $\mathbb{H}^n \setminus H_e$ is the disjoint union of its two connected components H_e^+ and H_e^-. The closures $\overline{H}_e^\pm$ are called *half-spaces*.

Definition 6.13

(1) A set $\{\Sigma_i \ : i \in I\}$, of subsets of a topological space X, is *locally finite*, if each point $x \in X$ has an open neighborhood U_x, such that the intersection $\Sigma_i \cap U_x$ is empty, for all but finitely many indices $i \in I$.

(2) A *decomposition* of $\mathbb{H}^n$ is a locally finite covering of $\mathbb{H}^n$ by closures of open connected subsets, no two of which have common interior points.

(3) A closure D of an open subset of $\mathbb{H}^n$ is said to be a *fundamental domain* of a discrete group of motions Γ, if $\{\gamma(D) \ : \ \gamma \in \Gamma\}$ is a decomposition of $\mathbb{H}^n$.

(4) ([AVS], Ch. 1, Definition 3.9) A *convex polyhedron* is an intersection of finitely many half-spaces, having a non-empty interior.

(5) ([VS], Ch 1, Definition 1.9) A closed subset $P \subset \mathbb{H}^n$ is a *generalized convex polyhedron*, if P is the closure of an open subset, and the intersection of P with every bounded convex polyhedron, containing at least one common interior point, is a convex polyhedron.

(6) A closed cone in $\mathscr{C}_V$ is a *generalized convex polyhedron*, if its image in $\mathbb{H}^n$ is a generalized convex polyhedron.

(7) A closed cone Π in $\mathscr{C}_V$ is a *rational convex polyhedron*, if its image in $\mathbb{H}^n$ is a convex polyhedron, which is the intersection of finitely many half spaces H_e^+ with $e \in \mathbb{Q}^{n+1}$.

Theorem 6.14

(1) ([VS], Ch. 1 Theorem 1.11) Any discrete group of motions of $\mathbb{H}^n$ has a fundamental domain, which is a generalized convex polyhedron.

(2) ([VS], Ch. 2 Theorem 2.5) The action on $\mathbb{H}^n$ of any arithmetic subgroup of $O^+(V)$ has a fundamental domain, which is a convex polyhedron.

The decomposition of $\mathbb{H}^n$, induced by translates of the fundamental domain in Theorem 6.14, is not canonical in general. A canonical decomposition exists, if the discrete group of motions is a reflection group. The hyperplanes of $n-1$ dimensional faces of a generalized convex polyhedron are called its *walls*.

Let Γ be a hyperbolic reflection group and $\mathscr{R}_\Gamma \subset \Gamma$ the subset of reflections. Given a reflection $\rho \in \mathscr{R}_\Gamma$, let $H_\rho \subset \mathbb{H}^n$ be the hyperplane fixed by ρ. Connected components of $\mathbb{H}^n \setminus \bigcup_{\rho \in \mathscr{R}_\Gamma} H_\rho$ are called *chambers*.

Theorem 6.15 *([VS], Ch. 5 Theorem 1.2 and Proposition 1.4)*

(1) The closure of each chamber of Γ in $\mathbb{H}^n$ is a generalized convex polyhedron,[7] *which is a fundamental domain for Γ.*

(2) Γ is generated by reflections in the walls of any of its chambers in $\mathbb{H}^n$.

Let Γ be any discrete group of motions of $\mathbb{H}^n$. Denote by Γ_r the subgroup of Γ generated by all reflections in Γ. We call Γ_r the *reflection subgroup of Γ*. Choose a chamber D of Γ_r. Let $\Gamma_D \subset \Gamma$ be the subgroup $\{\gamma \in \Gamma \ : \ \gamma(D) = D\}$.

Theorem 6.16 *([VS], Ch. 5 Proposition 1.5) Γ_r is a normal subgroup of Γ, and Γ is the semi-direct product of Γ_r and Γ_D.*

We refer the reader to the book [VS] and the interesting recent survey [Do] for detailed expositions of the subject of hyperbolic reflection groups.

Let X be a projective irreducible holomorphic symplectic manifold.

Theorem 6.17 *The fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, introduced in Definition 5.2, is equal to the connected component of*

$$\mathscr{C}_X \setminus \bigcup\{\ell^\perp \ : \ \ell \in \mathcal{S}pe\} \tag{6.2}$$

[7] This polyhedron is moreover a generalized Coxeter polyhedron ([VS], Ch. 5 Definition 1.1), but we will not use this fact.

containing the Kähler cone. In particular, $\mathcal{FE}_X$ is the interior of a generalized convex polyhedron (Definition 6.13).

Proof The group W_{Exc} is a hyperbolic reflection group and the set U in equation (6.2) is an open subset of $\mathcal{C}_X$, which is the union of the interiors of the fundamental chambers of the W_{Exc}-action on $\mathcal{C}_X$, by Theorem 6.15. The intersection of $\mathcal{FE}_X$ and U is the union of connected components of U, by the definitions of $\mathcal{FE}_X$ and W_{Exc}. $\mathcal{FE}_X$ is contained in U, by Proposition 6.10. $\mathcal{FE}_X$ is convex cone, hence a connected component of U. $\mathcal{FE}_X$ contains $\mathcal{K}_X$, by the definition of $\mathcal{FE}_X$. □

6.4 $Mon^2_{Hdg}(X)$ is a semi-direct product of W_{Exc} and $Mon^2_{Bir}(X)$

Denote by $\mathcal{P}ex$ the set of prime exceptional divisors in X. Given $E \in \mathcal{P}ex$, denote by R_E the corresponding reflection (Proposition 6.2).

Theorem 6.18

(1) The group $Mon^2_{Hdg}(X)$ acts transitively on the set of exceptional chambers, introduced in Definition 5.10, and the subgroup W_{Exc} acts simply-transitively on this set.

(2) The exceptional chambers are precisely the connected component of the open set in equation (6.2), i.e., each exceptional chamber is the interior of a fundamental domain of the W_{Exc} action on $\mathcal{C}_X$.

(3) The group W_{Exc} is generated by $\{R_e \,:\, e \in \mathcal{P}ex\}$.

(4) The subgroup of $Mon^2_{Hdg}(X)$ stabilizing the fundamental exceptional chamber $\mathcal{FE}_X$ is equal to $Mon^2_{Bir}(X)$.

(5) $Mon^2_{Hdg}(X)$ is the semi-direct product of its subgroups W_{Exc} and $Mon^2_{Bir}(X)$.

When X is a $K3$ surface $Mon^2_{Hdg}(X)$ is equal to the group of Hodge isometries of $H^2(X,\mathbb{Z})$ preserving the spinor norm and $Mon^2_{Bir}(X)$ is equal to the group of biregular automorphisms of X. Furthermore, the fundamental exceptional chamber is equal to the Kähler cone of the $K3$ surface. Theorem 6.18 is well known in the case of $K3$ surfaces [BR, PS], or ([LP], Proposition 1.9).

Proof Parts (1) and (2): $Mon^2_{Hdg}(X)$ acts transitively on the set of exceptional chambers, by their definition. The subgroup W_{Exc} acts simply-transitively on the set of connected components of the set U in equation (6.2), by Theorem 6.15. One of these is $\mathscr{F}\mathscr{E}_X$, by Theorem 6.17. Hence, every connected component of U is an exceptional chamber. $Mon^2_{Hdg}(X)$ acts on the set of connected component of U, by Corollary 6.9. Hence, every exceptional chamber is a connected component of U.

Part (3): The walls in the boundary of the fundamental exceptional chamber are all of the form $[E]^\perp \cap \mathscr{C}_X$, for some prime exceptional divisor E, by definition. $\mathscr{F}\mathscr{E}_X$ is the interior of a chamber of W_{Exc}, by Theorem 6.17. We conclude that W_{Exc} is generated by $\{R_e : e \in \mathscr{P}ex\}$, by Theorem 6.15.

Part (4): $Mon^2_{Bir}(X)$ is the subgroup of $Mon^2_{Hdg}(X)$ leaving $\mathscr{F}\mathscr{E}_X$ invariant, by Lemma 5.11 part 6.

Part (5): $Mon^2_{Hdg}(X)$ is generated by W_{Exc} and $Mon^2_{Bir}(X)$, by parts (1) and (4). The intersection $W_{Exc} \cap Mon^2_{Bir}(X)$ is trivial, since the action of W_{Exc} on the set of exceptional chambers is free. W_{Exc} is a normal subgroup of $Mon^2_{Hdg}(X)$, by Corollary 6.9. □

Caution 6.19 When X is a $K3$ surface, then W_{Exc} is the reflection subgroup of $Mon^2_{Hdg}(X)$, i.e., every reflection $g \in Mon^2_{Hdg}(X)$ is of the form R_ℓ, for a class ℓ satisfying $(\ell,\ell) = -2$. This follows easily from the fact that $H^2(X,\mathbb{Z})$ is a unimodular lattice. W_{Exc} may be strictly smaller than the reflection subgroup of $Mon^2_{Hdg}(X)$, for a higher dimensional irreducible holomorphic symplectic manifold X. In other words, there are examples of elements $\alpha \in H^{1,1}(X,\mathbb{Z})$, with $(\alpha,\alpha) < 0$, such that R_α belongs to $Mon^2_{Hdg}(X)$, but neither α, nor $-\alpha$, belongs to $\mathscr{S}pe$. Instead, R_α is induced by a bimeromorphic map from X to itself (see Example 9.20 below, and section 11 of [Ma7] for additional examples).

Let L be a stably prime-exceptional line bundle and set $\ell := c_1(L)$. The hyperplane $\ell^\perp$ intersects $\overline{\mathscr{F}\mathscr{E}}_X$ in a top dimensional cone in $\ell^\perp$, only if $L = \mathscr{O}_X(E)$ for some prime exceptional divisor E, by Proposition 6.10. We show next that the condition is also sufficient.

Lemma 6.20 *Let E be a prime exceptional divisor on X. Then $E^\perp \cap \overline{\mathscr{F}\mathscr{E}}_X$ is a top dimensional cone in the hyperplane $E^\perp$. Consequently, W_{Exc} can not be generated by any proper subset of $\{R_e : e \in \mathscr{P}ex\}$.*

Proof Let e be an element of $\mathscr{P}ex$. It suffices to show that $e^\perp \cap \overline{\mathscr{F}\mathscr{E}}_X \cap \mathscr{C}_X$ contains elements, which are not orthogonal to any other $e' \in \mathscr{P}ex$. Choose $x \in \mathscr{F}\mathscr{E}_X$ and set $y := x - \frac{(x,e)}{(e,e)}e$. Then $(y,e) = 0$. Given $e' \in \mathscr{P}ex$, $e' \neq e$, then $(e,e') \geq 0$ and $(x,e') > 0$. Now $(e,e) < 0$. We get the following inequalities.

$$(e',y) = (e',x) - \frac{(x,e)}{(e,e)}(e',e) \; > \; 0.$$
$$(y,y) = (x,x) - \frac{(x,e)^2}{(e,e)} \; > \; 0.$$

We conclude that y belongs to $e^{\perp} \cap \overline{\mathscr{F}\mathscr{E}}_X \cap \mathscr{C}_X$, and y does not belong to $(e')^{\perp}$, for any $e' \in \mathcal{P}ex \setminus \{e\}$. $\square$

Proof of Theorem 1.6: W_{Exc} is a normal subgroup of $Mon^2_{Hdg}(X)$, by Corollary 6.9. There exists a bimeromorphic map $h : X_1 \to X_2$, by Theorem 1.3, and h^* is a parallel transport operator, by Theorem 3.1. The composition $f \circ h^*$ belongs to $Mon^2_{Hdg}(X_2)$. There exists an element w of $W_{Exc}(X_2)$, such that $w^{-1} f \circ h^*$ belongs to $Mon^2_{Bir}(X_2)$, by Theorem 6.18. Let $\varphi : X_2 \to X_2$ be a bimeromorphic map, such that $\varphi_* = w^{-1} f \circ h^*$. Then $f = w(\varphi h)_*$. Set $g := \varphi h$ to obtain the desired decomposition $f = w \circ g_*$.

Assume that $\tilde{g} : X_1 \to X_2$ is a birational map and $\tilde{w}$ is an element of $W_{Exc}(X_2)$, such that $f = \tilde{w}\tilde{g}_*$. Then $w^{-1}\tilde{w} = (\tilde{g}^{-1}g)_*$ belongs to the intersection of $W_{Exc}(X_2)$ and $Mon^2_{Bir}(X_2)$, which is trivial, by Theorem 6.18. Thus, $w = \tilde{w}$ and $g_* = \tilde{g}_*$. Now, $\tilde{g} = g(g^{-1}\tilde{g})$, and $g^{-1}\tilde{g}$ is a birational map inducing the identity on $H^2(X_1, \mathbb{Z})$. In particular, $g^{-1}\tilde{g}$ maps $\mathscr{K}_{X_1}$ to itself, and hence is a biregular automorphism. $\square$

6.5 *Morrison's movable cone conjecture*

Let X be a projective irreducible holomorphic symplectic manifold. We describe first an analogy between results on the ample cone of a projective $K3$ surface and results on the movable cone of X. Set $\mathrm{NS} := H^{1,1}(X, \mathbb{Z})$, $\mathrm{NS}_{\mathbb{R}} := \mathrm{NS} \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathrm{NS}_{\mathbb{Q}} := \mathrm{NS} \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\mathscr{C}_{\mathrm{NS}}$ be the intersection $\mathscr{C}_X \cap \mathrm{NS}_{\mathbb{R}}$.

Definition 6.21

(1) A line bundle L on X is *movable*, if the base locus of the linear system $|L|$ has codimension ≥ 2.

(2) The *movable cone* $\mathscr{M}\mathscr{V}_X$ is the convex hull in $\mathrm{NS}_{\mathbb{R}}$ of all classes of movable line bundles.

Let $\mathscr{M}\mathscr{V}^0_X$ be the interior of $\mathscr{M}\mathscr{V}_X$ and $\overline{\mathscr{M}\mathscr{V}}_X$ the closure of $\mathscr{M}\mathscr{V}_X$ in $\mathrm{NS}_{\mathbb{R}}$.

Lemma 6.22 *The equality $\mathcal{MV}_X^0 = \mathcal{FE}_X \cap \mathrm{NS}_{\mathbb{R}}$ holds. W_{Exc} acts faithfully on $\mathcal{C}_{\mathrm{NS}}$ and the map $Ch \mapsto Ch \cap \mathrm{NS}_{\mathbb{R}}$ induces a one-to-one correspondence between the set of exceptional chambers and the chambers in $\mathcal{C}_{\mathrm{NS}}$ of the W_{Exc} action. In particular, the closure of $\mathcal{MV}_X$ in $\mathcal{C}_{\mathrm{NS}}$ is a fundamental domain for the action of W_{Exc} on $\mathcal{C}_{\mathrm{NS}}$.*

Proof The equality $\mathcal{MV}_X^0 = \mathcal{FE}_X \cap \mathrm{NS}_{\mathbb{R}}$ follows immediately from the Zariski decomposition (Theorem 5.8). The set $\mathcal{S}pe$ is contained in NS, hence the W_{Exc} action on $\mathcal{C}_{\mathrm{NS}}$ is faithful and the map $Ch \mapsto Ch \cap \mathrm{NS}_{\mathbb{R}}$ induces a bijection. □

Let $\rho : Mon^2_{Hdg}(X) \to O(\mathrm{NS})$ be the restriction homomorphism. We denote $\rho(W_{Exc})$ by W_{Exc} as well.

Lemma 6.23

(1) The image Γ of ρ is a finite index subgroup of $O^+(\mathrm{NS})$.

(2) The kernel of ρ is a subgroup of $Mon^2_{Bir}(X)$.

(3) Γ is a semi-direct product of its normal subgroup W_{Exc} and the quotient group $\Gamma_{Bir} := Mon^2_{Bir}(X)/\ker(\rho)$.

Proof (1) The positive cone $\mathcal{C}_X$ is $Mon^2_{Hdg}(X)$-invariant and $\mathcal{C}_{\mathrm{NS}} = \mathcal{C}_X \cap \mathrm{NS}$ is thus Γ-invariant. Hence, Γ is a subgroup of $O^+(\mathrm{NS})$. Let $O^+_{Hdg}\left(H^2(X,\mathbb{Z})\right)$ be the subgroup of $O^+\left(H^2(X,\mathbb{Z})\right)$ preserving the Hodge structure. Then $O^+_{Hdg}\left(H^2(X,\mathbb{Z})\right)$ maps onto a finite index subgroup of $O^+(\mathrm{NS})$. The index of $Mon^2(X)$ in $O^+H^2(X,\mathbb{Z})$ is finite, by a result of Sullivan [Su] (see also [Ver2], Theorem 3.4). Hence, $Mon^2_{Hdg}(X)$ is a finite index subgroup of $O^+_{Hdg}\left(H^2(X,\mathbb{Z})\right)$. Part (1) follows.

(2) Let g be an element of $\ker(\rho)$. Then g acts trivially on $\mathcal{S}pe$. Hence, g maps $\mathcal{FE}_X$ to itself. It follows that g belongs to $Mon^2_{Bir}(X)$, by Theorem 6.18 part 4.

Part (3) is an immediate consequence of part (2) and Theorem 6.18 part 5. □

Let $\mathcal{E}\mathrm{ff}_X \subset \mathrm{NS}_{\mathbb{R}}$ be the convex cone generated by classes of effective divisors on X. Set $\mathcal{MV}_X^e := \overline{\mathcal{MV}}_X \cap \mathcal{E}\mathrm{ff}_X$. Following is Morrison's movable cone conjecture.

Conjecture 6.24 *[Mor1, Mor2, Ka] There exists a rational convex polyhedral cone (Definition 6.13 part 7) Π, which is a fundamental domain for the action of* $\mathrm{Bir}(X)$ *on $\mathcal{MV}_X^e$.*

Morrison formulated a version of the conjecture for the ample cone as well. The two versions coincide in dimension 2 and for abelian varieties. The $K3$ surface case of the conjecture is proven by Looijenga and Sterk ([St], Lemma 2.4), the Enriques surfaces case by Namikawa ([Nam], Theorem 1.4), the case of abelian and hyper-elliptic surfaces by Kawamata ([Ka], Theorem 2.1), the case of two-dimensional Calabi-Yau pairs by Totaro [Tot], and the case of abelian varieties by Prendergast-Smith [Pre]. A version of the conjectures for fiber spaces was formulated by Kawamata and proven in dimension 3 in [Ka].

The following theorem is a weaker version of Morrison's movable cone conjecture, in the special case of projective irreducible holomorphic symplectic manifolds. Let $\mathcal{MV}_X^+$ be the convex hull of $\overline{\mathcal{MV}}_X \cap \mathrm{NS}_{\mathbb{Q}}$. Clearly, $\mathcal{MV}_X^0$ is equal to the interior of both $\mathcal{MV}_X^+$ and $\mathcal{MV}_X^e$. When X is a $K3$ surface the equality $\mathcal{MV}_X^+ = \mathcal{MV}_X^e$ holds. In the $K3$ case the inclusion $\mathcal{MV}_X^+ \subset \mathcal{MV}_X^e$ follows from ([BHPV], Proposition 3.6 part i) and the inclusion $\mathcal{MV}_X^+ \supset \mathcal{MV}_X^e$ is proven in ([Ka], Proposition 2.4).

Theorem 6.25 *There exists a rational convex polyhedral cone Π in $\mathcal{MV}_X^+$, such that Π is a fundamental domain for the action of Γ_{Bir} on $\mathcal{MV}_X^+$.*

Proof The proof is identical to that of Lemma 2.4 in [St], which proves the $K3$-surface case of the Theorem. When X is a $K3$ surface, $\mathcal{MV}_X^0$ is the ample cone and $\mathcal{P}ex$ is the set of nodal -2 classes. The proof is lattice theoretic. Following is the dictionary translating our notation to that of Sterk.

Our notation	$\mathcal{MV}_X^0$	$\mathcal{C}_{\mathrm{NS}}$	$\mathcal{MV}_X^+$	$\mathcal{P}ex$	$\mathcal{S}pe$	Γ	Γ_{Bir}	W_{Exc}
Sterk's notation	K	$\mathcal{C}$	$\overline{K} \cap \mathcal{C}_+$	B	Δ^+	Γ	Γ_B	W

One slight inaccuracy in the above dictionary is that Sterk chose Γ to be the subgroup of $O^+\left(H^2(X,\mathbb{Z})\right)$ acting trivially on the transcendental lattice $\mathrm{NS}^\perp$, while we consider (in case X is a $K3$ surface) the image of $O^+_{Hdg}\left(H^2(X,\mathbb{Z})\right)$ in $O^+(\mathrm{NS})$. So Sterk's Γ is the finite index subgroup of our Γ acting trivially on the finite discriminant group $\mathrm{NS}^*/\mathrm{NS}$. Both choices satisfy the following complete list of assertions needed for the Looijenga-Sterk argument (in Sterk's notation).

(1) NS is a lattice of signature $(1,*)$ and Γ is an arithmetic subgroup of $O^+(\mathrm{NS})$.

(2) $W \subset O^+(\mathrm{NS})$ is the reflection group generated by reflections in elements of $B \subset \mathrm{NS}$.

(3) Γ_B is equal to the subgroup $\{g \in \Gamma \ : \ g(B) = B\}$.

(4) W is a normal subgroup of Γ and $\Gamma = \Gamma_B \cdot W$ is a semi-direct product decomposition.

(5) $\overline{K} \cap \mathscr{C}$ is a fundamental domain for the action of W on $\mathscr{C}$, cut-out by closed half-spaces associated to elements of B.

Assertion (1) is verified in our case in Lemma 6.23 part 1. Assertion (2) is verified in Theorem 6.18 part 3. $Mon^2_{Bir}(X) = \{g \in Mon^2_{Hdg}(X) \ : \ g(\mathcal{P}ex) = \mathcal{P}ex\}$, by Theorem 6.18 part 4 and Lemma 6.20. Assertion (3) follows from the latter equality by Lemma 6.23 part 2. Assertion (4) is verified in Lemma 6.23 part 3. Assertion (5) is verified in Lemma 6.22.

The argument proceeds roughly as follows. Choose a rational element $x_0 \in \mathscr{MV}_X$ which is not fixed by any element of Γ. Let $\mathscr{C}_+$ be the convex hull of $\overline{\mathscr{C}}_{\mathrm{NS}} \cap \mathrm{NS}_{\mathbb{Q}}$ in $\mathrm{NS}_{\mathbb{R}}$. Set

$$\Pi := \{x \in \mathscr{C}_+ \ : \ (x_0, x) \leq (x_0, \gamma(x)), \text{ for all } \gamma \in \Gamma\}.$$

Then Π is a fundamental domain for the Γ action on $\mathscr{C}_+$, known as the *Dirichlet domain with center* x_0 (compare[8] with [VS], Ch. 1 Proposition 1.10). Π is shown to be a rational convex polyhedron ([St], Lemma 2.3, see also Theorem 6.14 part (2) above). The above depends only on Assertion (1). The interior of any fundamental domain for Γ can not intersect any hyperplane $e^{\perp}$, $e \in \mathcal{P}ex$. Hence, Π is contained in $\mathscr{MV}^+_X$, by Assertions (2) and (5). $\mathscr{MV}^+_X$ is a fundamental domain for the W_{Exc} action on $\mathscr{C}_+$, by Assertion (5). Hence, any fundamental domain for the Γ-action on $\mathscr{C}_+$ which is contained in $\mathscr{MV}^+_X$, is a fundamental domain for the Γ_{Bir} action on $\mathscr{MV}^+_X$, by Assertions (3) and (4). □

Proof (Of Theorem 1.7) Assume that D is an irreducible divisor on X. Then D is either prime exceptional, or the class $[D]$ belongs to $\overline{\mathscr{MV}_X}$, by Theorem 5.8. If D is prime exceptional, the statement follows by the same argument used in the $K3$ surface case ([St], Proposition 2.5). Otherwise, $[D]$ belongs to $\mathscr{MV}^+_X$, and there exists $g \in \Gamma_{Bir}$, such that $g([D])$ belongs to the rational convex polyhedron Π in Theorem 6.25. The intersection $\Pi \cap \mathrm{NS}$ is a finitely generated semi-group. Choose generators $\{x_1, \ldots, x_m\}$. Then $(x_i, x_i) \geq 0$, and $(x_i, x_j) > 0$, if x_i and x_j are linearly independent. It follows that $\Pi \cap \mathrm{NS}$ contains at most finitely many elements of any given positive Beauville-Bogomolov degree, and at most finitely many primitive isotropic classes. □

[8] The bilinear pairing (x_0, x) in the above definition of the Dirichlet domain is replaced with the hyperbolic distance $\rho(x_0, x)$ in Definition 1.8 in Ch. 1 of [VS]. However, the two definitions are equivalent, by the relation $cosh(\rho(x_0, x)) = (x_0, x)$ (see Ch. 1 section 4.2 in [AVS]).

7 The monodromy and polarized monodromy groups

In section 7.1 we prove Proposition 1.9, stating that the polarized monodromy group $Mon^2(X,H)$ is the stabilizer of $c_1(H)$ in $Mon^2(X)$. In section 7.2 we fix a lattice Λ and define the coarse moduli space of polarized Λ-marked pairs of a given deformation type.

7.1 *Polarized parallel transport operators*

Let Ω_Λ be a period domain as in equation (2.1). Choose a connected component $\mathfrak{M}^0_\Lambda$ of the moduli space of marked pairs, a class $h \in \Lambda$ with $(h,h) > 0$, and let $\Omega^+_{h^\perp}$ be the period domain given in equation (4.1). Let $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ be the period map. Denote the inverse image $P_0^{-1}\left(\Omega^+_{h^\perp}\right)$ by $\mathfrak{M}^+_{h^\perp}$. The discussion in section 4 provides the following modular description of $\mathfrak{M}^+_{h^\perp}$. A marked pair (X,η) belongs to $\mathfrak{M}^+_{h^\perp}$, if and only if (X,η) belongs to $\mathfrak{M}^0_\Lambda$, the class $\eta^{-1}(h)$ is of Hodge type $(1,1)$, and $\eta^{-1}(h)$ belongs to the positive cone $\mathscr{C}_X$.

Proposition 7.1 $\mathfrak{M}^+_{h^\perp}$ *is path-connected.*

Proof The proof is similar to that of Proposition 5.11 in [Ma7]. The proof relies on the Global Torelli Theorem 2.2 and the connectedness of $\Omega^+_{h^\perp}$. □

Definition 7.2 Let $Mon^2\left(\mathfrak{M}^0_\Lambda\right)$ be the subgroup $\eta \circ Mon^2(X) \circ \eta^{-1} \subset O(\Lambda)$, for some marked pair $(X,\eta) \in \mathfrak{M}^0_\Lambda$. Let $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$ be the subgroup of $Mon^2\left(\mathfrak{M}^0_\Lambda\right)$ stabilizing h.

The subgroup $Mon^2\left(\mathfrak{M}^0_\Lambda\right)$ is independent of the choice of (X,η), since $\mathfrak{M}^0_\Lambda$ is connected, by definition. $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$ naturally acts on $\mathfrak{M}^+_{h^\perp}$.

Let

$$\mathfrak{M}^a_{h^\perp} \tag{7.1}$$

be the subset of $\mathfrak{M}^+_{h^\perp}$, consisting of isomorphism classes of pairs (X,η), such that $\eta^{-1}(h)$ is an ample class of X. The stability of Kähler manifolds implies that $\mathfrak{M}^a_{h^\perp}$ is an open subset of $\mathfrak{M}^+_{h^\perp}$ ([Voi], Theorem 9.3.3). We refer to $\mathfrak{M}^a_{h^\perp}$ as a *connected component of the moduli space of polarized marked pairs.*

Corollary 7.3 *$\mathfrak{M}^a_{h^\perp}$ is a $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$-invariant path-connected open Hausdorff subset of $\mathfrak{M}^+_{h^\perp}$. The period map restricts as an injective open $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$-equivariant morphism from $\mathfrak{M}^a_{h^\perp}$ onto an open dense subset of $\Omega^+_{h^\perp}$.*

Proof Let us check first that $\mathfrak{M}^a_{h^\perp}$ is $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$-invariant. Indeed, let (X,η) belong to $\mathfrak{M}^a_{h^\perp}$ and let g be an element of $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$. Denote by H the line bundle with $c_1(H) = \eta^{-1}(h)$. Then $g = \eta f \eta^{-1}$, for some $f \in Mon^2(X)$ stabilizing $c_1(H)$, by definition of $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$. The pair $(X, g\eta) = (X, \eta f)$ belongs to $\mathfrak{M}^0_\Lambda$, since f is a monodromy-operator. We have

$$(g\eta)^{-1}(h) = f^{-1}(\eta^{-1}(h)) = f^{-1}(c_1(H)) = c_1(H).$$

Hence, $(g\eta)^{-1}(h)$ is an ample class in $H^{1,1}(X,\mathbb{Z})$.

Let (X,η) and (Y,ψ) be two inseparable points of $\mathfrak{M}^a_{h^\perp}$. Then $\psi^{-1}\eta$ is a parallel-transport operator, preserving the Hodge structure, by Theorem 3.2. Furthermore, $\psi^{-1}\eta$ maps the ample class $\eta^{-1}(h)$ to the ample class $\psi^{-1}(h)$, by definition. Hence, there exists an isomorphism $f : X \to Y$, such that $f_* = \psi^{-1}\eta$, by Theorem 1.3 part 2. The two pairs (X,η) and (Y,ψ) are thus isomorphic. Hence, $\mathfrak{M}^a_{h^\perp}$ is a Hausdorff subset of $\mathfrak{M}^+_{h^\perp}$.

$\mathfrak{M}^a_{h^\perp}$ is the complement of a countable union of closed complex analytic subsets of $\mathfrak{M}^+_{h^\perp}$. Hence, $\mathfrak{M}^a_{h^\perp}$ is path-connected (see, for example, [Ver2], Lemma 4.10).

The period map restricts to an injective map on any Hausdorff subset of a connected component of the moduli space of marked pairs, by Theorem 2.2. The image of $\mathfrak{M}^a_{h^\perp}$ contains the subset of $\Omega^+_{h^\perp}$, consisting of points p, such that $\Lambda^{1,1}(p) = \mathrm{span}_{\mathbb{Z}}\{h\}$, by Huybrechts' projectivity criterion [Hu1], and Theorem 2.2. Hence, the image of $\mathfrak{M}^a_{h^\perp}$ is dense in $\Omega^+_{h^\perp}$. The image is open, since $\mathfrak{M}^a_{h^\perp}$ is an open subset and the period map is open, being a local homeomorphism. □

Let (X_i, H_i), $i = 1,2$, be two pairs, each consisting of a projective irreducible holomorphic symplectic manifold X_i, and an ample line bundle H_i. Set $h_i := c_1(H_i)$.

Corollary 7.4 *A parallel transport operator $f : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is a polarized parallel transport operator from (X_1,H_1) to (X_2,H_2) (Definition 1.1), if and only if $f(h_1) = h_2$.*

Proof The 'only if' part is clear. We prove the 'if' part. Assume that $f(h_1) = h_2$. Choose a marking $\eta_2 : H^2(X_2,\mathbb{Z}) \to \Lambda$, and set $\eta_1 := \eta_2 \circ f$. Then $\eta_1(h_1) = \eta_2(h_2)$. Denote both $\eta_i(h_i)$ by h. Let $\mathfrak{M}^0_\Lambda$ be the connected component of (X_1,η_1). Then (X_2,η_2) belongs to $\mathfrak{M}^0_\Lambda$, by the assumption that f is a parallel transport operator.

Consequently, $P_0(X_i, \eta_i)$, $i = 1, 2$, both belong to the same connected component of $\Omega_{h^\perp}$. We may choose η_2, so that this connected component is $\Omega^+_{h^\perp}$. Then (X_1, η_1) and (X_2, η_2) both belong to $\mathfrak{M}^a_{h^\perp}$.

Choose a path $\gamma : [0,1] \to \mathfrak{M}^a_{h^\perp}$, with $\gamma(0) = (X_1, \eta_1)$ and $\gamma(1) = (X_2, \eta_2)$. This is possible, by Corollary 7.3. For each $t \in [0,1]$, there exists a simply-connected open neighborhood U_t of $\gamma(t)$ in $\mathfrak{M}^a_{h^\perp}$ and a semi-universal family $\pi_t : \mathscr{X}_t \to U_t$. The covering $\{U_t\}_{t\in[0,1]}$ of $\gamma([0,1])$ has a finite sub-covering $\{V_j\}_{j=1}^k$, for some integer[9] $k \geq 1$, with the property that $\gamma\left(\left[\frac{j-1}{k}, \frac{j}{k}\right]\right)$ is contained in V_j. Consider the analytic space B, obtained from the disjoint union of V_j, $1 \leq j \leq k$, by gluing V_j to V_{j+1} at the single point $\gamma\left(\frac{j}{k}\right)$ with transversal Zariski tangent spaces. Let $\pi_j : \mathscr{X}_j \to V_j$ be the universal family and denote its fiber over $v \in V_j$ by $\mathscr{X}_{j,v}$. Endow each fiber $\mathscr{X}_{j,v}$, of π_j over $v \in V_j$, with the marking $H^2(\mathscr{X}_{j,v}, \mathbb{Z}) \to \Lambda$ corresponding to the point v. For $1 \leq j \leq k$, choose an isomorphism of $\mathscr{X}_{j,\gamma\left(\frac{j}{k}\right)}$ with $\mathscr{X}_{j+1,\gamma\left(\frac{j}{k}\right)}$ compatible with the marking chosen, and use it to glue the family π_j to the family π_{j+1}. We get a family $\pi : \mathscr{X} \to B$. The paths $\gamma : \left[\frac{j-1}{k}, \frac{j}{k}\right] \to V_j$ can now be reglued to a path $\tilde{\gamma} : [0,1] \to B$. Parallel transport along $\tilde{\gamma}$ induces the isomorphism $\eta^{-1}_{\tilde{\gamma}(1)} \circ \eta_{\tilde{\gamma}(0)} = \eta^{-1}_{\gamma(1)} \circ \eta_{\gamma(0)} = \eta_2^{-1} \circ \eta_1 = f$. Hence, f is a polarized parallel transport operator from (X_1, H_1) to (X_2, H_2). □

7.2 Deformation types of polarized marked pairs

Fix an irreducible holomorphic symplectic manifold X_0 and let Λ be the lattice $H^2(X_0, \mathbb{Z})$, endowed with the Beauville-Bogomolov pairing. Let τ be the set of connected components of $\mathfrak{M}_\Lambda$, consisting of pairs (X, η), such that X is deformation equivalent to X_0.

Lemma 7.5 *The set τ is finite. The group $O(\Lambda)$ acts transitively on τ and the stabilizer of a connected component $\mathfrak{M}^0_\Lambda \in \tau$ is the subgroup $Mon^2(\mathfrak{M}^0_\Lambda)$, introduced in Definition 7.2.*

Proof The set $O[H^2(X, \mathbb{Z})]/Mon^2(X)$ is finite, by a result of Sullivan [Su] (see also [Ver2], Theorem 3.4). The rest of the statement is clear. □

Denote by $\mathfrak{M}^\tau_\Lambda$ the disjoint union of connected components parametrized by the set τ. We refer to $\mathfrak{M}^\tau_\Lambda$ as *the moduli space of marked pairs of deformation type τ.*

[9] We could take $k = 1$, if there exists a universal family over $\mathfrak{M}^a_{h^\perp}$, but such a family need not exist.

An example would be the moduli space of marked pairs of $K3^{[n]}$-type. Given a point $t \in \tau$, denote by $\mathfrak{M}^t_\Lambda$ the corresponding connected component of $\mathfrak{M}^\tau_\Lambda$.

Remark 7.6 If $Mon^2(X)$ is a normal subgroup of $O[H^2(X,\mathbb{Z})]$, then the subgroup $Mon^2(\mathfrak{M}^t_\Lambda)$ of $O(\Lambda)$ is equal to a fixed subgroup $Mon^2(\tau,\Lambda) \subset O(\Lambda)$, for all $t \in \tau$. This is the case when X is of $K3^{[n]}$-type (Theorem 9.1). The set τ is an $O(\Lambda)/Mon^2(\tau,\Lambda)$-torsor, by Lemma 7.5. We will identify the torsor τ with an explicit lattice theoretic $O(\Lambda)/Mon^2(\tau,\Lambda)$-torsor in Corollary 9.10.

We get the *refined period map*

$$\widetilde{P} : \mathfrak{M}^\tau_\Lambda \longrightarrow \Omega_\Lambda \times \tau, \tag{7.2}$$

sending a marked pair (X,η) to the pair $(P(X,\eta),t)$, where $\mathfrak{M}^t_\Lambda$ is the connected component containing (X,η). Then $\widetilde{P}$ is $O(\Lambda)$-equivariant with respect to the diagonal action of $O(\Lambda)$ on $\Omega_\Lambda \times \tau$.

Given $h \in \Lambda$, with $(h,h) > 0$, denote by $\Omega^{t,+}_{h^\perp}$ the period domain associated to $\mathfrak{M}^t_\Lambda$ in equation (4.1). Set $\mathfrak{M}^{t,+}_{h^\perp} := \widetilde{P}^{-1}(\Omega^{t,+}_{h^\perp})$. Let $\mathfrak{M}^{t,a}_{h^\perp} \subset \mathfrak{M}^{t,+}_{h^\perp}$ be the open subset of polarized pairs introduced in equation (7.1).

We construct next a polarized analogue of the refined period map. Given an $O(\Lambda)$-orbit $\bar{h} \subset \Lambda \times \tau$, of pairs (h,t) with $(h,h) > 0$, consider the disjoint unions

$$\mathfrak{M}^+_{\bar{h}} := \bigcup_{(h,t)\in\bar{h}} \mathfrak{M}^{t,+}_{h^\perp},$$

$$\Omega^+_{\bar{h}} := \bigcup_{(h,t)\in\bar{h}} \Omega^{t,+}_{h^\perp},$$

and let

$$\widetilde{P} : \mathfrak{M}^+_{\bar{h}} \longrightarrow \Omega^+_{\bar{h}} \tag{7.3}$$

be the map induced by the refined period map on each connected component. Then $\widetilde{P}$ is $O(\Lambda)$-equivariant and surjective. The disjoint union

$$\mathfrak{M}^a_{\bar{h}} := \bigcup_{(h,t)\in\bar{h}} \mathfrak{M}^{t,a}_{h^\perp} \tag{7.4}$$

is an $O(\Lambda)$-invariant open subset of $\mathfrak{M}^+_{\bar{h}}$. This open subset will be called *the moduli space of polarized marked pairs of deformation type* $\bar{h}$. Indeed, $\mathfrak{M}^a_{\bar{h}}$ coarsely represents a functor from the category of analytic spaces to sets, associating to a complex analytic space T the set of all equivalence classes of families of marked polarized triples (X, L, η), where X is of deformation type τ, L is an ample line bundle, and $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ is an isometry, such that the pair $[\eta(c_1(L)), t]$ belongs to the $O(\Lambda)$-orbit $\bar{h}$, where $\mathfrak{M}^t_\Lambda$ is the connected component of (X,η). A family $(\pi : \mathscr{X} \to T, \mathscr{L}, \tilde{\eta})$ consists of a family π, an element $\mathscr{L}$ of $\mathrm{Pic}(\mathscr{X}/T)$ and a trivialization $\tilde{\eta} : R^2\pi_*\mathbb{Z} \to (\Lambda)_T$, via isometries. Two families $(\mathscr{X} \to T, \mathscr{L}, \tilde{\eta})$ and $(\mathscr{X}' \to T, \mathscr{L}', \tilde{\eta}')$ are equivalent, if there exists a T-isomorphism $f : \mathscr{X} \to \mathscr{X}'$, such that $f^*\mathscr{L}' \cong \mathscr{L}$ and $\tilde{\eta}' = \tilde{\eta} \circ f^*$. We omit the detailed definition of this functor, as well as the proof that $\mathfrak{M}^a_{\bar{h}}$ coarsely represents it, as we will not use the latter fact below.

8 Monodromy quotients of type IV period domains

Fix a connected component $\mathfrak{M}^a_{h^\perp}$ of the moduli space $\mathfrak{M}^a_{\bar{h}}$ of polarized marked pairs of polarized deformation type $\bar{h}$. In the notation of section 7.2, $\mathfrak{M}^a_{h^\perp} := \mathfrak{M}^{t,a}_{h^\perp}$, for some $(h,t) \in \bar{h}$. Let $\mathfrak{M}^0_\Lambda$ be the connected component of $\mathfrak{M}_\Lambda$ containing $\mathfrak{M}^a_{h^\perp}$. Set $\Gamma := Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$ (Definition 7.2). The period domain $\Omega^+_{h^\perp}$ is a homogeneous domain of type IV ([Sa], Appendix, section 6). Γ is an arithmetic group, by ([Ver2], Theorem 3.5). The quotient $\Omega^+_{h^\perp}/\Gamma$ is thus a normal quasi-projective variety [BB].

Lemma 8.1 *There exist natural isomorphisms of complex analytic spaces*

$$\begin{aligned} \mathfrak{M}^+_{\bar{h}}/O(\Lambda) &\longrightarrow \mathfrak{M}^+_{h^\perp}/\Gamma, \\ \mathfrak{M}^a_{\bar{h}}/O(\Lambda) &\longrightarrow \mathfrak{M}^a_{h^\perp}/\Gamma, \\ \Omega^+_{\bar{h}}/O(\Lambda) &\longrightarrow \Omega^+_{h^\perp}/\Gamma. \end{aligned}$$

Furthermore, the period map descends to an open embedding

$$\overline{P} : \mathfrak{M}^a_{\bar{h}}/O(\Lambda) \hookrightarrow \Omega^+_{h^\perp}/\Gamma. \tag{8.1}$$

Proof We have the following commutative equivariant diagram

$$\begin{array}{ccccc} \mathfrak{M}^a_{\bar{h}} & \xrightarrow{\widetilde{P}} & \Omega^+_{\bar{h}} & \longrightarrow & \Omega^+_{\bar{h}}/O(\Lambda) \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{M}^a_{h^\perp} & \xrightarrow{P_0} & \Omega^+_{h^\perp} & \longrightarrow & \Omega^+_{h^\perp}/\Gamma, \end{array}$$

with respect to the $O(\Lambda)$ action on the top row, the Γ-action on the bottom, and the inclusion homomorphism $\Gamma \hookrightarrow O(\Lambda)$. $O(\Lambda)$ acts transitively on its orbit $\bar{h}$, and the stabilizer of the pair $(h, \mathfrak{M}^{+}_{h^{\perp}}) \in \bar{h}$ is precisely Γ, by Lemma 7.5 and Proposition 1.9.

The morphism (8.1) is an open embedding, since the Γ-equivariant open morphism $\mathfrak{M}^{a}_{h^{\perp}} \to \Omega^{+}_{h^{\perp}}$ is injective, by Corollary 7.3. □

A *polarized irreducible holomorphic symplectic manifold* is a pair (X,L), consisting of a smooth projective irreducible holomorphic symplectic variety X and an ample line bundle L. Consider the contravariant functor F' from the category of schemes over $\mathbb{C}$ to the category of sets, which associates to a scheme T the set of isomorphism classes of flat families of polarized irreducible holomorphic symplectic manifolds (X,L) over T, with a fixed Hilbert polynomial $p(n) := \chi(L^n)$. The coarse moduli space representing the functor F' was constructed by Viehweg as a quasi-projective scheme with quotient singularities [Vieh]. Fix a connected component $\mathscr{V}$ of this moduli space. Then $\mathscr{V}$ is a quasi-projective variety. Denote by F the functor represented by the connected component $\mathscr{V}$. The universal property of a coarse moduli space asserts that there is a natural transformation $\theta : F \to \mathrm{Hom}(\bullet, \mathscr{V})$, satisfying the following properties.

(1) $\theta(\mathrm{Spec}(\mathbb{C})) : F(\mathrm{Spec}(\mathbb{C})) \to \mathscr{V}$ is bijective.

(2) Given a scheme B and a natural transformation $\chi : F \to \mathrm{Hom}(\bullet, B)$, there is a unique morphism $\psi : \mathscr{V} \to B$, hence a natural transformation $\psi_* : Hom(\bullet, \mathscr{V}) \to \mathrm{Hom}(\bullet, B)$, with $\chi = (\psi_*) \circ \theta$.

Remark 8.2 Property (2) replaces the data of a universal family over $\mathscr{V}$, which may not exist when $\mathscr{V}$ fails to be a fine moduli space. When a universal family $\mathscr{U} \in F(\mathscr{V})$ exists, then the morphism ψ is the image of $\mathscr{U}$ via $\chi : F(\mathscr{V}) \to \mathrm{Hom}(\mathscr{V}, B)$.

Denote by $\bar{h}$ the deformation type of a polarized pair (X,L) in $\mathscr{V}$. We regard $\bar{h}$ both as a point in $[\Lambda \times \tau]/O(\Lambda)$ and as a subset of $\Lambda \times \tau$. Choose a point $(h,t) \in \bar{h}$ and set $\Omega^{+}_{h^{\perp}} := \Omega^{t,+}_{h^{\perp}}$.

Lemma 8.3 *There exists a natural injective and surjective morphism $\varphi : \mathscr{V} \to \mathfrak{M}^{a}_{\bar{h}}/O(\Lambda)$ in the category of analytic spaces.*

Proof The morphism $\Phi : \mathscr{V} \to \Omega^{+}_{\bar{h}}/O(\Lambda) \cong \Omega^{+}_{h^{\perp}}/\Gamma$, sending an isomorphism class of a polarized pair (X,L) to its period, is constructed in the proof of ([GHS1], Theorem 1.5). The morphism Φ is set-theoretically injective, by the Hodge theoretic Torelli Theorem 1.3. The image $\Phi(\mathscr{V})$ is the same subset as the image $P\left(\mathfrak{M}^{a}_{\bar{h}}\right)$, by definition of the two moduli spaces. The latter is the image also of the open immersion $\overline{P} : \mathfrak{M}^a_{\bar{h}}/O(\Lambda) \hookrightarrow \Omega^+_{\bar{h}}/O(\Lambda)$, by Lemma 8.1. Hence, the composition $\overline{P}^{-1} \circ \Phi : \mathscr{V} \to \mathfrak{M}^a_{\bar{h}}/O(\Lambda)$ is well defined and we denote it by φ. □

Theorem 8.4 *The composition Φ of*

$$\mathscr{V} \xrightarrow{\varphi} \mathfrak{M}^a_{\bar{h}}/O(\Lambda) \cong \mathfrak{M}^a_{h^\perp}/\Gamma \xrightarrow{\overline{P}} \Omega^+_{h^\perp}/\Gamma$$

is an open immersion in the category of algebraic varieties.

Proof The proof is similar to that of Theorem 1.5 in [GHS1] and Claim 5.4 in [O'G5]. If Γ happens to be torsion free, then any complex analytic morphism, from a complex algebraic variety to $\Omega^+_{h^\perp}/\Gamma$, is an algebraic morphism, as a consequence of Borel's extension theorem [Bo]. Γ need not be torsion free, but for sufficiently large positive integer N, the subgroup $\Gamma(N) \subset \Gamma$, acting trivially on $\Lambda/N\Lambda$, is torsion free, as a consequence of ([Sa], IV, Lemma 7.2). In our situation, where the domain $\mathscr{V}$ of Φ is a moduli space, one can apply Borel's extension theorem after passage to a finite cover $\widetilde{\mathscr{V}} \to \mathscr{V}$, where $\widetilde{\mathscr{V}}$ is a connected component of the moduli space of polarized irreducible holomorphic symplectic manifolds with a level-N structure, as done in the proofs of ([Has], Proposition 2.2.2) and ([GHS1], Theorem 1.5). The morphism Φ lifts to a morphism $\widetilde{\Phi} : \widetilde{\mathscr{V}} \to \Omega^+_{h^\perp}/\Gamma(N)$. $\widetilde{\Phi}$ is algebraic, by Borel's extension theorem, and a descent argument implies that so is Φ.

The morphism $P : \mathfrak{M}^a_{\bar{h}} \to \Omega^+_{\bar{h}}/O(\Lambda)$ is open. Hence, the image $\overline{P}(\mathfrak{M}^a_{h^\perp}/\Gamma)$ of Φ is an open subset of $\Omega^+_{h^\perp}/\Gamma$ in the analytic topology. The image of Φ is also a constructibe set, in the Zariski topology. The image is thus a Zariski dense open subset. Φ is thus an algebraic open immersion, by Zariski's Main Theorem. □

Remark 8.5 Theorem 8.4 answers Question 2.6 in the paper [GHS1], concerning the polarized $K3^{[n]}$-type moduli spaces. The map Φ in Theorem 8.4 is denoted by $\tilde{\varphi}$ in ([GHS1], Question 2.6) and is defined in ([GHS1], Theorem 2.3). There is a typo in the definition of $\tilde{\varphi}$ in [GHS1]; its target $\widetilde{O}^+(L_{2n-2}, h)\backslash\mathscr{D}_h$ should be replaced by $\widehat{O}^+(L_{2n-2}, h)\backslash\mathscr{D}_h$. When $n = 2$, these two quotients are the same, but for $n \geq 3$, the former is a branched double cover of the latter. Modulo this minor change, Theorem 8.4 provides an affirmative answer to ([GHS1], Question 2.6).

9 The $K3^{[n]}$ deformation type

In section 9.1 we review results about parallel-transport operators of $K3^{[n]}$-type. In section 9.2 we explicitly calculate the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$ of a projective manifold X of $K3^{[n]}$-type.

9.1 Characterization of parallel-transport operators of $K3^{[n]}$-type

In sections 9.1.1, 9.1.2, and 9.1.3, we provide three useful characterizations of the monodromy group $Mon^2(X)$ of an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type. Given X_1 and X_2 of $K3^{[n]}$-type, we state in section 9.1.4 a necessary and sufficient condition for an isometry $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ to be a parallel-transport operator.

9.1.1 First two characterizations of $Mon^2(K3^{[n]})$

Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type. If $n = 1$, then X is a $K3$ surface. In that case it is well known that $Mon^2(X) = O^+H^2(X,\mathbb{Z})$ (see [Bor]). From now on we assume that $n \geq 2$.

Given a class $u \in H^2(X,\mathbb{Z})$, with $(u,u) \neq 0$, let $R_u : H^2(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})$ be the reflection $R_u(\lambda) = \lambda - \frac{2(u,\lambda)}{(u,u)}u$. Set $\rho_u := \begin{cases} R_u & \text{if } (u,u) < 0, \\ -R_u & \text{if } (u,u) > 0. \end{cases}$ Then ρ_u belongs to $O^+H^2(X,\mathbb{Q})$. Note that ρ_u is an integral isometry, if $(u,u) = 2$ or -2. Let $\mathscr{N} \subset O^+H^2(X,\mathbb{Z})$ be the subgroup generated by such ρ_u.

$$\mathscr{N} \ := \ \langle \rho_u \ : \ u \in H^2(X,\mathbb{Z}) \text{ and } (u,u) = 2 \text{ or } (u,u) = -2 \rangle. \tag{9.1}$$

Clearly, $\mathscr{N}$ is a normal subgroup.

Theorem 9.1 *([Ma5], Theorem 1.2)* $Mon^2(X) = \mathscr{N}$.

A second useful description of $Mon^2(X)$ depends on the fact that the lattice $H^2(X,\mathbb{Z})$ is isometric to the orthogonal direct sum

$$\Lambda \ := \ E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U \oplus \mathbb{Z}\delta,$$

where $E_8(-1)$ is the negative definite (unimodular) E_8 root lattice, U is the rank 2 unimodular lattice of signature $(1,1)$, and $(\delta,\delta) = 2-2n$. See [Be1] for a proof of this fact.

Set $\Lambda^* := \mathrm{Hom}(\Lambda,\mathbb{Z})$. Then Λ^*/Λ is a cyclic group of order $2n-2$. Let $O(\Lambda^*/\Lambda)$ be the subgroup of $\mathrm{Aut}(\Lambda^*/\Lambda)$ consisting of multiplication by all elements of $t \in \mathbb{Z}/(2n-2)\mathbb{Z}$, such that $t^2 = 1$. Then $O(\Lambda^*/\Lambda)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where r is the number of distinct primes in the prime factorization $n-1 = p_1^{d_1}\cdots p_r^{d_r}$ of $n-1$ (see [Ogu]). The isometry group $O(\Lambda)$ acts on Λ^*/Λ and the image of $O^+(\Lambda)$ in $\mathrm{Aut}(\Lambda^*/\Lambda)$ is equal to $O(\Lambda^*/\Lambda)$ ([Ni], Theorem 1.14.2).

Let $\pi : O^+(\Lambda) \to O(\Lambda^*/\Lambda)$ be the natural homomorphism. The following characterization of the monodromy group follows from Theorem 9.1 via lattice theoretic arguments.

Lemma 9.2 *([Ma5], Lemma 4.2) $Mon^2(X)$ is equal to the inverse image via π of the subgroup $\{1,-1\} \subset O(\Lambda^*/\Lambda)$.*

We conclude that the index of $Mon^2(X)$ in $O^+H^2(X,\mathbb{Z})$ is 2^{r-1}, and $Mon^2(X) = O^+H^2(X,\mathbb{Z})$, if and only if $n=2$ or $n-1$ is a prime power. If $n=7$, for example, then $Mon^2(X)$ has index two in $O^+H^2(X,\mathbb{Z})$.

9.1.2 A third characterization of $Mon^2(K3^{[n]})$

The third characterization of $Mon^2(X)$ is more subtle, as it depends also on $H^4(X,\mathbb{Z})$. It is however this third characterization that will generalize to the case of parallel transport operators.

Given a $K3$ surface S, denote by $K(S)$ the integral K-ring generated by the classes of complex topological vector bundles over S. Let $\chi : K(S) \to \mathbb{Z}$ be the Euler characteristic $\chi(x) = \int_S ch(x)td_S$. Given classes $x,y \in K(S)$, let $x^\vee$ be the dual class and set

$$(x,y) \;:=\; -\chi(x^\vee \otimes y). \tag{9.2}$$

The above yields a unimodular symmetric bilinear pairing on $K(S)$, called the *Mukai pairing* [Mu1]. The lattice $K(S)$, endowed with the Mukai pairing, is isometric to the orthogonal direct sum

$$\widetilde{\Lambda} \;:=\; E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U \oplus U$$

and is called the *Mukai lattice*.

Let $Q^4(X,\mathbb{Z})$ be the quotient of $H^4(X,\mathbb{Z})$ by the image of the cup product homomorphism $\cup : H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z})$. Clearly, $Q^4(X,\mathbb{Z})$ is a $Mon(X)$-module, and it comes with a pure integral Hodge structure of weight 4. Let $q : H^4(X,\mathbb{Z}) \to Q^4(X,\mathbb{Z})$ be the natural homomorphism and set $\bar{c}_2(X) := q(c_2(TX))$.

Theorem 9.3 *([Ma5], Theorem 1.10) Let X be of $K3^{[n]}$-type, $n \geq 4$.*

(1) $Q^4(X,\mathbb{Z})$ is a free abelian group of rank 24.

(2) The element $\frac{1}{2}\bar{c}_2(X)$ is an integral and primitive class in $Q^4(X,\mathbb{Z})$.

(3) There exists a unique symmetric, even, integral, unimodular, $Mon(X)$-invariant bilinear pairing $(\bullet,\bullet)$ on $Q^4(X,\mathbb{Z})$, such that $\left(\frac{\bar{c}_2(X)}{2},\frac{\bar{c}_2(X)}{2}\right) = 2n-2$. The resulting lattice $\left[Q^4(X,\mathbb{Z}),(\bullet,\bullet)\right]$ is isometric to the Mukai lattice $\widetilde{\Lambda}$.

(4) The $Mon(X)$-module $\mathrm{Hom}\left[H^2(X,\mathbb{Z}),Q^4(X,\mathbb{Z})\right]$ contains a unique integral rank 1 saturated $Mon(X)$-submodule

$$\mathbb{E}(X),$$

which is a sub-Hodge structure of type $(1,1)$. A generator $e \in \mathbb{E}(X)$ induces a Hodge-isometry

$$e : H^2(X,\mathbb{Z}) \longrightarrow \bar{c}_2(X)^{\perp}$$

onto the co-rank 1 sublattice of $Q^4(X,\mathbb{Z})$ orthogonal to $\bar{c}_2(X)$.

Parts (1), (3), and (4) of the Theorem are explained in the following section 9.1.3.

Denote by $O(\Lambda,\widetilde{\Lambda})$ the set of primitive isometric embeddings of the $K3^{[n]}$-lattice Λ into the Mukai lattice $\widetilde{\Lambda}$. The isometry groups $O(\Lambda)$ and $O(\widetilde{\Lambda})$ act on $O(\Lambda,\widetilde{\Lambda})$. The action on $\iota \in O(\Lambda,\widetilde{\Lambda})$, of elements $g \in O(\Lambda)$, and $f \in O(\widetilde{\Lambda})$, is given by $(g,f)\iota = f \circ \iota \circ g^{-1}$.

Lemma 9.4 *([Ma5], Lemma 4.3) $O^+(\Lambda) \times O(\widetilde{\Lambda})$ acts transitively on $O(\Lambda,\widetilde{\Lambda})$. The subgroup $\mathscr{N} \subset O^+(\Lambda)$, given in (9.1), is equal to the stabilizer in $O^+(\Lambda)$ of every point in the orbit space $O(\Lambda,\widetilde{\Lambda})/O(\widetilde{\Lambda})$.*

The lemma implies that $O(\Lambda, \widetilde{\Lambda})$ is a finite set of order $[\mathcal{N} : O^+(\Lambda)]$. The following is our third characterization of $Mon^2(X)$.

Corollary 9.5

(1) An irreducible holomorphic symplectic manifold X of $K3^{[n]}$-type, $n \geq 2$, comes with a natural choice of an $O(\widetilde{\Lambda})$-orbit ι_X of primitive isometric embeddings of $H^2(X,\mathbb{Z})$ in the Mukai lattice $\widetilde{\Lambda}$.

(2) The subgroup $Mon^2(X)$ of $O^+[H^2(X,\mathbb{Z})]$ is equal to the stabilizer of ι_X as an element of the orbit space $O\left(H^2(X,\mathbb{Z}), \widetilde{\Lambda}\right)/O(\widetilde{\Lambda})$.

Proof Part (1): If $n = 2$, or $n = 3$, then $O(\Lambda, \widetilde{\Lambda})$ is a singleton, and there is nothing to prove. Assume that $n \geq 4$. Let $e : H^2(X,\mathbb{Z}) \to Q^4(X,\mathbb{Z})$ be one of the two generators of $\mathbb{E}(X)$. Choose an isometry $g : Q^4(X,\mathbb{Z}) \to \widetilde{\Lambda}$. This is possible by Theorem 9.3. Set $\iota := g \circ e : H^2(X,\mathbb{Z}) \to \widetilde{\Lambda}$ and let ι_X be the orbit $O(\widetilde{\Lambda})\iota$. Then ι_X is independent of the choice of g. If we choose $-e$ instead we get the same orbit, since -1 belongs to $O(\widetilde{\Lambda})$.

Part (2): Follows immediately from Theorem 9.1 and Lemma 9.4. □

Example 9.6 Let S be a projective $K3$ surface, H an ample line bundle on S, and $v \in K(S)$ a class in the K-group. Denote by $M_H(v)$ the moduli space of Gieseker-Maruyama-Simpson H-stable coherent sheaves on S of class v. A good reference about these moduli spaces is the book [HL]. Assume that $M_H(v)$ is smooth and projective (i.e., we assume that every H-semi-stable sheaf is automatically also H-stable). Then $M_H(v)$ is known to be connected and of $K3^{[n]}$-type, by a theorem due to Mukai, Huybrechts, O'Grady, and Yoshioka. It can be found in its final form in [Y2].

Let π_i be the projection from $S \times M_H(v)$ onto the i-th factor, $i = 1,2$. Denote by $\pi_{2_!} : K[S \times M_H(v)] \to K[M_H(v)]$ the Gysin map and by $\pi_1^! : K(S) \to K[S \times M_H(v)]$ the pull-back homomorphism. Assume, further, that there exists a universal sheaf $\mathscr{E}$ over $S \times M_H(v)$. Let $[\mathscr{E}] \in K[S \times M_H(v)]$ be the class of the universal sheaf in the K-group. We get the natural homomorphism

$$u : K(S) \to K(M_H(v)), \tag{9.3}$$

given by $u(x) := \pi_{2_!}\left\{\pi_1^!(x^\vee) \otimes [\mathscr{E}]\right\}$. Let $v^\perp \subset K(S)$ be the co-rank 1 sub-lattice of $K(S)$ orthogonal to the class v and consider Mukai's homomorphism

$$\theta : v^\perp \longrightarrow H^2(M_H(v),\mathbb{Z}), \tag{9.4}$$

given by $\theta(x) = c_1[u(x)]$. Then θ is an isometry, with respect to the Mukai pairing on $v^\perp$, and the Beauville-Bogomolov pairing on $H^2(\mathscr{M}_H(v), \mathbb{Z})$, by the work of Mukai, Huybrechts, O'Grady, and Yoshioka [Y2]. Furthermore, the orbit $\iota_{M_H(v)}$ of Corollary 9.5 is represented by the inverse of θ

$$\iota_{M_H(v)} = O[K(S)] \cdot \theta^{-1}, \tag{9.5}$$

by ([Ma5], Theorem 1.14).

9.1.3 Generators for the cohomology ring $H^*(X, \mathbb{Z})$

Part (1) of Theorem 9.3 is a simple consequence of the following result. Consider the case, where X is a moduli space M of H-stable sheaves on a $K3$ surface S and M is of $K3^{[n]}$-type, as in Example 9.6. Choose a basis $\{x_1, x_2, \ldots, x_{24}\}$ of $K(S)$. Let $u : K(S) \to K(M)$ be the homomorphism given in equation (9.3).

Theorem 9.7 *([Ma4], Theorem 1) The cohomology ring $H^*(M, \mathbb{Z})$ is generated by the Chern classes $c_j(u(x_i))$, for $1 \le i \le 24$, and for j an integer in the range $0 \le j \le 2n$.*

The map $\tilde{\varphi} : K(S) \to H^4(M, \mathbb{Z})$, given by $\tilde{\varphi}(x) = c_2(u(x))$, is not a group homomorphism. Nevertheless, the composition $\varphi := q \circ \tilde{\varphi} : K(S) \to Q^4(M, \mathbb{Z})$, of $\tilde{\varphi}$ with the projection $q : H^4(M, \mathbb{Z}) \to Q^4(M, \mathbb{Z})$, is a homomorphism of abelian groups ([Ma4], Proposition 2.6). We note here only that 2φ is clearly a group homomorphism, since $2c_2(y) = c_1^2(y) - 2ch_2(y)$, the map $2ch_2 : K(M) \to H^4(M, \mathbb{Z})$ is known to be a group homomorphism, and the term $c_1^2(y)$ is annihilated by the projection to $Q^4(M, \mathbb{Z})$.

Part (1) of Theorem 9.3 follows from the fact that φ is an isomorphism. The homomorphism φ is surjective, by Theorem 9.7. It remains to prove that φ is injective. Injectivity would follow, once we show that $Q^4(M, \mathbb{Z})$ has rank 24. Now cup product induces an injective homomorphism $\mathrm{Sym}^2 H^2(M, \mathbb{Q}) \to H^4(M, \mathbb{Q})$, for any irreducible holomorphic symplectic manifold of dimension ≥ 4, by a general result of Verbitsky [Ver1]. When $n \ge 4$, i.e., $\dim_{\mathbb{C}}(M) \ge 8$, then $\dim H^4(M, \mathbb{Q}) - \dim \mathrm{Sym}^2 H^2(M, \mathbb{Q}) = 24$, by Göttsche's formula for the Betti numbers of $S^{[n]}$ [Gö]. Hence, the rank of $Q^4(M, \mathbb{Z})$ is 24.

The bilinear pairing on $Q^4(M, \mathbb{Z})$, constructed in part (3) of Theorem 9.3, is simply the push-forward via the isomorphism φ of the Mukai pairing on $K(S)$. We then show that this bilinear pairing is monodromy invariant, hence it defines a bilinear pairing on $Q^4(X,\mathbb{Z})$, for any X of $K3^{[n]}$-type.

The isometric embedding $e : H^2(M,\mathbb{Z}) \to Q^4(M,\mathbb{Z})$, constructed in part (4) of Theorem 9.3, is simply the composition $\varphi \circ \theta^{-1}$, where θ is given in equation (9.4). We show that the composition is $Mon(M)$-equivariant, up to sign, hence defines the $Mon(X)$-submodule $\mathbb{E}(X)$ in part (4) of Theorem 9.3, for any X of $K3^{[n]}$-type.

9.1.4 Parallel transport operators of $K3^{[n]}$-type

Let X_1 and X_2 be irreducible holomorphic symplectic manifolds of $K3^{[n]}$-type. Denote by ι_{X_i} the natural $O(\widetilde{\Lambda})$-orbit of primitive isometric embedding of $H^2(X_i,\mathbb{Z})$ into the Mukai lattice $\widetilde{\Lambda}$, given in Corollary 9.5.

Theorem 9.8 *An isometry $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is a parallel-transport operator, if and only if g is orientation preserving and*

$$\iota_{X_1} = \iota_{X_2} \circ g. \tag{9.6}$$

Proof Assume first that g is a parallel-transport operator. Then g lifts to a parallel-transport operator $\tilde{g} : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$. Now $\tilde{g}$ induces a parallel-transport operators $\tilde{g}_4 : Q^4(X_1,\mathbb{Z}) \to Q^4(X_2,\mathbb{Z})$, as well as

$$Ad_{\tilde{g}} : \mathrm{Hom}\left[H^2(X_1,\mathbb{Z}), Q^4(X_1,\mathbb{Z})\right] \longrightarrow \mathrm{Hom}\left[H^2(X_2,\mathbb{Z}), Q^4(X_2,\mathbb{Z})\right],$$

given by $f \mapsto \tilde{g}_4 \circ f \circ g^{-1}$. We have the equality $Ad_{\tilde{g}}(\mathbb{E}_{X_1}) = \mathbb{E}_{X_2}$, by the characterization of the $Mon(X_i)$-module $\mathbb{E}(X_i)$ provided in Theorem 9.3. Hence, the equality (9.6) holds, by construction of ι_{X_i}.

Conversely, assume that the isometry g satisfies the equality (9.6). There exists a parallel-transport operator $f : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$, since X_1 and X_2 are deformation equivalent. Hence, the equality $\iota_{X_1} = \iota_{X_2} \circ f$ holds, as well. We get the equality $\iota_{X_1} = \iota_{X_1} \circ f^{-1}g$. We conclude that $f^{-1}g$ belongs to $Mon^2(X_1)$, by Corollary 9.5. The equality $g = f(f^{-1}g)$ represents g as a composition of two parallel-transport operators. Hence, g is a parallel-transport operator. $\square$

The following statement is an immediate corollary of Theorems 1.3 and 9.8.

Corollary 9.9 *Let X and Y be two manifolds of $K3^{[n]}$-type.*

(1) X and Y are bimeromorphic, if and only if there exists a Hodge-isometry $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, satisfying $\iota_X = \iota_Y \circ f$.

(2) X and Y are isomorphic, if and only if there exists a Hodge-isometry f as in part (1), which maps some Kähler class of X to a Kähler class of Y.

We do not require f in part (1) to be orientation preserving, since if it is not then $-f$ is, and the orbits $\iota_Y \circ f$ and $\iota_Y \circ (-f)$ are equal.

Let τ be the set of connected components of the moduli space of marked pairs (X,η), where X is of $K3^{[n]}$-type, and $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ is an isometry. Denote by $\mathfrak{M}_\Lambda^\tau$ the moduli space of isomorphism classes of marked pairs (X,η), where X is of $K3^{[n]}$-type. The group $O(\Lambda)$ acts on the set τ and the stabilizer of a connected component $\mathfrak{M}_\Lambda^t$, $t \in \tau$, is the monodromy group $Mon^2(\mathfrak{M}_\Lambda^t) \subset O(\Lambda)$ (Definition 7.2). Let

$$\text{orb} : \mathfrak{M}_\Lambda^\tau \rightarrow O(\Lambda,\widetilde{\Lambda})/O(\widetilde{\Lambda})$$

be the map given by $(X,\eta) \mapsto \iota_X \circ \eta^{-1}$. Let orient : $\mathfrak{M}_\Lambda^\tau \to \text{Orient}(\Lambda)$ be the map given in equation (4.2). The characterization of the monodromy group in Corollary 9.5 yields the following enumeration of τ.

Corollary 9.10 *The map* $(\text{orb},\text{orient}) : \mathfrak{M}_\Lambda^\tau \to O(\Lambda,\widetilde{\Lambda})/O(\widetilde{\Lambda}) \times \text{Orient}(\Lambda)$ *factors through a bijection*

$$\tau \rightarrow O(\Lambda,\widetilde{\Lambda})/O(\widetilde{\Lambda}) \times \text{Orient}(\Lambda).$$

9.2 A numerical determination of the fundamental exceptional chamber

Definition 9.11 A class $\ell \in H^{1,1}(X,\mathbb{Z})$ is called *monodromy-reflective*, if ℓ is a primitive class, $(\ell,\ell) < 0$, and R_ℓ is a monodromy operator. A holomorphic line bundle $L \in \text{Pic}(X)$ is called *monodromy-reflective*, if the class $c_1(L)$ is *monodromy-reflective*.

Let X be a manifold of $K3^{[n]}$-type, $n \geq 2$. In section 9.2.1 we classify monodromy-orbits of monodromy-reflective classes. This is done in terms of explicitly computable monodromy invariants. In section 9.2.2 we describe the values of the monodromy invariants, for which the monodromy-reflective class is stably prime-exceptional (Theorem 9.17). When X is projective Theorems 6.17 and 9.17

combine to provide a determination of the closure $\overline{\mathscr{BK}}_X$ of the birational Kähler cone in $\mathscr{C}_X$ in terms of explicitly computable invariants.

9.2.1 Monodromy-reflective classes of $K3^{[n]}$-type

Set $\Lambda := H^2(X,\mathbb{Z})$. Recall that if $\ell \in \Lambda$ is monodromy-reflective, then R_ℓ acts on Λ^*/Λ via multiplication by ± 1 (Lemma 9.2). The set of monodromy-reflective classes is determined by the following statement.

Proposition 9.12 *([Ma7], Proposition 1.5) Let $\ell \in H^2(X,\mathbb{Z})$ be a primitive class of negative degree $(\ell,\ell) < 0$. Then R_ℓ belongs to $Mon^2(X)$, if and only if ℓ has one of the following two properties.*

(1) $(\ell,\ell) = -2$.

(2) $(\ell,\ell) = 2-2n$*, and* $(n-1)$ *divides the class* $(\ell,\bullet) \in H^2(X,\mathbb{Z})^*$.

R_ℓ acts on Λ/Λ^* as the identity in case (1), and via multiplication by -1 in case (2).

Given a primitive class $e \in H^2(X,\mathbb{Z})$, we denote by $\mathrm{div}(e,\bullet)$ the largest positive integer dividing the class $(e,\bullet) \in H^2(X,\mathbb{Z})^*$. Let $\mathscr{R}_n(X) \subset H^2(X,\mathbb{Z})$ be the subset of primitive classes of degree $2-2n$, such that $n-1$ divides $\mathrm{div}(e,\bullet)$. Let $\ell \in \mathscr{R}_n(X)$ and choose an embedding $\iota : H^2(X,\mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$ in the natural orbit ι_X provided by Corollary 9.5. Choose a generator $v \in \widetilde{\Lambda}$ of the rank 1 sublattice orthogonal to the image of ι. Set $e := \iota(\ell)$ and let

$$L \subset \widetilde{\Lambda} \tag{9.7}$$

be the saturation of the rank 2 sublattice spanned by e and v.

Definition 9.13 Two pairs (L_i, e_i), $i = 1,2$, each consisting of a lattice L_i and a class $e_i \in L_i$, are said to be *isometric*, if there exists an isometry $g : L_1 \to L_2$, such that $g(e_1) = e_2$.

Given a rank 2 lattice L, let $I_n(L) \subset L$ be the subset of primitive classes e with $(e,e) = 2-2n$.

Lemma 9.14 *There exists a natural one-to-one correspondence between the orbit set $I_n(L)/O(L)$ and the set of isometry classes of pairs (L',e'), such that L' is isometric to L and e' is a primitive class in L' with $(e',e') = 2-2n$.*

Proof Let $\mathscr{P}(L,n)$ be the set of isometry classes of pairs (L',e') as above. Define the map $f : \mathscr{P}(L,n) \to I_n(L)/O(L)$ as follows. Given a pair (L',e') representing a class in $\mathscr{P}(L,n)$, choose an isometry $g : L' \to L$ and set $f(L',e') := O(L)g(e')$. The map f is well defined, since the orbit $O(L)g(e')$ is clearly independent of the choice of g. The map f is surjective, since given $e \in I_n(L)$, $f(L,e) = O(L)e$. If $f(L_1,e_1) = f(L_2,e_2)$, then there exist isometries $g_i : L_i \to L$ and an element $h \in O(L)$, such that $g_2(e_2) = hg_1(e_1)$. Then $g_2^{-1}hg_1$ is an isometry from (L_1,e_1) to (L_2,e_2). Hence, the map f is injective. □

Let U be the unimodular hyperbolic plane. Let $U(2)$ be the rank 2 lattice with Gram matrix $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ and let D be the rank 2 lattice with Gram matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.

Proposition 9.15 *([Ma7], Propositions 1.8 and 6.2)*

(1) If $(\ell,\ell) = -2$ then the $Mon^2(X)$-orbit of ℓ is determined by $\mathrm{div}(\ell,\bullet)$.

(2) Let $\ell \in \mathscr{R}_n(X)$.

  (1) The lattice L, given in (9.7), is isometric to one of the lattices U, $U(2)$, or D.

  (2) Let $f : \mathscr{R}_n(X) \longrightarrow I_n(U)/O(U) \cup I_n(U(2))/O(U(2)) \cup I_n(D)/O(D)$ be the function, sending a class ℓ to the isometry class of the pair $(L,\iota(\ell))$. Then the values $\mathrm{div}(\ell,\bullet)$ *and $f(\ell)$ determine the $Mon^2(X)$-orbit of ℓ.*

The values of the function f can be conveniently enumerated and calculated as follows. Set $e := \iota(\ell) \in L$. Let ρ be the largest integer, such that $(e+v)/\rho$ is an integral class of L. Let σ be the largest integer, such that $(e-v)/\sigma$ is an integral class of L. If $\mathrm{div}(\ell,\bullet) = n-1$ and n is even, set $\{r,s\}(\ell) = \{\rho,\sigma\}$. Otherwise, set $\{r,s\}(\ell) = \{\frac{\rho}{2},\frac{\sigma}{2}\}$. The unordered pair $\{r,s\} := \{r,s\}(\ell)$ has the following properties.

Proposition 9.16 *([Ma7], Lemma 6.4)*

(1) The isometry class of the lattice L and the product rs are determined in terms of (ℓ,ℓ), $\mathrm{div}(\ell,\bullet)$*, n, and $\{\rho,\sigma\}$ by the following table.*

	(ℓ,ℓ)	$\mathrm{div}(\ell,\bullet)$	n	$\rho\sigma$	L	$\{r,s\}$	$r\cdot s$
1)	$2-2n$	$2n-2$	≥ 2	$4n-4$	U	$\{\frac{\rho}{2},\frac{\sigma}{2}\}$	$n-1$
2)	$2-2n$	$n-1$	*even*	$n-1$	D	$\{\rho,\sigma\}$	$n-1$
3)	$2-2n$	$n-1$	*odd*	$2n-2$	$U(2)$	$\{\frac{\rho}{2},\frac{\sigma}{2}\}$	$(n-1)/2$
4)	$2-2n$	$n-1$	$\equiv 1$ *modulo* 8	$n-1$	D	$\{\frac{\rho}{2},\frac{\sigma}{2}\}$	$(n-1)/4$

(2) The pair $\{r,s\}$ consists of relatively prime positive integers. All four rows in the above table do occur, and every relatively prime decomposition $\{r,s\}$ of the integer in the rightmost column occurs, for some $\ell \in \mathscr{R}_n(X)$.

(3) If $\ell \in \mathscr{R}_n(X)$, then $\mathrm{div}(\ell,\bullet)$ *and $\{r,s\}(\ell)$ determine the $Mon^2(X)$-orbit of ℓ.*

9.2.2 Stably prime-exceptional classes of $K3^{[n]}$-type

Theorem 9.17 *([Ma7], Theorem 1.12). Let $\kappa \in H^{1,1}(X,\mathbb{R})$ be a Kähler class and L a monodromy reflective line bundle. Set $\ell := c_1(L)$. Assume that $(\kappa,\ell) > 0$.*

(1) If $(\ell,\ell) = -2$, then L^k is stably prime-exceptional, where

$$k = \begin{cases} 2, \textit{ if } \ \mathrm{div}(\ell,\bullet) = 2 \ \ \textit{and } n = 2, \\ 1, \textit{ if } \ \mathrm{div}(\ell,\bullet) = 2 \ \ \textit{and } n > 2, \\ 1 \ \textit{if } \ \mathrm{div}(\ell,\bullet) = 1. \end{cases}$$

(2) If $\mathrm{div}(\ell,\bullet) = 2n-2$ *and $\{r,s\}(\ell) = \{1,n-1\}$, then L^2 is stably prime-exceptional.*

(3) If $\mathrm{div}(\ell,\bullet) = 2n-2$ *and $\{r,s\}(\ell) = \{2,(n-1)/2\}$, then L is stably prime-exceptional.*

(4) If $\mathrm{div}(\ell,\bullet) = n-1$*, n is even, and $\{r,s\}(\ell) = \{1,n-1\}$, then L is stably prime-exceptional.*

(5) If $\mathrm{div}(\ell,\bullet) = n-1$*, n is odd, and $\{r,s\}(\ell) = \{1,(n-1)/2\}$, then L is stably prime-exceptional.*

(6) In all other cases, $H^0(L^k)$ vanishes, and so L^k is not stably prime-exceptional, for every non-zero integer k.

When X is projective Proposition 9.12 and Theorem 9.17 determine the set $\mathcal{S}pe \subset H^{1,1}(X,\mathbb{Z})$, of stably prime-exceptional classes, and hence also the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, by Proposition 6.10.

The proof of Theorem 9.17 has two ingredients. First we deform the pair (X,L) to a pair (M,L_1), where M is a moduli space of sheaves on a projective $K3$ surface, and L_1 is a monodromy-reflective line bundle with the same monodromy invariants. Then L is stably prime-exceptional, if and only if L_1 is, by Proposition 6.6. We then laboriously check an example, one for each value of the monodromy invariants n, (ℓ,ℓ), $\mathrm{div}(\ell,\bullet)$, and $\{r,s\}(\ell)$, and show that either R_ℓ is induced by a birational map $f: M \to M$, such that $f^*(L_1) = L_1^{-1}$, or that the linear system $|L_1^k|$ consists of a single prime exceptional divisor, for the power k prescribed by Theorem 9.17.

The two possible values of the degree -2 or $2-2n$, of a prime exceptional divisor, correspond to two types of well known constructions in the theory of moduli spaces of sheaves on a $K3$ surface S. We briefly describe these constructions below.

Pairs $(M,\mathscr{O}_M(E))$, where $M := M_H(v)$ is a moduli space of H-stable coherent sheaves of class $v \in K(S)$, and E is a prime exceptional divisor of Beauville-Bogomolov degree -2, arise as follows. The Mukai isometry (9.4) associates to the line bundle $\mathscr{O}_M(E)$ a class $e \in v^\perp$, with $(e,e) = -2$. In the examples considered in [Ma7], e is the class of an H-stable sheaf F on S. Such a sheaf is necessarily rigid, i.e., $\mathrm{Ext}^1(F,F) = 0$. Indeed,

$$\dim \mathrm{Ext}^1(F,F) = \dim \mathrm{Hom}(F,F) + \dim \mathrm{Ext}^2(F,F) - \chi(F^\vee \otimes F) = 1+1-2 = 0.$$

Furthermore, the moduli space $M_H(e)$ is connected, by a theorem of Mukai, and consists of the single point $\{F\}$ (see [Mu1]). The prime exceptional divisor E is the Brill-Noether locus

$$\{V \in M_H(v) \ : \ \dim \mathrm{Ext}^1(F,V) > 0\}.$$

Specific examples are easier to describe using Mukai's notation. Recall Mukai's isomorphism

$$ch(\bullet)\sqrt{td_S} \ : \ K(S) \ \longrightarrow \ H^*(S,\mathbb{Z}), \tag{9.8}$$

sending a class $v \in K(S)$ to the integral singular cohomology group. Let $D: H^*(S,\mathbb{Z}) \to H^*(S,\mathbb{Z})$ be the automorphism acting by $(-1)^i$ on $H^{2i}(S,\mathbb{Z})$. The homomorphism (9.8) is an isometry once we endow $H^*(S,\mathbb{Z})$ with the pairing

$$(x,y) \ := \ -\int_S D(x) \cup y,$$

by the Hirzebruch-Riemann-Roch theorem and the definition of the Mukai pairing in equation (9.2). We have $ch(v)\sqrt{td_S} = (r, c_1(v), s)$, where $r = \mathrm{rank}(v)$, $s = \chi(v) - r$, and we identify $H^0(S,\mathbb{Z})$ and $H^4(S,\mathbb{Z})$ with $\mathbb{Z}$, using the classes Poincaré-dual to S and to a point. Given two classes $v_i \in K(S)$, with $\mathrm{rank}(v_i) = r_i$, $c_1(v_i) = \alpha_i$, and

$s_i := \chi(v_i) - r_i$, then

$$(v_1, v_2) \;=\; \left(\int_S \alpha_1\alpha_2\right) - r_1 s_2 - r_2 s_1.$$

Example 9.18 Following is a simple example in which a prime exceptional divisor E of degree -2 and divisibility $\mathrm{div}([E], \bullet) = 1$ is realized as a Brill-Noether locus. Consider a $K3$ surface S, containing a smooth rational curve C. Consider the Hilbert scheme $M := S^{[n]}$ as the moduli space of ideal sheaves of length n subschemes. Let F be the torsion sheaf $\mathscr{O}_C(-1)$, supported on C as a line bundle of degree -1. Let $v \in K(S)$ be the class of an ideal sheaf in $S^{[n]}$ and e the class of F. The Mukai vector of v is $(1,0,1-n)$, that of e is $(0,[C],0)$, and $(v,e) = 0$. Let $E \subset M$ be the divisor of ideal sheaves I_Z of subscheme Z with non-empty intersection $Z \cap C$. The space $\mathrm{Hom}(F, I_Z)$ vanishes for all $I_Z \in M$, and so $\dim \mathrm{Ext}^1(F, I_Z) = \dim \mathrm{Ext}^2(F, I_Z)$, for all $I_Z \in M$. Now, $\mathrm{Ext}^2(F, I_Z) \cong \mathrm{Hom}(I_Z, F)^*$ vanishes, if and only if $Z \cap C = \emptyset$. Hence, $\mathrm{Ext}^1(F, I_Z) \neq 0$, if and only if I_Z belongs to E. See [Ma1, Y1] for many more examples of prime exceptional divisors E of degree -2 and $\mathrm{div}([E], \bullet) = 1$. See [Ma7], Lemma 10.7 for the case $(e,e) = -2$, $\mathrm{div}(e, \bullet) = 2$, and $n \equiv 2$ modulo 4.

Jun Li constructed a birational morphism from the moduli space of Gieseker-Maruyama H-stable sheaves on a $K3$ surface to the Uhlenbeck-Yau compactification of the moduli space of H-slope-stable locally-free sheaves [Li]. The examples of prime exceptional divisors of degree $2-2n$ on a moduli space of sheaves, provided in [Ma7], were all constructed as exceptional divisors for Jun Li's morphism.

Example 9.19 The simplest example is the Hilbert-Chow morphism, from the Hilbert scheme $S^{[n]}$, $n \geq 2$, to the symmetric product $S^{(n)}$ of a $K3$ surface S, where the exceptional divisor E is the big diagonal. The Mukai vector of the ideal sheaf is $v = (1,0,1-n)$. In this case $[E] = 2\delta$, where $\delta = (1,0,n-1)$. Note that $(\delta, \delta) = 2-2n$. The second cohomology of $S^{[n]}$ is an orthogonal direct sum $H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta$, by [Be1] or by Mukai's isometry (9.4). Hence, $\mathrm{div}(\delta, \bullet) = 2n-2$. The largest integer ρ dividing $\delta + v = (2,0,0)$ is 2 and the largest integer σ dividing $\delta - v = (0,0,2n-2)$ is $2n-2$. Hence, $\{r,s\}(\delta) = \{1, n-1\}$, by Proposition 9.16 and Equation (9.5).

Example 9.20 Consider, more generally, the moduli space $M_H(r,0,-s)$ of H-stable sheaves with Mukai vector $v = (r,0,-s)$, satisfying $s > r \geq 1$ and $\gcd(r,s) = 1$. Then $M_H(r,0,-s)$ is of $K3^{[n]}$-type, $n = rs+1$. The Mukai vector $e := (r,0,s) \in v^{\perp}$ maps to a monodromy-reflective class $\ell \in H^2(M_H(v),\mathbb{Z})$ of degree $(\ell,\ell) = 2-2n$, divisibility $\mathrm{div}(\ell, \bullet) = 2n-2$, and $\{r,s\}(\ell) = \{r,s\}$, by Proposition 9.16 and Equation

(9.5). When $r = 2$, ℓ is the class of the exceptional divisor E of Jun Li's morphism. E is the locus of sheaves, which are not locally-free or not H-slope-stable ([Ma7], Lemma 10.16). When $r > 2$, the exceptional locus has co-dimension ≥ 2, and no multiple of the class ℓ is effective. Instead, the reflection R_ℓ is induced by the birational map $f : M_H(r,0,-s) \to M_H(r,0,-s)$, sending a locally-free H-slope stable sheaf F of class $(r,0,-s)$ to the dual sheaf F^* ([Ma7], Proposition 11.1).

Remark 9.21 Fix an integer $n > 0$, such that $n-1$ is not a prime power, and consider all possible factorizations $n-1 = rs$, with $s > r \geq 1$ and $\gcd(r,s) = 1$. The sublattice $(r,0,-s)^\perp$ of the Mukai lattice of a $K3$ surface S is the orthogonal direct sum $H^2(S,\mathbb{Z}) \oplus \mathbb{Z}(r,0,s)$. We get the isometry

$$\theta \; : \; H^2(S,\mathbb{Z}) \oplus \mathbb{Z}(r,0,s) \;\; \longrightarrow \;\; H^2\left(M_H(r,0,-s),\mathbb{Z}\right),$$

using Mukai's isometry given in equation (9.4). Let $n-1 = r_1 s_1 = r_2 s_2$ be two different such factorizations. Then the two moduli spaces $M_H(r_1,0,-s_1)$ and $M_H(r_2,0,-s_2)$, considered in Example 9.20, come with a natural Hodge isometry

$$g \; : \; H^2(M_H(r_1,0,-s_1),\mathbb{Z}) \;\; \longrightarrow \;\; H^2(M_H(r_2,0,-s_2),\mathbb{Z}),$$

which restricts as the identity on the direct summand $\theta\left(H^2(S,\mathbb{Z})\right)$ and maps the class $\ell_1 := \theta(r_1,0,s_1) \in H^2(M_H(r_1,0,-s_1),\mathbb{Z})$ to the class $\ell_2 := \theta(r_2,0,s_2) \in H^2(M_H(r_2,0,-s_2),\mathbb{Z})$. The Hodge isometry g is not a parallel-transport operator, since the monodromy-invariants $\{r,s\}(\ell_i) = \{r_i,s_i\}$ are distinct. Indeed, these moduli spaces are not birational in general ([Ma5], Proposition 4.10). Furthermore, if $n-1 = rs$ is such a factorization with $r > 2$, then the birational Kähler cones $\mathscr{BK}_{S^{[n]}}$ and $\mathscr{BK}_{M_H(r,0,-s)}$ are not isometric in general. Indeed, $S^{[n]}$ admits a stably prime-exceptional class, while $M_H(r,0,-s)$ does not, for a $K3$ surface with a suitably chosen Picard lattice.

10 Open problems

Following is a very brief list of central open problems closely related to this survey. See [Be2] for a more complete recent survey of open problems in the subject of irreducible holomorphic symplectic manifolds.

Question 10.1 Let X be one of the known examples of irreducible holomorphic symplectic manifolds, i.e., of $K3^{[n]}$-type, a generalized Kummer variety, or one of the two exceptional examples of O'Grady [O'G2, O'G3]. Let Y be an irreducible holomorphic symplectic manifold, with $H^2(Y,\mathbb{Z})$ isometric to $H^2(X,\mathbb{Z})$. Is Y necessarily deformation equivalent to X?

Let Λ be a lattice isometric to $H^2(X,\mathbb{Z})$. At present it is only known that the number of deformation types of irreducible holomorphic symplectic manifolds of a given dimension $2n$, and with second cohomology lattice isometric to Λ, is finite [Hu4]. The moduli space $\mathfrak{M}_\Lambda$, of isomorphism classes of marked pairs (X,η), with X of dimension $2n$ and $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ an isometry, has finitely many connected components, by Huybrechts' result and Lemma 7.5. O'Grady has made substantial progress towards the proof of uniqueness of the deformation type in case the dimension is 4 and the lattice Λ is of $K3^{[2]}$-type [O'G5].

Problem 10.2 *Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type, $n \geq 2$. Determine the Kähler-type chamber (Definition 5.10) in the fundamental exceptional chamber $\mathcal{FE}_X$ of X, containing a given very general class $\alpha \in \mathcal{FE}_X$, in terms of the weight 2 integral Hodge structure $H^2(X,\mathbb{Z})$, the Beauville-Bogomolov pairing, and the orbit ι_X of isometric embeddings of $H^2(X,\mathbb{Z})$ in the Mukai lattice, given in Corollary 9.5.*

Note that the data specified in Problem 10.2 determines the isomorphism class of an irreducible holomorphic symplectic manifold Y, bimeromorphic to X, and an $\mathrm{Aut}(X) \times \mathrm{Aut}(Y)$-orbit[10] of a bimeromorphic map $f : Y \to X$, such that $f^*(\alpha)$ is a Kähler class on Y, by Corollaries 5.7 and 9.9. The homomorphism f^* takes the Kähler-type chamber in Problem 10.2 to $\mathcal{K}_Y$. Hassett and Tschinkel formulated a precise conjectural solution to problem 10.2 [HT4]. The Kähler cone, according to their conjecture, does not depend on the orbit ι_X. The birational Kähler cone does, as we saw in Remark 9.21.

Problem 10.3 *Find an explicit necessary and sufficient condition for a Hodge isometry $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ to be a parallel-transport operator, in the case X and Y are deformation equivalent to generalized Kummer varieties, or to O'Grady's two exceptional examples.*

Problem 10.4 *Let X be deformation equivalent to a generalized Kummer variety, or to one of O'Grady's two exceptional examples. Find an explicit necessary and sufficient condition for a class $\ell \in H^{1,1}(X,\mathbb{Z})$ to be stably prime-exceptional (Definition 6.4).*

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[10] The orbit of f is the set $\{g_1 f g_2^{-1} \ : \ g_1 \in \mathrm{Aut}(X),\ g_2 \in \mathrm{Aut}(Y)\}$.

Problem 10.4 is solved in the $K3^{[n]}$-type case (Proposition 9.12 and Theorem 9.17). A solution to problem 10.4 yields a determination of the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, by Proposition 1.8, and of the closure of the birational Kähler cone, by Proposition 5.6. Once solutions to Problems 10.3 and 10.4 are provided, the analogue of Problem 10.2 may be formulated as well.

Question 10.5 Is the monodromy group $Mon^2(X)$, of an irreducible holomorphic symplectic manifold X, necessarily a normal subgroup of the isometry group of $H^2(X,\mathbb{Z})$?

Let X be a generalized Kummer variety of dimension $2n$, $n \geq 2$. Then $H^2(X,\mathbb{Z})$ is isometric to the lattice $\Lambda := U \oplus U \oplus U \oplus \mathbb{Z}\delta$, where U is the unimodular rank 2 lattice of signature $(1,1)$, and $(\delta,\delta) = -2-2n$ (see [Be1, Y2]).

Conjecture 10.6 *$Mon^2(X)$ is equal to the subgroup $\mathscr{N}(X)$ of the signed isometry group $O^+H^2(X,\mathbb{Z})$, generated by products of an even number of reflections $R_{\ell_1}\cdots R_{\ell_{2k}}$, where $(\ell_i,\ell_i) = 2$, for an even number of indices i, and $(\ell_i,\ell_i) = -2$ for the rest of the indices i.*

The inclusion $\mathscr{N}(X) \subset Mon^2(X)$ was proven by the author in an unpublished work. When $n = 2$, the equality $\mathscr{N}(X) = Mon^2(X)$ follows from the Global Torelli Theorem 2.2 and Namikawa's counter example to the naive Hodge theoretic Torelli statement [Nam2].

Let X be an irreducible holomorphic symplectic manifold deformation equivalent to O'Grady's 10-dimensional exceptional example [O'G2]. Then $H^2(X,\mathbb{Z})$ is isometric to the orthogonal direct sum of $H^2(S,\mathbb{Z}) \oplus G_2$, where S is a $K3$ surface, and G_2 is the negative definite root lattice of type G_2, with Gram matrix $\begin{pmatrix} -2 & 3 \\ 3 & -6 \end{pmatrix}$ (see [R]). The isometry group $O(G_2)$ is equal to the Weyl group of G_2 and its extension to $H^2(X,\mathbb{Z})$, via the trivial action on $H^2(S,\mathbb{Z})$, is contained in $Mon^2(X)$, by ([Ma6], Lemma 5.1).

Conjecture 10.7 $Mon^2(X) = O^+H^2(X,\mathbb{Z})$.

There are many examples of non-isomorphic $K3$ surfaces with equivalent bounded derived categories of coherent sheaves [Or].

Question 10.8 Let X and Y be projective irreducible holomorphic symplectic manifolds, such that $H^2(X,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ are Hodge isometric. Are their bounded derived categories of coherent sheaves necessarily equivalent?

When $X = S_1^{[n]}$ and $Y = S_2^{[n]}$, where S_1 and S_2 are $K3$ surfaces, the answer to Question 10.8 is affirmative (see the proof of [Pl], Proposition 10). See [Hu5] for a survey on the topic of question 10.8.

Recall that a class $\ell \in H^{1,1}(X,\mathbb{Z})$ is monodromy-reflective, if it is a primitive class, and the reflection R_ℓ is a monodromy operator (Definition 9.11).

Question 10.9 Let $\ell \in H^{1,1}(X,\mathbb{Z})$ be a monodromy-reflective class. Is there always some non-zero integer λ, such that the class $\lambda(\ell,\bullet) \in H^2(X,\mathbb{Z})^* \cong H_2(X,\mathbb{Z})$ corresponds to an effective one-cycle?

An affirmative answer to the above question implies that the reflection R_ℓ can not be induced by a regular automorphism[11] of X. It follows that the Kähler cone is contained in a unique chamber of the subgroup of $Mon^2_{Hdg}(X)$ generated by all reflections in $Mon^2_{Hdg}(X)$ (see Theorem 6.15).

Problem 10.10 *Prove an analogue of Proposition 6.1, about birational contractibility of a prime exceptional divisor, for non-projective irreducible holomorphic symplectic manifolds.*

Druel's proof of Proposition 6.1 relies on results in the minimal model program, which are currently not available in the Kähler category [Dr].

Question 10.11 Let X be a projective irreducible holomorphic symplectic manifold. Is the semi-group Σ, of effective divisor classes on X, equal to the semi-group Σ' generated by the prime exceptional classes and integral points on the closure $\overline{\mathscr{BK}}_X$ of the birational Kähler cone in $H^{1,1}(X,\mathbb{R})$?

The answer is affirmative for any $K3$ surface, even without the projectivity assumption ([BHPV], Ch. IIIV, Proposition 3.7). Stronger results hold true for projective $K3$ surfaces [Kov]. The inclusion $\Sigma \subset \Sigma'$ is known in general, by the divisorial Zariski decomposition (Theorem 5.8). The integral points of $\mathscr{C}_X \cap \overline{\mathscr{BK}}_X$ are known to be contained in Σ. This is seen as follows. The integral points of the positive cone

[11] A weaker version of this assertion, namely the non-existence of a fixed-point free such automorphism g, is always true. Indeed, if $g^* = R_\ell$, and g is a fixed-point-free (necessarily symplectic) automorphism, then g^2 acts trivially on $H^2(X,\mathbb{Z})$. Hence, g^2 is an isometry with respect to a Kähler metric. It follows that g has finite order, since it generates a discrete subgroup of the compact isometry group. Thus, $X/\langle g\rangle$ is a non simply connected holomorphic symplectic Kähler manifold, with $h^{k,0}(X) = 1$, for even k in the range $0 \le k \le \dim_{\mathbb{C}}(X)$, and $h^{k,0}(X) = 0$, otherwise. Such X does not exist, by [HN], Proposition A.1.

are known to correspond to big line bundles, by ([Hu1], Corollary 3.10). Each integral point of $\mathscr{C}_X \cap \overline{\mathscr{B}\mathscr{K}}_X$ thus coresponds to a big and nef line bundle L on some birational irreducible holomorphic symplectic manifold Y, by Theorems 5.4 and 6.17, and so the cohomology groups $H^i(Y,L)$ vanish, for $i > 0$, by the Kawamata-Viehweg vanishing theorem. Set $\ell := c_1(L)$. If X is of $K3^{[n]}$-type or deformation equivalent to a generalized Kummer variety, then an explicit formula is known for the Euler characteristic $\chi(L)$ of a line bundle L, in terms of its Beauville-Bogomolov degree (ℓ,ℓ) ([Hu3], Examples 7 and 8). One sees, in particular, that $\chi(L) > 0$, if $(\ell,\ell) \geq 0$, and so L is effective.

An affirmative answer to Question 10.11 would thus follow, if one could prove that nef line bundles with $(\ell,\ell) = 0$ are effective. Some experts conjectured that such line bundles are related to Lagrangian fibrations ([Marku], Conjecture 1.7; [Saw], Conjecture 1, [Ver3], Conjecture 1.7). We refer the reader also to the important work of Matsushita on Lagrangian fibrations [Mat1, Mat2] and to the survey ([Be2], section 1.6).

Question 10.12 Which components, of the moduli spaces of polarized projective irreducible holomorphic symplectic manifolds, are unirational? Which are of general type?

Gritsenko, Hulek, and Sankaran had studied this question for fourfolds X of $K3^{[2]}$-type, and for primitive polarizations $h \in H^2(X,\mathbb{Z})$, with $\mathrm{div}(h,\bullet) = 2$. Let $(h,h) = 2d$. They show that for $d \geq 12$, the moduli space is of general type ([GHS1], Theorem 4.1). They use the theory of modular forms to show that the quotient of the period domain $\Omega^+_{h^\perp}$, given in equation (4.1), by the polarized monodromy group $Mon^2(X,h)$, is of general type.

On the other hand, unirational components are those likely to admit explicit and very beautiful geometric descriptions [BD, DV, IR, Mu2, O'G4].

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On singularities of generically immersive holomorphic maps between complex hyperbolic space forms

Ngaiming Mok

Abstract In 1965, Feder proved using a cohomological identity that any holomorphic immersion $\tau : \mathbb{P}^n \to \mathbb{P}^m$ between complex projective spaces is necessarily a linear embedding whenever $m < 2n$. In 1991, Cao-Mok adapted Feder's identity to study the dual situation of holomorphic immersions between compact complex hyperbolic space forms, proving that any holomorphic immersion $f : X \to Y$ from an n-dimensional compact complex hyperbolic space form X into any m-dimensional complex hyperbolic space form Y must necessarily be totally geodesic provided that $m < 2n$. We study in this article singularity loci of generically injective holomorphic immersions between complex hyperbolic space forms. Under dimension restrictions, we show that the open subset U over which the map is a holomorphic immersion cannot possibly contain compact complex-analytic subvarieties of large dimensions which are in some sense sufficiently deformable. While in the finite-volume case it is enough to apply the arguments of Cao-Mok, the main input of the current article is to introduce a geometric argument that is completely local. Such a method applies to $f : X \to Y$ in which the complex hyperbolic space form X is possibly of infinite volume. To start with we make use of the Ahlfors-Schwarz Lemma, as motivated by recent work of Koziarz-Mok, and reduce the problem to the local study of contracting leafwise holomorphic maps between open subsets of complex unit balls. Rigidity results are then derived from a commutation formula on the complex Hessian of the holomorphic map.

Keywords complex hyperbolic space form, holomorphic immersion, total geodesy, holomorphic isometry, leafwise contracting holomorphic map, complex Hessian, commutation formula.
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In 1965, Feder [Fe65] proved that any holomorphic immersion $\tau : \mathbb{P}^n \to \mathbb{P}^m$ between complex projective spaces is necessarily a linear embedding whenever $m < 2n$. He did this by using Whitney's formula on Chern classes associated to the tangent sequence of the holomorphic map, thereby proving that the degree of $\tau_* : H_2(\mathbb{P}^n, \mathbb{Z}) \to H_2(\mathbb{P}^m, \mathbb{Z})$ must be 1 under the dimension restriction, noting that the restriction $m < 2n$ forces the vanishing of the n-th Chern class of the normal bundle of the holomorphic immersion. An adaptation of Feder's identity was used by Cao-Mok [CM91] to study the Immersion Problem for the dual situation of holomorphic immersions between compact complex hyperbolic space forms. By an n-dimensional complex hyperbolic space form we mean the quotient of the n-dimensional complex unit ball B^n by a torsion-free discrete group of automorphisms equipped with the complete Kähler metric induced by the canonical complete Kähler-Einstein metric on B^n. By [CM91] any holomorphic immersion $f : X \to Y$ from an n-dimensional compact complex hyperbolic space form X into any m-dimensional complex hyperbolic space form Y must necessarily be totally geodesic provided that $m < 2n$. A generalization of the latter result to the case of complex hyperbolic space forms of finite volume was obtained by To [To93].

By the duality between the complex unit ball $(B^n, ds^2_{B^n})$ equipped with the unique complete Kähler-Einstein metric of constant holomorphic sectional curvature $-K$, $K > 0$, and the projective space $(\mathbb{P}^n, ds^2_{FS})$ equipped with the Fubini-Study metric of constant holomorphic sectional curvature equal to K, the total Chern class of a complex hyperbolic space form is determined by its first Chern class. Given a holomorphic immersion between complex hyperbolic space forms, the first Chern class can be represented by the first Chern form induced on the domain manifold from the canonical Kähler-Einstein metric of the target manifold via the immersion. The main entity in the first Chern form is a nonnegative closed $(1,1)$-form ρ which is derived from the second fundamental form σ on $(1,0)$-vectors of the holomorphic immersion and which enjoys the property that the vanishing of ρ means equivalently the vanishing of σ, i.e., the total geodesy of the immersion. The adaptation by Cao-Mok [CM91] of Feder's identity to the holomorphic immersion $f : X \to Y$ between complex hyperbolic space forms, applied to the tangent sequence $0 \to T_X \to f^*T_Y \to N \to 0$, where N stands for the normal bundle of the holomorphic immersion f, gives the vanishing $\rho^n \equiv 0$ when X is compact, $n := \dim(X)$ and $\dim(Y) := m < 2n$, and the same holds true when X is noncompact and of finite volume by To [To93]. At a general point x of X the kernels of ρ were shown to define a holomorphic foliation $\mathscr{E}$ on a neighborhood U of x the leaves of which are totally geodesic complex submanifolds. This was shown to lead to a contradiction to the fact that X is of finite volume unless the holomorphic foliation $\mathscr{E}$ is trivial.

Recently, Feder's identity has been applied by Koziarz-Mok [KM10] to the Submersion Problem concerning holomorphic submersions between compact complex hyperbolic space forms and more generally between complex hyperbolic space forms of finite-volume. There, given a holomorphic submersion $\pi : X \to Y$ between complex hyperbolic space forms, applying Feder's identity instead to the cotangent sequence $0 \to \pi^* T_Y^* \to T_X^* \to T_\pi^* \to 0$, where $T_\pi = \mathrm{Ker}\,(d\pi)$ stands for the relative tangent bundle $\pi : X \to Y$, yields the vanishing $\mu^{n-m+1} \equiv 0$ for the closed nonnegative $(1,1)$-form $\mu := \omega_X - \pi^*\omega_Y$, where ω_X resp. ω_Y stands for the Kähler form on X resp. on Y of the canonical Kähler-Einstein metric of constant holomorphic sectional curvature $-K$, and the nonnegativity of μ follows from the Ahlfors-Schwarz Lemma. Using the identity $\mu^{n-m+1} \equiv 0$ it was proven in [KM10] that there does not exist any holomorphic submersion between compact complex hyperbolic space forms, and the same was proven in the noncompact finite-volume case provided that the base manifold is of complex dimension ≥ 2.

Motivated by the use of the Ahlfors-Schwarz Lemma in [KM10], in the current article we re-visit the topic of holomorphic immersions $f : X \to Y$ between complex hyperbolic space forms. In [CM91] the closed nonnegative (1,1)-form ρ represents up to a positive constant the cohomology class $\frac{-c_1(X)}{n+1} + \frac{f^*c_1(Y)}{m+1}$. The possibility of representing the latter class by $\rho \geq 0$ results from the constancy of holomorphic sectional curvatures and from the monotonicity of holomorphic bisectional curvatures. The holomorphicity of the foliation defined by Ker(ρ) then follows from the holomorphicity of the second fundamental form σ on $(1,0)$-vectors. On the other hand, the cohomology class $\frac{-c_1(X)}{n+1} + \frac{f^*c_1(Y)}{m+1}$ is up to a positive constant represented by $\mu := \omega_X - f^*\omega_Y \geq 0$. We will make use simultaneously of the closed nonnegative (1,1)-forms ρ and μ. Motivated by results of [KM10] in the case of compact complex hyperbolic space forms concerning critical values of surjective holomorphic maps, we will study in this article singularity loci of generically injective holomorphic immersions between complex hyperbolic space forms. One of the main results is applicable also to complex hyperbolic space forms of infinite volume. Under dimension restrictions, we will show that the open subset U over which the map is a holomorphic immersion cannot possibly contain compact complex-analytic subvarieties of large dimensions which are in some sense sufficiently deformable.

For results in the finite-volume case it is enough to apply the arguments of Cao-Mok [CM91]. First of all, when X is compact, we observe that the arguments of Cao-Mok [CM91] already imply the estimate that dim(Sing(f)) $\geq 2n - m - 1$ unless f is totally geodesic. For the proof it suffices to restrict the tangent sequence to linear sections of X (with respect to a projective embedding) which avoid Sing(f) to deduce total geodesy of f whenever dim(Sing(f)) $< 2n - m - 1$. In the case of a noncompact complex hyperbolic space form of finite volume the abundant supply of linear sections avoiding $\mathrm{Sing}(f)$ is guaranteed in the arithmetic case by the existence of Satake-Borel-Baily compactifications [Sa60] and [BB66] obtained by adding a finite number of normal isolated singularities, and in the non-arithmetic case by the projective-algebraicity proven in [Mk10] of the complex-analytic compactification obtained in Siu-Yau [SY82] by adding a finite number of points corresponding to the finite number of ends.

As indicated in the above the dimension estimate on $\mathrm{Sing}(f)$ breaks up into two parts. The first half is cohomological. In more precise terms, assuming $\dim(\mathrm{Sing}(f)) < 2n-m-1$ there exists a q-dimensional compact complex submanifold S of $X' := X - \mathrm{Sing}(f)$ with $q = n-(2n-m-1) = m-n+1$ so that, denoting by N the normal bundle of the holomorphic immersion $f|_{X'} : X' \to Y$ we must have $c_{m-n+1}(N)|_S = 0$ since $\mathrm{rank}(N) = m-n$. Feder's identity and the compactness of S then forces the nonnegative (1,1)-form $\rho|_S$ to have a zero eigenvalue everywhere. By varying S obtained from taking linear sections with respect to a projective embedding one concludes that the closed nonnegative (1,1)-form ρ is degenerate everywhere on X. The second half of the argument is the same as in Cao-Mok [CM91] and [To93] where one derives from the degeneracy of X a holomorphic foliation on some nonempty connected open set by totally geodesic complex submanifolds consisting of maximal integral submanifolds of $\mathrm{Re}(\mathrm{Ker}(\rho))$, and where in the proof of the identical vanishing of ρ one requires the fact that the fundamental group of X is a lattice in $\mathrm{Aut}(B^n)$. For the sake of brevity we will call the second half the geometric argument.

The main input of the current article is to introduce a geometric argument that is completely local. Such a method applies to $f : X \to Y$ where the complex hyperbolic space form X is possibly of infinite volume, with a conclusion that X' cannot contain a sufficiently deformable $(m-n+1)$-dimensional compact complex-analytic subvariety, where by saying that a q-dimensional compact complex-analytic subvariety $S \subset X - \mathrm{Sing}(f)$ is sufficiently deformable we mean that points corresponding to tangent q-planes of deformations of S fill up a nonempty open subset of the Grassmann bundle of q-planes on X.

Making use of the fact that $\frac{-c_1(X)\big|_{X'}}{n+1} + \frac{f^*c_1(Y)\big|_{X'}}{m+1}$ can be represented by a closed nonnegative (1,1)-form ρ arising from the second fundamental form and another closed nonnegative (1,1)-form μ encoding the failure of f to be an isometry, reinforcing the cohomological argument we obtain a holomorphic foliation on a nonempty open subset U by totally geodesic submanifolds where f restricts to a totally geodesic isometric embedding on each of the totally geodesic leaves. Unless $\rho \equiv 0$ or equivalently $\mu \equiv 0$ we have obtained a nonempty open subset U of B^n, a holomorphic foliation $\mathscr{E}$ on U by totally geodesic complex submanifolds and a holomorphic embedding f of U into some B^m such that f is contracting (distance-decreasing) and it is a totally geodesic isometric embedding when restricted to any leaf of $\mathscr{E}$, and such that $\mathscr{E}_x = \mathrm{Re}\,(\mathrm{Ker}(\rho(x)))$ for any $x \in U$. We call such a map $f : U \to B^m$ a contracting leafwise totally geodesic holomorphic isometric embedding, where implicitly the leaves are assumed to be defined by Re(Ker(ρ)). Under the dimension restriction $m \leq 2n-4$ we prove that no contracting leafwise totally geodesic holomorphic isometric embedding exists unless $m = n$, in which case f is nothing other than a totally geodesic embedding. This is slightly short of giving a completely local proof for the geometric argument in the dimension estimate for $\mathrm{Sing}(f)$ even in the case where X is compact, where we need the local argument of $m \leq 2n-1$.

Crucial to our geometric argument is a commutation formula concerning the Hessian of the holomorphic map f, more precisely concerning $\nabla\partial f$, the vanishing of which is equivalent to the total geodesy of the map f. The commutation formula applies to any contracting leafwise totally geodesic holomorphic isometric embedding $f : U \to B^m$. However, in the application of the commutation formula, dimension counts are involved, which is the reason why the dimension restriction $m \leq 2n-4$ is imposed. We expect that there is no nontrivial holomorphic embedding $f : U \to B^m$ which is a contracting leafwise totally geodesic isometric embedding, but the latter remains unproved for $m \geq 2n-3$.

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1 Background

By a complex hyperbolic space form we mean the quotient of the n-dimensional complex unit ball B^n for some positive integer n by a torsion-free discrete group of automorphisms equipped with the complete Kähler metric induced by the canonical complete Kähler-Einstein metric on B^n. The total geodesy of holomorphic immersions between complex hyperbolic space forms under dimension restrictions was established in Cao-Mok [CM91] in the compact case and in [To93] in the noncompact finite-volume case. Here the requirement of compactness or of the finiteness of the volume is imposed only on the domain manifold.

Theorem (Cao-Mok [CM91], To [To93]). *Let n, m be positive integers such that $n \geq 2$ and $m < 2n$. Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice of biholomorphic automorphisms, $X := B^n/\Gamma$. Let Y be an n-dimensional complex hyperbolic space form. Let $f : X \to Y$ be a holomorphic immersion. Then, f is totally geodesic.*

The proofs of Cao-Mok [CM91] and To [To93] rely on a cohomological argument and a geometric argument. The starting point of the cohomological argument is the vanishing of the n-th Chern class of the normal bundle of the holomorphic immersion $f : X \to Y$, given that the normal bundle is of rank $m - n < n$. The crux of the cohomological argument is the following algebraic identity adapted from Feder [Fe65], in which it was proven that any holomorphic immersion $f : \mathbb{P}^n \to \mathbb{P}^m$ is linear whenever $m < 2n$.

Lemma 1. *For the compact complex hyperbolic space form $X = B^n/\Gamma$ let $\alpha, \beta \in H^2(X, \mathbb{R})$. Suppose for $1 \leq k \leq n - m$ there exists $\gamma_k \in H^{2k}(X, \mathbb{R})$ such that $(1+\alpha)^{n+1} = (1 + \gamma_1 + \cdots \gamma_{n-m})(1+\beta)^{m+1}$. Then, $(\alpha - \beta)^{n-m+1} = 0$.*

In the cohomological argument, the main entity is a closed nonnegative (1,1)-form ρ obtained from the second fundamental form of the holomorphic immersion and enjoying the property that the second fundamental form vanishes identically if and only if $\rho \equiv 0$. By the cohomological argument basing on Lemma 1 one concludes that $\rho^n \equiv 0$ on X. We lift ρ to $\widetilde{\rho}$ defined on some connected open subset $U \subset B^n$ holomorphically foliated by d-dimensional totally geodesic complex submanifolds for some d, $1 \leq d \leq n$. Completing these leaves to totally geodesic complex submanifolds (which are d-dimensional affine-linear sections of $B^n \subset \mathbb{C}^n$) we obtain a subset $S \subset B^n$ swept out by such submanifolds, where S contains $W \cap B^n$ for some neighborhood W of a boundary point $b \in \partial B^n$. The closed nonnegative (1,1)-form $\widetilde{\rho}$ can be extended to $W \cap B^n$. The proofs of the results of Cao-Mok [CM91] and To [To93] are completed by an argument by contradiction. This involves a geometric argument concerning the boundary behavior of $\widetilde{\rho}$ on $W \cap B^n$, where, assuming that $d < n$, $\widetilde{\rho} \geq 0$ is degenerate but not identically 0. From asymptotic properties of the canonical Kähler-Einstein metric on complex unit balls the latter is shown to be asymptotically of zero length as one approaches $W \cap \partial B^n$. Given that $\pi_1(X) = \Gamma$ is a lattice, the asymptotic vanishing of $\widetilde{\rho}$ implies $\rho \equiv 0$, yielding a proof of the theorem by contradiction.

In addition to holomorphic immersions there is naturally the problem of studying holomorphic submersions between complex hyperbolic space forms. In this regard Koziarz-Mok [KM10] has obtained recently the following result.

Theorem (part of Koziarz-Mok [KM10, Theorem 2]). *Let $n > m \geq 1$. Let Z be an m-dimensional compact complex hyperbolic space form. Let $f : X \to Z$ be a surjec-*

tive holomorphic map and denote by $E \subset Z$ the smallest subvariety such that f is a regular holomorphic fibration over $Z - E$. Then, $E \subset Z$ is of complex codimension 1.

Normalizing the canonical Kähler-Einstein metrics on X resp. Z with Kähler forms ω_X resp. ω_Z to be of constant holomorphic sectional curvature $-K$ for the same constant $K > 0$, the method of Koziarz-Mok [KM10] relies on Feder's identity as given in Lemma 1 and the use of a closed nonnegative (1,1)-form $\mu := \omega_X - f^*\omega_Z$, where the nonnegativity of μ follows from the Ahlfors-Schwarz Lemma.

Motivated by the above result of Koziarz-Mok [KM10] and the use of a different type of closed nonnegative (1,1)-form μ basing on the Ahlfors-Schwaz Lemma, we re-visit the study of holomorphic immersions between complex hyperbolic space forms, generalizing the context to the study of generically immersive holomorphic maps $f : X \to Y$ between complex hyperbolic space forms where neither X nor Y is required to be of finite volume with respect to the canonical Kähler-Einstein metric. Denoting by $\mathrm{Sing}(f)$ the singular locus of such a map, we are led to consider holomorphic immersions from $X - \mathrm{Sing}(f)$ into Y. In this article we present two main results. The first concerns a lower bound for the complex dimension of $\mathrm{Sing}(f)$ in the case where X is compact or noncompact but of finite volume. We will obtain such a result using essentially the arguments of Cao-Mok [CM91] and of To [To93] by considering furthermore the restriction of the tangent sequence to compact complex-analytic subvarieties of $X - \mathrm{Sing}(f)$. For the noncompact case of finite-volume, to obtain compact complex-analytic subvarieties of $X - \mathrm{Sing}(f)$ we make use of the following result on compactifying not necessarily arithmetic noncompact complex hyperbolic space forms of finite volume.

Theorem (Siu-Yau [SY82], Mok [Mk10]). *Let n be a positive integer, and let $\Gamma \subset \mathrm{Aut}(B^n)$ be a non-uniform torsion-free lattice; $X := B^n/\Gamma$. Then, X can be compactified to a normal projective-algebraic variety $\overline{X}_{\min}$ by adjoining a finite number of isolated normal singularities.*

Thus, in the case of a complex hyperbolic space form $X := B^n/\Gamma$, where Γ is a lattice, for our lower estimate on $\dim(\mathrm{Sing}(f))$ to be given in Theorem 1 we still rely on the use of the closed (1,1)-form $\rho \geq 0$ arising from the second fundamental form of the immersion on $X - \mathrm{Sing}(f)$. The second main result, to be given in Theorem 2 concerns the more general case where X may be of infinite volume, and we prove, under certain dimension restrictions, that the open set $X - \mathrm{Sing}(f)$ does not contain any irreducible compact complex-analytic subvariety of dimension $m - n + 1$ which is in some sense sufficiently deformable. In this result we make use of both the closed (1,1)-forms $\rho \geq 0$, which arises from the second fundamental form, and $\mu \geq 0$, where nonnegativity results from the Ahlfors-Schwarz Lemma, yielding on some holomorphically foliated connected open subset a contracting leafwise totally geodesic holomorphic isometric embedding. On the methodological plane we introduce a method which in principle replaces the geometric argument in Cao-Mok [Ca91] and To [To93] concerning $\rho \geq 0$, which relies on the fact that $\pi_1(X)$ is a lattice, by a local argument resulting from a commutation formula concerning the complex Hessian $\nabla\partial f$. For technical reasons we impose the slightly stronger dimension restriction $m \leq 2n-4$ for the local argument.

In the formulation of the second main result on complex hyperbolic space forms not necessarily of finite volume, we define the notion of sufficiently deformable compact complex-analytic subvarieties, as follows.

Definition 1 (sufficiently deformable subvariety). *Let N be a complex manifold of dimension n, $0 < q < n$. Let $S \subset N$ be a pure q-dimensional compact complex-analytic subvariety. We say that $S \subset N$ is sufficiently deformable if there exists an irreducible complex space B, $0 \in B$, a complex-analytic subvariety $\mathscr{S} \subset N \times B$ for which the canonical projection $\pi : \mathscr{S} \to B$ is proper with fibers being pure q-dimensional compact complex-analytic subvarieties $S_t := \pi^{-1}(t) \subset N$ for $t \in B$, $S_0 = S$, such that the following holds true. Denoting by $\tau : \mathscr{S} \to Gr(q, T(N))$ the canonical meromorphic map into the Grassmann bundle of q-dimensional vector subspaces of tangent spaces of N, where $\tau(x) = [T_x(S_{\pi(x)})] \in Gr(q, T_x(N))$ whenever x is a smooth point of $S_{\pi(x)}$, there is a point $y \in \mathscr{S}$ such that y is a smooth point of $\mathscr{S}$, $\pi(y)$ is a smooth point of B, π is a holomorphic submersion at y, and $\tau\big|_{U_y} : U_y \to Gr(q, T_x(N))$ is a holomorphic submersion on some open neighborhood U_y of y in $\mathscr{S}$.*

For an n-dimensional projective submanifold N by it is clear that whenever $0 < q < n$, any q-dimensional linear section cut out by $n-q$ hyperplanes is sufficiently deformable in N. The same is true for N being an n-dimensional quasi-projective manifold $N \subset \mathbb{P}^a$, and for any q-dimensional linear section $S \subset \overline{N}$ cut out by $n-q$ hyperplanes such that $S \subset N$, where $\overline{N} \subset \mathbb{P}^a$ denotes the topological closure of N in $\mathbb{P}^a$, $\overline{N} \subset \mathbb{P}^a$ being a projective-algebraic subvariety. Such q-dimensional linear sections S always exist whenever $q < n-d$, where $d = \dim(\overline{N} - N)$.

The first main result concerning singularities of generically immersive maps in the finite-volume case will be explained in §2. In §3-§5 we consider the more general situation in which the domain manifold $X := B^n/\Gamma$ may be of infinite volume. In §3, assuming the existence of a sufficiently deformable compact complex-analytic subvariety of $X - \mathrm{Sing}(f)$ of a certain dimension, we derive the existence of a contracting leafwise totally geodesic holomorphic isometric embedding from some open subset $U \subset B^n$ into B^m. In §4 we establish a commutation formula for the study of the complex Hessian $\nabla\partial f$ adapted to such maps, and in §5 we deduce consequences of the commutation formula, especially proving the second main result concerning compact complex-analytic subvarieties of $X - \mathrm{Sing}(f)$.

2 Singular loci in the finite-volume case

The first main result of the current article is given by the following theorem on the singular loci of generically immersive holomorphic maps between complex hyperbolic space forms in the case where the domain manifold is of finite volume.

Theorem 1. *Let n,m be positive integers such that $n \geq 2$ and $m < 2n$. Let $\Gamma \subset$ $\mathrm{Aut}(B^n)$ be a torsion-free lattice of automorphisms; $X := B^n/\Gamma$; and let Y be any m-dimensional complex hyperbolic space form. Suppose $f : X \to Y$ is a holomorphic map such that df is of rank n at a general point. Assume that the singular locus $\mathrm{Sing}(f)$ of f is of dimension strictly less than $2n-m-1$, then in fact $\mathrm{Sing}(f) = \emptyset$ and f is a totally geodesic map.*

As will be clear from the proof of Theorem 1, there is an obvious analogue of Theorem 1 for the dual case of nonconstant holomorphic maps $f : \mathbb{P}^n \to \mathbb{P}^m$. Such a holomorphic map is automatically an immersion at a general point since no algebraic curve on $\mathbb{P}^n$ can be collapsed to a point, $\mathbb{P}^n$ being of Picard number 1. The dual analogue of Theorem 1 says

Theorem 1'. *Let n,m be positive integers such that $n \geq 2$ and $m < 2n$. Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be a nonconstant holomorphic map. Then, $\mathrm{rank}(df(x))$ is equal to n at a general point $x \in \mathbb{P}^n$, and the singular locus $\mathrm{Sing}(f)$ must be of complex dimension $\geq 2n-m-1$ unless $f : \mathbb{P}^n \to \mathbb{P}^m$ is a projective-linear embedding.*

The inequality $\dim(\mathrm{Sing}(f)) \geq 2n-m-1$ is equivalent to the inequality $\mathrm{codim}(\mathrm{Sing}(f)) \leq n-(2n-m-1) = m-n+1$. For Theorem 1' it says in particular that a nonconstant holomorphic map $f : \mathbb{P}^n \to \mathbb{P}^{n+1}$ is either a projective-linear embedding, or its singular locus is of codimension at most equal to 2. We have the following example which shows in this case that the codimension may be exactly equal to 2.

EXAMPLE Let $f : \mathbb{P}^2 \to \mathbb{P}^3$ be defined by $f([z_0,z_1,z_2]) = [z_0^3, z_1^3, z_2^3, z_0z_1z_2]$ in terms of homogeneous coordinates. Then, f is holomorphic. By a straightforward computation, f is a holomorphic immersion excepting at the three points $[1,0,0],[0,1,0]$ and $[0,0,1]$. For any integer $n \geq 2$ the holomorphic map $f:\mathbb{P}^n \to \mathbb{P}^{n+1}$ defined by $f([z_0,z_1,\cdots,z_n]) = [z_0^{n+1},z_1^{n+1},\cdots z_n^{n+1},z_0z_1\cdots z_n]$ gives an example where $\mathrm{Sing}(f)$ is of codimension 2. In the latter case, $\mathrm{Sing}(f)$ is the union of the $\frac{n(n+1)}{2}$ projective-linear subspaces defined by $z_p = z_q = 0$, $0 \leq p < q \leq n$.

Proof of Theorem 1. Recall that n and m are positive integers, $n < m < 2n$, $\Gamma \subset \mathrm{Aut}(B^n)$ a torsion-free discrete group of automorphisms, $X = B^n/\Gamma$, and $f: X \to Y$ is a generically immersive holomorphic map. Write $X' := X - \mathrm{Sing}(f)$. Consider the tangent sequence $0 \to T_{X'} \to f^*T_Y \to N \to 0$ of X', where $N = f^*T_Y/T_{X'}$ denotes the normal bundle for the holomorphic immersion $f|_{X'}: X' \to Y$. Suppose $S \subset X'$ is an $(m-n+1)$-dimensional compact complex submanifold. Then $N|_S$ is a holomorphic vector bundle of rank $m-n$ and we have $c_{m-n+1}(N)|_S = c_{m-n+1}(N|_S) = 0$. By Feder's identity as given in Lemma 1 it follows that $[\nu^m] = 0$, where $[\cdots]$ denotes the de Rham cohomology class, for any closed smooth $(1,1)$-form ν representing the cohomology class $\frac{-c_1(X)}{n+1} + \frac{f^*c_1(Y)}{m+1}$. Since we have normalized the choice of the canonical Kähler-Einstein metric to be of constant holomorphic sectional curvature -4π, from Cao-Mok [CM91] we can take ν to be ρ, where, denoting by $g = \left(g_{\alpha\overline{\beta}}\right)$ resp. $h = \left(h_{i\overline{j}}\right)$ the canonical Kähler-Einstein metric on B^n resp. B^m of constant holomorphic sectional curvature -4π, we have

$$\rho_{\alpha\overline{\beta}} = \sum_{\gamma,\delta,k,\ell} g^{\gamma\overline{\delta}}\, \widehat{h}_{k\overline{\ell}}\sigma^k_{\alpha\gamma}\, \overline{\sigma^\ell_{\beta\delta}}, \tag{1}$$

where $\left(g^{\alpha\overline{\beta}}\right)$ denotes the conjugate inverse of $\left(g_{\alpha\overline{\beta}}\right)$, $\left(\widehat{h}_{k\ell}\right)$ denotes the Hermitian metric on N induced from h, $\sigma^k_{\alpha\gamma}$ denotes the (holomorphic) second fundamental form on (1,0)-vectors for the holomorphic immersion $f|_{X'}$, and the summation is performed over the ranges $1 \leq \alpha,\beta,\gamma,\delta \leq n$ and $1 \leq k,\ell \leq n-m$. From the curvature formula for Kähler submanifolds given by the Gauss equation we have in fact

$$\rho = \frac{-c_1(X,f^*h)}{n+1} + \frac{f^*c_1(Y,h)}{m+1}. \tag{2}$$

For the proof of Theorem 1 consider first of all the case where X is compact. Suppose the generically immersive holomorphic map $f: X \to Y$ is not totally geodesic and $\dim(\mathrm{Sing}(f)) < 2n-m-1$, i.e., $\mathrm{codim}(\mathrm{Sing}(f)) > n-(2n-m-1) = m-n+1$. Embedding X as a projective manifold and taking intersections of hyperplane sections for each $x \in X'$ there exists a smooth linear section $S \subset X$ passing through x and of dimension $m-n+1$ such that $S \cap \mathrm{Sing}(f) = \emptyset$. Then, by Feder's identity (Lemma 1) we have $[(\rho|_S)^{m-n+1}] = 0$. From $\rho \geq 0$ it follows that $\rho^{m-n+1}|_S = 0$, so that the smooth (1,1)-form $\rho|_S$ must have a positive-dimensional kernel at each point $s \in S$. Since $x \in X'$ is arbitrary, it follows that $\rho(x)$ must have a positive-dimensional kernel at each $x \in X'$. There is a real-analytic subvariety of $V \subsetneq X'$ such that $\dim(\mathrm{Ker}(\rho(x))$

is the same integer d, $1 \le d < n$. In particular there exists a non-empty connected open subset $U \subset X'$ such $\dim(\mathrm{Ker}(\rho(x)) = d$. Since Re(Ker($\rho$)) agrees with the kernel of the second fundamental form σ, the assignment of Re(Ker(ρ_x)) to $x \in X$ defines a holomorphic foliation $\mathscr{E}_x$ on U with d-dimensional leaves consisting of totally geodesic locally closed complex submanifolds. Taking U to be simply connected we can lift U in a univalent way to a connected open subset $\widetilde{U} \subset B^n$, ρ to a closed nonnegative (1,1)-form $\widetilde{\rho}$ on $\widetilde{U}$, and $\mathscr{E}$ to a holomorphic foliation $\widetilde{\mathscr{E}}$ on $\widetilde{U}$ consisting of totally geodesic complex submanifolds. By extending each leaf of $\widetilde{\mathscr{E}}$ to a complete totally geodesic complex submanifold of B^n, we sweep out $W \cap B^n$ for some neighborhood W of some boundary point $b \in \partial B^n$. We derive a contradiction exactly as in the argument of Cao-Mok [CM91] from the asymptotic behavior of an extension of $\widetilde{\rho}$ to $W \cap B^n$, which is based on the estimate that the extended closed (1,1)-form $\widetilde{\rho}$ is asymptotically of zero length as one approaches $W \cap \partial B^n$ and hence of zero length everywhere by the compactness of the fundamental domain of B^n modulo the action of Γ.

In the case where $X = B^n/\Gamma$ is noncompact and of finite volume, we adopt the arguments of To [To93], and the only thing that remains to be verified is that, under the assumption that $\dim(\mathrm{Sing}(f)) < 2n-m-1$ there still exists a compact complex submanifold $S \subset X - \mathrm{Sing}(f)$ of complex dimension exactly equal to $m-n+1$ obtained by taking the intersection of $2n-m-1$ hyperplane sections with respect to some projective embedding of X. That this is so follows readily from the existence of a projective-algebraic compactification X obtained by adding a finite number of normal isolated singularities, as follows from Siu-Yau [SY82] and Mok [Mk10] and stated in §1. ■

3 Contracting leafwise totally geodesic isometric embeddings

Motivated by the use of the Ahlfors-Schwarz Lemma in conjunction with Feder's identity (Lemma 1) in Koziarz-Mok [KM10], we examine further consequences that can be drawn from cohomological arguments by making use of both the Gauss equation (via the second fundamental form σ and hence ρ) and of the Ahlfors-Schwarz Lemma. Thus, in the notation of the proof of Theorem 1, the closed (1,1)-form ν can be taken to be $\mu = \omega_X - f^*\omega_Y$, where ω_X denotes the Kähler form of g on X, and ω_Y that of h on Y, so that $\mu = \frac{-c_1(X,g)}{n+1} + \frac{f^*c_1(Y,h)}{m+1}$. We have $\rho \ge 0$ from the definition of ρ in terms of σ, and $\mu \ge 0$ by the Ahlfors-Schwarz Lemma. When $S \subset X - \mathrm{Sing}(f)$ is smooth we have by Lemma 1

$$\rho^{m-n+1} = \mu^{m-n+1} = (\rho+\mu)^{m-n+1} = 0$$

identically on S, noting that Lemma 1 can be applied also to $\nu = \frac{\rho+\mu}{2} \geq 0$. When S is singular we can consider a desingularization $\zeta : \widetilde{S} \to S$ and the smooth closed (1,1)-form $\zeta^*\rho$ on $\widetilde{S}$, etc. in place of considering the restriction of ρ to the singular variety S. For the sake of brevity in place of specifying a desingularization we will speak of the restriction of ρ, etc., to the smooth part $\mathrm{Reg}(S)$ of S, written $\rho|_{\mathrm{Reg}(S)}$.

There is a smallest integer $r \geq 1$ such that $[\nu^r] = 0$ for $[\nu] = \frac{-c_1(X)}{n+1} + \frac{f^*c_1(Y)}{m+1} \geq 0$. The positive integer r is determined by the fact that the real-analytic semipositive closed $(1,1)-$form $\rho|_{\mathrm{Reg}(S)}$ has exactly $r-1$ non-zero eigenvalues on a dense open subset of S. Since ρ, μ and $\frac{\rho+\mu}{2}$ are cohomologous when pulled back to a desingularized model $\widetilde{S}$ we have

$$\rho^r = \mu^r = (\rho+\mu)^r = 0$$

on S. Thus, on a dense subset W of S, both ρ and μ have exactly $r-1$ non-zero eigenvalues over W, and they must have the same kernel over W. Note here that for $y \in W$, the vector subspaces $\mathrm{Ker}(\rho(y))$ and $\mathrm{Ker}(\mu(y))$ of $T_y(S)$ must agree with each other. Otherwise, $\dim(\mathrm{Ker}\,(\rho(y)) \cap \mathrm{Ker}\,(\mu(y))) < n-r+1$ and $(\rho(y)+\mu(y))^r \neq 0$, while $\rho^r \equiv 0$ over $\mathrm{Reg}(S)$, violating the fact that $\frac{\rho+\mu}{2}$ and ρ are cohomologous to each other when pulled back to a desingularized model $\widetilde{S}$ of S.

Suppose there exists a sufficiently deformable irreducible compact complex-analytic $(m-n+1)$-dimensional subvariety $S \subset X$. In the notations of the definition of such subvarieties as given in Definition 1, without loss of generality we may assume that there exists a holomorphic family $\pi : \mathscr{S} \to B$ of irreducible compact complex-analytic subvarieties $S_t \subset X = \pi^{-1}(t), t \in B$, parametrized by the complex unit ball B of a complex Eulcidean space, such that $S_0 = S$ and such that there exists a point $x \in W \subset S$ so that the holomorphic tangent spaces $T_x(S')$ of those $S' = S_t, t \in B$, passing through x wipes out an open neighborhood of $[T_x(S_0)]$ on $Gr(p, T_x(S))$. Thus $\mathrm{Ker}\,(\rho(x)|_{T_x(S')})$ and $\mathrm{Ker}\,((\rho(x)+\mu(x))|_{T_x(S')})$ are of codimension $r-1$ in $T_x(S')$. We conclude from the cohomological argument of the last paragraph that $\mathrm{Ker}\,(\rho(x)|_{T_x(S')}) = \mathrm{Ker}\,(\mu(x)|_{T_x(S')})$. For any q-plane $E \subset T_x(X)$ sufficiently close to $T_x(S)$ by assumption there exists some $t \in B$ such that $E = T_x(S')$ for $S' = S_t$. It follows that $\mathrm{Ker}\,(\rho(x)) = \mathrm{Ker}\,(\mu(x)) \subset T_x(X)$ is of codimension $r-1$, i.e., of dimension $n-r+1$. For a sufficiently small open neighborhood U of x in the ambient manifold X, the preceding discussion applies with x replaced by $y \in U$ and S replaced by some irreducible compact complex-analytic $(m-n+1)$ dimensional subvariety belonging to $(S_t)_{t\in B}$ and passing through y. Write $d = n-r+1$. Noting that $r \leq m-n+1$, we have $d \geq 2n-m$. Then, U is foliated by a holomorphic family of totally geodesic complex-analytic submanifolds Λ such that $\Lambda \subset X$ is totally geodesic, and such that $f|_\Lambda$ is a totally geodesic holomorphic isometric embedding. Lifting U to B^n and lifting Y locally to B^m, we have a holomorphic map $f : U \to B^m$ which is a leafwise totally geodesic holomorphic isometric embedding. It remains now to investigate whether such holomorphic maps can exist at all. In the next sections we will show that such maps do not exist under certain dimension restrictions, viz., we will show that leafwise totally geodesic holomorphic isometric embeddings are already totally geodesic. In other words, we will derive a contradiction unless $d = n$.

For the sake of convenience we introduce the notion of a contracting leafwise totally geodesic isometric embeddings, as follows.

Definition 2 (Contracting leafwise totally geodesic isometric embedding). *Let n, m be positive integers, $n < m$, $U \subset B^n$ be a connected open subset, and $f : U \to B^m$ be a holomorphic map. We say that f is contracting if and only if it is distance-decreasing when B^n resp. B^m are equipped with the canonical Kähler-Einstein metric $ds^2_{B^n}$ resp. $ds^2_{B^m}$ of constant holomorphic sectional curvature $-K$ for the same constant $K > 0$. Suppose f is an immersion and, denoting by σ the (holomorphic) second fundamental form on $(1,0)$-vectors for the immersion $f : U \to B^m$ with respect to $ds^2_{B^m}$, $\mathrm{Ker}(\sigma(x)) = \mathrm{Ker}(\rho(x)))$ is of the same rank d at every point $x \in U$. Denoting by $\mathscr{E} = \mathrm{Re}(\mathrm{Ker}(\rho))$ the associated integrable holomorphic foliation, assume that for each leaf Λ of $\mathscr{E}$, the restriction $f|_\Lambda : \Lambda \to B^m$ is a totally geodesic isometric embedding. Then, we say that $f : U \to B^m$ is a contracting leafwise totally geodesic isometric embedding (of leaf dimension d).*

REMARKS

(a) In place of the complex unit ball B^n resp. B^m we can consider the quotient manifold $X := B^n/\Gamma$ resp. $Y = B^m/\Psi$ with respect to a torsion-free discrete group of automorphisms Γ resp. Ψ, a connected open subset $U \subset X$, and a holomorphic immersion $f : U \to Y$. In this general situation we have analogously the notion of a contracting leafwise totally geodesic immersion, where the restriction of f to each totally geodesic leaf Λ of the analogously defined holomorphic foliation $\mathscr{E}$ is only assumed to be an isometric immersion.

(b) By Umehara [Um87] any isometric holomorphic immersion of an open subset of a complex hyperbolic space form into B^m is necessarily totally geodesic. In the terminology of a 'contracting leafwise totally geodesic isometric embedding (immersion)', it is implicit that the mapping is totally geodesic.

Summarizing in terms of the newly introduced terminology we have proven in this section

Proposition 1 *Let n, m be positive integers, $n < m$, $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free discrete group of automorphisms, $X := B^n/\Gamma$, and $\Psi \subset \mathrm{Aut}(B^m)$ be a torsion-free discrete group of automorphisms, $Y := B^n/\Psi$. Let $f : X \to Y$ be a generically immersive holomorphic map. Suppose there exists on $X - \mathrm{Sing}(f)$ a sufficiently deformable compact complex-analytic subvariety S of dimension $m-n+1$. Then, for a general point $x \in X - \mathrm{Sing}(f)$, there exists a connected open neighborhood W of x on $X - \mathrm{Sing}(f)$ and a positive integer d, $2n-m \le d \le n$, such that $f|_W : W \to Y$ is a contracting leafwise totally geodesic isometric immersion of leaf dimension d.*

4 A commutation formula

In this section we will derive a commutation formula for the Hessian of a contracting leafwise totally geodesic holomorphic isometric embedding $f : U \to B^m$ defined on a connected open subset $U \subset B^m$, where the underlying holomorphic foliation $\mathscr{E}$ is defined by $\mathrm{Re}(\mathrm{Ker}\,(\rho)) = \mathrm{Re}(\mathrm{Ker}\,(\sigma))$. We write $E \subset T_U$ for the holomorphic vector subbundle given by $E_x = \mathrm{Ker}\,(\rho(x))$. Following the conventions in the proof of [§2, Theorem 1] we will normalize holomorphic sectional curvatures to be -4π. Denote by g resp. h the canonical Kähler-Einstein metric on B^n resp. B^m of constant holomorphic sectional curvature -4π. We will be performing covariant differentiation on tensors fields on U. By ∇ we will denote covariant differentiation with respect to the canonical connections associated to g and h. Thus, ∂f is a holomorphic section of $\Omega_U \otimes f^*T_{B^m}$ over U, where Ω_U stands for the holomorphic cotangent bundle T_U^*, and $\nabla\partial f$ is a smooth section of $\Omega_U \otimes \Omega_U \otimes f^*T_{B^m}$ defined in terms of the affine connection on $T_U^* \otimes f^*T_{B^m}$ induced by the Riemannian connection of (B^n, g) and the pull-back of the Riemannian connection on (B^m, h). From the torsion-freeness of Riemannian connections it follows that the tensor field $\nabla\partial f$ takes values in $S^2\Omega_U \otimes f^*T_{B^m}$, i.e., $\nabla_\alpha\partial_\beta f$ is symmetric in α and β. We have the following commutation formula on $\nabla\partial f$.

Proposition 2 *Let $x \in U$, $\xi \in E_x$ and $\nu \in T_x(U)$. Denote by $\widetilde{\xi}$ an extension of $\xi \in E_x$ to a smooth E-valued vector field on some neighborhood of x on U. Then,*

$$\frac{1}{2\pi}\|\nabla_\xi\partial_\nu f\|^2 = \|\xi\|\left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right) - \left(\|\nabla_\nu\widetilde{\xi}\|^2 - \|\partial f(\nabla_\nu\widetilde{\xi})\|^2\right).$$

To simplify notations in what follows we will often write ξ for $\widetilde{\xi}$, etc. whenever there is no risk of confusion. Thus ξ will denote both a vector in E_x and a germ of smooth E-valued vector field at x extending that vector. Additional conditions may be imposed on the choice of smooth extensions for the computations. For the proof of Proposition 2 we start with a lemma regarding values of $\nabla\partial f$, as follows.

Lemma 2. *Let $x \in U$ and denote by σ, τ arbitrary smooth $(1,0)$-vector fields on a neighborhood of x, and ξ any smooth E-valued vector field on a neighborhood of x. Then, $h\left(\nabla_\sigma \partial_\tau f(x), \overline{\partial_\xi f}\right) = 0$.*

Proof. For the proof of Lemma 2 without loss of generality we may assume that $\widetilde{\xi}$ is a holomorphic E-valued holomorphic vector field on a neighborhood of s in U. Since f is isometric on each leaf Λ of $\mathscr{E}$, $E \subset \mathrm{Ker}\,(\mu)$ for $\mu := \omega_g - f^*\omega_h \geq 0$, where ω_g is the Kähler form (B^n, g) and ω_h is the Kähler form of (B^m, h). In particular, we have

$$h(\partial_\tau f, \overline{\partial_\xi f}) = g\left(\tau, \overline{\xi}\right). \tag{1}$$

Differentiating against the vector field σ we have

$$h\left(\nabla_\sigma \partial_\tau f, \overline{\partial_\xi f}\right) + h\left(\partial_{\nabla_\sigma \tau} f, \overline{\partial_\xi f}\right) + h\left(\partial_\tau f, \overline{\nabla_{\overline{\sigma}} \partial_\xi f}\right) + h\left(\partial_\tau f, \overline{\partial_{\nabla_{\overline{\sigma}}\xi} f}\right)$$
$$= g\left(\nabla_\sigma \tau, \overline{\xi}\right) + g\left(\tau, \overline{\nabla_{\overline{\sigma}} \xi}\right). \tag{2}$$

For the last term on the left-hand side of (2), by assumption ξ is a holomorphic vector field, hence $\nabla_{\overline{\sigma}}\xi = 0$ and we have

$$h\left(\partial_\tau f, \overline{\partial_{\nabla_{\overline{\sigma}}\xi} f}\right) = g\left(\tau, \overline{\nabla_{\overline{\sigma}}\xi}\right) = 0. \tag{3}$$

For the third term we have by symmetry

$$\nabla_{\overline{\sigma}} \partial_\xi f = \nabla_\xi \partial_{\overline{\sigma}} f = 0 \tag{4}$$

since f is holomorphic. For the second term, since $\xi(y) \in \mathrm{Ker}\,(\mu(y))$ where defined, we have

$$h\left(\partial_{\nabla_\sigma \tau} f, \overline{\partial_\xi f}\right) = g\left(\nabla_\sigma \tau, \overline{\xi}\right). \tag{5}$$

We conclude therefore from (2) that

$$h\left(\nabla_\sigma \partial_\tau f, \overline{\partial_\xi f}\right) = 0. \tag{6}$$

as desired. ■

Next, from standard commutation formulas for covariant differentiation on Hermitian holomorphic vector bundles on Kähler manifolds we have

Lemma 3. *Denote by R the curvature tensor on (U, g) and by S the curvature tensor of $(f^*T_{B^m}, f^*h)$ on U. Let σ, τ, ζ be smooth vector fields on U. Then, we have*

$$\nabla_{\overline{\zeta}}\nabla_\sigma\partial_\tau f^k=\sum_\mu R_{\sigma\overline{\zeta}\tau}{}^\mu\;\partial_\mu f^k-\sum_\ell S_{\sigma\overline{\zeta}\ell}{}^k\;\partial_\tau f^\ell\,,$$

where at $x\in U$, the symbol $\{\mu\}$ runs over the set of indexes of a basis $\{e_\mu\}$ of $T_{U,x}$, and the symbol ℓ runs over the set of indexes of a basis $\{\varepsilon_\ell\}$ of $f^*(T_{B^m,f(x)})$.

Proof. Since f is holomorphic we have $\nabla_{\overline{\zeta}}\partial_\tau f=\nabla_\tau\partial_{\overline{\zeta}}f=0$. Lemma 3 then follows from standard commutation formulas for Hermitian holomorphic vector bundles for the computation of $\nabla_\sigma\nabla_{\overline{\zeta}}\partial_\tau f^k-\nabla_{\overline{\zeta}}\nabla_\sigma\partial_\tau f^k=-\nabla_{\overline{\zeta}}\nabla_\sigma\partial_\tau f^k$.

We are now ready to derive Proposition 2.

Proof of Proposition 2. Let $x\in U$. We apply Lemma 2 to a special choice of vectors σ,τ,ζ at x, extended to smooth vector fields on U. Let ζ and σ be the same E-valued holomorphic vector field ξ on U such that $\xi(x)$ is of unit length, shrinking the neighborhood U of x if necessary. Let τ be an $E^\perp$-valued smooth vector field ν such that $\nu(x)$ is of unit length. Again shrinking U if necessary let $\{e_\mu\}$ be a smooth basis of T_U which is orthonormal at the point x and which includes at x the orthogonal unit vectors $\xi(x)$ and $\nu(x)$. Then, we have

$$\begin{aligned}R_{\sigma\overline{\zeta}\tau}{}^\mu(x)&=R_{\sigma\overline{\zeta}\tau\overline{\mu}}(x)=R_{\xi\overline{\xi}\nu\overline{\mu}}(x)\\&=\begin{cases}-2\pi & \text{if }\ e_\mu(x)=\nu(x)\\ 0 & \text{otherwise}\end{cases}.\end{aligned}\tag{1}$$

Denote by R' the curvature tensor of (B^m,h). For (1,0)-vectors α,β,γ at $x\in U$ we have

$$S_{\alpha\overline{\beta}\gamma}=f^*R'\left(\partial f(\alpha),\overline{\partial f(\beta)};\partial f(\gamma)\right)\in f^*T_{B^m,f(x)}\,.\tag{2}$$

At the point $x\in U$, for the subset $\{e_\lambda(x)\}$ of unit vectors in $\{e_\mu(x)\}$ belonging to E_x, define $\varepsilon_\lambda(x):=\partial f(e_\lambda(x))$. Since $\partial f(x):T_x(U)\to T_{f(x)}(B^m)$ restricts to a linear isometry on $E_x\subset T_x(U)$, the set $\{\varepsilon_\lambda(x)\}$ constitutes an orthonormal basis of $\partial f(E_x)\subset T_{B^m,f(x)}$. In the sequel for simplicity we will sometimes identify $T_{B^m,f(x)}$ with $f^*T_{B^m,f(x)}$ tautologically in the notation. Complete now $\{\varepsilon_\lambda\}$ to a smooth basis $\{\varepsilon_\ell\}$ of $f^*T_{B^m}$ on a neighborhood of x in such a way that $\{\varepsilon_\ell(x)\}$ is an orthonormal basis of $f^*T_{B^m,f(x)}$ with a further specification, as follows. For $\zeta\in T_{U,x}$ we will also write ζ' for $\partial f(\zeta)$. Since $E_x\subset T_{U,x}$ lies on $\mathrm{Ker}(\mu)$, $\nu'(x)=\partial f(\nu(x))$ is orthogonal to $\partial f(\xi)$ for any $\xi\in E_x$. Thus, the orthonormal basis $\{\varepsilon_\lambda(x)\}$ of $\partial f(E_x)$ can be completed to an orthonormal basis $\{\varepsilon_\ell(x)\}$ such that one of the basis vectors is the unit vector $\nu'':=\frac{\nu'(x)}{\|\nu'(x)\|}$, which is proportional to $\nu'(x)$. We will choose a smooth basis $\{\varepsilon_\ell\}$ of $f^*T_{B^m}$ on some neighborhood of x such that $\{\varepsilon_\ell(x)\}$ is an orthonormal basis of $f^*T_{B^m,f(x)}$ with the latter property. Furthermore such a basis will be chosen such that $\{\varepsilon_\ell\}$ corresponds on a neighborhood of $x\in U$ to $f^*\frac{\partial}{\partial w_\ell}$ for some holomorphic coordinates $(w_1,\cdots,w_m)$ on a neighborhood of $f(x)$ in $\mathbb{C}^m$. We have

$$\begin{aligned} S_{\sigma\overline{\zeta}\ell} &= S_{\nu\overline{\xi}\ell} \\ = R'_{\nu'\overline{\xi'}\ell}(x) &= \begin{cases} -2\pi\|\nu'(x)\|\nu'' & \text{if } \varepsilon_\ell(x)=\nu'' \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{3}$$

Let now ξ be an E-valued holomorphic vector field on U. By Lemma 2 we have

$$h\left(\nabla_\xi\partial_\nu f(x),\overline{\partial_\xi f}\right)=0\,. \tag{4}$$

Differentiating with respect to $\overline{\nu}$ we have

$$\begin{aligned} &h\left(\nabla_{\overline{\nu}}\nabla_\xi\partial_\nu f,\overline{\partial_\xi f}\right)+h\left(\nabla_{\partial_{\overline{\nu}}\xi}\partial_\nu f,\overline{\partial_\xi f}\right)+h\left(\nabla_\xi\partial_{\nabla_{\overline{\nu}}\nu}f,\overline{\partial_\xi f}\right) \\ &\quad+h\left(\nabla_\xi\partial_\nu f,\overline{\nabla_\nu\partial_\xi f}\right)+h\left(\nabla_\xi\partial_\nu f,\overline{\partial_{\nabla_\nu\xi}f}\right)=0\,. \end{aligned} \tag{5}$$

By Lemma 2 the second and the third terms on the left-hand side of (5) vanish. By the symmetry of the Hessian we have $\nabla_\nu\partial_\xi f=\nabla_\xi\partial_\nu f$ and hence

$$h\left(\nabla_{\overline{\nu}}\nabla_\xi\partial_\nu f,\overline{\partial_\xi f}\right)+\|\nabla_\xi\partial_\nu f\|^2+h\left(\nabla_\xi\partial_\nu f,\overline{\partial_{\nabla_\nu\xi}f}\right)=0\,. \tag{6}$$

We proceed to compute the first and the third terms of the left-hand side of (6). For the first term by Lemma 3 and by the symmetry of the Hessian we have

$$\begin{aligned} \nabla_{\overline{\nu}}\nabla_\xi\partial_\nu f^k(x) &= \nabla_{\overline{\nu}}\nabla_\nu\partial_\xi f^k(x)=\sum_\mu R_{\nu\overline{\nu}\xi}{}^\mu\,\partial_\mu f^k(x)-\sum_\ell S_{\nu\overline{\nu}\ell}{}^k\,\partial_\xi f^\ell(x) \\ &=\sum_\mu R_{\nu\overline{\nu}\xi}{}^\mu\,\partial_\mu f^k(x)-\sum_\ell R'_{\nu'\overline{\nu'}\ell}{}^k\,\partial_\xi f^\ell(x)\,. \end{aligned} \tag{7}$$

For the proof of Proposition 2 without loss of generality we may assume that $\xi(x)$ and $\nu(x)$ are (orthogonal) unit vectors. On a neighborhood of x, we use the same choice of a smooth basis $\{e_\mu\}$ and a smooth basis $\{\varepsilon_\ell\}$ of $f^*T_{B^m}$ as in the above, so that in particular $\{e_\mu(x)\}$ is an orthonormal basis of $T_{U,x}$ at x and $\{\varepsilon_\ell(x)\}$ is an orthonormal basis of $f^*T_{B^m,f(x)}$ at x. Write $\xi(x)=e_a$, $\nu(x)=e_b$. Recall the notation $\zeta':=\partial f(\zeta)$ for $(1,0)$-vectors ζ on U. We write also $\xi'(x)=\varepsilon_a$. Recall also $\nu'':=\frac{\nu'(x)}{\|\nu'(x)\|}=\varepsilon_b$. For the first summation on the last line of (7) the only possibly non-zero summand arises when $\mu=a$, giving $R_{\nu\overline{\nu}\xi\overline{\xi}}\partial_\xi f^k(x)=-2\pi\partial_\xi f^a$ when $k=a$ and 0 otherwise. For the second summation the only possibly non-zero summand arises when $\ell=a$, giving $R'_{\nu'\overline{\nu'}\ell}{}^k\,\partial_\xi f^\ell(x)=\|\nu'\|^2R'_{\nu''\overline{\nu''}\xi'\overline{\xi'}}\partial_\xi f^a(x)=-2\pi\partial_\xi f^a(x)$ when $k=a$ and 0 otherwise. It follows from (7) that

$$\nabla_{\overline{\nu}}\nabla_{\nu}\partial_{\xi}f(x) = \sum_k\left(\sum_{\mu}R_{\nu\overline{\nu}\xi}{}^{\mu}\ \partial_{\mu}f^k(x) - \sum_{\ell}R'_{\nu'\overline{\nu'}\ell}{}^{k}\ \partial_{\xi}f^{\ell}(x)\right)\otimes\varepsilon_k(x)$$
$$= -2\pi\left(\partial_{\xi}f(x) - \|\partial_{\nu}f\|^2\partial_{\xi}f(x)\right). \qquad (8)$$

Plugging into (5) and without assuming that $\xi(x)$ and $\nu(x)$ are of unit length the first term there on the left-hand side is given by

$$h\left(\nabla_{\overline{\nu}}\nabla_{\xi}\partial_{\nu}f, \overline{\partial_{\xi}f}\right) = -2\pi\|\xi\|^2\left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right). \qquad (9)$$

For the proof of Proposition 2 it remains to deal with the last term $h\left(\nabla_{\xi}\partial_{\nu}f, \overline{\partial_{\nabla_{\nu}\xi}f}\right)$ on the left-hand side of (5). Recall that

$$h\left(\partial_{\xi}f, \overline{\partial_{\nabla_{\nu}\xi}f}\right) = g\left(\xi, \overline{\nabla_{\nu}\xi}\right). \qquad (10)$$

Differentiating against ν we have

$$h\left(\nabla_{\nu}\partial_{\xi}f, \overline{\partial_{\xi}f}\right) + h\left(\partial_{\nabla_{\nu}\xi}f, \overline{\partial_{\nabla_{\nu}\xi}f}\right) + h\left(\partial_{\xi}f, \overline{\nabla_{\overline{\nu}}\partial_{\nabla_{\nu}\xi}f}\right) + h\left(\partial_{\xi}f, \overline{\partial_{\nabla_{\overline{\nu}}(\nabla_{\nu}\xi)}f}\right)$$
$$= g\left(\nabla_{\nu}\xi, \overline{\nabla_{\nu}\xi}\right) + g\left(\xi, \overline{\nabla_{\overline{\nu}}(\nabla_{\nu}\xi)}\right). \qquad (11)$$

By the symmetry of the Hessian, the pluriharmonicity of f, i.e., $\overline{\nabla}\partial f = 0$, and the identity $h\left(\partial_{\xi}f, \overline{\partial_{\tau}f}\right) = g(\xi,\tau)$ for any tangent vector field τ, the equation (11) gives

$$h\left(\nabla_{\xi}\partial_{\nu}f, \overline{\partial_{\xi}f}\right) + h\left(\partial_{\nabla_{\nu}\xi}f, \overline{\partial_{\nabla_{\nu}\xi}f}\right) = g\left(\nabla_{\nu}\xi, \overline{\nabla_{\nu}\xi}\right). \qquad (12)$$

In other words, we have

$$\begin{aligned} h\left(\nabla_{\xi}\partial_{\nu}f, \overline{\partial_{\xi}f}\right) &= g\left(\nabla_{\nu}\xi, \overline{\nabla_{\nu}\xi}\right) - h\left(\partial_{\nabla_{\nu}\xi}f, \overline{\partial_{\nabla_{\nu}\xi}f}\right) \\ &= \|\nabla_{\nu}\xi\|^2 - \|\partial f(\nabla_{\nu}\xi)\|^2. \end{aligned} \qquad (13)$$

Substituting (9) and (13) into (6) we deduce

$$\frac{1}{2\pi}\|\nabla_{\xi}\partial_{\nu}f\|^2 = \|\xi\|\left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right) - \left(\|\nabla_{\nu}\xi\|^2 - \|\partial f(\nabla_{\nu}\xi)\|^2\right), \qquad (14)$$

proving Proposition 2, as desired. ■

REMARKS In the proof of Proposition 2, the expression $\nabla_{\nu}\xi(x) = \nabla_{\nu}\widetilde{\xi}(x)$ depends on the choice of extension of the vector $\xi \in E_x$ to a germ of E-valued holomorphic section $\widetilde{\xi}$ at x, although the notation $\widetilde{\xi}$ is suppressed in the formulas. In the final outcome as given in the identity (14) there, if $\widetilde{\xi}$ is replaced by another smooth extension $\xi^{\sharp}$, then

$$\nabla_{\nu}\xi^{\sharp}(x) - \nabla_{\nu}\widetilde{\xi}(x) := \eta(x) \in E_x\ ; \quad \text{and} \quad \nabla_{\nu}\xi^{\sharp}(x) = \nabla_{\nu}\widetilde{\xi}(x) + \eta(x)$$

is an orthogonal decomposition such that $\partial f\big(\nabla_\nu \xi^\sharp(x)\big) = \partial f\big(\nabla_\nu \widetilde{\xi}(x)\big) + \partial f\big(\eta(x)\big)$ is again an orthogonal decomposition, while $\|\eta(x)\| = \|\partial f(\eta(x))\|$ since $\eta(x) \in E_x$ and $E_x \subset \operatorname{Ker}(\mu(x))$.

5 Consequences of the commutation formula

We start with the following general result on contracting leafwise totally geodesic holomorphic isometric embeddings from connected open subsets of the complex unit ball into complex unit balls.

Theorem 2. *Let n, m, d be positive integers, $m \leq 2n-4$, $3 \leq d \leq n$. Let $U \subset B^n$ be a nonempty connected open subset, and $\mathscr{E}$ be a holomorphic foliation on U by d-dimensional holomorphic totally geodesic complex submanifolds Λ. Let $f : U \to B^m$ be a contracting (distance-decreasing) holomorphic mapping such that $f|_\Lambda$ is a totally geodesic isometric embedding for each leaf Λ. Assume that the foliation $\mathscr{E}$ is defined by* $\mathrm{Re}(\mathrm{Ker}(\rho))$ *for the closed $(1,1)$-form given by $\rho = \frac{-c_1(X, f^*h)}{n+1} + \frac{f^*c_1(Y,h)}{m+1} \geq 0$, where g (resp. h) stands for the canonical Kähler-Einstein metric on B^n (resp. B^m) of constant holomorphic sectional curvature $-K$ for any fixed $K > 0$. Assume furthermore that* $\mathrm{Ker}(\rho) = \mathrm{Ker}(\mu)$ *for $\mu = \frac{-c_1(X,g)}{n+1} + \frac{f^*c_1(Y,h)}{m+1} \geq 0$. Then, $\rho \equiv 0$, $\mu \equiv 0$, $\mathscr{E}$ is trivial, and $f : U \to B^m$ is a totally geodesic isometric embedding.*

Proof. For the formulation and proof of Theorem 2 the choice of the constant $K > 0$ is unimportant. For the sake of uniformity we will choose K to be 4π as in the statement of Proposition 2. In this case μ agrees with the formula $\mu = \omega_g - f^*\omega_h$ given in the proof of Lemma 2. Recall that the holomorphic foliation $\mathscr{E}$ on U corresponds to a d-dimensional holomorphic distribution which we denote by $E \subset T_U$. Let now $x \in U$, $\xi \in E_x$ and $\nu \in T_{U,x}$. By Proposition 2, we have

$$\frac{1}{2\pi}\|\nabla_\xi \partial_\nu f\|^2 = \|\xi\|\left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right) - \left(\|\nabla_\nu \xi\|^2 - \|\partial f(\nabla_\nu \xi)\|^2\right). \tag{1}$$

Recall from the Remarks after the proof of Proposition 2 that in the commutation formula (1) it is understood that ξ is extended to a smooth vector field $\widetilde{\xi}$ on a neighborhood of x in U. The expression $\nabla_\nu \widetilde{\xi}(x)$ is then uniquely defined only modulo E_x, but the expression $\left(\|\nabla_\nu \widetilde{\xi}\|^2 - \|\partial f(\nabla_\nu \widetilde{\xi})\|^2\right)(x)$ is independent of the extension $\widetilde{\xi}$ since $\partial f(x)$ is an isometry on E_x, and since $\partial f(E_x)$ is orthogonal to $\partial f(E_x^\perp)$. Define now $T : E^\perp \otimes E \to E^\perp$ by $T(\nu \otimes \xi) = \mathrm{pr}_{E^\perp}(\nabla_\nu(\widetilde{\xi}))$ for $\nu \in E_x^\perp$, $\xi \in E_x$, where $\mathrm{pr}_{E^\perp} : T \to E^\perp$ denotes the orthogonal projection.

Then, $\left(\|\nabla_\nu\widetilde{\xi}\|^2 - \|\partial f(\nabla_\nu\widetilde{\xi})\|^2\right)(x) = \|T(\nu,\xi)(x)\|^2$. Here we write $T(\nu,\xi)$ for $T(\nu\otimes\xi)$ and the same notational convention will be adopted for linear maps on tensor products will be adopted elsewhere.

Consider now the linear map $Q : E \otimes E^\perp \to f^*T_{B^m}$ given at $x \in U$ by $Q(\xi\otimes\nu) = Q(\xi,\nu) = \nabla_\xi\partial_\nu f$ for $\xi \in E_x$, $\nu \in E_x^\perp$. The identity (1) translates into an identity of the form

$$\frac{1}{2\pi}\|Q(\xi\otimes\nu)\|^2 = P(\xi\otimes\nu,\overline{\xi\otimes\nu}) - \|T(\xi\otimes\nu)\|^2, \tag{2}$$

where $P(\cdot,\bar{\cdot})$ is the unique Hermitian bilinear form on $E\otimes E^\perp$ which satisfies $P\left(\xi\otimes\nu,\overline{\xi\otimes\nu}\right) = \|\xi\|^2\left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right)$. Denote by $\pi(\cdot,\bar{\cdot})$ the Hermitian bilinear form on E given by $\pi(\nu,\overline{\mu}) = g(\nu,\overline{\mu}) - h\big(\partial f(\nu),\overline{\partial f(\mu)}\big)$ for $\nu,\mu \in E_x, x\in U$. Then $\pi(\nu,\overline{\nu}) > 0$ whenever $\nu \in E^\perp$ is non-zero since ∂f is strictly distance-decreasing on $E^\perp$. For $x\in U$. Let now $\{\xi_1,\cdots\xi_d\}$ be an orthonormal basis of E_x and $\{\nu_1,\cdots\nu_{n-d}\}$ be an orthonormal basis of $E_x^\perp$ consisting of eigenvectors of the Hermitian form π_x. Thus, $\pi(\nu_j,\overline{\nu_\ell}) = 0$ for $j\neq\ell$, $1\le j,\ell\le n-d$ and $\pi(\nu_j,\overline{\nu_j}) = \lambda_j > 0$. Then, for

$$\tau = \sum_{i=1}^{d}\sum_{j=1}^{n-d} a_{ij}\xi_i\otimes\nu_j. \tag{3}$$

we have

$$P(\tau,\overline{\tau}) = \sum_{i=1}^{d}\sum_{j=1}^{n-d}\lambda_j^2|a_{ij}|^2. \tag{4}$$

In particular $P(\cdot,\bar{\cdot})$ is positive definite. From (2), for $\tau\in E\otimes E^\perp$ we have

$$\frac{1}{2\pi}\|Q(\tau)\|^2 = P(\tau,\overline{\tau}) - \|T(\tau)\|^2, \tag{5}$$

We examine further the identity (5). Now $\mathrm{rank}(E^\perp\otimes E) = (n-d)d$ and $\mathrm{rank}(E^\perp) = n-d$. By assumption $d\ge 3$ so that $\dim(\mathrm{Ker}(T_x)) \ge (n-d)(d-1) \ge 2(n-d) > 0$ whenever $3\le d<n$. (The case $d=n$ means precisely that f is totally geodesic.) By Lemma 2 we have $h\left(\nabla_\xi\partial_\nu f,\overline{\partial_\eta f}\right) = 0$ whenever $\eta\in E_x$, so that $\mathrm{Im}(Q_x)$ lies in the orthogonal complement H_x of $\partial f(E_x)$ in $T_{B^m,f(x)}$, where $\dim(H_x) = m-d$. By the preceding paragraph $\dim(\mathrm{Ker}(T_x)) = (n-d)(d-1)$, hence $\dim(\mathrm{Ker}(T_x)\cap\mathrm{Ker}(Q_x)) \ge (n-d)(d-1)-(m-d) \ge (n-d+1)(d-1)-(2n-5)$. Suppose $(3\le)\, d\ne n$. Then, $(n-d+1)(d-1)\ge 2(n-2)$, where equality is attained precisely when $d=3$ and $d=n-1$, and we have $\dim(\mathrm{Ker}(T_x)\cap\mathrm{Ker}(Q_x))\ge 1$. Let $\tau\in\mathrm{Ker}(T_x)\cap\mathrm{Ker}(Q_x)$ be a non-zero element. Then from the identity (5) we have

$$P(\tau,\overline{\tau}) = \frac{1}{2\pi}\|Q(\tau)\|^2 + \|T(\tau)\|^2 = 0\,, \tag{6}$$

violating the positivity of P. Thus, a contradiction arises if $3 \le d < n$. Since by assumption $d \ge 3$ it follows that the only possibility is that $d = n$. In other words, $f : U \to B^m$ is a totally geodesic embedding, as desired. ∎

Dimension restrictions have been placed on n, m and the leaf dimension d of the holomorphic foliation ξ. It is tempting to believe that such dimension restrictions are unnecessary. In the notations used in Theorem 2 we formulate a conjecture as follows.

Conjecture 1. *Let n,m be positive integers. Let $U \subset B^n$ be a nonempty connected open subset, $f : U \to B^m$ be a holomorphic immersion. Suppose there exists a nonzero integrable holomorphic distribution $E \subset T_U$ of rank $d > 0$ such that f is a contracting leafwise totally geodesic isometric embedding with respect to the holomorphic foliation $\mathscr{E}$ defined by* $\mathrm{Re}(E)$*. Assume furthermore that $E = \mathrm{Ker}(\rho) = \mathrm{Ker}(\mu)$ on U. Then, $E = T_U$ and f is totally geodesic.*

As a consequence of Theorem 2 we have the following general result about holomorphic mappings between complex unit balls equivariant with respect to a torsion-free discrete subgroup which is not necessarily a lattice. In particular, they are valid on domain complex hyperbolic space forms X of possibly infinite volume with respect to the canonical Kähler-Einstein metric.

Theorem 3. *Let n,m be positive integers, $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free discrete group of biholomorphic automorphisms, $X := B^n/\Gamma$. Let $\Phi : \Gamma \to \mathrm{Aut}(B^m)$ be a group homomorphism, and $f : B^n \to B^m$ be a holomorphic mapping which is equivariant with respect to Φ, i.e., $f(\gamma x) = \Phi(\gamma)(f(x))$ for any $x \in B^n$ and any $\gamma \in \Gamma$. Suppose $m \le 2n-4$, f is an immersion at a general point $x \in B^n$, and f is not totally geodesic. Then, writing* $\mathrm{Sing}(f)$ *for the singular locus of f (which is necessarily invariant under the action of Γ), E for the subvariety* $\mathrm{Sing}(f)/\Gamma \subset X$*, and $Z := X - E$, there does not exist any sufficiently deformable compact complex-analytic subvariety $S \subset Z$ of complex dimension $m-n+1$.*

Proof. Suppose there exists a sufficiently deformable compact complex-analytic subvariety $S \subset X$. By Proposition 1 there exists a non-empty connected open subset $U \subset B^n$ such that the restriction $f|_U : U \to B^m$ is a leafwise totally geodesic isometric embedding. Here the leaves of the underlying holomorphic foliation $\mathscr{E}$ by totally geodesic complex submanifolds are of complex dimension d, where d is the rank of $\mathrm{Ker}(\rho)$ and of $\mathrm{Ker}(\mu)$ at each point of U, and $d \ge 1 + (n-(m-n+1)) = 2n-m \ge 4$. In particular Theorem 2 applies and f is totally geodesic. ∎

Because of the dimension restriction $m \leq 2n-4$, Theorem 3 does not cover Theorem 2 for the cases where X is compact or noncompact and of finite volume. An affirmative solution of Conjecture 1 in the above would yield a geometric argument (after the global cohomological argument) for the proof of Theorem 2 which is completely local. For the latter purpose it would also suffice to establish a strengthened version of Theorem 2 in which the dimension restrictions in the hypothesis are relaxed to the conditions $m \leq 2n-1$ and $1 \leq d \leq n$.

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Generically nef vector bundles and geometric applications

Thomas Peternell

Abstract The cotangent bundle of a non-uniruled projective manifold is generically nef, due to a theorem of Miyaoka. We show that the cotangent bundle is actually generically ample, if the manifold is of general type and study in detail the case of intermediate Kodaira dimension. Moreover, manifolds with generically nef and ample tangent bundles are investigated as well as connections to classical theorems on vector fields on projective manifolds.

Keywords uniruled variety, generic nef bundle, movable cone, stability.
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1 Introduction

Given a vector field v on a complex projective manifold X, a classical theorem of Rosenlicht says that X is uniruled, i.e., X covered by rational curves, once v has a zero. If on the other hand v does not vanish at any point, Lieberman has shown that there is a finite étale cover $\pi : \tilde{X} \to X$ and a splitting

$$\tilde{X} \simeq A \times Y$$

with an abelian variety A such that the vector field $\pi^*(v)$ comes from a vector field on A. In particular, if X is of general type, then X does not carry any non-zero vector field.

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For various reasons it is interesting to ask what happens if v is a section in $S^m T_X$, or $(T_X)^{\otimes m}$, or even more general, in $(T_X)^{\otimes m} \otimes L$ with a numerically trivial line bundle L on X. In particular, one would like to have a vanishing

$$H^0(X, (T_X)^{\otimes m} \otimes L) = 0 \tag{1}$$

if X is of general type and ask possibly for structure results in case X is not uniruled. The question whether the above vanishing holds was communicated to me by N. Hitchin. The philosohical reason for the vanishing is quite clear: if X is of general type, then the cotangent bundle Ω^1_X should have some ampleness properties. One way to make this precise is to say that the restriction $\Omega^1_X|C$ is ample on sufficiently general curve $C \subset X$.
There are two things to be mentioned immediately. First, a fundamental theorem of Miyaoka says that $\Omega^1_X|C$ is nef on the general curve; we say shortly that Ω^1_X is *generically nef*. Second, if K_X is ample, then X admits a Kähler-Einstein metric, in particular Ω^1_X is stable and consequently $\Omega^1_X|C$ is stable, from which it is easy to deduce that $\Omega^1_X|C$ is ample.

We therefore ask under which conditions the cotangent bundle of a non-uniruled manifold is *generically ample.* We show, based on [BCHM09], [Ts88] and [En88], that generic ampleness indeed holds if X is of general type, implying the vanishing 1. We also give various results in case X is not of general type, pointing to a generalization of Lieberman's structure theorem. In fact, "most" non-uniruled varieties have generically ample cotangent bundles. Of course, if K_X is numerically trivial, then the cotangent bundle cannot be generically ample, and some vague sense, this should be the only reason, i.e. if Ω^1_X is not generically ample, then in some sense X should split off a variety with numerically trivial canonical sheaf. However certain birational transformations must be allowed as well as étale cover. Also it is advisable to deal with singular spaces as they occur in the minimal model program. One geometric reason for this picture is the fact that a non-uniruled manifold X, whose cotangent bundle is not generically ample, carries in a natural way a foliation $\mathscr{F}$ whose determinant $\det \mathscr{F}$ is numerically trivial (we assume that K_X is not numerically trivial). If $\mathscr{F}$ is chosen suitably, its leaves should be algebraic and lead to a decomposition of X. Taking determinants, we obtain a section in $\bigwedge^q T_X \otimes L$ for some numerically trivial line bundle L, giving the connection to the discussion we started with.

The organization of the paper is as follows. We start with a short section on the movable cone, because the difference between the movable cone and the "complete intersection cone" is very important in the framework of generic nefness. We also give an example where the movable cone and the complete intersection cone differ (worked out with J.P.Demailly). In section 3 we discuss in general the concept of generic nefness and its relation to stability. The following section is devoted to the study of generically ample cotangent bundles. In the last part we deal with generically nef tangent bundles and applications to manifolds with nef anticanonical bundles.

2 The movable cone

We fix a normal projective variety X of dimension n. Some notations first. Given ample line bundles $H_1,\dots,H_{n-1}$, we set $h=(H_1,\dots,H_{n-1})$ and simply say that h is an ample class. We let

$$NS(X)=N^1(X)\subset H^2(X,\mathbb{R})$$

be the subspace generated by the classes of divisors and

$$N_1(X)\subset H^{2n-2}(X,\mathbb{R})$$

be the subspace generated by the classes of curves.

Definition 1 (i) The *ample cone* $\mathscr{A}$ is the open cone in $N^1(X)$ generated by the classes of ample line bundles, its closure is the *nef cone.*

(ii) The *pseudo-effective cone* $\mathscr{PS}$ is the closed cone in $N^1(X)$ of classes of effective divisors.

(iii) The *movable cone* $\overline{ME}(X)\subset N_1(X)$ is the closed cone generated by classes of curves of the form

$$C=\mu_*(\tilde{H}_1\cap\ldots\cap\tilde{H}_{n-1});$$

here $\mu:\tilde{X}\to X$ is any modification from a projective manifold X and $\tilde{H}_i$ are very ample divisors in $\tilde{X}$. These curves C are called strongly movable.

(iv) $\overline{NE}(X)\subset N_1(X)$ is the closed cone generated by the classes of irreducible curves.

(v) An irreducible curve C is called movable, if $C=C_{t_0}$ is a member of a family (C_t) of curves such that $X=\bigcup_t C_t$. The closed cone generated by the classes of movable curves is denoted by $\overline{ME}(X)$.

(vi) The complete intersection cone $\overline{CI}(X)$ is the closed cone generated by classes $h = (H_1, \ldots, H_{n-1})$ with H_i ample.

Recall that a line bundle L is *pseudo-effective* if $c_1(L) \in \mathscr{PS}(X)$. The pseudo-effective line bundles are exactly those line bundles carrying a singular hermitian metric with positive curvature current; see [BDPP04] for further information.

Example 1 We construct a smooth projective threefold X with the property

$$\overline{ME}(X) \neq \overline{CI}(X).$$

This example has been worked out in [DP07]. We will do that by constructing on X a line bundle which is on the boundary of the pseudo-effective cone, but strictly positive on $\overline{CI}(X)$.

We choose two different points $p_1, p_2 \in \mathbb{P}_2$ and consider a rank 2-vector bundle E over $\mathbb{P}_2$, given as an extension

$$0 \to \mathscr{O} \to E \to \mathscr{I}_{\{p_1,p_2\}}(-2) \to 0 \tag{2}$$

(see e.g. [OSS80]). Observe $c_1(E) = -2$; $c_2(E) = 2$. Moreover, if $l \subset \mathbb{P}_2$ is the line through p_1 and p_2, then

$$E|l = \mathscr{O}(2) \oplus \mathscr{O}(-4). \tag{3}$$

Set

$$X = \mathbb{P}(E)$$

with tautological line bundle

$$L = \mathscr{O}_{\mathbb{P}(E)}(1).$$

First we show that L is strictly positive on $\overline{CI(X)}$. In fact, fix the unique positive real number a such that

$$L + \pi^*(\mathscr{O}(a))$$

is nef but not ample. Here $\pi : X \to \mathbb{P}_2$ is the projection. Notice that $a \geq 4$ by Equation 3. The nef cone of X is easily seen to be generated by $\pi^*\mathscr{O}(1)$ and $L + \pi^*\mathscr{O}(a)$, hence $\overline{CI}(X)$ is a priori spanned by the three classes $(L + \pi^*(\mathscr{O}(a))^2$, $\pi^*(\mathscr{O}(1))^2$ and $\pi^*(\mathscr{O}(1)) \cdot (L + \pi^*(\mathscr{O}(a))$. However

$$L^2 = c_1(E) \cdot L - c_2(E) = -2\pi^*\mathscr{O}(1) \cdot L - 2\pi^*\mathscr{O}(1)^2,$$

thus

$$(L + \pi^*(\mathscr{O}(a))^2 = (2a-2)\pi^*\mathscr{O}(1) \cdot L + (a^2-2)\pi^*\mathscr{O}(1)^2,$$

and as $(a^2-2)/(2a-2) < a$ we see that

$$\pi^*(\mathscr{O}(1)) \cdot (L + \pi^*(\mathscr{O}(a))$$

is a positive linear combination of $(L+\pi^*(\mathscr{O}(a))^2$ and $\pi^*(\mathscr{O}(1))^2$. Therefore the boundary of $\overline{CI}(X)$ is spanned by $(L+\pi^*(\mathscr{O}(a))^2$ and $\pi^*(\mathscr{O}(1))^2$. Now, using $a \geq 4$, we have

$$L \cdot (L + \pi^*(\mathscr{O}(a))^2 = 2 - 4a + a^2 \geq 2$$

and

$$L \cdot \pi^*(\mathscr{O}(1))^2 = 1,$$

hence L is strictly positive on $\overline{CI}(X)$.

On the other hand, L is effective since E has a section, but it is clear from the exact sequence 2 that L must be on the boundary of the pseudo-effective cone; otherwise $L - \pi^*(\mathscr{O}(\varepsilon))$ would be effective (actually big) for small positive ε. This is absurd because the tensor product of the exact sequence 2 by $\mathscr{O}(-\varepsilon)$ realizes the $\mathbb{Q}$-vector bundle $E \otimes \mathscr{O}(-\varepsilon)$ as an extension of two strictly negative sheaves (take symmetric products to avoid $\mathbb{Q}$ coefficients !). Therefore L cannot be strictly positive on $\overline{ME}(X)$.

The fact that $\overline{ME}(X)$ and $\overline{CI}(X)$ disagree in general is very unpleasant and creates a lot of technical troubles.

It is a classical fact that the dual cone of $\overline{NE}(X)$ is the nef cone; the main result of [BDPP04] determines the dual cone to the movable cone:

Theorem 1 *The dual cone to $\overline{ME}(X)$ is the pseudo-effective cone $\mathscr{PS}(X)$. Moreover $\overline{ME}(X)$ is the closed cone generated by the classes of movable curves.*

It is not clear whether the dual cone to $\overline{CI}(X)$ has a nice description. Nevertheless we make the following

Definition 2 A line bundle L is *generically nef* if $L \cdot h \geq 0$ for all ample classes h.

In the next section we extend this definition to vector bundles. Although generically nef line bundles are in general not pseudo-effective as seen in Example 1, this might still be true for the canonical bundle:

Problem 1 Let X be a projective manifold or a normal projective variety with (say) only canonical singularities. Suppose K_X is generically nef. Is K_X pseudo-effective?

In other words, suppose K_X not pseudo-effective, which is the same as to say that X is uniruled. Is there an ample class h such that $K_X \cdot h < 0$? This is open even in dimension 3; see [CP98] for some results.

3 Generically nef vector bundles

In this section we discuss generic nefness of general vector bundles and torsion free coherent sheaves.

Definition 3

(i) Let $h = (H_1, \dots, H_{n-1})$ be an ample class. A vector bundle $\mathscr{E}$ is said to be $h-$ generically nef (ample), if $\mathscr{E}|C$ is nef (ample) for a general curve $C = D_1 \cap \dots \cap D_{n-1}$ for general $D_i \in |m_i H_i|$ and $m_i \gg 0$. Such a curve is called MR-general, which is to say "general in the sense of Mehta-Ramanathan".

(ii) The vector bundle $\mathscr{E}$ is called generically nef (ample), if $\mathscr{E}$ is $(H_1, \dots, H_{n-1})-$ generically nef (ample) for all H_i.

(iii) $\mathscr{E}$ is almost nef [DPS01], if there is a countable union S of algebraic subvarieties such $\mathscr{E}|C$ is nef for all curves $C \not\subset S$.

Definition 4 Fix an ample class h on a projective variety X and let $\mathscr{E}$ be a vector bundle on X. Then we define the slope

$$\mu_h(\mathscr{E}) = c_1(\mathscr{E}) \cdot h$$

and obtain the usual notion of (semi-)stability w.r.t. h.

The importance of the notion of MR-generality comes from Mehta-Ranamathan's theorem [MR82]

Theorem 2 *Let X be a projective manifold (or a normal projective variety) and $\mathscr{E}$ a locally free sheaf on X. Then $\mathscr{E}$ is semi-stable w.r.t. h if and only $\mathscr{E}|C$ for C MR-general w.r.t. h.*

As a consequence one obtains

Corollary 1 *If $\mathscr{E}$ is semi-stable w.r.t. h and if $c_1(\mathscr{E}) \cdot h \geq 0$, then $\mathscr{E}$ is generically nef w.r.t. h; in case of stability $\mathscr{E}$ is even generically ample. If $c_1(\mathscr{E}) \cdot h = 0$, the converse also holds.*

The proof of Corollary 1 follows immediately from Miyaoka's characterization of semi-stable bundle on curves:

Proposition 1 *Let C be a smooth compact curve and $\mathscr{E}$ a vector bundle on C. Then $\mathscr{E}$ is semi-stable if and only if the $\mathbb{Q}-$bundle*

$$\mathscr{E} \otimes \frac{\det \mathscr{E}^*}{r}$$

is nef.

Remark 1 Everything we said in this section remains true for coherent sheaves $\mathscr{E}$ of positive rank r which are locally free in codimension 1, in particular for torsion free sheaves (the underlying variety being normal).
Recall that $\det \mathscr{E} := (\bigwedge^r)^{**}$.

For later use we note the following obvious

Lemma 3.1 *Let X be a normal projective variety, $\mathscr{E}$ a vector bundle or torsion free sheaf.*

(i) If $\mathscr{E}$ is $h-$generically ample for some h, then $H^0(X, (\mathscr{E}^*)^{\otimes m} \otimes L) = 0$ for all positive integers m and all numerically trivial line bundles L on X.

(ii) If $\mathscr{E}$ is $h-$generically nef for some h and $0 \neq s \in H^0(X, (\mathscr{E}^*)^{\otimes m} \otimes L) = 0$ for some positive integer m and some numerically trivial line bundle L, then s does not have zeroes in codimension 1.

Nef bundles satisfy many Chern class inequalities. Miyaoka [Mi87] has shown that at least one also holds for generically nef bundles, once the determinant is nef:

Theorem 3 *Let X be an $n-$dimensional normal projective variety which is smooth in codimension 2. Let $\mathscr{E}$ be a torsion free sheaf which is generically nef w.r.t. the polarization $(H_1, \ldots, H_{n-1})$. If $\det \mathscr{E}$ is $\mathbb{Q}-$Cartier and nef, then*

$$c_2(X) \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq 0.$$

This is not explicitly stated in [Mi87], but follows easily from ibid., Theorem 6.1. A Chern class inequality

$$c_1^2(\mathscr{E}) \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq c_2(\mathscr{E}) H_1 \cdot \ldots \cdot H_{n-2}$$

fails to be true: simply take a surface X with K_X ample and $c_1^2(X) < c_2(X)$ and let $\mathscr{E} = \Omega^1_X$ (which is a generically nef vector bundle, see the next section). Since generic nefness is a weak form of semi-stability, one might wonder whether there are Chern class inequalities of type

$$c_1(\mathscr{E})^2 \leq \frac{2r}{r-1} c_2(\mathscr{E}) \cdot h$$

(once $\det \mathscr{E}$ is nef). In case $\mathscr{E} = \Omega^1_X$, this is true, see again the next section.

If $\mathscr{E}$ is a generically nef vector bundle, then in general there will through any given point be many curves on which the bundle is not nef. For an *almost nef* bundle (see Definition 3), this will not be the case. Notice that in case $\mathscr{E}$ has rank 1, the notions "almost nefness" and "pseudo-effectivity" coincide. If a bundle is generically generated by its global sections, then $\mathscr{E}$ is almost nef. Conversely, one has

Theorem 4 *Let X be a projective manifold and $\mathscr{E}$ a vector bundle on X. If $\mathscr{E}$ is almost nef, then for any ample line bundle A, there are positive numbers m_0 and p_p0 such that*

$$H^0(X, S^p((S^m\mathscr{E}) \otimes A)) \neq 0$$

for $p \geq p_0$ and $m \geq m_0$.

For the proof we refer to [BDPP04]. The question remains whether the bundles $S^p((S^m\mathscr{E}) \otimes A)$ can even be generically generated. Here is a very special case, with a much stronger conclusion.

Theorem 5 *Let X be an almost nef bundle of rank at most* 3 *on a projective manifold X. If* $\det \mathscr{E} \equiv 0$, *then $\mathscr{E}$ is numerically flat.*

A vector bundle $\mathscr{E}$ is *numerically flat* if it admits a filtration by subbundles such that the graded pieces are unitary flat vector bundles, [DPS94]. For the proof we refer to [BDPP04],7.6. The idea of the proof is as follows. First notice that $\mathscr{E}$ is semi-stable for all polarizations by Corollary 1. This allows us to reduce to the case that $\dim X = 2$ and that $\mathscr{E}$ is stable for all polarizations. Now recall that if $\mathscr{E}$ is stable w.r.t. some polarization and if $c_1(\mathscr{E}) = c_2(\mathscr{E}) = 0$, then $\mathscr{E}$ is unitary flat, [Ko87]. Hence it suffices to show that $c_2(E) = 0$. This is done by direct calculations of intersection numbers on $\mathbb{P}(\mathscr{E})$. Of course there should be no reason why Theorem 5 should hold only in low dimensions, but in higher dimensions the calculations get tedious.

Corollary 2 *Let X be a K3 surface or a Calabi-Yau threefold. Then Ω_X^1 is not almost nef.*

A standard Hilbert scheme argument implies that there is a covering family (C_t) for curves (with C_t irreducible for general t), such that $\Omega_X^1|C_t$ is not nef for general t.

4 The cotangent bundle

In this section we discuss positivity properties of the cotangent bundles of non-uniruled varieties. At the beginning there is Miyaoka's

Theorem 6 *Let X be projective manifold or more generally, a normal projective variety. If X is not uniruled, then Ω_X^1 is generically nef.*

For a proof we refer to [Mi87] and to [SB92]. In [CP07] this was generalized in the following form

Theorem 7 *Let X be a projective manifold which is not uniruled. Let*

$$\Omega_X^1 \to Q \to 0$$

be a torsion free quotient. Then $\det Q$ *is pseudo-effective.*

Theorem 7 can be generalized to singular spaces as follows; the assumption on $\mathbb{Q}-$factoriality is needed in order to make sure that $\det Q$ is $\mathbb{Q}-$Cartier (so $\mathbb{Q}-$factoriality could be substituted by simply assuming that the bidual of $\bigwedge^r Q$ is $\mathbb{Q}-$Cartier).

Corollary 3 *Let X be a normal $\mathbb{Q}-$factorial variety. If X is not uniruled, then the conclusion of Theorem 7 still holds.*

Proof Choose a desingularization $\pi : \hat{X} \to X$ and let

$$\Omega_X^1 \to Q \to 0$$

be a torsion free quotient. We may assume that $\hat{Q} = \pi^*(Q)/\text{torsion}$ is locally free. Via the canonical morphism $\pi^*(\Omega^1_X) \to \Omega^1_{\hat{X}}$, we obtain a rational map $\Omega^1_{\hat{X}} \dashrightarrow \hat{Q}$. If E denotes exceptional divisor with irreducible components E_i, then this rational map yields a generically surjective map

$$\Omega^1_{\hat{X}} \to \hat{Q}(kE)$$

for some non-negative integer k. Appyling Theorem 7, $(\det \hat{Q})(mE)$ contains an pseudo-effective divisor for some m. Now

$$\det \hat{Q} = \pi^*(\det Q) + \sum a_i E_i,$$

with rational numbers a_i, hence $\det Q$ itself must be pseudo-effective (this can be easily seen in various ways). ■

Corollary 4 *Let $f : X \to Y$ be a fibration with X and Y normal $\mathbb{Q}-$Gorenstein. Suppose X not uniruled. Then the relative canonical bundle $K_{X/Y}$ (which is $\mathbb{Q}-$Cartier) is pseudo-effective.*

A much more general theorem has been proved by Berndtsson and Paun [BP07].

We consider a $\mathbb{Q}-$factorial normal projective variety which is not uniruled. The cotangent sheaf Ω^1_X being generically nef, we ask how far it is from being generically ample.

Proposition 2 *Let X be a $\mathbb{Q}-$factorial normal $n-$dimensional projective variety which is not uniruled. If Ω^1_X is not generically ample for some polarization h, then there exists a torsion free quotient*

$$\Omega^1_X \to Q \to 0$$

of rank $1 \leq p \leq n$ *such that* $\det Q \equiv 0$.
The case $p = n$ occurs exactly when $K_X \equiv 0$.

Proof Let C be MR-general w.r.t h. Let $\mathscr{S} \subset \Omega^1_X|C$ be the maximal ample subsheaf of the nef vector bundle $\Omega^1_X|C$, see [PS00],2.3, [PS04],p.636, [KST07], sect.6. Then the quotient Q_C is numerically flat and $\mathscr{S}_C$ is the maximal destabilizing subsheaf. By [MR82], $\mathscr{S}_C$ extends to a reflexive subsheaf $\mathscr{S} \subset \Omega^1_X$, which is $h-$maximal destabilizing. If $Q = \Omega^1_X/\mathscr{S}$ is the quotient, then obviously $Q|C = Q_C$. Now by Corollary 3, $\det Q$ is pseudo-effective. Since $c_1(Q) \cdot C = 0$, it follows that $\det Q \equiv 0$.
Finally assume $p = n$. Then $\Omega^1_X|C$ does not contain an ample subsheaf, hence $\Omega^1_X|C$

is numerically flat; in particular $K_X \cdot h = 0$. Since K_X is pseudo-effective, we conclude $K_X \equiv 0$. ■

So if X is not uniruled and Ω^1_X not generically ample, then $K_X \equiv 0$, or we have an exact sequence

$$0 \to \mathscr{S} \to \Omega^1_X \to Q \to 0$$

with Q torsion free of rank less than $n = \dim X$ and $\det Q \equiv 0$. Dually we obtain

$$0 \to \mathscr{F} \to T_X \to T_X/\mathscr{F} \to 0$$

with $\det \mathscr{F} \equiv 0$. Since $(T_X/\mathscr{F})|C$ is negative in the setting of the proof of the last proposition, the natural morphism

$$\bigwedge^2 \mathscr{F}/\text{torsion} \to T_X/\mathscr{F},$$

given by the Lie bracket, vanishes.Thus the subsheaf $\mathscr{F} \subset T_X$ is a singular foliation, which we call a *numerically trivial foliation.* So we may state

Corollary 5 *Let X be $\mathbb{Q}-$factorial normal $n-$dimensional projective variety. Suppose $K_X \not\equiv 0$. Then Ω^1_X is not generically ample if and only if X carries a numerically trivial foliation.*

If X is not uniruled, but Ω^1_X not generically ample, then we can take determinants in the setting of Proposition 2, and obtain

Corollary 6 *Let X be a $\mathbb{Q}-$factorial normal $n-$dimensional projective variety which is not uniruled. If Ω^1_X is not generically ample, then there exists a $\mathbb{Q}-$Cartier divisor $D \equiv 0$, a number q and a non-zero section in $H^0(X, (\bigwedge^q T_X)^{**} \otimes \mathscr{O}_X(D)^{**})$. In particular, if X is smooth, then there is a numerically flat line bundle L such that $H^0(X, \bigwedge^q T_X \otimes L) \neq 0$.*

Observe that the subsheaf $\mathscr{S} \subset \Omega^1_X$ constructed in the proof of Proposition 2 is $\alpha-$destabilizing for all $\alpha \in \overline{ME} \setminus \{0\}$. Therefore we obtain

Corollary 7 *Let X be a $\mathbb{Q}-$factorial normal projective variety which is not uniruled. If Ω^1_X is $\alpha-$semi-stable for some $\alpha \in \overline{ME} \setminus \{0\}$, then Ω^1_X is generically ample unless $K_X \equiv 0$.*

For various purposes which become clear immediately we need to consider certain singular varieties arising from minimal model theory. We will not try to prove things in the greatest possible generality, but restrict to the smallest class of singular varieties we need. We adopt the following notation.

Definition 5 A terminal $n-$fold X is a normal projective variety with at most terminal singularities which is also $\mathbb{Q}-$factorial. If additionally K_X is nef, X is called minimal.

Since the (co)tangent sheaf of a minimal variety X is always K_X-semi-stable [Ts88], [En88], we obtain

Corollary 8 *Let X be a minimal projective variety such that K_X is big. Then Ω^1_X is generically ample.*

Actually [En88] gives more: Ω^1_X is generically ample for all smooth X admitting a *holomorphic* map to a minimal variety. In general however a manifold of general type will not admit a holomorphic map to a minimal model. Nevertheless we can prove

Theorem 8 *Let X be a projective manifold or terminal variety of general type. Then Ω^1_X is generically ample.*

Proof If Ω^1_X would not be generically ample, then we obtain a reflexive subsheaf $\mathscr{S} \subset T_X$ such that $\det \mathscr{S} \equiv 0$. By [BCHM09] there exists a sequence of contractions and flips

$$f : X \dashrightarrow X' \tag{4}$$

such that X' is minimal. Since f consists only of contractions and flips, we obtain an induced subsheaf $\mathscr{S}' \subset T_{X'}$ such that $\det \mathscr{S}' \equiv 0$. Here it is important that no blow-up ("extraction") is involved in f. From Corollary 4 we obtain a contradiction. ■

Now Lemma 3.1 gives

Corollary 9 *Let X be a projective manifold of general type. Then*

$$H^0(X, (T_X)^{\otimes m}) = 0$$

for all positive integers m.

More generally, $H^0(X, (T_X)^{\otimes m} \otimes L^*) = 0$ if L is a pseudo-effective line bundle.

We now turn to the case that X is not of general type. We start in dimension 2.

Theorem 9 *Let X be a smooth projective surface with $\kappa(X) \geq 0$. Suppose that $H^0(X, T_X \otimes L) \neq 0$, where L is a numerically trivial line bundle. Then the non-trivial sections of $T_X \otimes L$ do not have any zeroes, in particular $c_2(X) = 0$ and one of the following holds up to finite étale cover.*

(i) X is a torus

(ii) $\kappa(X) = 1$ and $X = B \times E$ with $g(B) \geq 2$ and E elliptic.

In particular, X is minimal.
Conversely, if X is (up to finite étale cover) a torus or of the form $X = B \times E$ with $g(B) \geq 2$ and E elliptic, then $H^0(X, T_X \otimes L) \neq 0$ for some numerically trivial line bundle L.

Proof Fix a non vanishing section $s \in H^0(X, T_X \otimes L)$. Observe that due to Theorem 6 the section s cannot have zeroes in codimension 1. Thus $Z = \{s = 0\}$ is at most finite. Dualizing, we obtain an epimorphism

$$0 \to \mathscr{G} \to \Omega^1_X \to \mathscr{I}_Z \otimes L^* \to 0 \tag{5}$$

with a line bundle $\mathscr{G} \equiv K_X$. From Bogomolov's theorem [Bo79], we have $\kappa(\mathscr{G}) \leq 1$, hence $\kappa(X) \leq 1$. Next observe that if L is torsion, i.e. $L^{\otimes m} = \mathscr{O}_X$ for some m, then after finite étale cover, we may assume $L = \mathscr{O}_X$; hence X has a vector field s. This vector field cannot have a zero, otherwise X would be uniruled (see e.g. [Li78]. Then a theorem of Lieberman [Li78] applies and X is (up to finite étale cover) a torus or a poduct $E \times C$ with E elliptic and $g(C) \geq 2$.
So we may assume that L is not torsion; consequently $q(X) \geq 1$.

We first suppose that X is minimal. If $\kappa(X) = 0$, then clearly X is a torus up to finite étale cover. So let $\kappa(X) = 1$.
We start by ruling out $g(B) = 0$. In fact, if $B = \mathbb{P}_1$, then the semi-negativity of $R^1 f_*(\mathscr{O}_X)$ together with $q(X) \geq 1$ shows via the Leray spectral sequence that $q(X) = 1$. Let $g : X \to C$ be the Albanese map to an elliptic curve C. Then (possibly after finite étale cover of X), $L = g^*(L')$ with a numerically line bundle L' on C, which is not torsion. Since the general fiber F of f has an étale map to C, it follows that $L|F$ is not torsion. But then $H^0(F, T_X \otimes L|F) = 0$, a contradiction to the existence of the section s. Hence $g(B) \geq 1$.
Consider the natural map

$$\lambda : T_X \otimes L \to f^*(T_B) \otimes L.$$

Since L is not torsion, $\lambda(s) = 0$ (this property of L is of course only needed when $g(B) = 1$). Therefore $s = \mu(s')$, where

$$\mu : T_{X/B} \otimes L \to T_X \otimes L \tag{6}$$

is again the natural map. Recall that by definition $T_{X/B} = (\Omega^1_{X/B})^*$, which is a reflexive sheaf of rank 1, hence a line bundle. Now recall that s has zeroes at most in a finite set, so does s'. Consequently

$$T_{X/B} \otimes L = \mathscr{O}_X.$$

On the other hand

$$T_{X/B} = -K_X \otimes f^*(K_B) \otimes \mathscr{O}_X(\sum(m_i - 1)F_i),$$

where the F_i are the multiple fibers. Putting things together, we obtain

$$K_{X/B} = L \otimes \mathscr{O}_X(\sum(m_i - 1)F_i).$$

Since $K_{X/B}$ is pseudo-effective (see Corollary 4) we cannot have any multiple fibers, hence $K_{X/B} \equiv 0$. It follows that f must be locally trivial (see e.g. [BHPV04], III.18, and also that $g(B) \geq 2$. Then X becomes actually a product after finite étale cover.

We finally rule out the case that X is not minimal. So suppose X not minimal and let $\sigma : X \to X'$ be the blow-down of a $(-1)-$curve to a point p. Then we can write $L = \sigma^*(L')$ with some numerically trivial line bundle L' on X' and the section s induces a section $s' \in H^0(X', T_{X'} \otimes L')$. Notice that $\sigma_*(T_X) = \mathscr{I}_p \otimes T_{X'}$, hence $s'(p) = 0$. Therefore we are reduced to the case where X' is minimal and have to derive a contradiction. Now s' has no zeroes by what we have proved before. This gives the contradiction we are looking for.

■

Corollary 10 *Let X be a smooth projective surface with $\kappa(X) \geq 0$. The cotangent bundle Ω^1_X is not generically ample if and only if X is a minimal surface with $\kappa = 0$ (i.e., a torus, hyperelliptic, K3 or Enriques) or X is a minimal surface with $\kappa = 1$ and a locally trivial Iitaka fibration; in particular $c_2(X) = 0$ and X is a product after finite étale cover of the base.*

We now turn to the case of threefolds X - subject to the condition that Ω^1_X is not generically ample. By Theorem 8 X is not of general type; thus we need only to consider the cases $\kappa(X) = 0, 1, 2$. If $K_X \equiv 0$, then of course Ω^1_X cannot be generically ample. However it is still interesting to study numerically trivial foliations in this case.

Theorem 10 *Let X be a minimal projective threefold with $\kappa(X) = 0$. Let*

$$0 \to \mathscr{F} \to T_X \to Q \to 0$$

be a numerically trivial foliation, i.e., $\det \mathscr{F} \equiv 0$. *Then there exists a finite cover $X' \to X$, étale in codimension 2, such that X' is a torus or a product $A \times S$ with A an elliptic curve and S a K3-surface.*

Proof By abundance, $mK_X = \mathscr{O}_X$ for some positive integer m, since X is minimal. By passing to a cover which is étale in codimension 2 and applying Proposition 4 we may assume $K_X = \mathscr{O}_X$. We claim that

$$q(X) > 0,$$

possibly after finite cover étale in codimension 2.
If $\det Q$ is not torsion, then $q(X) > 0$ right away. If the $\mathbb{Q}-$Cartier divisor $\det Q$ is torsion, then, after a finite cover étale in codimension 2, we obtain a holomorphic form of degree 1 or 2. To be more precise, choose m such that $m \det Q$ is Cartier. Then choose m' such that $m'm \det Q = \mathscr{O}_X$. Then there exists a finite cover $h : \tilde{X} \to X$, étale in codimension 2, such that the pull-back $h^*(\det Q)$ is trivial. In the sheaf-theoretic language, $h^*(\det Q)^{**} = \mathscr{O}_X$. Now pull back the above exact sequence and conclude the existence of a holomorphic 1-form in case Q has rank 1 and a holomorphic 2-form in case Q has rank 2.
Since $\chi(X, \mathscr{O}_X) \leq 0$ by [Mi87], we conclude $q(X) \neq 0$.
Hence we have a non-trivial Albanese map

$$\alpha : X \to \mathrm{Alb}(X) =: A.$$

By [Ka85], sect. 8, α is surjective with connected fibers. Moreover, possibly after a finite étale base change, X is birational to $F \times A$ where F is a general fiber of α.
Suppose first that $\dim \alpha(X) = 1$, i.e., $q(X) = 1$. Then F must be a K3 surface (after another finite étale cover). Now X is birational to $F \times A$ via a sequence of flops [Ko89] and therefore X itself is smooth ([Ko89], 4.11). Hence by the Beauville-Bogomolov decomposition theorem, X itself is a product (up to finite étale cover).
The case $\dim \alpha(X) = 2$ cannot occur, since then X is birational to a product of an elliptic curve and a torus, so that $q(X) = 3$.
If finally $\dim \alpha(X) = 3$, then X is a torus.

■

In the situation of Theorem 10, it is also easy to see that the foliation $\mathscr{F}$ is induced by a foliation $\mathscr{F}'$ on X' in a natural way. Moreover $\mathscr{F}'$ is trivial sheaf in case X' is a torus and it is given by the relative tangent sheaf of a projection in case X' is a product.

From a variety X whose cotangent bundle is not generically ample, one can construct new examples by the following devices.

Proposition 3 *Let $f: X \dashrightarrow X'$ be a birational map of normal $\mathbb{Q}$–factorial varieties which is an isomorphism in codimension 1. Then Ω^1_X is generically ample if and only if $\Omega^1_{X'}$ is generically ample.*

Proof Suppose that Ω^1_X is generically ample and $\Omega^1_{X'}$ is not. Since X' is not uniruled, $\Omega^1_{X'}$ is generically nef and by Proposition 2 there is an exact sequence

$$0 \to \mathscr{S}' \to \Omega^1_{X'} \to Q' \to 0$$

such that $\det Q' \equiv 0$. Since f is an isomorphism in codimension 1, this sequence clearly induces a sequence

$$0 \to \mathscr{S} \to \Omega^1_X \to Q \to 0$$

such that $\det Q \equiv 0$. Since the problem is symmetric in X and X', this ends the proof.

Proposition 4 *Let $f: X \to X'$ be a finite surjective map between normal projective $\mathbb{Q}$–factorial varieties. Assume that f is étale in codimension 1. Then Ω^1_X is generically ample if and only if $\Omega^1_{X'}$ is generically ample.*

Proof If X' is not uniruled and $\Omega^1_{X'}$ is not generically ample, we lift a sequence

$$0 \to \mathscr{S}' \to \Omega^1_{X'} \to Q' \to 0$$

with $\det Q' \equiv 0$ and conclude that Ω^1_X is not generically ample.
Suppose now that Ω^1_X is not generically ample (and X not uniruled). Then we obtain a sequence

$$0 \to \mathscr{S} \to \Omega^1_X \to Q \to 0$$

with $\det Q \equiv 0$. If $\Omega^1_{X'}$ would be generically ample, then for a general complete intersection curve $C' \subset X'$ the bundle $\Omega^1_{X'}|C'$ is ample. Hence $\Omega^1_X|f^{-1}(C') = f^*(\Omega^1_{X'}|C')$ is ample, a contradiction.

In view of the minimal model program we are reduced to consider birational morphisms which are "divisorial" in the sense that their exceptional locus contains a divisor. In one direction, the situation is neat:

Proposition 5 *Let $\pi : \hat{X} \to X$ be a birational map of normal $\mathbb{Q}-$factorial varieties. If Ω^1_X is generically ample, so does $\Omega^1_{\hat{X}}$.*

Proof If Ω^1_X would not be generically ample, we obtain an epimorphism

$$\Omega^1_{\hat{X}} \to \hat{Q} \to 0 \tag{7}$$

with a torsion free sheaf $\hat{Q}$ such that $\det \hat{Q} \equiv 0$. Applying π_* yields a map

$$\mu : \pi_*(\Omega^1_{\hat{X}}) \to \pi_*(\hat{Q}),$$

which is an epimorphism in codimension 1. Since $\Omega^1_X = \pi_*(\Omega^1_{\hat{X}})$ outside a set of codimension at least 2, there exists a torsion free sheaf Q coinciding with $\pi_*(\hat{Q})$ outside a set of codimension at least 2 together with an epimorphism

$$\Omega^1_X \to Q \to 0.$$

Since $\det Q = \det \pi_*(\hat{Q}) \equiv 0$, the sheaf Ω^1_X cannot be generically ample. ∎

From a birational point of view, it remains to investigate the following situation. Let $\pi : \hat{X} \to X$ be a divisorial contraction of non-uniruled terminal varieties and suppose that Ω^1_X is not generically ample. Under which conditions is $\Omega^1_{\hat{X}}$ generically ample? Generic ampleness is not for free as shown in the following easy

Example 2 Let E be an elliptic curve and S an abelian surface, say. Let $\hat{S} \to S$ be the blow-up at $p \in S$ and set $\hat{X} = E \times \hat{S}$. Then $\hat{X}$ is the blow-up of $X = E \times S$ along the curve $E \times \{p\}$. Since $\Omega^1_{\hat{X}} = \mathscr{O}_{\hat{X}} \oplus p_2^*(\Omega^1_{\hat{S}})$, it cannot be generically ample.

We now study a special case of a point modification: the blow-up of a smooth point.

Proposition 6 *Let X be a non-uniruled $n-$dimensional projective manifold, $\pi : \hat{X} \to X$ the blow-up at the point p. If $\Omega^1_{\hat{X}}$ is not generically ample, then there exists a number $q < n$, a numerically trivial line bundle L and a non-zero section $v \in H^0(X, \bigwedge^q T_X \otimes L)$ vanishing at p: $v(p) = 0$.*

Proof By Corollary 6, we get a non-zero section $\hat{v} \in H^0(\hat{X}, \bigwedge^q T_{\hat{X}} \otimes \hat{L})$ for some numerically trivial line bundle $\hat{L}$. Notice that $\hat{L} = \pi^*(L)$ for some numerically trivial line bundle L on X. Since

$$\pi_*(\bigwedge^q T_{\hat{X}}) \subset \bigwedge^q T_X,$$

we obtain a section $v \in H^0(X, \bigwedge^q T_X \otimes L)$. It remains to show that $v(p) = 0$. This follows easily by taking π_* of the exact sequence

$$0 \to \bigwedge^q T_{\hat{X}} \to \pi^*(\bigwedge^q T_X) \to \bigwedge^q (T_E(-1)) \to 0.$$

Here E is the exceptional divisor of π. In fact, taking π_* gives

$$\pi_*(\bigwedge^q T_{\hat{X}}) = \mathscr{I}_p \otimes T_X.$$

■

From the Beauville-Bogomolov decomposition of projective manifolds X with $c_1(X) = 0$, we deduce immediately

Corollary 11 *Let $\hat{X}$ be the blow-up at a point p in a projective manifold X with $c_1(X) = 0$. Then $\Omega^1_{\hat{X}}$ is generically ample.*

Due to Conjecture 2 below this corollary should generalize to all non-uniruled manifolds X. Based on the results presented here, one might formulate the following

Conjecture 1 Let X be a non-uniruled terminal $n-$fold. Suppose that Ω^1_X is not generically ample and $K_X \not\equiv 0$. Then, up to taking finite covers $X' \to X$, étale in codimension 1, and birational maps $X' \dashrightarrow X''$,which are biholomorphic in codimension 1, X admits a locally trivial fibration, given by a numerically trivial foliation, which is trivialized after another finite cover, étale in codimension 1.

More generally, any numerical trivial foliation should yield the same conclusion.

This might require a minimal model program, a study of minimal models in higher dimensions and possibly also a study of the divisorial Mori contractions. In a subsequent paper we plan to study minimal threefolds X with $\kappa(X) = 1, 2$ whose cotangent bundles is not generically ample and then study the transition from a general threefold to a minimal model.

We saw that a non-uniruled manifold X whose cotangent bundle is not generically ample, admits a section v in some bundle $\bigwedge^q T_X \otimes L$, where L is numerically trivial. It is very plausible that v cannot have zeroes:

Conjecture 2 Let X be a projective manifold. Let $v \in H^0(X, \bigwedge^q T_X \otimes L)$ be a non-trivial section for some numerically trivial line bundle L. If v has a zero, then X is uniruled.

If $q = \dim X$, then the assertion is clear by [MM86]. If $q = 1$ and L is trivial, then the conjecture is a classical result, see e.g. [Li78]. We will come back to Conjecture 2 at the end of the next section.

A well-known, already mentioned theorem of Lieberman [Li78] says that if a vector field v has no zeroes, then some finite étale cover $\tilde{X}$ of X has the form $\tilde{X} = T \times Y$ with T a torus, and v comes from the torus. One might hope that this is simply a special case of a more general situation:

Conjecture 3 Let X be a projective manifold, L a numerically trivial line bundle and

$$v \in H^0(X, \bigwedge{}^q T_X \otimes L)$$

a non-zero section, where $q < \dim X$. Then X admits a finite étale cover $\tilde{X} \to X$ such that $\tilde{X} \simeq Y \times Z$ where Y is a projective manifold with trivial canonical bundle and v is induced by a section $v' \in H^0(Y, \bigwedge^q T_Y \otimes L')$.

5 The tangent bundle

In this section we discuss the dual case: varieties whose tangent bundles are generically nef or generically ample. If X is a projective manifold with generically nef tangent bundle T_X, then in particular $-K_X$ is generically nef. If K_X is pseudo-effective, then $K_X \equiv 0$ and the Bogomolov-Beauville decomposition applies. Therefore we will always assume that K_X is not pseudo-effective, hence X is uniruled. If moreover T_X is generically ample w.r.t some polarization, then X is rationally connected. Actually much more holds:

Theorem 11 *Let X be a projective manifold. Then X is rationally connected if and only if there exists an irreducible curve $C \subset X$ such that $T_X|C$ is ample.*

For the existence of C if X is rationally connected see [Ko96], IV.3.7; for the other direction we refer to [BM01], [KST07] and [Pe06].

The first class of varieties to consider are certainly Fano manifolds. One main problem here is the following standard

Conjecture 4 The tangent bundle of a Fano manifold X is stable w.r.t. $-K_X$.

This conjecture is known to be true in many cases, but open in general. Here is what is proved so far if $b_2(X) = 1$.

Theorem 12 *Let X be a Fano manifold of dimension n with $b_2(X) = 1$. Under one of the following conditions the tangent bundle is stable.*

- $n \leq 5$ *(and semi-stable if $n \leq 6$);*
- *X has index $> \frac{n+1}{2}$;*
- *X is homogeneous;*
- *X (of dimension at least 3) arises from a weighted projective space by performing the following operations: first take a smooth weighted complete intersection, then take a cyclic cover, take again a smooth complete intersections; finally stop ad libitum.*

For the first two assertions see [Hw01]; the third is classical; the last is in [PW95].

By Corollary 1, generic nefness, even generic ampleness, is a consequence of stability in case of Fano manifolds. Therefore generic nefness/ampleness is a weak version of stability. So it is natural to ask for generic nefness/ampleness of the tangent bundle of Fano manifolds:

Theorem 13 *Let X be a projective manifold with $-K_X$ big and nef. Then T_X is generically ample (with respect to any polarization).*

If $b_2(X) \geq 2$, then of course the tangent bundle might not be (semi-)stable w.r.t. $-K_X$; consider e.g. the product of projective spaces (of different dimensions).
The proof of Theorem 13 is given in [Pe08]. The key to the proof is the following observation. Fix a polarization $h = (H_1, \dots, H_{n-1})$, where $n = \dim X$. Suppose that T_X is not $h-$generically ample. Since $-K_X \cdot h > 0$, we may apply Corollary 1 and therefore T_X is not $h-$semi-stable More precisely, let C be MR-general w.r.t. h, then $T_X|C$ is not ample. Now we consider the Harder-Narasimhan filtration and find a piece $\mathscr{E}_C$ which is maximally ample, i.e., $\mathscr{E}_C$ contains all ample subsheaves of $T_X|C$. By the theory of Mehta-Ramanathan [MR82], the sheaf $\mathscr{E}_C$ extends to a saturated subsheaf $\mathscr{E} \subset T_X$. The maximal ampleness easily leads to the inequality

$$(K_X + \det \mathscr{E}) \cdot h > 0.$$

On the other hand, $K_X + \det \mathscr{E}$ is a subsheaf of Ω_X^{n-k}. If X is Fano with $b_2(X) = 1$, then we conclude that $K_X + \det \mathscr{E}$ must be ample, which is clearly impossible, e.g. by arguing via rational connectedness. In general we show, based on [BCHM09], that the movable $\overline{ME}(X)$ contains an extremal ray R such that

$$(K_X + \det \mathscr{E}) \cdot R > 0.$$

This eventually leads, possible after passing to a suitable birational model, to a Fano fibration $f : X \to Y$ such that $K_X + \det \mathscr{E}$ is relatively ample. This yields a contradiction in the same spirit as in the Fano case above.

With substantially more efforts, one can extend the last theorem in the following way.

Theorem 14 *Let X be a projective manifold with $-K_X$ semi-ample. Then T_X is generically nef.*

From Theorem 3 we therefore deduce

Corollary 12 *Let X be an $n-$dimensional projective manifold with $-K_X$ semi-ample. Then*

$$c_2(X) \cdot H_1 \ldots \cdot H_{n-2} \geq 0$$

for all ample line bundles H_j on X.

Of course Theorem 14 should hold for all manifolds X with $-K_X$ nef, and therefore also the inequality from the last corollary should be true in this case.

For biregular problems generic nefness is not enough; in fact, if $x \in X$ is a fixed point and T_X is generically nef, then it is not at all clear whether there is just one curve C passing through p such that $T_X|C$ is nef. Therefore we make the following

Definition 6 Let X be a projective manifold and E a vector bundle on X. We say that E is sufficiently nef if for any $x \in X$ there is a family (C_t) of curves through x covering X such that $E|C_t$ is nef for general t.

We want to apply this to the study of manifolds X with $-K_X$ nef:

Conjecture 5 Let X be a projective manifold with $-K_X$ nef. Then the Albanese map is a surjective submersion.

Surjectivity is known by Qi Zhang [Zh05] using char $p-$methods, smoothness of the Albanese map only in dimension at most 3 by [PS98]. The connection to the previous definition is given by

Proposition 7 *Suppose that T_X is sufficiently nef. Then the Albanese map is a surjective submersion.*

Proof (cp. [Pe08]). If the Albanese map would not be a surjective submersion, then there exists a holomorphic $1-$form ω on X vanishing at some point x. Now choose a general curve C from a covering family through x such that $T_X|C$ is nef. Then $\omega|C$ is a *non-zero* section of $T_X^*|C$ having a zero. This contradicts the nefness of $T_X|C$.

Of course, a part of the last proposition works more generally:

Proposition 8 *If E is sufficiently nef and if E^* has a section s, then s does not have any zeroes.*

We collect here some evidence that manifolds with nef anticanonical bundles have sufficiently nef tangent bundles and refer to [Pe08] for proofs.

Theorem 15 *Let X be a projective manifold.*

- *If E is a generically ample vector bundle, then E is sufficiently ample.*
- *If $-K_X$ is big and nef, then T_X is sufficiently ample.*
- *If $-K_X$ is hermitian semi-positive, then T_X is sufficiently nef.*

Notice however that a generically nef bundle need not be sufficiently nef; see [Pe08] for an example (a rank $2-$bundle on $\mathbb{P}_3$).

We finally come back to Conjecture 2. So suppose that X is a projective manifold, let L be numerically trivial and consider a non-zero section

$$v \in H^0(X, \bigwedge^q T_X \otimes L),$$

where $1 \leq q \leq \dim X - 1$. Applying Proposition 8, Conjecture 2 is therefore a consequence of

Conjecture 6 Let X be a non-uniruled projective manifold. Then Ω_X^1 is sufficiently nef.

Conjecture 6 is true in dimension 2 (using [Pe08], sect.7 and Corollary 10), and also if $K_X \equiv 0$ and if Ω^1_X is generically ample, again by [Pe08], sect.7.

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Dolbeault cohomology of nilmanifolds with left-invariant complex structure

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Abstract We discuss the known evidence for the conjecture that the Dolbeault cohomology of nilmanifolds with left-invariant complex structure can be computed as Lie-algebra cohomology and also mention some applications.

Keywords nilmanifold, Dolbeault cohomology, left-invariant complex structure.
Mathematics Subject Classification (2010) *Primary* 53C56. *Secondary* 22E25, 32G05, 17B56.

1 Introduction

Dolbeault cohomology is one of the most fundamental holomorphic invariants of a compact complex manifold X but in general it is quite hard to compute. If X is a compact Kähler manifold, then this amounts to describing the decomposition of the de Rham cohomology

$$H^k_{dR}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) = \bigoplus_{p+q=k} H^q(X,\Omega^p_X)$$

but in general there is only a spectral sequence connecting these invariants.

One case where at least de Rham cohomology is easily computable is the case of nilmanifolds, that is, compact quotients of real nilpotent Lie groups. If $M = \Gamma\backslash G$

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is a nilmanifold and $\mathfrak{g}$ is the associated nilpotent Lie algebra, Nomizu proved that we have a natural isomorphism

$$H^*(\mathfrak{g},\mathbb{R}) \cong H^*_{\mathrm{dR}}(M,\mathbb{R})$$

where the left hand side is the Lie-algebra cohomology of $\mathfrak{g}$. In other words, computing the cohomology of M has become a matter of linear algebra.

There is a natural way to endow an even-dimensional nilmanifold with an almost complex structure: choose any endomorphism $J:\mathfrak{g}\to\mathfrak{g}$ with $J^2=-\mathrm{id}$ and extend it to an endomorphism of TG, also denoted by J, by left-multiplication. Then J is invariant under the action of Γ and descends to an almost complex structure on M. If J satisfies the integrability condition

$$[x,y]-[Jx,Jy]+J[Jx,y]+J[x,Jy]=0 \text{ for all } x,y\in\mathfrak{g} \tag{1}$$

then, by Newlander–Nirenberg [28, p.145], it makes $M_J=(M,J)$ into a complex manifold.

In this survey we want to discuss the conjecture

The Dolbeault cohomology of a nilmanifold with left-invariant complex structure M_J can be computed using only left-invariant forms.

This was stated as a question in [14, 11] but we decided to call it Conjecture in the hope that it should motivate other people to come up with a proof or a counterexample. A more precise formulation in terms of Lie-algebra cohomology is given in Section 3.1.

Before concentrating on this topic, we would like to indicate why nilmanifolds have attracted much interest over the last years. Their main feature is that the construction and study of left-invariant geometric structures on them usually boils down to finite dimensional linear algebra. On the other hand, the structure is sufficiently flexible to allow the construction of many exotic examples. We only want to mention the three most prominent in complex geometry:

- If G is abelian then M_J is a complex torus.
- The Iwasawa manifold $X=\Gamma\backslash G$ is obtained as the quotient of the complex Lie group

$$G=\left\{\begin{pmatrix}1 & z_1 & z_3\\ 0 & 1 & z_2\\ 0 & 0 & 1\end{pmatrix}\right\}\subset \mathrm{Gl}(3,\mathbb{C})$$

by the lattice $\Gamma = G \cap \mathrm{Gl}(3, \mathbb{Z}[\mathrm{i}])$ and as such is complex parallelisable. Nakamura studied its small deformations and thus showed that a small deformation of a complex parallelisable manifold need not be complex parallelisable [35].

Observe that X cannot be Kähler since $dz_3 - z_2 dz_1$ is a holomorphic 1-form that is not closed.

- Kodaira surfaces, also known as Kodaira-Thurston manifolds, had appeared in Kodaira's classification of compact complex surfaces as non-trivial principal bundle of elliptic curves over an elliptic curve [29] and were later considered independently by Thurston as the first example of a manifold that admits both a symplectic and a complex structure but no Kähler structure. In our context it can be described as follows: let

$$G = \left\{ \begin{pmatrix} 1 & \bar{z}_1 & z_2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2 \in \mathbb{C} \right\} \subset \mathrm{Gl}(3, \mathbb{C})$$

and $\Gamma = G \cap \mathrm{Gl}(3, \mathbb{Z}[\mathrm{i}])$. Then $G \cong \mathbb{C}^2$ with coordinates z_1, z_2 and the action of Γ on the left is holomorphic; the quotient is a compact complex manifold. If we set $\alpha = dz_1 \wedge (d\bar{z}_2 - z_1 d\bar{z}_1)$ then $\alpha + \bar{\alpha}$ is a left-invariant symplectic form on G and thus descends to the quotient.

In fact, the first example is the only nilmanifold that can admit a Kähler structure [5], so none of the familiar techniques available for Kähler manifolds will be useful in our case.

Some more applications in complex geometry will be given in Section 4. Nilmanifolds also play a role in hermitian geometry [1, 4, 32], riemannian geometry [22, 6], ergodic theory [25], arithmetic combinatorics [21], and theoretical physics [19, 20].

In order to discuss the above conjecture on Dolbeault cohomology, we start by sketching the proof of Nomizu's theorem because some of the ideas carry over to the holomorphic setting. Then we recall the necessary details on Dolbeault cohomology to give a precise statement of the conjecture. It turns out that we are in a good position to prove the conjecture whenever we can inductively decompose the nilmanifold with left-invariant complex structure into simpler pieces. This is due to Console and Fino [11], generalising previous results of Cordero, Fernández, Gray and Ugarte [14].

Section 3.4 contains the only new result in this article. We prove that the conjecture always holds true if we pass to a suitable quotient of the nilmanifold with left-invariant complex structure and also discuss some possible approaches to attack the general case.

1.1 Notations

Throughout the paper G will be a simply connected nilpotent real Lie-group with Lie-algebra $\mathfrak{g}$. Every nilpotent Lie group can be realised as a subgroup of the group of upper triangular matrices with 1's on the diagonal.

We will always assume that G contains a lattice Γ, thus giving rise to a (compact) nilmanifold $M = \Gamma\backslash G$. Elements in $\mathfrak{g}$ will usually be interpreted as left-invariant vector fields on G or on M. We restrict our attention to those complex structures on M that are induced by an integrable left-invariant complex structure on G and are thus uniquely determined by an (integrable) complex structure $J : \mathfrak{g} \to \mathfrak{g}$. The resulting complex manifold is denoted M_J. Note that even on a real torus of even dimension at least 6 there are many complex structures that do not arise in this way [7].

The group G is determined up to isomorphism by the fundamental group of M [44, Corollary 2.8, p.45] and by abuse of notation we sometimes call $\mathfrak{g}$ the Lie-algebra of M.

2 Real nilmanifolds and Nomizu's result on de Rham cohomology

The aim of this section is to prove Nomizu's theorem.

Theorem 1 (Nomizu [36]) *Let M be a compact nilmanifold. Then the inclusion of left-invariant differential forms in the de Rham complex*

$$\Lambda^\bullet \mathfrak{g}^* \hookrightarrow \mathscr{A}^\bullet(M)$$

induces an isomorphism between the Lie-algebra cohomology of $\mathfrak{g}$ *and the de Rham cohomology of M,*

$$H^*(\mathfrak{g},\mathbb{R}) \cong H^*_{\mathrm{dR}}(M,\mathbb{R}).$$

Since some of the main results on Dolbeault cohomology discussed in the next section rely on similar ideas, we will examine the proof in some detail: at its heart lies an inductive argument.

Let $M = \Gamma\backslash G$ be a real nilmanifold with associated Lie algebra $\mathfrak{g}$ and let $\mathscr{Z}G$ be the centre of G. By [16, p. 208], $\mathscr{Z}\Gamma = \Gamma \cap \mathscr{Z}G$ is again a lattice and the projection $G \to G/\mathscr{Z}G$ descends to a fibration $M \to M'$. The fibres are real tori $T = \mathscr{Z}G/\mathscr{Z}\Gamma$. Since elements in $\mathscr{Z}G$ commute with elements in Γ, their action descends to the quotient and $M \to M'$ is a principal T-bundle.

To iterate this process we recall the following definition.

Definition 1 For a Lie-algebra $\mathfrak{g}$ we call

$$\mathscr{Z}^0\mathfrak{g} := 0, \qquad \mathscr{Z}^{i+1}\mathfrak{g} := \{x \in \mathfrak{g} \mid [x,\mathfrak{g}] \subset \mathscr{Z}^i\mathfrak{g}\}$$

the ascending central series and

$$\mathscr{C}^0\mathfrak{g} := \mathfrak{g}, \qquad \mathscr{C}^{i+1}\mathfrak{g} := [\mathscr{C}^i\mathfrak{g},\mathfrak{g}]$$

the descending central series of $\mathfrak{g}$.

The Lie-algebra is called nilpotent if there is a $\nu \in \mathbb{N}$ such that $\mathscr{Z}^\nu\mathfrak{g} = \mathfrak{g}$, or equivalently $\mathscr{C}^\nu\mathfrak{g} = 0$. The minimal such $\nu = \nu(\mathfrak{g})$ is called the index of nilpotency or step-length of $\mathfrak{g}$.

The same definition can be made on the level of the Lie-group G and the resulting sub-algebras and subgroups correspond to each other under the exponential map.

Proceeding inductively, we can use the first filtration on $\mathfrak{g}$ to decompose M geometrically; the second one induces a similar decomposition since $\mathscr{C}^i\mathfrak{g} \subset \mathscr{Z}^{\nu-i}\mathfrak{g}$. More precisely, if we denote by T_i the torus obtained as a quotient of $\mathscr{Z}^iG/\mathscr{Z}^{i+1}G$ by $\mathscr{Z}^i\Gamma/\mathscr{Z}^{i+1}\Gamma$, then there is a tower

$$\begin{tikzcd}
T_1 \arrow[r, hook] & M_1 \arrow[d, "\pi_1"] \\
T_2 \arrow[r, hook] & M_2 \arrow[d, "\pi_2"] \\
 & \vdots \arrow[d] \\
T_{\nu-1} \arrow[r, hook] & M_{\nu-1} \arrow[d, "\pi_{\nu-1}"] \\
 & M_\nu
\end{tikzcd} \tag{2}$$

and each $\pi_i : M_i \to M_{i+1}$ is a T_i-principal bundle.

This geometric description is crucial in the proof of Nomizu's Theorem. The underlying idea is quite simple: we perform induction over the index of nilpotency ν. If $\nu = 1$, i.e., $\mathfrak{g}$ is abelian, then M is a torus and the result is well known. For the induction step, we consider M as a principal torus bundle over a nilmanifold M' with lower nilpotency index. Then we have to combine our knowledge of the cohomology of the fibre and of the base to describe the cohomology of the total space M. This is achieved by means of two spectral sequences, the Leray-Serre spectral sequence and the Serre-Hochschild spectral sequence.

Let us work this out a bit more in detail starting on the geometric side: let $\mathscr{A}^k(M)$ be the the space of smooth differential k-forms on M and consider the de Rham complex

$$0 \to \mathscr{A}^0(M) \xrightarrow{d} \mathscr{A}^1(M) \xrightarrow{d} \dots \xrightarrow{d} \mathscr{A}^n(M) \to 0.$$

The principal bundle $\pi : M \to M'$ with fibre T induces an inclusion $\pi^*\mathscr{A}^1(M') \hookrightarrow \mathscr{A}^1(M)$ and thus a filtration of $\mathscr{A}^k(M)$ whose graded pieces are generated by forms of the type $(\pi^*\alpha) \wedge \beta$ where β is a differential form along the fibres. Decomposing also the differential and starting with the vertical component, we have constructed a version of the Leray Serre spectral sequence

$$E_2^{p,q} = H^p(M', H^q(T, \mathbb{R})) \Longrightarrow H_{dR}^{p+q}(M).$$

In the general case the E_2-term has to be interpreted as cohomology with values in a local system but since we have a principal bundle with connected structure group the monodromy action on $H^q(T, \mathbb{R})$ is trivial and we have $E_2^{p,q} = H_{dR}^p(M') \otimes H_{dR}^q(T)$.

Now we repeat the construction on the level of left-invariant forms. Consider $\Lambda^\bullet\mathfrak{g}^*$ as a subcomplex of the de Rham complex $(\mathscr{A}^\bullet, d)$. The differential of a k-form α can be defined entirely in terms of the Lie-bracket and the Lie-derivative as

$$\begin{aligned}(d_k\alpha)(x_1,\dots,x_{k+1}) := &\sum_{i=1}^{k+1}(-1)^{i+1}x_i(\alpha(x_1,\dots,\hat{x}_i,\dots,x_{k+1}))\\ &+\sum_{1\le i<j\le k+1}(-1)^{i+j}\alpha([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{k+1}).\end{aligned}$$

For left-invariant $\alpha \in \Lambda^k\mathfrak{g}^*$ and $x_i \in \mathfrak{g}$ it reduces to

$$(d_k\alpha)(x_1,\dots,x_{k+1}) = \sum_{1\le i<j\le k+1}(-1)^{i+j}\alpha([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{k+1})$$

and the complex $(\Lambda^\bullet\mathfrak{g}^*, d)$ is defined purely algebraically. It is known as Chevalley complex [10] and computes the Lie-algebra cohomology of $\mathfrak{g}$ (see also [45, Chapter 7]).

If the fibration $\pi : M \to M'$ corresponds to the short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

where $\mathfrak{h} = \mathscr{Z}\mathfrak{g}$ as explained above then the dual sequence induces a filtration on the exterior powers $\Lambda^k\mathfrak{g}^*$ and we can organise the graded pieces into a spectral sequence, the Hochschild-Serre spectral sequence (see [45, Section 7.5]), with

$$\begin{aligned}E_0^{p,q} &= \Lambda^p(\mathfrak{g}/\mathfrak{h})^*\otimes\Lambda^q\mathfrak{h}^*\\ E_2^{p,q} &= H^p(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h})) = H^p(\mathfrak{g}/\mathfrak{h})\otimes H^q(\mathfrak{h}) \Longrightarrow H^{p+q}(\mathfrak{g},\mathbb{R}).\end{aligned}$$

The second description of the E_2-term holds in our setting since $\mathfrak{h}$ is contained in the centre of $\mathfrak{g}$, which corresponds to π being a principal bundle.

Now we deduce a proof of Nomizu's theorem: we know the result for the torus and then proceed by induction on the nilpotency index. The inclusion $(\Lambda^\bullet\mathfrak{g}^*, d) \hookrightarrow (\mathscr{A}^\bullet(M), d)$ is compatible with the filtrations we introduced and thus we get an induced homomorphism of spectral sequences. At the E_2 level this is

$$H^p(\mathfrak{g}/\mathfrak{h})\otimes H^q(\mathfrak{h}) \to H^p_{dR}(M')\otimes H^q_{dR}(T)$$

which is an isomorphism by induction hypothesis. Thus also in the limit we have the desired isomorphism

$$H^*(\mathfrak{g}) \xrightarrow{\cong} H^*_{dR}(M).$$

Remark 1 The statement we just proved extends to solvmanifolds, i.e., compact quotients of solvable groups, that satisfy the so-called Mostow condition [34]. The de Rham cohomology of more general solvmanifolds can be studied via an auxiliary construction due to Guan [23] which was recently reconsidered by Console and Fino [12].

3 Left-invariant complex structures and Dolbeault cohomology

We start this section by recalling the definition of Dolbeault cohomology and giving the precise statement of the conjecture. Then we discuss to what extent the proof of Nomizu's result, discussed in the preceding section, carries over to the holomorphic setting. After mentioning the openness result of Console and Fino we will also give some new results and discuss directions of future research.

3.1 Reminder on Dolbeault cohomology

Recall that an (integrable) complex structure on a differentiable manifold M is a vector bundle endomorphism J of the tangent bundle which satisfies $J^2 = -\mathrm{id}$ and the integrability condition (1). The endomorphism J induces a decomposition of the complexified tangent bundle by letting pointwise $T^{1,0}M \subset T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ be the i-eigenspace of J. Then the $-$i-eigenspace is $T^{0,1}M = \overline{T^{1,0}M}$. Note that $T^{1,0}M$ is naturally isomorphic to (TM, J) as a complex vector bundle via the projection, and the integrability condition can be formulated as $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

The bundle of differential k-forms decomposes

$$\Lambda^k T^*_{\mathbb{C}}M = \bigoplus_{p+q=k} \Lambda^p T^{*1,0}M \otimes \Lambda^q T^{*0,1}M = \bigoplus_{p+q=k} \Lambda^{p,q}T^*M,$$

and we denote by $\mathscr{A}^{p,q}(M)$ the $\mathscr{C}^\infty$-sections of the bundle $\Lambda^{p,q}T^*M$, i.e., the global differential forms of type (p,q).

The integrability condition (1) is equivalent to the decomposition of the differential $d = \partial + \bar\partial$ and for all p we get the Dolbeault complex

$$(\mathscr{A}^{p,\bullet}(M_J), \bar\partial) : 0 \to \mathscr{A}^{p,0}(M) \xrightarrow{\bar\partial} \mathscr{A}^{p,1}(M) \xrightarrow{\bar\partial} \ldots$$

The Dolbeault cohomology groups $H^{p,q}(M) = H^q(\mathscr{A}^{p,\bullet}(M), \bar{\partial})$ are one of the most fundamental holomorphic invariants of M_J; from another point of view, the Dolbeault complex computes the cohomology groups of the sheaf $\Omega^p_{M_J}$ of holomorphic p-forms.

In case M is a nilmanifold and J is left-invariant, all of the above can be considered at the level of left-invariant forms. Decomposing $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^{*1,0} \oplus \mathfrak{g}^{*0,1}$ and setting $\Lambda^{p,q}\mathfrak{g}^* = \Lambda^p\mathfrak{g}^{*1,0} \otimes \Lambda^q\mathfrak{g}^{*0,1}$ we get subcomplexes

$$(\Lambda^{p,\bullet}\mathfrak{g}^*, \bar{\partial}) \hookrightarrow (\mathscr{A}^{p,\bullet}(M_J), \bar{\partial}). \tag{3}$$

In fact, the left hand side has a purely algebraic interpretation worked out in [41]: $\mathfrak{g}^{0,1}$ is a Lie-subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and the adjoint action followed by the projection to the $(1,0)$-part makes $\mathfrak{g}^{1,0}$ into an $\mathfrak{g}^{0,1}$-module. Then the complex $(\Lambda^{p,\bullet}\mathfrak{g}^*, \bar{\partial})$ computes the Lie-algebra cohomology of $\mathfrak{g}^{0,1}$ with values in $\Lambda^p\mathfrak{g}^{*1,0}$ and we call

$$H^{p,q}(\mathfrak{g}, J) = H^q(\mathfrak{g}^{0,1}, \Lambda^p\mathfrak{g}^{*1,0}) = H^q(\Lambda^{p,\bullet}\mathfrak{g}^*, \bar{\partial})$$

the Lie-algebra Dolbeault cohomology of $(\mathfrak{g}, J)$.

We can now formulate the analogue of Nomizu's theorem for Dolbeault cohomology as a conjecture.

Conjecture 1 Let M_J be a nilmanifold with left-invariant complex structure. Then the map

$$\varphi_J : H^{p,q}(\mathfrak{g}, J) \to H^{p,q}(M_J)$$

induced by (3) is an isomorphism.

It is known that φ_J is always injective (see [11] or [41]).

We will accumulate evidence for the conjecture over the next sections and also explain which are the open cases.

3.2 The inductive proof

In order to extend the idea of Nomizu's proof to Dolbeault cohomology, we need to have three ingredients:

(i) Can we start the induction, i.e., can we express the Dolbeault cohomology of a complex torus as a suitable Lie-algebra cohomology?

(ii) Does the complex geometry of nilmanifolds allow us to proceed by induction? For example, is every nilmanifold with left-invariant complex structure a holomorphic principal bundle?

(iii) Are there spectral sequences that play the role of the Leray-Serre and Hochschild-Serre spectral sequence for (Lie-algebra) Dolbeault cohomology?

It is well known that the first question has a positive answer (see e.g. [31, p.15]). In our language, assume that $\mathfrak{g}$ is abelian and J is a complex structure. Then the differential in the Lie-algebra Dolbeault complex $(\Lambda^{p,\bullet}\mathfrak{g}^*, \bar\partial)$ is trivial (being induced by the adjoint action) and thus

$$H^{p,q}(\mathfrak{g},J) = \Lambda^{p,q}\mathfrak{g}^* = \Lambda^p\mathfrak{g}^* \otimes \Lambda^q\bar{\mathfrak{g}}^* = H^{p,q}(M_J).$$

Unfortunately, the answer to the second question is negative. We will discuss the geometry of nilmanifolds with left-invariant complex structure in Section 3.2.1 and see that nevertheless the inductive approach works in many important special cases.

The positive answer to the third question, important for the induction step, has been worked out by Cordero, Fernández, Gray and Ugarte [14] for principal holomorphic torus bundles and in greater generality by Console and Fino [11]. The extra grading coming from the (p,q)-type of the differential forms makes the notation and the construction of the necessary spectral sequences more involved. For the usual Dolbeault cohomology of a holomorphic fibration the result goes back to Borel [24, Appendix II, Theorem 2.1].

Proposition 1 (Console, Fino) *Let M_J be a nilmanifold with left-invariant complex structure and let $\pi : M \to M'$ be a holomorphic fibration with typical fibre F induced by a Γ-rational and J-invariant ideal $\mathfrak{h} \subset \mathfrak{g}$ (as explained in Section 3.2.1). If for all p,q we have*

$$H^{p,q}(\mathfrak{h},J_{|\mathfrak{h}}) \cong H^{p,q}(F) \text{ and } H^{p,q}(\mathfrak{g}/\mathfrak{h},J') \cong H^{p,q}(M'),$$

where J' is the complex structure on $\mathfrak{g}/\mathfrak{h}$ induced by J, then also

$$H^{p,q}(\mathfrak{g},J) \cong H^{p,q}(M).$$

Clearly, with the above proposition we can proceed inductively to compute the Dolbeault cohomology of iterated holomorphic principal bundles, as we did in the real case. Unfortunately, considering principal holomorphic torus bundles is not enough, so we really need to decide when a nilmanifolds with left-invariant complex structure admits a suitable fibration.

3.2.1 When is a nilmanifold with left-invariant complex structure an iterated (principal) bundle?

We have seen that we need to understand the geometry of nilmanifolds with left-invariant complex structure, in particular whether there are natural fibrations over nilmanifolds of smaller dimension. In general, the projections in the tower of (real) principal bundles (2) will not be holomorphic, for example, the centre could be odd-dimensional.

It would be convenient if we could detect fibrations of M by studying only the Lie-algebra $\mathfrak{g}$. For a universal cover, i.e., the simply connected Lie group, this is easy: a fibration $G \to G'$ over another simply connected nilpotent Lie-group corresponds to a short exact sequence of Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}' \to 0$$

or, in other words, to an ideal $\mathfrak{h} \subset \mathfrak{g}$. Here we use that, by the Baker-Campell-Hausdorff formula (see e.g. [27, Section B.4]), the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism and hence every ideal induces a closed subgroup of G.

If we look at a 2-dimensional torus $M = \mathbb{R}^2/\mathbb{Z}^2$ then every 1-dimensional subspace $\mathfrak{h}$ in the abelian Lie-algebra $\mathfrak{g} = \mathbb{R}^2$ is an ideal. But there is some extra structure: a basis for the lattice (or, strictly speaking, the logarithm of this basis) generates a $\mathbb{Q}$-vector space $\mathfrak{g}_\mathbb{Q} \cong \mathbb{Q}^2 \subset \mathfrak{g}$ such that $\mathfrak{g}_\mathbb{Q} \otimes \mathbb{R} = \mathfrak{g}$. Clearly, a 1-dimensional subgroup corresponding to $\mathfrak{h} \subset \mathfrak{g}$ closes to a circle in the quotient if and only if it has rational slope, i.e., if and only if $\mathfrak{h} \cap \mathfrak{g}_\mathbb{Q}$ is a $\mathbb{Q}$-vector space of dimension 1.

The general case is captured in the following definition.

Definition 2 Let $\mathfrak{g}$ be a nilpotent Lie-algebra. A *rational structure* for $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_\mathbb{Q}$ defined over $\mathbb{Q}$ such that $\mathfrak{g}_\mathbb{Q} \otimes \mathbb{R} = \mathfrak{g}$.

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be rational with respect to a given rational structure $\mathfrak{g}_\mathbb{Q}$ if $\mathfrak{h}_\mathbb{Q} := \mathfrak{h} \cap \mathfrak{g}_\mathbb{Q}$ is a rational structure for $\mathfrak{h}$.

If Γ is a lattice in the corresponding simply connected Lie-group G then its associated rational structure is given by the $\mathbb{Q}$-span of $\log \Gamma$. A rational subspace with respect to this structure is called *Γ-rational.*

Remark 2 One has to check that this is well defined, i.e., that the $\mathbb{Q}$-span of $\log \Gamma$ gives a rational structure. Indeed more is true: a nilpotent Lie-algebra admits a $\mathbb{Q}$-structure if and only if the corresponding simply connected Lie-group contains a lattice [16, Theorem 5.1.8].

This criterion makes it particularly simple to produce examples: given a nilpotent Lie-algebra $\mathfrak{g}$ with rational structure constants, we know that there exists a lattice Γ in the corresponding Lie-group G and we get a compact nilmanifold $M = \Gamma\backslash G$. Since most properties of M are encoded in $\mathfrak{g}$, there is usually no need to specify the lattice concretely.

Coming back to the original problem we have [16, Lemma 5.1.4, Theorem 5.1.11]:

Lemma 3.1 *Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal. Then the fibration $G \to G/\exp\mathfrak{h}$ descends to a fibration of compact nilmanifolds $\pi : M \to M'$ if and only if $\mathfrak{h}$ is Γ-rational.*

In principle, all subspaces that are naturally associated to the Lie-algebra structure of $\mathfrak{g}$ are rational with respect to any rational structure in $\mathfrak{g}$. In particular this holds for the subspaces in the ascending and descending central series (Definition 1) and intersections thereof [16, p. 208].

If we add left-invariant complex structures, we would like the fibration $\pi : M_J \to M'_{J'}$ to be holomorphic as well, which, by left-invariance, is the same as to say that $\mathfrak{g} \to \mathfrak{g}'$ is complex linear or equivalently that $\mathfrak{h}$ is a complex subspace of $(\mathfrak{g}, J)$. We have proved

Proposition 2 *Let M_J be a nilmanifold with left-invariant complex structure. Then $\mathfrak{h} \subset \mathfrak{g}$ defines a holomorphic fibration $\pi : M_J \to M'_{J'}$ if and only if $\mathfrak{h}$ is a J-invariant and Γ-rational ideal in $\mathfrak{g}$.*

It is time for an example that shows what can go wrong:

Example 1 We define a 6-dimensional Lie algebra $\mathfrak{h}_7$ with basis $e_1, \dots, e_6$ where, up to anti-commutativity, the only non-zero brackets are

$$[e_1, e_2] = -e_4,\ [e_1, e_3] = -e_5,\ [e_2, e_3] = -e_6.$$

The vectors $e_4 \dots, e_6$ span the centre $\mathscr{Z}^1\mathfrak{h}_7 = \mathscr{C}^1\mathfrak{h}_7$.

Since the structure equations are rational, there is a lattice Γ in the corresponding simply connected Lie-group H_7 and we can consider the nilmanifold $M = \Gamma\backslash H_7$.

For $\lambda \in \mathbb{R}$ we give a left-invariant complex structure J_λ on M by specifying a basis for the space of $(1,0)$-vectors:

$$({\mathfrak{h}_7}^{1,0})_\lambda := \langle X_1 = e_1 - ie_2, X_2^\lambda = e_3 - i(e_4 - \lambda e_1), X_3^\lambda = -e_5 + \lambda e_4 + ie_6 \rangle$$

One can check that $[X_1, X_2^\lambda] = X_3^\lambda$ and, since X_3^λ is contained in the centre, the complex structure is integrable. The largest complex subspace of the centre is spanned by the real and imaginary part of X_3^λ since the centre has real dimension three.

The simply connected Lie-group H_7 has a filtration by subgroups induced by the filtration

$$\mathfrak{h}_7 \supset V_1 = \langle \lambda e_2 + e_3, e_4, Im(X_3^\lambda), Re(X_3^\lambda)\rangle \supset V_2 = \langle Im(X_3^\lambda), Re(X_3^\lambda)\rangle \supset 0$$

on the Lie-algebra and, since all these are J invariant, H_7 has the structure of a tower of principal holomorphic bundles with fibre $\mathbb{C}$. In fact, using the results of [43], a simple calculation shows that every complex structure on $\mathfrak{h}_7$ is equivalent to J_0.

Now we take the compatibility with the lattice into account. The rational structure induced by Γ coincides with the $\mathbb{Q}$-algebra generated by the basis vectors e_k and, by the criterion in Proposition 2, the fibrations on H_7 descends to the compact nilmanifold M if and only if λ is rational. In fact, one can check that for $\lambda \notin \mathbb{Q}$ the Lie-algebra $\mathfrak{h}_7$ does not contain any non-trivial J-invariant and Γ-rational ideals, so there is no holomorphic fibration at all over a nilmanifold of smaller dimension.

To understand when there is a suitable tower of fibrations on a nilmanifold, the following definitions turn out to be useful:

Definition 3 Let $\mathfrak{g}$ be a nilpotent Lie-algebra with rational structure $\mathfrak{g}_{\mathbb{Q}}$. We call an ascending filtration

$$0 = \mathcal{S}^0\mathfrak{g} \subset \mathcal{S}^1\mathfrak{g} \subset \cdots \subset \mathcal{S}^t\mathfrak{g} = \mathfrak{g}$$

a *(complex) torus bundle series* with respect to a complex structure J if for all $i = 1\ldots, t$

$$\mathcal{S}^i\mathfrak{g} \text{ is rational with respect to } \mathfrak{g}_{\mathbb{Q}} \text{ and an ideal in } \mathcal{S}^{i+1}\mathfrak{g}, \tag{a}$$

$$J\mathcal{S}^i\mathfrak{g} = \mathcal{S}^i\mathfrak{g}, \tag{b}$$

$$\mathcal{S}^{i+1}\mathfrak{g}/\mathcal{S}^i\mathfrak{g} \text{ is abelian .} \tag{c}$$

If in addition

$$\mathcal{S}^{i+1}\mathfrak{g}/\mathcal{S}^i\mathfrak{g} \subset \mathcal{Z}(\mathfrak{g}/\mathcal{S}^i\mathfrak{g}), \tag{c'}$$

then $(\mathcal{S}^i\mathfrak{g})_{i=0,\ldots,t}$ is called a *principal torus bundle series*.

An ascending filtration $(\mathcal{S}^i\mathfrak{g})_{i=0,\ldots,t}$ on $\mathfrak{g}$ is said to be a *stable torus bundle series* for $\mathfrak{g}$, if $(\mathcal{S}^i\mathfrak{g})_{i=0,\ldots,t}$ is a torus bundle series for every complex structure J

and every rational structure $\mathfrak{g}_{\mathbb{Q}}$ in $\mathfrak{g}$. If also condition (c') holds, then it is called a *stable principal torus bundle series*.

Geometrically, a principal torus bundle series induces the holomorphic analogue of the tower of real principal torus bundles described in (2).

With a torus bundle series we get in some sense the opposite picture: we start by fibring M over a complex torus with fibre a nilmanifold with left-invariant complex structure and then proceed by decomposing the fibre further. More precisely, the complex structure J restricts to each of the sub-algebras $\mathscr{S}^i\mathfrak{g}$, and since they are rational we get a nilmanifold with left-invariant complex structure $M_i = \mathscr{S}^i\Gamma\backslash\mathscr{S}^iG$ where $\mathscr{S}^iG = \exp\mathscr{S}^i\mathfrak{g}$ and $\mathscr{S}^i\Gamma = \Gamma\cap\mathscr{S}_iG$. Let T_i be the complex torus associated to $\mathscr{S}^i\mathfrak{g}/\mathscr{S}^{i-1}\mathfrak{g}$ with the induced complex structure and lattice. The short exact sequences

$$0\to\mathscr{S}^{i-1}\mathfrak{g}\to\mathscr{S}^i\mathfrak{g}\to\mathscr{S}^i\mathfrak{g}/\mathscr{S}^{i-1}\mathfrak{g}\to 0$$

give rise to holomorphic fibre bundles

$$\begin{tikzcd} M_{i-1} \arrow[r, hook] & M_i \arrow[d, "\pi_i"] \\ & T_i \end{tikzcd} \qquad \text{for } i = 1,\dots,t \tag{4}$$

with $M_t = M$ and $M_1 = T_1$. Note that these bundles cannot be principal bundles in general since the fibre is not a complex Lie group.

Thus a torus bundle series gives an inductive decomposition of M_J into complex tori. Considering the complex structure J_0 in Example 1, we see that the length of a (principal) torus bundle series may be larger than the nilpotency index.

The notions of stable (principal) torus bundle series appear to be quite strong but in [40] many examples of such have been produced. For example, the classification of complex structures on Lie-algebras with $\dim\mathscr{C}^1\mathfrak{g} = 1$, worked out independently by several authors, shows that $0\subset\mathscr{Z}\mathfrak{g}\subset\mathfrak{g}$ is a stable principal torus bundle series [40, Propostion 3.6]. The notion has the advantage to be independent of the chosen lattice and complex structure and allows to give structural information valid for all nilmanifolds with left-invariant complex structure and Lie-algebra $\mathfrak{g}$.

If we have a holomorphic decomposition as (2) on page 374 or (4), then, by Proposition 1, the inductive approach works and we obtain

Theorem 2 (Console, Fino) *If M_J is a nilmanifold with left-invariant complex structure such that $\mathfrak{g}$ admits a (principal) torus bundle series with respect to J, then Conjecture 1 holds for M_J.*

Corollary 1 *If $\mathfrak{g}$ admits a stable (principal) torus bundle series then Conjecture 1 holds for every nilmanifold with left-invariant structure with Lie-algebra $\mathfrak{g}$.*

All possible types of nilmanifolds with left-invariant complex structure up to real dimension 4 were mentioned in the introduction – there are only complex tori and Kodaira surfaces for which the conjecture is well known. In real dimension 6 there are only 34 isomorphism classes of nilpotent Lie-algebras and the 18 classes admitting a complex structure have been classified by Salamon [42]. We already met the Lie-algebra $\mathfrak{h}_7$ in Example 1. The first part of the following result, which implies the second, is contained in [40, Section 4.2].

Corollary 2 *If M_J is a nilmanifold of dimension at most six with Lie-algebra $\mathfrak{g} \not\cong \mathfrak{h}_7$ then $\mathfrak{g}$ admits a stable (principal) torus bundle series and Conjecture 1 holds for M_J.*

Roughly half of the Hodge numbers of a nilmanifold $(\Gamma\backslash H_7, J)$ can be checked by hand to coincide with the predictions but the ones in the middle are not immediately accessible.

The conjecture is known to be true in other important special cases. If M_J is the quotient of a complex Lie group, i.e., $(\mathfrak{g}, J)$ is a complex Lie algebra, then the tangent bundle of M_J is holomorphically trivial and M_J is complex parallelisable. This can be reformulated as $[Jx, y] = J[x, y]$ for all $x, y \in \mathfrak{g}$ or equivalently as $[\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}] = 0$.

Complex structures satisfying the opposite condition $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] = 0$ are called abelian (because $\mathfrak{g}^{1,0}$ is an abelian subalgebra of $\mathfrak{g}_{\mathbb{C}}$). Such complex structures were introduced by Barberis [3] and come up in different contexts [2, 17].

In both cases it is straightforward to check that the ascending central series is a principal torus bundle series and thus we have

Corollary 3 *If M_J is a nilmanifold with left-invariant complex structure and J is abelian or if M_J is complex parallelisable then M_J is an iterated principal holomorphic torus bundle and Conjecture 1 holds for M_J.*

It was another insight of Console and Fino that the essential issue here is rationality of ideals: consider the descending central series adapted to J defined by

$$\mathscr{C}_J^i(\mathfrak{g}) = \mathscr{C}^i\mathfrak{g} + J\mathscr{C}^i\mathfrak{g},$$

in other words $\mathscr{C}_J^i\mathfrak{g}$ is the smallest J-invariant subspace of $\mathfrak{g}$ containing $\mathscr{C}^i\mathfrak{g}$. Then, by [11, Lemma 1], these subspaces satisfy condition (b) and (c) of Definition 3.

Thus they induce a decomposition of the universal cover (G,J) as an iterated holomorphic bundle over complex vector spaces similar to (4).

The decomposition of the universal cover descends to the compact manifold M_J if and only if the subspaces $\mathscr{C}_J^i\mathfrak{g}$ are rational. In particular this is the case, if J itself is rational, i.e., if J maps $\mathfrak{g}_{\mathbb{Q}}$ to itself. Thus we have

Corollary 4 (Console, Fino) *If J is rational, then $\mathfrak{g}$ admits a torus bundle series adapted to J and Conjecture 1 holds for M_J.*

This result is very useful, since if one is looking for specific examples usually everthing can be chosen to be rational.

3.3 Console and Fino's result on openness

In the last section we have seen that we can compute Dolbeault cohomology with left-invariant forms whenever we have some control over the geometry of M_J. Using deformation theoretic methods one can go further.

Recall that the datum of a complex structure $J : \mathfrak{g} \to \mathfrak{g}$ is equivalent to specifying the subspace $\mathfrak{g}^{1,0} \subset \mathfrak{g}_{\mathbb{C}}$. So the set of left-invariant complex structures can be identified with the subset

$$\mathscr{C}(\mathfrak{g}) = \{V \in \mathbb{G}r(n, \mathfrak{g}_{\mathbb{C}}) \mid V \cap \bar{V} = 0, [V,V] \subset V\}$$

of the Grassmannian of half-dimensional subspaces of $\mathfrak{g}_{\mathbb{C}}$. The first condition ensures that $\mathfrak{g}_{\mathbb{C}} = V \oplus \bar{V}$ and the second that the complex structure J_V with the corresponding eigenspace decomposition is integrable.

The question when the universal cover decomposes as an iterated principal bundle as in (2) has been studied by Cordero, Fernández, Gray and Ugarte. Such left-invariant complex structures are called nilpotent and an algebraic characterisation has been given in [14].

Note that it is a hard problem to decide whether $\mathscr{C}(\mathfrak{g}) \neq \varnothing$ for a given nilpotent Lie-algebra $\mathfrak{g}$.

Theorem 3 ([11, Theorem A]) *Let $U \subset \mathscr{C}(\mathfrak{g})$ be the subset of left-invariant complex structures J for which the inclusion*

$$\varphi_J : H^{p,q}(\mathfrak{g}, J) \hookrightarrow H^{p,q}(M_J)$$

is an isomorphism. Then U is an open subset of $\mathscr{C}(\mathfrak{g})$.

The strategy of the proof is to show that the dimension of the complement of $H^{p,q}(\mathfrak{g},J)$ in $H^{p,q}(M_J)$ is upper-semi-continuous and thus remains equal to zero in an open neighbourhood of any point J where φ_J is an isomorphism.

So to prove Conjecture 1 it would be sufficient to show that, for each connected component of $\mathscr{C}(\mathfrak{g})$, the subset U as in the Theorem is non-empty and closed. Unfortunately Hodge-numbers do behave badly when going to the limit, especially for non-Kähler manifolds, so closedness is very difficult.

The set of rational complex structures is a good candidate to show that U is non-empty and dense but it is not clear to me whether $\mathscr{C}(\mathfrak{g})$ does always contain rational complex structures provided it is non-empty. Calculations suggest that this will not be the case but a concrete counterexample is complicated to write down.

Remark 3 In Corollary 3 we saw that the conjecture holds for abelian complex structure and complex parallelisable nilmanifolds. Small deformations of such structures have been studied in some detail and deformations of these are again left-invariant but in general neither abelian nor complex parallelisable (see Section 4.2 and [13, 33, 39]). In this way we can get more examples of interesting complex structures where the conjecture still holds.

3.4 Some new results and open questions

In this section we first present a result that any nilmanifold with left-invariant complex structure is not too far away from satisfying Conjecture 1, it suffices to take a finite quotient. This result is new and might lead to a complete proof; we will discuss some possible approaches below.

We first need a lemma that exploits the especially simple arithmetics of lattices in nilpotent Lie groups.

Lemma 3.2 *Let $\mathfrak{g}$ be a nilpotent real Lie algebra, $\Gamma \subset G$ a lattice and $\mathfrak{g}_{\mathbb{Q}}$ the rational structure associated to* $\log\Gamma$. *Then for any $x \in \mathfrak{g}_{\mathbb{Q}}$ there exists a lattice Γ' such that $\Gamma \subset \Gamma'$ of finite index and* $\exp(x) \in \Gamma'$.

Proof Pick any lattice $\tilde{\Gamma}$ containing $\exp(x)$ and inducing the same rational structure in $\mathfrak{g}$ as Γ. This is possible by [16, Lemma 5.1.10]. Then by [16, Theorem 5.1.12]

$\Gamma \cap \tilde{\Gamma}$ is a lattice in G which is of finite index in both Γ and $\tilde{\Gamma}$. If we define Γ' to be the subgroup of G generated by Γ and $\tilde{\Gamma}$ then Γ' is again discrete and contains both $\exp(x)$ and Γ. ■

Proposition 3 *Let $M_J = (\Gamma \backslash G, J)$ be a nilmanifold with left-invariant complex structure. Then there exists a lattice $\Gamma' \subset G$ with Γ of finite index in Γ' such that*

$$\varphi_J : H^{p,q}(\mathfrak{g}) \cong H^{p,q}(\Gamma' \backslash G, J).$$

In other words, given any nilmanifold with left-invariant complex structure M_J there is a finite regular covering $\pi : M_J \to M'_J$ such that the conjecture holds for M'_J.

Proof Endow all involved bundles with left-invariant hermitian metrics. Then the Laplacian $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is a left-invariant elliptic differential operator on G. Let $\mathscr{H}(G) := \ker(\Delta_{\bar{\partial}})$ be the space of harmonic forms of type (p,q) on G. We can take invariants under G and Γ respectively and get

$$H^{p,q}(M) \cong \mathscr{H}(G)^{\Gamma} \supset \mathscr{H}(G)^{G} = H^{p,q}(\mathfrak{g}, J).$$

The last equality comes from the compatibility of the Hodge-decomposition with the subspace of left-invariant forms; this had been worked out in detail in [41].

We prove our claim by induction on $d := \dim \mathscr{H}(G)^{\Gamma} - \dim \mathscr{H}(G)^{G}$. If $d = 0$ we can take $\Gamma' = \Gamma$.

If $d > 0$ there exists an $\alpha \in \mathscr{H}(G)^{\Gamma}$ and an open subset $U \subset G$ such that

$$g^*\alpha \neq \alpha$$

for $g \in U$. Let $\mathfrak{g}_{\mathbb{Q}}$ be the rational structure induced by $\log(\Gamma) \subset \mathfrak{g}$. Since the exponential map is a diffeomorphism, the image of $\mathfrak{g}_{\mathbb{Q}}$ is dense in G and we can find an $x \in \mathfrak{g}_{\mathbb{Q}}$ such that $\exp(x) \in U$.

By Lemma 3.2, we can find a lattice $\Gamma' \subset G$ such that $\Gamma \subset \Gamma'$ of finite index and $\exp(x) \in \Gamma'$; then $\alpha \notin \mathscr{H}(G)^{\Gamma'} = H^{p,q}(\Gamma' \backslash G, J)$ and we conclude by induction. ■

Remark 4 Proposition 3 suggested an approach that unfortunately did not prove successful. Assume we have constructed for a nilmanifold with left-invariant complex structure M_J a lattice $\Gamma \subset \Gamma'$ as above and then manage to find a way to scale it down, i.e., to find a contracting automorphism μ of G such that $\mu(\Gamma') = \tilde{\Gamma}' \subset \Gamma$. This is possible if $\mathfrak{g}$ is naturally graded but not in general [18]. On the level of real manifolds this corresponds to two regular coverings

$$\tilde{M}' = \tilde{\Gamma}'\backslash G \to M \to M' = \Gamma'\backslash G$$

and a (different) isomorphism $\mu : M' \cong \tilde{M}'$.

If μ preserves the complex structure, i.e., M'_J and $\tilde{M}'_J$ are isomorphic as complex manifolds, then the injections

$$H^{p,q}(\mathfrak{g},J) = H^{p,q}(M'_J) \hookrightarrow H^{p,q}(M_J) \hookrightarrow H^{p,q}(\tilde{M}'_J) = H^{p,q}(\mathfrak{g},J)$$

prove the conjecture for M_J. But this will generally not be the case, as can be worked out for the Lie-algebra given in Example 1.

Remark 5 We have seen that Conjecture 1 holds if we understand the complex geometry of a nilmanifold with left-invariant complex structure M_J. In addition we have the openness result of Console and Fino. Nevertheless the general case remains open.

There are two other approaches one could try: in the proof of Proposition 3 we compared G-invariant and Γ-invariant $\Delta_{\bar\partial}$-harmonic differential forms on the universal cover G after choosing some left-invariant hermitian structure. The study of this elliptic operator falls into the realm of harmonic analysis but there does not seem to be a general result that shows that Γ-invariant harmonic forms are G-invariant. One problem is again that $\Delta_{\bar\partial}$ does not need to have any compatibility with the natural filtrations on $\mathfrak{g}$ but working on G we might avoid the issue of rationality.

Going back to the compact manifold M_J one might try to use some Weitzenböck formula to express $\Delta_{\bar\partial}$ in a different way. But since M_J is in general not Kähler the Chern-connection compatible with the hermitian structure will differ from the Levi-Civita connection and again there does not seem to be an applicable general formula at the moment. In this context Gromov's characterisation of nilmanifolds as almost flat manifolds [22] might play an important role.

4 Applications

As mentioned in the introduction, nilmanifolds can be a convenient source of examples in many contexts. Integrability conditions for additional left-invariant geometric structures usually boil down to linear algebra and thus one easily writes down interesting examples of complex, riemannian, hermitian or symplectic structures. Proceeding from the examples to general results is more difficult.

Here we will discuss two further applications related to complex structures. References to other areas have already been given in the introduction.

4.1 Prescribing cohomology behaviour and the Frölicher spectral sequence

If Conjecture 1 holds for a nilmanifold with left-invariant complex structure M_J, the computation of its Dolbeault cohomology $H^{p,q}(M_J) = H^{p,q}(\mathfrak{g}, J)$ is a matter of finite-dimensional linear algebra and can be taught to a computer algebra system. In addition this makes it possible to study the Frölicher spectral sequence

$$E_2^{p,q} = H^{p,q}(M_J) \to H_{dR}^{p+q}(M, \mathbb{C}),$$

that measures the difference between Dolbeault cohomology and de Rham cohomology. This spectral sequence degenerates at E_1 for all compact complex surfaces but Cordero, Fernández, Gray and Ugarte showed in [15], studying nilmanifolds, that for complex 3-folds the maximal non-degeneracy $E_2 \not\cong E_3 = E_\infty$ is possible. Later we constructed a family $X_n \to T_n$ of principal torus bundles over tori such that $d_n \neq 0$ for X_n (see [38]). Probably, starting from dimension 3, the maximal non-degeneracy is possible but concrete examples are still missing. If we ask in addition for simply connected manifolds, there are only very few examples with non-zero higher differentials known [37].

The idea behind these examples is that if we write down some 1-forms and their differentials carefully enough we get a nilmanifold supporting these forms for free. For example, let V, W be two complex vector spaces and give an arbitrary map

$$\delta : W^* \to \Lambda^2 V^* \otimes (V^* \otimes \bar{V}^*).$$

Setting $\mathfrak{g}^{1,0} = V \oplus W$ and $\mathfrak{g}_\mathbb{C} := \mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}}$ we extend δ to a map

$$d : \mathfrak{g}_\mathbb{C}^* \to \Lambda^2 \mathfrak{g}_\mathbb{C}^*$$

which is zero on $V^* \oplus \bar{V}^*$ and $\delta + \bar{\delta}$ on $W^* \oplus \bar{W}^*$. There is a natural real vector space $\mathfrak{g} = \{z + \bar{z} \mid z \in \mathfrak{g}^{1,0}\} \subset \mathfrak{g}_\mathbb{C}$ and via the identity

$$d\alpha(x,y) = -\alpha([x,y]) \quad \text{for } \alpha \in \mathfrak{g}^* \text{ and } x, y \in \mathfrak{g}$$

the vector space $\mathfrak{g}$ becomes a 2-step nilpotent Lie-algebra. The decomposition of $\mathfrak{g}_{\mathbb{C}}$ defines an almost complex structure J on $\mathfrak{g}$, which is integrable by our choice that δ has no component mapping to $\Lambda^2\bar{V}$. If we have chosen δ such that the structure constants of $\mathfrak{g}$ turn out to be rational, there exists a lattice in the associated nilpotent Lie-group and we have constructed a nilmanifold M_J with left-invariant complex structure.

Nearly by definition M_J is a principal holomorphic torus bundle over a torus and thus we not only have prescribed the differential of some 1-forms quite arbitrarily but our datum encodes in fact the whole cohomology algebra.

Constructing nilmanifolds with higher nilpotency index in a similar way is more tedious since one has to take care of the Jacobi identity, equivalent to $d^2 = 0$, as well.

4.2 Deformations of complex structures

Our main motivation to study Conjecture 1 was the question if small deformations of left-invariant complex structures remain left-invariant. Generalising results of Console, Fino and Poon [13] (see also [33]) we proved

Theorem 4 ([41, Theorem 2.6]) *If Conjecture 1 holds for a nilmanifold with left-invariant complex structure M_J then all sufficiently small deformations of J are again left-invariant complex structures.*

The idea of the proof is that small deformations of J are controlled by the first and second cohomology groups of the holomorphic tangent bundle. By constructing a version of Serre-duality that works purely on the level of Lie-algebra cohomology one can represent the elements of $H^i(M_J, \mathscr{T}_{M_J})$ by left-invariant forms and the result follows by the standard inductive construction of the Kuranishi space [30].

The space of all integrable complex structures on a nilmanifold M modulo orientation preserving diffeomorphisms isotopic to the identity is called Teichmüller space $\mathfrak{T}(M)$. It is (locally) a complex analytic space, the germ at a fixed complex structure J being the Kuranishi space of (M,J). Thus the theorem says that, under the assumption of Conjecture 1, the set of left-invariant complex structures is open in $\mathfrak{T}(M)$.

If the Lie algebra $\mathfrak{g}$ of M admits a stable (principal) torus bundle series (see Definition 3), then Conjecture 1 holds for all left-invariant complex structures on $\mathfrak{g}$

and it is natural to ask if the set of left-invariant complex structures is also closed. The starting point in this direction is Catanese's result that all deformations in the large of a complex torus are complex tori [7]. Generalising results of Catanese and Frediani [8, 9], this was extended in [40] to a large class of nilmanifolds with left-invariant complex structure. As an example we would like to mention that every deformation in the large of the Iwasawa manifold is a nilmanifold with left-invariant complex structure; in this case the topology of the space of left-invariant complex structures is known [26].

In this area many interesting questions remain open, we hope to address some of these in future work. Progress in the direction of Conjecture 1 would encourage our belief that the complex geometry of nilmanifolds with left-invariant complex structure can be completely understood via linear algebra.

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Smooth rationally connected threefolds contain all smooth curves

G.K. Sankaran

Abstract We show that if X is a smooth rationally connected threefold and C is a smooth projective curve then C can be embedded in X. Furthermore, a version of this property characterises rationally connected varieties of dimension at least 3. We give some details about the toric case.

Keywords Rationally connected variety, toric variety, embedding.
Mathematics Subject Classification (2010) *Primary* 14M22. *Secondary* 14E25, 14M25.

It is very easy to see that every smooth projective curve can be embedded in $\mathbb{P}^3$. Eisenbud and Harris asked whether the same is true if $\mathbb{P}^3$ is replaced by an arbitrary smooth rational projective 3-fold X and Eisenbud suggested starting with the case where X is toric. In that case the answer is yes, and one can see that in a very explicit way, as was done in my preprint [Sa2].

In response to [Sa2], it was pointed out by János Kollár that much more is true: the property of containing every curve sufficiently often actually characterises rationally connected 3-folds over the complex numbers. In fact, this is already implicit in the work of Kollár and others on rational curves in algebraic varieties, but had apparently not been directly noticed.

The purpose of this note is to explain these facts. In the first part I follow Kollár's hints and show how to assemble a proof of the characterisation of rationally connected 3-folds (Theorem 1). In the second part, which is a shortened ver-

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sion of [Sa2], I show explicitly (Theorem 3) how to construct an embedding of a given curve into a given smooth projective toric 3-fold by toric methods.

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1 Rationally connected varieties

In this section X is a smooth projective variety over an algebraically closed field.

1.1 RC and SRC

We recall some standard definitions from [Ko2] and [AK].

Definition 1 [Ko2, IV.3.2.3] X is *separably rationally connected*, abbreviated *SRC*, if there exists a variety W and a morphism $u\colon \mathbb{P}^1 \times W \to X$ such that

$$u^{(2)}\colon (\mathbb{P}^1 \times W) \times_W (\mathbb{P}^1 \times W) \cong \mathbb{P}^1 \times \mathbb{P}^1 \times W \longrightarrow X \times X$$

is dominant.

In other words, X is SRC if for all $x_1,\ x_2$ in some Zariski-dense subset of X we can find $w \in W$ and $P_1,\ P_2 \in \mathbb{P}^1$ such that $u(P_i, w) = x_i$ for $i = 1,\ 2$.

Definition 2 [Ko2, IV.3.2.2] X is *rationally connected*, abbreviated *RC*, if there exists a variety W, a proper family $U \to W$ whose fibres are irreducible rational curves, and a morphism $u\colon U \to X$ such that

$$u^{(2)}\colon U \times_W U \longrightarrow X \times X$$

is dominant.

Clearly SRC $\Longrightarrow$ RC, and the converse is also true in characteristic zero ([Ko2, IV.3.3.1]).

Definition 3 [AK, Definition 8] A morphism $f\colon \mathbb{P}^1 \to X$ is said to be *very free* if f^*T_X is an ample vector bundle.

Recall that a vector bundle $\mathscr{E}$ on $\mathbb{P}^1$ is ample if and only if $\mathscr{E} = \bigoplus \mathscr{O}(a_j)$ with all $a_j > 0$.

Lemma 1.1 *[Ko2, IV.3.9] If X is a smooth projective SRC variety then there exists a very free map $g_0\colon \mathbb{P}^1 \to X$.*

1.2 Maps from curves

We can use Lemma 1.1 to obtain results about maps from curves to SRC varieties.

Lemma 1.2 *If X is a smooth projective SRC variety, then for any smooth projective curve C there is a map $g\colon C \to X$ such that $H^1\big(g^*T_X(-P-Q)\big) = 0$ for any two distinct points $P,\ Q \in C$.*

Proof Suppose C has genus g. We choose a map $g_1\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $g_1^*g_0^*T_X$ is sufficiently ample: it is enough to require that $g_1^*g_0^*T_X \cong \bigoplus \mathscr{O}(a_i)$ with each $a_i > 2g$, which can be achieved by taking g_1 to have sufficiently high degree. Take any surjection $g_2\colon C \to \mathbb{P}^1$ and let F be a fibre of $g_1g_2\colon C \to \mathbb{P}^1$. If we put $g = g_0g_1g_2\colon C \to X$, we have

$$g^*T_X(-P-Q) = \bigoplus \mathscr{O}_C(a_iF - P - Q).$$

This is the direct sum of line bundles of degree $> 2g-2$ and therefore $H^1\big(g^*T_X(-P-Q)\big) = 0$.

Proposition 1 *If C is any smooth projective curve and X is any smooth SRC projective variety of dimension ≥ 3, then C can be embedded in X.*

Proof Choose $g\colon C \to X$ as in Lemma 1.2. According to [Ko2, II.1.8.2] (with $B = \emptyset$), a general deformation of g is an embedding because

$$\dim H^1\big(g^*T_X(-P-Q)\big) \leq \dim X - 3 = 0.$$

Over the complex numbers we can show more.

Lemma 1.3 *Let X be any smooth quasi-projective variety over $\mathbb{C}$, and suppose $x \in X$. Then there exists a subset $X_1(x) \subset X$, the complement of a countable union of Zariski-closed sets, such that if $y \in X_1(x)$ and if the image of $f\colon \mathbb{P}^1 \to X$ passes through both x and y then f is very free.*

Proof This follows from [AK, Proposition 13], exactly as [AK, Remark 10] follows from [AK, Proposition 10]. We consider one of the countably many irreducible components R of $\mathrm{Hom}_x(\mathbb{P}^1, X) = \{f\colon \mathbb{P}^1 \to X \mid f(0) = x\}$ and the evaluation morphism $u_R\colon \mathbb{P}^1 \times R \to X$ given by $u_R(P, f) = f(P)$. The morphisms that are not very free form a closed subscheme $R' \subseteq R$: but $u_R|_{\mathbb{P}^1 \times R'}$ is not dominant because of [AK, Proposition 13(2)], so its image lies in a proper closed subset $X_R \subset X$. So any f that is not very free has image contained in some X_R, and we take $X_1 = X \setminus \bigcup_R X_R$.

This yields a characterisation of RC varieties of dimension ≥ 3 in terms of maps from curves.

Theorem 1 *If X is a smooth projective variety of dimension ≥ 3 over $\mathbb{C}$, then X is rationally connected if and only if the following holds: for every smooth projective curve C and zero-dimensional subscheme $Z \subset C$, and every embedding $f_Z\colon Z \hookrightarrow X$, there is an embedding $f_C\colon C \hookrightarrow X$ such that $f_C|_Z = f_Z$.*

Proof One direction is trivial: if every f_Z extends then taking $C = \mathbb{P}^1$ and $Z = \{0, 1\}$ we recover the definition of RC.

Conversely, suppose that X is RC of dimension at least 3 and suppose first that $Z = \{P_1, \ldots, P_n\}$ is reduced, and write $x_i = f(P_i)$. If $Z = \emptyset$ then the statement reduces to Proposition 1. Otherwise, we may choose $x_0 \in X_1(x_1)$ as in Lemma 1.3. By [Ko1, (4.1.2.4)] we can find a map $f_0\colon \mathbb{P}^1 \to X$ such that $x_0, \ldots, x_n$ are all in the image of f_0. See also [Ko1, (5.2)]. If Z is not reduced, we can still arrange for $f_C|_Z = f_Z$ because [Ko1, (4.1.2.4)] allows us to specify the Taylor expansion of f_0 as far as we like.

The map f_0 is very free by Lemma 1.3. Exactly as in Lemma 1.2 we may compose f_0 with suitable maps $f_2\colon C \to \mathbb{P}^1$ and $f_1\colon \mathbb{P}^1 \to \mathbb{P}^1$ so as to get a map $f\colon C \to X$ such that $f|_Z = f_Z$ and $\dim H^1\big(C, f^*T_X \otimes I_Z(-P-Q)\big) = 0$ for every P, $Q \in C$. To do this we choose f_2 first to be a surjection such that $f_2|_Z$ is an isomorphism. Then we choose f_1, of sufficiently large degree, so that $f|_Z = f_Z$: to do so we need only choose a polynomial with prescribed values at each point of $f_2(Z)$ and injective on

$f_2(Z)$, which is trivial to do as the degree of f_1 may be as large as we please. Although f need not be an embedding, we may take f_C to be a general deformation of f preserving $f|_Z$, and this is an embedding by [Ko2, II.1.8.2].

Remark 1 The condition in Theorem 1 that C be smooth is not strictly necessary. It is enough for C to be a reduced curve whose singularities have embedding dimension $\leq \dim X$.

Indeed, let $\nu\colon \tilde{C} \to C$ be the normalisation. Consider the subscheme $Z' = Z \cup \operatorname{Sing} C \subset C$ and let $\tilde{Z}$ be the subscheme of $\tilde{C}$ given by $I_{\tilde{Z}} = \mathscr{H}om_{\mathscr{O}_C}(\mathscr{O}_{\tilde{C}}, I_{Z'}) \cdot \mathscr{O}_{\tilde{C}}$. If $f_{Z'}$ is an embedding of Z' in X, extending f_Z, then the argument above allows us to extend $f_{\tilde{Z}} = f_{Z'}\nu\colon \tilde{Z} \to X$ to $f_{\tilde{C}}\colon \tilde{C} \to X$ in such a way that $f_{\tilde{C}}$ is an embedding away from $\tilde{Z}$. The image of $f_{\tilde{C}}$ is then isomorphic to C.

2 Toric varieties

In this section we look at the particular case in which X is a smooth projective toric 3-fold over $\mathbb{C}$. As toric varieties are rational, they are in particular SRC, so by Proposition 1 a smooth projective toric 3-fold contains all curves. However, in the toric case it is possible to give a more direct proof, and one that shows rather more concretely how to construct an embedding of a given curve in a given toric variety X.

2.1 *Maps to toric varieties*

We need a good description of maps to a smooth projective toric variety. Several descriptions of maps to toric varieties exist, due to Cox [Co], Kajiwara [Ka] and others. The version that we use here appeared in [Sa1, Section 2][1] but the proof, which is largely due to Tadao Oda, is very short, so we give it here. We refer to [Od] for general background on toric varieties.

Let Δ be a finite (but not necessarily complete) smooth fan for a free $\mathbb{Z}$-module N of rank r. Denote the corresponding toric variety by X, and write M for the dual lat-

[1] Warning: some other unrelated parts of [Sa1] are incorrect.

tice $\mathrm{Hom}(N,\mathbb{Z})$, with pairing $\langle\ ,\ \rangle\colon M\times N\to\mathbb{Z}$. The torus is then $\mathbb{T}=\mathrm{Spec}(\mathbb{C}[M])$, where $\mathbb{C}[M]=\bigoplus_{m\in M}\mathbb{C}\mathbf{e}(m)$ is the semigroup ring of M over $\mathbb{C}$. Here, as in [Od, 1.2], the character $\mathbf{e}(m)\colon\mathbb{T}\to\mathbb{C}^*$ may be thought of after identifying M with $\mathbb{Z}^n$ as $(u_1,\dots,u_n)\mapsto u_1^{m_1}\dots u_n^{m_n}$ for suitable coordinates u_i on $\mathbb{T}\cong(\mathbb{C}^*)^n$.

As usual, $\Delta(d)$ denotes the set of d-dimensional cones in Δ. For each $\rho\in\Delta(1)$, we denote by $V(\rho)$ the corresponding irreducible Weil divisor on X and by n_ρ the generator of the semigroup $N\cap\rho$.

Theorem 2 *Let Y be a normal algebraic variety over $\mathbb{C}$. A morphism $f\colon Y\to X$ such that $f(Y)\cap\mathbb{T}\neq\emptyset$ corresponds to a collection of effective reduced Weil divisors $D(\rho)$ on Y indexed by $\rho\in\Delta(1)$ and a group homomorphism $\varepsilon\colon M\to\mathscr{O}_Y(Y_0)^\times$ to the multiplicative group of invertible regular functions on $Y_0=Y\setminus\bigcup_{\rho\in\Delta(1)}D(\rho)$, such that*

$$D(\rho_1)\cap D(\rho_2)\cap\cdots\cap D(\rho_s)=\emptyset\qquad \textit{if } \rho_1+\rho_2+\cdots+\rho_s\notin\Delta \tag{1}$$

and

$$\mathrm{div}(\varepsilon(m))=\sum_{\rho\in\Delta(1)}\langle m,n_\rho\rangle D(\rho)\qquad \textit{for all}\quad m\in M. \tag{2}$$

Proof Suppose $f\colon Y\to X$ is a morphism with $f(Y)\cap\mathbb{T}\neq\emptyset$. For each $\rho\in\Delta(1)$, we take $D(\rho)$ to be the pull-back Weil divisor $f^{-1}(V(\rho))$, which is well-defined since Y is normal, X is smooth and $f(Y)\not\subset V(\rho)$.

If $\rho_1+\cdots+\rho_s\notin\Delta$, then $V(\rho_1)\cap\cdots\cap V(\rho_s)=\emptyset$ so $D(\rho_1)\cap\cdots\cap D(\rho_s)=\emptyset$. In this case $Y_0=f^{-1}(\mathbb{T})$ is nonempty by assumption, and $f|_{Y_0}$ induces

$$f|_{Y_0}^*\colon\mathbb{C}[M]\to\mathscr{O}_Y(Y_0)^\times.$$

The composite $\varepsilon:=f|_{Y_0}^*\circ\mathbf{e}$ satisfies (2), since

$$\mathrm{div}(\mathbf{e}(m))=\sum_{\rho\in\Delta(1)}\langle m,n_\rho\rangle V(\rho)\qquad\text{for all}\quad m\in M.$$

Conversely, suppose $\{D(\rho)\}_{\rho\in\Delta(1)}$ and ε satisfy (1) and (2). For $\sigma\in\Delta$, put $\hat{\sigma}=\{\rho\in\Delta(1)\mid\rho\not<\sigma\}$. Then the corresponding open piece U_σ of X satisfies

$$\begin{aligned} U_\sigma &= X \setminus \bigcup_{\rho \in \hat{\sigma}} V(\rho) \\ &= \bigcap_{\rho \in \hat{\sigma}} (X \setminus V(\rho)) \\ &\cong \operatorname{Spec}(\mathbb{C}[M \cap \sigma^\vee]). \end{aligned}$$

Put $Y_\sigma = f^{-1}(U_\sigma) = Y \setminus \bigcup_{\rho \in \hat{\sigma}} D(\rho)$. Then $Y = \bigcup_{\sigma \in \Delta} Y_\sigma$ since the U_σ cover X (or one can check this directly).

For each $\sigma \in \Delta$, $M \cap \sigma^\vee$ is the semigroup consisting of $m \in M$ such that $\mathbf{e}(m)$ is regular on U_σ. Thus $\varepsilon(M \cap \sigma^\vee)$ consists of regular functions on Y_σ, and defines a morphism $f_\sigma \colon Y_\sigma \to U_\sigma$. These morphisms glue together to give a morphism $f \colon Y \to X$.

In choosing the collection of divisors $\{D(\rho)\}$ we determine how the toric divisors are to intersect the image $f(Y)$. Not all choices are possible: if $\{D_\rho\}$ is chosen arbitrarily then possibly no map corresponding to that collection exists. Although the $D(\rho)$ themselves are not required to be Cartier divisors, the left-hand side of (2) is Cartier, so one necessary condition for such an f to exist is that the right-hand side of (2) is Cartier.

The condition that X be smooth is stronger than we need. See [Ka] for related results for singular toric varieties.

2.2 *Embedding a curve*

Now we apply Theorem 2 to the case where $Y = C$ is a smooth projective curve and X is projective of dimension 3.

Let L be an effective (hence ample) divisor on C. Let $\Delta(1) = \{\rho_1, \dots, \rho_r\}$, so $r - 3 = \operatorname{rk} \operatorname{Pic} X > 0$. We write n_j and V_j (rather than n_{ρ_j} and $V(\rho_j)$) for the generator and the divisor corresponding to $\rho_j \in \Delta(1)$.

Let $\{m_1, m_2, m_3\}$ be a $\mathbb{Z}$-basis for M and put $a_{ij} = \langle m_i, n_j \rangle$. The system of linear equations $\sum_{j=1}^r a_{ij}\xi_j = 0$ has rank at most 3, so we can find nontrivial integer solutions. In the projective case we can do better.

Lemma 2.4 *If X is projective, then $\sum_{j=1}^r a_{ij}\xi_j = 0$ has integer solutions with $\xi_j > 0$ for all j.*

Proof Let H be a very ample divisor on X. We have $\sum_j a_{ij}V_j = 0$ in $\operatorname{Pic} X$, since it is the divisor of $\mathbf{e}(m_i)$. But H^2V_j is the degree of the surface V_j in the projective embedding of X under $|H|$ and is therefore positive, so it is enough to take $\xi_j = H^2V_j$.

On C we take the line bundles $\mathscr{D}_j = \mathscr{O}_C(\xi_j H)$, with ξ_j as in Lemma 2.4. We may assume that $\xi_j > 2g(C)$ for all j, so that any nonzero linear combination of the $\mathscr{D}_j$ with nonnegative integer coefficients is very ample.

We want to specify a map $f\colon C \to X$ by means of data as in Theorem 2. Thus we must give elements D_j of the linear system $|\mathscr{D}_j|$.

Lemma 2.5 *If the D_j are general in $|\mathscr{D}_j|$ then they are reduced divisors and $\bigcap_j D_j = \emptyset$. In particular they satisfy* (1) *from Theorem 2.*

Proof This follows from the very ampleness of the linear systems $|\mathscr{D}_j|$.

To specify a map $f\colon C \to X$ we now need only choose ε according to Theorem 2. This amounts to choosing suitable trivialisations of each of the three bundles $\mathscr{O}_C(\sum a_{ij}\mathscr{D}_j)$, i.e. non-vanishing sections of $\mathscr{O}_C(\sum a_{ij}\mathscr{D}_j)$ with order $-a_{ij}$ along D_j. Such trivialisations are unique up to multiplication by nonzero scalars. This means that the map $f = f_{\mathbb{D},\mathbf{t}}$ is determined by choices of $\mathbb{D} = (D_1,\ldots,D_r) \in |\mathscr{D}_1| \times \cdots \times |\mathscr{D}_r|$ together with a choice of an element $\mathbf{t} \in (\mathbb{C}^*)^3 = \mathbb{T} \subset \operatorname{Aut} X$. In other words, choosing the D_j determines f up to composition with an element of $\mathbb{T}$ acting as an automorphism of X.

We note that the action of $\mathbb{T}$ has no effect on the question of whether or not the map is an embedding, and accordingly we suppress $\mathbf{t}$ in the notation.

Later we shall see that $f_{\mathbb{D}}$ will turn out to be an embedding for all sufficiently general $\mathbb{D} \in |\mathscr{D}_1| \times \cdots \times |\mathscr{D}_r|$. The next lemma shows that in order to determine whether the general $f_{\mathbb{D}}$ is an immersion, it is enough to check it over the standard affine pieces of X.

Lemma 2.6 *Suppose that, for every $\tau \in \Delta(3)$, there is a nonempty open subset $A_\tau \subset \prod_j |\mathscr{D}_j|$ such that*

$$f_{\mathbb{D}}\colon C_\tau = f_{\mathbb{D}}^{-1}(U_\tau) \to U_\tau$$

is a closed immersion if $\mathbb{D} \in A_\tau$. Then $f_{\mathbb{D}}\colon C \to X$ is a closed immersion for general $\mathbb{D} \in \prod_j |\mathscr{D}_j|$.

Proof It is enough to take $\mathbb{D} \in \bigcap_{\tau\in\Delta(3)} A_\tau$.

Theorem 3 *If X is a projective smooth toric 3-fold, C is a smooth projective curve and $\mathscr{D}_j$ are as above, the map $f_{\mathbb{D},\mathbf{t}}\colon C \to X$ is an embedding for almost all $\mathbb{D} \in \prod_j |\mathscr{D}_j|$.*

Proof In view of Lemma 2.6 it remains to check that the set A_τ for which $f_{\mathbb{D}}$ is an embedding above U_τ is indeed nonempty.

After renumbering, we have $\tau = \rho_1 + \rho_2 + \rho_3$ and we consider the semigroup $M \cap \tau^\vee$. It is generated by l_1, l_2, $l_3 \in M$ with the property that $\langle l_i, n_i \rangle > 0$ and $\langle l_i, n_k \rangle = 0$ if $1 \le k \le 3$ and $k \ne i$. The function $p_i = \varepsilon_{\mathbb{D},\mathbf{t}}(l_i) = f_{\mathbb{D},\mathbf{t}}|_{C_\tau} \circ \mathbf{e}(l_i) f_{\mathbb{D},\mathbf{t}}$ is the ith coordinate function: it takes the value 0 on D_i and is nonzero on D_k for $1 \le k \le 3$, $k \ne i$.

We first pick D_j for $j > 3$ once and for all, only requiring them to be general in the sense of Lemma 2.5. Now choose D_3 so that D_3 is also reduced and disjoint from the other D_j chosen so far. This is enough to determine p_3 up to the torus action, since $\operatorname{div}(p_3) = \langle l_3, n_3 \rangle D_3 + \sum_{j>3} \langle l_3, n_j \rangle D_j$ is independent of D_1 and D_2. Similarly a choice of D_1 or of D_2 determines p_1 or p_2 up to the torus action, independently of the choice of the other two.

After making such a choice of D_3, we claim that for general $D_2 \in |\mathscr{D}_2|$ the map $(p_2, p_3)\colon C_\tau \to \mathbb{A}^2$ is generically injective. We shall check this by exhibiting a choice of D_2 which makes this map injective near D_3. Observe that for any pair P, $Q \in D_3$ (so $p_3(P) = p_3(Q) = 0$) we can find $D_2 \in |\mathscr{D}_2|$ such that $P \in D_2$ but $Q \notin D_2$ (although such a choice of D_2 will not be general in the sense of Lemma 2.5), because $\mathscr{D}_2$ is sufficiently ample. For this choice of D_2, we have $0 = p_2(P) \ne p_2(Q)$, so $p_2(P) \ne p_2(Q)$ for general D_2 and hence for general D_2 the values of p_2 on the points of D_3 are all different from one another. In particular (p_2, p_3) corresponding to a general D_2 is injective at any point of D_3 and is therefore injective generically.

By exactly the same argument, a general choice of $D_1 \in |\mathscr{D}_1|$ separates points not separated by the other choices. If P' and Q' are (possibly infinitely near) points such that $p_2(P') = p_2(Q')$ and $p_3(P') = p_3(Q')$, then $p_1(P') \ne p_1(Q')$ if $P' \in D_1$ and $Q' \notin D_1$. Such D_1 exist if $\mathscr{D}_1$ is sufficiently ample. So for general D_1 we also have $p_1(P') \ne p_1(Q')$, as required.

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Submanifolds in Poisson geometry: a survey

Marco Zambon

Abstract We describe various classes of submanifolds of a Poisson manifold M, both in terms of tensors on M and of constraints: coisotropic submanifolds, Poisson-Dirac submanifolds (which inherit a Poisson structure), and the very general class of pre-Poisson submanifolds. We discuss embedding results for these classes of submanifolds, quotient Poisson algebras associated to them, and their relationship to subgroupoids of the symplectic groupoid of M.

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1 Poisson geometry

The phase space of a physical system, in the hamiltonian formalism, is usually given the structure of a symplectic manifold. When the system is invariant under symmetries, it makes sense to consider the "reduced" phase space obtained quotienting the original phase space by the symmetries. The reduced phase space in general is no longer symplectic, but rather has the structure of a Poisson manifold. We recall some basic facts about Poisson manifolds (see Weinstein's seminal 1983 paper [14] or the book [12] for detailed expositions).

The algebraic definition of Poisson manifold is the following:

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Definition 1 A *Poisson manifold* is a manifold M such that the algebra of functions $C^\infty(M)$ is endowed with a Lie bracket $\{\cdot,\cdot\}$ satisfying $\{f,gh\} = \{f,g\}h + g\{f,h\}$ for all f,g,h.

Often it is convenient to use a more geometric definition:

Definition 2 A *Poisson manifold* is a manifold M endowed with a bivector field $\pi \in \Gamma(\wedge^2 TM)$ satisfying $[\pi,\pi] = 0$.

Here $[\cdot,\cdot]$ denotes the Schouten bracket of multivector fields, which extends the Lie bracket of vector fields on M. The Poisson bracket $\{\cdot,\cdot\}$ and π are related by $\{f,g\} = \pi(df,dg)$.

Let M be a Poisson manifold. The bivector field $\pi \in \Gamma(\wedge^2 TM)$ can be equivalentely described by

$$\sharp\colon T^*M \to TM,\ \ \sharp\xi = \pi(\xi,\cdot),$$

a bundle map which is skew-symmetric (i.e., $\sharp^* = -\sharp$). One can show that $Im(\sharp) \subset TM$ is an involutive singular distribution[1], so M is foliated by leaves (immersed submanifolds of varying dimensions) whose tangent spaces are exactly given by $Im(\sharp)$.

At every $p \in M$ the kernel of $\sharp_p\colon T_p^*M \to Im(\pi_p)$ is the annihilator $(Im(\sharp_p))^\circ := \{\xi \in T_p^*M : \xi|_{Im(\sharp_p)} = 0\}$, hence inverting the induced isomorphism

$$T_p^*M/(Im(\sharp_p))^\circ \cong Im(\sharp_p)^* \to Im(\sharp_p)$$

we obtain a linear symplectic from ω_p on $Im(\sharp_p)$. One can show that the 2-form ω on each leaf $\mathcal{O}$ is actually symplectic. So we conclude that an equivalent characterization of Poisson manifold is the following: *a manifold foliated by leaves of varying dimensions, each of which carries a symplectic form varying smoothly with the leaf.*

Example 1 a) A symplectic form ω on a manifold M can be regarded as a Poisson bivector field, by the requirement $\sharp = -\tilde{\omega}^{-1}$ where $\tilde{\omega}\colon TM \to T^*M, v \mapsto \omega(v,\cdot)$.

b) If $\mathfrak{g}$ is a finite dimensional real Lie algebra, then $\mathfrak{g}^*$ has a natural Poisson structure, determined by $\{v,w\} = [v,w]$ where $v,w \in \mathfrak{g}$ are also viewed as linear functions on $\mathfrak{g}^*$. The symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits.

For instance, the symplectic leaves in $\mathfrak{su}(2)^*$ are spheres centered at the origin, with symplectic form growing linearly with the radius. The Poisson bivector field

[1] Indeed T^*M is a Lie algebroid (see §1.2), and the image of the anchor of any Lie algebroid is an involutive singular distribution.

is given by $x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}+x_1\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}+x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1}$ in suitable linear coordinates on $\mathfrak{su}(2)^*$.

c) On every manifold M, setting $\pi = 0$ one obtains a Poisson bivector field. Each point of M is a symplectic leaf.

1.1 Submanifolds and symplectic leaves

Let (M,π) be a Poisson manifold. In this Subsection we use the symplectic foliation described above as a guide to determine classes of submanifolds.

A natural class of submanifolds are *symplectic leaves* (leaves $\mathscr{O}$ endowed with the symplectic form ω as above). Generalizing this slightly, we obtain Poisson submanifolds, which are just unions of (open subsets of) symplectic leaves.

Definition 3 [12, §6.6] $N \subset (M,\pi)$ is a *Poisson submanifold* iff $\pi_p \in \wedge^2 T_pN$ for every $p \in N$.

Equivalent conditions are $\sharp TN^\circ = \{0\}$ or $Im(\sharp|_N) \subset TN$.

Given a symplectic vector space (V,ω), an interesting class of subspaces W are the coisotropic ones, i.e. those for which $W^\omega \subset W$. Another interesting class is given by the symplectic subspaces, i.e. those for which $W^\omega \cap W = \{0\}$.

It is natural to consider submanifolds of the Poisson manifold (M,π) whose intersections with the symplectic leaves are coisotropic or symplectic submanifolds of the leaves. Since these intersections are usually not smooth, we are lead to consider tangent spaces.

Lemma 1.1 *Let N be a submanifold of (M,π). For all $p \in N$ denote by $(\mathscr{O},\omega)$ the symplectic leaf through p. The symplectic orthogonal of $T_pN\cap T_p\mathscr{O}$ in $(T_p\mathscr{O},\omega_p)$ is $(T_pN\cap T_p\mathscr{O})^{\omega_p} = \sharp T_pN^\circ$. Hence:*

- *$T_pN\cap T_p\mathscr{O}$ is a coisotropic subspace of $(T_p\mathscr{O},\omega_p) \Leftrightarrow \sharp T_pN^\circ \subset T_pN$,*
- *$T_pN\cap T_p\mathscr{O}$ is a symplectic subspace of $(T_p\mathscr{O},\omega_p) \Leftrightarrow \sharp T_pN^\circ \cap T_pN = \{0\}$.*

The above lemma follows from a simple computation and from $(T_pN\cap T_p\mathscr{O})\cap(T_pN\cap T_p\mathscr{O})^{\omega_p} = \sharp T_pN^\circ \cap T_pN$.

Submanifolds satisfying the first condition above are called *coisotropic*. In some cases they are the replacement in Poisson geometry of the symplectic-geometric notion of "Lagrangian", see Ex. 2 b). Those that satisfy the second condition and an additional smoothness requirement are called *Poisson-Dirac* submanifolds. We will elaborate on them in §2 and §3 respectively. The intersection between the classes of coisotropic and Poisson-Dirac submanifolds are exactly the Poisson submanifolds.

1.2 Lie algebroids and Dirac manifolds

In order to determine further classes of submanifolds of a Poisson manifold, we introduce two notions that are canonically associated to Poisson geometry.

A *Lie algebroid* [12, §16] consists of a vector bundle $A \to N$ together with a Lie bracket on the space of sections $\Gamma(A)$ and a bundle map $\rho : A \to TN$ (called anchor) satisfying $[a, f \cdot b] = \rho(a) f \cdot b + f \cdot [a,b]$ for all sections a, b and functions f. When N is a point, this notion reduces to that of Lie algebra.

For any Poisson manifold (M,π), T^*M is naturally a Lie algebroid [12, §13] with anchor $-\sharp \colon T^*M \to TM$, and bracket determined by $[df, dg] := d\{f,g\}$.

A *Dirac manifold* [9, §2.2] is a manifold P together with a subbundle $L \subset TP \oplus T^*P$ which is maximal isotropic w.r.t. the pairing $\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2)$ and whose sections are closed under the Courant bracket

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2] = \big([X_1, X_2] \oplus \mathscr{L}_{X_1}\xi_2 - \mathscr{L}_{X_2}\xi_1 + \frac{1}{2} d(i_{X_2}\xi_1 - i_{X_1}\xi_2)\big). \quad (1)$$

on $\Gamma(TP \oplus T^*P)$. Given any submanifold $N \subset (P,L)$, one can pull back the Dirac structure L to N, by defining $L_N = L \cap (TN \oplus T^*M|_N)/L \cap TN^\circ$. This subset of $TN \oplus T^*N$ is not necessarily a smooth subbundle, but when it is, it is automatically a Dirac structure on N [9, §3.1].

For any Poisson manifold (M,π), $L := graph(\pi) := \{(\sharp\xi, \xi) : \xi \in T^*M\}$ is a Dirac structure. Viewing a Poisson manifold as a Dirac manifold has the advantage that, even though we can not restrict the Poisson bivector field π to a submanifold $N \subset M$ (except when N is a Poisson submanifold), N is always endowed with the geometric structure L_N, which is a Dirac structure whenever L_N is a smooth subbundle.

2 Coisotropic submanifolds

In this section we elaborate on coisotropic submanifolds [12, §6.4].

Definition 4 $N \subset (M,\pi)$ is a *coisotropic submanifold* iff $\sharp TN^\circ \subset TN$.

Let N be a submanifold of (M,π). Then

$$\mathscr{I} := \{f \in C^\infty(M) : f|_N = 0\}$$

is a multiplicative ideal of the Poisson algebra $C^\infty(M)$. The submanifolds for which $\mathscr{I}$ is also a Poisson subalgebra (i.e., $\{\mathscr{I},\mathscr{I}\} \subset \mathscr{I}$) are exactly the coisotropic submanifolds. Those satisfying the stronger condition that $\mathscr{I}$ is a Poisson ideal (i.e., $\{\mathscr{I},C^\infty(M)\} \subset \mathscr{I}$) are exactly the Poisson submanifolds.

In the physics literature, sometimes submanifolds are specified by constraints, i.e. open subsets $\{U_\alpha\}$ of M and, for each α, independent functions $\varphi^1_\alpha,\dots,\varphi^k_\alpha$ defined on U_α such that $N \cap U_\alpha$ is the common zero set of $\varphi^1_\alpha,\dots,\varphi^k_\alpha$. As we just saw, the coisotropic submanifolds are exactly those given by so-called *first class constraints*, i.e. constraints satisfying $\{\varphi^i,\varphi^j\}|_N = 0$.

Example 2

a) Poisson submanifolds are coisotropic.

b) If $\varphi\colon (M_1,\pi_1) \to (M_2,\pi_2)$ is a Poisson morphism, then its graph is coisotropic in $(M_1 \times M_2, \pi_1 - \pi_2)$.

c) If $\mathfrak{h}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}$, then $\mathfrak{h}^\circ$ is a coisotropic submanifold of $\mathfrak{g}^*$. (See Ex. 5 for an extension of this example.)

Remark 1 The intersection of a coisotropic submanifold with the symplectic leaves $\mathscr{O}$ of M is usually not clean: for instance, the symplectic foliation of $\mathfrak{su}(2)^*$ is given by concentric spheres in $\mathbb{R}^3$. Any plane N in $\mathfrak{su}(2)^*$ not containing the origin is coisotropic, has a (unique) point p at which N is tangent to a symplectic sphere $\mathscr{O}$, and at that point $T_p(N \cap \mathscr{O}) \neq T_pN \cap T_p\mathscr{O}$.

Coisotropic submanifolds enjoy nice properties: their conormal bundle TN° is a Lie subalgebroid of T^*M, and they admit a natural quotient which is again a Poisson manifold, provided it is smooth. We will discuss these properties for the more general class of pre-Poisson submanifolds in §4.3 and §4.2 respectively.

3 Poisson-Dirac submanifolds

Poisson-Dirac submanifolds, introduced by Crainic and Fernandes [10, §8] in 2002, are the submanifolds of (M,π) which have a canonically induced Poisson structure.

Definition 5 [10, Def. 4, §8]. N is a *Poisson-Dirac submanifold* of (M,π) if it has a Poisson structure such that:

(i) N intersects cleanly[2] the symplectic leaves of M, and the symplectic leaves of N are the connected components of $N\cap\mathscr{O}$ as $\mathscr{O}$ ranges over all symplectic leaves of M,

(ii)$N\cap\mathscr{O}$ is a symplectic submanifold of $\mathscr{O}$, for every symplectic leaf $\mathscr{O}$ of M .

An alternative characterization, along the lines of our reasoning at the end of §1, is the following. Let N be a submanifold of M such that, for any $p\in N$, $T_pN\cap T_p\mathscr{O}$ is a symplectic subspace of $(T_p\mathscr{O},\omega_p)$. Here $(\mathscr{O},\omega)$ denotes the symplectic leaf through p. Then the restriction of ω_p to $T_pN\cap T_p\mathscr{O}$ is a non-degenerate bilinear form, and inverting it we obtain a bivector $(\pi_N)_p\in\wedge^2T_pN$. Notice that in general π_N is not a smooth section of $\wedge^2TN$ (see [10, Ex. 3, §8]).

Definition 6 [10, Cor. 11, §8] $N\subset(M,\pi)$ is a *Poisson-Dirac submanifold* iff $\sharp TN^\circ\cap TN=\{0\}$ and the induced tensor π_N on N is smooth.

In that case, π_N is automatically a Poisson tensor [10, Prop. 6, §8]. The name "Poisson-Dirac" derives from the fact that $graph(\pi_N)$ is equal to L_N, the Dirac structure obtained pulling back $graph(\pi)$ via the inclusion $N\hookrightarrow M$.

Any submanifold N such that $\sharp TN^\circ\cap TN=\{0\}$ and for which $\sharp TN^\circ$ has constant rank, is automatically Poisson-Dirac[3]. Indeed the latter condition implies that pulling back the Dirac structure $graph(\pi)$ we obtain a smooth subbundle of $TN\oplus T^*N$ (see §1.2), which hence is the graph of a smooth bivector field on N. In this case the Poisson bracket on N is computed as follows:

$$\{f,g\}:=\{\hat{f},\hat{g}\}|_N$$

where $\hat{f},\hat{g}\in C^\infty(M)$ are extensions of $f,g\in C^\infty(N)$ such that $df|_{\sharp TN^\circ}=0$.

---

[2] This means that $N\cap\mathscr{O}$ is a manifold with $T(N\cap\mathscr{O})=TN\cap T\mathscr{O}$.

[3] It also falls into the more restrictive class of quasi-Dirac submanifolds [13, Def. 2.2], see also [10, Prop. 7, §8].

Examples of Poisson-Dirac submanifolds are:

Example 3

a) If (M,ω) is a symplectic manifold, then a submanifold N is Poisson-Dirac iff it is a symplectic submanifold.

b) Poisson submanifolds.

c) Lie-Dirac submanifolds, in particular cosymplectic submanifolds. We will elaborate on them in §3.1 and §3.2.

Within the class of Poisson-Dirac submanifolds, the cosymplectic ones and the Poisson submanifolds lie at opposite extremes: for the former the rank of $\sharp TN^\circ$ is maximized, for the latter it is zero.

3.1 Lie-Dirac submanifolds

Lie-Dirac submanifolds were introduced by Xu[4] in 2001 [15]. They are special cases of Poisson-Dirac submanifolds [10, §8.3].

Definition 7 [15, Def. 2.1] $N \subset (M,\pi)$ is a *Lie-Dirac submanifold* iff there exists a subbundle E with $TM|_N = TN \oplus E$ such that E° is a Lie subalgebroid of T^*M.

Recall [11, Def. 4.3.14] that if $A \to M$ is a Lie algebroid with anchor ρ, a subbundle $B \to N$ is a *Lie subalgebroid* if $\rho(B) \subset TN$ and for all sections X,Y of A one has $(X|_N, Y|_N \in \Gamma(B) \Rightarrow [X,Y]|_N \in \Gamma(B))$ and $(X|_N = 0, Y|_N \in \Gamma(B) \Rightarrow [X,Y]|_N = 0)$.

The embedding $T^*N \to T^*M$, given by the canoncial identification between the vector bundles T^*N and E°, is actually a morphism of Lie algebroids, giving rise to a Lie subalgebroid of T^*M [15, Thm. 2.3 iii)]. (Here the Lie algebroid structures or T^*N and T^*M are those given by the Poisson bivector fields on N and M.) The fact that Lie-Poisson submanifolds come with a canonical Lie subalgebroid of T^*M accounts for several good properties of Lie-Poisson submanifolds, see for example Prop. 4.

A characterization in terms of functions is

[4] Xu introduced them with the name "Dirac submanifolds"; the name "Lie-Dirac" was proposed in [10].

Definition 8 [13, Def. 2.1] $N \subset (M,\pi)$ is a *Lie-Dirac submanifold* iff there exists a subbundle E containing $\sharp TN^\circ$ for which $TM|_N = TN \oplus E$, such that for all $f,g \in C^\infty(M)$

$$df|_E = 0, dg|_E = 0 \Rightarrow d\{f,g\}|_E = 0.$$

Being a Lie-Dirac submanifold is global property of the submanifold N: if we can find subbundles as above on open subsets of N, in general we can not glue them into a subbundle E over N as above.

Example 4 a) Points of Poisson manifolds are Lie-Dirac submanifolds.

b) Cosymplectic submanifolds, which we will introduce in §3.2, are Lie-Dirac submanifolds [15, Cor 2.11].

c) Symplectic leaves of Poisson manifolds are usually *not* Lie-Dirac submanifolds. For instance, the symplectic foliation of $\mathfrak{su}(2)^*$ consist of concentric spheres, and among these only the origin is a Lie-Dirac submanifold. The exact obstruction for regular[5] symplectic leaves is given in [10, Cor 13, §8], see also [15, Ex. 2.17].

3.2 Cosymplectic submanifolds

The notion of cosymplectic submanifold is much older than that of Poisson-Dirac or Lie-Dirac submanifold.

Definition 9 [14, §1] $N \subset (M,\pi)$ is a *cosymplectic submanifold* iff $\sharp TN^\circ \oplus TN = TM|_N$.

Hence cosymplectic submanifolds are exactly the submanifolds given by *second class constraints*, i.e. constraints $\{\varphi^A\}$ such that $\{\varphi^A, \varphi^B\}|_p$ is an invertible matrix at all points $p \in N$. This follows from the fact for any Poisson-Dirac submanifold $\sharp TN^\circ$ is a symplectic subbundle, see Lemma 1.1.

Cosymplectic submanifolds constitute a useful tool in hamiltonian mechanics. Let (M,π) be the Poisson manifold representing the phase space of a physical system. Sometimes the physical system is constrained to a submanifold $N \subset M$ with an induced Poisson structure (a Poisson-Dirac submanifold), and one would like to

[5] I.e., leaves such that all the symplectic leaves in a neighborhood have the same dimension.

express the induced Poisson bracket $\{\cdot,\cdot\}_N$ on N in terms of the Poisson bracket $\{\cdot,\cdot\}$ on M.

The case when N is cosymplectic is well-known in the physics literature, and has been threated using the so-called Dirac bracket. We describe it as follows. Let $\varphi^1,\dots,\varphi^k$ be constraints for the cosympectic submanifold N defined on an open subset $U \subset M$. Since the matrix $C^{AB} := \{\varphi^A,\varphi^B\}$ is invertible on $N \cap U$, we may assume that it is invertible on U, shrinking U if necessary. We denote its inverse by C_{AB}. The *Dirac bracket* is the bracket on $C^\infty(U)$ defined by

$$\{f,g\}_{Dirac} := \{f,g\} - \{f,\varphi^A\}C_{AB}\{\varphi^B,g\}. \tag{2}$$

It is a Poisson bracket, and it allows to recover easily the bracket $\{\cdot,\cdot\}_N$ on N: the latter is computed extending *in any arbitrary way* functions on N to functions on M and taking their Dirac bracket. (Notice that computing $\{\cdot,\cdot\}_N$ by means of the Poisson bracket $\{\cdot,\cdot\}$ on M, as in §3, requires *specific* extensions of the functions on N: the extensions must annihilate $\sharp TN^\circ$.)

We explain the above statement as follows. Denote by π_{Dirac} the Poisson bivector field U given by the Dirac bracket. It can be shown [4, §5.1] that the level sets of the constraints (in particular N) are cosymplectic submanifolds of (M,π) and also Poisson submanifolds of (U,π_{Dirac}), and that the Poisson structures on the level sets induced by π and π_{Dirac} coincide.

4 Pre-Poisson submanifolds

Given a symplectic manifold (X,Ω), a submanifold $\iota: C \hookrightarrow X$ is called *presymplectic* if the characteristic distribution $ker(\iota^*\Omega) = TC \cap TC^\Omega$ has constant rank along C, or equivalently if $TC + TC^\Omega$ has constant rank. In this Section we consider an extension of the notion of presymplectic submanifold to Poisson geometry, and in the three Subsections we establish various interesting properties.

Let (M,π) be a Poisson manifold and N a submanifold. It is natural to consider the kernel of $\omega|_{T_pN\cap T_p\mathscr{O}}$, where ω is the symplectic form on the symplectic leaf $\mathscr{O}$ through p, and impose that it have constant rank for all $p \in N$. By Lemma 1.1 this amounts to asking that $char(N) := TN \cap \sharp TN^\circ$ has constant rank along N. This turns out not to be a good notion. For instance, $char(N) \subset TM|_N$ may have constant rank but fail to be a smooth subbundle of $TM|_N$ (see Ex. 5.7 of [7]).

Instead of the intersection of TN and $\sharp TN^\circ$, it is better to consider their sum:

Definition 10 [7, Def. 2.2] A submanifold N of a Poisson manifold (M,π) is called *pre-Poisson* if the rank of $TN+\sharp TN^{\circ}$ is constant along N.

Such submanifolds were first considered by Calvo and Falceto [2, 3] in 2004, and studied by Cattaneo and the author in [6],[7]. A first good property of $TN+\sharp TN^{\circ}$ is the following: if the rank of $\sharp TN^{\circ}+TN$ is constant, then it is automatically a smooth subbundle of $TM|_N$, because smooth sections spanning $\sharp TN^{\circ}+TN$ can be easily constructed from a smooth frame for TN and the image under $\sharp|_N$ of a smooth frame for TN°.

Example 5 1) If (M,ω) is a symplectic manifold, a submanifold N is pre-Poisson iff it is presymplectic.

2) Coisotropic submanifolds (see §2) are pre-Poisson.

3) Poisson-Dirac submanifolds (see §3) or even Lie-Dirac submanifolds (see §3.1) are usually *not* pre-Poisson, but cosymplectic submanifolds (see §3.2) are.

4) Let $\mathfrak{h}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$ and fix $\lambda\in\mathfrak{g}^*$. Then the affine subspace $\mathfrak{h}^{\circ}+\lambda$ is pre-Poisson [6, §5].

To put into perspective Def. 10, let N be an arbitrary submanifold of (M,π) and consider three "singular subbundles":

- $\sharp TN^{\circ}$
- $char(N)=TN\cap\sharp TN^{\circ}$
- $TN+\sharp TN^{\circ}$.

The first two are the domain and kernel respectively of

$$\varphi\colon \sharp TN^{\circ}\to \nu N,$$

the restriction of the projection $pr_{\nu N}\colon TM|_N\to \nu N:=TM|_N/TN$. The image of φ is $pr_{\nu N}(\sharp TN^{\circ}+TN)$. Hence it is clear that

Lemma 4.2 *Let N be a submanifold of M. Whenever any two of $char(N)$, $\sharp TN^{\circ}$, $\sharp TN^{\circ}+TN$ have constant rank, then the remaining one also does.*

We elaborate on the properties that N has when one of the three above "singular subbundles" has constant rank. By definition Pre-Poisson submanifolds are exactly those for which $\sharp TN^\circ + TN$, or equivalently the image $Im(\varphi)$, has constant rank. The following table taken from [7][6] characterizes submanifolds of symplectic and Poisson manifolds in terms of $Im(\varphi)$:

	M symplectic	M Poisson
$Im(\varphi) = 0$	N coisotropic	N coisotropic
$Im(\varphi) = \nu N$	N symplectic	N cosymplectic
$Rk(Im(\varphi)) =$const	N presymplectic	N pre-Poisson

If $\sharp TN^\circ$ has constant rank, then pulling back the Dirac structure $graph(\pi)$ via the inclusion $N \hookrightarrow M$ one obtains a smooth Dirac structure on N.

When $char(N)$ has constant rank and is smooth, then $char(N)$ is an involutive distribution on N, whose quotient (when smooth) has nice properties, see §4.2.

4.1 Embeddings of pre-Poisson submanifolds

In this Subsection we show that pre-Poisson submanifolds of (M,π) can be regarded as coisotropic ones (in some other Poisson manifold), and hence share many properties of coisotropic submanifolds.

Given a pre-Poisson submanifold N, one can find constraints (defined on some open subset $U \subset M$) that are split into first and second class constraints [3, §2.1]. More precisely, choose constraints $\{\varphi^\nu\}$ such that $d\varphi^\nu|_{\sharp TN^\circ} = 0$, and complete by adding other constraints $\{\varphi^A\}$. The map $\sharp\colon TN^\circ \to \sharp TN^\circ$ maps $\sharp d\varphi^\nu$ into TN, so the φ^ν are first class constraints (i.e., $\{\varphi^\nu, \varphi^\mu\}$ and $\{\varphi^\nu, \varphi^A\}$ vanish along $U \cap N$). Further it maps $span\{d\varphi^A\}$ isomorphically onto a complement W of $TN \cap \sharp TN^\circ$ in $\sharp TN^\circ$, and in the basis of W dual to $d\varphi^A|_W$ the isomorphism is represented by the matrix $\{\varphi^A, \varphi^B\}$. So the φ^A are second class constraints (i.e., the matrix $\{\varphi^A, \varphi^B\}$ is non-degenerate along $U \cap N$).

The zero level set of the second class constraints φ^A is a cosymplectic submanifold $\tilde{M}$ of (M,π), see §3.2. The submanifold $U \cap N \subset \tilde{M}$ is given by the remaining constraints $\varphi^\nu|_{\tilde{M}}$, which are first class, hence $U \cap N$ is a coisotropic submanifold of $\tilde{M}$.

The above argument is a local one. One can show that the result holds globally, with a uniqueness statement:

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[6] [7] considers the map $pr_{\nu N} \circ \sharp\colon TN^\circ \to \nu N$, whose image is of course the same as the one of φ.

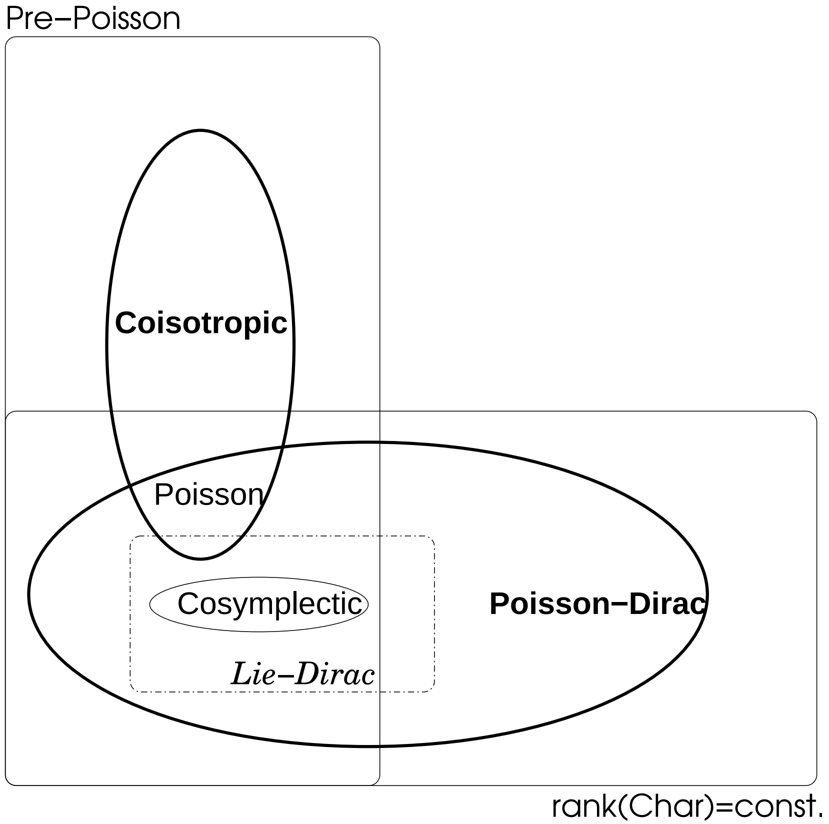

Fig. 1 Relation between the classes of submanifolds considered in this note. Recall that a submanifold N is Pre-Poisson iff $rank(\sharp TN^\circ + TN) = const.$ and that $char(N) = \sharp TN^\circ \cap TN$.

Proposition 1 *[7, Thm 3.3 and Thm. 4.3] Let N be a pre-Poisson submanifold of a Poisson manifold (M,π). Then there exists a cosymplectic submanifold $\tilde{M}$ containing N such that N is coisotropic in $\tilde{M}$.*

Further $\tilde{M}$ is unique up to neighborhood equivalence: if $\tilde{M}_0$, $\tilde{M}_1$ are cosymplectic submanifolds that contain N as a coisotropic submanifold then, shrinking $\tilde{M}_0$ and $\tilde{M}_1$ to a smaller tubular neighborhood of N if necessary, there is a Poisson diffeomorphism from $\tilde{M}_0$ to $\tilde{M}_1$ which is the identity on N.

The above proposition does not imply that all questions involving pre-Poisson submanifolds can be reduced to questions about coisotropic ones. For instance in [2, §6] the authors consider two distinct pre-Poisson submanifolds N_1 and N_2 with non-empty intersection, and in general it is not possible to find a cosymplectic submanifold containing coisotropically both N_1 and N_2.

4.2 Quotients of pre-Poisson submanifolds

In this Subsection we show that every submanifold N of a Poisson manifold has an associated "reduced" Poisson algebra which – when certain assumptions on N are satisfied – corresponds to the quotient of N by $char(N) = TN \cap \sharp TN^\circ$. We follow [7, §6].

For any submanifold N of (M,π), consider again the multiplicative ideal $\mathscr{I} := \{f \in C^\infty(M) : f|_N = 0\}$ of the Poisson algebra $C^\infty(M)$. Its Poisson normalizer

$$\mathscr{F} := \{\hat{f} \in C^\infty(M) : \{\hat{f}, \mathscr{I}\} \subset \mathscr{I}\}$$

is a Poisson subalgebra of $C^\infty(M)$, and by construction $\mathscr{F} \cap \mathscr{I}$ is a Poisson ideal in $\mathscr{F}$. Hence the quotient $\mathscr{F}/(\mathscr{F} \cap \mathscr{I})$ is a Poisson algebra. Notice that $\mathscr{F}/(\mathscr{F} \cap \mathscr{I})$ is exactly the subset of functions f on N which admits an extension to some function $\hat{f} \in C^\infty(M)$ whose differential annihilates $\sharp TN^\circ$ (or equivalently $X_{\hat{f}}|_N \subset TN$). In geometric terms, the induced Poisson bracket on $\mathscr{F}/(\mathscr{F} \cap \mathscr{I})$ is computed as follows:

$$\{f,g\} = \{\hat{f}, \hat{g}\}|_C = X_{\hat{f}}(g)|_N$$

for extensions $\hat{f}, \hat{g} \in \mathscr{F}$, where the second Poisson bracket is the one on $C^\infty(M)$.

On the other hand $char(N)_p \subset T_pN$ is the kernel of the bilinear form $\iota^*\omega_p$, where $(\mathscr{O}, \omega)$ is the symplectic leaf of M through p and $\iota \colon N \cap \mathscr{O} \hookrightarrow \mathscr{O}$ the inclusion. Hence, from a geometric point of view, it is natural to consider the set of basic functions on N, i.e.

$$C^\infty_{bas}(N) = \{f \in C^\infty(N) : df|_{\sharp TN^\circ \cap TN} = 0\}.$$

When $char(N)$ is regular and smooth and the quotient $\underline{N}$ is a smooth manifold, then $C^\infty_{bas}(N)$ is isomorphic to $C^\infty(\underline{N})$.

In general we have $\mathscr{F}/(\mathscr{F} \cap \mathscr{I}) \subset C^\infty_{bas}(N)$. When N is a pre-Poisson submanifold one has equality [2, Thm. 3]. Hence, for pre-Poisson submanifolds, the set of basic functions has a Poisson algebra structure, and whenever the quotient $\underline{N}$ is a smooth manifold, it has an induced Poisson structure.

4.3 Relation to subgroupoids of $\Gamma(M)$

Generalizing the fact that Lie algebras are the infinitesimal objects associated to Lie groups, Lie algebroids (see §1) are associated to so-called Lie groupoids [12, §13]. A *groupoid* is a category (so in particular it consists of a set of arrows with two maps $\mathbf{s}$ and $\mathbf{t}$ to the set of objects) where every arrow is invertible. For *Lie groupoids* we require that the sets involved in the definition be manifolds, the maps be smooth, and $\mathbf{s},\mathbf{t}$ surjective submersions.

Let (M,π) be a Poisson manifold. When certain obstructions vanish [10, Thm. 2], there exists a Lie groupoid whose Lie algebroid is T^*M. There exists a unique (up to isomorphism) such Lie groupoid $\Gamma(M)$ whose $\mathbf{s}$-fibers are simply connected. $\Gamma(M)$ is actually a *symplectic groupoid* [8], i.e. it carries a symplectic form Ω such that the graph of the multiplication (composition of arrows) in $(\Gamma(M)\times\Gamma(M)\times\Gamma(M),\Omega\times\Omega\times(-\Omega))$ is Lagrangian, and so that the target map $\mathbf{t}\colon\Gamma(M)\to M$ is a Poisson map. For instance, if (M,ω) is a simply connected symplectic manifold, then $(\Gamma(M),\Omega)=(M\times M,\omega\times(-\omega))$, and the groupoid multiplication of $\Gamma(M)$ is given by $(x,y)\cdot(y,z)=(x,z)$.

Assume that the Poisson manifold (M,π) admits a symplectic groupoid $\Gamma(M)$. "Nice" classes of (immersed) subgroupoids of $\Gamma(M)$ are given by the subgroupoids which are coisotropic or symplectic submanifolds, or more generally presymplectic submanifolds. It is natural to ask which classes of submanifolds of M are the bases (sets of objects) of "nice" subgroupoids of $\Gamma(M)$. Given a submanifold $N\subset M$, any Lie subalgebroid of T^*M over N must be contained in $\sharp^{-1}TN$ (otherwise there is no induced anchor). Further, the only subbundle of T^*M naturally associated to the submanifold N is TN°. Hence we are lead to consider

- $TN^\circ\cap\sharp^{-1}TN$ (it has constant rank iff N a pre-Poisson submanifold)
- $\sharp^{-1}TN$ (it has constant rank iff its annihilator $\sharp TN^\circ$ does).

When they have constant rank, they are automatically Lie subalgebroids of T^*M [7, Prop. 3.6]. Now we look at the corresponding subgroupoids of $\Gamma(M)$.

Considering the Lie subalgebroid $TN^\circ\cap\sharp^{-1}TN$ we have:

Proposition 2 *[7, Prop. 7.2] Let N be a pre-Poisson submanifold of (M,π). Then the subgroupoid of $\Gamma(M)$ integrating $TN^\circ\cap\sharp^{-1}TN$ is an isotropic subgroupoid of $\Gamma(M)$.*

The above subgroupoid is Lagrangian exactly when N is coisotropic [5, §5]. (In [5] this correspondence is the main tool to show that the integration of Poisson manifolds can be derived from the one of Lie algebroids). When N is cosymplectic, then the above subgroupoid is the trivial groupoid $N \rightrightarrows N$.

Next assume that $\sharp TN^\circ$ has constant rank and consider the Lie subalgebroid $\sharp^{-1}TN$. A subgroupoid of $\Gamma(M)$ integrating it is $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$.

Remark 2 We saw that the graph of π pulls back to a smooth Dirac structure on N. It can be shown [7, Rem. 7.3] that $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$, with the restriction of the sympletic form Ω on $\Gamma(M)$, is an over-pre-symplectic groupoid inducing the same Dirac structure on N [1, Ex. 6.7].

Proposition 3 *[7, Prop. 7.5] Let N be any submanifold of M. Then $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$ is a presymplectic[7] submanifold of $\Gamma(M)$ iff N is pre-Poisson and $char(N)$ has constant rank. In this case the characteristic distribution of $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$ has rank $2rk(char(N))+rk(TN^\circ\cap T\mathcal{O}^\circ)$, where $\mathcal{O}$ denotes the symplectic leaves of M intersecting N.*

We have the following special cases: if N is coisotropic and $\sharp TN^\circ$ has constant rank, then $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$ is also coisotropic; if N is cosymplectic, then $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$ is also cosymplectic [7, Lemma. 7.1].

When N is pre-Poisson and $char(N)$ has constant rank, the quotient $\underline{N}$ of N by $char(N)$ (when smooth) is a Poisson manifold, see §4.2. As seen in Prop. 3, $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$ is a Lie subgroupoid and a presymplectic submanifold of $\Gamma(M)$. When the quotient $\underline{\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)}$ by its characteristic distribution (i.e., the leaf-space) is smooth, one expects it[8] to be a symplectic groupoid for $\underline{N}$.

The following example, which is the only original contribution of present note, shows that this is *not* the case: $\underline{\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)}$ usually is not even a set-theoretic groupoid.

Example 6 Let N be the trivial circle bundle over the open 2-disk D, but with one point removed in the fiber over $0\in D$. We write suggestively $N = D\times\hat{S^1}$, where $\hat{S^1}_p$ denotes the circle for all non-zero $p\in D$, while $\hat{S^1}_0$ is the circle with a point

[7] Recall that a submanifold S of the symplectic manifold $(\Gamma(M),\Omega)$ is presymplectic iff its characteristic distribution $TS\cap TS^\Omega$ has constant rank.

[8] For N a Poisson-Dirac submanifold this was already pointed out in [10, §8].

deleted. Notice that $\pi_1(N) = \mathbb{Z}$, generated by any of the circle fibers. It is easy to see that the universal cover of N is $\tilde{N} = (D \times \mathbb{R}) - (\{0\} \times \mathbb{Z})$. To emphatize the fact that $\tilde{N}$ is a bundle over D we write $N = D \times \hat{\mathbb{R}}$, where $\hat{\mathbb{R}}_p = \mathbb{R}$ for non-zero $p \in D$ and $\hat{\mathbb{R}}_0 = \mathbb{R} - \mathbb{Z}$.

Now we bring in Poisson structures. Let $M = D \times \hat{S^1} \times I$, where I is the open interval, and endow it with the symplectic structure Ω obtained as the product of the symplectic structure on the disk D and the (restriction of) the symplectic structure on $S^1 \times I$. The symplectic groupoid of (M, Ω) is $\Gamma(M) = (\tilde{M} \times_{\mathbb{Z}} \tilde{M}, \mathbf{t}^*\Omega - \mathbf{s}^*\Omega)$, where $\tilde{M}$ denotes the universal cover of M and the action of $\pi_1(M) = \mathbb{Z}$ is by diagonal deck transformations.

We view N as a submanifold of M; it is a presymplectic submanifold, and clearly $\underline{N} \cong D$. We have

$$\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N) = (D \times \hat{\mathbb{R}}) \times_{\mathbb{Z}} (D \times \hat{\mathbb{R}}).$$

The characteristic leaves of $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ almost coincide with the fibers of the natural projection onto $D \times D$: the characteristic leaves are[9] $F_{(p_1,p_2)} = \{[(p_1, t_1, p_2, t_2)] : t_1, t_2 \in \mathbb{R}\}$ if $(p_1, p_2) \neq (0,0) \in D \times D$ (topologically these are either cylinders or rectangles), whereas sitting over $(0,0) \in D \times D$ we have the quotient of $(\mathbb{R} - \mathbb{Z}) \times (\mathbb{R} - \mathbb{Z})$ by the diagonal $\mathbb{Z}$ action, which consists of countably many leaves. Hence the leaf space is

$$\underline{\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)} = D \hat{\times} D,$$

where the latter denotes the non-Hausdorff manifold obtained from $D \times D$ replacing $(0,0)$ with a copy of $\mathbb{Z}$.

We ask whether the projection $pr\colon \mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N) \to \underline{\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)}$ induces a groupoid structure (over $\underline{N}$) on the quotient. We have well-defined source and target maps for $\underline{\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)}$, but the groupoid multiplication of $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ does not descend to the quotient. Indeed, consider $(0, p) \in D \hat{\times} D$ where p is non-zero. A preimage under pr is $[(0, \mu_1), (p, \lambda)]$ where $\mu_1 \in \mathbb{R} - \mathbb{Z}$ and $\lambda \in \mathbb{R}$ are arbitrary. Similarly, we consider $(p, 0) \in D \hat{\times} D$ and as a preimage we pick $[(p, \lambda), (0, \mu_2)]$ where again $\mu_2 \in \mathbb{R} - \mathbb{Z}$ is arbitrary. Now multiplying these two elements of $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ we obtain $[(0, \mu_1), (0, \mu_2)]$. The value of its projection under pr depends on the concrete choice of μ_1 and μ_2. This shows that $\underline{\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)}$ does not have an induced groupoid structure.

[9] Square brackets denote equivalence classes under the $\mathbb{Z}$-action on $(D \times \hat{\mathbb{R}}) \times (D \times \hat{\mathbb{R}})$.

In order to obtain a groupoid as a quotient of $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$, one would need to identify all the countably many characteristic leaves sitting over $(0,0) \in D \times D$.

To end with, we consider a Lie-Dirac submanifold N. By the very definition (see §3.1) there is a canonical embedding[10] of Lie algebroids $\varphi \colon T^*N \to T^*M$, giving rise to a subgroupoid of $\Gamma(M)$. We have

Proposition 4 *[15, Thm. 3.7] If $\Gamma' \rightrightarrows N$ is a symplectic subgroupoid of $\Gamma(M)$ then N is a Lie-Dirac submanifold of (M,π). Conversely, if N is a Lie-Dirac submanifold of (M,π), then $\varphi(T^*N)$ integrates to a symplectic subgroupoid of $\Gamma(M)$.*

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