Tricky Asymptotics Fixed Point Notes.
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Abstract

In this notes we analyze an example of a linearly degenerate critical point, illustrating some of the standard techniques one must use when dealing with nonlinear systems near a critical point. For a particular value of a parameter, these techniques fail and we show how to get around them. For ODE's the situations where standard approximations fail are reasonably well understood, but this is not the case for more general systems. Thus we do the exposition here trying to emphasize generic ideas and techniques, useful beyond the context of ODE's.

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1 Introduction.

Here we consider some subtle issues that arise while analyzing the behavior of the orbits near the (single, thus isolated) critical point at the origin of the *Dipole Fixed Point* system (see problem 6.1.9 in Strogatz book)

$$\frac{dx}{dt} = \frac{2}{n}xy\,, \qquad \text{and} \qquad \frac{dy}{dt} = y^2 - x^2\,, \tag{1.1}$$

where $0 < n \leq 2$ is a constant. Our objective is to illustrate how one can analyze the behavior of the orbits near this **linearly degenerate critical point** and arrive at a qualitatively[1] correct description of the phase portrait. We will use for this "standard" asymptotic analysis techniques.

The *case $n = 2$ is of particular interest, because then the* **standard techniques fail**, and some extra tricks are needed to make things work.

Just so we know what we are dealing with, a computer made phase portrait for the system[2] (case $n = 1$) is shown in figure 1.1. Other values of $0 < n \leq 2$ give qualitatively similar pictures. However, for $n > 2$ there is a qualitative change in the picture. We will not deal with the case $n > 2$ here, but the analysis will show how it is that things change then. The threshold between the two behaviors is precisely the tricky case where "standard" asymptotic analysis techniques do not work.

2 Qualitative analysis.

We begin by searching for invariant curves, symmetries, nullclines, and general "orbit shape" properties for the system in (1.1).

A. Symmetries. The equations in (1.1) are invariant under the transformations:[3]

A1. $x \longrightarrow -x$

A2. $y \longrightarrow -y$ and $t \longrightarrow -t$

A1 and **A2** show that **we need only study the behavior of the equation in the quadrant** $x \geq 0$, $y \geq 0$.

A3. $x \longrightarrow ax$, $y \longrightarrow at$, and $t \longrightarrow t/a$, for any constant $a > 0$.

[1] **With quantitative extra information.**

[2] The analysis will, however, proceed in a form that is independent of the information shown in this picture.

[3] Notice that these types of invariances occur as a rule when analyzing the "leading order" behavior near degenerate critical points; because such systems tend to have homogeneous simple structures.

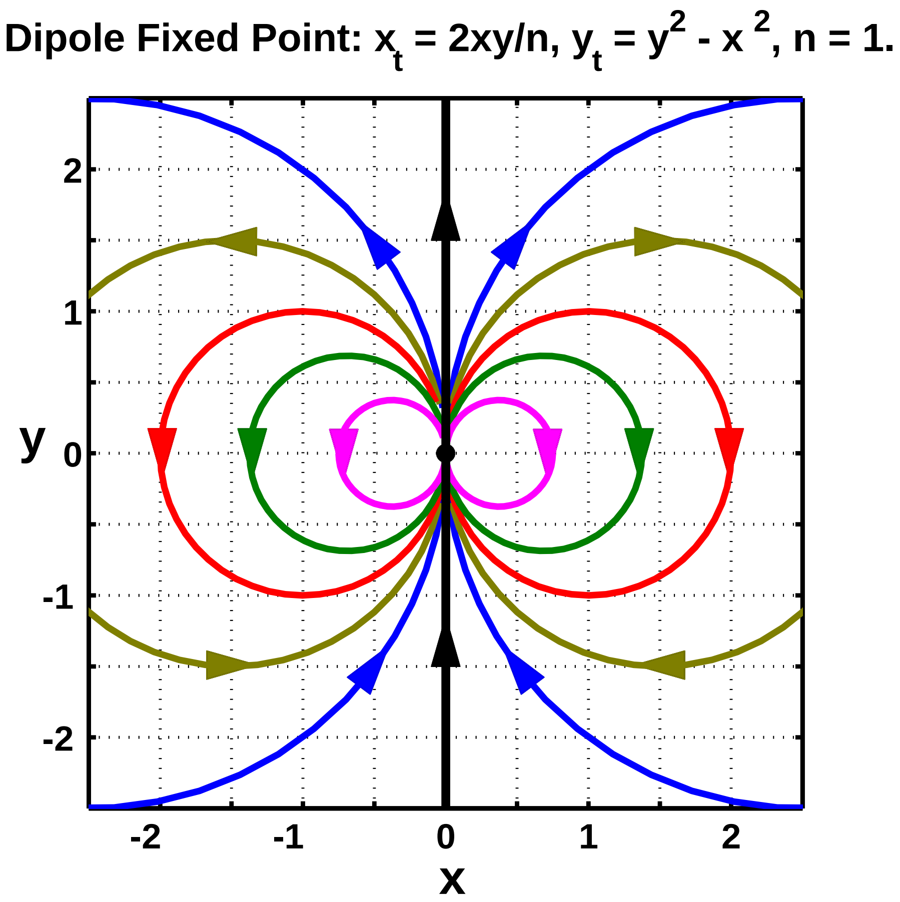

Figure 1.1: Phase plane portrait for the *Dipole Fixed Point* system (1.1) for $n = 1$. The qualitative details of the portrait do not change in the range $0 < n \leq 2$. However, for $n > 2$ differences arise.

The last set of symmetries (**A3**) shows that we need only compute a few orbits, since **once we have one orbit, we can get others by expanding/contracting it by arbitrary factors** $a > 0$**.** *Note that we say "a few" here, not "one"!* This is because the expansion/contractions of a single orbit **need not** fill up the whole phase space, but just some fraction of it. A particularly extreme example of this can be seen in figure 1.1, where the orbit given by $y > 0$ and $x \equiv 0$ simply gives back itself upon expansion. On the other hand, we will show that any of the orbits on $x > 0$ (or $x < 0$) gives all the orbits on $x > 0$ (respectively, $x < 0$) upon expansion/contraction.[4] Actually: *this is, precisely, the property that is lost for* $n > 2$*!*

Note: **(A2)** shows that this system is reversible. On the other hand, because there are open

[4]It even gives the special orbits on the y-axis by taking $a = \infty$, and the critical point by taking $a = 0$.

sets of orbits that are attracted by the critical point (we will show this later), the system is not conservative. In fact, this is an example of a reversible, non-conservative system with a minimum number of critical points.

B. Simple invariant curves. The y-axis ($x \equiv 0$) is an invariant line. Along it the flow is in the direction of increasing y, with vanishing derivative at the origin only. This invariant line is clearly seen in figure 1.1.

For $n > 2$, two further (simple) invariant lines are: $y = \pm \dfrac{\sqrt{n}}{\sqrt{n-2}}\, x$.

Whenever a one parameter family of symmetries exist (such as **(A3)**), you should look for invariant curves that are invariant under the whole family. In this case, this means looking for straight lines (which is what we just did.)

C. Nullclines. The nullclines are given by

C1. The x-axis ($y \equiv 0$), where $\dot{x} = 0$ (and, for $x \neq 0$, $\dot{y} < 0$.)

C2. The y-axis ($x \equiv 0$), where $\dot{x} = 0$ (and, for $y \neq 0$, $\dot{y} > 0$.)

C3. The lines $y = \pm x$, where $\dot{y} = 0$. In the first quadrant we also have $\dot{x} > 0$ here.

D. Orbit shape properties. In the *first quadrant* (from (**A**) above, it is enough to study this $x > 0$ and $y > 0$ quadrant only), consider the equation for the orbits

$$\frac{dy}{dx} = \frac{n(y^2 - x^2)}{2xy} = \frac{n}{2}\left(\frac{y}{x} - \frac{x}{y}\right). \tag{2.1}$$

A simple computation then shows that:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{n}{2}\left(\frac{1}{x} + \frac{x}{y^2}\right)\frac{dy}{dx} - \frac{n}{2}\left(\frac{y}{x^2} + \frac{1}{y}\right) \\ &= -\frac{n}{4x^2y^3}\Big((2-n)y^2 + nx^2\Big)\Big(y^2 + x^2\Big) < 0\,. \end{aligned} \tag{2.2}$$

This shows that the **orbits are (strictly) concave in this quadrant.** Note, however, that the inequality breaks down for $n > 2$. Then the orbits are concave for $(n-2)y^2 < nx^2$ and convex for $(n-2)y^2 > nx^2$.

All this information can now be put together, to obtain a **first approximation to what the phase portrait must look like,** as follows:

I. Region $0 < y < x$ $(\dot{y} < 0$ **and** $\dot{x} > 0.)$ The orbits enter this region (horizontally) across the nullcline $y = x$, bend down, and **must** eventually exit the region (vertically) across the nullcline $y = 0$. It should be clear that, once we show that one orbit exhibiting this behavior occurs, then all the others will be expansion/contractions of this one and, in particular, of each other (see **(A3).**)

The only point that must be clarified here is why we say above that the orbit "must eventually exit the region"? Why are we excluding the possibility that y will decrease, and x will increase, but in such a fashion that the orbit diverges to infinity, without ever making it to the x-axis? The answer to this is very simple: this would require the orbit to have an inflection point, which it cannot have.[5]

II. Region $0 < x < y$ $(\dot{y} > 0$ **and** $\dot{x} > 0.)$ Considering the flow backwards in time, we see that all the orbits that exit this region (horizontally, entering region I) across the nullcline $y = x$, must originate at the critical point.

However: *do all the orbits that originate at the critical point, exit this region across the nullcline $y = x$? Or is it possible for such an orbit to reach infinity without ever leaving this region?* — in fact, this is precisely what happens when $n > 2$, when all the orbits in the region $\sqrt{n-2}\,y > \sqrt{n}\,x$ do this. Figure 1.1 seems to indicate that this is not the case, but: how can be sure that a very thin pencil of orbits hugging the y-axis does not exist?

In section 3 we will show that **all the orbits leave the critical point with infinite slope (i.e.: vertically).** Consider now any orbit that exits this region through the nullcline $y = x$, and (we know) starts vertically at the critical point. We also know that all the expansions/contractions of this orbit must also be orbits (see **(A)**), and it should be clear that these will fill up this region completely (the fact that the orbit starts vertically is crucial for this.) But then there is no space left for the alternative type of orbits suggested in the prior paragraph, thus there are none. **This clarifies the point in the prior paragraph.**

[5] See **(D)** — notice that the orbits are always concave in this region, for *all* values of $n > 0$.

III. Conclusion. With this information, and using the symmetries in **(A)**, we can draw a qualitatively correct phase plane portrait, which will look as the one shown in figure 1.1. It should be clear from this figure that:

$$\boxed{\textbf{The index of the critical point is } I = 2.}$$

3 Quantitative analysis, and failure for $n = 2$.

Our aim in this section is to get some quantitative information about the orbits near the critical point. In particular, exactly how they approach or leave it.

Our approach below is "semi-rigorous", in the sense that we try to justify all the steps as best as possible, without going to "extremes" (whatever this means). 100% mathematical rigor in calculations like the ones that follow is possible in simple examples like the one we are doing — and not even very hard — but quickly becomes prohibitive as the complexity of the problems increases. But the type of techniques and way of thinking that we follow below remain useful well beyond the point where full mathematical rigor is currently achievable. Thus, provided one is willing to pay the price of not having the "absolute" certainty that full mathematical rigor gives, large gains can be made — while maintaining a "reasonable" level of certainty. This point of view is pretty close to the one adopted by Strogatz in his book.

We begin by showing the result announced (and used) towards the end of section 2, namely: that **all the orbits leave/approach the critical point vertically.** As before,

$$\boxed{\textbf{we restrict out attention to the first quadrant, and assume } x,\ y > 0.}$$

a. All the orbits must have a tangent limit direction as they approach the origin. This follows easily from the concavity of the orbits (see **(D)**): as $t \to -\infty$, the slope $\dfrac{dy}{dx}$ increases monotonically. Thus, it must have a well defined limit (which may be ∞; in fact, the aim here is to show that this limit is ∞.)

b. Suppose that there is an orbit that does not approach the critical point vertically. Then, the result in item **(a)** shows that we should be able to write

$$y \approx \alpha x\,, \quad \text{for } 0 < x \ll x\,, \tag{3.1}$$

where $1 \leq \alpha < \infty$ is a constant,[6] in fact $\alpha = \lim_{x \to 0} \dfrac{dy}{dx}$. Substitution of this into equation (2.1) then yields (upon taking the limit $x \to 0$)

$$\alpha = \frac{n}{2}\left(\alpha - \frac{1}{\alpha}\right) \quad \Longleftrightarrow (n-2)\alpha^2 = n\,, \tag{3.2}$$

which has no solution for $0 < n \leq 2$! It follows that an orbit approaching the critical point at a finite slope cannot occur — which is precisely what we wanted to show.

We now become more ambitious and ask the question: **How exactly do the orbits leave the critical point?** — that is to say: **What is the leading order behavior in their shape for** $0 < x << 1$**?** As we will show later (see remark 3.2), the answer to this question is useful in calculating the rate (in time) at which the solutions approach the critical point.

To answer this last question we proceed as follows: We know that the orbits have infinite slope near the critical point, thus we can write

$$y \gg x \quad \text{for } \; 0 < x \ll 1\,. \tag{3.3}$$

Using this, we should be able to replace equation (2.1) by the approximation

$$\frac{dy}{dx} \approx \frac{n\,y^2}{2xy} = \frac{ny}{2x}\,. \tag{3.4}$$

This yields

$$y \approx \beta x^{n/2}\,, \tag{3.5}$$

where β is a constant. This last step is not rigorous, by a long shot, and we must be a bit careful before accepting it. Equation (3.4) is correct (the neglected terms are smaller than the ones kept), but it is not clear that (upon integration) the neglected terms will not end up having a significant contribution to the solution of the equation.

Thus **before we accept equation (3.5) we must make some basic checks** (these sort of checks are important, you must always try to do as much as it is reasonable and you can do along these lines), **such as:**

[6]We know that $\alpha \geq 1$ because the orbit must leave the critical point staying above the line $y = x$.

c. Consistency with known facts. For example:

- **c1.** For $0 < n < 2$, (3.5) is consistent with (3.3).
- **c2.** For $n > 2$, (3.5) is **not** consistent with (3.3). However, our proof that the orbits approach the critical point vertically (which is what (3.3) is based on) does not apply for $n > 2$. In fact, for $n > 2$, (3.2) provides a very definite (neither infinite nor zero) direction of approach — which happens to agree with the invariant lines mentioned in **(B)** earlier. So, there is no contradiction (see remark 3.1 below for a brief description of what the situation is when $n > 2$.)
- **c3.** For $n = 2$, (3.5) is **not** consistent with (3.3). Since our proof that the orbits approach the critical point vertically (which implies (3.3)) does apply for $n = 2$, **we have a problem here, a rather tricky one, which we will address in section 4 below.**

d. Self-consistency (plug in the proposed approximation into the full equation and check that the neglected terms are indeed small). In this case the neglected term in the equation is $\dfrac{nx}{2y}$, which has size (using (3.5))

$$\frac{nx}{2y} = O(x^{(2-n)/2}), \quad \text{while} \quad \frac{dy}{dx} = \frac{ny}{2x} = O(x^{(n-2)/2}).$$

For the retained terms to be smaller than the neglected terms, we need $(2-n)/2 > (n-2)/2$, which is true only for $n < 2$. Thus **(3.5) is self-consistent only for $n < 2$.**

e. Estimate the error. That is, write the solution as

$$y = \beta x^{n/2} + y_1 ,$$

and assume $y_1 \ll \beta x^{n/2}$. Then use this to get an approximate equation for y_1, solve it, and check that, indeed: $y_1 \ll \beta x^{n/2}$.

In the case $0 < n < 2$ (the only one worth doing this for, since the other cases have already failed the two prior tests) one can do not only this, but repeat the process over and over again, obtaining at each stage higher order asymptotic approximations to the solution. That is, an asymptotic series of the form

$$y = \beta x^{n/2} + y_1 + y_2 + y_3 + \dots , \tag{3.6}$$

where $y_{n+1} \ll y_n$, can be systematically computed.

Remark 3.1 **What happens when $n > 2$.**

The same methods that work for $0 < n < 2$ can be used to study this case (but a bit more work is needed). The main difference in the phase portrait occurs because all the orbits (except for the special ones along the y-axis) approach the critical point along the lines $\sqrt{n-2}\, y = \pm\sqrt{n}\, x$.

For $\sqrt{n-2}\, |y| < \pm\sqrt{n}\, |x|$, the orbits look rather similar to the orbits in the case $0 < n < 2$, that is to say: closed loops starting and ending at the critical point, except that they approach the critical point along the lines $\sqrt{n-2}\, y = \pm\sqrt{n}\, x$, not the y-axis.

For $\sqrt{n-2}\, |y| > \pm\sqrt{n}\, |x|$, the orbits approach the critical point at one end (along the lines $\sqrt{n-2}\, y = \pm\sqrt{n}\, x$) and infinity at the other (ending parallel to the y-axis there). In between their slopes vary steadily (no inflection points) from one limit to the other.

Figure 3.1 shows a typical phase plane portrait for the $n > 2$ case. From the figure it should be clear that we still **have for the index: $I = 2$.**

Remark 3.2 **Rate of approach to the critical point** $(0 < n < 2.)$

Substituting (3.5) into (1.1), we obtain (near the critical point, where both x and y are small)

$$\frac{dx}{dt} \approx \frac{2\beta}{n} x^{(n+2)/2}\,, \qquad \text{and} \qquad \frac{dy}{dt} \approx y^2\,,$$

where (in the second equation) we simply used the fact that $y \gg x$. Thus

$$x = O\left(\frac{1}{(-t)^{2/n}}\right) \qquad \text{and} \qquad y = O\left(\frac{1}{t}\right)\,, \qquad \text{as} \qquad t \to -\infty\,.$$

4 Resolution of the difficulty in the case $n = 2$.

Again **we restrict out attention to the first quadrant, and assume $x,\ y > 0$.**

The results of section 3 are quite contradictory, when it comes to the case when $n = 2$. On the one hand, we showed that (3.3) must apply. But, on the other hand, when we implemented the consequences of this result (in (3.4)) we arrived at the contradictory result in (3.5). As we pointed out, the step from (3.4) to (3.5), is not foolproof and need not work. On the other hand, it usually does, and when it does not, things can get very subtle.[7] We will show next a simple approach that works in fixing some problems like the one we have.

[7] In fact, there are some open research problems that have to do with failures of this type, albeit in contexts quite a bit more complicated than this one.

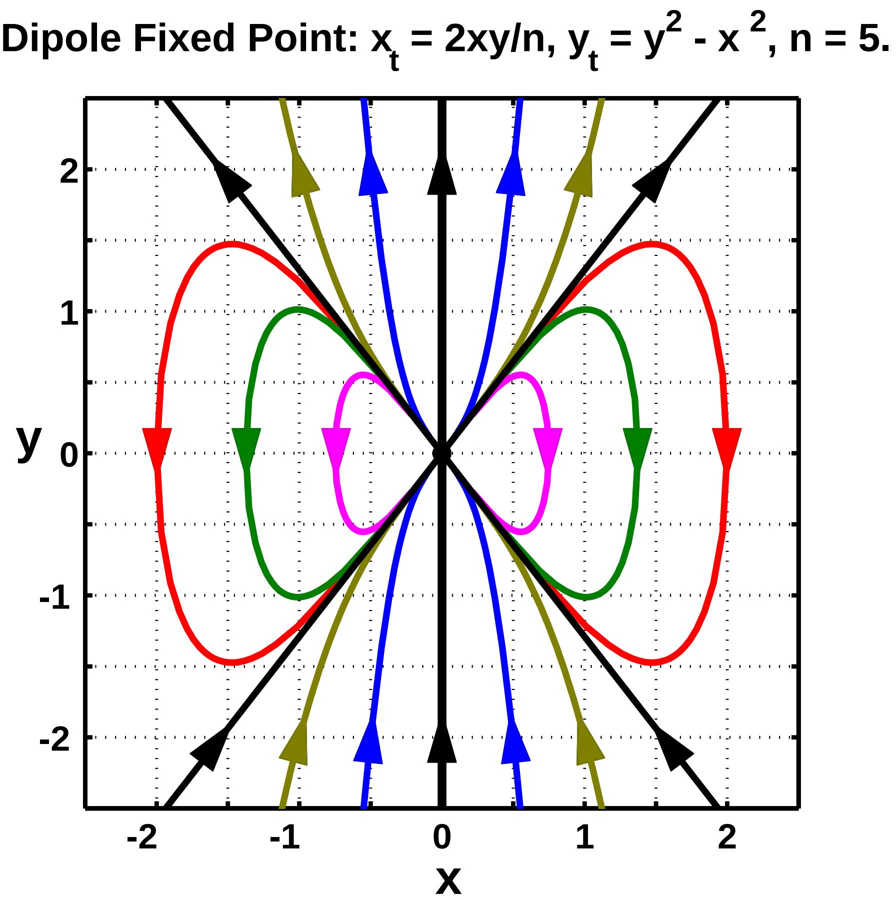

Figure 3.1: Phase plane portrait for the *Dipole Fixed Point* system (1.1) for $n = 5$. The qualitative details of the portrait do not change in the range $2 < n$, but differ from those that apply in the range $0 < n < 2$ (see figure 1.1.)

What happens for $n = 2$ must be, in same sense, a limit of the behavior for $n < 2$, as $n \to 2$. Now, look at (3.5) in this limit: it is clear that the behavior must become closer and closer to that of a straight line (since the exponent approaches 1), at least locally (i.e.: near any fixed value of x). On the other hand, it would be incorrect to assume that this implies that the orbits become straight in this limit, because this ignores that fact that β will depend on n too. In fact, we know that the limit behavior is not a straight line, but this argument shows that is must be very, very close to one. Thus we propose to seek solutions of the form

$$y = \alpha x \,, \tag{4.1}$$

where $\alpha = \alpha(x)$ is not a constant, but behaves very much like one as $x \to 0$. By this we mean that, when we calculate the derivative

$$\frac{dy}{dx} = \alpha + \frac{d\alpha}{dx} x \,, \tag{4.2}$$

we can neglect the second term. That is

$$\alpha \gg \frac{d\alpha}{dx} x \,, \quad \text{as} \quad x \to 0 \,. \tag{4.3}$$

We also expect that $\boxed{\alpha \to \infty \ \text{ as } \ x \to 0,}$ since we know that the orbits must approach the critical point vertically.

Notice that **this proposal provides a very clean explanation of how it is that the step from (3.3) to (3.5), via (3.4), fails (and provides a way out):** In writing (3.4) some small terms are neglected, and what is left is (when writing the solution in the form (4.1)) is α. Comparing this with (4.2), we see that the neglected terms are, precisely, those that make α non-constant. Thus, by neglecting them we end predicting that α is a constant,[8] which leads to all the contradictions pointed out in section 3.

What we need to do, therefore, is calculate the leading order correction[9] to the right hand side in (3.4), and equate it to the second term in (4.2). This will then give an equation for $\frac{d\alpha}{dx}$, which we must then solve. If the solution is then consistent with the assumption above in (4.3), we will have our answer and the mystery will be solved.[10]

We now implement the process described in the prior paragraph. The leading order correction to the right hand side in (3.4) is (recall $n = 2$ now)

$$\mathsf{correction} = -\frac{x}{y} = -\frac{1}{\alpha} \,, \tag{4.4}$$

which is small, since α is large for $0 < x \ll 1$. Thus the equation for α is:

$$x\frac{d\alpha}{dx} = -\frac{1}{\alpha} \quad \Longrightarrow \quad \alpha = \sqrt{c - 2\ln(x)} \,, \tag{4.5}$$

where c is a constant. It is easy to see that this is consistent with (4.3).

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[8] That is, $\alpha = \beta$ in (3.5).

[9] That is to say: plug (4.1) into equation (2.1) and then expand, using the fact that α is large.

[10] Note that this answer must be subject to the same type of basic checks we went through in **(c)**, **(d)**, and **(e)** of section 3, before we accepted (3.5) in the case $0 < n < 2$.

Remark 4.1 *It turns out that this problem is so simple that the "leading order" correction — i.e.:* $\left(\dfrac{-1}{\alpha}\right)$ *above in (4.4) — is everything! Thus (4.1 – 4.5) in fact provides not just an approximation near the critical point, but an* **exact solution!** *It follows that we do not need to check for any "consistencies" to make sure that the "approximation" can be trusted (in the manner of* **(c)**, **(d)**, *and* **(e)** *of section 3.)*

Of course, in more complicated problems this will (generally) not happen, and expressions like the one in (4.1) — with α *given by (4.5) — will end up being just the first term in an asymptotic approximation for the orbit shape.*

At this point you may wonder: **what exactly is the "method" proposed here?** Well, as usual with these kind of things, there is no precise recipe that can be given — just as there is no precise recipe that can be given to explain the "standard" methods. However, just as in the standard methods one can give a vague — and rather short — list of things to do (e.g.: balance terms and look for pairs that may dominate, therefore simplifying the problem[11]) we provide below a list of hints as to what one can do when faced with problems like the one we treat in this section. In the end, though, each problem is its own thing and (at least with our present level of understanding) the only way to learn how to do these things "well" is by painfully acquired experience.

When faced with a problem of this type, you may try this:

1. See if you can add a parameter to the equations (say: n), in such a way that the difficult problem corresponds to some critical value $n = n_c$, and you can do the problem for $n \neq n_c$. In the example here $n_c = 2$.

2. Look at the behavior of the solution for the "easier" problems as $n \to n_c$. This limit will, almost certainly, be singular. What you should then do is try to extract a functional form (by looking at these limits) with appropriate properties.[12] The aim is to "guess" what the "right" form to try for the solution is, by looking at the behavior of the solutions of the nearby problems on each side of n_c (these ought to "sandwich" the right behavior between them.)

[11]Books in asymptotic expansions deal with these and other ideas at length; see (for example) Bender, C. M., and Orszag, S. A. (1978) *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York.)

[12]Sorry if this sounds very vague; it is very vague, but it is the best I can do!

In the example studied here we had (β is a constant):

$$y \approx \beta\, x^{1-\epsilon}\,, \qquad \text{where} \quad \epsilon = \frac{n_c - n}{2}\,, \qquad \text{when} \quad n < n_c = 2$$

and

$$y \approx \frac{\sqrt{1-\epsilon}}{\sqrt{\epsilon}}\, x\,, \qquad \text{where} \quad \epsilon = \frac{n - n_c}{2}\,, \qquad \text{when} \quad n > n_c = 2\,.$$

In the first case the limit behavior is βx, but it is a very non-uniform limit near $x = 0$ (see what happens with the derivatives.) In the second case there is not even a limit.

The solutions for both cases, however, have the common form αx, where the bad behavior is restricted to α. Thus we picked this common form, and assumed properties for α "intermediate" between the behaviors on each side: a constant, but not quite one, and going to infinity as $x \to 0$.

3. Alternatively, look at the solution[13] that fails for $n = n_c$. This solution will satisfy an approximate form of the equations (where some small terms have been neglected), but will be inconsistent with the assumptions made in arriving to it — e.g.: the small terms end up not being as small as assumed. The failure must occur because the neglected small terms have some important effect. Therefore, try the following: assume a form of the solution equal to the one that fails, but allow any free parameters in this solution to be "slow" functions, rather than constants (this means: when taking derivatives, the terms involving derivatives of the parameters will be higher order.[14]) Then use this "slow" dependence to eliminate the leading order terms in the errors to the approximations that lead to the failed solution in the first place. If you are lucky, and clever enough, this might fix the problem.

In the example studied here, the failure occurs for $n = 2$, when equation (3.4) becomes

$$\frac{dy}{dx} = \frac{y}{x}\,, \qquad \text{with solution} \quad y = \alpha x \;\; (\alpha \text{ a constant.})$$

This solution is inconsistent with the assumption $y \gg x$ used in deriving (3.4). Thus we took this form, but made the free constant parameter in the solution (α) a slow function of x, with

[13]Given by "standard" techniques.

[14]These functions should also have properties (e.g.: large, small, ... in some limit) that make the assumed form consistent with the approximations that lead to the equations they solve.

the additional property $\alpha \gg 1$ as $x \to 0$ (so that $y \gg x$ still applies.) This then works, in this case so well that it gives an exact solution.

The three hints outlined above will work straightforwardly for relatively simple problems, both in ODE's and PDE's. Beyond that ...

5 Exact solution of the orbit equation.

Equation (2.1) is simple enough that one can solve it exactly (for all values of n.) We can then use this exact solution to verify that everything done earlier (using approximate arguments) is absolutely correct. This is not a luxury one can afford too often; generally exact solutions are not available and rigorous arguments are either too expensive or impossible — thus, the only tools one is left with are numerical computations, approximate analysis, and experimental observations.[15]

Let us now solve (2.1). Multiply both sides of the equation by $2y$ and integrate. This yields a *linear* equation in y^2, namely:

$$\frac{dy^2}{dx} - \frac{n}{x} y^2 = -nx\,.$$

Now multiply the equation by x^{-n}, and integrate again, to obtain (assume $x > 0$):

$$\frac{dy^2 x^{-n}}{dx} = -nx^{1-n}\,.$$

From this the following solutions follow:

- **Case** $0 < n < 2$**.**

$$y^2 = 2Rx^n - \frac{n}{2-n} x^2\,, \qquad \text{for} \quad 0 \le x \le \left(\frac{2R(2-n)}{n} \right)^{\dfrac{1}{2-n}}\,, \tag{5.1}$$

  where $R > 0$ is a constant. For $n = 1$ these are circles of radius R, centered at $(x,\, y) = (R,\, 0)$.

- **Case** $n = 2$**.**

$$y^2 = \left(2\ln(x_0) - 2\ln(x)\right) x^2\,, \qquad \text{for} \quad 0 \le x \le x_0\,, \tag{5.2}$$

  where $x_0 > 0$ is a constant.

[15]For 2-D problems all sorts of theoretician luxuries are available. But real problems are seldom this simple.

- **Case $n > 2$.**

$$y^2 = -Cx^n + \frac{n}{n-2}x^2\,, \qquad \text{for} \quad 0 \le x \le \left(\frac{n}{(n-2)C}\right)^{\frac{1}{n-2}}\,, \tag{5.3}$$

where $C > 0$ is a constant (these are the orbits giving closed loops in figure 3.1), or

$$y^2 = Cx^n + \frac{n}{n-2}x^2\,, \qquad \text{for} \quad 0 \le x\,, \tag{5.4}$$

where $C \ge 0$ is a constant (these are the orbits that diverge to infinity in the sectors around the y-axis in figure 3.1.)

6 Commented Bibliography.

Below I list a few books that I think might be of use to you.

1. Cole, J. D. (1968). *Perturbation Methods in Applied Mathematics,* Blaisdell, Waltham, Mass.
   Very nice and concise book (unfortunately, out of print.) It introduces the fundamental concepts in asymptotic methods, using examples from applications (fluid dynamics, mostly.) It aims at realistic scientific problems, so it deals mostly with PDE's (not ODE's).

2. Bender, C. M., and Orszag, S. A. (1978). *Advanced Mathematical Methods for Scientists and Engineers,* McGraw-Hill, New York.
   This book has an extensive treatment of many of the ideas in asymptotic (and other) methods, with many comparisons between the asymptotic approximations and numerical solutions. It introduces the methods using simple examples, so it deals (mostly) with ODE's.

3. Coddington, E. A., and Levinson, N. (1955). *Theory of Ordinary Differential Equations,* McGraw-Hill, New York.
   A rigorous treatment of the theory of ODE's, and a classic for this. This book proves everything, but it does so with minimum use of jargon. It has several chapters dedicated to asymptotic properties of ODE's, a complete treatment of the Poincaré Bendixson theorem, and many other things. If you want hard core proofs, without excuses or unnecessary jargon, this is the place to go. Of course, it is a bit old, and a lot of the new theory is not here —

but you cannot really appreciate (or understand) any proof in the newer theory without this background.

4. Ince, E. L. (1926). *Ordinary Differential Equations,* Longmans, Green, London.
There is also a Dover edition!
Old, perhaps, but very good. A hard core exposition of the classical theory of ODE's.

THE END.

18.385 MIT

Hopf Bifurcations.

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Abstract

In two dimensions a Hopf bifurcation occurs as a Spiral Point switches from stable to unstable (or vice versa) and a periodic solution appears. There are, however, more details to the story than this: The **fact that a critical point switches from stable to unstable spiral** (or vice versa) **alone does not guarantee that a periodic solution will arise,**[1] though one almost always does. Here we will explore these questions in some detail, using the method of multiple scales to find precise conditions for a limit cycle to occur and to calculate its size. We will use a second order scalar equation to illustrate the situation, but the results and methods are quite general and easy to generalize to any number of dimensions and general dynamical systems.

[1]Extra conditions have to be satisfied. For example, in the damped pendulum equation: $\ddot{x}+\mu\dot{x}+\sin x = 0$, there are **no** periodic solutions for $\mu \neq 0$ **!**

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1 Hopf bifurcation for second order scalar equations.

1.1 Reduction of general phase plane case to second order scalar.

We will consider here equations of the form

$$\ddot{x} + h(\dot{x}, x, \mu) = 0\,, \tag{1.1}$$

where h is a smooth and μ is a parameter.

Note 1 *There is* **not much loss of generality** *in studying an equation like (1.1), as opposed to a phase plane general system. For let:*

$$\dot{x} = f(x, y, \mu) \quad \textit{and} \quad \dot{y} = g(x, y, \mu)\,. \tag{1.2}$$

Then we have

$$\ddot{x} = f_x \dot{x} + f_y \dot{y} = f_x f + f_y g = F(x, y, \mu)\,. \tag{1.3}$$

Now, from $\dot{x} = f(x, y, \mu)$ we can, at least in principle,[2] *write*

$$y = G(\dot{x}, x, \mu)\,. \tag{1.4}$$

Substituting then (1.4) into (1.3) we get an equation of the form (1.1).[3]

1.2 Equilibrium solution and linearization.

Consider now an **equilibrium solution**[4] for (1.1), that is:

$$x = X(\mu) \quad \text{such that} \quad h(0, X, \mu) = 0\,, \tag{1.5}$$

[2]We can do this in a neighborhood of any point (x_*, y_*) (say,a critical point) such that $f_y(x_*, y_*, \mu) \neq 0$, as follows from the Implicit Function theorem. If $f_y = 0$, but $g_x \neq 0$, then the same ideas yield an equation of the form $\ddot{y} + \widetilde{h}(\dot{y}, y, \mu) = 0$ for some $\widetilde{h}$. The approach will fail only if both $f_y = g_x = 0$. But, for a critical point this last situation implies that the eigenvalues are f_x and g_y, that is: **both real** ! Since we are interested in studying the behavior of phase plane systems near a **non–degenerate** critical point switching from stable to unstable spiral behavior, this **cannot happen.**

[3]Vice versa, if we have an equation of the form (1.1), then defining y by $y = G(\dot{x}, x, \mu)$, for any G such that the equation can be solved to yield $\dot{x} = f(x, y, \mu)$ (for example: $G = \dot{x}$), then $\dot{y} = G_{\dot{x}}\ddot{x} + G_x \dot{x} = g(x, y)$ upon replacing $\dot{x} = f$ and $\ddot{x} = -h$.

[4]i.e.: a critical point.

so that $x \equiv X$ is a solution for any fixed μ. There is $\boxed{\textbf{no loss of generality}}$ in assuming

$$X(\mu) \equiv 0 \quad \text{for all values of } \mu\,, \tag{1.6}$$

since we can always change variables as follows: $x_{\text{old}} = X(\mu) + x_{\text{new}}$.

The linearized equation near the equilibrium solution $x \equiv 0$ (that is, the equation for x infinitesimal) is now:

$$\ddot{x} - 2\alpha\dot{x} + \beta x = 0\,, \tag{1.7}$$

where $\boxed{\alpha = \alpha(\mu) = -\frac{1}{2}h_{\dot{x}}(0,0,\mu)}$ and $\boxed{\beta = \beta(\mu) = h_x(0,0,\mu)\,.}$

The critical point is a **spiral point** if $\beta > \alpha^2$. The eigenvalues and linearized solution are then

$$\lambda = \alpha \pm i\widetilde{\omega} \tag{1.8}$$

(where $\widetilde{\omega} = \sqrt{\beta - \alpha^2}$) and

$$x = ae^{\alpha t}\cos\left(\widetilde{\omega}(t - t_0)\right)\,, \tag{1.9}$$

where a and t_0 are constants.

1.3 Assumptions on the linear eigenvalues needed for a Hopf bifurcation.

Assume now: At $\mu = 0$ the **critical point changes from a stable to an unstable spiral point** (if the change occurs for some other $\mu = \mu_c$, one can always redefine $\mu_{\text{old}} = \mu_c + \mu_{\text{new}}$). Thus

$$\boxed{\alpha < 0 \text{ for } \mu < 0 \text{ and } \alpha > 0 \text{ for } \mu > 0, \text{ with } \beta > 0 \text{ for } \mu \text{ small.}}$$

In fact, **assume:**

$$\left.\begin{array}{ll} \bullet\ \textbf{I.} & h \text{ is smooth.} \\ \bullet\ \textbf{II.} & \alpha(0) = 0,\ \ \beta(0) > 0 \ \text{ and } \ \dfrac{d}{d\mu}\alpha(0) > 0\,.^{5} \end{array}\right\} \tag{1.10}$$

We point out that, in addition, there are some restrictions on the behavior of the nonlinear terms near the critical point that are needed for a Hopf bifurcation to occur. See equation (1.22).

[5] This last is known as the Transversality condition. It guarantees that the eigenvalues **cross** the imaginary axis as μ varies.

1.4 Weakly Nonlinear things and expansion of the equation near equilibrium.

Our objective is to study what happens **near** the critical point, for μ small. Since for $\mu = 0$ the critical point is a **linear center**, the *nonlinear terms will be important in this study.* Since we will be considering the **region near** the critical point, the **nonlinearity will be weak**. Thus we will **use the methods** introduced in the *Weakly Nonlinear Things* notes.

For $x, \dot{x}$**, and** μ **small we can expand** h **in (1.1)**. This yields

$$\boxed{\begin{aligned} \ddot{x} + \omega_0^2 x \;&+\; \left\{\tfrac{1}{2}A\dot{x}^2 + B\dot{x}x + \tfrac{1}{2}Cx^2\right\} + \\ &+\; \tfrac{1}{6}\left\{D\dot{x}^3 + 3E\dot{x}^2x + 3F\dot{x}x^2 + Gx^3\right\} \\ &-\; 2p^2\dot{x}\mu + \Omega x\mu + O(\epsilon^4, \epsilon^2\mu, \epsilon\mu^2) \qquad = \; 0\,, \end{aligned}} \tag{1.11}$$

where **we have used that** $h(0, 0, \mu) \equiv 0$ and $\alpha(0) = 0$. In this equation we have:

A. $\omega_0^2 = \dfrac{\partial}{\partial x}h(0, 0, 0) = \beta(0) > 0,$ with $\omega_0 > 0$,

B. $A = \dfrac{\partial^2}{\partial \dot{x}^2}h(0, 0, 0), \quad B = \dfrac{\partial^2}{\partial \dot{x}\partial x}h(0, 0, 0)\,, \;\ldots,$

C. $p^2 = -\dfrac{1}{2}\dfrac{\partial^2}{\partial \dot{x}\partial \mu}h(0, 0, 0) = \dfrac{d}{d\mu}\alpha(0) > 0,$ with $p > 0$,

D. $\Omega = \dfrac{\partial^2}{\partial x\partial \mu}h(0, 0, 0) = \dfrac{d}{d\mu}\beta(0),$

E. ϵ **is a measure of the size of** $(x, \dot{x})$**.** Further: **both** ϵ **and** μ **are small**.

1.5 Explanation of the idea behind the calculation.

We now want to study the solutions of (1.11). The idea is, again: for ϵ and μ small the solutions are going to be dominated by the center in the linearized equation $\ddot{x} + \omega_0^2 x = 0$, with a *slow drift* in the amplitude and small changes to the period[6] caused by the higher order terms. Thus we will use an approximation for the solution like the ones in section 2.1 of the *Weakly Nonlinear Things* notes.

[6] We will not model these period changes here. See section 2.3 of the *Weakly Nonlinear Things* notes for how to do so.

1.6 Calculation of the limit cycle size.

$$\text{\underline{\underline{\textbf{An important point to be answered is}}}\textbf{:}} \qquad \boxed{\text{What is epsilon?}} \tag{1.12}$$

This is a parameter that does not appear in (1.1) or, equivalently, (1.11). In fact, the only parameter in the equation is μ (assumed small as we are close to the bifurcation point $\mu = 0$). Thus:

$$\boxed{\epsilon \text{ must be related to } \mu.} \tag{1.13}$$

In fact, ϵ **will be a measure of the size of the limit cycle**, which is a **property of the equation** (and thus a function of μ and not arbitrary all).

However: $\boxed{\text{We do not know } \epsilon \text{ a priori! How do we go about determining it?}}$

$\boxed{\textbf{The idea is:}}$ If we choose ϵ "too small" in our scaling of $(x, \dot{x})$, then we will be looking "too close" to the critical point and thus will find only spiral-like behavior, with no limit cycle at all. Thus, we **must choose** ϵ **just large enough** so that the terms involving μ in (1.11) (**specifically** $2p^2\mu\dot{x}$, *which is the* **leading order term** *in producing the* **stable/unstable spiral behavior**) are "balanced" by the nonlinearity in such a fashion that a limit cycle is allowed. In the context of Two–Timing this means we **want** μ **to "kick in"** the damping/amplification term $2p^2\mu\dot{x}$ at **"just the right level"** in the sequence of solvability conditions the method produces. Thus, going back to (1.11), we see that[7]

- The linear leading order terms $\ddot{x} + \omega_0^2 x$ appear at $O(\epsilon)$.
- The first nonlinear terms (quadratic) appear at $O(\epsilon^2)$.

  However: Quadratic terms produce **no resonances,** since $\sin^2\theta = \dfrac{1}{2}(1 - \cos 2\theta)$ and there are no sine or cosine terms. The same applies to $\cos^2\theta$ and to $\sin\theta\cos\theta$.
- Thus, the first resonances will occur when the cubic terms in x play a role $\Rightarrow$ **we must have the balance**

$$O(x^3) = O(\mu\dot{x})\,, \tag{1.14}$$

$\Rightarrow$ $\boxed{\mu = O(\epsilon^2).}$

[7]This is a crucial argument that must be well understood. Else things look like a bunch of miracles!

1.7 The Two Timing expansion up to $O(\epsilon^3)$.

We are now ready to start. The expansion to use in (1.11) is

$$x = \epsilon x_1(\tau, T) + \epsilon^2 x_2(\tau, T) + \epsilon^3 x_3(\tau, T) + \dots, \tag{1.15}$$

where $0 < \epsilon \ll 1$, 2π–periodicity in T is required, $T = \omega_0 t$, ω_0 is as in (1.11)[8], τ is a *slow time* variable and ϵ is related to μ by $\boxed{\mu = \nu\epsilon^2,}$ where $\boxed{\nu = \pm 1}$ (which ν we take depends on which "side" of $\mu = 0$ we want to investigate).

What exactly is τ? Well, we need τ to resolve resonances, which will not occur until the cubic terms kick in into the expansion $\Rightarrow$ $\boxed{\tau = \epsilon^2 t.}$ (This is exactly the same argument used to get (1.14)).

Then, with $\boxed{{}' = \frac{\partial}{\partial T}}$, (1.11) becomes:

$$\begin{aligned} &\omega_0^2 x'' + \omega_0^2 x + \left\{\tfrac{1}{2} A\omega_0^2 (x')^2 + B\omega_0 x x' + \tfrac{1}{2} C x^2\right\} \quad + \\ &\tfrac{1}{6}\left\{D\omega_0^3 (x')^3 + 3E\omega_0^2 (x')^2 x + 3F\omega_0 x' x^2 + G x^3\right\} \quad + \\ &\qquad\qquad 2\epsilon^2 \omega_0 x'_\tau - 2\epsilon^2 \nu p^2 \omega_0 x' + \epsilon^2 \nu \Omega x \quad + \quad O(\epsilon^4) = 0\,. \end{aligned} \tag{1.16}$$

The rest is now a computational nightmare, but it is fairly straightforward. **Without getting into any of the messy algebra, this is what will happen:**

$\boxed{\textbf{At } O(\epsilon)}$ $\qquad \omega_0^2\{x_1'' + x_1\} = 0$. Thus

$$x_1 = a_1(\tau) e^{iT} + c.c. \tag{1.17}$$

for some complex valued function $a_1(\tau)$. We use complex notation, as in the *Weakly Non-linear Things* notes.

$\boxed{\textbf{At } O(\epsilon^2)}$

$$\omega_0^2\{x_2'' + x_2\} + \underbrace{\{\text{quadratic terms in} \quad x_1 \text{ and } x_1'\}} = 0\,. \tag{1.18}$$

From the first bracket in (1.16), the quadratic terms here have the form:

$$C_1 a_1^2 e^{i2T} + C_2 \left|a_1^2\right| + C_1^* (a_1^*)^2 e^{-2iT}\,,$$

where C_1 and C_2 are **constants that can be computed in terms of** ω_0, A, B and C. Since the solution and equation are real valued, C_2 is real. Here, as usual, $*$ indicates the complex conjugate.

[8]Same as the linear (at $\mu = 0$) frequency. No attempt is made in this expansion to include higher order nonlinear corrections to the frequency.

No resonances occur and we have

$$x_2 = \left\{ \left(a_2(\tau)e^{iT} + \frac{1}{3}\omega_0^{-2}C_1 a_1^2 e^{i2T} \right) + c.c. \right\} - \omega_0^{-2}C_2 \left| a_1^2 \right| . \tag{1.19}$$

$\boxed{\textbf{At } O(\epsilon^3)}$

$$\omega_0^2 \left(x_3'' + x_3 \right) + 2\omega_0 x_{1\tau}' - 2\nu p^2 \omega_0 x_1' + \nu\Omega x_1 + \mathbf{CNLT} = 0 \,, \tag{1.20}$$

where **CNLT** stands for **C**ubic **N**on **L**inear **T**erms, involving products of the form $x_2 x_1$, $x_2' x_1$, $x_2 x_1'$, $x_2' x_1'$, $(x_1')^3$, $(x_1')^2 x_1$, $x_1' x_1^2$ and x_1^3. These will produce a term of the form $\boxed{d a_1^2 a_1^* e^{iT} + c.c.}$ plus other terms whose T dependencies are: 1, $e^{\pm 2iT}$ and $e^{\pm 3iT}$, none of which is resonant (forces a non periodic response in x_3). Here

$$\boxed{d \textbf{ is a constant} \text{ that can be computed in terms of } \omega_0, A, B, C, D, E, F \text{ and } G \,.} \tag{1.21}$$

This is a big and messy calculation, but it involves only sweat. **In general**, of course, $\mathrm{Im}(d) \neq 0$. The case $\mathrm{Im}(d) = 0$ is very particular, as it requires h in equation (1.1)to be just right, so that the particular combination of its derivatives at $x = 0$, $\dot{x} = 0$ and $\mu = 0$ that yields $\mathrm{Im}(d)$ just happens to vanish. Thus

$$\boxed{\text{Assume a } \underline{\underline{\textbf{nondegenerate case}}}\text{:} \quad \mathrm{Im}(d) \neq 0 \,.} \tag{1.22}$$

For equation (1.20) to have solutions x_3 periodic in T, the forcing terms proportional to $e^{\pm iT}$ must vanish. This leads to the equation:

$$\boxed{2\omega_0 i \frac{d}{d\tau} a_1 - 2\nu p^2 \omega_0 i a_1 + \nu\Omega a_1 + d \left| a_1^2 \right| a_1 = 0.} \tag{1.23}$$

Then write

$$a_1 = \rho e^{i\theta} \,, \quad \text{with } \rho \text{ and } \theta \text{ real} \,, \; \rho > 0 \,.$$

This yields

$$\boxed{\frac{d}{d\tau}\theta = \frac{1}{2}\nu\omega_0^{-1}\Omega + \frac{1}{2}\omega_0^{-1}\mathrm{Re}(d)\rho^3} \tag{1.24}$$

and

$$\boxed{\frac{d}{d\tau}\rho = \nu p^2 (1 - \nu q \rho^2)\rho \,,} \tag{1.25}$$

where $\boxed{q = \frac{1}{2}\omega_0^{-1} p^{-2} \mathrm{Im}(d).}$

Equation(1.24) provides a correction to the phase of x_1, since $x_1 = 2\rho\cos(T+\theta)$. The first term on the right of (1.24) corresponds to the changes in the linear part of the phase due to $\mu \neq 0$, away from the phase $T = \omega_0 t$ at $\mu = 0$. The second term accounts for the nonlinear effects.

The second equation (1.25) above is more interesting. First of all, it reconfirms that for $\mu < 0$ (that is, $\nu = -1$) the critical point ($\rho = 0$) is a stable spiral, and that for $\mu > 0$ (that is, $\nu = 1$) it is an unstable spiral. **Further**

$$\left.\begin{array}{ll} \boxed{\text{If } \mathrm{Im}(d) > 0.} & \text{Then a } \textbf{stable limit cycle} \text{ exists for} \\ & \mu > 0 \ (\text{i.e.}\, \nu = 1) \text{ with } \rho = \sqrt{2\omega_0 p^2 (\mathrm{Im}(d))^{-1}}\,. \\ & \textbf{Supercritical (Soft) Hopf Bifurcation.} \\ & \\ \boxed{\text{If } \mathrm{Im}(d) < 0.} & \text{Then an } \textbf{unstable limit cycle} \text{ exists for} \\ & \mu < 0 \ (\text{i.e.}\, \nu = -1) \text{ with } \rho = \sqrt{-2\omega_0 p^2 (\mathrm{Im}(d))^{-1}}\,. \\ & \textbf{Subcritical (Hard) Hopf Bifurcation.} \end{array}\right\} \tag{1.26}$$

Notice that ρ here is equal to $\dfrac{1}{2\epsilon}$ the radius of the limit cycle.

1.7.1 Remark on the situation at the critical bifurcation value.

Notice that, **for** $\mu = 0$ (critical value of the bifurcation parameter)[9] we can do a two timing analysis as above to verify what the nonlinear terms do to the center.[10] The calculations are exactly as the ones leading to equations (1.23)–(1.25), except that $\nu = 0$ and ϵ is now a small parameter (unrelated to μ, as $\mu = 0$ now) simply measuring the strength of the nonlinearity near the critical point. Then we get for $\rho = \dfrac{1}{2\epsilon}$ radius of orbit around the critical point

$$\frac{d}{d\tau}\rho = -\frac{1}{2}\omega_0^{-1}\mathrm{Im}(d)\rho^3\,. \tag{1.27}$$

From this the behavior near the critical point follows.

[9] **Then the critical point is a center in the linearized regime.**

[10] This is the way one would normally go about deciding if a linear center is actually a spiral point and what stability it has.

Clearly
$$\begin{cases} \bullet\ \mathsf{Im}(d) > 0 \iff \text{Soft bifurcation} \iff \text{Nonlinear terms stabilize.} \\ \quad \textit{For } \mu = 0 \textit{ critical point is a stable spiral.} \\ \bullet\ \mathsf{Im}(d) < 0 \iff \text{Hard bifurcation} \iff \text{Nonlinear terms de-stabilize.} \\ \quad \textit{For } \mu = 0 \textit{ critical point is an unstable stable spiral.} \end{cases}$$

1.7.2 Remark on higher orders and two timing validity limits.

As pointed out in the *Weakly Nonlinear Things* notes, Two Timing is generally valid for some "limited" range in time, here probably $|\tau| \ll \epsilon^{-1}$. This is because we have no mechanism for incorporating the higher order corrections to the period the nonlinearity produces. If we are only interested in calculating the limit cycle in a Hopf bifurcation (not it's stability characteristics), we can always do so using the Poincaré–Lindsteadt Method. In particular, then we can get the period to as high an order as wanted.

1.7.3 Remark on the problem when the nonlinearity is degenerate.

What about the **degenerate case** $\boxed{\mathrm{Im}(d) = 0}$ **?**

In this case there may be a limit cycle, or there may not be one. To decide the question one must look at the effects of nonlinearities higher than cubic (going beyond $O(\epsilon^3)$ in the expansion) and see if they stabilize or destabilize. If a limit cycle exists, then its size will not be given by $\sqrt{|\mu|}$, but something else entirely different (given by the appropriate balance between nonlinearity and the linear damping/amplification produced by $\alpha \neq 0$ when $\mu \neq 0$ in equation (1.7)). The details of the calculation needed in a case like this can be quite hairy. One must use methods like the ones in Section 2.3 of the *Weakly Nonlinear Things* notes because: even though the nonlinearity may require a high order before it decides the issue of stability, modifications to the frequency of oscillation will occur at lower orders.[11] We will not get into this sort of stuff here.

[11]Note that $\mathrm{Re}(d) \neq 0$ in (1.24) produces such a change, even if $\mathrm{Im}(d) = 0$ and there are no nonlinear effects in (1.25).

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Weakly Nonlinear Things: *Oscillators.*

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Abstract

When nonlinearities are "small" there are **various ways one can exploit this fact** — and the fact that the linearized problem can be solved exactly[1] — to produce useful approximations to the solutions.

We **illustrate two of these techniques here**, with examples from phase plane analysis: **The Poincaré–Lindstedt method** and the (more flexible) **Two Timing method.** This second method is a particular case of the **Multiple Scales** approximation technique, which is useful whenever the solution of a problem involves effects that occur on very different scales. In the particular examples we consider, the different scales arise from the basic vibration frequency induced by the linear terms (fast scale) and from the (slow) scale over which the small nonlinear effects accumulate.

The material in these notes is intended to amplify the topics covered in section 7.6 and problems 7.6.13–7.6.22 of the book "Nonlinear Dynamics and Chaos" by S. Strogatz.

[1]Actually, one can also use these ideas when one has a **nonlinear** problem with known solution, and wishes to solve a slightly different one. But we will not talk about this here.

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1 Poincaré-Lindstedt Method (PLM). **3**

General ideas behind the method.

1.1 Duffing Equation. 3

Periodic solutions and amplitude dependence of their periods.

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2 Two Timing, Multiple Scales method (TTMS) for the van der Pol equation. **8**

2.1 Calculation of the limit cycle and stability. 8

2.2 Higher orders and limitations of TTMS. 11

This topic is fairly technical.†

2.3 Generalization of TTMS to extend the range of validity. 14

This topic is fairly technical.†

A Appendix. **16**

A.1 Some details regarding section 1.1. 16

A.2 More details regarding section 1.1. 17

This topic is fairly technical.†

A.3 Some details regarding section 1.2. 17

† The material here is for completeness, but not actually needed to get a "basic" understanding.

1 Poincaré-Lindstedt Method (PLM).

PLM is a technique for calculating *periodic solutions*. The **idea is that**, if the linearized equations have periodic solutions and $0 < \epsilon \ll 1$ is a measure of the *size* of the nonlinear terms then:

I. For any *finite* time period $t_0 \le t \le t_0 + T_f \quad (T_f > 0)$, the trajectories for the full system will remain pretty close to those of the linearized system (errors no worse than $O(\epsilon T_f)$, typically).

II. On the other hand, even a small error is enough to destroy periodicity. An orbit that "closes on itself" after some time period, will generally fail to do so if slightly perturbed. Thus, typically, nonlinearity will destroy most periodic orbits the linearized system might have. Some, however, may survive[2] $\longrightarrow$ **PLM is designed to pick those up.**

Even if a periodic orbit of the linearized system survives:

III. The **nonlinearity** will **change** (slightly) **the shape of the orbit**.

IV. The **speed of "travel"** along the orbit will be **affected** by the **nonlinearity.** In particular the **period will change** (slightly.)

PLM takes care of these effects as follows:

A. The *solution is approximated at leading order* by the linear solution, but small corrections at higher orders are introduced to take care of the (small) shape changes.

B. The linear solution is evaluated at a stretched time, to account for the change in period.

The two examples that follow illustrate the ideas.

1.1 Duffing Equation.

The equation can be written in the form

$$\ddot{x} + x + \epsilon \nu x^3 = 0\,, \tag{1.1}$$

[2] That is, if $\vec{u} = \vec{u}(t)$ is a periodic solution of the linearized system, then so is $a\vec{u}$, for any scalar constant a. But for only a few values of a will periodicity "survive" the effect of the nonlinearity.

where $0 < \epsilon \ll 1$ and $\nu = \pm 1$. This equation is actually a conservative system, with (conserved) energy

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}\epsilon\nu x^4 \,. \tag{1.2}$$

Thus all orbits for x bounded will be periodic.[3] PLM will allow us to calculate corrections to the linear period of 2π and sinusoidal orbit shape (for the bounded orbits).

The **PLM expansion** is given by:

$$x(t) = x_0(T) + \epsilon x_1(T) + \epsilon^2 x_2(T) + \cdots \,, \tag{1.3}$$

where $x_j = x_j(T)$ is periodic of period 2π in T and **does not depend on** ϵ. $T = \omega t$ is the **stretched time variable**, where

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \,, \tag{1.4}$$

is a (real, positive) constant to be computed. The **nonlinear period is then** $2\pi/\omega$.

Note 1 *$x_0(T)$ will be the solution to the* **linearized problem***, so (1.3) will reduce to the right answer when $\epsilon = 0$.*

We now **proceed as follows**:

- **First:** Rewrite (1.1) in terms of the new independent variable T, replacing $\boxed{\dot{} = \frac{d}{dt}}$ by $\boxed{{}' = \frac{d}{dT}}$ via $\boxed{\frac{d}{dt} = \omega\frac{d}{dT}}$. Thus:

$$\omega^2 x'' + x + \epsilon\,\nu\,x^3 = 0 \,. \tag{1.5}$$

- **Second:** Substitute (1.3) and (1.4) into (1.5) and collect equal powers[4] of ϵ. Then require that the equation be satisfied at each level in ϵ. Thus we get an equation for each order ϵ^p, which determine higher and higher orders of approximation in the expansion (1.3), as follows.

[3]Notice that, **for $\nu = 1$ ALL orbits are periodic.** However, for $\nu = -1$, orbits where $|x| > \epsilon^{-\frac{1}{2}}$ are not periodic. This follows from looking at the level curves for E in the $(x, \dot{x})$ phase plane. Of course, when $|x| = O(\epsilon^{-\frac{1}{2}})$, the nonlinear term in equation (1.1) has the same size as the linear terms: the problem is no longer "weakly nonlinear". Thus, we should not be surprised if the solution exhibits behavior not close to the linearized one.

[4]This is the **messy part.** It means you have to plug (1.3) and (1.4) into (1.5), then do all the products, etc. ... so as to end with the equation written as: $\boxed{\{\cdots\} + \epsilon\,\{\cdots\} + \epsilon^2\,\{\cdots\} + \cdots = 0.}$

$\boxed{O(1)\ \textbf{equation:}}$

$$x_0'' + x_0 = 0\,. \tag{1.6}$$

Clearly then

$$x_0 = a\cos T\,, \tag{1.7}$$

where a is, at this stage, an arbitrary constant.[5]

$\boxed{O(\epsilon)\ \textbf{equation:}}$ $\quad x_1'' + 2\omega_1 x_0'' + x_1 + \nu x_0^3 = 0,\quad$ that is:

$$\begin{aligned} x_1'' + x_1 &= 2\omega_1 a\cos T - \nu a^3\cos^3 T = \\ &= \left\{2\omega_1 a - \tfrac{3}{4}\nu a^3\right\}\cos T - \tfrac{1}{4}\nu a^3\cos 3T\,. \end{aligned} \tag{1.8}$$

> **The form of equation (1.8) is typical of all the higher order equations.** Namely, we get the linear equation for the new term in x at that order — x_1 here — forced by terms involving the lower orders already solved for.

The solution x_1 to (1.8) will be 2π-periodic in T **only** if the coefficient of the $\cos T$ term on the right hand side (terms between the brackets) vanishes. This is because this term will produce a response in x_1 proportional to $T\sin T$, which is **clearly** not periodic. Since we are interested in a **nontrivial solution** (that is $a \neq 0$) we conclude that:

$$\begin{aligned} \omega_1 &= \tfrac{3}{8}\nu a^2\,, \\ x_1 &= \tfrac{1}{32}\nu a^3\cos 3T + \underbrace{A\cos T + B\sin T}\,, \end{aligned} \tag{1.9}$$

where the term marked by the brace in the second equation is the arbitrary homogeneous solution, with A and B arbitrary constants. The first equation here determines the first frequency correction, in terms of the amplitude[6] of the oscillations a, which remains arbitrary at this level.[7] We note also that the homogeneous solution in the second equation above

[5] In fact, in this case, a will remain arbitrary. There is also a *phase shift* we could include in (1.7). But this is just a matter of where we put the time origin (see appendix A.1).

[6] This is **typical of nonlinear oscillators:** the frequency depends on the amplitude.

[7] That is, no restrictions have been imposed by the expansion on it. In fact, it can be shown that no restrictions on a will appear at any level of the expansion. This is because there is in fact a whole one parameter set of periodic solutions, which can be parameterized by the amplitude a.

amounts to no more than a small change in the amplitude and phase of the leading order solution. That is:

$$a\cos T \longrightarrow (a+\epsilon A)\cos T + \epsilon B \sin T = \widetilde{a}\cos(T-\widetilde{T})\,,$$

for some $\widetilde{a}$ and $\widetilde{T}$. Thus (see appendix A.1)

$$\boxed{\textbf{Without Loss of Generality: } \text{we can set } A = B = 0 \text{ in (1.9).}} \tag{1.10}$$

$\boxed{O(\epsilon^2)\ \textbf{equation:}}$ $\quad x_2'' + 2\omega_1 x_1'' + (2\omega_2 + \omega_1^2)\, x_0'' + x_2 + 3\nu x_0^2 x_1 = 0,\quad$ that is:

$$x_2'' + x_2 = \left(2\omega_2 + \omega_1^2\right) a\cos T + \frac{9}{16}\omega_1 \nu a^3 \cos 3T - \frac{3}{32} a^5 \cos^2 T \cos 3T\,, \tag{1.11}$$

where $\cos^2 T \cos 3T = \frac{1}{4}\cos T + \frac{1}{2}\cos 3T + \frac{1}{4}\cos 5T$. Again: x_2 will be periodic only if the coefficient of the $\cos T$ forcing term on the right hand side here vanishes. This yields

$$\omega_2 = -\frac{1}{2}\omega_1^2 + \frac{3}{256}a^4 = -\frac{15}{256}a^4 \tag{1.12}$$

and an explicit formula for x_2, which we do not display here. Clearly, this **process can be carried to any desired order** (see appendix A.2).

In summary, we have found for the solutions[8] of the Duffing equation:

$$\boxed{\begin{aligned} x &\sim a\cos T + \tfrac{1}{32}\epsilon\nu a^3 \cos 3T + O(\epsilon^2)\,, \\ T &= \omega t\,, \\ \omega &\sim 1 + \tfrac{3}{8}\epsilon\nu a^2 - \tfrac{15}{256}\epsilon^2 a^4 + O(\epsilon^3)\,. \end{aligned}} \tag{1.13}$$

1.2 van der Pol equation.

The equation has the form

$$\ddot{x} - \epsilon\nu(1-x^2)\dot{x} + x = 0\,, \tag{1.14}$$

where $0 < \epsilon \ll 1$ and $\nu = \pm 1$. We use now the same ideas of section 1.1, so that (1.3) and (1.4) still apply. Instead of (1.5) we get now

$$\omega^2 x'' + x - \epsilon\nu\omega(1-x^2)x' = 0\,. \tag{1.15}$$

[8]Notice that this is valid only as long as $0 \le a^2 \ll \epsilon^{-1}$. When $|a| = O(\epsilon^{-\frac{1}{2}})$, the "corrections" cease to be smaller than the leading order and the expansion fails. This agrees with our observations in footnote 3.

We proceed now to look at the expansion order by order.

At $O(1)$ we get, as before (see appendix A.3):

$$x_0 = a\cos T. \tag{1.16}$$

$O(\epsilon)$ equation: $\quad x_1'' + 2\omega_1 x_0'' + x_1 - \nu(1 - x_0^2)x_0' = 0,$ that is:

$$\begin{aligned} x'' + x_1 &= 2\omega_1 a\cos T - \nu a \sin T + \nu a^3 \cos^2 T \sin T \\ &= 2\omega_1 a \cos T + \nu a \left(\tfrac{1}{4}a^2 - 1\right)\sin T + \tfrac{1}{4}\nu a^3 \sin 3T\,. \end{aligned} \tag{1.17}$$

To get a periodic solution x_1, both the coefficients of $\cos T$ and $\sin T$ must vanish on the right hand side $\Longrightarrow$ For a nontrivial solution ($a \neq 0$) we must have[9]:

$$a = 2\,, \quad \omega_1 = 0 \quad \text{and} \quad x_1 = -\frac{1}{32}\nu a^3 \sin 3T + \underbrace{A\cos T + B\sin T}\,. \tag{1.18}$$

Note 2 *There is an important difference here with the situation in the analog equations (1.8) and (1.9). Now both sines and cosines appear on the right hand side of equation (1.17). Thus we* **end up with TWO conditions** *that must be satisfied if equation (1.17) is to have a periodic solution for x_1. These conditions are generally called* **Solvability Conditions**. *Thus now* **BOTH** *a and ω_1 are determined. There is* **NO FREE PARAMETER** *left and there is just* **one periodic orbit: the LIMIT CYCLE.**

Since now a is fixed to be $a = 2$, we **can no longer argue** that by a slight change in the amplitude and phase of x_0, we can set $A = B = 0$ (homogeneous part of the solution, marked by the brace above), as we did in (1.10). It is still true, however, that the phase of the leading order x_0 can be changed slightly. We can then use this to conclude (see appendix A.3)

Without Loss of Generality: we can set $B = 0$ in (1.18). (1.19)

On the other hand, we point out that A **remains to be determined.** That is, the circular part of the limit cycle orbit does not have a radius exactly equal to 2, but rather equal to $2 + \epsilon A + \ldots$

[9] We could take $a = -2$ also. This, however, is just a phase change $T \to T + \pi$. Thus, we may as well assume $a > 0$.

At the **next order** (that is, $O(\epsilon^2)$) we will get an equation of the form:

$$x_2'' + x_2 = \text{Forcing}\,. \tag{1.20}$$

Again (see note **3**) sine and cosine forcing terms on the right will have to be eliminated. This *will produce* **two conditions**, that will determine both A and ω_2 uniquely. In x_2 an homogeneous term of the form $\alpha \cos T$ will appear,[10] with α and ω_3 determined at $O(\epsilon^3)$. And **so on to higher and higher orders.**

$$\boxed{\begin{array}{l}\text{A unique solution, up to a phase shift, is produced}\\ \text{this way to all orders:}\qquad \textbf{The LIMIT CYCLE.}\end{array}} \tag{1.21}$$

Note 3 *In fact, after some calculation — using (1.16), (1.18) and (1.19) — we can see that (1.20) is:*

$$\begin{aligned} x_2'' + x_2 \;\; = \;\; & \left(2\omega_2 + \tfrac{1}{128}a^4\right) a\cos T + \left(\tfrac{3}{4}a^2 - 1\right)\nu A \sin T \\ & -\tfrac{3}{64}a^3\left(2 - a^2\right)\cos 3T + \tfrac{3}{4}\nu A a^2 \sin 3T + \tfrac{5}{128}a^5 \cos 5T\,. \end{aligned} \tag{1.22}$$

Thus we conclude

$$\omega_2 = -\frac{1}{256}a^4\,, \quad A = 0 \quad \text{and} \quad x_2 = \alpha\cos T + \frac{3}{512}a^3(2-a^2)\cos 3T - \frac{5}{3072}a^5\cos 5T\,, \tag{1.23}$$

where we recall that $a = 2$.

2 Two Timing, Multiple Scales method (TTMS) for the van der Pol equation.

2.1 Calculation of the limit cycle and stability.

In section 1.1 we basically obtained **all the solutions** to the Duffing equation (1.1) — since we ended up with two free parameters: the amplitude a and an arbitrary phase shift $T \to T - T_0$. On the other hand, in section 1.2 we only obtained the limit cycle solution. Now, suppose **we want all the solutions to the van der Pol equation (1.14)** — this will

[10] With a "$\beta sinT$" homogeneous part of the solution eliminated just as above in (1.19)

allow us to determine, in particular, the stability of the limit cycle. The method we introduce in this section (TTMS) will allow us to do this.

The main idea is that, if the **solution is not periodic**, then we *cannot represent it with a* **single solution of the linearized equation** (as we did in section 1, with its time dependence stretched by ω from t to $T = \omega t$ — to allow for nonlinear corrections to the period.[11]) For any "short" time period this will be O.K., but over long periods **large errors** may result because they **accumulate**. To resolve this difficulty we will **allow ALL the parameters of the linear solution to change SLOWLY in time**, so as to *track the true evolution of the solution.* Thus, for equation (1.14), we **expand**[12]**:**

$$x \sim x_0(\tau, t) + \epsilon x_1(\tau, t) + \epsilon^2 x_2(\tau, t) + \cdots \,, \tag{2.1}$$

where t takes care of the "normal" 2π-periodic dependence induced by the linear solution and $\boxed{\tau = \epsilon t}$ is the *slow time* (that will allow the linear solution being used to drift (slowly) as time evolves, from one linear orbit to the next.[13])

Remark 1 *Note that now the solution depends explicitly on* **two times***, thus the name for the method. In this case the "slow" time is $\tau = \epsilon t$, but in other problems it may be $\tau = \epsilon^2 t$ — or something else. Figuring out what the exact dependence should be need not be trivial and usually requires some thinking: it is related to the rate at which the nonlinearity causes drift in the orbits — as opposed to just shape changes. We will talk about this later.*

We now rewrite equation (1.14) in terms of the increased set of "independent" variables τ and t to obtain (here a dot indicates differentiation with respect to t):

$$\ddot{x} + 2\epsilon\dot{x}_\tau + \epsilon^2 x_{\tau\tau} + x - \epsilon\nu(1 - x^2)\dot{x} - \epsilon^2\nu(1 - x^2)x_\tau = 0\,. \tag{2.2}$$

Note that the equation is now a P. D. E. ! This method appears to complicate things! **However,** the extra terms are multiplied by ϵ and ϵ^2 and so at leading order we only get the linear O. D. E. In fact: we will only have to solve linear O. D. E.'s at each order in the approximation!

[11]**Namely**: the orbits in phase space are quite close to the linear ones, but the speed at which they are tracked is slightly different $\Longrightarrow$ Over long times a big error will accumulate, unless we correct for it.

[12]This is only a first, very simple, implementation. We will introduce a more refined one in section 2.3.

[13]This description, strictly, only applies to x_0 above. The higher order terms $\epsilon x_1 \ldots$ are there to account for the fact that the nonlinear orbits will have slightly different shapes than the linear ones.

As usual, we now substitute the expansion (2.1) into equation (2.2) and *collect equal powers of* ϵ to obtain

$O(1)$ equation:

$$x_0'' + x_0 = 0\,. \tag{2.3}$$

This is the same as in section 1.2, except that now the arbitrary "constants" in the solution of (2.3) will depend on τ. We thus have

$$x_0 = A_0(\tau)e^{it} + c.c.\,, \tag{2.4}$$

where *c.c.* **denotes complex conjugate** and A_0 is complex valued.

Remark 2 *Alternatively, we could write $x_0 = a(\tau)\cos t + b(\tau)\sin t$, where $A = \frac{1}{2}(a - ib)$. We cannot now argue, as we did before, that it is O.K. to set $b = 0$ using the fact that a change of time origin $t \to t + t_0$ is allowed. This is because t_0 has to be constant, while setting an arbitrary $b(\tau)$ to zero would require $t_0 = t_0(\tau)$, at least in principle.*[14]

Remark 3 *The use of complex notation in (2.4) makes life simpler. The kind of expansions we are doing require at each level of approximation that one expand things like x_0^3 in Fourier modes. This is much easier to do with exponentials than with sine and cosines!*

At $O(\epsilon)$ we obtain:

$$\begin{aligned} \ddot{x}_1 + x_1 &= -2\dot{x}_{0\,\tau} + \nu(1 - x_0^2)\dot{x}_0 \\ &= \left\{-2i\left(\frac{d}{d\tau}A_0 - \frac{1}{2}\nu A_0\left(1 - |A_0|^2\right)\right)e^{it} - i\nu A_0^3 e^{3it}\right\} + c.c.\,. \end{aligned} \tag{2.5}$$

This equation is very similar to (1.17), except that now: **(i)** We are using complex notation, **(ii)** There is no ω_1 term and **(iii)** A new term in $\frac{d}{d\tau}A_0$ appears because of the allowed τ dependence. The solution x_1 will be periodic in t provided the coefficient of the e^{it} forcing on the right hand side of (2.5) vanishes. This yields the equation

$$\boxed{\frac{d}{d\tau}A_0 = \frac{1}{2}\nu\left(1 - |A_0|^2\right)A_0,} \tag{2.6}$$

[14] Actually, an argument to set $b = 0$ can be made, namely: we expect the solutions of equation (1.14) to be basically oscillatory. Thus, they will have maximums and minimums. If we set $t = 0$ to occur at a local maximum, then $\dot{x} = 0$ at $t = 0$, which yields $b = 0$. But this argument will not work at higher orders.

which governs the evolution[15] of the amplitude A_0 for the linear (circular) orbits under the effect of the weak nonlinearity.

If we let $\boxed{A_0 = \frac{1}{2} a e^{i\varphi},}$ where a and φ are a real amplitude and phase, respectively, then[16]

$$\frac{d}{d\tau}\varphi = 0 \quad \text{and} \quad \frac{d}{d\tau} a = \frac{1}{8}\nu(4 - a^2)a\,. \tag{2.7}$$

These formulas show that the orbits in the phase plane are nearly circular, with a slowly changing radius a that evolves following the second equation in (2.7) and **a limit cycle for $a = 2$.** In particular:

$$\boxed{\textbf{For } \nu = 1 \textbf{ the limit cycle is stable and it is unstable for } \nu = -1.} \tag{2.8}$$

If we let $\mu = \epsilon\nu$ in (1.14) and write the equation as

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0\,, \tag{2.9}$$

then we see that our calculations here show that at $\mu = 0$ we have a **bifurcation**, with an exchange of stability between the limit cycle and the critical point at the origin.

$$\boxed{\begin{array}{l} \mu < 0.\ \text{Unstable limit cycle and stable spiral point.} \\ \mu > 0.\ \text{Stable limit cycle and unstable spiral point.} \\ \mu = 0.\ \text{Center with continuoum of periodic orbits. } \textit{(There is no limit cycle.)} \end{array}} \tag{2.10}$$

2.2 Higher orders and limitations of TTMS.

We us now finish the $O(\epsilon)$ calculation and solve equation (2.5) using (2.6). We have

$$x_1 = \left\{ \frac{1}{8} i\nu A_0^3 e^{3it} + A_1(\tau) e^{it} \right\} + c.c.\,, \tag{2.11}$$

where A_1 is complex valued.

Let us now continue the expansion to one more order, as there is an **important detail to be learned from doing this.**

[15] *Drift* in phase space

[16] Since this shows that φ is a constant, we could have taken $b = 0$ in remark **2 !**

The $O(\epsilon^2)$ equation is:

$$\begin{aligned}\ddot{x}_2 + x_2 &= -2\dot{x}_{1\tau} - x_{0\tau\tau} + \nu\dot{x}_1 + \nu x_{0\tau} - \nu x_0^2\dot{x}_1 - 2\nu x_0 x_1 \dot{x}_0 - \nu x_0^2 x_{0\tau} \\ &= \Big\{-2i\Big(A_1' - \tfrac{1}{2}\nu A_1 + \nu\,|A_0^2|\,A_1 + \tfrac{1}{2}\nu A_0^2 A_1^* + \tfrac{1}{2}i\nu A_0' - \tfrac{1}{2}iA_0'' \\ &\quad -\tfrac{1}{2}i\nu\,(|A_0^2|\,A_0)' + \tfrac{1}{16}i\,|A_0^4|\,A_0\Big)\,e^{it} + (\ldots)e^{3it} + (\ldots)e^{5it}\Big\} + c.c.\,,\end{aligned} \tag{2.12}$$

where $\boxed{{}' = \frac{d}{d\tau}}$ and A_1^* denotes the complex conjugate of A_1. Thus, *to avoid secular terms* in x_2 (namely: terms proportional to $t\,e^{it}$, that destroy the periodicity in t) the coefficient of e^{it} on the right hand side of this last equation must vanish. Thus

$$A_1' - \frac{1}{2}\nu A_1 + \nu\left|A_0^2\right|A_1 + \frac{1}{2}\nu A_0^2 A_1^* = -\frac{1}{2}i\nu A_0' + \frac{1}{2}iA_0'' + \frac{1}{2}i\nu\left(\left|A_0^2\right|A_0\right)' - \frac{1}{16}i\left|A_0^4\right|A_0\,. \tag{2.13}$$

This is a rather messy equation. We do not aim to solve it here; but only to analyze its behavior for τ large.

Assume $\nu = 1$: In this case the limit cycle is stable and, **for τ large** — see equation (2.7) — $A_0 \sim e^{i\varphi}$**, for some constant φ.** Then equation (2.13) reduces to

$$A_1' + \frac{1}{2}A_1 + \frac{1}{2}e^{2i\varphi}A_1^* = -\frac{1}{16}ie^{i\varphi}\,. \tag{2.14}$$

This is much simpler and can be solved explicitly[17]

$$A_1 = \left(C_1 e^{-\tau} + iC_2 - \frac{1}{16}i\tau\right)e^{i\varphi}\,, \tag{2.15}$$

where C_1 and C_2 are real constants. This means that **the solution of equation (2.13) will behave, for large τ, like**

$$A_1 \sim -\frac{1}{16}i\tau e^{i\varphi}\,. \tag{2.16}$$

This is "bad". Notice that the expansion (2.1) for the solution of (1.14) — use equations (2.4) and (2.11) — is

$$x \sim 2\,\mathrm{Re}\left(A_0(\tau)e^{i\tau}\right) - \frac{1}{4}\epsilon\,\mathrm{Im}\left(A_0^3(\tau)e^{3it}\right) + 2\epsilon\,\mathrm{Re}\left(A_1(\tau)e^{it}\right) + \cdots\,.$$

But, when $\epsilon\tau = O(1)$ **the second term in the expansion will not be small at all** (as $\epsilon A_1 \sim -\frac{1}{16}i\epsilon\tau e^{i\varphi}$) **!** Thus

$$\boxed{\text{The two timing expansion (2.1) is only valid as long as } |\tau| \ll \epsilon^{-1}.} \tag{2.17}$$

[17] Write $A_1 = ze^{i\varphi}$. Then $z' + \mathrm{Re}(z) = -\frac{1}{16}i$.

This is **pretty typical for TTMS expansions:** Usually they are valid for a time range where the "slow" time can be taken large — but not arbitrary large. Beyond some ϵ^{-p}, for some p, they fail.

In the current situation (2.17) is not terribly upsetting. It still allows us to take τ fairly large. Once τ is large and the limit cycle is reached $\Longrightarrow$ can switch to the expansion in section 1.2 **!!**

However: suppose that (2.17) makes us terribly unhappy, for whatever reasons. Then

$$\boxed{\textbf{Can we fix the problem posed by (2.17)?}} \tag{2.18}$$

The answer to this question is **YES**, but **first we must understand why (2.17) occurs!** This is clarified next; **for simplicity we CONSIDER ONLY the STABLE LIMIT CYCLE case, when $\nu = 1$.**

Note 4 *Equations (2.1)–(2.7) lead to an approximation of the limit cycle (for large τ, so that $A_0 \sim e^{i\varphi}$) given by*

$$x \sim 2\,Re(e^{i(t+\varphi)}) = 2\,\cos(t+\varphi)\,. \tag{2.19}$$

On the other hand, the **PLM calculation of section 1.2 tells us that we should use**

$$x \sim 2\,\cos(\omega t+\varphi) = 2\,Re(e^{i(\omega t+\varphi)})\,,$$

where $\omega = 1 - \frac{1}{16}\epsilon^2 + \cdots$. Now, since (expand in Taylor series)

$$e^{i(\omega t+\varphi)} = e^{i(t+\varphi)}e^{-i\frac{1}{16}\epsilon^2 t+\cdots} = e^{i(t+\varphi)} - \frac{1}{16}i\epsilon^2 t e^{i(t+\varphi)} + \cdots\,, \tag{2.20}$$

we see that the error in (2.19) is $-\frac{1}{16}i\epsilon^2 t e^{i(t+\varphi)} + \cdots$, which is precisely the "bad" behavior arising in A_1 earlier in equation (2.16). *Thus*

The TTMS expansion goes bad because it does not properly take into account the fact that the nonlinearity affects the *phase* — i.e. the position along the linear orbit of the solution.	(2.21)

• *It follows that,* **to achieve (2.18) we must fix the problem pointed out by (2.21).** THIS WE DO NEXT.

2.3 Generalization of TTMS to extend the range of validity.

Let ϕ be the *phase* of the solution — namely: its position along the orbit — and $\boxed{\omega = \frac{d}{dt}\phi}$ its *angular velocity.* The phase increases with time and, **for the linearized equation, we have**

$$\frac{d}{dt}\phi = \omega = 1\,. \tag{2.22}$$

However, once **nonlinear effects** kick in, **there is no reason for** ω *to remain equal to* 1, *or in fact even constant!*

Now, when considering a **periodic orbit**, as long as ω is *approximated by its correct average value* things will be O.K. (as then errors will not accumulate over time). This is what **PLM does**, by taking $\boxed{\phi = T = \omega t}$ with $\boxed{\omega = 1 + \epsilon\omega_1 + \cdots}$. We **cannot use this idea of PLM in TTMS,** because now the orbit (thus the average value of ω) varies slowly as time changes. We **must then allow ω to be a function of τ.** Thus
To fix the type of problem discussed in the previous section 2.2 we must replace the expansion (2.1) by a subtler type, where the phase (fast time) itself is to be determined. Generally we must deal then with expansions of the form

$$x \sim X_0(\tau, \phi) + \epsilon\, X_1(\tau, \phi) + \epsilon^2\, X_2(\tau, \phi) + \cdots\,, \tag{2.23}$$

where 2π–periodic dependence on the phase ϕ is required, $\tau = \epsilon t$ and

$$\frac{d}{dt}\phi = \omega = 1 + \epsilon\,\omega_1(\tau) + \epsilon^2\,\omega_2(\tau) + \cdots\,.$$

This amounts to writing: $\phi = \frac{1}{\epsilon}\,(\tau + \epsilon\,\phi_1(\tau) + \epsilon^2\,\phi_2(\tau) + \cdots)$, where $\boxed{\frac{d}{d\tau}\phi_j = \omega_j}$.

When no τ dependence is allowed, this reduces to PLM. We will not carry out the details of this expansion here — they are quite messy and some technicalities are involved in selecting the ω_j's so that the X_j's behave "properly" as functions of τ (that is, no *secular* growth in τ occurs). On the other hand, in the particular case of the van der Pol equation (1.14), when the limit cycle is stable[18]: all solutions eventually approach the limit cycle, and they do so on time scales where $\tau \ll \epsilon^{-1}$ (as follows from our results in section 1.2). Thus, as long as no cumulative errors occur in tracking the limit cycle, there should be no problems. We can conclude thus, <u>without doing any calculations</u>, that:

[18] That is, $\nu = 1$.

> For equation (1.14), in the case $\nu = 1$:
>
> - The ω_j's in equation (2.23) are constant and equal to the values calculated for the expansion in section 1.2.
> - The functional form of $X_0(\tau, \phi)$ in equation (2.23) is the same as that we obtained for $x_0(\tau, t)$ in equation (2.1), with t replaced by ϕ. That is: $X_0(\tau, \phi) = x_0(\tau, \phi)$.

In particular, note that from this we learn that the *TTMS approximation for the behavior of the van der Pol equation is quite good.* The secular growth displayed by A_1 in equation (2.16) for very long times is nothing to worry about. It is simply a manifestation of the fact that we have some small (very small, $O(\epsilon^2)$) errors on the velocity at which the solution moves along the limit cycle, but of nothing else. *No important qualitative or quantitative effect is missing.*

Note 5 *Other ways to fix the problem in (2.17) can be devised. For example, some people advocate introducing ever slower time scales, such as $\epsilon^2 t$, $\epsilon^3 t$ and so on — in addition to the ϵt of equation (2.1).* **This is not a good idea, unless the problem truly depends on that many scales!** *For example: if the difficulty arises because the true slow time dependence*[19] *is on something like (say) $\frac{\epsilon}{1+\epsilon^2} t$ and not ϵt, then this "lots of scales" approach will just complicate things for no real gain at all. For an expansion to be useful, it has to zero into the real behavior of the solution. The aim of doing an asymptotic expansion should be to learn something useful about the solution, not to produce a massive amount of algebra (even if this is, sometimes, an unfortunate byproduct, it is not the aim). In particular, producing an "approximation" that fools us into believing that the solution depends on very many different time scales (when in fact it does not), is exactly opposite to this objective.*

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[19]Notice that the van der Pol equation is exactly an example of this type.

A Appendix.

A.1 Some details regarding section 1.1.

Generally, **asymptotic expansions** — like the ones in these notes — **require** at each level the **solution** of a **linear equation** with some forcing made up from the prior terms. The solution of this linear equation is then required to satisfy some condition (periodicity in the examples here) and this imposes restrictions on the forcing terms. These restrictions are then used to determine free parameters, slow time evolutions, etc.

When solving the linear equations in the expansion, it is very important to include in the solution **ALL** the free parameters consistent with the conditions imposed on the solution. This is because parameters that are "arbitrary" at some level, may later be needed to satisfy the restrictions at a higher order.[20] Failure to include a particular parameter — which boils down to setting it to some arbitrary fixed value — will typically cause trouble at higher order, when a restriction on a forcing term will be found impossible to satisfy.

On the other hand, practical considerations dictate that we carry as few free parameters in a calculation as feasible. Thus, one must always look at the equations involved and ask if there is some argument that would allow for the elimination of a parameter — but *never* must one eliminate a parameter without a good reason.[21]

Consider now equation (1.1) — or (1.5). This *equation is invariant under time translation:* if $x = X(t)$ is a solution, then so it is $x = X(t - t_0)$. Thus, we can always pick the origin of the time coordinate to simplify the solution and eliminate parameters.

For example: The general solution of (1.6) is: $a\cos(T - T_0)$, where a and T_0 are constants. But the invariance under time translation shows that we can set $T_0 = 0$.

Furthermore: At the level of (1.9) we know that in fact a is arbitrary. Then, since A and B in (1.9) amount to making small $O(\epsilon)$ changes to a and T_0 at the $O(1)$ level — thus they are not true "new" free parameters — we can again set $A = B = 0$, as in (1.10), without any fear.

[20] For example, in section 1.2, the amplitude a in (1.16) is eventually set to $a = 2$ in (1.18).

[21] Conversely: if an expansion fails at some level, one should **always** check to see if somehow an important degree of freedom (some parameter) was ignored!

In fact, the same argument shows that **we can conclude:**

$$\boxed{\begin{array}{l}\text{At any level } O(\epsilon^n) \text{ in the expansion, for } n > 1, \text{ we can} \\ \text{take } x_n \text{ in (1.3) with } \textbf{NO} \cos T \text{ or } \sin T \text{ components.}\end{array}} \tag{A.1}$$

A.2 More details regarding section 1.1.

It is clear that, in the expansion of section 1.1, the $O(\epsilon^n)$ equations — for $n > 1$ — have the form

$$x_n'' + x_n = P_n(x_0, \ldots, x_{n-1}) - \sum_{\ell=1}^{n} \alpha_\ell x_{n-\ell}''\,, \tag{A.2}$$

where P_n is a cubic polynomial and the α_ℓ's are constants defined by $\omega^2 = \sum_{\ell=0}^{\infty} \alpha_\ell \epsilon^\ell$. Thus $\alpha_0 = 1$, $\alpha_1 = 2\omega_1$, $\alpha_2 = 2\omega_2 + \omega_1^2$, $\alpha_3 = 2\omega_3 + 2\omega_1\omega_2$, $\ldots$. In general we can see that $\boxed{\alpha_n = 2\omega_n + f_n(\omega_1, \ldots, \omega_{n-1}),}$ where f_n is a quadratic polynomial.

Because x_0 is even, the forcing on the right hand side of (1.8) is also even. Then (1.10) gives x_1 even. The same type of argument shows then that x_2 is also even. More generally, one can show using (A.1) that **all the x_n's are even.**

Now, the condition on (A.2) to get x_n periodic in T is that the right hand side should not have any forcing proportional to either $\sin T$ or $\cos T$. But the right hand side is even, thus there is **NO** $\sin T$ forcing ever. On the other hand, the coefficient of the $\cos T$ forcing has the form: $2a\omega_n + G_n(a, \omega_1, \ldots, \omega_{n-1})$, where G_n is some polynomial function. Thus, **one can always choose ω_n so as to make the coefficient of $\cos T$ vanish.** We have thus shown that

$$\boxed{\text{The expansion in equation (1.3) works up to any order.}} \tag{A.3}$$

A.3 Some details regarding section 1.2.

Equation (1.14) is invariant under time translation. Thus, just as we did in appendix A.1, we have a phase to play with and can use to eliminate parameters.

We used this fact in (1.16) to eliminate the sine component in $x_0(T)$. But now a is no longer a free parameter in the solution, as equation (1.18) shows that $a = 2$. Thus, in order to eliminate spurious parameters in $x_1(T)$ (from the two – A and B – that appear in (1.18)), we only have a phase to play with.

Since $2\cos(T - \frac{1}{2}\epsilon B) = 2\cos T + \epsilon B \sin T + \ldots$, it follows that a small phase change can be used to eliminate B in $x_1(T)$ as given in (1.18). But A cannot and should not be eliminated from the formula. In fact, at $O(\epsilon^2)$ the solvability requirement on the equations (periodicity of $x_2(T)$) will determine A in the same fashion that $a = 2$ followed from the $O(\epsilon)$ equation. At this level it will be possible to argue that no term in $\sin T$ is needed in $x_2(T)$, but a term $\alpha \cos T$ must be kept (with α determined at $O(\epsilon^3)$). Clearly the same pattern will be repeated over and over. In this fashion the **expansion can be continued to any desired order**.

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First Problem Set

Suggested Readings (textbook): Chapters 1-2-3.
Suggested Problems (textbook):

Ch. 2:	2.2.9	2.2.12	2.2.13	2.3.3	2.4.9	2.6.1	2.8.3	2.8.5
Ch. 3:	3.3.1	3.4.5	3.4.7	3.4.8	3.4.9	3.4.10		

Problems to hand in for grading (textbook):

Ch. 2:	2.2.8	2.2.10	2.3.2	2.5.4	2.5.5	2.5.6
Ch. 3:	3.2.6	3.2.7	3.3.2	3.4.6		

PROBLEM TO HAND IN FOR GRADING (not in textbook):

PDE_Blow_Up

In the lectures we considered the PDE problem initial value problem:

$$u_t + u*u_x = 0; \; u(x, 0) = F(x).$$

Notation:
1) u_t and u_x are the partial derivatives, with respect to t and x (resp.).
2) t is time and x is space.
3) * is the multiplication operator.
4) ^ denotes taking a power [u^2 is the square of u].
4) u = u(x, t) is a function of x and t.

We showed that the solution to this problem ceased to exist at a finite time (the derivatives of u become infinite and, beyond that, u becomes multiple valued) whenever dF/dx was negative anywhere.

This was shown "graphically". It can be shown analytically as follows:

--- A. Consider the CHARACTERISTIC CURVES dx/dt = u(x, t), as instroduced in the lecture.

--- B. Along each characteristic curve, one has du/dt = 0, as shown in class. Now, let v = u_x. Then v satisfies the equation [obtained by taking the partial derivative with respect to x of the equation for u]:

$$v_t + u*v_x + v^2 = 0.$$

Thus, along characteristics: dv/dt + v^2 = 0. Thus, if v is negative anywhere, v develops an infinity in finite time. But the initial conditions for v, along the characteristic such that x(0) = x_0, is v(0) = dF/dx(x_0).

Hence the conclusion follows: the solution u = u(x, t) to the problem ceases to exist at a finite time (with the derivative u_x of u becoming infinite somewhere) whenever dF/dx is negative anywhere.

CONSIDER NOW THE PROBLEM:

$$u_t + u*u_x = -u; \; u(x, 0) = F(x).$$

Show that the solution to this second problem ceases to exist at a finite time, provided that dF/dx < C < 0, where C is a finite (non-zero) constant. Again, what happens is that the derivatives become infinite. Calculate C.

Hint: Use an approach analog to the one used above: get an ODE for the derivative v = u_x along the characteristics, and study the conditions under which the solutions of the ODE blow-up in a finite time.

A graphical approach for how the solution to u_t + u*u_x = -u
behaves in time will also work, but the approach using the ODE
for v along characteristics turns out to be simpler.

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Second Problem Set

Suggested Readings (textbook): Chapters 4-5-6.
Suggested Problems (textbook): Strongly Recommended: SR
Ch. 3: 3.4.11 3.5.6 3.5.7 3.6.6 3.7.5 3.7.6
Ch. 4: 4.3.2 4.3.3 4.6.4 4.6.5
Ch. 5: 5.1.10 5.2.14 .... SR: 5.1.10
Ch. 6: 6.1.8 6.1.9 6.1.11 6.1.13 6.2.2 .... SR: 6.2.2

Problems to hand in for grading (textbook):
Ch. 3: 3.4.14 3.4.15 3.5.8 3.6.3
Ch. 4: 4.1.1 4.1.8
Ch. 5: 5.2.11
Ch. 6: 6.1.7 6.1.10 6.1.12
Special Problem Below.

NOTE: you can use the MatLab® scripts:
PHPLdemoA, PHPLdemoB, PHPLplot or PHPLplot_v2
with the problems requiring computer plotting.

SPECIAL PROBLEM

Consider a system in the plane: dx/dt = f(x, y),
dy/dt = g(x, y)

such that the origin P = (x, y) = (0, 0)} is an isolated critical point, with the linearized system there having a stable star. Now consider the following two alternatives for the complete behavior of the system:

(a) Linearized: stable star -----> Fully nonlinear: stable spiral.
(b) Linearized: stable star -----> Fully nonlinear: stable proper node.

Which ones are possible? For each one that is possible, give an example of a system with the desired behavior. Otherwise, explain why you think the particular alternative cannot happen. In this case, how close can you get (produce an example that ``almost'' does it)?

OPTIONAL: Give thought to the nature of the perturbation you need:

smooth (smooth means that the perturbation has infinitely many derivatives) perturbations will not do the job, why?

It turns out that the perturbations needed cannot even have a second derivative at the origin (you need at least one derivative to have the linearization make sense). Can you give some argument in the direction of what is the ``minimum'' amount of singularity needed for the job?

RECALL THE DEFINITIONS:

(1) For a linear system, a stable star is a point with a double eigenvalue of equal algebraic and geometric multiplicities. Thus its associated matrix is a multiple of the identity.
(2) We say that a critical point for a nonlinear system is a node (spiral, whatever) if the phase portrait NEAR the critical point can be ``deformed'' by a continuous transformation into the phase portrait for the corresponding linear system. That is: the two phase portraits ``look'' qualitatively the same.[For the purposes of this problem use this second ``definition'' --- i.e.: do not worry about continuous transformations, just show that the key properties are the same.

Thus, the origin is:

(I) A (stable) spiral point if the orbits near the origin satisfy:

r ---> 0 and \theta ---> infinity (or \theta ---> -infinity)
as t ---> infinity.

(II) A (stable) proper node} if all the orbits near the origin
approach it as t ---> infinity, and there are two special
directions
theta = +/- theta_1
and
theta = +/- theta_2
such that:
--- There is exactly one orbit such that
theta ---> + theta_1 as t ---> infinity.
--- There is exactly one orbit such that
theta ---> - theta_1 as t ---> infinity.
--- For all other orbits: As t ---> infinity,
either theta ---> + theta_2,
or theta ---> - theta_2.

HINT:
Consider first small linear perturbations to a linear systems
that cause the appropriate changes. Then write systems where
perturbations of the same form are introduced by a nonlinearity.
The nonlinearity will have to be small, so that it vanishes
faster than the linear terms as the origin is approached; but do not
make it vanish too fast, else it will not do the job! In fact, you
should find that it must vanish so "slowly", that the resulting
function has second derivatives that "blow up" at the origin.

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Third Problem Set

Suggested Readings (textbook): Chapters 6-7.
Suggested Problems (textbook):
Ch. 6: 6.3.10 6.3.11 6.5.8 6.5.13 6.6.3 6.8.7
Ch. 7: 7.1.9 7.2.5 7.2.7 7.2.16
Note: Part b of 6.5.13 is actually wrong. Figure out what actually happens when epsilon < 0.
Note: For 6.8.7 you will need index theory, but it will not be enough. Dulac's criterion (for example) will also be needed.

Problems to hand in for grading (textbook)
Ch. 6: 6.3.13 6.3.16 6.5.7 6.5.19 6.8.9
Ch. 7: 7.2.6

NOTE: you can use the MatLab® scripts:
PHPLdemoA, PHPLdemoB, PHPLplot or PHPLplot_v2
with the problems requiring computer plotting.

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Fourth Problem Set

Suggested Readings (textbook): Chapters 7-8.
Suggested Problems (textbook):
  Ch. 7: 7.2.17 7.3.1 7.3.2 7.3.8 7.3.11 7.5.5 7.5.7 7.6.10
  Ch. 8:

Problems to hand in for grading (textbook)
  Ch. 7: 7.3.10 7.5.6 7.6.14
  Ch. 8:

Note: For 7.3.10; think before you do anything. It's quite easy if you go at it the right way, else ... think of the component of \dot{\vect{x}} in the radial direction for large r.

Note: 7.5.5 may sound confusing at first; but it can be done using exactly the same set of tricks that work for van de Pol equation.

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Fifth Problem Set

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Suggested Readings (textbook): Chapter 8-9.
Suggested Problems (textbook):
   Ch. 8:  8.1.11  8.2.6   8.2.11  8.2.12
           8.3.1   8.5.2   8.6.7   8.7.5
   Note: For 8.2.12; see if you can derive the criteria for
         a Hopf Bifurcation using the tools in the Handouts.

Strongly Suggested Problems:
   ---- "Fourier Series Problem".
   ---- "Variable Length Pendulum Problem".
   To do these you will have to download the problem statements,and the
   MatLab® scripts in the "MatLab® for Fourier Series" link, namely:
      readmeFouSer.m
      fourierSC.m
      FSFun.m
      FSoption.m
      FSoptionP.m
      heatSln.m
   All this can be found in the 18.385 WEB page.
```

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Problems to hand in for grading (textbook):
   Ch. 8:  8.1.6   8.2.5   8.6.5   8.7.3
```

Exam Number 1 for 18.385, MIT (Fall 2002).

Due at the Lecture of Thursday November 14, 2002.

Rodolfo R. Rosales*   November 7, 2002.

Course TA: Boguk Kim, MIT, Dept. of Mathematics.

Contents

- **Instructions.** **1**

- **Problems.** **2**

  1 Problem 2002.1.1. 2

  2 Problem 2002.1.2. 3

  3 Problem 2002.1.3. 3

  4 Problem 2002.1.4. 3

• Instructions.

1. **Read all the instructions and read each problem statement completely <u>before</u> you begin the process of solving it.**
2. **Do it alone.** You **can consult only** with the lecturer and/or the TA.
3. You can use the textbook, your class notes, hand-outs and the problem set answers posted in the course WEB page. **Nothing else,** in particular: you **cannot** use the results in the answers to the problems at the end of the book. When referencing a result, be **specific** (e.g.: "Using Theorem 787, page 9986 in the book"). **Explain** how the thing being referred to fits into your answer.
4. There is no time limit, but a few hours should be enough.
5. **Write the answers to each problem on separate pages. Write your name and the problem solved** (as in: *18.385, J. Doe, Exam #1, Problem #77, page 2 of 7329.*) **at the head of each page** (meaning that your exam answer has 7329 pages, this is page 2 and you are doing problem # 77). **This is important, in case the answers get mixed up, which will probably happen!**
6. **Staple** the whole exam.
7. In all your solutions show your reasoning, explaining carefully what you are doing. **Use English, not just mathematical symbols.** You play dice with unjustified steps. Maybe I'll buy them, maybe not. If not: tough luck. **Note:** this does not mean that you have to justify $2 + 2 = 4$!

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IMPORTANT POINT: a computer plot is not enough! For example, in cases where a phase plane portrait, or classification of critical points, or something like this is required, an explanation and justification of your conclusions is expected. The computer should be taken merely as a way to produce a pretty plot, give you hints as to what might be going on, check out your ideas and so. I have no interest in testing your ability to use PHPLdemoB or equivalent software! At any rate, beware: some of the problems may have subtleties that are easy to figure out with a good analysis, but can fool you in a computer plot (if you do not know what to look for).

Please: no "chicken scratches" or arrows on the side of the page leading from one piece of an argument to another and so on. *If a particular thing is illegible,* **write it again.** The **answers MUST be readable.**

8. Start **early.** Do not wait till the night before it is due.
9. By the way: you are not expected to produce answers at the level of the ones provided in the web page; you are, however, expected to read them.

• Problems.

Note: I will be looking for <u>REAL</u> arguments. I do not require, nor expect mathematical rigor, but do expect what may be called a "reasonable person" justification of your conclusions — of the type used in both the lectures, the notes on the WEB page, and the answers to the problem sets provided.

1 Problem 2002.1.1.

Statement:

Consider the equation

$$\frac{d^2 x}{dt^2} - \epsilon \cos(x) \frac{dx}{dt} + x = 0\,, \tag{1.1}$$

where $0 < \epsilon \ll 1$. Using a *two-timing* approach, calculate (to leading order) the limit cycles [and their stability] for this equation.

Hint 1.1 *In this particular example it is convenient to write the leading order solution in the form* $x_0 = a(\tau)\cos(t)$*, where* τ *is the slow time and the origin of time is selected so that there is no sine term. What you are being asked to do is to find an equation for the amplitude* $a = a(\tau)$*. In this endeavor you may want to check the integral representations for the integer order Bessel functions.*

Note 1.1 *This* **problem is worth 25 points.**

2 Problem 2002.1.2.

Statement:

Do problem 3.5.4 in the book by Strogatz [bead on an horizontal wire].

Note 2.1 *This* **problem is worth 25 points.**

3 Problem 2002.1.3.

Statement:

Consider problem 3.5.4 in the book by Strogatz [bead on an horizontal wire], and change it as follows:

- Assume that the relaxed length of the spring L, is much smaller than the distance h from the spring support to the wire. That is $L \ll h$. Thus you can make the approximation $L = 0$ and eliminate a parameter from the problem.
- Assume that the whole contraption is rotating, at angular velocity Ω, around a vertical axis going through the support of the spring.

Note 3.1 *This* **problem is worth 25 points.**

4 Problem 2002.1.4.

Statement:

Do problem 3.6.6 in the book by Strogatz [patterns in fluids].

Note 4.1 *This* **problem is worth 25 points.**

THE END.

Exam Number 2 for 18.385, MIT (Fall 2002).

Due on the last day of classes, Fall 2002.

MIT, Department of Mathematics, Cambridge, MA 02139.

INSTRUCTIONS and RULES: same as those for the first exam.

Problem #1 (35 POINTS).
--- Do the "Balancing a Broom" problem (posted on the WEB page).

Problem #2 (20 POINTS)
--- Do the "Coastline Fractal" problem (posted on the WEB page).

Problem #3 (25 POINTS)
--- Do problem 8.6.3 of the book by Strogatz.

Problem #4 (20 POINTS)
--- Do problem 9.6.2 of the book by Strogatz.

The points are assigned more-or-less on the basis of the degree of
difficulty, and/or length of the problem (in my estimation).

18.385 Problem. Rodolfo R. Rosales.[1] (December 5, 2002).

• Balancing a broom.

Statement:

Consider the problem of balancing a broom upright, by placing it on a surface that moves up and down in some prescribed manner. Specifically:

Assume a rough flat horizontal surface, which oscillates up and down following some prescribed law (that is, at any time the surface can be described by the equation $y = Y(t)$, where y is the vertical coordinate, and Y is some oscillatory function). On this surface we place a broom, in upright position, with the sweeping side pointing up.[2] **Question: Can we prescribe Y in such a way that the broom remains upright — i.e.: the position is stable?**

In order to answer the question, consider the following idealized situation:

A) Replace the broom by a mass m, placed at the upper end of a (massless) rigid rod of length L. Let the displacement of the rod from the vertical position be given by the angle θ, with $\theta = 0$ corresponding to the rod standing vertical, and the mass on the upper end.

B) The bottom of the rod is attached to a hinge that allows it to rotate in a plane. Thus the motion of the rod is restricted to occur on a plane.

C) Assume that friction can be neglected.

D) The hinge to which the rod is attached oscillates up and down, with position $x = 0$ and $y = Y(t)$ — x is the horizontal coordinate on the plane where the rod moves. The mass is then at $x = L\,\sin(\theta)$ and $y = Y + L\,\cos(\theta)$ — we measure angles clockwise from the top.

Now, do the following:

(1)

Use Newton's laws to derive the equation of motion for the mass m. You should obtain a second order ODE for the angle θ, with coefficients depending on the parameters g (the acceleration of gravity) and the length of the rod L — in addition to the forcing function $Y = Y(t)$.

Hint: Only two forces act on the mass m, namely: gravity and a force $F = F(t)$ along the rod. The force F has just the right strength to keep the (rigid) rod at constant length L — this is enough to determine F, though you do not need to calculate it.

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[2]Because the surface is rough, the contact point of the broom with the surface will not move relative to the surface.

(2)

You should notice that adding a constant velocity to the hinge motion (that is: $Y \to Y + v\,t$, where v is a constant) does not change the equation of motion. Why should this be so? What physical principle is involved?

(3)

Write down the (linearized) equations for small perturbations of the equilibrium position ($\theta = 0$) that we wish stabilized. **Stability occurs if and only if $Y = Y(t)$ can be selected so that the solutions of this linear equation do not grow in time** — strictly speaking we should also consider the possible effects of nonlinearity, but we will ignore this issue here.

(4)

You should notice that it is possible to stabilize $\theta = 0$ by taking $Y = -a\,t^2$, where $a > 0$ is a constant acceleration. How large does a have to be for this to happen? Give a justification of this result based on physical reasoning, without involving any equations (this is something you should have been able to predict before you wrote a single equation).

(5)

Of course, the "solution" found in **(4)** is not very satisfactory, since Y grows without bound in it. Consider now oscillatory forcing functions of the form:

$$Y = \ell\,\cos(\omega\,t)\,, \tag{1}$$

where $\ell > 0$ and $\omega > 0$ are constants (with dimensions of length and time^{-1}, respectively).

$$\boxed{\textbf{The objective is to find conditions on } (\ell, \omega) \textbf{ that guarantee stability.}} \tag{2}$$

The next steps will lead you through this process, but first: **Nondimensionalize the (linearized) stability equation.** In doing so it is convenient to use the time scale provided by the forcing to nondimensionalize time — i.e.: let the nondimensional time be $\boxed{\tau = \omega\,t\,.}$

This step should lead you to an equation describing the evolution of the angle θ (valid for small angles), involving two nondimensional parameters. One of them, $\boxed{\epsilon = \ell/L,}$ measures the amplitude of the oscillations in terms of the length of the rod. The other measures the time scale of the forcing (as given by $1/\omega$) in terms of the time scale of the gravitational instability — a function of g and L. **Call this second parameter** $\boxed{\mu}$ — note that in the equation only μ^2 appears, not μ itself.

(6)

Find the stability range for μ as a function of ϵ, for the values $0 < \epsilon \leq 0.6$ — it is enough to pick a few values of ϵ, say $\epsilon = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, and then to compute the stability range for each of them.

Note/hint: This step will require not just analysis, but some numerical computation. So as not to be forced to explore all possible values of μ when looking for the stability ranges (numerically an impossible task), you should notice that the analysis for $\epsilon = 0$ can be done exactly — and should provide you with a good hint as to where to look.

(7)

Write the period $p = \dfrac{2\pi}{\omega}$ of the forcing, in terms of the nondimensional parameter μ, and the parameters g and L. The results of **part (6)** should provide you with the period ranges (for a given oscillation amplitude) where stability occurs. Use this information to provide a rough explanation of why it is relatively easy to balance a broom on the palm of your hand (using the strategy outlined in this problem — try it), and why you will not be able to balance a pencil.

(8)

For $0 \leq \epsilon \ll 1$ and $0 \leq \mu \ll 1$ you should be able to obtain analytical approximations for the stable ranges. Do so, and compare your results with those of **part (6)**.

Hint: Floquet theory provides a function (the Floquet Trace $\alpha = \alpha(\mu, \epsilon)$) that characterizes linearized stability — stability if and only if $|\alpha| \leq 1$. Compute this function for μ and ϵ small.

THE END.

- ## Coastline Fractal.

Statement:

In this problem we construct a fractal that is a *very idealized* caricature of what a coastline looks like. The construction proceeds by iteration of a basic process, which we describe next.

We start with a simple curve, Γ_0, and apply to it a simple process, that yields a new curve Γ_1. This new curve is made up of several parts, each of which is a scaled down copy of Γ_0. The same simple process is then applied to each of these parts, yielding Γ_2. Then we iterate, to obtain in this fashion a series of curves Γ_n, for $n = 0, 1, 2, 3 \ldots$. The fractal is then the limit of this process: $\Gamma = \lim_{n\to\infty} \Gamma_n$ — provided the limit exists.

For the "coastline fractal" we **start by picking an angle** $0 < \theta < \pi$**, and a length** $R_0 > 0$**.** Then the first curve is:

$$\boxed{\Gamma_0 = \textbf{Circular arc of radius } R_0\textbf{, subtending an angle } \theta.} \tag{1}$$

Next **divide** Γ_0 **into three equal sub-arcs, each subtending an angle** $\theta/3$**, and replace each of these pieces by a properly scaled version of** Γ_0**. This yields** Γ_1**.** The process is then repeated on each of the three pieces making up Γ_1, so as to obtain Γ_2, and so on ad infinitum. The first two steps in this construction are illustrated in figure 1.

The issue of whether or not the limit $\lim_{n\to\infty} \Gamma_n$ exists is easy to settle. Consider an arbitrary radial line within the circle sector associated with Γ_0, and the intersection of this line with Γ_n. It should be clear that this intersection is unique. Let d_n be the distance of this intersection from the origin of the radial line. Then $\{d_n\}$ is an increasing, bounded sequence — so it has a limit. This limit defines a point along the radial line. The set of all these points is the fractal Γ.

Now do the following:

(1)

For each $n = 0, 1, 2, 3 \ldots$, calculate the length ℓ_n of the curve Γ_n. What is the "length" of Γ?

(2)

Calculate the fractal dimension (self-similar or box) of Γ.

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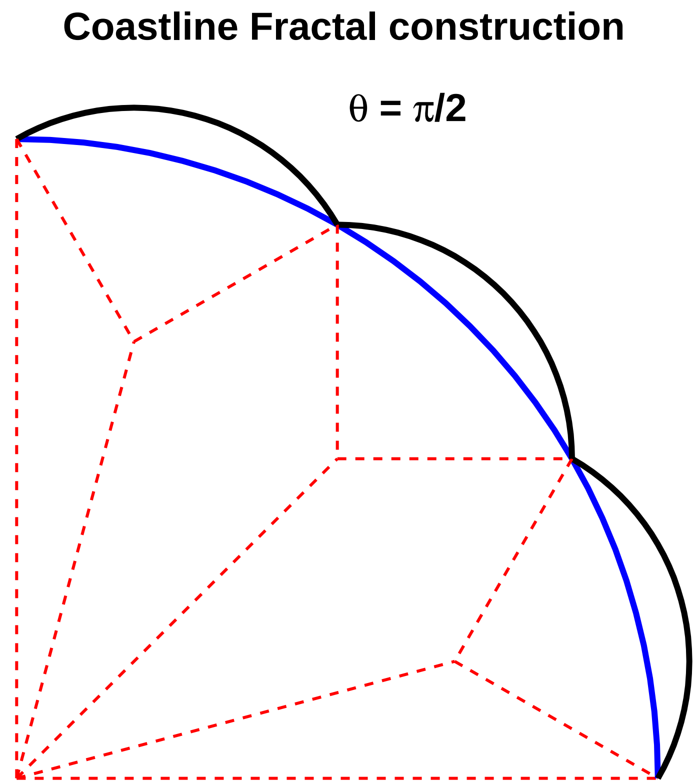

Figure 1: This figure illustrates the first two steps in the construction of the coastline fractal. That is, the curve Γ_0 (solid blue) and the curve Γ_1 (solid black). The dashed red lines indicate various radial lines useful in the construction.

Hint:

The first thing you will need to calculate is the "scaling" factor between Γ_0 and each of the three parts that make up Γ_1. With this scaling factor $0 < S_c = S_c(\theta) < 1$, everything else follows.

Notes:

Real coastlines are not this simple, of course. At the very least the number of parts into which each sector is divided should not be a constant (3 here), nor should the parts be equal in size, nor should they all subtend the same angle θ. But further: the sectors need not be exactly circular — though, this is probably not a terrible approximation.

THE END.

• Variable Length Pendulum.

Statement:

Consider a pendulum (in a plane), whose arm length $L > 0$ changes in time (i.e.: $L = L(t)$). To make matters more precise:

(a) Let the **hinge** for the pendulum be at origin in the plane: $x = y = 0$.

(b) Let the mass M for the pendulum be at $x = L\sin\theta$ and $y = -L\cos\theta$, where θ is the angle measured (counter-clockwise) from the down-rest position of the pendulum.

(c) Let g be the acceleration of gravity, and assume that frictional forces can be neglected.

(d) Assume that the mass of the pendulum arm can be neglected.

Now **do the following**

A Using Newton's laws, **derive the equations for the pendulum.**

Hint: There are two forces acting on the mass M:

- The force of gravity (of magnitude Mg, pointing downwards).
- A force (of magnitude $F = F(t)$) acting along the arm of the pendulum.

The force F is not known a-priori, but it must have the exact magnitude to keep the distance from the mass to the pendulum hinge at the length $L = L(t)$. This is enough to determine this force.

B Consider the following situation: you have a mass tied up at the end of a string. The string goes through a small hole somewhere — say, the hole at the end of a fishing rod. Now, pull steadily on the string, shortening the string length from the hole to the mass (do not move the hole while this happens). You should observe that, quite often, you end up with the mass going around the "fishing rod", wrapping the string there. **Explain this behavior using the equations derived in A.** *(Note that real life is neither 2-D, nor frictionless: the equations tend to over-predict what happens).*

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C Study the stability of the $\theta = 0$ equilibrium position for the pendulum. Linearize the equations near this solution, and obtain an equation of the form

$$\frac{d^2\varphi}{dt^2} + V(t)\varphi = 0\,, \tag{1}$$

where $\varphi = L\theta$ and $V = V(t)$ is some "potential" obtained from L and its derivatives.

D Argue that, if L is sinusoidal, with small amplitude variations, then one can take

$$V = \Omega^2(1 + \epsilon\cos(\omega t))\,, \tag{2}$$

in (1), where ϵ is small. Then (1) becomes Mathieu's equation.

E Take $\Omega = 1$ in Mathieu's equation and use Floquet theory to study the stability of the pendulum. That is, calculate (numerically) the trace of the Floquet matrix as a function of ϵ and ω (say, for $0 \le \epsilon \le 0.3$ and $0.5 \le \omega \le 5$). Note that the period to use in the calculation is $2\pi/\omega$ — i.e.: the period of $V = V(t)$ — and that instability corresponds to $\alpha = \text{trace}/2$ having magnitude bigger than one.

Alternatively: you can take $\omega = 1$, and then vary Ω and ϵ.

Answers:

Answer to Part A: Derivation of the equations.

Using Newton's law, we can write — for the position of the mass M — the equations

$$\left.\begin{aligned} M\,\ddot{x} &= -F\,\sin(\theta)\,, \\ M\,\ddot{y} &= +F\,\cos(\theta) - M\,g\,, \end{aligned}\right\} \tag{3}$$

where $F = F(t)$ is the (unknown at this stage) force along the pendulum arm (we use the convention that $F > 0$ corresponds to tension on the pendulum arm). We also have that:

$$\left.\begin{aligned} x &= +L\,\sin(\theta)\,, \\ y &= -L\,\cos(\theta)\,, \end{aligned}\right\} \tag{4}$$

where $L = L(t)$ is the (given) variable length of the pendulum. At this stage it is convenient to **introduce complex notation** (since it simplifies the algebra considerably), with $\boxed{z = x + i\,y.}$ Then equations (3) and (4) take the form:

$$M\,\ddot{z} = i\,F\,e^{i\,\theta} - i\,M\,g\,, \quad \text{with} \quad z = -i\,L\,e^{i\,\theta}\,. \tag{5}$$

From the second equation in (5) we obtain: $\boxed{\ddot{z} = (\ddot{\theta}\, L + 2\,\dot{\theta}\,\dot{L} - i\,\ddot{L} + i\,(\dot{\theta})^2\, L)\, e^{i\,\theta}}$ (intermediate step: $\dot{z} = (\dot{\theta}\, L - i\,\dot{L})\, e^{i\,\theta}$). Thus:

$$M\,(\ddot{\theta}\, L + 2\,\dot{\theta}\,\dot{L}) + i\, M\,((\dot{\theta})^2\, L - \ddot{L}) = i\, F - i\, M\, g\, e^{-i\,\theta}\,. \tag{6}$$

From the imaginary part of this equation we obtain a formula for F, namely:

$$F = M\, g\, \cos(\theta) + M\,((\dot{\theta})^2\, L - \ddot{L})\,. \tag{7}$$

The real part gives the equation of motion for θ, namely:

$$\ddot{\theta}\, L + 2\,\dot{\theta}\,\dot{L} + g\, \sin(\theta) = 0\,. \tag{8}$$

Alternatively, in terms of $\boxed{\varphi = L\,\theta,}$ this last equation can be written in the form

$$\ddot{\varphi} - \frac{\ddot{L}}{L}\,\varphi + g\, \sin(\frac{\varphi}{L}) = 0\,. \tag{9}$$

Remark 1 Derivation of the equations using Lagrangian Mechanics.
Equation (8) is straightforward to derive using Lagrangians, since:

$$\begin{aligned} \mathcal{L} &= \text{Lagrangian} \\ &= \text{Kinetic Energy} - \text{Potential Energy} \\ &= \frac{M}{2}(\dot{x}^2 + \dot{y}^2) - g\, M\, y \\ &= \frac{M}{2}(\dot{\theta}^2\, \dot{L}^2 + \dot{L}^2) + g\, M\, L\, \cos(\theta)\,. \end{aligned} \tag{10}$$

Then the Euler-Lagrange equation for $\mathcal{L}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0\,, \tag{11}$$

is precisely equation (8).

Answer to Part B: Steady pulling on the pendulum mass.

In this case $\boxed{L = L_0\,(1 - \omega\, t),}$ where $L_0 > 0$ and $\omega > 0$ are constants. Then equation (9) becomes

$$\ddot{\varphi} + g\, \sin(\frac{\varphi}{L}) = 0\,. \tag{12}$$

Let us write this equation in nondimensional form, with $\phi = \dfrac{\varphi}{L_0}$ and $\tau = \omega\, t$. Then

$$\frac{d^2\,\phi}{d\tau^2} + \gamma \sin(\frac{\phi}{1-\tau}) = 0\,, \quad \text{where} \quad \gamma = \frac{g}{L_0\,\omega^2} > 0 \tag{13}$$

is a nondimensional parameter. Near $\tau = 1$, the general solution to this equation behaves like

$$\phi = \phi_0 + \phi_1\,(1-\tau) + \gamma \int_\tau^1 (\tau - s)\, \sin\left(\frac{\phi_0}{1-s} + \phi_1\right)\, ds + \dots \tag{14}$$

where ϕ_0 and ϕ_1 are constants. Since (generally) $\phi_0 \neq 0$, it follows that

$$\theta = \frac{\varphi}{L} = \frac{L_0\,\phi}{L} \approx \frac{\phi_0}{1-\omega\,t} \quad \text{as} \quad t \to \frac{1}{\omega}\,. \tag{15}$$

Thus θ grows unboundedly as the string is pulled. Of course: the mathematical model breaks down way before $\theta = \infty$ can occur, but it does give an explanation for the observed behavior.

Answer to Part C: Linearized stability equations.

Near equilibrium, both θ and $\varphi = L\,\theta$ are small.[2] Using (9) and linearizing, we obtain:

$$\ddot{\varphi} + V(t)\,\varphi = 0\,, \quad \text{where} \quad V = \frac{g - \ddot{L}}{L}\,. \tag{16}$$

Answer to Part D: Mathieu's equation.

We now take $L = L(t)$ sinusoidal, of the form $\boxed{L = L_0\,(1 + \delta\,\cos(\omega\,t)),}$ where $L_0 > 0$, $\omega > 0$, and δ are constants, with δ small. Then

$$V = \frac{g - \ddot{L}}{L} = \frac{1}{1 + \delta\,\cos(\omega\,t)}\,\left(\Omega^2 + \delta\,\omega^2\,\cos(\omega\,t)\right) \approx \Omega^2\,(1 + \epsilon\,\cos(\omega\,t)) \tag{17}$$

where $\boxed{\Omega = \sqrt{g/L_0},}$ $\boxed{\epsilon = \delta\,\omega^2/\Omega^2,}$ and we have used the fact that δ is small.

Let us now nondimensionalize the equations, using $\boxed{\phi = \varphi/L_0}$ and $\boxed{\tau = \omega t.}$ Then

$$\frac{d^2\,\phi}{d\,\tau^2} + \left(\mu^2 + \delta\,\cos(\tau)\right)\,\phi = 0\,, \tag{18}$$

where $\boxed{\mu = \Omega/\omega}$ **is the ratio of the angular frequencies (pendulum to forcing).**

[2] Assume that L is oscillatory and stays away from zero. Thus the singular behavior studied in part (B) is avoided.

Answer to Part E: Floquet analysis and stability.

The stability question (are the solutions of equation (18) bounded, or do they grow?) can be decided using **Floquet Theory.** For this purpose first we introduce the **Floquet Matrix**, defined by:

$$\mathcal{F}_{\mathcal{M}} = \mathcal{F}_{\mathcal{M}}(\mu, \delta) = \begin{bmatrix} \phi_1(\tau = 2\pi) & \phi_2(\tau = 2\pi) \\ \dfrac{d\,\phi_1}{d\tau}(\tau = 2\pi) & \dfrac{d\,\phi_2}{d\tau}(\tau = 2\pi) \end{bmatrix}, \tag{19}$$

where $\phi_1$and ϕ_2 are the solutions of (18) defined by the initial conditions (at $\tau = 0$)

$$\phi_1 = 1\,, \quad \frac{d\,\phi_1}{d\tau} = 0\,, \qquad \text{and} \qquad \phi_2 = 0\,, \quad \frac{d\,\phi_2}{d\tau} = 1\,, \tag{20}$$

and 2π is the period of the coefficients in equation (18). The **Floquet Trace** is then given by

$$\mathcal{F}_{\mathcal{T}} = \mathcal{F}_{\mathcal{T}}(\mu, \delta) = \frac{1}{2}\,\mathsf{Trace}(\mathcal{F}_{\mathcal{M}}) = \frac{1}{2}\left(\phi_1(\tau = 2\pi) + \frac{d\,\phi_2}{d\tau}(\tau = 2\pi)\right). \tag{21}$$

The conditions for stability/instability are then

$$|\mathcal{F}_{\mathcal{T}}| \leq 1 \;\; \text{(stability)} \qquad \text{and} \qquad |\mathcal{F}_{\mathcal{T}}| > 1 \;\; \text{(instability)}\,. \tag{22}$$

One of the questions we would like to answer is: **can the pendulum be de-stabilized by selecting the frequency and amplitude of the forcing appropriately?** Of course, in general $\mathcal{F}_{\mathcal{T}}$ can only be computed numerically. However, we note that for $\delta = 0$ an analytic solution is possible (since then equation (18) is just the linear harmonic oscillator). In this case:

$$\mathcal{F}_{\mathcal{T}}(\mu, 0) = \cos(2\pi\mu)\,, \tag{23}$$

so $\mathcal{F}_{\mathcal{T}}(n/2, 0) = (-1)^n$ for n a natural number. Thus, **for δ small, we should explore near $\mu = n/2$ to find ranges where the pendulum is destabilized by the forcing** ($\mathcal{F}_{\mathcal{T}}$ is a continuous function of its arguments). Since $\mu = n/2$ yields $\omega = 2\,\Omega/n$, the unstable parameter values occur in situations where the forcing frequency is a subharmonic of twice the the unperturbed pendulum frequency. *Why this is so can be easily understood in terms of resonances.* **For δ small:**

- At leading order (0-th), the solutions to equation (18) is a sinusoidal of angular frequency μ.
- At 1-st order, the term $\cos(\tau)\,\phi$ in the equation creates the frequencies $\mu - 1$ and $\mu + 1$.
- At 2-nd order, the frequencies $\mu + n$, with $-2 \leq n \leq 2$, appear.

- In general, at m-th order, the frequencies $\mu + n$, with $-m \le n \le m$, appear.
- A **resonance will occur** if $\mu + n = \pm\mu$, for some n. That is, if $\boxed{\mu = n/2.}$
- The larger n is, the further up the expansion the resonance occurs. Thus, the instabilities that occur for larger values of n should be weaker. The figures below confirm this expectation: both the ranges where instability occurs, and the deviations there above absolute value one of $\mathcal{F}_T$, decrease very fast as n grows. *Finally: note that a large value of μ corresponds to very slow forcing. It is natural to expect instabilities in this regime to be very hard to produce!*

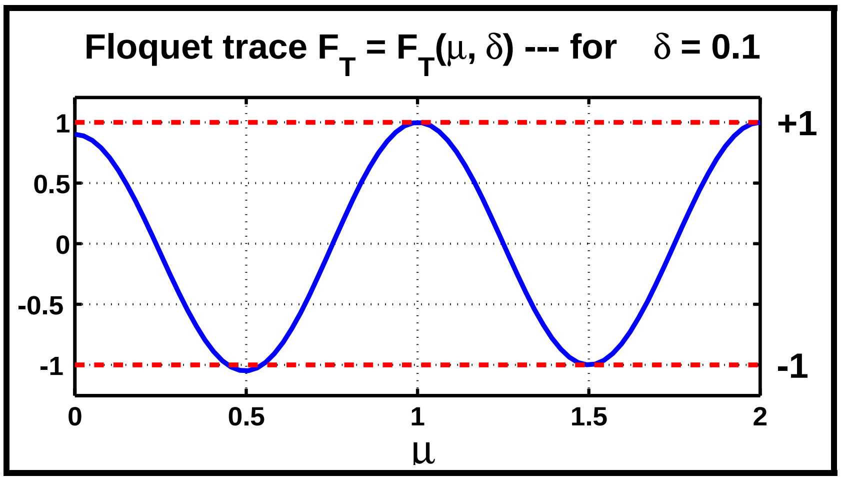

Figure 1:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.1$ and $0 \le \mu \le 2$.

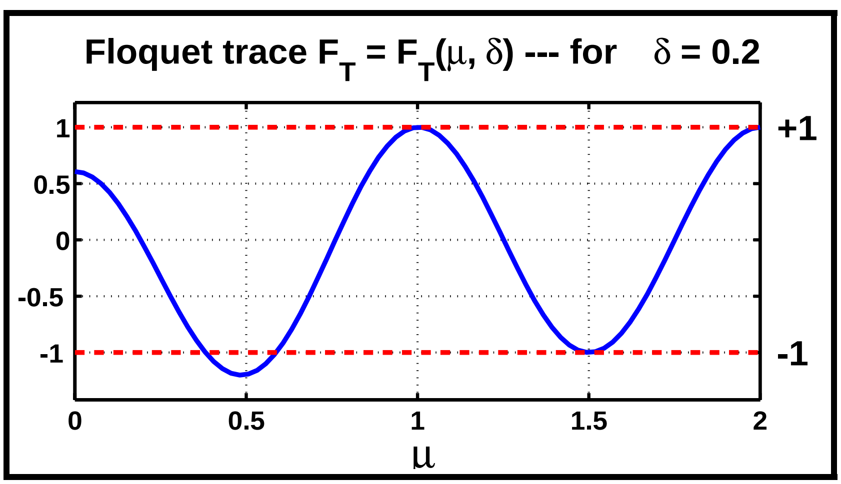

Figure 2:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.2$ and $0 \le \mu \le 2$.

Description of the Figures:

The figures in this problem illustrate the behavior of the Floquet Trace $\mathcal{F}_T(\mu, \delta)$, as a function of μ, for a sequence of increasingly larger values of (small) δ. We note how windows of instability arise

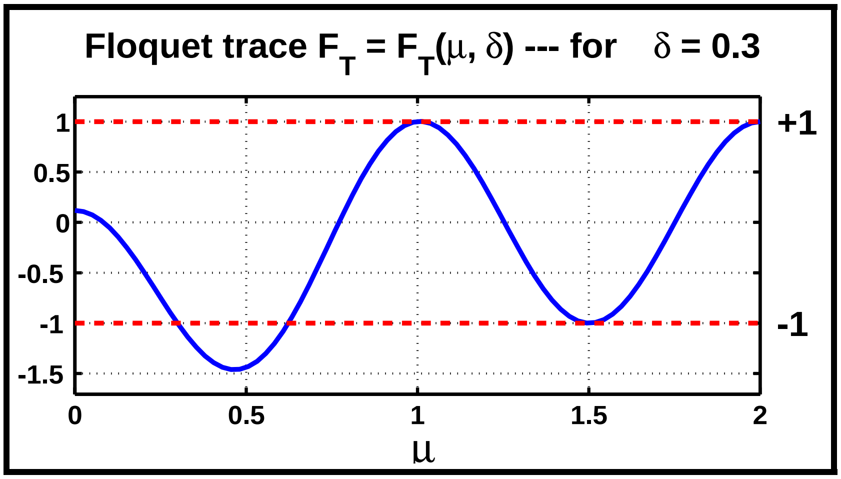

Figure 3:

Floquet Trace $\mathcal{F}_T$, for $\delta = 0.3$ and $0 \le \mu \le 2$.

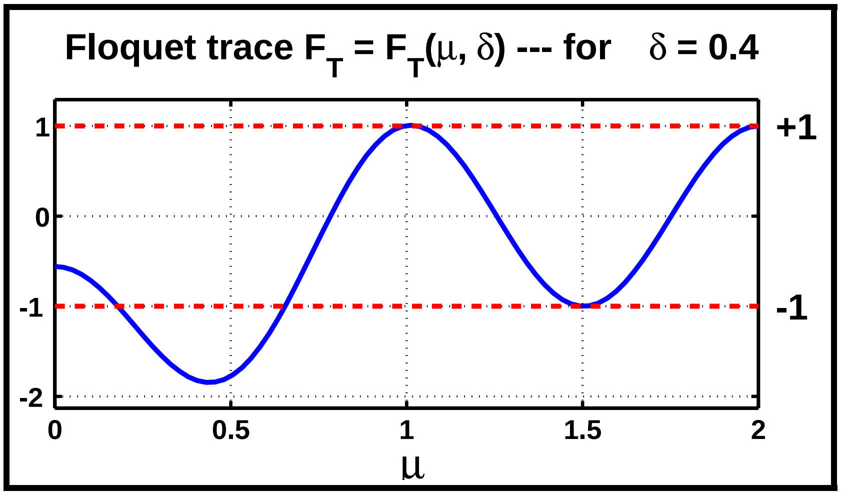

Figure 4:

Floquet Trace $\mathcal{F}_T$, for $\delta = 0.4$ and $0 \le \mu \le 2$.

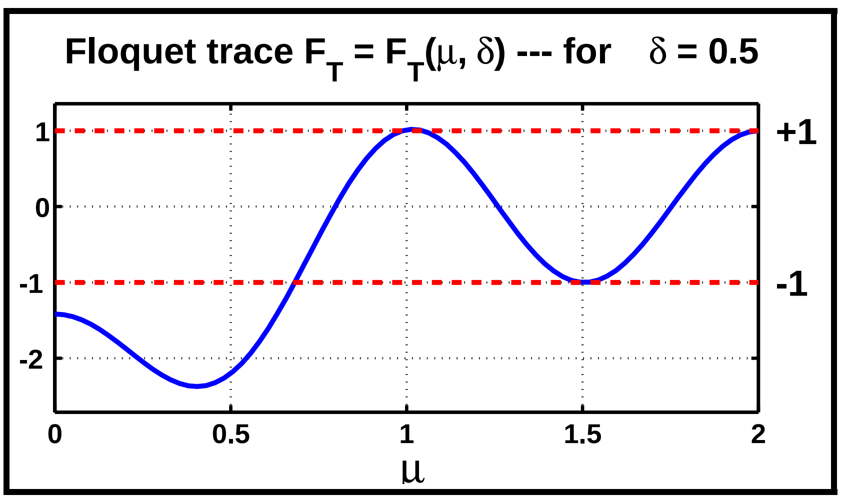

Figure 5:

Floquet Trace $\mathcal{F}_T$, for $\delta = 0.5$ and $0 \le \mu \le 2$.

near each of the critical values of μ (i.e.: $\mu = n/2$), and grow in width as δ grows. We also note that, for a given δ, the windows widths decrease very fast as n gets larger.

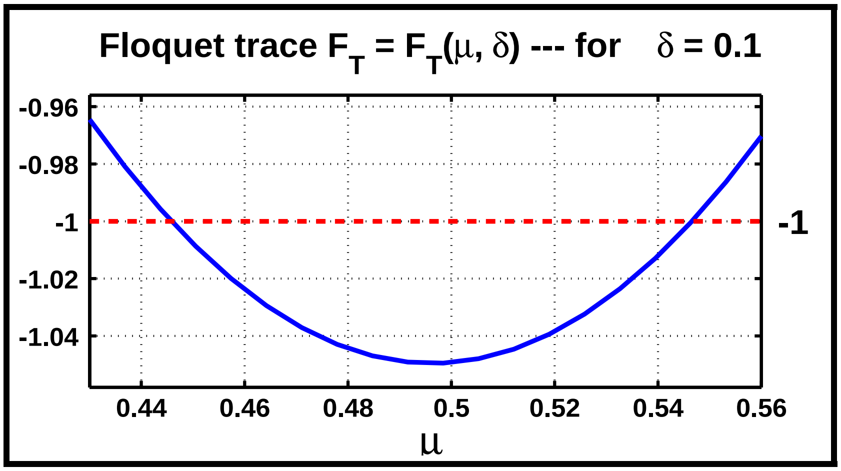

Figure 6:

Floquet Trace $\mathcal{F}_\mathcal{T}$, for $\delta = 0.1$ and $\mu \approx 0.5$.

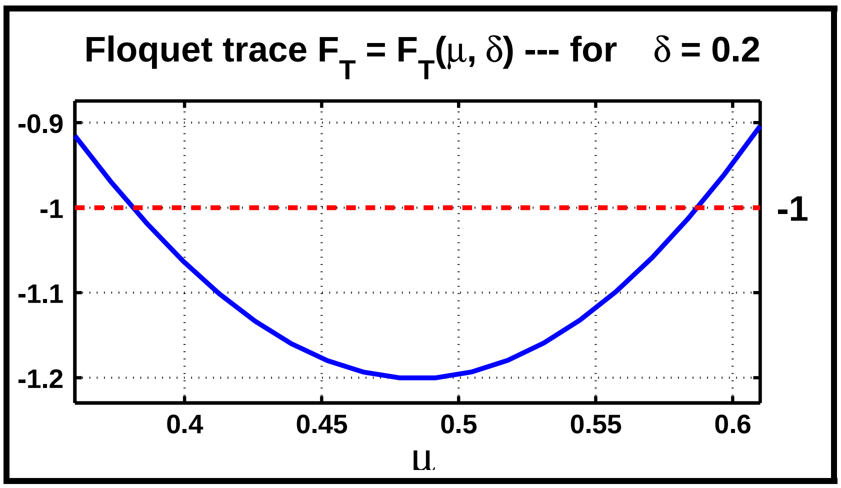

Figure 7:

Floquet Trace $\mathcal{F}_\mathcal{T}$, for $\delta = 0.2$ and $\mu \approx 0.5$.

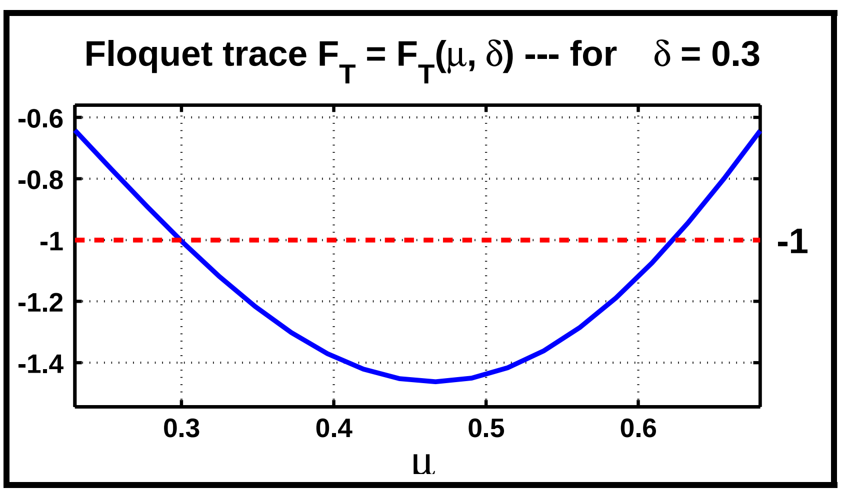

Figure 8:

Floquet Trace $\mathcal{F}_\mathcal{T}$, for $\delta = 0.3$ and $\mu \approx 0.5$.

First consider the plots of the Floquet Trace $\mathcal{F}_\mathcal{T}$ — as a function in the range $0 \leq \mu \leq 2$ — for the values $\delta = 0.1$, 0.2, 0.3, 0.4 and 0.5 (see Figures 1 through 5). On this scale the instability window near $\mu = 0.5$ is clearly visible for $\delta \geq 0.1$, while the other windows (near

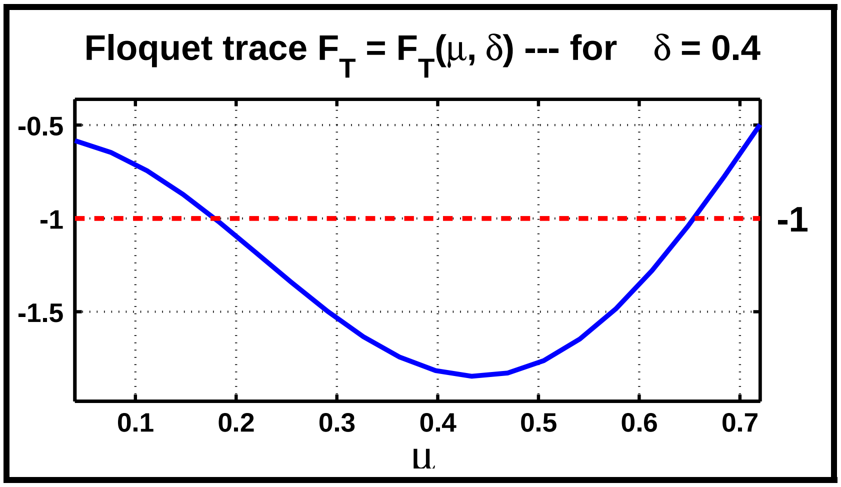

Figure 9:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.4$ and $\mu \approx 0.5$.

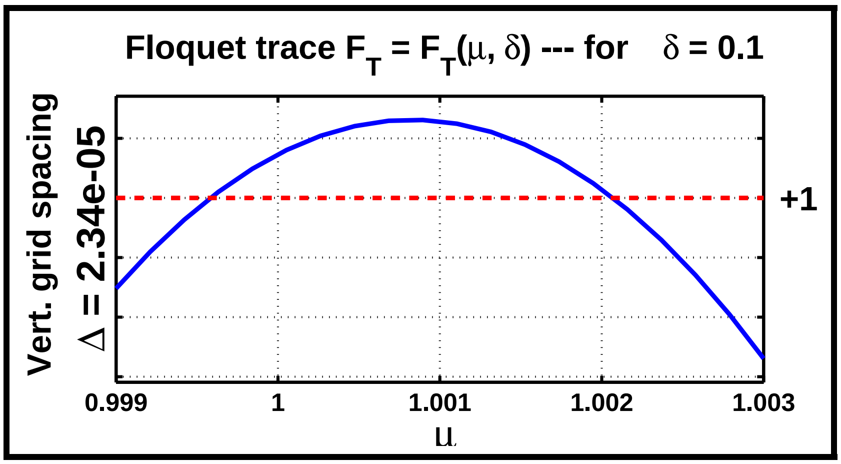

Figure 10:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.1$ and $\mu \approx 1.0$.

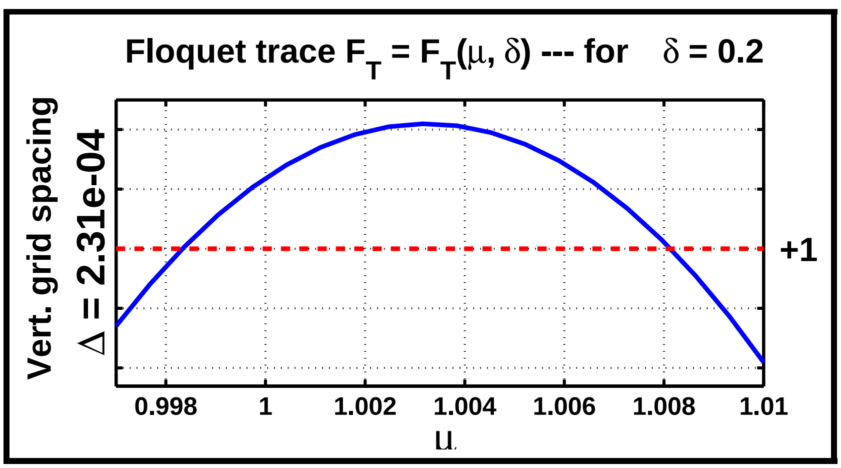

Figure 11:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.2$ and $\mu \approx 1.0$.

$\mu = 1,\ 1.5,$ and 2) are too small to be seen.[3] In particular, note that by $\delta = 0.5$ the instability window near $\mu = 0.5$ has grown so much that there is no longer a stable range for μ small — note

[3]These windows can be seen in the plots involving small ranges of μ; see Figures 6 through 18.

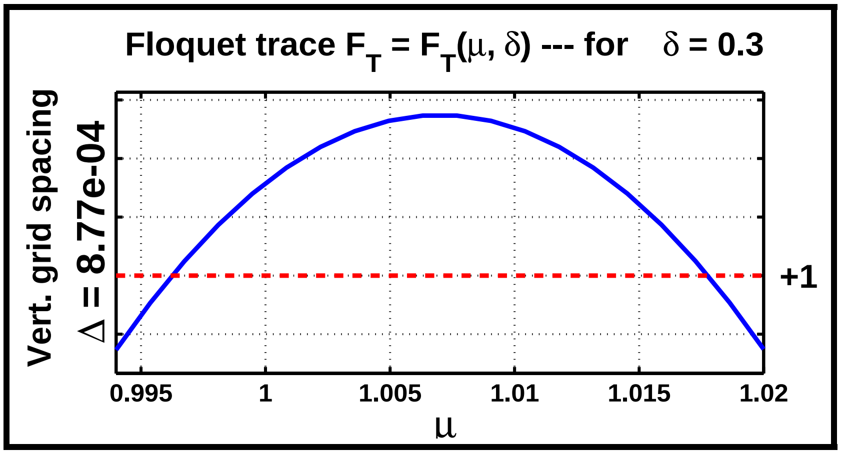

Figure 12: **Floquet Trace $\mathcal{F}_T$, for** $\delta = 0.3$ **and** $\mu \approx 1.0$**.**

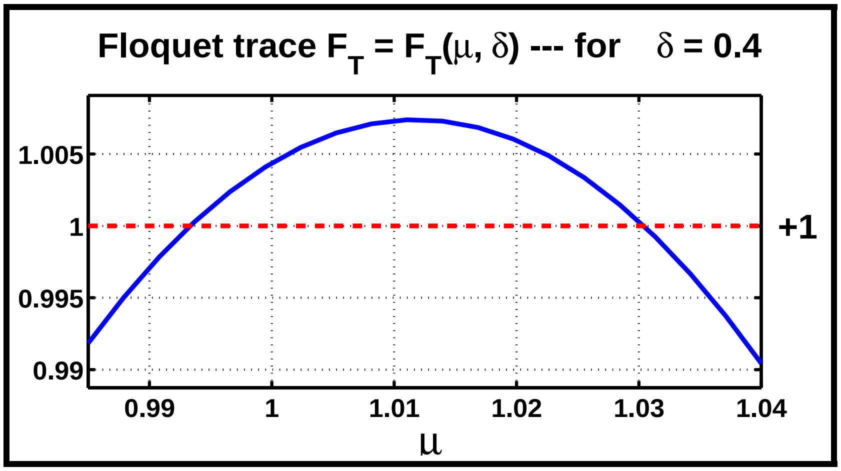

Figure 13: **Floquet Trace $\mathcal{F}_T$, for** $\delta = 0.4$ **and** $\mu \approx 1.0$**.**

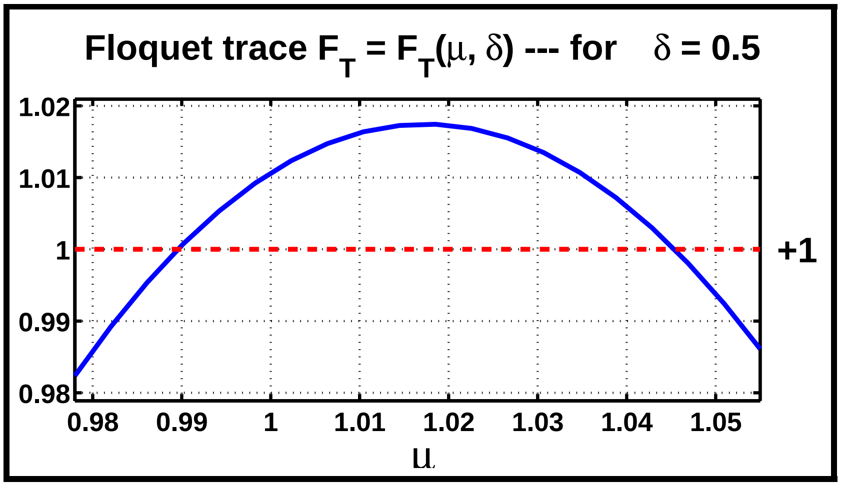

Figure 14: **Floquet Trace $\mathcal{F}_T$, for** $\delta = 0.5$ **and** $\mu \approx 1.0$**.**

that μ small corresponds to a forcing frequency that is much faster than the natural pendulum frequency. A fairly large forcing amplitude is required to de-stabilize the equilibrium position under such conditions.

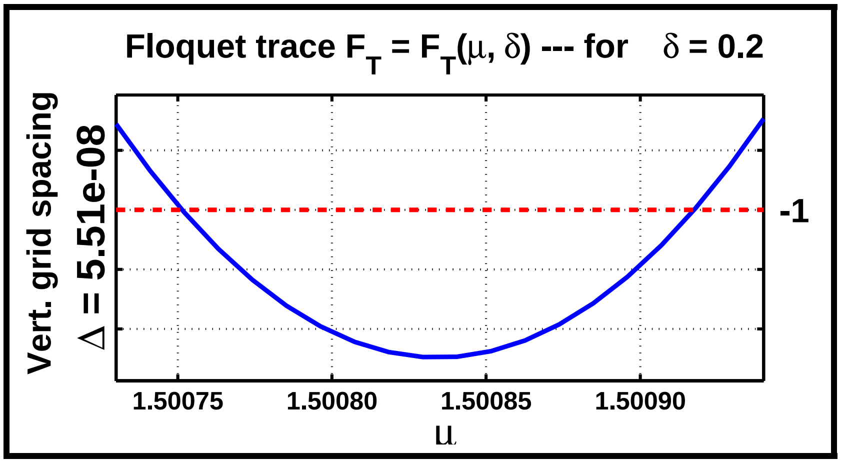

Figure 15:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.2$ and $\mu \approx 1.5$.

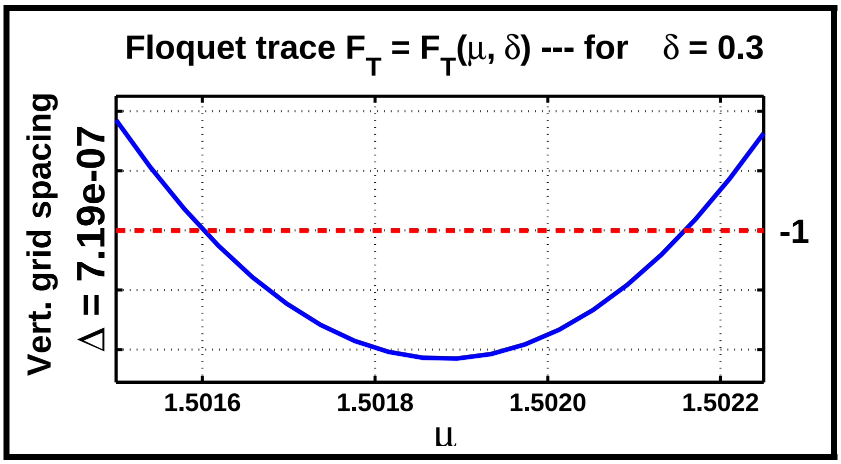

Figure 16:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.3$ and $\mu \approx 1.5$.

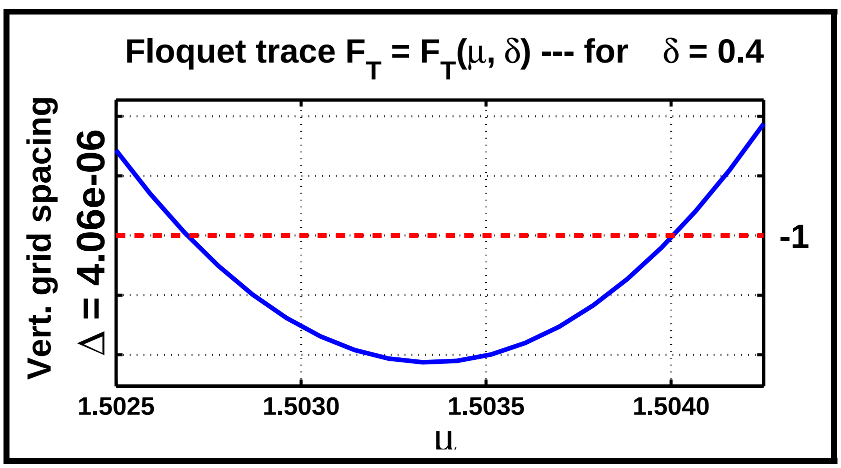

Figure 17:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.4$ and $\mu \approx 1.5$.

Figures 6 through 9 show plots of the Floquet Trace $\mathcal{F}_T$ in a neighborhood of $\mu = 0.5$, for the values $\delta = 0.1$, 0.2, 0.3, and 0.4. Thus **these figures show details of the lowest, and largest, instability window, for δ small and μ near** $1/2$**.** Note that the width of this window grows roughly linearly with δ (for small δ this can be shown using asymptotic expansion techniques). Of

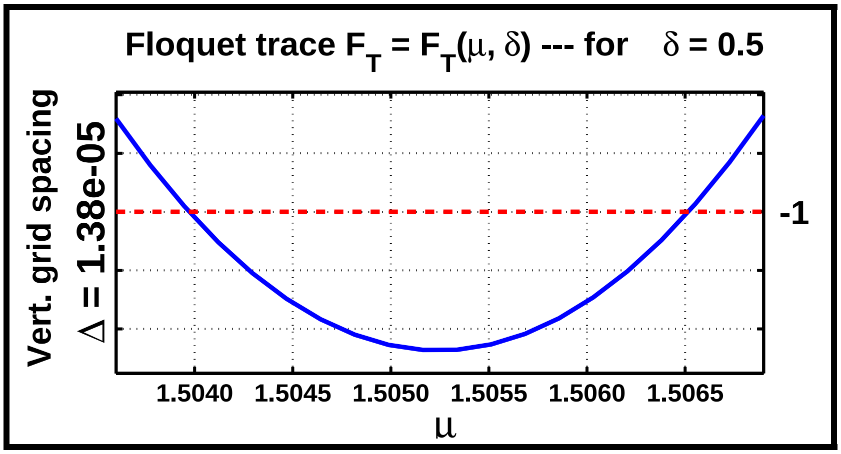

Figure 18:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.5$ and $\mu \approx 1.5$.

course, by $\delta = 0.5$ this is no longer true, and the window is so large as to have completely absorbed the stable "μ small" range.

Figures 10 through 14 show plots of the **Floquet Trace $\mathcal{F}_T$ in a neighborhood of $\mu = 1.0$,** for the values $\delta = 0.1$, 0.2, 0.3, 0.4, and 0.5. This window is much smaller than the $\mu \approx 0.5$ window, and it grows much more slowly. In fact, note that the width of this window grows roughly quadratically with δ (for small δ this can be shown using asymptotic expansion techniques).

Figures 15 through 18 show plots of the **Floquet Trace $\mathcal{F}_T$ in a neighborhood of $\mu = 1.5$,** for the values $\delta = 0.2$, 0.3, 0.4, and 0.5. This window is still smaller than the prior ones — so small, in fact, that I was un-able to resolve it for $\delta = 0.1$. The width of this window grows roughly cubically with δ (for small δ this can be shown using asymptotic expansion techniques).

Note: The figures were done using MatLab. To calculate the Floquet Trace $\mathcal{F}_T$, the ode solver ode113 was used to solve for the functions ϕ_1 and ϕ_2. To speed up the process, the calculation was "vectorized": for each value of δ, the solutions for all the calculated values of μ were calculated simultaneously.

THE END.