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Yuli Eidelman
Israel Gohberg
Iulian Haimovici

Separable Type Representations of Matrices and Fast Algorithms

Volume 1
Basics. Completion Problems. Multiplication and Inversion Algorithms

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Yuli Eidelman
Israel Gohberg
Iulian Haimovici
School of Mathematical Sciences
Raymond and Beverly Sackler
Faculty of Exact Sciences
Tel Aviv University
Tel Aviv
Israel

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Part III Quasiseparable Representations of Matrices, Descriptor Systems with Boundary Conditions and First Applications

Preface

Our interest in structured matrices was inspired by the outstanding mathematician Professor Israel Gohberg, our older friend, colleague and teacher. His ideas, projects and our joint results lie at the basis of this book. His supervision and participation were crucial in the preparation of the manuscript. On 12 October 2009 Israel Gohberg passed away and we finished the book alone. Our work is devoted to his blessed memory.

October 31, 2012 Yuli Eidelman, Iulian Haimovici

Introduction

The book. The majority of the basic algorithms of computations with matrices are expressed via the entries of the matrices and are not taking into account the individual properties or the specific structure of these matrices. This often results in a unjustified high complexity of the algorithms.

For instance, the multiplication of two matrices of order N via the entries of the matrices requires in general N^3 operations. For many classes of structured matrices this complexity can be reduced by an appropriate presentation of the factors and the product as well as the algorithm. For this purpose we have to represent the matrices and the algorithm not in terms of the entries of the matrices, but in terms of other parameters (generators) which are essentially involved in the description of the structure of these matrices. For matrices of the form

$$A = \{a_{ik}\}_{i,k=1}^N, \quad a_{ik} = x_i^T y_k, \; i,k = 1,2,\dots,N$$

with $x_i, y_k \in \mathbb{C}^n, \; n \ll N$, which are often called separable matrices, the natural parameters (generators) are the n-dimensional vectors x_i, y_k $(i,k = 1,\dots,N)$. The computations for matrices of this form in terms of the natural parameters are of a much lower complexity. So for the product of two such matrices, A and

$$B = \{b_{kj}\}_{k,j=1}^N, \quad b_{kj} = v_k^T u_j, \quad k,j = 1,2,\dots,N,$$

we get $C = AB = \{c_{ij}\}_{i,j=1}^N$ with

$$c_{ij} = \sum_{k=1}^N x_i^T y_k v_k^T u_j = x_i^T \left(\sum_{k=1}^N y_k v_k^T \right) u_j.$$

Hence the product C is a matrix with separable generators x_i and

$$w_j = \left(\sum_{k=1}^N y_k v_k^T \right) u_j.$$

To compute the sum $a = \sum_{k=1}^N y_k v_k^T$ one requires Nn^2 operations and the products $w_j = au_j$ $(j = 1,\dots,N)$ cost Nn^2 operations. Thus for the multiplication of two matrices in separable form one needs only $2n^2N$ operations. If n is fixed, the complexity is asymptotically equal to $O(N)$. A similar situation appears also for the inversion of matrices of this type.

This book contains a systematic theoretical and computational study of several types of generalizations of separable matrices. It is related to semiseparable, quasiseparable, band and companion representations of matrices. For them their natural parameters, called generators, are analyzed and algorithms are expressed in terms of generators. Connections between matrices and boundary value problems for discrete systems play an important role. The book is focused on algorithms of multiplication, inversion and description of eigenstructure of matrices. A large number of illustrations are provided in the text. The book consists of eight parts.

Description of parts. The first part is mainly of a theoretical character. Here we introduce the notions of quasiseparable and semiseparable structure. These notions are illustrated on some well-known examples of tridiagonal matrices, band matrices, diagonal plus semiseparable matrices, scalar and block companion matrices. We derive various properties of quasiseparable and semiseparable structure which are used in the sequel. An essential part of the material concerns the minimal rank completion problem.

The second part is devoted to completion to Green matrices and to unitary matrices and also to the completion of mutually inverse matrices.

Discrete systems with boundary conditions allow to present a transparent description of various algorithms which is started in the third part. We begin the presentation of algorithms with multiplication by vectors and then with algorithms which are based on some well-known inversion formulas via quasiseparable structure. An essential role in this part plays the interplay between the quasiseparable structure and discrete-time varying linear systems with boundary conditions.

The fourth part contains factorization and inversion algorithms for matrices via quasiseparable and semiseparable structure. We present the LDU factorization and inversion algorithms for strongly regular matrices. Algorithms of this type are extended to arbitrary matrices with quasiseparable representations of the first order. In the last chapter algorithms for the QR factorization and the QR based solver for linear algebraic systems are presented.

The second volume is divided into Parts V–VIII. The titles are as follows. Part V: The eigenvalue structure of order one quasiseparable matrices; Part VI: Divide and conquer method for eigenproblem; Part VII: Algorithms for QR iterations and for reduction to Hessenberg form; Part VIII: QR iterations for companion matrices.

To whom this book is addressed. The book belongs to the area of theoretical and computational Linear Algebra. It is a self-contained monograph which has the structure of a graduate text. The main material was developed the last 30–40 years and is presented here following the lines and principles of a course in Linear Algebra. The book is based mostly on the relatively recent results obtained by the authors and their coauthors. All these features together with many significant applications and accessible style will make it widely useful for engineers, scientists, numerical analysts, computer scientists and mathematicians alike.

Acknowledgment. We would like to express our gratitude the late Israel Koltracht and also Harry Dym, Rien Kaashoek, Thomas Kailath and Peter Lancaster with whom the work on semiseparable matrices has been started. It is also a pleasure to thank our colleagues Tom Bella, Dario Bini, Paola Boito, Patrick Dewilde, Luca Gemignani, Vadim Olshevsky, Victor Pan, Eugene Tyrtyshnikov, Marc Van Barel, Raf Vandebril, Hugo Woerdeman, Jianlin Xia and Pavel Zhlobich for fruitful discussions and cooperation. The authors acknowledge the help and understanding of the School of Mathematical Sciences at Tel-Aviv University and of the Nathan and Lilly Silver Family Foundation. We thank also the Israel Science Foundation for partial support of our work by a grant in the period from 1997 till 2000.

Part I

Basics on Separable, Semi-separable and Quasiseparable Representations of Matrices

Introduction to Part I

In this part we introduce and study three types of representations of matrices: pure separable, semi-separable and quasi-separable. Each representation is defined via its generators and each representation has its order. For $N \times N$ matrices, using these representations, we derive algorithms of linear $O(N)$ complexity for some important procedures and operations. These fast algorithms are expressed in terms of the representations or generators. Generators are in fact the parameters that allow to reduce essentially the complexity of the main algorithms. Important topics in this part are minimal rank completion of matrices and necessary preparations for the next parts. In particular, here are introduced the classes of tridiagonal, scalar and block companion, Green, band, Hessenberg and other matrices. This part contains also a review of different factorizations and theorems of inversion of matrices. For each class of representations of matrices we introduce the notion of order r of a representation. These orders play an essential role in the estimates for complexity of the fast algorithms. More precisely, for an algorithm with complexity c we obtain an estimate of the form $c \leq w(r)N$, where $w(r)$ is a polynomial in r.

Chapter 1

Matrices with Separable Representation and Low Complexity Algorithms

One of the simplest representations of matrices used for a reduction of complexity of algorithms is the separable representation. The term separable comes from the fact that the (block) entries A_{kj} of such an $N \times N$ matrix A can be presented in a separated form

$$A_{kj} = b_k \cdot c_j, \quad j, k = 1, \ldots, N,$$

where b_k and c_j are matrices of certain sizes. The latter matrices are now considered as the main parameters (they form the so-called separable generators). Our aim is to represent the main operations, such as multiplication, inversion, different factorizations and others, on separable represented matrices in terms of their generators. In the cases when the generators have small sizes this leads to a considerable reduction of the complexity of the algorithms. The notion of separable order that we introduce in this chapter is used essentially in estimates of the complexity of the algorithms.

One of our main tools is to reduce operations on separable matrices to operations on time-dependent linear systems with boundary conditions. The latter systems represent certain recurrences which are often convenient in computations. In the next chapters more complicated representations of matrices are considered, namely the semiseparable and quasiseparable ones. The material of this chapter serves as a model for the developments in the other ones.

This chapter consists of the following sections: The first section contains some basic formulas of factorization of matrices, while basic formulas for LDU factorizations and inversion are presented in the sixth and in the seventh sections. The second section contains the definitions of separable and diagonal plus separable representations of matrices, with some examples. In the third it is studied the multiplication of a separable matrix with a vector. Systems with boundary conditions

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and, associated to them, matrices in diagonal plus separable form are considered in the forth section. In the fifth section are presented algorithms of multiplication of matrices via generators, and in the last three sections inversion and factorization methods are studied.

§1.1 Rank and related factorizations

Here we present some well-known facts on factorization of matrices in a form convenient for the subsequent use.

Let A be an $m \times n$ matrix of rank r. Let $b_1, b_2, \dots, b_r$ be m-dimensional columns which are a basis of the column space $\mathrm{Im}(A)$ of the matrix A. For every column of the matrix A one has

$$A(:, j) = \sum_{k=1}^{r} b_k \gamma_{kj}, \quad j = 1, \dots, n$$

for some complex numbers γ_{kj}. Setting $B = \begin{bmatrix} b_1 & b_2 & \dots & b_r \end{bmatrix}$, $\Gamma = (\gamma_{kj})_{k,j=1}^{r,n}$ one gets

$$A = B \cdot \Gamma \tag{1.1}$$

with matrices B, Γ of sizes $m \times r, r \times n$ respectively. Moreover, using the inequalities

$$r = \operatorname{rank} A \leq \operatorname{rank} B \leq r, \quad r = \operatorname{rank} A \leq \operatorname{rank} \Gamma \leq r$$

one obtains

$$\operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} \Gamma = r. \tag{1.2}$$

The factorization (1.1) with matrices B, Γ satisfying the condition (1.2) is called *the rank factorization* of the matrix A.

Let V be an $r \times n$ matrix with $\operatorname{rank} V = r$ and let for $i = 1, \dots, r$ the symbol $l(i)$ mean the first nonzero element in the ith row of V. We say that the matrix V is in *the canonical form* if the condition

$$l(1) < l(2) < \dots < l(r) \tag{1.3}$$

holds. From (1.3) it follows that

$$V(i + 1 : r, 1 : l(i + 1) - 1) = 0, \quad i = 1, \dots, r - 1. \tag{1.4}$$

Applying some elementary transformations to the $r \times n$ matrix Γ in (1.1) one can obtain the representation $\Gamma = P \cdot V$ with an invertible $r \times r$ matrix P and a matrix V in the canonical form. Setting $B_1 = BP$ one gets instead of (1.1) the factorization

$$A = B_1 \cdot V \tag{1.5}$$

with matrices B_1, V of sizes $m \times r, r \times n$ such that

$$\operatorname{rank} A = \operatorname{rank} B_1 = \operatorname{rank} V = r, \quad V \text{ in the canonical form.} \tag{1.6}$$

The factorization (1.5) with matrices B_1, V satisfying the condition (1.6) is called *the rank canonical factorization* of the matrix A.

Taking the basis $b_1, b_2, \ldots, b_r$ in the column space $\operatorname{Im}(A)$ to be orthonormal, one obtains the factorization (1.1) with matrices B, Γ of sizes $m \times r, r \times n$ such that

$$\operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} \Gamma = r, \quad B^* B = I_r. \tag{1.7}$$

The factorization (1.1) with matrices B, Γ satisfying the condition (1.7) is called *the orthogonal rank factorization* of the matrix A.

Applying unitary transformations to the $r \times n$ matrix Γ one obtains the representation $\Gamma = P \cdot R$ with a unitary $r \times r$ matrix P and an upper triangular $r \times n$ matrix R, i.e., satisfying the condition $R_{ij} = 0,\ i > j$. Setting $Q = BP$ one gets instead of (1.1) the factorization

$$A = Q \cdot R \tag{1.8}$$

with matrices Q, R of sizes $m \times r, r \times n$ such that

$$\operatorname{rank} A = \operatorname{rank} Q = \operatorname{rank} R = r, \quad Q^* Q = I_r, \quad R_{ij} = 0,\ i > j. \tag{1.9}$$

The factorization (1.8) with matrices Q, R satisfying the condition (1.9) is called *the orthogonal rank upper triangular factorization* of the matrix A.

Another form of orthogonal rank factorization is *the singular value decomposition* (SVD), i.e., the factorization of an $m \times n$ matrix A of rank r in the form

$$A = Q \cdot \Sigma \cdot U$$

with matrices Q, U of sizes $m \times r, r \times n$ such that

$$\operatorname{rank} A = \operatorname{rank} Q = \operatorname{rank} U = r, \quad Q^* Q = U U^* = I_r$$

and the $r \times r$ diagonal matrix Σ has positive diagonal entries. The diagonal entries of Σ are supposed to satisfy

$$\Sigma_{11} \geq \Sigma_{22} \geq \cdots \geq \Sigma_{rr}$$

which is always true up to a change of rows and columns.

Rank factorization of a matrix is not unique. However there is a simple connection between two different rank factorizations of the same matrix.

Lemma 1.1. *Let P, Q and B, Γ be two pairs of matrices of sizes $m \times r, r \times n$ such that*

$$P \cdot Q = B \cdot \Gamma, \quad \operatorname{rank} P = \operatorname{rank} Q = \operatorname{rank} B = \operatorname{rank} \Gamma = r.$$

Then there exists an invertible matrix S of size $r \times r$ such that $P = BS^{-1}$ and $Q = S\Gamma$.

Proof. The matrix P has a left inverse $\widehat{P}$ and the matrix Q has a right inverse $\widetilde{Q}$. Multiplying the equality

$$PQ = B\Gamma \tag{1.10}$$

by $\widehat{P}$ on the left one obtains

$$Q = (\widehat{P}B)\Gamma \tag{1.11}$$

while multiplying (1.10) by $\widetilde{Q}$ on the right one gets

$$P = B(\Gamma\widetilde{Q}). \tag{1.12}$$

Furthermore, multiplying (1.12) by $\widehat{P}$ on the left we conclude that

$$(\widehat{P}B)(\Gamma\widetilde{Q}) = I_r. \tag{1.13}$$

Define the $r \times r$ matrix S by $S = \widehat{P}B$. The equality (1.13) implies that S is invertible and $S^{-1} = \Gamma\tilde{Q}$. It now follows from (1.12), (1.11) that $P = BS^{-1}$, $Q = S\Gamma$. □

§1.2 Definitions and first examples

Let $A = (A_{ij})_{i,j=1}^N$ be an $N \times N$ block matrix with blocks of sizes $m_i \times n_j$ and with rank ρ. Consider the rank factorization

$$A = PQ \tag{1.14}$$

with the matrices P, Q of sizes $\sum_{i=1}^N m_i \times \rho$ and $\rho \times \sum_{j=1}^N n_j$, respectively, such that

$$\operatorname{rank} A = \operatorname{rank} P = \operatorname{rank} Q = \rho.$$

Based on factorizations of the form (1.14) we can define representations for any block matrix.

Definiton 1.2. Let $A = (A_{ij})_{i,j=1}^N$ be an $N \times N$ block matrix with blocks $A_{i,j}$, $i, j = 1, 2, \dots, N$ of size $m_i \times n_j$ represented in the form $A = PQ$, where $P = \operatorname{col}(p(i))_{i=1}^N$, $Q = \operatorname{row}(q(i))_{i=1}^N$ are matrices with blocks $p(i), q(i)$ $(i = 1, \dots, N)$ of sizes $m_i \times r$ and $r \times n_i$ respectively. Then the representation $A = PQ$, or which is equivalent, the representation

$$A_{ij} = p(i)q(j), \quad i, j = 1, \dots, N \tag{1.15}$$

is called a *separable representation of order* r of the matrix A. We also say that A is represented in separable of order r form.

The matrices $p(i), q(i)$ $(i = 1, \dots, N)$ are called the *separable generators* of the matrix A. The matrices P and Q are called *the matrices of generators of* A.

The number r is called the *order* of these generators.

The rank ρ of the matrix A is called the *separable order* of A.

Theorem 1.3. *The separable order of a matrix is equal to the minimal order of its separable generators.*

The proof is obvious. □

Let A be a matrix represented in separable of order r form. The rank of A is at most r. The $2N$ matrices which form the generators of the matrix A have in total less than $2mrN$ entries, where $m = \max_{i=1}^{N}\{m_i, n_i\}$. Compared with the number of entries of A there are fewer entries whenever $r \leq \frac{(\sum_{i=1}^{N} m_i)(\sum_{i=1}^{N} n_i)}{2}$. In fact r is assumed to be much smaller, so that less data are needed to store and work with the matrix A.

Example 1.4. Consider the $N \times N$ matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & N-1 & N \\ 2 & 4 & 6 & \cdots & 2(N-1) & 2N \\ 3 & 6 & 9 & \cdots & 3(N-1) & 3N \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ N-1 & 2(N-1) & 3(N-1) & \cdots & (N-1)^2 & (N-1)N \\ N & 2N & 3N & \cdots & (N-1)N & N^2 \end{pmatrix}.$$

The following separable generators of order one can be used for A:

$$p(i) = i, \quad q(i) = i, \ i = 1, \dots, N. \qquad \diamondsuit$$

Next we consider perturbations of block matrices represented in separable of order r form by block diagonal matrices.

Definiton 1.5. Let $A = (A_{ij})_{i,j=1}^{N} = D + B$ be an $N \times N$ block matrix with blocks $A_{i,j}$, $i, j = 1, 2, \dots, N$ of sizes $m_i \times n_i$, which is the sum of a block diagonal matrix $D = \operatorname{diag}(d(i))_{i=1}^{N}$ with blocks $d(i)$ of sizes $m_i \times n_i$ and a matrix B of rank ρ. The number ρ is called the *separable order* of the matrix A. Assume that the matrix B is represented in separable of order r form: $B = PQ$ with the matrices of generators $P = \operatorname{col}(p(i))_{i=1}^{N}$, $Q = \operatorname{row}(q(i))_{i=1}^{N}$, which are matrices with blocks $p(i), q(i)$ $(i = 1, \dots, N)$ of sizes $m_i \times r$ and $r \times n_i$, respectively. Then A is called a matrix with *diagonal plus separable of order r representation.*

The matrices $p(i), q(i)$ $(i = 1, \dots, N)$ are called the *separable generators* of the matrix A. The matrices $d(i)$ $(i = 1, \dots, N)$ are called the *diagonal generators* of A.

The above definition means that the matrix $A = (A_{ij})_{i,j=1}^{N}$ has the representation $A_{ij} = p(i)q(j) + \delta_{ij}d(i)$, $1 \leq i, j \leq N$, where δ_{ij} is the Kronecker symbol.

Example 1.6. Consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 6 & 18 & 8 \\ 7 & 20 & 18 \end{pmatrix}.$$

We have $A = 2I + B$, where

$$B = \begin{pmatrix} -1 & -2 & 2 \\ 6 & 16 & 8 \\ 7 & 20 & 16 \end{pmatrix}.$$

The 3×3 matrix B has a zero determinant, which means that it admits a separable representation of an order less than 3 and because $\operatorname{rank} B \neq 1$, its minimal separable order is greater than one. Indeed,

$$B = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 2 & 5 & 1 \end{pmatrix},$$

hence the matrix A has the diagonal generators $d(1) = d(2) = d(3) = 2$ and the separable of order 2 generators

$$p(1) = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad p(2) = \begin{pmatrix} 2 & 2 \end{pmatrix}, \quad p(3) = \begin{pmatrix} 5 & 1 \end{pmatrix}$$

and

$$q(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad q(2) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad q(3) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \qquad \diamondsuit$$

§1.3 The algorithm of multiplication by a vector

Let $A = (A_{ij})_{i,j=1}^N = D + PQ$ be an $N \times N$ block matrix with blocks $A_{i,j}$, $i, j = 1, 2, \ldots, N$ of size $m_i \times n_i$, in diagonal plus separable of order r representation with separable generators $p(i), q(i)$ $(i = 1, \ldots, N)$ of order r. Let $x = \operatorname{col}(x(i))_{i=1}^N$ be a vector with column coordinates $x(i)$ of sizes n_i. The product $y = Ax$ of the matrix A by the vector x is a vector $y = \operatorname{col}(y(i))_{i=1}^N$ with column coordinates $y(i)$ of sizes m_i. We consider various procedures to compute the components $y(i)$.

§1.3.1 Forward and backward computation of y

For y one obtains

$$y(i) = \sum_{j=1}^N A_{ij} x(j) = \sum_{j=1}^N p(i) q(j) x(j) + d(i) x(i) = p(i) \chi_{N+1} + d(i) x(i),$$

where

$$\chi_i = \sum_{j=1}^{i-1} q(j) x(j), \quad i = 1, \ldots, N+1.$$

One can see that the variable χ_i satisfies the following relations: $\chi_1 = 0$ and for $i = 1, \dots, N$

$$\chi_{i+1} = \sum_{j=1}^{i} q(j)x(j),$$

hence one obtains the recursion

$$\chi_{i+1} = \chi_i + q(i)x(i), \quad i = 1, \dots, N.$$

Alternatively, for y one obtains

$$y(i) = \sum_{j=1}^{N} A_{ij}x(j) = \sum_{j=1}^{N} p(i)q(j)x(j) + d(i)x(i) = p(i)\eta_0 + d(i)x(i),$$

where

$$\eta_i = \sum_{j=i+1}^{N} q(j)x(j).$$

One has $\eta_N = 0$ and it follows that η_i satisfies the recurrence relations

$$\eta_{i-1} = \eta_i + q(i)x(i), \quad i = N, \dots, 1.$$

From these relations one obtains the following algorithms for computing the product $y = Ax$.

Algorithm 1.7. (Multiplication by a vector with forward computation of data)

1. Start with

$$\chi_1 = 0 \tag{1.16}$$

and for $i = 2, \dots, N+1$ compute recursively

$$\chi_i = \chi_{i-1} + q(i-1)x(i-1). \tag{1.17}$$

2. Compute for $k = 1, \dots, N$ the components of the vector y:

$$y(k) = p(k)\chi_{N+1} + d(k)x(k). \tag{1.18}$$

Algorithm 1.8. (Multiplication by a vector with backward computation of data)

1. Start with

$$\eta_N = 0 \tag{1.19}$$

and for $i = N-1, \dots, 0$ compute recursively

$$\eta_i = \eta_{i+1} + q(i+1)x(i+1). \tag{1.20}$$

2. Compute for $k = N, \dots, 1$ the components of the vector y:

$$y(k) = p(k)\eta_0 + d(k)x(k). \tag{1.21}$$

The complexity of the arithmetic operations used in Algorithms 1.7 and 1.8 is as follows.

1. The formula (1.17): r additions plus the matrix multiplication, which comprises rn_{i-1} multiplications and $r(n_{i-1}-1)$ additions.
2. The formula (1.18) or (1.21): $m_k r + m_k n_k$ multiplications and $m_k(r-1) + m_k + m_k(n_k - 1)$ additions.
3. The formula (1.20): r additions plus the matrix multiplication, which comprises rn_{i+1} multiplications and $r(n_{i+1}-1)$ additions.

For instance, the matrix operation $q(i-1)x(i-1)$ in formula (1.17) is a product of an $r \times n_{i-1}$ matrix $q(i-1)$ by an n_{i-1}-dimensional vector $x(i-1)$ and hence it requires rn_{i-1} multiplications and $r(n_{i-1}-1)$ additions. Thus the total complexity for computation of the value χ_i is $2rn_{i-1}$ arithmetical operations. In the same way one obtains complexities for the computation of the other variables of the algorithm.

Hence, the total complexity of Algorithm 1.7 is estimated as

$$c < 2\sum_{k=1}^{N}(rn_{k-1} + m_k r + m_k n_k). \tag{1.22}$$

Similarly the complexity of Algorithm 1.8 is estimated as

$$c < 2\sum_{k=1}^{N}(rn_{k+1} + m_k r + m_k n_k). \tag{1.23}$$

Let the block sizes $m_k,\ n_k$ be bounded by the number m, i.e., $m_k, n_k \leq m$. In this case inequality (1.22) or inequality (1.23) yield the estimate

$$c < (2r + m)2mN.$$

Thus, for a matrix with diagonal plus separable representation the multiplication by a vector costs $O(N)$ arithmetic operations in contrast with $O(N^2)$ for a matrix in general form. It is clear that in this form the best estimate is obtained when r equals the separable order of the matrix.

§1.3.2 Forward-backward computation of y

Here we consider another procedure which will be extended later to various types of representations. In this case the vector y is found as $y = y_L + y_D + y_U$, where $y_L = A_L x$, $y_D = A_D x$, $y_U = A_U x$ and $A_L,\ A_D,\ A_U$ are correspondingly the strictly lower triangular, the diagonal and the strictly upper triangular parts of the matrix A.

For y_L one has $y_L(1) = 0$ and for $i \geq 2$ one obtains

$$y_L(i) = \sum_{j=1}^{i-1} A_{ij}x(j) = \sum_{j=1}^{i-1} p(i)q(j)x(j) = p(i)\chi_i$$

where

$$\chi_i = \sum_{j=1}^{i-1} q(j)x(j).$$

One can see that the variable χ_i satisfies the following relations: $\chi_1 = 0$ and for $i = 1, \ldots, N-1$ one obtains the recursion

$$\chi_{i+1} = \chi_i + q(i)x(i).$$

For y_U one has $y_U(N) = 0$ and for $i \leq N-1$ one obtains

$$y_U(i) = \sum_{j=i+1}^{N} A_{ij}x(j) = \sum_{j=i+1}^{N} p(i)q(j)x(j) = p(i)\eta_i,$$

where

$$\eta_i = \sum_{j=i+1}^{N} q(j)x(j).$$

One has $\eta_N = 0$ and it follows that for $i = N, \ldots, 2$ the variable η_i satisfies the recurrence relation

$$\eta_{i-1} = \eta_i + q(i)x(i).$$

For y_D it is obvious that $y_D(i) = p(i)q(i)x(i) + d(i)x(i),\ i = 1, \ldots, N.$

From these relations one obtains the following algorithm for computing the product $y = Ax$.

Algorithm 1.9. (Forward-backward multiplication by a vector)

1. Start with $y_L(1) = 0,\ \chi_1 = 0$ and for $i = 2, \ldots, N$ compute recursively

$$\chi_i = \chi_{i-1} + q(i-1)x(i-1), \tag{1.24}$$
$$y_L(i) = p(i)\chi_i. \tag{1.25}$$

2. Compute for $i = 1, \ldots, N$

$$y_D(i) = (p(i)q(i) + d(i))x(i). \tag{1.26}$$

3. Start with $y_U(N) = 0,\ \eta_N = 0$ and for $i = N-1, \ldots, 1$ compute recursively

$$\eta_i = \eta_{i+1} + q(i+1)x(i+1), \tag{1.27}$$
$$y_U(i) = p(i)\eta_i. \tag{1.28}$$

4. Compute the vector y

$$y = y_L + y_D + y_U. \tag{1.29}$$

The complexity of the arithmetic operations used in Algorithm 1.9 is as follows.

1. The formula (1.24): r additions and a matrix vector multiplication which comprises another rn_{i-1} multiplications and $r(n_{i-1}-1)$ additions.
2. The formula (1.25): $m_i r$ multiplications and $m_i(r-1)$ additions.
3. The formula (1.26): $m_i r n_i$ multiplications and $m_i(r-1)n_i$ additions plus $m_i n_i$ additions inside the brackets and then another $m_i n_i$ multiplications and $m_i(n_i-1)$ additions.
4. The formula (1.27): r additions and a matrix vector multiplication which comprises another rn_{i+1} multiplications and $r(n_{i+1}-1)$ additions.
5. The formula (1.28): $m_i r$ multiplications and $m_i(r-1)$ additions.
6. The formula (1.29): $2m_k$ additions.

Hence the total complexity of Algorithm 1.9 is estimated as follows:

$$c < 2\sum_{k=1}^{N}[m_k(2r + n_k + rn_k) + n_{k-1}r + rn_{k+1} + m_k]. \tag{1.30}$$

Let the block sizes m_k, n_k be bounded by the number m, i.e., $m_k, n_k \leq m$. In this case using the inequality (1.30) one obtains the estimate

$$c < (rm + 4r + m + 1)2mN. \tag{1.31}$$

Thus for a matrix with separable representation the forward multiplication by a vector in Algorithm 1.7, the backward multiplication by a vector in Algorithm 1.8, as well as the forward-backward multiplication by a vector in Algorithm 1.9 costs $O(N)$ arithmetic operations in contrast to $O(N^2)$ for a matrix in general form. It is clear that in the form (1.31) the best estimate is obtained when r equals the separable order of the matrix.

§1.4 Systems with homogeneous boundary conditions associated with matrices in diagonal plus separable form

Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be an $N \times N$ block matrix with diagonal plus separable representation with block entries of sizes $m_i \times n_j$, with separable generators $p(i)$ $(i = 1, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N)$ of order r.

§1.4.1 Forward and backward systems

Consider in detail the arithmetic operations used in Algorithm 1.7. The formulas (1.16), (1.17) and (1.18) together describe what is called a discrete-time forward system with homogeneous boundary conditions:

$$\begin{cases} \chi_{k+1} = \chi_k + q(k)x(k), & k = 1, \dots, N \\ y(k) = p(k)\chi_{N+1} + d(k)x(k), & k = 1, \dots, N \\ \chi_1 = 0. \end{cases} \tag{1.32}$$

Consider in detail the arithmetic operations used in Algorithm 1.8. The formulas (1.19), (1.20) and (1.21) together describe what is called a discrete-time backward system with homogeneous boundary conditions:

$$\begin{cases} \eta_{k-1} = \eta_k + q(k)x(k), & k = N, \dots, 1 \\ y(k) = p(k)\eta_0 + d(k)x(k), & k = N, \dots, 1 \\ \eta_N = 0. \end{cases} \tag{1.33}$$

Here the vectors $x(k)$ $(k = 1, \dots, N)$ are called the input, the vectors $y(k)$ $(k = 1, \dots, N)$ are called the output, and the vectors χ_k and η_k of size r are called the state space variables of the respective systems. The transformation from $x = (x(k))_{k=1}^N$ to $y = (y(k))_{k=1}^N$ is a linear transformation which maps the input of the system into the output. This transformation is called the *input-output operator of the system.*

Thus one obtains the following.

Theorem 1.10. *Let A be an $N \times N$ block matrix with diagonal plus separable of order r representation with separable generators $p(k), q(k)$ $(k = 1, \dots, N)$ and diagonal generators $d(k)$ $(k = 1, \dots, N)$.*

Then A is the matrix of the input-output operator of the forward system (1.32) *and of the backward system* (1.33) *with coefficients equal to the corresponding generators of the matrix A.*

The inverse statement is also true.

Theorem 1.11. *Let there be given a forward system* (1.32) *or a backward system* (1.33).

Then the matrix A with separable of order r generators $p(k), q(k)$ $(k = 1, \dots, N)$ and diagonal generators $d(k)$ $(k = 1, \dots, N)$ which are equal to the corresponding coefficients of the system, is the matrix of the input-output operator of the system (1.32)*, respectively* (1.33).

Proof. Let x be an input of the system. One can easily prove by induction that the solution of the first equation in (1.32) is given by

$$\chi_k = \sum_{j=1}^{k-1} q(j)x(j), \quad k = 1, \dots, N+1. \tag{1.34}$$

Indeed, for $k = 1$ the relation (1.34) follows directly from $\chi_1 = 0$. Suppose that (1.34) holds for some $k, k \geq 1$. Using the first equation from (1.32) one gets

$$\chi_{k+1} = \sum_{j=1}^{k-1} q(j)x(j) + q(k)x(k) = \sum_{j=1}^{k} q(j)x(j).$$

Similarly, the solution of the first equation in the backward system (1.33) is given by

$$\eta_k = \sum_{j=k+1}^{N} q(j)x(j), \quad k = N, \dots, 0. \tag{1.35}$$

Indeed, for $k = N$ the relation (1.35) follows directly from $\eta_N = 0$. Suppose that (1.35) holds for some $k, k \leq N$. Using the first equation from (1.33) one gets

$$\eta_{k-1} = \sum_{j=k+1}^{N} q(j)x(j) + q(k)x(k) = \sum_{j=k}^{N} q(j)x(j).$$

Thus for the output y one obtains in both cases

$$y(k) = p(k) \sum_{j=1}^{N} q(j)x(j) + d(k)x(k), \quad k = 1, \dots, N.$$

From here one obtains $y = Ax$ with the matrix A of the form $A = D + PQ$. $\square$

Example 1.12. Consider the $N \times N$ matrix A from Example 1.4 with the separable generators $p(k) = q(k) = k, \ k = 1, \dots, N$ and the diagonal generators $d(k) = 0, \ k = 1, \dots, N$.

Then the forward system with homogenous boundary conditions (1.32) becomes

$$\begin{cases} \chi_{k+1} = \chi_k + kx(k), \quad k = 1, \dots, N, \\ y(k) = p(k)\chi_{N+1} + d(k)x(k) = k\chi_{N+1}, \quad k = 1, \dots, N, \\ \chi_1 = 0 \end{cases} \tag{1.36}$$

and the backward system with homogeneous boundary conditions (1.33) becomes

$$\begin{cases} \eta_{k-1} = \eta_k + kx(k), \quad k = N, \dots, 1, \\ y(k) = p(k)\eta_0 + d(k)x(k) = k\eta_0, \quad k = 1, \dots, N, \\ \eta_N = 0. \end{cases} \tag{1.37}$$

$\diamondsuit$

§1.4.2 Forward-backward descriptor systems

Consider in detail the arithmetic operations used in Algorithm 1.9. Using formulas (1.25), (1.26), (1.28) one has

$$y(k) = p(k)\chi_k + p(k)\eta_k + p(k)q(k)x(k) + d(k)x(k), \quad k = 1, \dots, N. \tag{1.38}$$

Here the auxiliary variables χ_k, η_k are determined via the recurrence relations

$$\chi_1 = 0, \quad \chi_i = \chi_{i-1} + q(i-1)x(i-1), \quad i = 2, \dots, N; \tag{1.39}$$

$$\eta_N = 0, \quad \eta_i = \eta_{i+1} + q(i+1)x(i+1), \quad i = N-1, \dots, 1. \tag{1.40}$$

One can rewrite relations (1.39), (1.40) in the form

$$\chi_1 = 0_r, \quad \chi_{k+1} = \chi_k + q(k)x(k), \quad k = 1, \dots, N-1; \tag{1.41}$$

$$\eta_N = 0_r, \quad \eta_{k-1} = \eta_k + q(k)x(k), \quad k = N, \dots, 2. \tag{1.42}$$

The formulas (1.38), (1.41), (1.42) together are called the forward-backward descriptor system with homogeneous boundary conditions:

$$\begin{cases} \chi_{k+1} = \chi_k + q(k)x(k), & k = 1, \dots, N-1 \\ \eta_{k-1} = \eta_k + q(k)x(k), & k = N, \dots, 2 \\ y(k) = p(k)\chi_k + p(k)\eta_k + (p(k)q(k) + d(k))x(k), & k = 1, \dots, N \\ \chi_1 = 0, \quad \eta_N = 0. \end{cases} \tag{1.43}$$

Alternatively, consider the system (1.32). From the recursions

$$\chi_{k+1} = \chi_k + q(k)x(k), \quad k = 1, \dots, N, \quad \chi_1 = 0, \tag{1.44}$$

it follows that

$$\chi_{N+1} = \sum_{k=1}^{N} q(k)x(k).$$

Denote

$$\eta_k = \sum_{i=k+1}^{N} q(i)x(i).$$

Then the following recursion takes place

$$\eta_{k-1} = \eta_k + q(k)x(k), \quad k = N, \dots, 2, \quad \eta_N = 0, \tag{1.45}$$

while

$$\chi_{N+1} = \chi_k + q(k)x(k) + \eta_k.$$

Inserting this relation in the second equation of the system (1.32) we get

$$y(k) = p(k)\chi_k + p(k)\eta_k + (p(k)q(k) + d(k))x(k), \quad k = 1, \dots, N. \tag{1.46}$$

The equations (1.44)–(1.46) form again the system (1.43).

In the system (1.43) as above the vectors $x(k)$ $(k = 1, \dots, N)$ are called the input, the vectors $y(k)$ $(k = 1, \dots, N)$ are called the output, and the vectors χ_k and η_k of size r are called the state space variables of the system. The transformation from $x = (x(k))_{k=1}^N$ to $y = (y(k))_{k=1}^N$ is a linear transformation which maps the input of the system into the output. This transformation is called the *input-output operator of the system.*

Thus one obtains the following.

Theorem 1.13. *Let A be an $N \times N$ block matrix with diagonal plus separable of order r representation, with separable generators $p(k), q(k)$ $(k = 1, \dots, N)$ and diagonal generators $d(k)$ $(k = 1, \dots, N)$.*

Then A is the matrix of the input-output operator of the system (1.43) *with coefficients equal to the corresponding generators of A.*

The inverse statement is also true.

Theorem 1.14. *Let there be given a system* (1.43)*. Then the matrix A with separable of order r generators $p(k), q(k)$ $(k = 1, \dots, N)$ and diagonal generators $d(k)$ $(k = 1, \dots, N)$ which are equal to the corresponding coefficients of the system, is the matrix of the input-output operator of the system* (1.43)*.*

Example 1.15. Consider the $N \times N$ matrix from Example 1.4 with the same separable generators $p(k) = q(k) = k,\ k = 1, \dots, N$ and with the diagonal generators $d(k) = 0,\ k = 1, \dots, N$. Then the descriptor system with boundary conditions (1.43) becomes

$$\begin{gathered}
\chi_{k+1} = \chi_k + q(k)x(k) = \chi_k + kx(k), \quad k = 1, \dots, N-1, \\
\eta_{k-1} = \eta_k + q(k)x(k) = \eta_k + kx(k), \quad k = N, \dots, 2, \\
y(k) = p(k)\chi_k + p(k)\eta_k + p(k)q(k)x(k) = k\chi_k + k\eta_k + k^2x(k), \quad k = 1, \dots, N, \\
\chi_1 = 0, \quad \eta_N = 0.
\end{gathered} \qquad \diamondsuit$$

§1.5 Multiplication of matrices

In this section we consider products of matrices with separable or diagonal plus separable representations. We derive formulas to compute generators of the product.

§1.5.1 Product of matrices with separable representations

In this subsection it is shown that the product of two suitable matrices with given separable representations of orders r and s, respectively, is a matrix with separable representation of the lesser order. Formulas for separable generators of the product are derived.

Theorem 1.16. *Let $A^{(1)}$ and $A^{(2)}$ be two $N \times N$ block matrices of total scalar sizes of $(\sum_{i=1}^N m_i) \times (\sum_{i=1}^N \nu_i)$ and $(\sum_{i=1}^N \nu_i) \times (\sum_{i=1}^N n_i)$ respectively, represented in separable form of order r and respectively s, with separable generators $p^{(1)}(i), q^{(1)}(i),\ i = 1, \dots, N$ and $p^{(2)}(i), q^{(2)}(i),\ i = 1, \dots, N$ which are matrices of sizes $m_i \times r$, $r \times \nu_i$, $\nu_i \times s$ and $s \times n_i$ respectively. In matrix form, if we consider the matrices $P^{(1)} = \operatorname{col}(p^{(1)}(i))_{i=1}^N$, $Q^{(1)} = \operatorname{row}(q^{(1)}(i))_{i=1}^N$, $P^{(2)} = \operatorname{col}(p^{(2)}(i))_{i=1}^N$, $Q^{(2)} = \operatorname{row}(q^{(2)}(i))_{i=1}^N$, then $A^{(1)} = P^{(1)}Q^{(1)}$ and $A^{(2)} = P^{(2)}Q^{(2)}$.*

Denote by Z the $r \times s$ matrix

$$Z = Q^{(1)}P^{(2)} = \sum_{k=1}^N q^{(1)}(k)p^{(2)}(k).$$

Then the product block matrix $A = A^{(1)}A^{(2)}$ admits the order s separable representation $A = PQ^{(2)}$, where $P = P^{(1)}Z$, with separable generators $p^{(1)}(i)Z$, $q^{(2)}(i)$, $i = 1, \dots, N$ of sizes $m_i \times s$ and $s \times n_i$, respectively, as well as the order r separable representation $A = P^{(1)}Q$ where $Q = ZQ^{(2)}$, with separable generators $p^{(1)}(i)$, $Zq^{(2)}(i)$, $i = 1, \dots, N$ of sizes $m_i \times r$ and $r \times n_i$, respectively.

Proof. For any $i, j = 1, \dots, N$

$$A = A^{(1)}A^{(2)} = P^{(1)}Q^{(1)}P^{(2)}Q^{(2)} = P^{(1)}ZQ^{(2)} = (P^{(1)}Z)Q^{(2)} = P^{(1)}(ZQ^{(2)}).$$

Therefore, the product block matrix $A = A^{(1)}A^{(2)}$ admits the order s separable representation $A = PQ^{(2)}$, where $P = P^{(1)}Z$, with separable generators $p^{(1)}(i)Z$, $q^{(2)}(i)$, $i = 1, \dots, N$ as well as the order r separable representation $A = P^{(1)}Q$, where $Q = ZQ^{(2)}$, with separable generators $p^{(1)}(i)$, $Zq^{(2)}(i)$, $i = 1, \dots, N$. □

Let m be the maximal block size of the matrices $A^{(1)}, A^{(2)}$, i.e.,

$$m = \max_{1 \le k \le N} (m_k, \nu_k, n_k).$$

The complexity of the arithmetic operations of the algorithm in Theorem 1.16 is calculated as follows.

1. Computation of Z: $\sum_{k=1}^N r\nu_k s$ multiplications and $\sum_{k=1}^N r(\nu_k - 1)s$ additions, thus less than $2mrsN$ arithmetic operations.
2. Computation of $P = P^{(1)}Z$: $\sum_{i=1}^N m_i rs$ multiplications and $\sum_{i=1}^N m_i(r-1)s$ additions, thus less than $2mrsN$ arithmetic operations.
3. Computation of $A = PQ^{(2)}$: $\sum_{j=1}^N rsn_j$ multiplications and $\sum_{j=1}^N r(s-1)n_j$ additions, thus less than $2mrsN$ arithmetic operations.

Thus the total complexity of the algorithm for the multiplication of two block matrices with separable representation is linear, namely

$$c < 6mrsN$$

arithmetic operations.

§1.5.2 Product of matrices with diagonal plus separable representations

In this subsection it is shown that the product of two suitable matrices with given diagonal plus separable representations of orders r and s, respectively, is a matrix with diagonal plus separable representation of order at most $r+s$. Formulas for the separable and diagonal generators of the product are derived.

Theorem 1.17. *Let $A^{(1)}$ and $A^{(2)}$ be two $N \times N$ block matrices of total scalar sizes $(\sum_{i=1}^{N} m_i) \times (\sum_{i=1}^{N} \nu_i)$ and $(\sum_{i=1}^{N} \nu_i) \times (\sum_{i=1}^{N} n_i)$, respectively, given in diagonal plus separable form of orders r and s, respectively, namely $A^{(1)}$ has the diagonal generators $d^{(1)}(i),\ i=1,\dots,N$ of size $m_i \times \nu_i$ and the separable generators $p^{(1)}(i), q^{(1)}(i),\ i=1,\dots,N$ of sizes $m_i \times r$ and $r \times \nu_i$, respectively, while $A^{(2)}$ has the diagonal generators $d^{(2)}(i),\ i=1,\dots,N$ of size $\nu_i \times n_i$ and the separable generators $p^{(2)}(i), q^{(2)}(i),\ i=1,\dots,N$ of sizes $\nu_i \times s$ and $s \times n_i$, respectively. In matrix form, if we consider also the matrices*

$$D^{(1)} = \operatorname{diag}(d^{(1)}(i))_{i=1}^{N}, \quad P^{(1)} = \operatorname{col}(p^{(1)}(i))_{i=1}^{N}, \quad Q^{(1)} = \operatorname{row}(q^{(1)}(i))_{i=1}^{N}$$

and

$$D^{(2)} = \operatorname{diag}(d^{(2)}(i))_{i=1}^{N}, \quad P^{(2)} = \operatorname{col}(p^{(2)}(i))_{i=1}^{N}, \quad Q^{(2)} = \operatorname{row}(q^{(2)}(i))_{i=1}^{N},$$

then $A^{(1)} = D^{(1)} + P^{(1)}Q^{(1)}$ and $A^{(2)} = D^{(2)} + P^{(2)}Q^{(2)}$.

Denote by Z the $r \times s$ matrix

$$Z = Q^{(1)}P^{(2)} = \sum_{k=1}^{N} q^{(1)}(k)p^{(2)}(k). \tag{1.47}$$

Then the product block matrix $A = A^{(1)}A^{(2)}$ admits the order $r+s$ diagonal plus separable representation $A = D + PQ$ with the diagonal generators $d(i),\ i = 1,\dots,N$ of size $m_i \times n_i$ and the separable generators $p(i), q(i),\ i = 1,\dots,N$ of sizes $m_i \times (r+s)$ and $(r+s) \times n_i$, respectively. These generators are given by the formulas

$$d(i) = d^{(1)}(i)d^{(2)}(i), \qquad i = 1,\dots,N, \tag{1.48}$$

$$p(i) = \begin{pmatrix} p^{(1)}(i) & d^{(1)}(i)p^{(2)}(i) \end{pmatrix}, \qquad i = 1,\dots,N, \tag{1.49}$$

$$q(i) = \begin{pmatrix} q^{(1)}(i)d^{(2)}(i) + Zq^{(2)}(i) \\ q^{(2)}(i) \end{pmatrix}, \qquad i = 1,\dots,N. \tag{1.50}$$

Proof. We have

$$A = A^{(1)}A^{(2)} = (D^{(1)} + P^{(1)}Q^{(1)})(D^{(2)} + P^{(2)}Q^{(2)}) = D + PQ$$

with

$$D = D^{(1)}D^{(2)}, \quad P = \begin{pmatrix} P^{(1)} & D^{(1)}P^{(2)} \end{pmatrix}, \quad Q = \begin{pmatrix} Q^{(1)}D^{(2)} + ZQ^{(2)}) \\ Q^{(2)} \end{pmatrix}.$$

Here Z is the $r \times s$ matrix defined in (1.47).

The formula $D = D^{(1)}D^{(2)}$ means (1.48) for diagonal entries. For the computation of $p(i)$, note that

$$\begin{pmatrix} P^{(1)} & D^{(1)}P^{(2)} \end{pmatrix} = \operatorname{col}\begin{pmatrix} p^{(1)}(i) & d^{(1)}(i)p^{(2)(i)} \end{pmatrix}_{i=1}^{N},$$

which implies (1.49). Finally, using the fact that

$$Q^{(1)}D^{(2)} = \operatorname{row}(q^{(1)}(i)d^{(2)}(i))_{i=1}^{N}, \quad Q^{(2)} = \operatorname{row}(q^{(2)}(i))_{i=1}^{N},$$

we obtain (1.50). □

Let m be the maximal block size of the matrices $A^{(1)}, A^{(2)}$, i.e.,

$$m = \max_{1 \le k \le N}(m_k, \nu_k, n_k).$$

The complexity of the arithmetic operations of the algorithm in Theorem 1.17 is calculated as follows.

1. The computation of $Z = Q^{(1)}P^{(2)}$ costs less than $rmNs$ multiplications and less than $r(mN-1)s$ additions, thus less than $2rmNs$ arithmetic operations.
2. Computation of all the formulas (1.48) costs $\sum_{i=1}^{N} m_i\nu_i n_i$ multiplications and $\sum_{i=1}^{N} m_i(\nu_i - 1)n_i$ additions, thus less than $2m^3N$ arithmetic operations.
3. Computation of all the formulas (1.49): $\sum_{i=1}^{N} m_i\nu_i s$ multiplications and $\sum_{i=1}^{N} m_i(\nu_i - 1)s$ additions, thus less than $2m^2sN$ arithmetic operations.
4. Computation of all the formulas (1.50): the computation of $q^{(1)}(i)d^{(2)}(i)$ requires for $\sum_{i=1}^{N} r\nu_i n_i$ multiplications and $\sum_{i=1}^{N} r(\nu_i - 1)n_i$ additions, thus less than $2m^2rN$ arithmetic operations. The computation of all $Zq^{(2)}(i)$ costs at most $rsNm$ multiplications and $r(s-1)Nm$ additions, thus less than $2rsNm$ arithmetic operations.

Thus the total complexity of the algorithm for the multiplication of two block matrices with diagonal plus separable representation is linear, namely

$$c < (2m^2 + 2sm + 2rm + 4rs)mN$$

arithmetic operations. For a scalar matrix there are $c = (2rs+r+s+1)N$ arithmetic operations.

Example 1.18. Consider the 3×3 matrix $A^{(1)}$ which has a diagonal plus order 1 separable representation:

$$A^{(1)} = \begin{pmatrix} 1 & 3 & -5 \\ 6 & -15 & 30 \\ 3 & -9 & 20 \end{pmatrix} = D + P^{(1)}Q^{(1)}$$
$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 5 \end{pmatrix}.$$

It follows from this computation that the diagonal generators are $d(1) = 2, d(2) = 3, d(3) = 5$ and the separable generators are $p(1) = -1, p(2) = 6, p(3) = 3, q(1) = 1, q(2) = -3, q(3) = 5$.

Multiply the matrix $A^{(1)}$ on the right by the matrix

$$A^{(2)} = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 7 & 6 \\ 3 & 6 & 12 \end{pmatrix} = 3I + \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix},$$

which has the following separable generators

$$p^{(2)}(i) = q^{(2)}(i) = i, \; i = 1, 2, 3,$$

as Example 1.4 for $N = 3$ shows, and diagonal generators $d^{(2)}(1) = d^{(2)}(2) = d^{(2)}(3) = 3$.

Theorem 1.17 gives the following diagonal plus order 2 separable representation of the product matrix. Compute first according to (1.47)

$$Z = \begin{pmatrix} 1 & -3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 10.$$

Compute $d(i), \; i = 1, 2, 3$ using (1.48)

$$d(1) = d^{(1)}(1)d^{(2)}(1) = 6, \; d(2) = 3 \cdot 3 = 9, \; d(3) = 5 \cdot 3 = 15.$$

Compute $p(i), \; i = 1, 2, 3$ using (1.49)

$$p(1) = \begin{pmatrix} p^{(1)}(1) & d^{(1)}(1)p^{(2)}(1) \end{pmatrix} = \begin{pmatrix} -1 & 2 \cdot 1 \end{pmatrix},$$
$$p(2) = \begin{pmatrix} 6 & 6 \end{pmatrix}, \quad p(3) = \begin{pmatrix} 3 & 15 \end{pmatrix}.$$

Compute $q(i), \; i = 1, 2, 3$ using (1.50)

$$q(1) = \begin{pmatrix} q^{(1)}(1)d^{(2)}(1) + Zq^{(2)}(1) \\ q^{(2)}(1) \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 10 \cdot 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 1 \end{pmatrix},$$
$$q(2) = \begin{pmatrix} -3 \cdot 3 + 10 \cdot 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix}, \quad q(3) = \begin{pmatrix} 5 \cdot 3 + 10 \cdot 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 45 \\ 3 \end{pmatrix}.$$

As a check, compute the product matrix A as $A = D + PQ$ where $D = \operatorname{diag}(d(i))_{i=1}^3$, $P = \operatorname{col}(p(i))_{i=1}^3$, $Q = \operatorname{row}(q(i))_{i=1}^3$, namely

$$A = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 15 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 6 & 6 \\ 3 & 15 \end{pmatrix} \begin{pmatrix} 13 & 11 & 45 \\ 1 & 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 15 \end{pmatrix} + \begin{pmatrix} -11 & -7 & -39 \\ 84 & 78 & 288 \\ 54 & 63 & 180 \end{pmatrix}$$

and note that the result of the direct multiplication is the matrix

$$A = A^{(1)}A^{(2)} = \begin{pmatrix} -5 & -7 & -39 \\ 84 & 87 & 288 \\ 54 & 63 & 195 \end{pmatrix}. \qquad \Diamond$$

§1.6 Schur factorization and inversion of block matrices

Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a block matrix with block entries of sizes $m_i \times m_j$ which has invertible principal leading submatrices $\{A_{ij}\}_{i,j=1}^{k}$, $k = 1, 2, \ldots, N$. Such a matrix A is called *strongly regular*. Here we derive some general results concerning Schur factorizations of strongly regular matrices. It will be proved that every strongly regular matrix A admits the LDU factorization

$$A = LDU, \tag{1.51}$$

where L, U, D are block matrices with the same sizes of blocks as A, L and U are block lower and respectively upper triangular matrices with only identities on the main diagonals and D is a block diagonal matrix.

We start with a detailed study of factorizations of 2×2 block matrices with invertible principal submatrices. We also derive inversion formulas for such matrices.

Theorem 1.19. *Let A be an $(m_1 + m_2) \times (m_1 + m_2)$ matrix partitioned in the form*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with matrices $A_{11}, A_{12}, A_{21}, A_{22}$ of sizes $m_1 \times m_1, m_1 \times m_2, m_2 \times m_1, m_2 \times m_2$, respectively.

Assume that the matrix A_{11} is invertible. Then the factorization

$$A = \begin{pmatrix} I_{m_1} & 0 \\ A_{21}A_{11}^{-1} & I_{m_2} \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & \Gamma \end{pmatrix} \begin{pmatrix} I_{m_1} & A_{11}^{-1}A_{12} \\ 0 & I_{m_2} \end{pmatrix}, \tag{1.52}$$

with $\Gamma = A_{22} - A_{21}A_{11}^{-1}A_{12}$, holds. Moreover the formula

$$\det A = \det A_{11} \det \Gamma \tag{1.53}$$

is valid. Furthermore the matrix A is invertible if and only if the matrix Γ is invertible and in this case the inversion formula

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}\Gamma^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}\Gamma^{-1} \\ -\Gamma^{-1}A_{21}A_{11}^{-1} & \Gamma^{-1} \end{pmatrix} \tag{1.54}$$

holds.

Assume that the matrix A_{22} is invertible. Then the factorization

$$A = \begin{pmatrix} I_{m_1} & A_{12}A_{22}^{-1} \\ 0 & I_{m_2} \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} I_{m_1} & 0 \\ A_{22}^{-1}A_{21} & I_{m_2} \end{pmatrix}, \tag{1.55}$$

with $\Theta = A_{11} - A_{12}A_{22}^{-1}A_{21}$, holds. Moreover the formula

$$\det A = \det A_{22} \det \Theta \tag{1.56}$$

is valid. Furthermore the matrix A is invertible if and only if the matrix Θ is invertible and in this case the inversion formula

$$A^{-1} = \begin{pmatrix} \Theta^{-1} & -\Theta^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}\Theta^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}\Theta^{-1}A_{12}A_{22}^{-1} \end{pmatrix} \tag{1.57}$$

holds.

Proof. Assume that the matrix A_{11} is invertible. Applying (block) Gauss elimination to the matrix A we obtain the formula (1.52). The formula (1.53) follows directly from (1.52). Moreover from (1.53) it follows that A is invertible if and only if Γ is invertible. If the last condition is valid, then the matrix A^{-1} may be represented in the form

$$A^{-1} = \begin{pmatrix} I_{m_1} & -A_{11}^{-1}A_{12} \\ 0 & I_{m_2} \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & \Gamma^{-1} \end{pmatrix} \begin{pmatrix} I_{m_1} & 0 \\ -A_{21}A_{11}^{-1} & I_{m_2} \end{pmatrix}.$$

Multiplication of the factors yields (1.54).

Now assume that the matrix A_{22} is invertible. The formula (1.55) is obtained by applying (1.52) to the matrix

$$\begin{pmatrix} 0 & I_{m_2} \\ I_{m_1} & 0 \end{pmatrix} A \begin{pmatrix} 0 & I_{m_1} \\ I_{m_2} & 0 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{21} \\ A_{12} & A_{11} \end{pmatrix}.$$

The formula (1.56) follows directly from (1.55). Furthermore, from (1.55) it follows that A is invertible if and only if Θ is invertible and in this case the matrix A^{-1} may be represented in the form

$$A^{-1} = \begin{pmatrix} I_{m_1} & 0 \\ -A_{22}^{-1}A_{21} & I_{m_2} \end{pmatrix} \begin{pmatrix} \Theta^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} \begin{pmatrix} I_{m_1} & -A_{12}A_{22}^{-1} \\ 0 & I_{m_2} \end{pmatrix}.$$

Multiplication of the factors in the last equality yields (1.57). □

Now we consider the LDU factorization for an arbitrary matrix with invertible principal leading submatrices.

Theorem 1.20. *Let* $A = \{A_{ij}\}_{i,j=1}^{N}$ *be a block matrix with block entries of sizes* $m_i \times m_j$ *and with invertible principal leading submatrices* $A_k = \{A_{ij}\}_{i,j=1}^{k}$, $k = 1, 2, \dots, N$.

Then A admits a unique LDU factorization (1.51). *Moreover the factors in* (1.51) *may be determined as follows. Let the matrix A be partitioned in the form*

$$A = \begin{pmatrix} A_{k-1} & B_k \\ C_k & M_k \end{pmatrix}, \quad k = 2, \dots, N \tag{1.58}$$

with the submatrices $A_k = A(1:k, 1:k)$ *partitioned in the form*

$$A_k = \begin{pmatrix} A_{k-1} & b_k \\ c_k & d_k \end{pmatrix}, \quad k = 2, \dots, N. \tag{1.59}$$

The matrix D in (1.51) *is a block diagonal matrix*

$$D = \mathrm{diag}\{\gamma_1, \dots, \gamma_N\},$$

with invertible diagonal blocks γ_k $(k = 1, \dots, N)$ *of sizes* $m_k \times m_k$ *obtained by the formulas*

$$\gamma_1 = d_1, \quad \gamma_k = d_k - c_k A_{k-1}^{-1} b_k, \quad k = 2, \dots, N. \tag{1.60}$$

Furthermore, the matrices L, U in (1.51) *are determined via the relations*

$$L(k:N, k) = \Delta_k(:, 1)\gamma_k^{-1}, \quad k = 1, \dots, N-1 \tag{1.61}$$

$$U(k, k:N) = \gamma_k^{-1}\Delta_k(1, :), \quad k = 1, \dots, N-1, \tag{1.62}$$

where

$$\Delta_1 = A, \quad \Delta_k = M_k - C_k A_{k-1}^{-1} B_k, \quad k = 2, \dots, N-1. \tag{1.63}$$

Proof. The existence of the factorization (1.51) is established by induction on k. For $k = 1$ one obviously obtains $A_1 = L_1 D_1 U_1$ with $L_1 = I, D_1 = A_1, U_1 = I$. Suppose by induction that for some k with $2 \le k \le N$ the factorization

$$A_{k-1} = L_{k-1} D_{k-1} U_{k-1} \tag{1.64}$$

holds, with block lower and upper triangular L_{k-1} and U_{k-1} having identities on the main diagonals, and with block diagonal D_{k-1}. Consider the matrix A_k partitioned in the form (1.59). Applying formula (1.52) one gets

$$A_k = \begin{pmatrix} I & 0 \\ c_k A_{k-1}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{k-1} & 0 \\ 0 & d_k - c_k A_{k-1}^{-1} b_k \end{pmatrix} \begin{pmatrix} I & A_{k-1}^{-1} b_k \\ 0 & I \end{pmatrix}. \tag{1.65}$$

Furthermore, using (1.64) one obtains

$$A_k = \begin{pmatrix} L_{k-1} & 0 \\ c_k A_{k-1}^{-1} L_{k-1} & I \end{pmatrix} \begin{pmatrix} D_{k-1} & 0 \\ 0 & d_k - c_k A_{k-1}^{-1} b_k \end{pmatrix} \begin{pmatrix} U_{k-1} & U_{k-1} A_{k-1}^{-1} b_k \\ 0 & I \end{pmatrix}.$$

Here the first and the last factors are (block) lower and upper triangular matrices with identities on the main diagonals and the middle factor is a block diagonal matrix. Thus one obtains the factorization of the form (1.51) for the matrix A_k. Taking $k = N$ one obtains the factorization (1.51) for the matrix A.

To prove the uniqueness, let

$$A = L_1 D_1 U_1 = L_2 D_2 U_2$$

be two LDU factorizations of the matrix A. All the matrices here are invertible. Moreover, one has

$$L_2^{-1} L_1 = (D_2 U_2)(D_1 U_1)^{-1}.$$

Since the matrix $L_2^{-1} L_1$ is lower triangular with only identities on the main diagonal and the matrix $(D_2 U_2)(D_1 U_1)^{-1}$ is upper triangular, one gets $L_2^{-1} L_1 = (D_2 U_2)(D_1 U_1)^{-1} = I$ and therefore $L_1 = L_2$ and $D_1 U_1 = D_2 U_2$. From the second equality since D_1, D_2 are diagonal and U_1, U_2 are upper triangular with only identities on the main diagonals one obtains $D_1 = D_2$, $U_1 = U_2$.

To derive the formulas for the factors, consider at the beginning the first column of the matrix L, the first row of the matrix U and the first diagonal entry of the matrix D. From $A = LDU$ one obviously gets

$$\gamma_1 = A(1,1) = d_1, \quad A(:,1) = L(:,1)\gamma_1, \quad A(1,:) = \gamma_1 U(1,:),$$

which means

$$D(1,1) = \gamma_1, \quad L(:,1) = A(:,1)\gamma_1^{-1}, \quad U(1,:) = \gamma_1^{-1} A(1,:). \tag{1.66}$$

Next let A be partitioned in the form (1.58). By the formula (1.52),

$$A = \begin{pmatrix} I & 0 \\ C_k A_{k-1}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{k-1} & 0 \\ 0 & \Delta_k \end{pmatrix} \begin{pmatrix} I & A_{k-1}^{-1} B_k \\ 0 & I \end{pmatrix}, \quad k = 2, \dots, N. \tag{1.67}$$

This implies in particular that

$$\begin{aligned} A_{k-1+m} &= A(1:k-1+m, 1:k-1+m) \\ &= \begin{pmatrix} I_{k-1} & 0 \\ C_k(1:m,:)A_{k-1}^{-1} & I_m \end{pmatrix} \begin{pmatrix} A_{k-1} & 0 \\ 0 & \Delta_k(1:m,1:m) \end{pmatrix} \begin{pmatrix} I & A_{k-1}^{-1} B_k(:,1:m) \\ 0 & I_m \end{pmatrix}, \\ & \qquad m = 1, \dots, N-k+1, \; k = 2, \dots, N. \end{aligned} \tag{1.68}$$

Every matrix A_k is strongly regular and therefore admits the LDU factorization. Moreover from (1.68) it follows that every matrix Δ_k has the LDU factorization. Let $A_{k-1} = L_{k-1} D_{k-1} U_{k-1}$ and $\Delta_k = X_k D'_k Y_k$ be the corresponding LDU factorizations. Substituting this in (1.67) one obtains the LDU factorization $A = LDU$ with

$$L = \begin{pmatrix} L_{k-1} & 0 \\ C_k A_{k-1}^{-1} L_{k-1} & X_k \end{pmatrix}, \quad D = \begin{pmatrix} D_{k-1} & 0 \\ 0 & D'_k \end{pmatrix}, \quad U = \begin{pmatrix} U_{k-1} & U_{k-1} A_{k-1}^{-1} B_k \\ 0 & Y_k \end{pmatrix}.$$

Using (1.68) with $m = 1$ and the equalities $C_k(1,:) = c_k,\ B_k(:,1) = b_k$ we have

$$A_k = \begin{pmatrix} I & 0 \\ c_k A_{k-1}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{k-1} & 0 \\ 0 & \Delta_k(1,1) \end{pmatrix} \begin{pmatrix} I & A_{k-1}^{-1} b_k \\ 0 & I \end{pmatrix}.$$

Comparing with (1.65), we get $\Delta_k(1,1) = \gamma_k$. Now applying (1.66) to the matrix Δ_k we obtain

$$D(k,k) = D'_k(1,1) = \gamma_k, \quad L(k:N,k) = X_k(:,1) = \Delta_k(:,1)\gamma_k^{-1},$$
$$U(k,k:N) = Y_k(1,:) = \gamma_k^{-1}\Delta_k(1,:),$$

which completes the proof. □

If a square $N \times N$ matrix A is symmetric ($A_{ij} = A_{ji},\ i,j, = 1,\dots,N$) and positive definite (namely the scalar product $< Ax, x >> 0$ for any N-dimensional nonzero block vector x), then the matrix A admits the Cholesky decomposition, which is identical with the LDU decomposition in which $U = L^T$ and this $A = LDL^T$ is obtained much faster.

§1.7 A general inversion formula

Here we present a well-known inversion formula with a complete proof.

Theorem 1.21. *Let A be an $m \times m$ invertible matrix and B and C be $m \times n$ and $n \times m$ matrices.*

The matrix $A - BC$ is invertible if and only if the matrix $V = I_n - CA^{-1}B$ is invertible. Moreover, if this is the case the inversion formula

$$(A - BC)^{-1} = A^{-1} + A^{-1}BV^{-1}CA^{-1} \tag{1.69}$$

holds.

Proof. We start with the case $A = I_m$, i.e., with the proof that the matrix $I_m - BC$ is invertible if and only if the matrix $I_n - CB$ is invertible and moreover if this is the case the inversion formula

$$(I_m - BC)^{-1} = I + B(I_n - CB)^{-1}C \tag{1.70}$$

holds.

Applying the formulas (1.55) and (1.52) to the matrix $R = \begin{pmatrix} I_m & B \\ C & I_n \end{pmatrix}$ one has

$$\begin{pmatrix} I_m & B \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_m - BC & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_m & 0 \\ C & I_n \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ C & I_n \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & I_n - CB \end{pmatrix} \begin{pmatrix} I_m & B \\ 0 & I_n \end{pmatrix}. \tag{1.71}$$

Hence it follows that $I_m - BC$ is invertible if and only if $I_n - CB$ is invertible. Moreover, using the fact that

$$\begin{pmatrix} I_m & B \\ 0 & I_n \end{pmatrix}^{-1} = \begin{pmatrix} I_m & -B \\ 0 & I_n \end{pmatrix}, \quad \begin{pmatrix} I_m & 0 \\ C & I_n \end{pmatrix}^{-1} = \begin{pmatrix} I_m & 0 \\ -C & I_n \end{pmatrix}$$

one obtains

$$\begin{pmatrix} I_m - BC & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & -B \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_m & 0 \\ C & I_n \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & I_n - CB \end{pmatrix}\begin{pmatrix} I_m & B \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_m & 0 \\ -C & I_n \end{pmatrix}.$$

Furthermore, assuming that the matrix $I_m - BC$ (or the matrix $I_n - CB$) is invertible one obtains

$$\begin{pmatrix} (I_m - BC)^{-1} & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ C & I_n \end{pmatrix}\begin{pmatrix} I_m & -B \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & (I_n - CB)^{-1} \end{pmatrix}\begin{pmatrix} I_m & 0 \\ -C & I_n \end{pmatrix}\begin{pmatrix} I_m & B \\ 0 & I_n \end{pmatrix}.$$

Consequently,

$$(I_m - BC)^{-1} = \begin{pmatrix} I_m & 0 \end{pmatrix}\begin{pmatrix} I_m & -B \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & (I_n - CB)^{-1} \end{pmatrix}\begin{pmatrix} I_m & 0 \\ -C & I_n \end{pmatrix}\begin{pmatrix} I_m \\ 0 \end{pmatrix},$$

i.e.,

$$(I_m - BC)^{-1} = \begin{pmatrix} I_m & -B \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & (I_n - CB)^{-1} \end{pmatrix}\begin{pmatrix} I_m \\ -C \end{pmatrix} = I_m + B(I_n - CB)^{-1}C,$$

which completes the proof of (1.70).

Now consider the case of an arbitrary $m \times m$ invertible matrix A. One has $A - BC = A(I_m - (A^{-1}B)C)$. Therefore the matrix $A - BC$ is invertible if and only if the matrix $I_m - (A^{-1}B)C$ is. But as it was shown above, the matrix $I_m - (A^{-1}B)C$ is invertible if and only if the matrix $I_n - CA^{-1}B = V$ is and moreover, if this is the case, using the formula (1.70) one gets

$$(I_m - A^{-1}BC)^{-1} = I_m + A^{-1}BV^{-1}C,$$

which implies (1.69). □

Corollary 1.22. *Let B and C be $m \times n$ and $n \times m$ matrices. Then*

$$\det(I_m - BC) = \det(I_n - CB). \tag{1.72}$$

The proof follows directly from the formula (1.71).

§1.8 Inversion of matrices with diagonal plus separable representation

In this section it is shown that inverting a matrix in separable representation of order r amounts to inverting an $r \times r$ matrix.

Theorem 1.23. *Let A be an $N \times N$ block matrix with diagonal plus separable of order r representation and let the diagonal generators $d(i)$, $i = 1, \dots, N$ of sizes $m_i \times m_i$ be invertible. Consider the diagonal matrix $D = \operatorname{diag}(d(i))_{i=1}^N$ and the matrices $P = \operatorname{col}(p(i))_{i=1}^N$, $Q = \operatorname{row}(q(i))_{i=1}^N$ which are formed with the separable generators $p(i), q(i)$ $(i = 1, \dots, N)$ of sizes $m_i \times r$ and $r \times m_i$, respectively, such that $A = D + PQ$. Define the $r \times r$ matrix $V = I_r + QD^{-1}P$. Then obviously*

$$V = I_r + \sum_{k=1}^{N} q(k)(d(k))^{-1}p(k). \tag{1.73}$$

The matrix $A = D + PQ$ is invertible if and only if the $r \times r$ matrix $V = I_r + QD^{-1}P$ is invertible. Moreover, if this is the case, then entries of the inverse matrix A^{-1} are given by the formulas

$$A^{-1}(i,j) = (d(i))^{-1}\delta_{ij} - (d(i))^{-1}p(i)V^{-1}q(j)(d(j))^{-1}, \quad i,j = 1, \dots, N. \tag{1.74}$$

Proof. Applying Theorem 1.21 we conclude that the matrix A is invertible if and only if the matrix V is and if this is the case the inversion formula

$$A^{-1} = D^{-1} - D^{-1}PV^{-1}QD^{-1}$$

holds, i.e.,

$$A^{\times} = D^{\times} + P^{\times}Q^{\times},$$

with $D^{\times} = D^{-1}$, $P^{\times} = -D^{-1}P$, $Q^{\times} = V^{-1}QD^{-1}$. Here $D^{\times}$ is a block diagonal matrix and $P^{\times}, Q^{\times}$ are matrices of sizes $\left(\sum_{i=1}^N m_i\right) \times r$, $r \times \left(\sum_{i=1}^N m_i\right)$, respectively. Hence we have obtained a diagonal plus separable of order r representation of the matrix A^{-1}. Moreover, we have $D^{\times} = \operatorname{diag}(d^{\times}(i))_{i=1}^N$ with $d^{\times}(i) = (d(i))^{-1}$, $i = 1, \dots, N$, $P^{\times} = \operatorname{col}(p^{\times}(i))_{i=1}^N$ with $p^{\times}(i) = -(d(i))^{-1}p(i)$, $i = 1, \dots, N$ and $Q^{\times} = \operatorname{row}(q^{\times}(i))_{i=1}^N$ with $q^{\times}(i) = V^{-1}q(i)(d(i))^{-1}$, $i = 1, \dots, N$. This means that the matrices $d^{\times}(i)$ $(i = 1, \dots, N)$ and $p^{\times}(i), q^{\times}(i)$ $(i = 1, \dots, N)$ are the diagonal and separable generators of the matrix A^{-1}. Hence the formula (1.74) follows.

Another proof of the theorem can be obtained using the system (1.32)

$$\begin{cases} \chi_{k+1} = \chi_k + q(k)x(k), \quad k = 1, \dots, N \\ y(k) = p(k)\chi_{N+1} + d(k)x(k), \quad k = 1, \dots, N, \\ \chi_1 = 0. \end{cases}$$

Since the blocks $d(k),\ k=1,\dots,N$ are invertible one obtains from the second equation that

$$x(k)=d^{\times}(k)y(k)-p^{\times}(k)\chi_{N+1},\quad k=1,\dots,N.$$

Substituting this in the first equation we get

$$\chi_{k+1}=\chi_k+q(k)(d(k))^{-1}y(k)-q(k)(d(k))^{-1}p(k)\chi_{N+1}$$

for $k=1,\dots,N$. Hence, using also that $\chi_1=0$ it follows that

$$\chi_{N+1}=\sum_{k=1}^{N}q(k)(d(k))^{-1}y(k)-\left(\sum_{k=1}^{N}q(k)(d(k))^{-1}p(k)\right)\chi_{N+1},$$

which means that

$$V\chi_{N+1}=\sum_{k=1}^{N}q(k)(d(k))^{-1}y(k), \tag{1.75}$$

with V given by (1.73). From (1.75) and the invertibility of V we obtain

$$\chi_{N+1}=\sum_{k=1}^{N}V^{-1}q^{\times}(k)y(k).$$

Define the state space variables $\chi_k^{\times},\ k=1,\dots,N+1$ via

$$\chi_1^{\times}=0,\quad \chi_{k+1}^{\times}=\chi_k^{\times}+q^{\times}(k)y(k),\ k=1,\dots,N.$$

It follows that $\chi_{N+1}^{\times}=\chi_{N+1}$. Thus we obtain the system

$$\begin{cases}\chi_{k+1}^{\times}=\chi_k^{\times}+q^{\times}(k)x(k), & k=1,\dots,N\\ x(k)=p^{\times}(k)\chi_{N+1}^{\times}+d^{\times}(k)x(k), & k=1,\dots,N,\\ \chi_1^{\times}=0.\end{cases} \tag{1.76}$$

By Theorem 1.11 we obtain that A^{-1} has the diagonal generators $d^{\times}(i),\ i=1,\dots,N$ and the separable generators $p^{\times}(i),\ q^{\times}(i),\ i=1,\dots,N$. □

Example 1.24. Consider the $N\times N$ matrix

$$A=\begin{pmatrix}2 & 2 & 3 & \cdots & N-1 & N\\ 2 & 8 & 6 & \cdots & 2(N-1) & 2N\\ 3 & 6 & 18 & \cdots & 3(N-1) & 3N\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ N-1 & 2(N-1) & 3(N-1) & \cdots & 2(N-1)^2 & (N-1)N\\ N & 2N & 3N & \cdots & (N-1)N & 2N^2\end{pmatrix}$$

which resembles the matrix from Example 1.4, having the same separable generators $p(i)=q(i)=i,\ i=1,\dots,N$, but it also has non zero diagonal generators

$$d(i)=i^2,\ i=1,\dots,N.$$

If we define the diagonal matrix $D = \operatorname{diag}(d(i))_{i=1}^N$ and the matrices

$$P = \operatorname{col}(p(i))_{i=1}^N \quad \text{and} \quad Q = \operatorname{row}(q(i))_{i=1}^N$$

which are formed with the separable generators, then $A = D + PQ$.

Theorem 1.23 above can be used to invert the matrix A as follows.

Define the $r \times r$ matrix $V = I_r + QD^{-1}P$, which for $r = 1$ is a scalar. One has, by (1.73),

$$V = 1 + \sum_{k=1}^{N} q(k)(d(k))^{-1}p(k) = 1 + \sum_{k=1}^{N} k \cdot \frac{1}{k^2} \cdot k = 1 + N.$$

The matrix $A = D + PQ$ is therefore invertible for any positive integer N, since $V = 1 + N \neq 0$. Moreover, the entries of the inverse matrix A^{-1} are given by the formulas (1.74). Therefore, for any $i, j = 1, \dots, N$ the corresponding entry of A^{-1} is

$$\begin{aligned} A^{-1}(i,j) &= (d(i))^{-1}\delta_{ij} - (d(i))^{-1}p(i)V^{-1}q(j)(d(j))^{-1} \\ &= \frac{1}{i^2} \cdot \delta_{ij} - \frac{1}{i^2} \cdot i \cdot \frac{1}{N+1} \cdot j \cdot \frac{1}{j^2} \\ &= \frac{1}{i^2} \cdot \delta_{ij} - \frac{1}{ij(N+1)} = -\frac{1}{N+1} \cdot \frac{(-N)^{\delta_{ij}}}{ij}. \end{aligned}$$

It follows that

$$A^{-1} = \frac{1}{N+1} \cdot \begin{pmatrix} N & -\frac{1}{2} & -\frac{1}{3} & \cdots & -\frac{1}{N-1} & -\frac{1}{N} \\ -\frac{1}{2} & \frac{1}{4}N & -\frac{1}{6} & \cdots & -\frac{1}{2(N-1)} & -\frac{1}{2N} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{1}{9}N & \cdots & -\frac{1}{3(N-1)} & -\frac{1}{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{N-1} & -\frac{1}{2(N-1)} & -\frac{1}{3(N-1)} & \cdots & \frac{1}{(N-1)^2}N & -\frac{1}{(N-1)N} \\ -\frac{1}{N} & -\frac{1}{2N} & -\frac{1}{3N} & \cdots & -\frac{1}{(N-1)N} & \frac{1}{N} \end{pmatrix}$$

as one can also check directly . ◊

Based on the formulas (1.73), (1.74) one obtains the following fast algorithm for the solution of the system of linear algebraic equations $Ax = y$ with x and y partitioned in the form $x = \operatorname{col}(x(i))_{i=1}^N$, $y = \operatorname{col}(y(i))_{i=1}^N$, where $x(i)$ and $y(i)$ are m_i-dimensional columns.

Algorithm 1.25.

1) Compute

$$(d(k))^{-1}, \quad k = 1, \dots, N$$

2) Compute the $r \times r$ matrix

$$V = I_r + \sum_{k=1}^{N} (q(k))(d(k))^{-1} p(k).$$

3) Compute the r-dimensional column

$$z = \sum_{j=1}^{N} q(j)(d(j))^{-1} y(j).$$

4) Solve the $r \times r$ linear system

$$Vw = z.$$

5) Compute the vector $x = (x(i))_{i=1}^{N}$:

$$x(i) = (d(i))^{-1}(y(i) + p(i)w).$$

The complexity of the arithmetic operations used in Algorithm 1.25 is calculated as follows.

1. Step 1: $\sum_{k=1}^{N} \rho(m_k)$ arithmetic operations of addition or multiplication.
2. Step 2: consists in the addition of an $r \times r$ matrix to a sum of two matrix multiplications which comprise each rm_k^2 multiplications and $r(m_k - 1)m_k$ additions, thus this step costs less than $r^2 + \sum_{k=1}^{N} 4rm_k^2$ arithmetic operations.
3. Step 3: is a sum of matrix matrix multiplications which comprise each rm_k^2 multiplications and $r(m_k - 1)m_k$ additions and matrix vector multiplications which comprise each rm_k multiplications and $r(m_k - 1)$ additions, thus this step costs less than $\sum_{k=1}^{N} 2rm_k(m_k + 1)$ arithmetic operations.
4. Step 4: $\rho(r)$ arithmetic operations.
5. Step 5: inside the brackets m_i additions and a matrix vector multiplication which costs $m_i r$ multiplications and $m_i(r - 1)$ additions, plus m_i^2 multiplications and $m_i(m_i - 1)$ additions outside the brackets, thus in total less than $\sum_{k=1}^{N} 2m_k(r + m_k)$ arithmetic operations.

Here $\rho(m)$ is the complexity of the solution of an $m \times m$ system of linear algebraic equations using a standard method.

Thus the total complexity of the algorithm is

$$c = r^2 + \rho(r) + \sum_{k=1}^{N} ((3rm_k r + 2r + m_k)2m_k + \rho(m_k))$$

arithmetic operations. Setting $m = \max_{1 \le k \le N}(m_k)$ one obtains the estimate

$$c \le (\rho(m) + m(3mr + 2r + m))N + r^2 + \rho(r).$$

§1.9 LDU factorization of matrices with diagonal plus separable representation

Let A be an $N \times N$ block matrix with block entries of sizes $m_i \times m_j$. By Theorem 1.20 if A is a strongly regular matrix, then it admits the LDU factorization

$$A = LDU, \tag{1.77}$$

where L, U, D are block matrices with the same sizes of blocks as A, and L and U are block lower and upper triangular matrices with identities on the main diagonals, while D is a block diagonal matrix. Next we derive a specification of Theorem 1.20 for matrices with diagonal plus separable representations.

Theorem 1.26. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be an $N \times N$ block matrix with block entries of sizes $m_i \times m_j$ with invertible principal leading block submatrices $A_k = \{A_{ij}\}_{i,j=1}^k$, $k = 1, \dots, N$. Assume that for A it is given a diagonal plus separable representation of order r, i.e., A has the diagonal generators $d_0(i)$, $i = 1, \dots, N$ of sizes $m_i \times m_i$ and the separable generators $p(i), q(i)$, $i = 1, \dots, N$ of sizes $m_i \times r$ and $r \times m_i$ respectively. In matrix form, $A = D_0 + PQ$ with $D_0 = \operatorname{diag}(d_0(i))_{i=1}^N$, $P = \operatorname{col}(p(i))_{i=1}^N$ and $Q = \operatorname{row}(q(i))_{i=1}^N$.*

Then in the factorization (1.77) the matrix L is the identity matrix plus the strictly lower triangular part of a matrix PQ^L in separable form of order r, namely L has the lower separable generators $p(i)$, $i = 2, \dots, N$, which are the same as for the matrix A, and $q^L(i)$, $i = 1, \dots, N-1$ of size $r \times m_i$, while the matrix U is the identity matrix plus the strictly upper triangular part of a matrix $P^U Q$ in separable form of order r, namely U has the upper separable generators $q(i)$, $i = 2, \dots, N$, which are the same as for the matrix A, and $p^U(i)$, $i = 1, \dots, N-1$ of size $m_i \times r$. Finally the matrix D has the form

$$D = \operatorname{diag}\{\gamma_1, \dots, \gamma_N\}$$

with blocks γ_i, $i = 1, \dots, N$ of sizes $m_i \times m_i$. Here $P^U = \operatorname{col}(p^U(i))_{i=1}^N$ and $Q^L = \operatorname{row}(q^L(i))_{i=1}^N$. These ingredients are obtained as follows:

1. *Compute*

$$\gamma_1 = d_0(1) + p(1)q(1), \quad q^L(1) = q(1)\gamma_1^{-1}, \quad p^U(1) = \gamma_1^{-1}p(1), \tag{1.78}$$

$$\alpha_1 = I_r - q^L(1)\gamma_1 p^U(1). \tag{1.79}$$

2. *For $k = 2, \dots, N-1$, compute*

$$\beta_k = \alpha_{k-1}q(k), \tag{1.80}$$

$$\gamma_k = d_0(k) + p(k)\beta_k, \tag{1.81}$$

$$q^L(k) = \beta_k \gamma_k^{-1}, \tag{1.82}$$

$$p^U(k) = \gamma_k^{-1}p(k)\alpha_{k-1}, \tag{1.83}$$

$$\alpha_k = \alpha_{k-1} - q^L(k)\gamma_k p^U(k). \tag{1.84}$$

3. *Compute*

$$\gamma_N = d_0(N) + p(N)\alpha_{N-1}q(N). \tag{1.85}$$

Here α_k $(k = 1, \dots, N-1)$, β_k $(k = 2, \dots, N-1)$ *are auxiliary variables which are* $r \times r$ *and respectively* $r \times m_k$ *matrices.*

Proof. One must check that

$$D = \operatorname{diag}\{\gamma_1, \dots, \gamma_N\}$$

and moreover that the matrices L, U satisfy the relations

$$L(k+1:N, k) = P_{k+1}q^L(k), \quad k = 1, \dots, N-1, \tag{1.86}$$

$$U(k, k+1:N) = p^U(k)H_{k+1}, \quad k = 1, \dots, N-1, \tag{1.87}$$

with the matrices P_k, H_k defined by

$$P_k = \operatorname{col}(p(i))_{i=k}^N, \quad H_k = \operatorname{row}(q(i))_{i=k}^N, \quad k = 1, \dots, N$$

and the elements $\gamma_k, q^L(k), p^U(k)$ determined in the statement of the theorem.

Denote also

$$G_k = \operatorname{col}(p(i))_{i=1}^k, \quad Q_k = \operatorname{row}(q(i))_{i=1}^k, \quad k = 1, \dots, N$$

and introduce the matrices

$$\alpha_k = I_r - Q_k A_k^{-1} G_k, \quad k = 1, \dots, N-1. \tag{1.88}$$

In the formulas (1.58), (1.59) one gets

$$B_k = G_{k-1}H_k, \quad C_k = P_k Q_{k-1}, \quad b_k = G_{k-1}h(k), \quad c_k = p(k)Q_{k-1}.$$

Using the formulas (1.60) one gets

$$\gamma_1 = d_0(1) + p(1)q(1), \quad \gamma_k = d_0(k) + p(k)\alpha_{k-1}q(k), \quad k = 2, \dots, N, \tag{1.89}$$

which means that the formulas (1.78), (1.81), (1.85) for γ_k $(k = 1, \dots, N)$ hold, and moreover using (1.61)–(1.63) we obtain

$$L(k:N, k) = \Delta_k(:, 1)\gamma_k^{-1}, \quad k = 1, \dots, N, \tag{1.90}$$

$$U(k, k:N) = \gamma_k^{-1}\Delta_k(1, :), \quad k = 1, \dots, N, \tag{1.91}$$

where

$$\Delta_1 = A, \quad \Delta_k = M_k + P_k\alpha_{k-1}H_k - P_kH_k, \quad k = 2, \dots, N. \tag{1.92}$$

Now we will prove the relations (1.86), (1.87). For $k = 1$ one has $\gamma_1 = d_0(1) + p(1)q(1)$ and one gets

$$\Delta_1(2:N,1) = A(2:N,1) = P_2 q(1),$$
$$\Delta_1(1,2:N) = A(1,2:N) = p(1)H_2$$

and hence using (1.90), (1.91) one obtains

$$L(2:N,1) = P_2 q^L(1), \quad U(1,2:N) = p^U(1)H_2$$

with the elements $q^L(1), p^U(1)$ defined in (1.78).

For $k > 1$ one has the following. One obtains the representations

$$M_k(:,1) = A(k:N,k) = \begin{pmatrix} d_0(k) + p(k)q(k) \\ P_{k+1}q(k) \end{pmatrix}, \quad k = 2,\dots,N-1 \tag{1.93}$$

and

$$M_k(1,:) = A(k,k:N) = \begin{pmatrix} d_0(k) + p(k)q(k) & p(k)H_{k+1} \end{pmatrix}, \quad k = 2,\dots,N-1. \tag{1.94}$$

Taking the first columns in (1.92) and using (1.93) one obtains

$$\Delta_k(:,1) = \begin{pmatrix} \gamma_k \\ P_{k+1}\alpha_{k-1}q(k) \end{pmatrix}.$$

Similarly, taking the first rows in (1.92) and using (1.94) one obtains

$$\Delta_k(1,:) = \begin{pmatrix} \gamma_k & p(k)\alpha_{k-1}H_{k+1} \end{pmatrix}.$$

Thus one obtains the relations

$$\Delta_k(:,1) = \begin{pmatrix} \gamma_k \\ P_{k+1}\alpha_{k-1}q(k) \end{pmatrix} = \begin{pmatrix} \gamma_k \\ P_{k+1}q'(k) \end{pmatrix}, \quad k = 2,\dots,N$$

and

$$\Delta_k(1,:) = \begin{pmatrix} \gamma_k & p(k)\alpha_{k-1}H_{k+1} \end{pmatrix} = \begin{pmatrix} \gamma_k & p'(k)H_{k+1} \end{pmatrix}, \quad k = 2,\dots,N,$$

with the elements γ_k from (1.81) and

$$q'(k) = \alpha_{k-1}q(k), \quad p'(k) = p(k)\alpha_{k-1}.$$

Furthermore, using (1.90), (1.91) and (1.80), (1.82), (1.83) one obtains (1.86) and (1.87).

It remains to prove the relations (1.79), (1.84). The equality (1.79) follows directly from the definition (1.88) and the relations

$$Q_1 = q(1), \quad G_1 = p(1), \quad \gamma_1 = d_0(1) + p(1)q(1).$$

For $k > 1$, applying the factorization (1.65) one obtains

$$A_k = \begin{pmatrix} I & 0 \\ p(k)Q_{k-1}A_{k-1}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{k-1} & 0 \\ 0 & \gamma_k \end{pmatrix} \begin{pmatrix} I & A_{k-1}^{-1}G_{k-1}q(k) \\ 0 & I \end{pmatrix},$$

which implies

$$A_k^{-1} = \begin{pmatrix} I & -A_{k-1}^{-1}G_{k-1}q(k) \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{k-1}^{-1} & 0 \\ 0 & \gamma_k^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -p(k)Q_{k-1}A_{k-1}^{-1} & I \end{pmatrix}. \tag{1.95}$$

It follows that

$$\begin{aligned} Q_k \begin{pmatrix} I & -A_{k-1}^{-1}G_{k-1}q(k) \\ 0 & I \end{pmatrix} &= \begin{pmatrix} Q_{k-1} & q(k) \end{pmatrix} \begin{pmatrix} I & -A_{k-1}^{-1}G_{k-1}q(k) \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} Q_{k-1} & \alpha_{k-1}q(k) \end{pmatrix} = \begin{pmatrix} Q_{k-1} & q'(k) \end{pmatrix} \end{aligned} \tag{1.96}$$

and

$$\begin{aligned} \begin{pmatrix} I & 0 \\ -p(k)Q_{k-1}A_{k-1}^{-1} & I \end{pmatrix} G_k &= \begin{pmatrix} I & 0 \\ -p(k)Q_{k-1}A_{k-1}^{-1} & I \end{pmatrix} \begin{pmatrix} G_{k-1} \\ p(k) \end{pmatrix} \\ &= \begin{pmatrix} G_{k-1} \\ p(k)\alpha_{k-1} \end{pmatrix} = \begin{pmatrix} G_{k-1} \\ p'(k) \end{pmatrix}. \end{aligned} \tag{1.97}$$

Now from the definition (1.88) and the relations (1.95)–(1.97) one gets

$$\alpha_k = \alpha_{k-1} - q'(k)\gamma_k^{-1}p'(k),$$

which completes the proof. □

In order to compute the complexity of the algorithm in Theorem 1.26, denote by $\rho(m_k)$ the complexity of inverting an $m_k \times m_k$ matrix and put $m = \max_{1 \le k \le N}(m_k)$.

The first computation in (1.78) is the multiplication of an $m_1 \times r$ matrix by an $r \times m_1$ matrix, which comprises rm_1^2 multiplications and $(r-1)m_1^2$ additions, therefore less than $2rm_1^2$ arithmetic operations. It then adds two $m_1 \times m_1$ matrices. The second computation in (1.78) costs less than $\rho(m_1) + 2rm_1^2$ arithmetic operations, while the last computation in (1.78) adds another $2rm_1^2$ arithmetic operations. Computing (1.79) involves less than $r^2 + 2rm_1^2 + 2rm_1^2$ arithmetic operations.

For each $k = 2, \dots, N-1$ Step 2 of the algorithm in the theorem requires less than $2r^2m_k$ arithmetic operations for (1.80), less than $m_k^2 + 2m_k^2r$ for (1.81), less than $\rho(m_k) + 2m_k^2r$ for (1.82), while (1.83) adds less than $2m_k^2r + 2m_kr^2$ arithmetic operations. Finally, (1.84) is analogous to (1.79), namely it costs less than $r^2 + 4rm_k^2$ arithmetic operations.

In Step 3 of the algorithm the computation of (1.85) is analogous to (1.80) and (1.81), namely it costs less than $m_N^2 + 2rm_N^2 + 2r^2m_N$ arithmetic operations.

The total complexity of the algorithm in Theorem 1.26 is

$$c < \sum_{k=1}^{N}(m_k(m_k + 6r^2 + 10rm_k) + r^2 + \rho(m_k)),$$

therefore

$$c < (m(m + 6r^2 + 10rm) + r^2 + \rho(m))N.$$

It is easy to compute that for scalar matrices the algorithm costs less than $(5r^2 + 5r + 3)N$ arithmetic operations.

Example 1.27. Consider the 3×3 matrix A from Example 1.6,

$$A = 2I + PQ = 2I + \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 2 & 5 & 1 \end{pmatrix}.$$

Let us compute for A the LDU factorization using Theorem 1.26.

Compute in Step 1, by (1.78),

$$\gamma_1 = d_0(1) + p(1)q(1) = A_{11} = 2 + \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1,$$

$$q^L(1) = q(1)\gamma_1^{-1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad p^U(1) = \gamma_1^{-1}p(1) = \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

By (1.79),

$$\alpha_1 = I_r - q^L(1)\gamma_1 p^U(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

Step 2. For $k = 2$ compute, using (1.80),

$$\beta_k = \beta_2 = \alpha_{2-1}q(2) = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix},$$

compute using (1.81)

$$\gamma_2 = d_0(2) + p(2)\beta_2 = 2 + \begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 9 \end{pmatrix} = 30,$$

compute using (1.82)

$$q^L(2) = \beta_2\gamma_2^{-1} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} \frac{1}{30} = \begin{pmatrix} \frac{1}{6} \\ \frac{3}{10} \end{pmatrix},$$

compute using (1.83)

$$p^U(2) = \gamma_2^{-1}p(2)\alpha_{2-1} = \frac{1}{30} \begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{15} & \frac{4}{15} \end{pmatrix}$$

and by (1.84) compute

$$\alpha_2 = \alpha_{2-1} - q^L(2)\gamma_2 p^U(2)$$
$$= \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{6} & \frac{3}{10} \end{pmatrix} \cdot 30 \cdot \begin{pmatrix} -\frac{2}{15} & \frac{4}{15} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

Finally, in Step 3 compute by (1.85)

$$\gamma_N = \gamma_3 = d_0(3) + p(3)\alpha_{3-1}q(3) = 2 + \begin{pmatrix} 5 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{128}{15}.$$

As a check that the generators obtained above give the LDU decomposition of A, the following L, D, U matrices are computed using these generators:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ p(2)q^L(1) & 1 & 0 \\ p(3)q^L(1) & p(3)q^L(2) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 7 & \frac{17}{15} & 1 \end{pmatrix},$$
$$D = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & \frac{128}{15} \end{pmatrix},$$
$$U = \begin{pmatrix} 1 & p^U(1)q(2) & p^U(1)q(3) \\ 0 & 1 & p^U(2)q(3) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -\frac{2}{15} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then a direct multiplication gives

$$DU = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 30 & -4 \\ 0 & 0 & \frac{128}{15} \end{pmatrix},$$

and so $L(DU) = A$. ◊

Example 1.28. Consider the $N \times N$ matrix A from Example 1.24 with the same separable generators $p(i) = q(i) = i,\ i = 1, \dots, N$ and diagonal generators

$$d_0(i) = i^2,\ i = 1, \dots, N.$$

Theorem 1.26 above can be used to compute separable generators for the matrices in the LDU decomposition of A.

1. Compute using (1.78)

$$\gamma_1 = d_0(1) + p(1)q(1) = 1^2 + 1 \cdot 1 = 2, \quad q^L(1) = q(1)\gamma_1^{-1} = \frac{1}{2},$$
$$p^U(1) = \gamma_1^{-1}p(1) = \frac{1}{2}$$

and using (1.79)

$$\alpha_1 = I_r - q^L(1)\gamma_1 p^U(1) = 1 - \frac{1}{2}\cdot 2\cdot\frac{1}{2} = \frac{1}{2}.$$

2. One can prove by induction that for any $1 \le k \le N-1$,

$$\alpha_k = \frac{1}{k+1}, \quad \gamma_k = k(k+1), \quad q^L(k) = p^U(k) = \frac{1}{k(k+1)}. \tag{1.98}$$

Indeed, this has been proved for $k=1$. Suppose that it is true for a certain $k \ge 1$ and let us verify (1.98) for $k+1$. First, by (1.80),

$$\beta_{k+1} = \alpha_k q(k+1) = \frac{1}{k+1}\cdot(k+1) = 1.$$

Then, using (1.81)

$$\gamma_{k+1} = d_0(k+1) + p(k+1)\beta_{k+1} = (k+1)^2 + (k+1)\cdot 1 = (k+1)(k+2),$$

using (1.82)

$$q^L(k+1) = \beta_{k+1}\gamma_{k+1}^{-1} = \frac{1}{(k+1)(k+2)},$$

and using (1.83)

$$\begin{aligned} p^U(k+1) = \gamma_{k+1}^{-1}p(k+1)\alpha_{k+1-1} &= \frac{1}{(k+1)(k+2)}\cdot(k+1)\cdot\frac{1}{k+1} \\ &= \frac{1}{(k+1)(k+2)}. \end{aligned}$$

Finally, by (1.84)

$$\begin{aligned} \alpha_{k+1} &= \alpha_k - q^L(k+1)\gamma_{k+1}p^U(k+1) \\ &= \frac{1}{k+1} - \frac{1}{(k+1)(k+2)}\cdot(k+1)(k+2)\cdot\frac{1}{(k+1)(k+2)} = \frac{1}{k+2}, \end{aligned}$$

which completes the induction.

3. Compute using (1.85)

$$\gamma_N = d_0(N) + p(N)\alpha_{N-1}q(N) = N^2 + N\cdot\frac{1}{N}\cdot N = N^2 + N = N(N+1).$$

Therefore the generators of the lower triangular matrix L are

$$p(i) = i, \quad q^L(j) = \frac{1}{j(j+1)}, \quad i = 2,\dots,N, \quad j = 1,\dots,N-1,$$

the generators of the upper triangular matrix U are

$$p^U(i) = \frac{1}{i(i+1)}, \quad q(j) = j, \quad i = 1, \dots, N-1, \quad j = 2, \dots, N,$$

and the entries of the diagonal matrix D are

$$\gamma_i = i(i+1), \quad i = 1, \dots, N. \qquad \diamondsuit$$

One can obtain LDU factorizations also for matrices in separable form which are not strongly regular. In this case the matrix D is not invertible.

Example 1.29. Consider the $N \times N$ block matrix A in separable of order r form with the separable generators $p(i), q(i), i = 1, \dots, N$ of sizes $m_i \times r$ and $r \times m_i$, respectively, and suppose that

$$\operatorname{rank} A = \operatorname{rank} A_{11} = r = m_1.$$

It follows that

$$A = \begin{pmatrix} p(1)q(1) & p(1)q(2) & \cdots & p(1)q(N) \\ p(2)q(1) & p(2)q(2) & \cdots & p(2)q(N) \\ \vdots & \vdots & \ddots & \vdots \\ p(N)q(1) & p(N)q(2) & \cdots & p(N)q(N) \end{pmatrix}$$

and that $p(1)$ and $q(1)$ are invertible matrices of sizes $r \times r$.

Applying the formula (1.52) with

$$A_{11} = p(1)q(1), \quad A_{21} = \begin{pmatrix} p(2) \\ \vdots \\ p(N) \end{pmatrix} q(1), \quad A_{12} = p(1) \begin{pmatrix} q(2) & \dots & q(N) \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} p(2)q(2) & \cdots & p(2)q(N) \\ \vdots & \ddots & \vdots \\ p(N)q(2) & \cdots & p(N)q(N) \end{pmatrix}$$

we get $\Gamma = 0$ and therefore we obtain the factorization $A = LDU$ with

$$L = \begin{pmatrix} I_{m_1} & 0 & 0 & \cdots & 0 & 0 \\ p(2)(p(1))^{-1} & I_{m_2} & 0 & \cdots & 0 & 0 \\ p(3)(p(1))^{-1} & 0 & I_{m_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p(N-1)(p(1))^{-1} & 0 & 0 & \cdots & I_{m_{N-1}} & 0 \\ p(N)(p(1))^{-1} & 0 & 0 & \cdots & 0 & I_{m_N} \end{pmatrix},$$

$$U = \begin{pmatrix} I_{m_1} & (q(1))^{-1}q(2) & (q(1))^{-1}q(3) & \cdots & (q(1))^{-1}q(N-1) & (q(1))^{-1}q(N) \\ 0 & I_{m_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{m_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m_{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{m_N} \end{pmatrix},$$

and

$$D = \mathrm{diag}\{p(1)q(1), 0, 0, \dots, 0\}.$$

§1.10 Solution of linear systems in the presence of the *LDU* factorization of the matrix of the system in diagonal plus separable form

Suppose that the conditions in the statement of Theorem 1.26 are fulfilled. Then solving the system $Ax = y$ with $A = LDU$, i.e., $LDUx = y$ amounts to solving linear systems with two triangular matrices and a diagonal matrix in three steps: solve $Lz = y$, solve $Dw = z$ and solve $Ux = w$.

1. In the first step one has to solve the equation $Lz = y$ with a lower $N \times N$ triangular block matrix $L = (L_{ij})_{i,j=1}^N$ having only identities on the main diagonal, i.e.,

$$\sum_{j=1}^{i-1} L_{ij} z(j) + z(i) = y(i), \quad i = 1, \dots, N. \tag{1.99}$$

In the presence of generators of order r for the matrix L, namely $p(i)$, $i = 2, \dots, N$ of size $m_i \times r$ which are the same as for the matrix A and $q^L(i)$, $i = 1, \dots, N-1$ of size $r \times m_i$, one has

$$L_{ij} = p(i)q^L(j), \quad 1 \le j \le i \le N$$

hence the equation (1.99) becomes

$$p(i) \sum_{j=1}^{i-1} q^L(j)z(j) + z(i) = y(i), \quad i = 1, \dots, N. \tag{1.100}$$

Denote

$$\chi_i = \sum_{j=1}^{i-1} q^L(j)z(j), \quad i = 1, \dots, N.$$

Then the variables χ_i satisfy the recurrence relation

$$\chi_{i+1} = \chi_i + q^L(i)z(i), \quad i = 1, \dots, N-1 \tag{1.101}$$

and the initial condition

$$\chi_1 = 0. \tag{1.102}$$

Equation (1.100) becomes

$$p(i)\chi_i + z(i) = y(i), \quad i = 1, \dots, N,$$

therefore together with (1.102), which gives $z(1)$, one obtains

$$z(1) = y(1), \quad z(i) = y(i) - p(i)\chi_i, \quad i = 2, \dots, N. \tag{1.103}$$

Adding the relations (1.102), (1.103) and (1.101), we get

$$z(1) = y(1), \quad \chi_1 = 0$$

and

$$\chi_i = \chi_{i-1} + q^L(i-1)z(i-1), \quad z(i) = y(i) - p(i)\chi_i, \quad i = 2, \dots, N.$$

2. In the second step one has to solve the equation $Dw = z$, where D is a diagonal matrix with invertible $m_i \times m_i$ blocks γ_i, $i = 1, \dots, N$, on the main diagonal. It follows that

$$\gamma_i w(i) = z(i), \qquad i = 1, \dots, N,$$

hence

$$w(i) = \gamma_i^{-1} z(i), \quad i = 1, \dots, N.$$

3. In the third step one has to solve the equation $Ux = w$ with an upper triangular $N \times N$ block matrix $U = (U_{ij})_{i,j=1}^N$ having only identities on the main diagonal. This amounts to

$$\sum_{j=i+1}^{N} U_{ij} x(j) + x(i) = w(i), \quad i = 1, \dots, N. \tag{1.104}$$

When generators of order r for the matrix U are known, namely $p^U(i)$, $i = 1, \dots, N-1$ of size $m_i \times r$ and $q(i)$, $i = 2, \dots, N$ of size $r \times m_i$ which are the same as for the matrix A, one has

$$U_{ij} = p^U(i) q(j), \quad 1 \le i \le j \le N,$$

hence the equation (1.104) becomes

$$p^U(i) \sum_{j=i+1}^{N} q(j) x(j) + x(i) = w(i), \quad i = 1, \dots, N. \tag{1.105}$$

Denote

$$\eta_i = \sum_{j=i+1}^{N} q(j)x(j), \quad i = 1, \dots, N.$$

Then the variables η_i satisfy the recurrence relation

$$\eta_{i-1} = \eta_i + q(i)x(i), \quad i = 2, \dots, N \tag{1.106}$$

and the initial condition

$$\eta_N = 0. \tag{1.107}$$

Equation (1.105) becomes

$$p^U(i)\eta_i + x(i) = w(i), \quad i = 1, \dots, N,$$

therefore together with (1.107), which gives $x(N)$, one obtains

$$x(N) = w(N), \quad x(i) = w(i) - p^U(i)\eta_i, \quad i = N-1, \dots, 1. \tag{1.108}$$

Summing the relations (1.107), (1.108) and (1.106) it follows that

$$x(N) = w(N), \quad \eta_N = 0$$

and

$$\eta_i = \eta_{i+1} + q(i+1)x(i+1), \quad x(i) = w(i) - p^U(i)\eta_i, \quad i = N-1, \dots, 1.$$

The above considerations lead to the following algorithm. Note that the simpler Step 2 above can be performed inside Step 3 and together with it, and this is how we are going to proceed.

Algorithm 1.30. Solution of linear systems using the LDU factorization of a separable matrix

First compute with the LDU algorithm from Theorem 1.26 above the lower separable generators $q^L(j), \quad j = 1, \dots, N-1$, of the lower triangular matrix L, the upper separable generators $p^U(i), \quad i = 1, \dots, N-1$, of the upper triangular matrix U, and the entries of γ_i, $i = 1, \dots, N$, of the diagonal matrix D.

1 Start with

$$\chi_1 = 0, \quad z(1) = y(1)$$

and for $i = 2, \dots, N$ compute recursively

$$\chi_i = \chi_{i-1} + q^L(i-1)z(i-1) \tag{1.109}$$

and

$$z(i) = y(i) - p(i)\chi_i. \tag{1.110}$$

2 Start with

$$\eta_N = 0, \quad x(N) = \gamma_N^{-1} z(N)$$

and for $i = N-1, \dots, 1$ compute recursively

$$\eta_i = \eta_{i+1} + q(i+1)x(i+1) \tag{1.111}$$

and

$$x(i) = \gamma_i^{-1} z(i) - p^U(i)\eta_i. \tag{1.112}$$

The complexity of this algorithm is linear in N and therefore much lower than that of the usual solution of a linear system with an arbitrary matrix. Indeed, denote by $\upsilon(n)$ the complexity of inverting an $n \times n$ matrix.

The computation of (1.109) and of (1.111) for each iteration costs each r additions and a matrix vector multiplication which comprises rm_i multiplications and $r(m_i - 1)$ additions, thus in total $2rm_i$ arithmetic operations.

The computation of (1.110) costs m_i additions and a matrix vector multiplication which comprises $m_i r$ multiplications and $m_i(r-1)$ additions, thus in total $2m_i r$ arithmetic operations.

Finally, the computation of (1.112) costs m_i additions between two matrix vector products, the first of which comprises $\upsilon(m_i)$ arithmetic operations together with $(m_i)^2$ multiplications and $m_i(m_i - 1)$ additions, while the second product uses $m_i r$ multiplications and $m_i(r-1)$ additions, thus in total less than $2m_i(m_i + r)$ arithmetic operations.

It follows that the total complexity is

$$c < \sum_{i=1}^{N} 2m_i(4r + m_i),$$

so if we set $m = \max_{1 \le i \le N} m_i$, then

$$c < 2m(4r + m)N.$$

Example 1.31. Let $N > 3$ and consider the $N \times N$ matrix A from Example 1.24, with the same separable generators and diagonal generators. In the present example Algorithm 1.30 will be used to solve the linear system $Ax = y$, where

$$y = \begin{pmatrix} 5 & 14 & 24 & 20 & 25 & 30 & \cdots & 5N \end{pmatrix}^T,$$

with $y(1) = 5,\ y(2) = 14,\ y(3) = 24$ and $y(i) = 5i,\ i = 4, \dots, N$.

In Example 1.28 the separable generators and diagonal entries of the matrices in the LDU factorization of A are computed and it follows that the generators of the lower triangular matrix L are

$$p(i) = i, \quad q^L(j) = \frac{1}{j(j+1)}, \quad i = 2, \dots, N, \quad j = 1, \dots, N-1,$$

the generators of the upper triangular matrix U are

$$p^U(i) = \frac{1}{i(i+1)}, \quad q(j) = j, \quad i = 1, \dots, N-1, \quad j = 2, \dots, N$$

and the entries of the diagonal matrix D are

$$\gamma_i = i(i+1), \quad i = 1, \dots, N.$$

First perform Step 1 of the Algorithm 1.30. Set $\chi_1 = 0, z(1) = y(1) = 5$ and for $i = 2$ compute (1.109)

$$\chi_2 = \chi_i = \chi_{i-1} + q^L(i-1)z(i-1) = 0 + \frac{1}{2} \cdot 5 = \frac{5}{2},$$

and (1.110)

$$z(2) = z(i) = y(i) - p(i)\chi_i = 14 - 2 \cdot \frac{5}{2} = 9.$$

For $i = 3$ compute (1.109)

$$\chi_3 = \chi_i = \chi_2 + q^L(2)z(2) = \frac{5}{2} + \frac{1}{6} \cdot 9 = 4,$$

and (1.110)

$$z(3) = z(i) = y(3) - p(3)\chi_3 = 24 - 3 \cdot 4 = 12.$$

For $i = 4$ compute (1.109) and (1.110)

$$\chi_4 = \chi_3 + q^L(3)z(3) = 4 + \frac{1}{12} \cdot 12 = 5, \quad z(4) = y(4) - p(4)\chi_4 = 20 - 4 \cdot 5 = 0.$$

One can prove by induction that for $4 \leq i \leq N$ one has

$$\chi_i = 5, \ z(i) = 0. \tag{1.113}$$

Indeed, this is true for $i = 4$. Suppose that (1.113) is true for a certain i. Then for $i+1$ (1.109) becomes

$$\chi_{i+1} = \chi_i + q^L(i)z(i) = 5 + \frac{1}{i(i+1)} \cdot 0 = 5$$

again and (1.110) becomes

$$z(i+1) = y(i+1) - p(i+1)\chi_{i+1} = 5k - k \cdot 5 = 0,$$

which completes the induction. It follows that

$$z = \begin{pmatrix} 5 & 9 & 12 & 0 & \cdots & 0 \end{pmatrix}^T.$$

Next perform Step 2 of the Algorithm 1.30. Set $\eta_N = 0$, $x(N) = \gamma_N^{-1} z(N) = 0$ and for $i = N-1$ compute (1.111)

$$\eta_{N-1} = \eta_i = \eta_{i+1} + q(i+1)x(i+1) = 0 + N \cdot 0 = 0,$$

and (1.112)

$$x(N-1) = x(i) = \gamma_i^{-1} z(i) - p^U(i)\eta_i = \frac{1}{(N-1)N} \cdot 0 - \frac{1}{(N-1)N} \cdot 0 = 0.$$

Then compute the same formulas for $i = N-2, \ldots, 5, 4$. As long as $x(i+1) = 0$ and $\eta_{i+1} = 0$, formula (1.111) gives $\eta_i = 0$. And as long as $\eta_i = 0$ and also $z(i) = 0$, which is true for $i > 3$, formula (1.112) gives $x(i) = 0$ too.

It remains to compute $\eta_i, x(i)$ for $i = 3, 2, 1$. For $i = 3$, by (1.111) and (1.112), one has that

$$\eta_3 = \eta_4 + q(4)x(4) = 0 + 4 \cdot 0 = 0, \quad x(3) = \gamma_3^{-1} z(3) - p^U(3)\eta_3 = \frac{1}{3 \cdot 4} \cdot 12 - \frac{1}{3 \cdot 4} \cdot 0 = 1.$$

For $i = 2$, by (1.111) and (1.112), one has that

$$\eta_2 = \eta_3 + q(3)x(3) = 0 + 3 \cdot 1 = 3, \quad x(2) = \gamma_2^{-1} z(2) - p^U(2)\eta_2 = \frac{1}{2 \cdot 3} \cdot 9 - \frac{1}{2 \cdot 3} \cdot 3 = 1.$$

Finally, for $i = 1$, by (1.111) and (1.112), one has that

$$\eta_1 = \eta_2 + q(2)x(2) = 3 + 2 \cdot 1 = 5, \quad x(1) = \gamma_1^{-1} z(1) - p^U(1)\eta_1 = \frac{1}{2} \cdot 5 - \frac{1}{2} \cdot 5 = 0.$$

Therefore the solution of the system is

$$x = \begin{pmatrix} 0 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}^T. \qquad \diamondsuit$$

We also presented earlier Algorithm 1.25 for solving linear systems. The analysis of their complexity shows that the *LDU* algorithm above is more expensive then the other one. However, the stability of Algorithm 1.25 for large matrices is problematic.

§1.11 Comments

This chapter contains mostly well-known results, but in this form they appear here for the first time.

Chapter 2

The Minimal Rank Completion Problem

Here we study the problem of completion of a partially specified matrix with a given lower triangular part to a matrix with minimal rank. This chapter contains a formula for the rank of a minimal completion and an algorithm to build such a completion, first in the case of a 2×2 block matrix, which is then of help for the proof of the general case. For further purposes we also find a condition equivalent with the uniqueness of the minimal completion. Examples which analyze scalar and block matrices are given for all the procedures and for the use of the main formulas.

§2.1 The definition. The case of a 2×2 block matrix

Let

$$\mathcal{A} = \begin{pmatrix} A_{11} & ? & \dots & ? \\ A_{21} & A_{22} & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{pmatrix}$$

be a partially specified block matrix with elements of sizes $m_i \times n_j,\ i, j = 1, \dots, N$, with a given lower triangular part $\tilde{A} = \{A_{ij}\}_{1 \le j \le i \le N}$. The completion of $\mathcal{A}$ (or $\tilde{A}$) to a matrix with the smallest possible rank is called a *minimal rank completion* and its rank $\hat{r}_L$ is called the *minimal completion rank* of $\mathcal{A}$.

We start with the simplest case of a partially specified 2×2 block matrix.

Lemma 2.1. *Let*

$$\mathcal{A} = \begin{pmatrix} A_{11} & ? \\ A_{21} & A_{22} \end{pmatrix} \tag{2.1}$$

with the specified blocks A_{11}, A_{21}, A_{22} *of sizes* $m_1 \times n_1, m_2 \times n_1, m_2 \times n_2$*, respectively.*

Y. Eidelman et al., *Separable Type Representations of Matrices and Fast Algorithms: Volume 1 Basics. Completion Problems. Multiplication and Inversion Algorithms*, Operator Theory: Advances and Applications 234, DOI 10.1007/978-3-0348-0606-0_2, © Springer Basel 2014

The minimal completion rank of $\mathcal{A}$ is given by the formula

$$\hat{r}_L = \operatorname{rank}\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} - \operatorname{rank} A_{21}. \tag{2.2}$$

Moreover, a minimal rank completion A_L of a partially specified matrix $\mathcal{A}$ has the form $A_L = PQ$, where the matrices P, Q of sizes $(m_1+m_2)\times\hat{r}_L, \hat{r}_L\times(n_1+n_2)$, which satisfy the condition

$$\operatorname{rank} A_L = \operatorname{rank} P = \operatorname{rank} Q = \hat{r}_L, \tag{2.3}$$

can be obtained via the following algorithm.

Set

$$\rho_1 = \operatorname{rank}\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}, \quad \rho_2 = \operatorname{rank}\begin{pmatrix} A_{21} & A_{22} \end{pmatrix}, \quad s = \operatorname{rank} A_{21}. \tag{2.4}$$

Applying the rank factorization to the matrix $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$ determine the matrices P_1, q of sizes $(m_1+m_2)\times\rho_1, \rho_1\times n_1$, respectively, such that

$$\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = P_1\cdot q, \quad \operatorname{rank} P_1 = \operatorname{rank} q = \rho_1. \tag{2.5}$$

Determine the matrices p, P' of sizes $m_1\times\rho_1, m_2\times\rho_1$ from the partition

$$P_1 = \begin{pmatrix} p \\ P' \end{pmatrix}. \tag{2.6}$$

Applying the rank canonical factorization to the matrix $\begin{pmatrix} P' & A_{22} \end{pmatrix}$ of the size $m_2\times(\rho_1+n_2)$, determine the matrix $\hat{P}$ of size $m_2\times\rho_2$ and the matrix V of size $\rho_2\times(\rho_1+n_2)$ in the canonical form such that

$$\begin{pmatrix} P' & A_{22} \end{pmatrix} = \hat{P}V, \quad \operatorname{rank}\hat{P} = \operatorname{rank} V = \rho_2. \tag{2.7}$$

Determine the matrices Z, P'' of sizes $m_2\times s, m_2\times(\rho_2-s)$ from the partition

$$\hat{P} = \begin{pmatrix} Z & P'' \end{pmatrix} \tag{2.8}$$

and the matrices a, v, q'' of the sizes $s\times\rho_1, s\times n_2, (\rho_2-s)\times n_2$ from the partition

$$V = \begin{pmatrix} a & v \\ 0_{(\rho_2-s)\times\rho_1} & q'' \end{pmatrix}. \tag{2.9}$$

The matrix a has a right inverse $\hat{a}$. Compute the matrix $q' = \hat{a}v$ and take y to be an arbitrary $m_1\times(\rho_2-s)$ matrix.

Set

$$P = \begin{pmatrix} p & y \\ P' & P'' \end{pmatrix}, \quad Q = \begin{pmatrix} q & q' \\ 0_{(\rho_2-s)\times n_1} & q'' \end{pmatrix}. \tag{2.10}$$

Proof. Let ρ_1, ρ_2, s be the numbers given by (2.4). Starting with the rank factorization for the matrix $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$ one determines the matrices P_1, q via (2.5). Next one determines the matrices p, P' from the partition (2.6). Using (2.5), (2.6) one gets $A_{21} = P'q$ and since the matrix q has full row rank one obtains

$$\operatorname{rank} A_{21} = \operatorname{rank} P' = s. \tag{2.11}$$

Moreover, one gets

$$\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} P' & A_{22} \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & I \end{pmatrix}$$

and therefore

$$\operatorname{rank} \begin{pmatrix} P' & A_{22} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} = \rho_2.$$

Applying the rank canonical factorization to the matrix $\begin{pmatrix} P' & A_{22} \end{pmatrix}$ one determines the matrix $\hat{P}$ and the matrix V in the canonical form such that (2.7) holds. One needs to check that the representation (2.9) of the matrix V is possible. Let $l(i),\ i = 1, \dots, \rho_2$ be indices of the first nonzero entries in the rows of V, we set also $l(\rho_2 + 1) = \rho_1 + n_2 + 1$. One can easily derive that

$$V(:, 1:j) = \begin{pmatrix} a_j \\ 0_{(\rho_2 - k) \times j} \end{pmatrix}, \quad j = l(k), \dots, l(k+1) - 1, \quad k = 1, \dots, \rho_2, \tag{2.12}$$

with $k \times j$ matrices a_j with full row rank. Indeed, since $j \leq l(k+1) - 1$, using (1.4) we get

$$V(k+1 : \rho_2, 1 : l(k+1) - 1) = 0.$$

Since $j \geq l(k)$, using the fact that V is in the canonical form we conclude that the matrix $a_j = V(1 : j, 1 : l(k+1) - 1)$ has full row rank. Now using (2.7) we have $P' = \hat{P}V(:, 1 : \rho_1)$ and using (2.12) with $j = \rho_1$ we obtain

$$V(:, 1 : \rho_1) = \begin{pmatrix} a_{\rho_1} \\ 0_{(\rho_2 - k) \times \rho_1} \end{pmatrix}, \tag{2.13}$$

where k is a number such that $l(k) \leq \rho_1 \leq l(k+1) - 1$ and the $k \times \rho_1$ matrix a_{ρ_1} has full row rank k. Thus we get $P' = \hat{P}(:, 1 : k)a_{\rho_1}$. Here $\hat{P}(:, 1 : k)$ is a matrix with full column rank k and since $\operatorname{rank} P' = s$ we conclude that $k = s$. Inserting $k = s$ in (2.13) we obtain the partition (2.9).

Next one determines the matrices Z, P'' from the partition (2.8) and the matrices a, v, q'' from the partition (2.9). Furthermore, using (2.7), (2.8), (2.9) one gets

$$P' = Za, \tag{2.14}$$

$$A_{22} = Zv + P''q''. \tag{2.15}$$

The matrix a of size $s \times \rho_1$ has rank s and therefore has a right inverse $\hat{a}$. Thus, from (2.14) one obtains $Z = P'\hat{a}$. Substituting the last expression in (2.15) and setting $q' = \hat{a}v$ one obtains

$$A_{22} = P'q' + P''q''. \tag{2.16}$$

Next one determines the matrices P, Q via (2.10). Setting $A_L = PQ$ and using (2.5), (2.16) and (2.10) one obtains

$$A_L = \begin{pmatrix} A_{11} & * \\ A_{21} & A_{22} \end{pmatrix},$$

i.e., the matrix A_L is a completion of $\mathcal{A}$.

Now it will be proved that the number $\hat{r}_L$ defined in (2.2) is the minimal completion rank of $\mathcal{A}$ and moreover (2.3) holds. Let

$$A_0 = \begin{pmatrix} A_{11} & X \\ A_{21} & A_{22} \end{pmatrix},$$

with a matrix X of the size $m_1 \times n_2$, be a completion of $\mathcal{A}$. Using the formulas (2.5), (2.6), (2.16) one gets

$$A_0 = \begin{pmatrix} pq & X \\ P'q & P'q' + P''q'' \end{pmatrix}. \tag{2.17}$$

The matrix P_1 of size $(m_1 + m_2) \times \rho_1$ has full column rank ρ_1. Applying the orthogonal rank factorization to P_1 one determines the matrix P_0 of size $(m_1 + m_2) \times \rho_1$ and the invertible matrix R_0 of size $\rho_1 \times \rho_1$ such that the representation

$$P_1 = P_0 R_0, \quad P_0^* P_0 = I_{\rho_1} \tag{2.18}$$

holds. Next one determines the matrices p_0, P_0' of sizes $m_1 \times \rho_1, m_2 \times \rho_1$ from the partition

$$P_0 = \begin{pmatrix} p_0 \\ P_0' \end{pmatrix}. \tag{2.19}$$

Combining the relations (2.6), (2.18) and (2.19) one obtains

$$p = p_0 R_0, \quad P' = P_0' R_0,$$

and moreover

$$p_0^* p + (P_0')^* P' = R_0. \tag{2.20}$$

Further, the matrix $\hat{P}$ from (2.7) has full column rank. Applying the orthogonal rank upper triangular factorization to $\hat{P}$ one obtains the matrix $\hat{P}_0$ of size $m_2 \times \rho_2$ with $\hat{P}_0^* \hat{P}_0 = I_{\rho_2}$ and the upper triangular invertible matrix R of size $\rho_2 \times \rho_2$ such that

$$\hat{P} = \hat{P}_0 R. \tag{2.21}$$

We determine the matrices Z_0, P_0'' of sizes $m_2 \times s, m_2 \times (\rho_2 - s)$ and R_{11}, R_{12}, R_{22} of sizes $s \times s, s \times (\rho_2 - s), (\rho_2 - s) \times (\rho_2 - s)$ from the partitions

$$\hat{P}_0 = \begin{pmatrix} Z_0 & P_0'' \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}. \tag{2.22}$$

Note that here the matrix R_{22} is invertible. Using the relation $\hat{P}_0^* \hat{P}_0 = I_{\rho_2}$ one gets

$$Z_0^* Z_0 = I_s, \quad (P_0'')^* P_0'' = I_{\rho_2 - s}, \quad (P_0'')^* Z_0 = 0_{(\rho_2 - s) \times s}. \tag{2.23}$$

Using (2.21), (2.22) and (2.8) one obtains

$$\begin{pmatrix} Z & P'' \end{pmatrix} = \begin{pmatrix} Z_0 & P_0'' \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix},$$

whence

$$Z = Z_0 R_{11}, \quad P'' = Z_0 R_{12} + P_0'' R_{22}. \tag{2.24}$$

Next, relations (2.14), (2.23), (2.24) yield

$$(P_0'')^* P' = (P_0'')^* Z a = (P_0'')^* Z_0 R_{11} a = 0 \tag{2.25}$$

and

$$(P_0'')^* P'' = (P_0'')^* Z_0 R_{12} + (P_0'')^* P_0'' R_{22} = R_{22}. \tag{2.26}$$

Now setting

$$R = \begin{pmatrix} R_0^{-1} p_0^* & R_0^{-1} (P_0')^* \\ 0 & R_{22}^{-1} (P_0'')^* \end{pmatrix}$$

and using relations (2.17), (2.20), (2.25), (2.26) one obtains

$$R A_0 = \begin{pmatrix} q & * \\ 0 & q'' \end{pmatrix}.$$

Note that since the matrix V in (2.9) has full row rank one gets

$$\operatorname{rank} q'' = \rho_2 - s. \tag{2.27}$$

Thus one obtains

$$\operatorname{rank} R A_0 = \operatorname{rank} q + \operatorname{rank} q'' = \rho_1 + \rho_2 - s = \hat{r}_L$$

and thus we conclude that $\operatorname{rank} A_0 \geq \hat{r}_L$. But as it was shown above, the product $A_L = PQ$ of the matrices P, Q of sizes $(m_1 + m_2) \times \hat{r}_L, \hat{r}_L \times (n_1 + n_2)$ is a completion of $\mathcal{A}$. Hence we conclude that

$$\hat{r}_L \leq \operatorname{rank} A_L \leq \operatorname{rank} P, \operatorname{rank} Q \leq \hat{r}_L.$$

Hence it follows that (2.3) holds, the number $\hat{r}_L$ is the minimal completion rank of $\mathcal{A}$ and the matrix A_L is a completion of $\mathcal{A}$ of this rank. □

Example 2.2. We consider the partially specified matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & ? & ? \\ 1 & 1 & 2 & ? & ? \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

of the form (2.1). One has

$$A_{11} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and therefore

$$\rho_1 = \operatorname{rank} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 2, \quad \rho_2 = \operatorname{rank} \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} = 2, \quad s = \operatorname{rank} A_{21} = 1. \tag{2.28}$$

Hence, by formula (2.2), the minimal completion rank $\hat{r}_L$ of $\mathcal{A}$ equals 3.

Next we apply the algorithm from Lemma 2.1 to compute a minimal completion of $\mathcal{A}$. Using (2.5) one gets

$$P_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and using (2.6) one obtains

$$p = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad P' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Next we apply the factorization (2.7) to the matrix

$$\begin{pmatrix} P' & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and determine the matrices

$$\hat{P} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Using partitions (2.8), (2.9) one gets

$$a = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad q'' = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad P'' = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We take the right inverse of a in the form $\hat{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and obtain

$$q' = \hat{a}v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Finally by the formulas (2.10) one determines the matrices

$$P = \begin{pmatrix} 1 & 1 & y_1 \\ 1 & 2 & y_2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

with arbitrary numbers y_1, y_2 and obtain the minimal completion in the form

$$A_L = PQ = \begin{pmatrix} 1 & 1 & 1 & y_1 & 1 \\ 1 & 1 & 2 & y_2 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{2.29}$$

Note that if we take the right inverse of a in the form $\hat{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we obtain

$$q' = \hat{a}v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_L = PQ = \begin{pmatrix} 1 & 1 & 1 & y_1 & 1 \\ 1 & 1 & 2 & y_2 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which is another minimal rank completion of $\mathcal{A}$. ◊

§2.2 Solution of the general minimal rank completion problem. Examples

Now we consider the minimal rank completion problem for an arbitrary partially specified matrix with a given lower triangular part.

Theorem 2.3. *Let $\mathcal{A}$ be a partially specified block matrix with elements of sizes $m_i \times n_j,\ i, j = 1, \dots, N$ and a given lower triangular part $\tilde{A} = \{A_{ij}\}_{1 \le j \le i \le N}$.*

The minimal completion rank of $\mathcal{A}$ is given by the formula

$$\hat{r}_L = \sum_{k=1}^{N} \operatorname{rank} \tilde{A}(k:N, 1:k) - \sum_{k=2}^{N} \operatorname{rank} \tilde{A}(k:N, 1:k-1). \tag{2.30}$$

Moreover, a minimal rank completion A_L of $\mathcal{A}$ has the form $A_L = PQ$, where the matrices P, Q of sizes $\sum_{i=1}^N m_i \times \hat{r}_L, \hat{r}_L \times \sum_{i=1}^N n_i$ which satisfy the condition

$$\operatorname{rank} A_L = \operatorname{rank} P = \operatorname{rank} Q \hat{r}_L, \tag{2.31}$$

may be obtained via the following algorithm.

1. *Set* $A^{(1)} = \tilde{A}(:,1),\ \rho_1 = \operatorname{rank} A^{(1)},\ \eta_1 = \sum_{i=1}^{N} m_i$.

Using the rank factorization of the matrix $A^{(1)}$ *determine the matrices* P_1, Q_1 *of sizes* $\eta_1 \times \rho_1$ *and* $\rho_1 \times n_1$*, respectively, such that*

$$A^{(1)} = P_1 Q_1, \tag{2.32}$$

with

$$\operatorname{rank} P_1 = \operatorname{rank} Q_1 = \rho_1. \tag{2.33}$$

Set $r_1 = \rho_1$.

2. *For* $k = 2, \dots, N$ *perform the following.*

Set $\eta_k = \sum_{i=k}^{N} m_i,\ \lambda_k = \sum_{i=1}^{k-1} m_i,\ \nu_k = \sum_{i=1}^{k-1} n_i$. *Determine the matrices* $\tilde{P}_{k-1}, P'_k$ *of sizes* $\lambda_k \times r_{k-1}, \eta_k \times r_{k-1}$*, respectively, from the partition*

$$P_{k-1} = \begin{pmatrix} \tilde{P}_{k-1} \\ P'_k \end{pmatrix}. \tag{2.34}$$

Set

$$A^{(k)} = \tilde{A}(k:N,k), \quad s_k = \operatorname{rank}\tilde{A}(k:N,1:k-1), \quad \rho_k = \operatorname{rank}\tilde{A}(k:N,1:k).$$

Using the rank canonical factorization of the matrix $\begin{pmatrix} P'_k & A^{(k)} \end{pmatrix}$ *determine the matrices* $\hat{P}_k, V_k$ *of sizes* $\eta_k \times \rho_k, \rho_k \times (r_{k-1}+n_k)$*, respectively, such that*

$$\begin{pmatrix} P'_k & A^{(k)} \end{pmatrix} = \hat{P}_k V_k, \quad \operatorname{rank} \hat{P}_k = \operatorname{rank} V_k = \rho_k. \tag{2.35}$$

Determine the matrices Z_k, P''_k *of sizes* $\eta_k \times s_k, \eta_k \times (\rho_k - s_k)$ *and the matrices* a_k, v_k, q''_k *of sizes* $s_k \times r_{k-1}, s_k \times n_k, (\rho_k - s_k) \times n_k$*, respectively, from the partitions*

$$\hat{P}_k = \begin{bmatrix} Z_k & P''_k \end{bmatrix}, \quad V_k = \begin{bmatrix} a_k & v_k \\ 0_{(\rho_k - s_k)\times r_{k-1}} & q''_k \end{bmatrix}. \tag{2.36}$$

The matrix a_k *has a right inverse* $\hat{a}_k$. *Compute* $q'_k = \hat{a}_k v_k$ *and take* y_k *to be arbitrary* $\lambda_k \times (\rho_k - s_k)$ *matrix. Next, set*

$$P_k = \begin{pmatrix} \tilde{P}_{k-1} & y_k \\ P'_k & P''_k \end{pmatrix}, \quad Q_k = \begin{pmatrix} Q_{k-1} & q'_k \\ 0_{(\rho_k - s_k)\times \nu_k} & q''_k \end{pmatrix}, \quad r_k = r_{k-1} + \rho_k - s_k. \tag{2.37}$$

3. *Set* $\hat{r}_L = r_N,\ P = P_N,\ Q = Q_N$.

Proof. We consider the sequence of block $N \times k$ matrices $A_k = P_k Q_k,\ k = 1, \dots, N$ and the sequence of partially specified matrices

$$\mathcal{A}_k = \begin{pmatrix} A_{11} & ? & \dots & ? \\ A_{21} & A_{22} & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \dots & A_{Nk} \end{pmatrix}, \quad k = 1, \dots, N$$

with the given parts $\tilde{A}_k = \{A_{ij}, 1 \le j \le k, j \le i \le N\},\ k = 1, \dots, N$. We prove by induction that for any $k = 1, \dots, N$ the matrix A_k is a completion of $\mathcal{A}_k$ with

$$\operatorname{rank} A_k = \operatorname{rank} P_k = \operatorname{rank} Q_k = r_k. \tag{2.38}$$

For $k = 1$ this follows from (2.32), (2.33). Let for some k with $2 \le k \le N$ the matrix $A_{k-1} = P_{k-1}Q_{k-1}$ with $\operatorname{rank} A_{k-1} = \operatorname{rank} P_{k-1} = \operatorname{rank} Q_{k-1} = r_{k-1}$ be a completion of $\mathcal{A}_{k-1}$. Consider the partially specified matrix

$$\hat{\mathcal{A}}_k = \begin{pmatrix} A_{k-1}(1:k-1, 1:k-1) & ? \\ A_{k-1}(k:N, 1:k-1) & \tilde{A}(k:N, k) \end{pmatrix}.$$

One has

$$\operatorname{rank} \begin{pmatrix} A_{k-1}(1:k-1, 1:k-1) \\ A_{k-1}(k:N, 1:k-1) \end{pmatrix} = \operatorname{rank} A_{k-1} = r_{k-1}.$$

Next, the part $\tilde{A}_{k-1}$ contains the submatrix $A(k:N, 1:k-1)$ and since A_{k-1} is a completion of $\mathcal{A}_{k-1}$ one gets $A_{k-1}(k:N, 1:k-1) = \tilde{A}(k:N, 1:k-1)$. Thus one has

$$\operatorname{rank} \begin{pmatrix} A_{k-1}(k:N, 1:k-1) & \tilde{A}(k:N, k) \end{pmatrix} = \operatorname{rank} \tilde{A}(k:N, 1:k) = \rho_k,$$
$$\operatorname{rank} A_{k-1}(k:N, 1:k-1) = \operatorname{rank} \tilde{A}(k:N, 1:k-1) = s_k.$$

Now applying the algorithm from Lemma 2.1 to the partially specified matrix $\hat{\mathcal{A}}_k$ one obtains the formulas (2.34)–(2.37) to compute the matrices P_k, Q_k satisfying (2.38) such that $A_k = P_k Q_k$ is a completion of $\hat{\mathcal{A}}_k$. Since $\hat{\mathcal{A}}_k$ contains the part $\tilde{A}_k$, the matrix A_k is a completion of $\mathcal{A}_k$.

For $k = N$ we conclude that $A_L = P_N Q_N = PQ$ is a completion of $\mathcal{A}$ and the equality (2.31) holds.

It remains to show that the number $\hat{r}_L$ defined in (2.30) is the minimal completion rank of $\mathcal{A}$. Let A_0 be completion of $\mathcal{A}$. Set

$$\begin{aligned} t_k &= \operatorname{rank} A_0(:, 1:k), & k &= 1, \dots, N, \\ \rho_k &= \operatorname{rank} A_0(k:N, 1:k) = \operatorname{rank} \tilde{A}(k:N, 1:k), & k &= 1, \dots, N, \\ s_k &= \operatorname{rank} A_0(k:N, 1:k-1) = \operatorname{rank} \tilde{A}(k:N, 1:k-1), & k &= 2, \dots, N. \end{aligned}$$

One obviously has $t_1 = \rho_1$. Next, for $k = 2, \dots, N$ we use the partitions

$$A_0(:, 1:k) = \begin{pmatrix} A_0(1:k-1, 1:k-1) & A_0(1:k-1, k) \\ A(k:N, 1:k-1) & A(k:N, k) \end{pmatrix}$$

and applying the formula (2.2) from Lemma 2.1 we get

$$t_k \ge t_{k-1} + \rho_k - s_k, \quad k = 2, \dots, N. \tag{2.39}$$

It is clear that $t_N = \operatorname{rank} A_0$ and from (2.39) it follows that

$$t_N \geq t_{N-1} + \rho_N - s_N \geq t_{N-2} + (\rho_{N-1} + \rho_N) - (s_{N-1} + s_N) \geq \cdots$$
$$\geq t_1 + \sum_{k=2}^{N} \rho_k - \sum_{k=2}^{N} s_k = \hat{r}_L. \qquad \square$$

Example 2.4. Let $\mathcal{A}$ be a partially specified matrix with scalar elements and given lower triangular part

$$\tilde{A} = \{A_{ij}\}_{1 \leq j \leq i \leq 4} = \begin{pmatrix} 2 & * & * & * \\ 1 & 2 & * & * \\ 1 & 1 & 2 & * \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

The minimal completion rank of $\mathcal{A}$ is given by the formula

$$\hat{r}_L = \sum_{k=1}^{4} \operatorname{rank} \tilde{A}(k:4, 1:k) - \sum_{k=2}^{4} \operatorname{rank} \tilde{A}(k:4, 1:k-1)$$
$$= 1 + 2 + 2 + 1 - (1 + 1 + 1) = 3.$$

A minimal rank completion A_L of $\mathcal{A}$ has the form $A_L = PQ$, where the matrices P, Q of sizes $\sum_{i=1}^{N} m_i \times \hat{r}_L = 4 \times 3, \hat{r}_L \times \sum_{i=1}^{N} n_i = 3 \times 4$ which satisfy the condition

$$\operatorname{rank} A_L = \operatorname{rank} P = \operatorname{rank} Q = 3,$$

can be obtained via the following algorithm.

Step 1. Set $A^{(1)} = A(:,1) = \begin{pmatrix} 2 & 1 & 1 & 1 \end{pmatrix}^T$, $\rho_1 = \operatorname{rank} A^{(1)} = 1$, $\eta_1 = \sum_{i=1}^{N} m_i = 4$.

Determine the matrices P_1, Q_1 of the sizes $\eta_1 \times \rho_1 = 4 \times 1$ and $\rho_1 \times n_1 = 1 \times 1$, respectively, such that (2.32) and (2.33) hold. In this example $P_1 = A^{(1)}, Q_1 = 1$.

Set $r_1 = \rho_1 = 1$.

Step 2. For $k = 2$ perform the following.

Set $\eta_2 = \sum_{i=2}^{4} m_i = 3$, $\lambda_2 = \sum_{i=1}^{2-1} m_i = 1$, $\nu_2 = \sum_{i=1}^{2-1} n_i = 1$. Determine the matrices $\tilde{P}_{2-1}, P'_2$ of sizes $\lambda_2 \times r_{2-1} = 1 \times 1, \eta_2 \times r_{2-1} = 3 \times 1$, respectively, from the partition (2.34). In this example $\tilde{P}_1 = 2, P'_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Set

$$A^{(2)} = \tilde{A}(2:4, 2) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad s_2 = \operatorname{rank} \tilde{A}(2:4, 1:2-1) = 1,$$
$$\rho_2 = \operatorname{rank} \tilde{A}(2:4, 1:2) = 2.$$

Using the rank canonical factorization of the matrix $\begin{pmatrix} P_2' & A^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ determine the matrices $\hat{P}_2, V_2$ of the sizes $\eta_2 \times \rho_2 = 3 \times 2, \rho_2 \times (r_{2-1} + n_2) = 2 \times 2$, respectively, such that (2.35) takes place. In this example

$$\hat{P}_2 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Determine the matrices Z_2, P_2'' of sizes $\eta_2 \times s_2 = 3 \times 1, \eta_2 \times (\rho_2 - s_2) = 3 \times 1$ and the matrices a_2, v_2, q_2'' of the sizes $s_2 \times r_{2-1}, s_2 \times n_2, (\rho_2 - s_2) \times n_2$, thus all of them 1×1, from the partitions (2.36). Namely, $P_2'' = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $a_2 = q_2'' = 1, v_2 = 0$.

The matrix a_2 has a right inverse $\hat{a}_2 = 1$. Compute $q_2' = \hat{a}_2 v_2 = 0$ and take y_2 to be an arbitrary $\lambda_2 \times (\rho_2 - s_2) = 1 \times 1$ matrix, for instance $y_2 = 0$.

Next, from (2.37):

$$P_2 = \begin{pmatrix} \tilde{P}_1 & y_2 \\ P_2' & P_2'' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} Q_{2-1} & q_2' \\ 0_{(\rho_2 - s_2) \times \nu_2} & q_2'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$r_2 = r_1 + \rho_2 - s_2 = 1 + 2 - 1 = 2.$$

For $k = 3$ perform again Step 2. Set $\eta_3 = 2,\ \lambda_3 = 2,\ \nu_3 = 2$. Determine the matrices $\tilde{P}_{3-1}, P_3'$ of sizes $\lambda_3 \times r_{3-1} = 2 \times 2, \eta_3 \times r_{3-1} = 2 \times 2$, respectively, from the partition (2.34). It follows that $P_3' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Set

$$A^{(3)} = \tilde{A}(3:4, 3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad s_3 = \operatorname{rank} \tilde{A}(3:4, 1:3-1) = 1,$$
$$\rho_3 = \operatorname{rank} \tilde{A}(3:4, 1:3) = 2.$$

Using the rank canonical factorization of the matrix $\begin{pmatrix} P_3' & A^{(3)} \end{pmatrix}$ determine the matrices $\hat{P}_3, V_3$ of the sizes $2 \times 2, 2 \times 3$, respectively, such that (2.35) takes place. It follows that

$$\hat{P}_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Determine the matrices Z_3, P_3'' and the matrices a_3, v_3, q_3'' of proper sizes from the partitions (2.36). It follows that $P_3'' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 & 1 \end{pmatrix}, q_3'' = 1, v_3 = 1$.

The matrix a_3 has a right inverse $\hat{a}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Compute $q_3' = \hat{a}_3 v_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and take y_3 to be an arbitrary $\lambda_3 \times (\rho_3 - s_3)$ matrix, for instance $y_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Next, from (2.37):

$$P_3 = \begin{pmatrix} \tilde{P}_2 & y_3 \\ P_3' & P_3'' \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} Q_{3-1} & q_3' \\ 0_{(\rho_3 - s_3)\times\nu_3} & q_3'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_3 = 3.$$

Finally, for $k = 4$ perform again Step 2. Set $\eta_4 = 1$, $\lambda_4 = 3$, $\nu_4 = 3$. Determine the matrix P_4' of size $\eta_4 \times r_{4-1} = 1 \times 3$ from the partition (2.34). It follows that $P_4' = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$.

Set

$$A^{(4)} = \tilde{A}(4:4,4) = 2, \quad s_4 = \operatorname{rank} \tilde{A}(4:4,1:3) = 1, \quad \rho_4 = \operatorname{rank} \tilde{A}(4:4,1:4) = 1.$$

Determine the matrices $\hat{P}_4, V_4$ of the sizes $1 \times 1, 1 \times 4$, respectively, such that (2.35) holds. It follows that

$$\hat{P}_4 = 1, \quad V_4 = \begin{pmatrix} 1 & 1 & 0 & 2 \end{pmatrix}.$$

Determine the matrices Z_4, P_4'' and the matrices a_4, v_4, q_4'' of proper sizes from the partitions (2.36). It follows that P_4'' and q_4'' do not exist, $a_4 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$, $v_4 = 2$.

The matrix a_4 has a right inverse $\hat{a}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Compute $q_4' = \hat{a}_4 v_4$ and take y_4 to be an arbitrary $\lambda_4 \times (\rho_4 - s_4)$ matrix. Since $\rho_4 = s_4$, the matrix y_4 does not exist.

Next, from (2.37),

$$P_4 = \begin{pmatrix} \tilde{P}_3 & y_4 \\ P_4' & P_4'' \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} Q_3 & q_4' \\ 0_{(\rho_4 - s_4)\times\nu_4} & q_4'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad r_4 = 3.$$

Step 3. Set $\hat{r}_L = r_4 = 3$, $P = P_4$, $Q = Q_4$.

The minimal rank completion obtained here is

$$A_L = PQ = \begin{pmatrix} 2 & 0 & 2 & 4 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

◊

Example 2.5. Let $\mathcal{A}$ be a partially specified block matrix with scalar elements and given lower triangular part

$$\tilde{A} = \{A_{ij}\}_{1\le j\le i\le 6} = \begin{pmatrix} \beta & * & * & * & * & * \\ \gamma & \beta & * & * & * & * \\ 0 & \gamma & \beta & * & * & * \\ 0 & 0 & \gamma & \beta & * & * \\ 0 & 0 & 0 & \gamma & \beta & * \\ 0 & 0 & 0 & 0 & \gamma & \beta \end{pmatrix}$$

with $\gamma \neq 0$ and β scalars. The given 6×6 part can be seen as the strictly lower triangular part of a scalar 5-band Toeplitz Hermitian matrix of size 7×7.

The minimal completion rank of $\mathcal{A}$ is given by the formula

$$\begin{aligned} \hat{r}_L &= \sum_{k=1}^{6} \operatorname{rank} \tilde{A}(k:6, 1:k) - \sum_{k=2}^{6} \operatorname{rank} \tilde{A}(k:6, 1:k-1) \\ &= (1+2+2+2+2+1) - (1+1+1+1+1) = 5. \end{aligned}$$

Using algorithm from Theorem 2.3 one can determine a minimal rank completion A_L of $\mathcal{A}$ in the form $A_L = PQ$, where the matrices P, Q of sizes $\sum_{i=1}^{6} m_i \times \hat{r}_L = 6 \times 5, \hat{r}_L \times \sum_{i=1}^{6} n_i = 5 \times 6$ are given by

$$P = \begin{pmatrix} \beta & 0 & 0 & 0 & 0 \\ \gamma & \beta & 0 & 0 & 0 \\ 0 & \gamma & \beta & 0 & 0 \\ 0 & 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\beta}{\gamma} & 0 \\ 0 & 0 & 0 & 0 & \gamma & \beta \end{pmatrix},$$

and therefore

$$A_L = \begin{pmatrix} \beta & 0 & 0 & 0 & 0 & 0 \\ \gamma & \beta & 0 & 0 & 0 & 0 \\ 0 & \gamma & \beta & 0 & 0 & 0 \\ 0 & 0 & \gamma & \beta & \frac{\beta^2}{\gamma} & 0 \\ 0 & 0 & 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & 0 & \gamma & \beta \end{pmatrix}.$$

◊

Example 2.6. Consider the 4×4 partially specified matrix

$$\mathcal{A} = \begin{pmatrix} 1 & ? & ? & ? \\ 1 & 2 & ? & ? \\ 1 & 1 & 1 & ? \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

with given lower triangular part $\tilde{A} = \{A_{ij}\}_{1 \le j \le i \le 4}$. Set

$$\rho'_k = \text{rank}\tilde{A}(k:4, 1:k),\ k = 1,2,3,4; \quad s'_k = \text{rank}\tilde{A}(k:4, 1:k-1),\ k = 2,3,4.$$

One has

$$\rho'_k = 1,\ k = 1,3,4, \quad \rho'_2 = 2, \quad s'_k = 1,\ k = 2,3,4$$

and therefore, by Theorem 2.3, the minimal completion rank of $\mathcal{A}$ equals two. Moreover, by Theorem 2.3, a minimal rank completion of $\mathcal{A}$ is given by $A_L = PQ$, with

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

i.e.,

$$A_L = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

◊

In a similar way one can solve the minimal rank completion problem for a partially specified block matrix

$$\mathcal{B} = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1N} \\ ? & B_{22} & \dots & B_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \dots & B_{NN} \end{pmatrix}$$

with given upper triangular part $\tilde{B} = \{B_{ij}\}_{1 \le i \le j \le N}$. Applying formula (2.30) to the matrix $\mathcal{B}^t$ one obtains the expression for the minimal completion rank of $\mathcal{B}$:

$$\hat{r}_U = \sum_{k=1}^{N} \text{rank}\,\tilde{B}(1:k, k:N) - \sum_{k=2}^{N} \text{rank}\,\tilde{B}(1:k-1, k:N). \tag{2.40}$$

Moreover applying to $\mathcal{B}^t$ the algorithm from Theorem 2.3 and taking transposed matrices we may compute a minimal rank completion of the matrix $\mathcal{B}$.

Example 2.7. Consider the 4×4 partially specified matrix $\mathcal{B}$

$$\mathcal{B} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ ? & 1 & 1 & 1 \\ ? & ? & 1 & 1 \\ ? & ? & ? & 1 \end{pmatrix}.$$

with given upper triangular part $\tilde{B} = \{B_{ij}\}_{1\le i\le j\le 4}$. Set

$$\rho_k'' = \operatorname{rank} \tilde{B}(1:k, k:4),\ k = 1,2,3,4; \quad s_k'' = \operatorname{rank} \tilde{B}(1:k-1, k:4),\ k = 2,3,4.$$

One has

$$\rho_k'' = 1,\ k = 1,4, \quad \rho_k'' = 2,\ k = 2,3, \quad s_k'' = 1,\ k = 2,4, \quad s_3'' = 2$$

and by formula (2.40) one obtains $\hat{r}_U = 2$. Next, by Theorem 2.3, a minimal rank completion of $\mathcal{B}^t$ is given by $H^t G^t$, with

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},\ H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

From here one obtains the minimal rank completion B_U of $\mathcal{B}$:

$$B_U = GH = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

◊

Example 2.8. Consider the 4×4 partially specified matrix $\mathcal{B}$

$$\mathcal{B} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ ? & 2 & 1 & 1 \\ ? & ? & 2 & 1 \\ ? & ? & ? & 2 \end{pmatrix}.$$

with given upper triangular part $\tilde{B} = \{B_{ij}\}_{1\le i\le j\le 4}$. Set

$$\rho_k'' = \operatorname{rank} \tilde{B}(1:k, k:4),\ k = 1,2,3,4; \quad s_k'' = \operatorname{rank} \tilde{B}(1:k-1, k:4),\ k = 2,3,4.$$

By formula (2.40), one obtains $\hat{r}_U = 3$. Next, by Theorem 2.3 and Example 2.4, a minimal rank completion of $\mathcal{B}^T$ is given by $H^T G^T$, with

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

From here one obtains the minimal rank completion B_U of $\mathcal{B}$:

$$B_U = GH = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 4 & 2 & 2 & 2 \end{pmatrix}.$$

◊

Example 2.9. Consider the 6×6 partially specified matrix $\mathcal{B}$

$$\mathcal{B} = \begin{pmatrix} \beta & \gamma & 0 & 0 & 0 & 0 \\ ? & \beta & \gamma & 0 & 0 & 0 \\ ? & ? & \beta & \gamma & 0 & 0 \\ ? & ? & ? & \beta & \gamma & 0 \\ ? & ? & ? & ? & \beta & \gamma \\ ? & ? & ? & ? & ? & \beta \end{pmatrix}.$$

with given upper triangular part $\tilde{B} = \{B_{ij}\}_{1 \le i \le j \le 6}$. Here $\gamma \neq 0$ and β are scalars. Set

$$\rho''_k = \operatorname{rank} \tilde{B}(1:k, k:6),\ k = 1, \ldots, 6; \quad s''_k = \operatorname{rank} \tilde{B}(1:k-1, k:6),\ k = 2, \ldots, 6.$$

By formula (2.40), one obtains $\hat{r}_U = 5$. Next, by Theorem 2.3 and Example 2.5, a minimal rank completion of $\mathcal{B}^T$ is given by $H^T G^T$, with

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\beta}{\gamma} & \gamma \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \qquad H = \begin{pmatrix} \beta & \gamma & 0 & 0 & 0 & 0 \\ 0 & \beta & \gamma & 0 & 0 & 0 \\ 0 & 0 & \beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & \beta & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

namely $G = Q^T, H = P^T$, where P and Q have been computed in Example 2.5.

From here one obtains the minimal rank completion B_U of $\mathcal{B}$:

$$B_U = GH = \begin{pmatrix} \beta & \gamma & 0 & 0 & 0 & 0 \\ 0 & \beta & \gamma & 0 & 0 & 0 \\ 0 & 0 & \beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & \beta & \gamma & 0 \\ 0 & 0 & 0 & \frac{\beta^2}{\gamma} & \beta & \gamma \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}.$$

◊

§2.3 Uniqueness of the minimal rank completion

Here we consider a condition for the uniqueness of a minimal rank completion. As above we start with a partially specified 2×2 block matrix.

Lemma 2.10. *Let*

$$\mathcal{A} = \begin{pmatrix} A_{11} & ? \\ A_{21} & A_{22} \end{pmatrix} \tag{2.41}$$

with the specified blocks A_{11}, A_{21}, A_{22} of sizes $m_1 \times n_1, m_2 \times n_1, m_2 \times n_2$, respectively, such that $m_1, n_2 > 0$.

A minimal rank completion of $\mathcal{A}$ is unique if and only if the condition

$$\operatorname{rank} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} = \operatorname{rank} A_{21} := s \tag{2.42}$$

holds. Moreover, if this is the case, the minimal completion rank $\hat{r}_L$ of $\mathcal{A}$ equals s.

Proof. To prove the sufficiency, let the condition (2.42) hold. Formula (2.2) implies $\hat{r}_L = s$. Let A_L, B_L be two minimal rank completions of $\mathcal{A}$. We show that $A_L = B_L$. Consider the rank factorizations

$$A_L = P \cdot Q, \quad B_L = F \cdot G,$$

with the matrices P, F of size $(m_1 + m_2) \times s$ and the matrices Q, G of size $s \times (n_1 + n_2)$ such that

$$\operatorname{rank} P = \operatorname{rank} Q = \operatorname{rank} F = \operatorname{rank} G = s.$$

We use the partitions

$$P = \begin{pmatrix} p \\ P' \end{pmatrix}, \; Q = \begin{pmatrix} q & q' \end{pmatrix}, \quad F = \begin{pmatrix} f \\ F' \end{pmatrix}, \; G = \begin{pmatrix} g & g' \end{pmatrix}$$

with $m_1 \times s$ submatrices p, f, $m_2 \times s$ submatrices P', F', $s \times n_1$ submatrices q, g and $s \times n_2$ submatrices q', g'. One has

$$\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = P \cdot q = F \cdot g.$$

By Lemma 1.1, there is an invertible matrix S of size $s \times s$ such that $P = FS$. In particular, $P' = F'S$. Next one has

$$\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} = P'Q = F'G = P'S^{-1}G.$$

But also one has $A_{21} = P'q$ with the matrices P', q of sizes $m_2 \times s, s \times n_1$ and since $\operatorname{rank} A_{21} = s$ one gets $\operatorname{rank} P' = s$. Hence the equality $P'Q = P'S^{-1}G$ implies $Q = S^{-1}G$. We conclude that

$$A_L = PQ = FSS^{-1}G = FG = B_L,$$

as claimed.

To prove the necessity, suppose that the condition (2.42) does not hold. Set $s = \operatorname{rank} A_{21}$. Assume that

$$\rho_2 := \operatorname{rank} \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} > s.$$

Applying the algorithm from Lemma 2.1 one obtains a minimal rank completion of $\mathcal{A}$ in the form

$$A_L = \begin{pmatrix} p & y \\ P' & P'' \end{pmatrix} \begin{pmatrix} q & q' \\ 0 & q'' \end{pmatrix} = \begin{pmatrix} A_{11} & pq' + yq'' \\ A_{21} & A_{22} \end{pmatrix}, \tag{2.43}$$

with an arbitrary $m_1 \times (\rho_2 - s)$ matrix y. From (2.27) we see that the matrix q'' is not zero. Hence, we can take different matrices y to obtain different matrices yq''. Substituting this in (2.43), we obtain different minimal rank completions of $\mathcal{A}$.

Assume now that $\rho_1 := \operatorname{rank} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} > s = \rho_2$. Using the algorithm from Lemma 2.1 with $\rho_2 - s = 0$ one obtains a minimal rank completion in the form

$$A_L = \begin{pmatrix} p \\ P' \end{pmatrix} \begin{pmatrix} q & q' \end{pmatrix},$$

with

$$\operatorname{rank} \begin{pmatrix} p \\ P' \end{pmatrix} = \operatorname{rank} \begin{pmatrix} q & q' \end{pmatrix} = \rho_1.$$

Here the matrix $\begin{pmatrix} p \\ P' \end{pmatrix}$ has full column rank ρ_1. But using (2.11) we see that $\operatorname{rank} P' = s < \rho_1$ and therefore there is a nonzero $\rho_1 \times n_2$ matrix Q' such that $P'Q' = 0$. At the same time one has $\begin{pmatrix} p \\ P' \end{pmatrix} Q' \neq 0$ and therefore $pQ' \neq 0$. Define A'_L by

$$A'_L = \begin{pmatrix} p \\ P' \end{pmatrix} \begin{pmatrix} q & q' + Q' \end{pmatrix}.$$

We see that

$$A'_L = \begin{pmatrix} pq & pq' + pQ' \\ P'q & P'q' \end{pmatrix} = \begin{pmatrix} A_{11} & pq' + pQ' \\ A_{21} & A_{22} \end{pmatrix}.$$

Hence, the matrix A'_L is a minimal rank completion of $\mathcal{A}$ that is different from A_L. □

Corollary 2.11. *Let the partially specified matrix $\mathcal{A}$ from* (2.41) *satisfy the condition*

$$\operatorname{rank} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \operatorname{rank} A_{21}.$$

Then there is a matrix S such that $A_{11} = SA_{21}$ and furthermore the unspecified entry in (2.41) *can be chosen as $A_{12} = SA_{22}$ to obtain a minimal rank completion. Moreover, if the condition* (2.42) *is valid, then the unique minimal rank completion of* (2.41) *is given by the formula $A_{12} = SA_{22}$.*

Proof. Set

$$s := \operatorname{rank}\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \operatorname{rank} A_{21}$$

and consider the rank factorization

$$\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \begin{pmatrix} p \\ P' \end{pmatrix} q,$$

with matrices p, P', q of sizes $m_1 \times s, m_2 \times s, s \times n_1$, respectively, such that

$$\operatorname{rank}\begin{pmatrix} p \\ P' \end{pmatrix} = \operatorname{rank} q = s.$$

We have $A_{21} = P'q$ and since $\operatorname{rank} A_{21} = s$, the matrix P' has full column rank and therefore has a left inverse $\hat{P}'$. Setting $S = p\hat{P}'$ one gets

$$A_{11} = pq = p\hat{P}'P'q = SP'q = SA_{21}.$$

Hence taking $A_{12} = SA_{22}$ and $A_0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ we get

$$A_0 = \begin{pmatrix} SA_{21} & SA_{22} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} S \\ I \end{pmatrix} \begin{pmatrix} A_{21} & A_{22} \end{pmatrix}.$$

Consequently,

$$\operatorname{rank} A_0 \leq \operatorname{rank}\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} \leq \hat{r}_L,$$

where $\hat{r}_L$ is the minimal completion rank of (2.41), which implies that A_0 is a minimal rank completion of (2.41). Moreover, if the condition (2.42) is satisfied, then by Lemma 2.10 this minimal rank completion is unique. □

Example 2.12. Consider the partially specified matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & ? & ? \\ 1 & 1 & 1 & ? & ? \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

of the form (2.41). One has

$$A_{11} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and therefore

$$\operatorname{rank}\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} = \operatorname{rank} A_{21} = 2.$$

Hence the minimal rank completion of $\mathcal{A}$ is unique. Moreover this completion is the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

◊

Example 2.13. Consider the partially specified matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & ? & ? \\ 1 & 1 & 2 & ? & ? \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

from Example 2.2. From (2.28) we see that a minimal rank completion of $\mathcal{A}$ is not unique. Moreover, one can obtain different minimal rank completions of $\mathcal{A}$ by taking different values of y_1, y_2 in the formula (2.29). Furthermore, another minimal rank completion of $\mathcal{A}$ is for instance the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

◊

Now we derive a uniqueness criterion for the minimal rank completion of an arbitrary partially specified block matrix with a given lower triangular part.

Theorem 2.14. *Let $\mathcal{A}$ be a partially specified block matrix with elements of sizes $m_i \times n_j,\ i,j = 1,\dots,N$ with $m_1 > 0, n_N > 0$ and with a given lower triangular part $\tilde{A} = \{A_{ij}\}_{1\le j\le i\le N}$.*

Then the following are equivalent:

(i) *the minimal rank completion of $\mathcal{A}$ is unique;*

(ii) $\operatorname{rank}\tilde{A}(k:N,1:k) = \operatorname{rank}\tilde{A}(k+1:N,k:N) = \operatorname{rank}\tilde{A}(k+1:N,k+1:N)$, $k = 1,\dots,N-1$;

(iii) *the numbers* $\operatorname{rank}\tilde{A}(i:N,1:j)$, $1 \le j \le i \le N$ *are all equal;*

(iv) $\operatorname{rank}\tilde{A}(N,1) = \hat{r}_L$, *where $\hat{r}_L$ is the minimal completion rank of $\mathcal{A}$.*

Proof. Clearly $\operatorname{rank}\tilde{A}(N,1) \le \operatorname{rank}\tilde{A}(i:N,1:j) \le \hat{r}_L,\ 1 \le j \le i \le N$. So (iv) implies (iii). The implication (iii)⇒(ii) is also trivial. Let us prove (ii)⇒(i) and (ii)⇒(iv). Assume that (ii) holds. Using the formula (2.30) one gets

$$\hat{r}_L = \operatorname{rank}\tilde{A}(1:N,1) = \operatorname{rank}\tilde{A}(N,1:N).$$

Consider the partially specified matrix

$$\mathcal{A}_0 = \begin{bmatrix} \tilde{A}(1:N-1,1) & ? \\ \tilde{A}(N,1) & \tilde{A}(N,2:N) \end{bmatrix}.$$

Every completion of $\mathcal{A}$ is a completion of $\mathcal{A}_0$. If $\operatorname{rank} \tilde{A}(N,1) < \hat{r}_L$ then by (2.2) the minimal completion rank of $\mathcal{A}_0$ is given by

$$\operatorname{rank} \tilde{A}(1:N,1) + \operatorname{rank} \tilde{A}(N,1:N) - \operatorname{rank} \tilde{A}(N,1) > \hat{r}_L$$

which is a contradiction. Thus one obtains (iv).

Furthermore,

$$\hat{r}_L = \operatorname{rank} \tilde{A}(1:N,1) = \operatorname{rank} \tilde{A}(N,1:N) = \operatorname{rank} \tilde{A}(N,1)$$

and therefore, by (2.2), the minimal completion rank of $\mathcal{A}_0$ equals $\hat{r}_L$ and by Lemma 2.10 the minimal rank completion of $\mathcal{A}_0$ is unique. Hence, if $\mathcal{A}$ has two different minimal rank completions one obtains two different minimal rank completions of $\mathcal{A}_0$, a contradiction. Thus, one has proved (i).

We finish by proving the implication (i)⇒(ii). Assume that (i) holds and let A_L be the unique minimal rank completion of $\mathcal{A}$. For each $k = 2, \dots, N$ consider the partially specified matrix

$$\begin{aligned} \mathcal{A}_k &= \begin{bmatrix} A_L(1:k-1,1:k-1) & ? \\ A_L(k:N,1:k-1) & A_L(k:N,k:N) \end{bmatrix} \\ &= \begin{bmatrix} A_L(1:k-1,1:k-1) & ? \\ \tilde{A}(k:N,1:k-1) & A_L(k:N,k:N) \end{bmatrix}. \end{aligned}$$

Every completion of $\mathcal{A}_k$ is a completion of $\mathcal{A}$. Hence $\mathcal{A}_k$ cannot have completions of rank less than $\hat{r}_L$. At the same time, the matrix A_L is a completion of $\mathcal{A}_k$ of rank $\hat{r}_L$. Hence, $\hat{r}_L$ is the minimal completion rank of $\mathcal{A}_k$. Moreover, A_L is the unique minimal rank completion of $\mathcal{A}_k$, otherwise one obtains different minimal rank completions of $\mathcal{A}$. By Lemma 2.10,

$$\begin{aligned} \operatorname{rank} \begin{pmatrix} A_L(1:k-1,1:k-1) \\ \tilde{A}(k:N,1:k-1) \end{pmatrix} &= \operatorname{rank} \begin{pmatrix} \tilde{A}(k:N,1:k-1) & A_L(k:N,k:N) \end{pmatrix} \\ &= \operatorname{rank} \tilde{A}(k:N,1:k-1) = \hat{r}_L. \end{aligned}$$

Now using the inequalities

$$\begin{aligned} \hat{r}_L &= \operatorname{rank} \tilde{A}(k:N,1:k-1) \le \operatorname{rank} \tilde{A}(k-1:N,1:k-1) \le \operatorname{rank} A_L = \hat{r}_L, \\ \hat{r}_L &= \operatorname{rank} \tilde{A}(k:N,1:k-1) \le \operatorname{rank} \tilde{A}(k:N,1:k) \le \operatorname{rank} A_L = \hat{r}_L \end{aligned}$$

one obtains (ii). □

Example 2.15. The partially specified matrix

$$\mathcal{A} = \begin{pmatrix} 1 & ? & ? & ? \\ 1 & 2 & ? & ? \\ 1 & 2 & 3 & ? \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

has the unique minimal rank completion

$$A_L = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

◊

§2.4 Comments

The material of this chapter is borrowed from the Ph.D. Thesis of H. Woerdeman [48], but our proofs are different.

Chapter 3

Matrices in Diagonal Plus Semiseparable Form

Here we consider diagonal plus semiseparable representations of matrices. This is a direct generalization of diagonal plus separable representations studied in Chapter 1. Note that every matrix may be represented in the diagonal plus semiseparable form. The problem is to obtain such a representation with minimal orders. This may be treated as the problem of completing strictly lower triangular and strictly upper triangular parts of a matrix to matrices with minimal ranks, since it will be proved that minimal orders of the generators are equal to the ranks of those minimal completions. Thus one can apply results of Chapter 2 to determine diagonal plus semiseparable representation of a matrix. An algorithm for finding minimal generators of a semiseparable representation of a given matrix is presented.

§3.1 Definitions

Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a matrix with block entries of sizes $m_i \times n_j$. Denote by $\tilde{A}_L = \{A_{ij}\}_{1\le j<i\le N}$ the strictly lower triangular part of A. We treat $\tilde{A}_L$ as a given lower triangular part of a partially specified $(N-1)\times(N-1)$ block matrix $\mathcal{B} = \{B_{ij}\}_{i=2,j=1}^{N,N-1}$ with $B_{ij} = A_{ij}$ for $1 \le j < i \le N$. The minimal completion rank $\hat{r}_L$ of $\tilde{A}_L$ is called the *lower semiseparable order* of the matrix A.

Similarly denote by $\tilde{A}_U = \{A_{ij}\}_{1\le i<j\le N}$ the strictly upper triangular part of A. We treat $\tilde{A}_U$ as a given upper triangular part of a partially specified $(N-1)\times(N-1)$ block matrix $\mathcal{C} = \{C_{ij}\}_{i=1,j=2}^{N-1,N}$ with $B_{ij} = A_{ij}$ for $1 \le i < j \le N$. The minimal completion rank $\hat{r}_U$ of $\tilde{A}_U$ is called the *upper semiseparable order* of the matrix A.

We say also that A has *semiseparable order* $(\hat{r}_L, \hat{r}_U)$.

Let $S = \{S_{ij}\}_{i,j=1}^{N}$ be a matrix with block entries S_{ij} of sizes $m_i \times n_j$ and

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with zero diagonal. Assume that the entries of S are represented in the form

$$S_{ij} = \begin{cases} p(i)q(j), & 1 \le j < i \le N, \\ 0, & 1 \le i = j \le N, \\ g(i)h(j), & 1 \le i < j \le N. \end{cases} \tag{3.1}$$

Here $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ are matrices of sizes $m_i \times r_L$, $r_L \times n_j$, respectively, $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ are matrices of sizes $m_i \times r_U$, $r_U \times n_j$, respectively. The representation of the matrix S in the form (3.1) is called a *semiseparable representation.* The elements $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ and $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ are called *semiseparable generators* of the matrix S. The numbers r_L and r_U are called the *orders* of these generators.

The representation (3.1) means that if we introduce the matrices

$$P = \mathrm{col}(p(i))_{i=2}^{N}, \quad Q = \mathrm{row}(q(j))_{j=1}^{N-1}$$

of sizes $(\sum_{i=2}^{N} m_i) \times r_L, r_L \times (\sum_{j=1}^{N-1} n_j)$ and the matrices

$$G = \mathrm{col}(g(i))_{i=1}^{N-1}, \quad H = \mathrm{row}(h(j))_{j=2}^{N}$$

of sizes $(\sum_{i=1}^{N-1} m_i) \times r_U, r_U \times (\sum_{j=2}^{N} n_j)$ and we define the $(N-1) \times (N-1)$ block matrices $S_L = PQ$ and $S_U = GH$ of ranks r_L and r_U at most, then the relations

$$\mathrm{trils}(S) = \mathrm{tril}(S_L), \quad \mathrm{trius}(S) = \mathrm{triu}(S_U),$$

hold. Here $\mathrm{tril}(X), \mathrm{triu}(X)$ denote the lower triangular and upper triangular parts of a matrix X and $\mathrm{trils}(S), \mathrm{trius}(S)$ denote strictly lower triangular and strictly upper triangular parts of the matrix S. In other words, the strictly lower triangular and the strictly upper triangular parts of the matrix S may be completed to some matrices S_L and S_U with the ranks not greater than r_L and r_U, respectively.

Let A be a matrix with block entries A_{ij} of sizes $m_i \times n_j$ represented as a sum $A = D + S$ of a block diagonal matrix $D = \mathrm{diag}(d(i))_{i=1}^{N}$ with the entries $d(i)$ of sizes $m_i \times n_i$ and a matrix S with semiseparable representation (3.1). The entries of A are specified as follows:

$$A_{ij} = \begin{cases} p(i)q(j), & 1 \le j < i \le N, \\ d(i), & 1 \le i = j \le N, \\ g(i)h(j), & 1 \le i < j \le N. \end{cases} \tag{3.2}$$

The representation of the matrix A in the form (3.2) is called *diagonal plus semiseparable representation.*

The elements $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ and $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ are called lower and upper semiseparable generators of the matrix A. The most interesting case for us is when for a given matrix A the orders r_L and r_U are minimal. Lower and upper semiseparable generators of A with minimal orders are called *minimal semiseparable generators.*

§3.2 Semiseparable order and minimal semiseparable generators

Here we derive a formula to compute the lower semiseparable order of a matrix. We show that the lower semiseparable order is the minimal order of semiseparable generators and we derive an algorithm to compute a set of such generators.

Theorem 3.1. *Let A be a matrix with block entries of sizes $m_i \times n_j,\ i, j = 1, \dots, N$, with specified strictly lower triangular part $\tilde{A} = \{A_{ij}\}_{1 \le j < i \le N}$.*

Then the lower semiseparable order of A is given by the formula

$$\hat{r}_L = \sum_{k=1}^{N-1} \operatorname{rank} A(k+1:N, 1:k) - \sum_{k=1}^{N-2} \operatorname{rank} A(k+2:N, 1:k). \tag{3.3}$$

Proof. We treat the strictly lower triangular part $\tilde{A} = \{A_{ij}\}_{1 \le j < i \le N}$ of A as a given lower triangular part of a partially specified $(N-1) \times (N-1)$ block matrix $\mathcal{B} = \{B_{ij}\}_{i=2,j=1}^{N,N-1}$ with $B_{ij} = A_{ij}$ for $1 \le j < i \le N$. By Theorem 2.3 the minimal completion rank of $\tilde{A}$ is given by the formula (3.3). □

Theorem 3.2. *Let A be a matrix with block entries of sizes $m_i \times n_j,\ i, j = 1, \dots, N$, with specified strictly lower triangular part $\tilde{A} = \{A_{ij}\}_{1 \le j < i \le N}$.*

Then the lower semiseparable order $\hat{r}_L$ is the minimal order of lower semiseparable generators of the matrix A. Moreover, a set of lower semiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ of order $\hat{r}_L$ may be obtained via the following algorithm.

1.1. *Set* $A^{(1)} = A(2:N, 1),\ \operatorname{rank} A^{(1)} = \rho_1,\ \eta_2 = \sum_{i=2}^{N} m_i$.

Using the rank factorization of the matrix $A^{(1)}$ determine the matrices P_2, Q_1 of the sizes $\eta_2 \times \rho_1$ and $\rho_1 \times n_1$, respectively, such that

$$A^{(1)} = P_2 Q_1,$$

with

$$\operatorname{rank} P_2 = \operatorname{rank} Q_1 = \rho_1.$$

Set $r_1 = \rho_1$.

1.2. *For $k = 2, \dots, N-1$ perform the following.*

Set $\eta_{k+1} = \sum_{i=k+1}^{N} m_i,\ \lambda_{k+1} = \sum_{i=2}^{k} m_i,\ \nu_k = \sum_{i=1}^{k-1} n_i$. Determine the matrices $\tilde{P}_k, P'_{k+1}$ of sizes $\lambda_{k+1} \times r_{k-1}, \eta_{k+1} \times r_{k-1}$, respectively, from the partition

$$P_k = \begin{pmatrix} \tilde{P}_k \\ P'_{k+1} \end{pmatrix}.$$

Set

$$A^{(k)} = A(k+1:N, k), \quad \operatorname{rank} P'_{k+1} = s_k, \quad \operatorname{rank} \begin{pmatrix} P'_{k+1} & A^{(k)} \end{pmatrix} = \rho_k.$$

Using the rank canonical factorization of the matrix $\begin{pmatrix} P'_{k+1} & A^{(k)} \end{pmatrix}$ *determine the matrices* $\hat{P}_{k+1}, V_k$ *of the sizes* $\eta_{k+1} \times \rho_k, \rho_k \times (r_{k-1} + n_k)$*, respectively, such that*

$$\begin{pmatrix} P'_{k+1} & A^{(k)} \end{pmatrix} = \hat{P}_{k+1} V_k, \quad \operatorname{rank} \hat{P}_{k+1} = \operatorname{rank} V_k = \rho_k.$$

Determine the matrices Z_{k+1}, P''_{k+1} *of sizes* $\eta_{k+1} \times s_k, \eta_{k+1} \times (\rho_k - s_k)$ *and the matrices* a_k, v_k, q''_k *of the sizes* $s_k \times r_{k-1}, s_k \times n_k, (\rho_k - s_k) \times n_k$ *from the partitions*

$$\hat{P}_{k+1} = \begin{pmatrix} Z_{k+1} & P''_{k+1} \end{pmatrix}, \quad V_k = \begin{pmatrix} a_k & v_k \\ 0_{(\rho_k - s_k) \times r_{k-1}} & q''_k \end{pmatrix}.$$

The matrix a_k *has a right inverse* $\hat{a}_k$*. Compute* $q'_k = \hat{a}_k v_k$*, take* y_k *to be arbitrary* $\lambda_{k+1} \times (\rho_k - s_k)$ *matrix and set*

$$P_{k+1} = \begin{pmatrix} \tilde{P}_k & y_k \\ P'_{k+1} & P''_{k+1} \end{pmatrix}, \quad Q_k = \begin{pmatrix} Q_{k-1} & q'_k \\ 0_{(\rho_k - s_k) \times \nu_k} & q''_k \end{pmatrix}, \quad r_k = r_{k-1} + \rho_k - s_k.$$

1.3. *Set* $\hat{r}_L = r_{N-1},\ P = P_N,\ Q = Q_{N-1}$.

2. *Determine the matrices* $p(i),\ i = 2, \ldots, N$ *of sizes* $m_i \times \hat{r}_L$ *and the matrices* $q(j),\ j = 1, \ldots, N-1$ *of sizes* $\hat{r}_L \times n_j$ *from the partitions*

$$P = \operatorname{col}(p(i))_{i=2}^{N}, \quad Q = \operatorname{row}(q(j))_{j=1}^{N-1}.$$

Proof. By Theorem 2.3, a completion A_L of $\tilde{A}$ with the minimal rank $\hat{r}_L$ has the form $A_L = PQ$, where the matrices P, Q of sizes $\sum_{i=2}^{N} m_i \times \hat{r}_L, \hat{r}_L \times \sum_{i=1}^{N-1} n_i$ may be obtained via Step 1 of the algorithm.

Since $A_L = PQ$ is a completion of $\tilde{A}$, one obtains $A_{ij} = (A_L)_{i,j} = p(i)q(j)$, $1 \le j < i \le N$, i.e., $p(i)$ $(i = 2, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N-1)$ are lower semiseparable generators of the matrix A.

To prove that the semiseparable order $\hat{r}_L$ is the minimal order of lower semiseparable generators, assume that $p'(i)$ $(i = 2, \ldots, N)$, $q'(j)$ $(j = 1, \ldots, N-1)$ are lower semiseparable generators of A of some order τ'_L. We set $P' = \operatorname{col}(p'(i))_{i=2}^{N}$, $Q' = \operatorname{row}(q'(j))_{j=1}^{N-1}$ and then $A'_L = P'Q'$. The matrix A'_L is a completion of $\tilde{A}$. Moreover the rank ρ'_L of A'_L is not greater than τ'_L and since by Theorem 2.3 the number $\hat{r}_L$ is the minimal completion rank of $\tilde{A}$, we conclude that $\hat{r}_L \le \rho'_L \le \tau'_L$. □

The same relations are valid for the upper semiseparable order and minimal upper semiseparable generators of a matrix. More precisely, for a matrix A with a specified upper triangular part the upper semiseparable order is given by the formula

$$\hat{r}_U = \sum_{k=1}^{N-1} \operatorname{rank} A(1:k, k+1:N) - \sum_{k=1}^{N-2} \operatorname{rank} A(1:k, k+2:N).$$

Moreover, a set of minimal upper semiseparable generators may be obtained in much the same way as in Theorem 3.2.

Example 3.3. We consider the 5×5 matrix A with the specified strictly lower triangular part

$$A = \begin{pmatrix} ? & ? & ? & ? & ? \\ 1 & ? & ? & ? & ? \\ 1 & 2 & ? & ? & ? \\ 1 & 1 & 1 & ? & ? \\ 1 & 1 & 1 & 1 & ? \end{pmatrix}.$$

Using Example 2.6, one obtains a set of minimal lower semiseparable generators of the matrix A:

$$p(2) = \begin{pmatrix} 1 & 0 \end{pmatrix}, p(3) = \begin{pmatrix} 1 & 2 \end{pmatrix}, p(4) = \begin{pmatrix} 1 & 1 \end{pmatrix}, p(5) = \begin{pmatrix} 1 & 1 \end{pmatrix},$$
$$q(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad q(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad q(3) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad q(4) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Similarly, consider the 5×5 matrix B with the specified strictly upper triangular part

$$B = \begin{pmatrix} ? & 1 & 1 & 1 & 0 \\ ? & ? & 1 & 1 & 1 \\ ? & ? & ? & 1 & 1 \\ ? & ? & ? & ? & 1 \\ ? & ? & ? & ? & ? \end{pmatrix}.$$

Using Example 2.7, one obtains a set of minimal upper semiseparable generators of the matrix B:

$$g(1) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ g(2) = \begin{pmatrix} 0 & 1 \end{pmatrix}, g(3) = \begin{pmatrix} 0 & 1 \end{pmatrix}, \ g(4) = \begin{pmatrix} 0 & 1 \end{pmatrix},$$
$$h(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h(3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad h(4) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad h(5) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad \Diamond$$

Example 3.4. Consider the 5×5 matrix

$$A = \begin{pmatrix} d & 2 & 1 & 1 & 1 \\ 2 & d & 2 & 1 & 1 \\ 1 & 2 & d & 2 & 1 \\ 1 & 1 & 2 & d & 2 \\ 1 & 1 & 1 & 2 & d \end{pmatrix}.$$

Using Example 2.4, one obtains a set of minimal lower semiseparable generators of the matrix A:

$$p(2) = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix}, \ p(3) = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}, p(4) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \ p(5) = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix},$$
$$q(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad q(2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad q(3) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad q(4) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly, considering the 5×5 matrix B with the specified strictly upper triangular part from Example 2.8 one obtains a set of minimal upper semiseparable generators:

$$g(1) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, g(2) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, g(3) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, g(4) = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix},$$
$$h(2) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad h(3) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad h(4) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad h(5) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \qquad \diamondsuit$$

Example 3.5. Consider the 7×7 scalar Toeplitz Hermitian 5-band matrix

$$A = \begin{pmatrix} \alpha & \beta & \gamma & 0 & 0 & 0 & 0 \\ \beta & \alpha & \beta & \gamma & 0 & 0 & 0 \\ \gamma & \beta & \alpha & \beta & \gamma & 0 & 0 \\ 0 & \gamma & \beta & \alpha & \beta & \gamma & 0 \\ 0 & 0 & \gamma & \beta & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \gamma & \beta & \alpha & \beta \\ 0 & 0 & 0 & 0 & \gamma & \beta & \alpha \end{pmatrix}$$

where $\gamma \neq 0$, α and β are scalars.

Using Example 2.5, one obtains a set of minimal lower semiseparable generators of the matrix A:

$$p(2) = \begin{pmatrix} \beta & 0 & 0 & 0 & 0 \end{pmatrix}, \; p(3) = \begin{pmatrix} \gamma & \beta & 0 & 0 & 0 \end{pmatrix}, \; p(4) = \begin{pmatrix} 0 & \gamma & \beta & 0 & 0 \end{pmatrix},$$
$$p(5) = \begin{pmatrix} 0 & 0 & \gamma & \beta & 0 \end{pmatrix}, \; p(6) = \begin{pmatrix} 0 & 0 & 0 & \gamma & 0 \end{pmatrix}, \; p(7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$q(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; q(2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; q(3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$
$$q(4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; q(5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\beta}{\gamma} \\ \gamma \end{pmatrix}, \; q(6) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \beta \end{pmatrix}.$$

Similarly, considering the 7×7 matrix B with the specified strictly upper triangular part from Example 2.9, one obtains a set of minimal upper semiseparable generators

$$g(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \; g(2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad g(3) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$
$$g(4) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \; g(5) = \begin{pmatrix} 0 & 0 & 0 & \frac{\beta}{\gamma} & \gamma \end{pmatrix}, \; g(6) = \begin{pmatrix} 0 & 0 & 0 & 0 & \beta \end{pmatrix},$$

$$h(2) = \begin{pmatrix} \beta \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad h(3) = \begin{pmatrix} \gamma \\ \beta \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad h(4) = \begin{pmatrix} 0 \\ \gamma \\ \beta \\ 0 \\ 0 \end{pmatrix},$$

$$h(5) = \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \beta \\ 0 \end{pmatrix}, \qquad h(6) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \gamma \\ 0 \end{pmatrix}, \qquad h(7) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

◊

Example 3.6. Consider the companion matrix

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & -\alpha_{N-2} \\ 0 & 0 & \dots & 1 & -\alpha_{N-1} \end{pmatrix}.$$

Here one has $\hat{r}_L = N-1, \quad \hat{r}_U = 1$ and minimal rank completions of the strictly lower and strictly upper triangular parts are

$$A_L = I_{N-1}, \quad A_U = \begin{pmatrix} 0 & \dots & 0 & -\alpha_0 \\ 0 & \dots & 0 & -\alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -\alpha_{N-2} \end{pmatrix}.$$

Furthermore, lower and upper semiseparable generators of C are

$$p(i) = e_i,\ i = 2, \dots, N, \quad q(j) = e_j^t,\ j = 1, \dots, N-1,$$
$$g(i) = -\alpha_{i-1},\ i = 1, \dots, N-1, \quad h(j) = 0,\ j = 2, \dots, N-1,\ h(N) = 1.$$

Here e_i is the ith column vector of the standard basis in $\mathbb{C}^{N-1}$. ◊

§3.3 Comments

For the first time semiseparable representations of matrices were used for symmetric matrices in the monograph by F.R. Gantmacher and M.G. Krein [36]. The relationships between semiseparable representations and minimal rank completions are presented in detail for the first time in this chapter.

Chapter 4

Quasiseparable Representations: The Basics

This chapter is of introductory character. Here for any block matrix the lower and the upper rank numbers are defined as the ranks of the maximal (block) submatrices entirely located in the strictly lower triangular part and respectively in the strictly upper triangular part of the matrix. Using these ranks we define the quasiseparable orders of a matrix. These notions are illustrated on various examples. The chapter contains also the definition and examples of quasiseparable representations of matrices and their basic properties. The connection of rank numbers with the orders of quasiseparable representations is studied in detail in the next section.

§4.1 The rank numbers and quasiseparable order. Examples

§4.1.1 The definitions

Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a matrix with block entries. Consider the ranks of the maximal submatrices of A entirely located in the strictly lower triangular part and in the strictly upper triangular part:

$$\operatorname{rank} A(k+1:N, 1:k) = \rho_k^L, \quad k = 1, \dots, N-1 \tag{4.1}$$

and

$$\operatorname{rank} A(1:k, k+1:N) = \rho_k^U, \quad k = 1, \dots, N-1. \tag{4.2}$$

The numbers ρ_k^L $(k = 1, \dots, N-1)$ are called the *lower rank numbers* of the matrix A. The numbers ρ_k^U $(k = 1, \dots, N-1)$ are called the *upper rank numbers* of the matrix A.

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Let $A = \{A_{ij}\}_{i,j=1}^N$ be a matrix with block entries with lower rank numbers ρ_k^L $(k = 1, \dots, N-1)$ and upper rank numbers ρ_k^U $(k = 1, \dots, N-1)$. Set

$$\rho_L = \max_{1 \le k \le N-1} \rho_k^L, \quad \rho_U = \max_{1 \le k \le N-1} \rho_k^U.$$

We say that the matrix A has *lower quasiseparable order* ρ_L and *upper quasiseparable order* ρ_U. We say also that A has *quasiseparable order* (ρ_L, ρ_U).

§4.1.2 The companion matrix

For the monic polynomial $p(x) = x^N + \alpha_{N-1}x^{N-1} + \dots + \alpha_1 x + \alpha_0$, the *companion matrix* of $p(x)$ is defined to be

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & -\alpha_{N-2} \\ 0 & 0 & \dots & 1 & -\alpha_{N-1} \end{pmatrix}. \tag{4.3}$$

The polynomial $p(x)$ is the characteristic polynomial for the matrix C.

The maximal submatrices of C from the strictly lower triangular part are

$$C(k+1:N, 1:k) = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \quad k = 1, \dots, N-1$$

and therefore the lower rank numbers of the matrix C are equal to one. Further, the maximal submatrices of C from the strictly upper triangular part are

$$C(1:k, k+1:N) = \begin{pmatrix} 0 & \dots & -\alpha_0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -\alpha_{k-1} \end{pmatrix}, \quad k = 1, \dots, N-1$$

and therefore the upper rank numbers of C are not greater than one. Moreover, if the coefficient α_0 is not zero, then all upper rank numbers of the matrix C equal one.

The companion matrix C defined in (4.3) has lower quasiseparable order one and upper quasiseparable order at most one. If at least one of the numbers α_k, $k = 0, \dots, N-2$ is not equal to zero, C has quasiseparable order $(1, 1)$.

§4.1.3 The block companion matrix

For the monic matrix polynomial $p(x) = Ix^N + \alpha_{N-1}x^{N-1} + \dots + \alpha_1 x + \alpha_0$ with $n \times n$ matrix coefficients α_k $(k = 0, \dots, N-1)$, the *block companion matrix* of

$p(x)$ is defined similarly as above, i.e., via

$$C' = \begin{pmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ I_n & 0 & \dots & 0 & -\alpha_1 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & I_n & 0 & -\alpha_{N-2} \\ 0 & 0 & \dots & I_n & -\alpha_{N-1} \end{pmatrix}. \tag{4.4}$$

The polynomial $\det[p(x)]$ of degree Nn is the characteristic polynomial for the matrix C'.

For $k = 1, \dots, N-1$ one has

$$C'(k+1:N, 1:k) = \begin{pmatrix} 0 & \dots & I_n \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \quad C'(1:k, k+1:N) = \begin{pmatrix} 0 & \dots & -\alpha_0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -\alpha_{k-1} \end{pmatrix}.$$

Therefore the lower rank numbers of C' equal n and the upper rank numbers are not greater than n.

The block companion matrix C' defined in (4.4) has lower quasiseparable order n and upper quasiseparable order n at most.

§4.1.4 Tridiagonal matrices and band matrices

Consider a scalar matrix $A = \{A_{ij}\}_{i,j=1}^{N}$ with $A_{ij} = 0$ for $|i-j| > 1$. This is a tridiagonal scalar matrix. For $k = 1, \dots, N-1$ one has

$$\begin{aligned} A(k+1:N, 1:k) &= \begin{pmatrix} 0 & \dots & A_{k+1,k} \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \\ A(1:k, k+1:N) &= \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ A_{k,k+1} & \dots & 0 \end{pmatrix}. \end{aligned} \tag{4.5}$$

Hence it follows that the lower and upper rank numbers of this matrix are not greater than one. If A is irreducible, i.e., all subdiagonal and superdiagonal entries $A_{k+1,k}, A_{k,k+1}$ are not equal to zero, all lower and upper rank numbers of A are equal to one.

Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a block tridiagonal matrix with block entries A_{ij} of sizes $m_i \times n_j$, i.e., $A_{ij} = 0$ for $|i-j| > 1$. In this case for $k = 1, \dots, N-1$ the relations (4.5) hold with block entries. Since the blocks $A_{k+1,k}$ and $A_{k,k+1}$ have respective the sizes $m_{k+1} \times n_k$ and $m_k \times n_{k+1}$ one gets

$$\rho_k^L \le \min\{m_{k+1}, n_k\}, \quad \rho_k^U \le \min\{m_k, n_{k+1}\}, \quad k = 1, \dots, N-1.$$

Consider a (b_L, b_U)-band scalar matrix $B = \{B_{ij}\}_{i,j=1}^N$, i.e., $B_{ij} = 0$ for $i - j > b_L, j - i > b_U$. One has

$$B(k+1:N, 1:k) = \begin{pmatrix} 0 & \dots & B'_k \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \quad k = 1, \dots, N-1$$

with

$$B'_k = \begin{pmatrix} B_{k+1,1} & \dots & B_{k+1,k} \\ \vdots & \ddots & \vdots \\ B_{b_L+1,1} & \dots & B_{b_L+1,k} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{k+b_L,k} \end{pmatrix}, \qquad k = 1, \dots, b_L,$$

$$B'_k = \begin{pmatrix} B_{k+1,k-b_L+1} & \dots & B_{k+1,k} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{k+b_L,k} \end{pmatrix}, \quad k = b_L+1, \dots, N-b_L-1,$$

$$B'_k = \begin{pmatrix} B_{k+1,k-b_L+1} & \dots & B_{k+1,k} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{N,k} \end{pmatrix}, \quad k = N-b_L, \dots, N-1.$$

Since the sizes of the scalar matrices B'_k are not greater than b_L, the lower rank numbers of the matrix B are not greater than b_L. In the same way one obtains that the upper rank numbers of B are not greater than b_U. Similar relations hold for band matrices with block entries.

A tridiagonal scalar matrix $A = \{A_{ij}\}_{i,j=1}^N$ has quasiseparable order $(1,1)$ at most. If for some i and j with $(1 \leq i, j \leq N-1)$ one has $A_{i+1,i} \neq 0$ $A_{j,j+1} \neq 0$, then A has quasiseparable order $(1,1)$. A (b_L, b_U)-band scalar matrix $B = \{B_{ij}\}_{i,j=1}^N$ has quasiseparable order (b_L, b_U) at most.

§4.1.5 Matrices with diagonal plus semiseparable representations

Let A be a matrix with block entries A_{ij} of sizes $m_i \times n_j$ with the diagonal plus semiseparable representation

$$A_{ij} = \begin{cases} p(i)q(j), & 1 \leq j < i \leq N, \\ d(i), & 1 \leq i = j \leq N, \\ g(i)h(j), & 1 \leq i < j \leq N. \end{cases} \tag{4.6}$$

Here $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ are matrices of sizes $m_i \times r_L$, $r_L \times n_j$, respectively, and $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ are matrices of sizes $m_i \times r_U$, $r_U \times n_j$, respectively. The lower rank numbers of the matrix A of the form (4.6) are always not greater than r_L and the upper rank numbers are always not greater than r_U. Indeed, one has

$$A(k+1:N, 1:k) = P_{k+1}Q_k, \quad A(1:k, k+1:N) = G_k H_{k+1}, \quad k = 1, \dots, N-1$$

with $P_k = \mathrm{col}(p(i))_{i=k}^N, Q_k = \mathrm{row}(q(i))_{i=1}^k, G_k = \mathrm{col}(g(i))_{i=1}^k, H_k = \mathrm{row}(h(i))_{i=k}^N$. The matrices P_{k+1} and Q_k have r_L columns and rows, respectively, and so

$$\rho_k^L \leq r_L, \quad k = 1, \dots, N-1.$$

Similarly, the matrices G_k and H_{k+1} have r_U columns and rows respectively and so

$$\rho_k^U \leq r_U, \quad k = 1, \dots, N-1.$$

The matrix defined in (4.6) has quasiseparable order (r_L, r_U) at most.

§4.2 Quasiseparable generators

Here we define the main representation of matrices used in this book.

Let $\{a(k)\}$ be a family of matrices of sizes $r_k \times r_{k-1}$. For positive integers $i, j,\ i > j$ define the operation $a_{ij}^{>}$ as follows: $a_{ij}^{>} = a(i-1) \cdot \dots \cdot a(j+1)$ for $i > j+1$, $a_{j+1,j}^{>} = I_{r_j}$.

Let $\{b(k)\}$ be a family of matrices of sizes $r_{k-1} \times r_k$. For positive integers $i, j,\ j > i$ define the operation $b_{ij}^{<}$ as follows: $b_{ij}^{<} = b(i+1) \cdot \dots \cdot b(j-1)$ for $j > i+1$, $b_{i,i+1}^{<} = I_{r_i}$.

It is easy to see that

$$a_{ij}^{>} = a_{ik}^{>} a_{k+1,j}^{>}, \quad i > k \geq j \tag{4.7}$$

and

$$b_{ij}^{<} = b_{i,k+1}^{<} b_{k,j}^{<}, \quad i \leq k < j. \tag{4.8}$$

Let $A = \{A_{ij}\}_{i,j=1}^N$ be a matrix with block entries A_{ij} of sizes $m_i \times n_j$. Assume that the entries of A are represented in the form

$$A_{ij} = \begin{cases} p(i) a_{ij}^{>} q(j), & 1 \leq j < i \leq N, \\ d(i), & 1 \leq i = j \leq N, \\ g(i) b_{ij}^{<} h(j), & 1 \leq i < j \leq N. \end{cases} \tag{4.9}$$

Here $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ are matrices of sizes $m_i \times r_{i-1}^L$, $r_j^L \times n_j$, $r_k^L \times r_{k-1}^L$, respectively, $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ are matrices of sizes $m_i \times r_i^U$, $r_{j-1}^U \times n_j$, $r_{k-1}^U \times r_k^U$, respectively, $d(i)$ $(i = 1, \dots, N)$ are $m_i \times n_i$ matrices.

The representation of a matrix A in the form (4.9) is called a *quasiseparable representation.* The elements $p(i)$ $(i = 2, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N-1)$, $a(k)$ $(k = 2, \ldots, N-1)$; $g(i)$ $(i = 1, \ldots, N-1)$, $h(j)$ $(j = 2, \ldots, N)$, $b(k)$ $(k = 2, \ldots, N-1)$; $d(i)$ $(i = 1, \ldots, N)$ are called *quasiseparable generators* of the matrix A. The numbers r_k^L, r_k^U $(k = 1, \ldots, N-1)$ are called the *orders* of these generators. The elements $p(i)$ $(i = 2, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N-1)$, $a(k)$ $(k = 2, \ldots, N-1)$ and $g(i)$ $(i = 1, \ldots, N-1)$, $h(j)$ $(j = 2, \ldots, N)$, $b(k)$ $(k = 2, \ldots, N-1)$ are called also *lower quasiseparable generators* and *upper quasiseparable generators* of the matrix A. For scalar matrices the elements $d(i)$ are numbers and the generators $p(i), g(i)$ and $q(j), h(j)$ are rows and columns of the corresponding sizes.

Sometimes we define also lower and upper quasiseparable generators for the indices $k = 0$ and $k = N$. More precisely we define the numbers $r_0^L, r_N^L, r_0^U, r_N^U$ to be arbitrary nonnegative integers and $p(1), q(N), a(1), a(N), g(N), h(1), b(1), b(N)$ to be arbitrary matrices of sizes $m_1 \times r_0^L, r_N^L \times n_N, r_1^L \times r_0^L, r_N^L \times r_{N-1}^L, m_N \times r_N^U, r_0^U \times n_1, r_0^U \times r_1^U, r_{N-1}^U \times r_N^U$. In this case we use the name quasiseparable generators of orders r_k^L, r_k^U $(k = 0, \ldots, N)$ for the set $p(k), q(k), a(k); g(k), h(k), b(k);$ $k = 1, \ldots, N$ and the names lower quasiseparable generators and upper quasiseparable generators for the sets $p(k), q(k), a(k)$, $k = 1, \ldots, N$ and $g(k), h(k), b(k)$, $k = 1, \ldots, N$, respectively.

In the examples above one has the following.

1. For the companion matrix C defined in (4.3) quasiseparable generators may be taken in the form

$$\begin{aligned}
&p(i) = 1,\ i = 2, \ldots, N, && q(j) = 1, \quad j = 1, \ldots, N-1,\\
&a(k) = 0,\ k = 2, \ldots, N-1, && g(i) = -\alpha_{i-1},\ i = 1, \ldots, N-1,\\
&h(j) = 0,\ j = 2, \ldots, N-1, && h(N) = 1,\\
&b(k) = 1,\ k = 2, \ldots, N-1, &&\\
&d(i) = 0,\ i = 1, \ldots, N-1, && d(N) = -\alpha_{N-1}.
\end{aligned} \tag{4.10}$$

2. Similarly, for the block companion matrix C' defined in (4.4) one can take

$$\begin{aligned}
&p(i) = I_n,\ i = 2, \ldots, N, && q(j) = I_n, \quad j = 1, \ldots, N-1,\\
&a(k) = 0,\ \ k = 2, \ldots, N-1, && g(i) = -\alpha_{i-1},\ i = 1, \ldots, N-1,\\
&h(j) = 0,\ \ j = 2, \ldots, N-1, && h(N) = I_n,\\
&b(k) = I_n\ \ k = 2, \ldots, N-1, && d(i) = 0, \quad i = 1, \ldots, N-1,\ d(N) = -\alpha_{N-1}.
\end{aligned}$$

3. For a scalar tridiagonal matrix $A = \{A_{ij}\}_{i,j=1}^N$ one has

$$\begin{aligned}
&p(i) = 1, \quad i = 2, \ldots, N, && q(j) = A_{j+1,j},\ j = 1, \ldots, N-1,\\
&a(k) = b(k) = 0,\ k = 2, \ldots, N-1, &&\\
&g(i) = A_{i,i+1}, \quad i = 1, \ldots, N-1, && h(j) = 1, \quad j = 2, \ldots, N,\\
&d(k) = A_{kk}, \quad k = 1, \ldots, N.
\end{aligned} \tag{4.11}$$

Similarly,b for a block tridiagonal matrix with entries A_{ij} of sizes $m_i \times n_j$ one has

$$\begin{array}{llll} p(i) = I_{m_i}, & i = 2, \dots, N, & q(j) = A_{j+1,j}, & j = 1, \dots, N-1; \\ a(k) = 0_{m_{k+1}\times m_k}, & k = 2, \dots, N-1 & b(k) = 0_{n_k \times n_{k+1}}, & k = 2, \dots, N-1; \\ g(i) = A_{i,i+1}, & i = 1, \dots, N-1, & h(j) = I_{n_j}, & j = 2, \dots, N; \\ d(k) = A_{kk}, & k = 1, \dots, N. & & \end{array}$$

4. To determine quasiseparable generators of a (b_L, b_U)-band scalar matrix

$$B = \{B_{ij}\}_{i,j=1}^{N}$$

we use the shift $n \times n$ matrices

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and the n-dimensional rows $e_n = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$. The lower and upper quasiseparable generators of the matrix B may be taken in the form

$$p(i) = e_{b_L},\ i = 2, \dots, N, \quad q(j) = \begin{pmatrix} B_{j+1,j} \\ \vdots \\ B_{j+b_L,j} \end{pmatrix},\ j = 1, \dots, N-1;$$

$$a(k) = J_{b_L},\ k = 2, \dots, N-1;$$

$$g(i) = \begin{pmatrix} B_{i,i+1} & \dots & B_{i,i+b_U} \end{pmatrix},\ i = 1, \dots, N-1, \quad h(j) = e_{b_U}^T,\ j = 2, \dots, N;$$

$$b(k) = J_{b_U}^T,\ k = 2, \dots, N-1.$$

Here the entries B_{ij} for $i > N$ or $j > N$ are assumed to be zero.

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5. Let A be a matrix with the given diagonal plus semiseparable representation (4.6), with lower and upper semiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ and $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ of orders r_L and r_U, respectively. For A one gets the quasiseparable representation with lower and upper quasiseparable generators by setting

$$\begin{array}{lll} p(i),\ i = 2, \dots, N, & q(j),\ j = 1, \dots, N-1, & a(k) = I_{r_L},\ k = 2, \dots, N-1; \\ g(i),\ i = 1, \dots, N-1, & h(j),\ j = 2, \dots, N, & b(k) = I_{r_U},\ k = 2, \dots, N-1. \end{array}$$

§4.3 Minimal completion rank, rank numbers, and quasiseparable order

In this section we compare the minimal completion rank and the rank numbers of a matrix. For a matrix A the minimal completion rank $\hat{r}_L$ is greater than or equal to the lower quasiseparable order ρ_L. The difference between these two numbers may be essential, as one can see, for instance, for an irreducible tridiagonal matrix, i.e., for a tridiagonal matrix with all the entries non-zeros on the subdiagonal and the superdiagonal. Necessary and sufficient conditions for the equality of these two numbers and for the equality of rank numbers and the minimal completion rank are given as follows.

Theorem 4.1. *For a matrix A the inequality $\hat{r}_L \geq \rho_L$ holds and $\rho_L = \hat{r}_L$ if and only if*

$$\begin{aligned} \operatorname{rank} A(k+1:N, 1:k) &= \operatorname{rank} A(k+2:N, 1:k), \quad k = 1, \dots, k_0 - 1; \\ \operatorname{rank} A(k+1:N, 1:k-1) &= \operatorname{rank} A(k+1:N, 1:k), \quad k = k_0 + 1, \dots, N-1, \end{aligned} \tag{4.12}$$

where $k_0, 1 \leq k_0 \leq N-1$ is such that $\rho_L = \operatorname{rank} A(k_0+1 : N, 1 : k_0)$. Furthermore, the rank numbers ρ_k^L $(k = 1, \dots, N-1)$ of A are not greater than the minimal completion rank $\hat{r}_L$ and moreover

$$\rho_k^L = \hat{r}_L, \quad k = 1, \dots, N-1 \tag{4.13}$$

valid if and only if

$$\rho_k^L = \operatorname{rank} A(N, 1), \quad k = 1, \dots, N-1. \tag{4.14}$$

Proof. We write the equality (3.3) in the form

$$\begin{aligned} \hat{r}_L = &\operatorname{rank} A(k_0+1 : N, 1 : k_0) \\ &+ \sum_{k=1}^{k_0-1} [\operatorname{rank} A(k+1 : N, 1 : k) - \operatorname{rank} A(k+2 : N, 1 : k)] \\ &+ \sum_{k=k_0+1}^{N-1} [\operatorname{rank} A(k+1 : N, 1 : k) - \operatorname{rank} A(k+1 : N, 1 : k-1)]. \end{aligned}$$

Since

$$\begin{aligned} &\operatorname{rank} A(k+1 : N, 1 : k) \geq \operatorname{rank} A(k+2 : N, 1 : k), \\ &\operatorname{rank} A(k+1 : N, 1 : k) \geq \operatorname{rank} A(k+1 : N, 1 : k-1) \end{aligned}$$

and $\rho_L = \operatorname{rank} A(k_0 + 1 : N, 1 : k_0)$, we conclude that $\hat{r}_L \geq \rho_L$ and moreover $\hat{r}_L = \rho_L$ if and only if the condition (4.12) holds.

From the inequality $\hat{r}_L \geq \rho_L$ and the fact that ρ_L is the maximal lower rank number of the matrix A we conclude that

$$\rho_k^L \leq \hat{r}_L, \quad k = 1, \ldots, N-1.$$

Let the condition (4.14) hold. Using the inequalities

$$\begin{aligned}\operatorname{rank} A(N,1) &\leq \operatorname{rank} A(k+1:N, 1:k-1)\\ &\leq \operatorname{rank} A(k+1:N, 1:k) = \rho_k^L, \quad k = 2, \ldots, N-1,\end{aligned}$$

we conclude that

$$\operatorname{rank} A(k+1:N, 1:k-1) = \operatorname{rank} A(k+1:N, 1:k), \quad k = 2, \ldots, N-1.$$

Hence, using (3.3) one obtains $\hat{r}_L = \operatorname{rank} A(2:N,1) = \rho_1^L$ and using (4.14) again one obtains (4.13)

Assume that the condition (4.13) holds. Recall that the number $\hat{r}_L$ is the minimal completion rank of the strictly lower triangular part $\tilde{A} = \{A_{ij}\}_{1 \leq j < i \leq N}$ of the matrix A. Consider the partially specified matrix

$$\mathcal{A}_0 = \begin{pmatrix} A(2:N-1,1) & ? \\ A(N,1) & A(N, 2:N-1) \end{pmatrix}.$$

Let r_0 be the minimal completion rank of $\mathcal{A}_0$. Since the given part of $\mathcal{A}_0$ is a submatrix of $\tilde{A}$, one gets $r_0 \leq \hat{r}_L$. Formula (2.2) yields

$$r_0 = \operatorname{rank} A(2:N,1) + \operatorname{rank} A(N, 1:N-1) - \operatorname{rank} A(N,1).$$

Now if $\operatorname{rank} A(N,1) < \hat{r}_L$ one gets $r_0 > \hat{r}_L$, a contradiction. □

§4.4 The quasiseparable and semiseparable generators

Let A be a matrix with lower semiseparable generators $p(i)$ $(i = 2, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N-1)$ of order r_L. As mentioned above, A has lower quasiseparable generators

$$p(i),\ i = 2, \ldots, N, \quad q(j),\ j = 1, \ldots, N-1, \quad a(k) = I_{r_L},\ k = 2, \ldots, N-1$$

of orders $r_k^L = r_L$ $(k = 1, \ldots, N-1)$.

Under some conditions the converse statement holds.

Theorem 4.2. *Let A be a block matrix with lower quasiseparable generators $p(i)$ $(i = 2, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N-1)$, $a(k)$ $(k = 2, \ldots, N-1)$ such that the orders of these generators are constant $r_k^L = r_0$ $(k = 1, \ldots, N-1)$ and the $r_0 \times r_0$ matrices $a(k)$ $(k = 2, \ldots, N-1)$ are invertible.*

Then the matrix A has lower semiseparable generators $\tilde{p}(i)$ $(i = 2, \ldots, N)$, $\tilde{q}(j)$ $(j = 1, \ldots, N-1)$ of order r_0 which are given by the formulas

$$\tilde{p}(i) = p(i) a_{i1}^{>},\ i = 2, \ldots, N, \quad \tilde{q}(j) = (a_{j+1,1}^{>})^{-1} q(j),\ j = 1, \ldots, N-1. \quad (4.15)$$

Proof. One has the quasiseparable representation

$$A_{ij} = p(i)a^{>}_{ij}q(j), \quad 1 \leq j < i \leq N.$$

For any i, j with $1 \leq j < i \leq N$, using the formula $a^{>}_{i1} = a^{>}_{ij}a^{>}_{j+1,1}$ and the fact that the matrix $a^{>}_{j+1,1}$ is invertible one gets $a^{>}_{ij} = a^{>}_{i1}(a^{>}_{j+1,1})^{-1}$ and therefore

$$A_{ij} = (p(i)a^{>}_{i1})((a^{>}_{j+1,1})^{-1}q(j)). \tag{4.16}$$

Define the matrices $\tilde{p}(i)$ $(i = 2, \dots, N)$, $\tilde{q}(j)$ $(j = 1, \dots, N-1)$ by (4.15). Every matrix $\tilde{p}(i)$ has r_0 columns and every matrix $\tilde{q}(j)$ has r_0 rows. Moreover using (4.16) one gets

$$A_{ij} = \tilde{p}(i)\tilde{q}(j), \quad 1 \leq j < i \leq N$$

and therefore $\tilde{p}(i)$ $(i = 2, \dots, N)$, $\tilde{q}(j)$ $(j = 1, \dots, N-1)$ are lower semiseparable generators of order r_0 of the matrix A. □

In the same way one obtains the corresponding statement for upper generators.

Theorem 4.3. *Let A be a block matrix with lower quasiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ such that the orders of these generators are constant $r^U_k = r_0$ $(k = 1, \dots, N-1)$ and the $r_0 \times r_0$ matrices $b(k)$ $(k = 2, \dots, N-1)$ are invertible.*

Then the matrix A has lower semiseparable generators $\tilde{g}(i)$ $(i = 1, \dots, N-1)$, $\tilde{h}(j)$ $(j = 2, \dots, N)$ of order r_0 which are given by the formulas

$$\tilde{g}(i) = g(i)(b^{<}_{1,i+1})^{-1}, \ i = 1, \dots, N-1, \quad \tilde{h}(j) = b^{<}_{1j}h(j), \ j = 2, \dots, N. \tag{4.17}$$

§4.5 Comments

The rank numbers of a matrix have been defined and used in by M. Fiedler and T.L. Markham in [30], [29], see also a note by E. Tyrtyshnikov [44]. The quasiseparable representations were used in the paper [38] and in the monograph by P.M. Dewilde and A.J. van der Veen [15] in the study of discrete time-varying systems. The connection between quasiseparable and semiseparable representations was discussed by R. Vandebril, M. Van Barel and N. Mastronardi in the note [45] and in the book [46].

The material of this chapter is taken mainly from [20] and [27].

Chapter 5

Quasiseparable Generators

It is clear from the preceding chapter that any matrix has quasiseparable representations. By padding given quasiseparable generators with zero matrices of large sizes one can arrange that they have arbitrarily large orders. However, one is looking for generators of minimal orders, because they will give better computational complexity in applications.

This chapter defines minimality for quasiseparable generators in a natural way. Algorithms to build them from a given triangular part or from a general set of generators are given. A notion of similarity for two sets of generators of the same matrix readily emerges and it turns out that all minimal generators are similar. Also, notions of normality for sets of generators are defined and minimal and general normal generators are computed.

Approximations of a given matrix by another matrix with predefined small quasiseparable order conclude this chapter.

To be more precise, the first section has a preliminary character; in it, using quasiseparable generators of a matrix we define some auxiliary matrices which are employed in the sequel. In the second section it is shown that the minimal orders of the quasiseparable generators coincide with the corresponding rank numbers of the matrix. An algorithm to build a set of minimal lower quasiseparable generators of a matrix with the strictly lower triangular part given is also presented. As a corollary one obtains that the maximal orders of minimal quasiseparable generators coincide with the quasiseparable orders of a matrix. The third section contains some examples of computing of quasiseparable generators. A special example of the block companion matrix is considered in the fourth section. In Section 5 a criterion for minimality of generators is proved and some examples show its usefulness. In Section 6 we consider relations between different sets of generators of the same matrix. Here we define a notion of similarity for candidate quasiseparable generator sets of the same matrix. It turns out that sets of matrices similar to a given set of generators are also quasiseparable generators. Also, any two sets of minimal quasiseparable generators of the same matrix are similar. A counterexample shows that non-minimal generators are not always similar. The seventh section computes

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a set of minimal generators from a set of given presumably non-minimal generators. In Section 8 we consider special quasiseparable generators satisfying some orthonormality conditions. The last section gives the approximations of a matrix by a quasiseparable matrix with orders not exceeding a specified maximum, or with a matrix for which the middle (diagonal) factor in the SVD decomposition has only entries which are larger than a specified tolerance.

§5.1 Auxiliary matrices and relations

In this section we derive some connections between a quasiseparable representation of a matrix and the structure of its submatrices.

Let $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ be matrices of sizes $m_i \times r^L_{i-1}$, $r^L_j \times n_j$, $r^L_k \times r^L_{k-1}$, respectively. We define the matrices Q_k $(k = 1, \dots, N-1)$ of sizes $r^L_k \times \sum_{j=1}^k n_j$ and the matrices P_k $(k = N, \dots, 2)$ of sizes $\sum_{i=k}^N m_i \times r^L_{k-1}$ via the relations

$$Q_k = \mathrm{row}(a^{>}_{k+1,i} q(i))_{i=1}^k, \quad k = 1, \dots, N-1; \tag{5.1}$$

$$P_k = \mathrm{col}(p(i) a^{>}_{i,k-1})_{i=k}^N, \quad k = N, \dots, 2. \tag{5.2}$$

One can check directly that the matrices P_k, Q_k satisfy the recursion relations

$$Q_1 = q(1), \quad Q_k = \begin{pmatrix} a(k)Q_{k-1} & q(k) \end{pmatrix}, \quad k = 2, \dots, N-1; \tag{5.3}$$

$$P_N = p(N), \quad P_k = \begin{pmatrix} p(k) \\ P_{k+1} a(k) \end{pmatrix}, \quad k = N-1, \dots, 2. \tag{5.4}$$

Similarly, let $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ be matrices of sizes $m_i \times r^U_i$, $r^U_{j-1} \times n_j$, $r^U_{k-1} \times r^U_k$, respectively. We define the matrices G_k $(k = 1, \dots, N-1)$ of sizes $\sum_{j=1}^k m_j \times r^U_k$ and the matrices H_k $(k = N, \dots, 2)$ of sizes $r^U_{k-1} \times \sum_{i=k}^N n_i$ via the relations

$$G_k = \mathrm{col}(g(i) b^{<}_{i,k+1})_{i=1}^k, \quad k = 1, \dots, N-1; \tag{5.5}$$

$$H_k = \mathrm{row}(b^{<}_{k-1,i} h(i))_{i=k}^N, \quad k = N, \dots, 2. \tag{5.6}$$

One can check directly that the matrices G_k, H_k satisfy the recursion relations

$$G_1 = g(1), \quad G_k = \begin{pmatrix} G_{k-1} b(k) \\ g(k) \end{pmatrix}, \quad k = 2, \dots, N-1; \tag{5.7}$$

$$H_N = h(N), \quad H_k = \begin{pmatrix} h(k) & b(k) H_{k+1} \end{pmatrix}, \quad k = N-1, \dots, 2. \tag{5.8}$$

The following relations for the corresponding submatrices of a quasiseparable matrix follow directly from the definition (4.9).

Lemma 5.1. *Let* $A = \{A_{ij}\}_{i,j=1}^{N}$ *be a matrix with lower quasiseparable generators* $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ *of orders* r_k^L $(k = 1, \dots, N-1)$. *Using these generators define the matrices* Q_k $(k = 1, \dots, N-1)$, P_k $(k = N, \dots, 2)$ *via the formulas* (5.1), (5.2).

Then the equalities

$$A(k+1:N, 1:k) = P_{k+1}Q_k, \quad k = 1, \dots, N-1, \tag{5.9}$$

hold.

Proof. The first formula from (4.9) yields

$$A(k+1:N, 1:k) = \begin{pmatrix} p(k+1)a^{>}_{k+1,1}q(1) & \dots & p(k+1)a^{>}_{k+1,k}q(k) \\ \vdots & \ddots & \vdots \\ p(N)a^{>}_{N,1}q(1) & \dots & p(N)a^{>}_{N,k}q(k) \end{pmatrix},$$
$$k = 1, \dots, N-1.$$

Furthermore using the equalities (4.7) one obtains

$$A(k+1:N, 1:k) = \begin{pmatrix} p(k+1) \\ p(k+2)a^{>}_{k+2,k} \\ \vdots \\ p(N)a^{>}_{N,k} \end{pmatrix} \cdot \begin{pmatrix} a^{>}_{k+1,1}q(1) & \dots & a^{>}_{k+1,k-1}q(k-1) & q(k) \end{pmatrix} = P_{k+1}Q_k.$$

□

Corollary 5.2. *Under the conditions of Lemma* 5.1, *the equalities*

$$A(k+1:N, k) = P_{k+1}q(k), \qquad k = 1, \dots, N-1, \tag{5.10}$$

and

$$A(k+1, 1:k) = p(k+1)Q_k, \quad k = 1, \dots, N-1, \tag{5.11}$$

hold.

Proof. Using (5.3) we see that $Q_k(:, k) = q(k)$ and therefore using (5.9) we obtain

$$A(k+1:N, k) = P_{k+1}q(k), \quad k = 1, \dots, N-1.$$

Similarly using (5.9) and (5.4) one obtains (5.11). □

Conversely, it is easy to check that the relations (5.10) or (5.11) define a quasiseparable representation of the strictly lower triangular part of a matrix.

Lemma 5.3. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be a block matrix. Consider the relations* (5.10) *or the relations* (5.11)*, where* P_k $(k = 2, \dots, N)$ *and* Q_k $(k = 1, \dots, N-1)$ *are matrices defined via the relations* (5.2) *and* (5.1) *with some matrices* $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$.

Then $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ *are lower quasiseparable generators of the matrix* A.

Proof. Let the equalities (5.10) hold. Using (5.2) one obtains

$$A(k+1:N,k) = P_{k+1}q(k) = \mathrm{col}(p(i)a^{>}_{i,k})_{i=k+1}^N q(k), \quad k = 1, \dots, N-1$$

which implies the first formula from (4.9).

Now let the equalities (5.11) hold. Using (5.1) one obtains

$$A(k,1:k-1) = p(k)Q_{k-1} = p(k)\mathrm{row}(a^{>}_{kj}q(j))_{j=1}^{k-1}, \quad k = 2, \dots, N$$

from which the first formula from (4.9) follows. □

Similarly one can prove the following assertions concerning the strictly upper triangular part of the matrix A.

Lemma 5.4. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be a matrix with upper quasiseparable generators* $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ *of orders* r_k^U $(k = 1, \dots, N-1)$*. Using these generators define the matrices* G_k $(k = 1, \dots, N-1)$, H_k $(k = N, \dots, 2)$ *via* (5.5), (5.6).

Then the equalities

$$A(1:k, k+1:N) = G_k H_{k+1}, \quad k = 1, \dots, N-1, \tag{5.12}$$

hold.

Corollary 5.5. *Under the conditions of Lemma* 5.4*, the equalities*

$$A(k, k+1:N) = g(k)H_{k+1}, \qquad k = 1, \dots, N-1 \tag{5.13}$$

and

$$A(1:k, k+1) = G_k h(k+1), \quad k = 1, \dots, N-1 \tag{5.14}$$

hold.

Lemma 5.6. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be a block matrix. Let the relations* (5.13) *or the relations* (5.14)*, where* H_k $(k = 2, \dots, N)$ *and* G_k $(k = 1, \dots, N-1)$ *are matrices defined via the relations* (5.6) *and* (5.5) *with some matrices* $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$*, hold.*

Then $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ *are upper quasiseparable generators of the matrix* A.

§5.2 Existence and minimality of quasiseparable generators

Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$. The matrices $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ defined by

$$p'(i) = \begin{pmatrix} p(i) & 0 \end{pmatrix}, \quad q'(j) = \begin{pmatrix} q(j) \\ 0 \end{pmatrix}, \quad a'(k) = \begin{pmatrix} a(k) & 0 \\ 0 & 0 \end{pmatrix},$$

with zeros of the corresponding sizes, are also a set of lower quasiseparable generators of the same matrix A. Indeed one can check easily that

$$p(i)a^{>}_{ij}q(j) = p'(i)(a')^{>}_{ij}q'(j), \quad 1 \le j < i \le N.$$

We see that the orders of quasiseparable generators of a matrix can be arbitrarily large. We will be interested in generators of minimal orders.

Definiton 5.7. Let $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ be lower quasiseparable generators of a matrix A with the orders r_k $(k = 1, \dots, N-1)$. These generators are called minimal if for any other set of lower quasiseparable generators of A with orders t_k $(k = 1, \dots, N-1)$ the inequalities

$$r_k \le t_k, \quad k = 1, \dots, N-1$$

hold. The orders r_k $(k = 1, \dots, N-1)$ are called the minimal orders of lower quasiseparable generators of the matrix A.

At first we show that the orders of quasiseparable generators are not smaller than the corresponding rank numbers.

Lemma 5.8. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with the lower rank numbers ρ_k^L $(k = 1, \dots, N-1)$ and let t_k $(k = 1, \dots, N-1)$ be the orders of some lower quasiseparable generators of the matrix A.*

Then

$$\rho_k^L \le t_k, \quad k = 1, \dots, N-1.$$

Proof. Assume that $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the matrix A with the orders t_k $(k = 1, \dots, N-1)$. Define the matrices P'_k $(k = 2, \dots, N)$, Q'_k $(k = 1, \dots, N-1)$ by the formulas (5.1), (5.2). By Lemma 5.1,

$$A(k+1 : N, 1 : k) = P'_{k+1}Q'_k, \quad k = 1, \dots, N-1.$$

Since the number of the columns of the matrix P'_{k+1} and that of the rows in the matrix Q'_k are both equal to t_k, one obtains

$$\rho_k^L = \operatorname{rank} A(k+1 : N, 1 : k) \le t_k, \quad k = 1, \dots, N-1. \qquad \square$$

Now we give an algorithm to compute a set of minimal quasiseparable generators of a matrix with a given strictly lower triangular part.

Theorem 5.9. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be a block matrix with lower rank numbers* ρ_k^L $(k = 1, \dots, N-1)$.

Then the matrix A *has lower quasiseparable generators with orders* ρ_k^L $(k = 1, \dots, N-1)$. *Moreover, a set* $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ *of lower quasiseparable generators of the matrix* A *with these orders is obtained by means of the following algorithm.*

1. *Set* $A^{(1)} = A(2:N, 1)$. *Using rank factorization of the matrix* $A^{(1)}$ *determine the matrices* $P_2, q(1)$ *of the sizes* $\left(\sum_{i=2}^N m_i\right) \times r_1$, $r_1 \times n_1$, *respectively such that*

$$A^{(1)} = P_2 q(1) \tag{5.15}$$

with

$$\operatorname{rank} P_2 = \operatorname{rank} q(1) = \operatorname{rank} A^{(1)} = r_1. \tag{5.16}$$

2. *For* $k = 2, \dots, N-1$ *perform the following.*

Set $A^{(k)} = A(k+1:N, k)$.

Determine the matrices $p(k), P_k''$ *of sizes* $m_k \times r_{k-1}$, $\left(\sum_{i=k+1}^N m_i\right) \times r_{k-1}$, *respectively, from the partition*

$$P_k = \begin{pmatrix} p(k) \\ P_k'' \end{pmatrix}. \tag{5.17}$$

Using the rank factorization of the matrix $\begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix}$, *determine the number* r_k *and the matrices* P_{k+1}, V_k *of sizes* $\left(\sum_{i=k+1}^N m_i\right) \times r_k, r_k \times (r_{k-1} + n_k)$, *respectively, such that*

$$\begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix} = P_{k+1} V_k, \tag{5.18}$$

$$\operatorname{rank} P_{k+1} = \operatorname{rank} V_k = \operatorname{rank} \begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix} = r_k. \tag{5.19}$$

Determine the matrices $a(k), q(k)$ *of sizes* $r_k \times r_{k-1}, r_k \times n_k$, *respectively, from the partition*

$$V_k = \begin{pmatrix} a(k) & q(k) \end{pmatrix}. \tag{5.20}$$

3. *Set*

$$p(N) = P_N. \tag{5.21}$$

Proof. Consider the matrices P_k $(k = 2, \dots, N-1)$ defined in the algorithm. Comparing the corresponding entries in (5.18) and (5.20) one gets

$$P_k'' = P_{k+1} a(k), \quad k = 2, \dots, N-1.$$

Hence from the relations (5.21) and (5.17) it follows that the matrices P_k $(k = 2,\dots,N-1)$ satisfy the recursion (5.4). Furthermore, using (5.15), (5.18), (5.20) one gets

$$A(k+1:N,k) = P_{k+1}q(k), \quad k = 1,\dots,N-1.$$

Thus, by Lemma 5.3, $p(i)$ $(i = 2,\dots,N)$, $q(j)$ $(j = 1,\dots,N-1)$, $a(k)$ $(k = 2,\dots,N-1)$ are lower quasiseparable generators of the matrix A.

Next we prove that the orders r_k $(k = 1,\dots,N-1)$ are equal to the corresponding rank numbers ρ_k^L $(k = 1,\dots,N-1)$. Define the matrices Q_k $(k = 1,\dots,N-1)$ via the relations (5.3). By Lemma 5.1,

$$A(k+1:N,1:k) = P_{k+1}Q_k, \quad k = 1,\dots,N-1.$$

We should check that

$$\operatorname{rank} P_{k+1} = \operatorname{rank} Q_k = r_k, \quad k = 1,\dots,N-1.$$

This means that the matrices P_{k+1}, Q_k have full column rank and row rank, respectively, and therefore

$$\rho_k^L = r_k, \quad k = 1,\dots,N-1.$$

The relations (5.16), (5.19) imply that the conditions

$$\operatorname{rank} P_{k+1} = r_k, \quad k = 1,\dots,N-1$$

hold. Next, using the relation (5.15) one gets $\operatorname{rank}(q(1)) = r_1$. Assume that for some k with $N-1 \geq k \geq 2$ the relation $\operatorname{rank} Q_{k-1} = r_{k-1}$ holds. Using (5.3) one has

$$Q_k = \begin{pmatrix} a(k) & q(k) \end{pmatrix} \begin{pmatrix} Q_{k-1} & 0 \\ 0 & I \end{pmatrix}. \tag{5.22}$$

The matrix $\begin{pmatrix} Q_{k-1} & 0 \\ 0 & I \end{pmatrix}$ has full row rank. Moreover using (5.18) one gets

$$\operatorname{rank} \begin{pmatrix} a(k) & q(k) \end{pmatrix} = r_k$$

and furthermore, by using (5.22), one obtains

$$\operatorname{rank} Q_k = \operatorname{rank} \begin{pmatrix} a(k) & q(k) \end{pmatrix} = r_k. \qquad \square$$

In order to compute the complexity of the algorithm from Theorem 5.9 one has to compute only the number of operations in (5.15), (5.18). Every such factorization costs $O(N)$ operations and hence the total complexity of the algorithm is $O(N^2)$.

Corollary 5.10. *The lower rank numbers of a matrix are equal to the minimal orders of its lower quasiseparable generators. Moreover, a set of minimal lower generators can be obtained by means of the algorithm from Theorem* 5.9.

Corollary 5.11. *Let A be a matrix with quasiseparable order (ρ_L, ρ_U).*

The minimal orders r_k^L $(k = 1, \dots, N-1)$ and r_k^U $(k = 1, \dots, N-1)$ of lower and upper quasiseparable generators of A satisfy the relations

$$\max_{1 \le k \le N-1} r_k^L = \rho_L, \qquad \max_{1 \le k \le N-1} r_k^U = \rho_U,$$

i.e., the maximal orders of minimal quasiseparable generators of a matrix are equal to its quasiseparable orders.

§5.3 Examples

Example 5.12. We compute lower quasiseparable generators for the 6×6 scalar matrix

$$A = \begin{pmatrix} * & * & * & * & * & * \\ 1 & * & * & * & * & * \\ 1 & 2 & * & * & * & * \\ 1 & 1 & 1 & * & * & * \\ 1 & 1 & 1 & 2 & * & * \\ 1 & 1 & 1 & 1 & 1 & * \end{pmatrix}.$$

By formula (5.15), one obtains

$$P_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad q(1) = 1, \quad r_1 = 1.$$

Next, for $k = 2$ formula (5.17) yields $p(2) = 1$ and, moreover, using formula (5.18) one gets

$$\begin{pmatrix} P_2'' & A^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = P_3 V_2$$

with

$$P_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_2 = 2.$$

Using the partition (5.20) one obtains $a(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For $k = 3$ formula (5.17) yields $p(3) = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and, moreover, using formula (5.18) one gets

$$\begin{pmatrix} P''_3 & A^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = P_4 V_3$$

with

$$P_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad r_3 = 1.$$

Using the partition (5.20) one obtains $a(3) = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $q(3) = 1$.

For $k = 4$ formula (5.17) yields $p(4) = 1$ and, moreover, using formula (5.18) one gets

$$\begin{pmatrix} P''_4 & A^{(4)} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = P_5 V_4$$

with

$$P_5 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_4 = 2.$$

Using the partition (5.20) one obtains $a(4) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q(4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For $k = 5$ formula (5.17) yields $p(5) = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and, moreover, using formula (5.18) one gets

$$\begin{pmatrix} P''_5 & A^{(5)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = P_6 V_5$$

with

$$P_6 = 1, \quad V_5 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad r_5 = 1.$$

Using the partition (5.20) one obtains $a(5) = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $q(5) = 1$.

Finally, by formula (5.21), one gets $p(6) = P_6 = 1$.

Thus one obtains a set of minimal lower quasiseparable generators of the matrix A:

$$\begin{array}{lllll} p(2) = 1, & p(3) = \begin{pmatrix} 1 & 2 \end{pmatrix}, & p(4) = 1, & p(5) = \begin{pmatrix} 1 & 2 \end{pmatrix}, & p(6) = 1, \\ q(1) = 1, & q(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & q(3) = 1, & q(4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & q(5) = 1, \end{array}$$

$$a(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, a(3) = \begin{pmatrix} 1 & 1 \end{pmatrix}, a(4) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, a(5) = \begin{pmatrix} 1 & 1 \end{pmatrix}. \qquad \Diamond$$

Example 5.13. Consider the $N \times N$ matrix

$$A = \begin{pmatrix} d & 1 & 2 & 3 & \cdots & N-2 & N-1 \\ 1 & d & 4 & 6 & \cdots & 2(N-2) & 2(N-1) \\ 2 & 4 & d & 9 & \cdots & 3(N-2) & 3(N-1) \\ 3 & 6 & 9 & d & \cdots & 4(N-2) & 4(N-1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ N-2 & 2(N-2) & 3(N-2) & 4(N-2) & \cdots & d & (N-1)^2 \\ N-1 & 2(N-1) & 3(N-1) & 4(N-1) & \cdots & (N-1)^2 & d \end{pmatrix}.$$

A set $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of lower quasiseparable generators of the matrix A with order one is obtained via the following algorithm.

Step 1. Set $A^{(1)} = A(2:N, 1) = \begin{pmatrix} 1 & 2 & 3 & \cdots & N-2 & N-1 \end{pmatrix}^T$. Using rank factorization of the matrix $A^{(1)}$ determine the matrices $P_2, q(1)$ of the sizes $\left(\sum_{i=2}^N m_i\right) \times r_1 = (N-1) \times 1$, $r_1 \times n_1 = 1 \times 1$, respectively, such that (5.15),(5.16) hold. Indeed, take $P_2 = A^{(1)}$, $q(1) = 1$.

Step 2. For $k = 2, \dots, N-1$ perform the following.

We have

$$A^{(k)} = A(k+1:N, k) = \begin{pmatrix} k^2 & k(k+1) & \cdots & k(N-2) & k(N-1) \end{pmatrix}^T.$$

Determine the matrices $p(k), P''_k$ of sizes 1×1, $(N-k) \times 1$, respectively, from the partition (5.17), namely

$$P_k = \begin{pmatrix} p(k) \\ P''_k \end{pmatrix}.$$

It follows that $p(k) = k-1, P''_k = \begin{pmatrix} k & k+1 & \cdots & N-2 & N-1 \end{pmatrix}^T$.

Using rank factorization of the matrix $\begin{pmatrix} P''_k & A^{(k)} \end{pmatrix}$ determine the matrices P_{k+1}, V_k such that (5.18) and (5.19) hold. For instance,

$$P_{k+1} = \begin{pmatrix} k & k+1 & \cdots & N-2 & N-1 \end{pmatrix}^T, \quad V_k = \begin{pmatrix} 1 & k \end{pmatrix}, \quad r_k = 1.$$

Determine the matrices $a(k), q(k)$ from the partition (5.20), namely

$$V_k = \begin{pmatrix} a(k) & q(k) \end{pmatrix}.$$

This means that $a(k) = 1,\ q(k) = k$.

Finally, $p(N) = P_N = N-1$.

In this way, we obtained the following lower quasiseparable generators:

$$p(i) = i-1,\ i = 2, \dots, N, \quad q(j) = j,\ j = 1, \dots, N-1,$$
$$a(k) = 1,\ k = 2, \dots, N-1. \qquad \diamondsuit$$

Example 5.14. Consider the $N \times N$ matrix

$$A = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{N-2} & a^{N-1} \\ b & 1 & a & \cdots & a^{N-3} & a^{N-2} \\ b^2 & b & 1 & \cdots & a^{N-4} & a^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b^{N-2} & b^{N-3} & b^{N-4} & \cdots & 1 & a \\ b^{N-1} & b^{N-2} & b^{N-3} & \cdots & b & 1 \end{pmatrix}.$$

The algorithm yields the quasiseparable generators

$$\begin{aligned} p(i) &= b,\ i = 2, \dots, N, & q(j) &= 1,\ j = 1, \dots, N-1, \\ a(k) &= b,\ k = 2, \dots, N-1, & g(i) &= 1,\ i = 1, \dots, N-1, \\ h(j) &= a,\ j = 2, \dots, N, & b(k) &= a,\ k = 2, \dots, N-1, \\ d(k) &= 1,\ k = 1, \dots, N. \end{aligned}$$

◊

Example 5.15. Consider the 7×7 scalar Toeplitz Hermitian 5-band matrix A from Example 3.5, namely

$$A = \begin{pmatrix} \alpha & \beta & \gamma & 0 & 0 & 0 & 0 \\ \beta & \alpha & \beta & \gamma & 0 & 0 & 0 \\ \gamma & \beta & \alpha & \beta & \gamma & 0 & 0 \\ 0 & \gamma & \beta & \alpha & \beta & \gamma & 0 \\ 0 & 0 & \gamma & \beta & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \gamma & \beta & \alpha & \beta \\ 0 & 0 & 0 & 0 & \gamma & \beta & \alpha \end{pmatrix},$$

where $\gamma \neq 0$. We use Theorem 5.9 to obtain quasiseparable generators for this matrix.

The matrix A has lower rank numbers ρ_k^L $(k = 1, \dots, 6)$ not exceeding 2. Then A has lower quasiseparable generators with orders ρ_k^L $(k = 1, \dots, 6)$. Moreover a set $p(i)$ $(i = 2, \dots, 7)$, $q(j)$ $(j = 1, \dots, 6)$, $a(k)$ $(k = 2, \dots, 6)$ of lower quasiseparable generators of A with these orders are obtained by means of the following algorithm.

Step 1. Set $A^{(1)} = A(2:7, 1) = \begin{pmatrix} \beta & \gamma & 0 & 0 & 0 & 0 \end{pmatrix}^T$. Using rank factorization of the matrix $A^{(1)}$ determine the matrices $P_2, q(1)$ of the sizes $\left(\sum_{i=2}^{1} m_i\right) \times r_1 = 6 \times 1$, $r_1 \times n_1 = 1 \times 1$, respectively, such that (5.15),(5.16) take place. Indeed, take $P_2 = A^{(1)}$, $q(1) = 1$.

Step 2. For $k = 2, \dots, 6$ perform the following:

Set $k = 2$ and $A^{(2)} = A(2+1:7, 2) = \begin{pmatrix} \beta & \gamma & 0 & 0 & 0 \end{pmatrix}^T$.

Determine the matrices $p(k), P_k''$ of sizes

$$m_2 \times r_{2-1} = 1 \times 1, \left(\sum_{i=2+1}^{7} m_i\right) \times r_{2-1} = 5 \times 1,$$

respectively, from the partition (5.17), namely

$$P_2 = \begin{pmatrix} p(2) \\ P''_2 \end{pmatrix}.$$

It follows that $p(2) = \beta, P''_2 = \begin{pmatrix} \gamma & 0 & 0 & 0 & 0 \end{pmatrix}^T$.

Using rank factorization of the matrix $\begin{pmatrix} P''_2 & A^{(2)} \end{pmatrix}$ determine the matrices P_{2+1}, V_2 of sizes $\left(\sum_{i=2+1}^{7} m_i\right) \times r_2 = 5 \times 2, r_2 \times (r_{2-1} + n_2) = 2 \times 2$, respectively, such that (5.18) and (5.19) hold. For instance,

$$P_{2+1} = \begin{pmatrix} \gamma & \beta \\ 0 & \gamma \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_2 = 2.$$

Determine the matrices $a(2), q(2)$ of sizes $r_2 \times r_{2-1} = 2 \times 1, r_2 \times n_2 = 2 \times 1$, respectively, from the partition (5.20), namely

$$V_2 = \begin{pmatrix} a(2) & q(2) \end{pmatrix}.$$

This means that $a(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Set $k = 3$ and $A^{(3)} = A(3+1:7, 3) = \begin{pmatrix} \beta & \gamma & 0 & 0 \end{pmatrix}^T$.

Determine the matrices $p(3), P''_3$ of sizes $1 \times 2,\ 4 \times 2$, respectively, from the partition (5.17). It follows that $p(3) = \begin{pmatrix} \gamma & \beta \end{pmatrix}$, $P''_3 = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Using rank factorization of the matrix $\begin{pmatrix} P''_3 & A^{(3)} \end{pmatrix}$ determine the matrices P_{3+1}, V_3 of sizes $4 \times 2,\ 2 \times 3$, respectively, such that (5.18) and (5.19) hold. For instance,

$$P_{3+1} = \begin{pmatrix} \gamma & \beta \\ 0 & \gamma \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_3 = 2.$$

Determine the matrices $a(3), q(3)$ of sizes $2 \times 2,\ 2 \times 1$, respectively, from the partition (5.20). It follows that $a(3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $q(3) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Set $k = 4$ and $A^{(4)} = A(4+1:7, 4) = \begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix}$.

Determine the matrices $p(4), P''_4$ of sizes $1 \times 2,\ 3 \times 2$, respectively, from the partition (5.17). It follows that $p(4) = \begin{pmatrix} \gamma & \beta \end{pmatrix},\ P''_4 = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Using rank factorization of the matrix $\begin{pmatrix} P''_4 & A^{(4)} \end{pmatrix}$ determine the matrices P_{4+1}, V_4 of sizes $3 \times 2,\ 2 \times 3$, respectively, such that (5.18) and (5.19) hold. For instance

$$P_{4+1} = \begin{pmatrix} \gamma & \beta \\ 0 & \gamma \\ 0 & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_4 = 2.$$

Determine the matrices $a(4), q(4)$ of sizes $2 \times 2,\ 2 \times 1$, respectively, from the partition (5.20). It follows that $a(4) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ q(4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Set $k = 5$ and $A^{(5)} = A(5+1:7, 5) = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$.

Determine the matrices $p(5), P''_5$ of sizes $1 \times 2,\ 2 \times 2$, respectively, from the partition (5.17). It follows that $p(5) = \begin{pmatrix} \gamma & \beta \end{pmatrix},\ P''_5 = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$.

Using rank factorization of the matrix $\begin{pmatrix} P''_5 & A^{(5)} \end{pmatrix}$ determine the matrices P_{5+1}, V_5 of sizes $2 \times 2,\ 2 \times 3$, respectively, such that (5.18) and (5.19) hold. For instance,

$$P_{5+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_5 = \begin{pmatrix} 0 & \gamma & \beta \\ 0 & 0 & \gamma \end{pmatrix}, \quad r_5 = 2.$$

Determine the matrices $a(5), q(5)$ of sizes $2 \times 2,\ 2 \times 1$, respectively, from the partition (5.20). It follows that $a(5) = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix},\ q(5) = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$.

Finally, set $k = 6$ and $A^{(6)} = A(6+1:7, 6) = \beta$. Determine the matrices $p(6), P''_6$ of sizes $1 \times 2,\ 1 \times 2$, respectively, from the partition (5.17). It follows that $p(6) = \begin{pmatrix} 1 & 0 \end{pmatrix},\ P''_6 = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Using rank factorization of the matrix $\begin{pmatrix} P''_6 & A^{(6)} \end{pmatrix}$ determine the matrices P_7, V_6 of sizes $1 \times 1,\ 1 \times 3$, respectively, such that (5.18) and (5.19) hold. For instance,

$$P_7 = 1, \quad V_6 = \begin{pmatrix} 0 & 1 & \beta \end{pmatrix}, \quad r_6 = 1.$$

Determine the matrices $a(6), q(6)$ of sizes $1 \times 2,\ 1 \times 1$, respectively, from the partition (5.20). It follows that $a(6) = \begin{pmatrix} 0 & 1 \end{pmatrix},\ q(6) = \beta$.

In Step 3 of the algorithm set $p(7) = P_7 = 1$.

Summing up, the following lower quasiseparable generators have been obtained:

$$p(2) = \beta,\ p(3) = p(4) = p(5) = \begin{pmatrix} \gamma & \beta \end{pmatrix},\ p(6) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad p(7) = 1,$$
$$q(1) = 1,\ q(2) = q(3) = q(4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad q(5) = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \quad q(6) = \beta,$$

$$a(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ a(3) = a(4) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ a(5) = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix},\ a(6) = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Since the matrix A is Hermitian and for a symmetric matrix the generators $g(j) = q(j)^T,\ j = 1, \dots, N-1,\ h(i) = p(i)^T,\ i = 2, \dots, N,\ b(k) = a(k)^T,\ k = 2, \dots, N-1$, the following upper quasiseparable generators can been obtained:

$$h(2) = \beta,\ h(3) = h(4) = h(5) = \begin{pmatrix} \gamma \\ \beta \end{pmatrix}, \quad h(6) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h(7) = 1,$$

$$g(1) = 1,\ g(2) = g(3) = g(4) = \begin{pmatrix} 0 & 1 \end{pmatrix},\ \ g(5) = \begin{pmatrix} \beta & \gamma \end{pmatrix},\ \ g(6) = \beta,$$

$$b(2) = \begin{pmatrix} 1 & 0 \end{pmatrix},\ b(3) = b(4) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ b(5) = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix},\ b(6) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In order to describe completely the matrix A note that the diagonal entries are $d(k) = \alpha,\ k = 1, \dots, N$

The order of the above quasiseparable generators is much lower than 5, which is the order of the minimal semiseparable generators obtained for the same matrix in Example 3.5 . ◊

Example 5.16. Consider the 5×5 matrix

$$A = \begin{pmatrix} d & 2 & 1 & 1 & 1 \\ 2 & d & 2 & 1 & 1 \\ 1 & 2 & d & 2 & 1 \\ 1 & 1 & 2 & d & 2 \\ 1 & 1 & 1 & 2 & d \end{pmatrix}.$$

Then $A^{(1)} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $q(1) = 1$, $A^{(2)} = A(3:5, 2) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $p(2) = 2$, $P_2'' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the rank factorization gives $P_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and their rank is $r_2 = 2$. Therefore $a(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$A^{(3)} = A(4:5, 3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $p(3) = \begin{pmatrix} 1 & 2 \end{pmatrix}$, $P_2'' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and the rank factorization gives $P_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $V_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and their rank is $r_3 = 2$. Therefore $a(3) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $q(3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$A^{(4)} = A(5:5, 4) = 2$, $p(4) = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $P_2'' = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and the rank factorization gives $P_5 = 1$, $V_4 = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$ and their rank is $r_4 = 1$. Therefore $a(4) = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $q(4) = 2$.

Summing up, a set of lower quasiseparable generators of the matrix A is

$$p(2) = 2, \quad p(3) = \begin{pmatrix} 1 & 2 \end{pmatrix}, \quad p(4) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad p(5) = 1,$$
$$q(1) = 1, \quad q(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad q(3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad q(4) = 2,$$
$$a(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a(3) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad a(4) = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Since the matrix A is hermitian, a set of upper quasiseparable generators of this matrix is

$$g(1) = 1, \quad g(2) = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad g(3) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad g(4) = 2,$$
$$h(2) = 2, \quad h(3) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad h(4) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad h(5) = 1,$$
$$b(2) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad b(3) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad b(4) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Notice that for a symmetric matrix the generators satisfy $g(j) = q(j)^T,\ j = 1, \dots, N-1$, $h(i) = p(i)^T,\ i = 2, \dots, N$, $b(k) = a(k)^T,\ k = 2, \dots, N-1$.

Note also that the semiseparable generators obtained for the same matrix in Example 3.4 have a higher order than the quasiseparable generators obtained here. ◊

§5.4 Quasiseparable generators of block companion matrices viewed as scalar matrices

For the monic $n \times n$ matrix polynomial of order N

$$P(x) = I_N x^N + \alpha_{N-1} x^{N-1} + \cdots + \alpha_1 x + \alpha_0$$

with coefficients α_k $(k = 0, \dots, N-1)$, the block companion $N \times N$ matrix associated to it is

$$C' = \begin{pmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ I_n & 0 & \dots & 0 & -\alpha_1 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & I_n & 0 & -\alpha_{N-2} \\ 0 & 0 & \dots & I_n & -\alpha_{N-1} \end{pmatrix}. \tag{5.23}$$

We will regard the matrix $C' = \{C'_{ij}\}_{i,j=1}^{nN}$ as an $nN \times nN$ scalar matrix. Consider the submatrices of the form $C'(k+1 : Nn, 1 : k)$. We have

$$C'(k+1 : Nn, 1 : k) = \begin{pmatrix} 0_{(n-k)\times k} \\ I_k \\ 0_{(nN-n-k)\times k} \end{pmatrix}, \quad k = 1, \dots, n, \tag{5.24}$$

$$C'(k+1:Nn,1:k) = \begin{pmatrix} 0_{n\times(k-n)} & I_n \\ 0_{(nN-n-k)\times(k-n)} & 0 \end{pmatrix}, \quad k = n+1,\dots,nN-n, \tag{5.25}$$

$$\begin{aligned} &C'(k+1:Nn,1:k) \\ &\quad = \begin{pmatrix} 0_{(nN-k)\times(k-n)} & I_{nN-k} & -\alpha_{N-1}(k-nN+n+1:n,1:k-nN+n) \end{pmatrix}, \\ &\qquad k = nN-n+1,\dots,nN-1. \end{aligned} \tag{5.26}$$

By formula (4.1), C' is a scalar matrix with lower rank numbers

$$\begin{aligned} \rho_k^L &= k, & k &= 1,\dots,n, \\ \rho_k^L &= n, & k &= n+1,\dots,nN-n, \\ \rho_k^L &= nN-k, & k &= nN-n+1,\dots,nN-1. \end{aligned}$$

By Theorem 5.9 the matrix C' has lower quasiseparable generators with orders ρ_k^L $(k=1,\dots,nN-1)$ and a set $p(i)$ $(i=2,\dots,nN)$, $q(j)$ $(j=1,\dots,nN-1)$, $a(k)$ $(k=2,\dots,nN-1)$ of lower quasiseparable generators of C' with these orders are given via the algorithm in the theorem. Thus we obtain the following

1. Set $C'^{(1)} = C'(2:nN,1) = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}^T$, which is an $(nN-1)\times 1$ matrix with $n-1$ zeroes before the 1. Using rank factorization of the matrix $C'^{(1)}$ determine the matrices $P_2, q(1)$ of sizes $(nN-1)\times r_1 = (nN-1)\times 1$, $r_1\times 1 = 1\times 1$, respectively, such that (5.15), (5.16) take place. We get

$$P_2 = C'^{(1)}, \quad q(1) = 1. \tag{5.27}$$

2. For $k=2,\dots,nN-1$ perform the following.

For $k=2,\dots,n$ we get

$$P_k = C'(k:Nn,1:k-1). \tag{5.28}$$

Indeed, for $k=2$ the relation (5.28) follows from (5.27). Let for some k with $2\le k\le n$ the relation (5.28) hold. Using (5.17) we get

$$p(k) = C'(k,1:k-1), \quad P_k'' = C'(k+1:Nn,1:k-1). \tag{5.29}$$

Taking $C'^{(k)} = C'(k+1:Nn,k)$ we get $\begin{pmatrix} P_k'' & C'^{(k)} \end{pmatrix} = C'(k+1:Nn,1:k)$ and inserting (5.24) in (5.18), (5.19) we get

$$P_{k+1} = C'(k+1:Nn,1:k), \quad V_k = I_k, \tag{5.30}$$

which completes the proof of (5.28).

For $k=n+1,\dots,Nn-n$ we get

$$P_k = C'(k:Nn,k-n:k-1). \tag{5.31}$$

Indeed for $k = n+1$ the relation (5.28) follows from (5.30). with $k = n$. Let for some k with $n \le k \le Nn - n - 1$ the relation (5.31) hold. Using (5.17) we get

$$p(k) = C'(k, k-n : k-1), \quad P''_k = C'(k+1 : Nn, k-n : k-1). \tag{5.32}$$

Taking $C'^{(k)} = C'(k+1 : Nn, k)$ we get $\begin{pmatrix} P''_k & C'^{(k)} \end{pmatrix} = C'(k+1 : Nn, 1 : k)$, and using (5.25) we get

$$\begin{pmatrix} P''_k & C'^{(k)} \end{pmatrix} = \begin{pmatrix} 0_{n\times 1} & I_n \\ 0 & 0_{(nN-n-k)\times n} \end{pmatrix}.$$

Hence, using the factorization (5.18), (5.19) we get

$$P_{k+1} = C'(k+1 : Nn, k-n : k) = \begin{pmatrix} I_n \\ 0_{(nN-n-k)\times n} \end{pmatrix}, \quad V_k = \begin{pmatrix} 0_{n\times 1} & I_n \end{pmatrix} \tag{5.33}$$

which completes the proof of (5.31).

For $k = Nn - n + 1, \dots, Nn - 1$ we get

$$P_k = I_{Nn-k+1} \tag{5.34}$$

Indeed, for $k = Nn-n+1$ the relation (5.34) follows from (5.33) with $k = Nn-n$. Let for some k with $Nn - n + 1 \le k \le Nn - 2$ the relation (5.34) hold. Using (5.17) we get

$$p(k) = \begin{pmatrix} 1 & 0_{1\times(Nn-k)} \end{pmatrix}, \quad P''_k = \begin{pmatrix} 0_{(nN-k)\times 1} & I_{nN-k} \end{pmatrix}. \tag{5.35}$$

Using (5.26) we have $C'^{(k)} = L_k$, where

$$L_k = -\alpha_{N-1}(k - nN + n + 1 : n, k - nN + n),$$

and therefore we get

$$\begin{pmatrix} P''_k & C'^{(k)} \end{pmatrix} = \begin{pmatrix} 0_{(nN-k)\times 1} & I_{nN-k} & L_k \end{pmatrix}.$$

Hence in the rank factorization (5.18) one can take

$$P_{k+1} = I_{nN-k}, \quad V_k = \begin{pmatrix} 0_{(nN-k)\times 1} & I_{nN-k} & L_k \end{pmatrix}. \tag{5.36}$$

This in particular implies (5.34).

From this it follows that the quasiseparable generators $p(k)$, $k = 2, \dots, nN$ which are the first row of P_k are vectors of length $\min(k-1, n, nN-k+1)$ given by:

$$\begin{aligned} p(k) &= 0_{1\times(k-1)}, & k &= 2, \dots, n, \\ p(k) &= \begin{pmatrix} 1 & 0_{1\times(n-1)} \end{pmatrix}, & k &= n+1, \dots, nN-n, \\ p(k) &= \begin{pmatrix} 1 & 0_{1\times(nN-k)} \end{pmatrix}, & k &= nN-n+1, \dots, nN. \end{aligned} \tag{5.37}$$

It also follows that the quasiseparable generators $q(k),\ k = 1,\dots,nN-1$, which are the last column of V_k, are vectors of length $\min(k, n, nN-k)$ given by:

$$\begin{aligned}
q(k) &= \begin{pmatrix} 0_{(k-1)\times 1} \\ 1 \end{pmatrix}, && k = 1,\dots,n, && (5.38)\\
q(k) &= \begin{pmatrix} 0_{(n-1)\times 1} \\ 1 \end{pmatrix}, && k = n+1,\dots,nN-n,\\
q(k) &= -\alpha_{N-1}(k-nN+n+1:n, k-nN+n), && k = nN-n+1,\dots,nN-1.
\end{aligned}$$

Further, the quasiseparable generators $a(k),\ k = 2,\dots,nN-1$, which are the all columns of V_k except its last column, are matrices given by:

$$\begin{aligned}
a(k) &= \begin{pmatrix} I_{k-1} \\ 0_{1\times(k-1)} \end{pmatrix}, && k = 2,\dots,n,\\
a(k) &= \begin{pmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{pmatrix}, && k = n+1,\dots,nN-n, && (5.39)\\
a(k) &= \begin{pmatrix} 0_{(nN-k)\times 1} & I_{nN-k} \end{pmatrix}, && k = nN-n+1,\dots,nN-1.
\end{aligned}$$

In order to find upper quasiseparable generators for C' just note that its upper triangular part is the same as the upper triangular part of the separable block matrix GH, where

$$G = \begin{pmatrix} -\alpha_0^T & -\alpha_1^T & \cdots & -\alpha_{N-1}^T \end{pmatrix}^T, \quad H = \begin{pmatrix} 0_n & \cdots & 0_n & I_n \end{pmatrix}.$$

Therefore one can take as quasiseparable generators of the scalar matrix C' the matrices $b(k) = I_n,\ k = 2,\dots,nN-1$ and the following vectors of length n:

$$g(k) = -\alpha_s(t, 1:n),$$

where s and t are the quotient and the remainder of the integer division of k by n, i.e., $k = sn + t,\ 0 \le t < n-1$, and

$$h(k) = 0_{n\times 1}, \quad k = 1,\dots,nN-n$$

and

$$h(k) = \begin{pmatrix} 0_{(k-nN+n-1)\times 1} \\ 0_{(nN-k)\times 1} \end{pmatrix}, \quad k = nN-n+1,\dots,nN.$$

§5.5 Minimality conditions

Now we proceed with a careful study of minimality conditions for quasiseparable generators.

Theorem 5.17. *Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a block matrix with the lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of the orders r_k $(k = 1, \dots, N-1)$. Define the matrices Q_k $(k = 1, \dots, N-1)$, and P_k $(k = 2, \dots, N)$ by the formulas* (5.1) *and* (5.2).

The quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ are minimal if and only if the following relations hold:

$$\operatorname{rank} P_{k+1} = \operatorname{rank} Q_k = r_k, \; k = 1, \dots, N-1. \tag{5.40}$$

Proof. Lemma 5.1 yields the representations (5.9), where the matrix P_{k+1} has r_k columns and the matrix Q_k has r_k rows.

Assume that the quasiseparable generators $p(i), q(j), a(k)$ are minimal. Then, by Corollary 5.10, the relations

$$\operatorname{rank} A(k+1:N, 1:k) = r_k, \quad k = 1, \dots, N-1 \tag{5.41}$$

hold. Hence using the inequalities

$$\operatorname{rank} Q_k \leq r_k, \; \operatorname{rank} P_{k+1} \leq r_k, \; \operatorname{rank} A(k+1:N, 1:k) \leq \min(\operatorname{rank} P_{k+1}, \operatorname{rank} Q_k)$$

one obtains (5.40).

Assume that the relations (5.40) hold, which means that the matrices P_k have full column rank and the matrices Q_k full row rank. Then using (5.9) one obtains (5.41) and therefore, by Corollary 5.10, the quasiseparable generators $p(i), q(j), a(k)$ are minimal. □

Next we present a minimality criterion for quasiseparable generators without using the matrices P_k, Q_k.

Theorem 5.18. *Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a block matrix with the lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of the orders r_k $(k = 1, \dots, N-1)$.*

These generators are minimal if and only if the following relations hold:

$$\operatorname{rank}(q(1)) = r_1; \tag{5.42}$$

$$\operatorname{rank} \begin{pmatrix} p(k) \\ a(k) \end{pmatrix} = r_{k-1}, \quad \operatorname{rank} \begin{pmatrix} a(k) & q(k) \end{pmatrix} = r_k, \quad k = 2, \dots, N-1; \tag{5.43}$$

$$\operatorname{rank}(p(N)) = r_{N-1}. \tag{5.44}$$

Proof. By Theorem 5.17, the minimality of the quasiseparable generators $p(i)$, $q(j)$, $a(k)$ is equivalent to the relations (5.40).

Assume that the relations (5.40) hold. This already implies (5.42) by (5.3) and (5.44) by (5.4). For $k = 2, \dots, N-1$ using (5.3) and (5.4) one obtains the representations

$$Q_k = \begin{pmatrix} a(k) & q(k) \end{pmatrix} \begin{pmatrix} Q_{k-1} & 0 \\ 0 & I \end{pmatrix}, \tag{5.45}$$

$$P_k = \begin{pmatrix} I & 0 \\ 0 & P_{k+1} \end{pmatrix} \begin{pmatrix} p(k) \\ a(k) \end{pmatrix}. \tag{5.46}$$

The matrix $\begin{pmatrix} a(k) & q(k) \end{pmatrix}$ contains r_k rows and the matrix $\begin{pmatrix} p(k) \\ a(k) \end{pmatrix}$ contains r_{k-1} columns. Hence (5.45) and (5.46) imply

$$\operatorname{rank} Q_k \le \operatorname{rank} \begin{pmatrix} a(k) & q(k) \end{pmatrix} \le r_k, \quad \operatorname{rank} P_k \le \operatorname{rank} \begin{pmatrix} p(k) \\ a(k) \end{pmatrix} \le r_{k-1}.$$

From here using (5.40) one obtains (5.43).

Assume that the relations (5.42)–(5.44) hold. Let us prove by induction that from the relations

$$\operatorname{rank}(q(1)) = r_1, \quad \operatorname{rank} \begin{pmatrix} a(k) & q(k) \end{pmatrix} = \mathrm{r_k},\ \mathrm{k = 2, \dots, N-1}$$

it follows that

$$\operatorname{rank} Q_k = r_k, \quad k = 1, \dots, N-1.$$

The case $k = 1$ is clear. Assume that for some k, $2 \le k \le N-1$ one has $\operatorname{rank} Q_{k-1} = r_{k-1}$. This means that the matrix $\begin{pmatrix} Q_{k-1} & 0 \\ 0 & I \end{pmatrix}$ has full row rank. Hence using (5.45) one obtains

$$\operatorname{rank} Q_k = \operatorname{rank} \begin{pmatrix} a(k) & q(k) \end{pmatrix} = r_k.$$

Similarly, using (5.46) one obtains the relations

$$\operatorname{rank} P_k = r_{k-1}, \quad k = 2, \dots, N. \qquad \square$$

It is also the case that upper quasiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ of the orders ρ_k $(k = 1, \dots, N-1)$ of the matrix A are minimal if and only if

$$\operatorname{rank}(g(1)) = \rho_1; \tag{5.47}$$

$$\operatorname{rank} \begin{pmatrix} g(k) & b(k) \end{pmatrix} = \rho_k,\ \operatorname{rank} \begin{pmatrix} b(k) \\ h(k) \end{pmatrix} = \rho_{k-1},\ k = 2, \dots, N-1; \tag{5.48}$$

$$\operatorname{rank}(h(N)) = \rho_{N-1}. \tag{5.49}$$

Now we apply the minimality criterion obtained in Theorem 5.18 to some examples.

Example 5.19. Let A be a matrix with block entries of sizes $l \times l$ and with diagonal plus semiseparable representation with lower semiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ of order l. The corresponding lower quasiseparable generators of A are $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k) = I_l$ $(k = 2, \dots, N-1)$. One has

$$\begin{pmatrix} p(k) \\ a(k) \end{pmatrix} = \begin{pmatrix} p(k) \\ I_l \end{pmatrix}, \ \begin{pmatrix} a(k) & q(k) \end{pmatrix} = \begin{pmatrix} I_l & q(k) \end{pmatrix}, \ k = 2, \dots, N-1,$$

and therefore the conditions (5.43) are valid automatically. Hence these generators are minimal if and only if

$$\operatorname{rank}(q(1)) = \operatorname{rank}(p(N)) = l.$$

For scalar matrices with quasiseparable generators of order one the minimality conditions of lower generators (5.42)–(5.44) have the form

$$|q(1)|^2 > 0; \quad |p(k)|^2 + |a(k)|^2 > 0, \quad |q(k)|^2 + |a(k)|^2 > 0,$$
$$k = 2, \dots, N-1; \quad |p(N)|^2 > 0$$

and the minimality conditions (5.47)–(5.49) for upper generators are equivalent to

$$|g(1)|^2 > 0; \quad |g(k)|^2 + |b(k)|^2 > 0, \quad |h(k)|^2 + |b(k)|^2 > 0,$$
$$k = 2, \dots, N-1; \quad |h(N)|^2 > 0. \qquad \Diamond$$

§5.6 Sets of generators. Minimality and similarity

As was mentioned above, quasiseparable generators of a matrix are not unique. We consider here relations between different quasiseparable generators of a given matrix.

Theorem 5.20. *Let A be a block matrix with lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r_k $(k = 1, \dots, N-1)$ and suppose that for the matrices S_k $(k = 1, \dots, N-1)$ of sizes $r_k \times r'_k$ and matrices $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ of corresponding sizes the relations*

$$q(k) = S_k q'(k), \qquad k = 1, \dots, N-1, \tag{5.50}$$
$$a(k) S_{k-1} = S_k a'(k), \qquad k = 2, \dots, N-1, \tag{5.51}$$
$$p'(k) = p(k) S_{k-1}, \quad k = 2, \dots, N \tag{5.52}$$

hold.

Then the elements $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the matrix A of orders r'_k $(k = 1, \dots, N-1)$.

Proof. For any $j = 1, \dots, N-1$, using (5.50), (5.52) one has

$$p(j+1)q(j) = p(j+1)S_j q'(j) = p'(j+1)q'(j). \tag{5.53}$$

Next for any $1 \le j < i-1 \le N-1$, using (5.50), (5.51) one gets

$$\begin{aligned} p(i)a(i-1)\cdot\dots\cdot a(j+1)q(j) &= p(i)a(i-1)\cdot\dots\cdot a(j+1)S_j q'(j) \\ &= p(i)a(i-1)\cdot\dots\cdot a(j+2)S_{j+1}a'(j+1)q'(j) \\ = \dots &= p(i)S_{i-1}a'(i-1)a'(i-2)\cdot\dots\cdot a'(j+1)q'(j) \end{aligned}$$

and moreover using (5.52) one obtains

$$p(i)a(i-1)\cdot\dots\cdot a(j+1)q(j) = p'(i)a'(i-1)\cdot\dots\cdot a'(j+1)q'(j). \tag{5.54}$$

Thus from the relations (5.53), (5.54) one obtains

$$p(i)a^{>}_{ij}q(j) = p'(i)(a')^{>}_{ij}q'(j), \quad 1 \le j < i \le N,$$

which implies that $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the same matrix A. □

We also use another version of Theorem 5.20, which is proved in a similar way.

Theorem 5.21. *Let A be a block matrix with lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r_k $(k = 1, \dots, N-1)$ and let for matrices S_k $(k = 1, \dots, N-1)$ of sizes $r'_k \times r_k$ and matrices $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ of corresponding sizes the relations*

$$S_k q(k) = q'(k), \qquad k = 1, \dots, N-1, \tag{5.55}$$
$$a'(k)S_{k-1} = S_k a(k), \qquad k = 2, \dots, N-1, \tag{5.56}$$
$$p(k) = p'(k)S_{k-1}, \quad k = 2, \dots, N \tag{5.57}$$

hold.

Then the elements $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the matrix A of orders r'_k $(k = 1, \dots, N-1)$.

Theorem 5.21 admits the following generalization.

Theorem 5.22. *Let A be a block matrix with lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r_k $(k = 1, \dots, N-1)$. Using the generators $q(j), a(k)$ define the matrices Q_k $(k = 1, \dots, N-1)$ via relations* (5.3). *Let for matrices S_k $(k = 1, \dots, N-1)$ of sizes*

$r'_k \times r_k$ *and matrices* $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ *of corresponding sizes the relations*

$$S_k q(k) = q'(k), \qquad k = 1, \dots, N-1, \tag{5.58}$$
$$a'(k) S_{k-1} Q_{k-1} = S_k a(k) Q_{k-1}, \qquad k = 2, \dots, N-1, \tag{5.59}$$
$$p(k) Q_{k-1} = p'(k) S_{k-1} Q_{k-1}, \quad k = 2, \dots, N \tag{5.60}$$

hold.

Then the elements $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ *are lower quasiseparable generators of the matrix* A *of orders* r'_k $(k = 1, \dots, N-1)$.

Proof. Using the equalities (5.11) we have

$$A(k+1, 1:k) = p(k+1) Q_k, \ k = 1, \dots, N-1. \tag{5.61}$$

Using the elements $q'(j), a'(k)$, define the matrices Q'_k $(k = 1, \dots, N-1)$ via relations (5.3). One proves by induction that

$$Q'_k = S_k Q_k, \quad k = 1, \dots, N-1.$$

Indeed for $k = 1$ (5.58) yields

$$Q'_1 = q'(1) = S_1 q(1) = S_1 Q_1.$$

Let for some k with $2 \le k \le N-1$ the equality $Q'_{k-1} = S_{k-1} Q_{k-1}$ hold. Using (5.3) and(5.58),(5.59) we get

$$\begin{aligned} Q'_k &= \begin{pmatrix} a'(k) Q'_{k-1} & q'(k) \end{pmatrix} = \begin{pmatrix} a'(k) S_{k-1} Q_{k-1} & q'(k) \end{pmatrix} \\ &= \begin{pmatrix} S_k a(k) Q_{k-1} & S_k q(k) \end{pmatrix} = S_k Q_k. \end{aligned}$$

Thus, (5.60) and (5.61) yield

$$A(k+1, 1:k) = p(k+1) Q_k = p'(k+1) S_k Q_k = p'(k+1) Q'_k, \quad k = 1, \dots, N-1.$$

From here by Lemma 5.3 we conclude that $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the matrix A. It is clear that $a'(k)$ $(k = 2, \dots, N-1)$ are matrices of sizes $r'_k \times r'_{k-1}$ and hence the numbers r'_k $(k = 1, \dots, N-1)$ are the orders of these generators. □

Applying Theorem 5.22 to the transposed matrix A^T we obtain the corresponding result for upper quasiseparable generators.

Lemma 5.23. *Let* A *be a block matrix with upper quasiseparable generators*

$$g(i) \ (i = 1, \dots, N-1), \ h(j) \ (j = 2, \dots, N), \ b(k) \ (k = 2, \dots, N-1)$$

of orders r_k $(k = 1, \dots, N-1)$. *Using the generators* $g(k), b(k)$ *define the matrices* G_k $(k = 1, \dots, N-1)$ *via relations* (5.7). *Suppose that for matrices* S_k $(k = 1, \dots, N-1)$ *of sizes* $r_k \times r'_k$ *and matrices* $g'(i)$ $(i = 1, \dots, N-1)$, $h'(j)$ $(j = 2, \dots, N)$, $b'(k)$ $(k = 2, \dots, N-1)$ *of corresponding sizes the relations*

$$g(k)S_k = g'(k), \qquad k = 1, \dots, N-1, \tag{5.62}$$

$$G_{k-1}S_{k-1}b'(k) = G_{k-1}b(k)S_k, \quad k = 2, \dots, N-1, \tag{5.63}$$

$$G_{k-1}S_{k-1}h'(k) = G_{k-1}h(k), \qquad k = 2, \dots, N \tag{5.64}$$

hold.

Then the elements $g'(i)$ $(i = 1, \dots, N-1)$, $h'(j)$ $(j = 2, \dots, N)$, $b'(k)$ $(k = 2, \dots, N-1)$ *are upper quasiseparable generators of the matrix* A *of orders* r'_k $(k = 1, \dots, N-1)$.

Next we introduce a notion of similarity for sets of quasiseparable generators of a matrix.

Definiton 5.24. Let $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ be lower quasiseparable generators of a matrix A and let $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ be also lower quasiseparable generators of A of the same orders. These generators are called similar if there exist invertible matrices S_k $(k = 1, \dots, N-1)$ such that the relations (5.50)–(5.52) hold.

Next it will be proved that any two sets of minimal quasiseparable generators of a matrix are similar.

Theorem 5.25. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be a block matrix and let*

$$p(i)\ (i = 2, \dots, N),\ q(j)\ (j = 1, \dots, N-1),\ a(k)\ (k = 2, \dots, N-1)$$

and

$$p'(i)\ (i = 2, \dots, N),\ q'(j)\ (j = 1, \dots, N-1),\ a'(k)\ (k = 2, \dots, N-1)$$

be two sets of lower quasiseparable generators of the matrix A *with the minimal orders* r_k $(k = 1, \dots, N-1)$.

Then the quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ *and* $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ *are similar.*

Proof. Using $p(i), q(j), a(k)$ and $p'(i), q'(j), a'(k)$ define the matrices P_{k+1}, Q_k and P'_{k+1}, Q'_k, $(k = 1, \dots, N-1)$, respectively, via the formulas (5.1) and (5.2). The matrices P_{k+1}, P'_{k+1} contain r_k columns, the matrices Q_k, Q'_k contain r_k rows. Moreover, by Theorem 5.17, the ranks of these matrices are equal to r_k. Lemma 5.1 implies that

$$A(k+1:N, 1:k) = P_{k+1}Q_k = P'_{k+1}Q'_k, \quad k = 1, \dots, N-1.$$

Hence, by Lemma 1.1, there exist invertible $r_k \times r_k$ matrices S_k $(k = 1, \dots, N-1)$ such that

$$Q_k = S_k Q'_k, \quad k = 1, \dots, N-1, \tag{5.65}$$

and $P_{k+1} = P'_{k+1} S_k^{-1}$, $k = 1, \dots, N-1$, which implies that

$$P'_k = P_k S_{k-1}, \quad k = 2, \dots, N. \tag{5.66}$$

Using (5.65) and (5.3) one gets

$$q(1) = S_1 q'(1) \tag{5.67}$$

and

$$\begin{pmatrix} a(k)Q_{k-1} & q(k) \end{pmatrix} = S_k \begin{pmatrix} a'(k)Q'_{k-1} & q'(k) \end{pmatrix}, \quad k = 2, \dots, N-1. \tag{5.68}$$

Equating the corresponding entries in (5.68) one obtains

$$q(k) = S_k q'(k), \qquad k = 2, \dots, N-1 \tag{5.69}$$

and

$$a(k)Q_{k-1} = S_k a'(k) Q'_{k-1}, \quad k = 2, \dots, N-1. \tag{5.70}$$

The relations (5.67), (5.69) mean (5.50). Furthermore using (5.70) and (5.65) one gets

$$a(k)S_{k-1}Q'_{k-1} = a(k)Q_{k-1} = S_k a'(k) Q'_{k-1}, \quad k = 2, \dots, N-1.$$

From here since each matrix Q'_{k-1} has full row rank one gets

$$a(k)S_{k-1} = S_k a'(k), \quad k = 2, \dots, N-1,$$

which is (5.51). Finally, using (5.66) and (5.4) one obtains

$$p'(N) = p(N)S_{N-1}, \quad \begin{pmatrix} p(k) \\ * \end{pmatrix} S_{k-1} = \begin{pmatrix} p'(k) \\ * \end{pmatrix}, \quad k = N-1, \dots, 2,$$

whence

$$p'(N) = p(N)S_{N-1}, \quad p'(k) = p(k)S_{k-1}, \quad k = N-1, \dots, 2,$$

i.e., the relations (5.52) hold. □

The statement of Theorem 5.25 is not true without assuming the minimality of quasiseparable generators. For instance, taking the sets of quasiseparable generators

$$p(i) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad q(j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$p'(i) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad q'(j) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad a'(k) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

one can easily check that

$$p(i)a^{>}_{ij}q(j) = p'(i)(a')^{>}_{ij}q'(j), \quad 1 \le j < i \le N.$$

However, it is clear that in this example the equalities (5.51) with invertible matrices S_{k-1}, S_k are impossible.

§5.7 Reduction to minimal quasiseparable generators

At first we consider the case when the quasiseparable generators of a matrix, not necessarily minimal, are given. Our goal is using the given quasiseparable generators to compute other quasiseparable generators of the same matrix, but with minimal orders.

Theorem 5.26. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with block entries of sizes $m_i \times n_j$ and lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders l_k $(k = 1, \dots, N-1)$.*

Then a set $\tilde{p}(i)$ $(i = 2, \dots, N)$, $\tilde{q}(j)$ $(j = 1, \dots, N-1)$, $\tilde{a}(k)$ $(k = 2, \dots, N-1)$ of minimal lower quasiseparable generators of the matrix A is determined from the quasiseparable generators $p(i), q(j), a(k)$ via the following algorithm.

1.1. *Using rank factorization of the matrix $q(1)$ determine the matrices $L_1, q'(1)$ of sizes $l_1 \times r'_1, r'_1 \times n_1$, respectively, such that*

$$q(1) = L_1 q'(1), \tag{5.71}$$
$$\operatorname{rank} L_1 = \operatorname{rank}(q'(1)) = \operatorname{rank}(q(1)) = r'_1. \tag{5.72}$$

1.2. *For $k = 2, \dots, N-1$ perform the following. Using rank factorization of the matrix $\begin{pmatrix} a(k)L_{k-1} & q(k) \end{pmatrix}$ determine the matrices L_k, V_k of sizes $l_k \times r'_k, r'_k \times (r'_{k-1} + n_k)$, respectively, such that*

$$\begin{pmatrix} a(k)L_{k-1} & q(k) \end{pmatrix} = L_k V_k, \tag{5.73}$$
$$\operatorname{rank} L_k = \operatorname{rank} V_k = \operatorname{rank} \begin{pmatrix} a(k)L_{k-1} & q(k) \end{pmatrix} = r'_k. \tag{5.74}$$

Determine the matrices $a'(k), q'(k)$ of sizes $r'_k \times r'_{k-1}, r'_k \times n_k$, respectively, from the partition

$$V_k = \begin{bmatrix} a'(k) & q'(k) \end{bmatrix}. \tag{5.75}$$

1.3. *For $k = 2, \dots, N$ compute*

$$p'(k) = p(k)L_{k-1}. \tag{5.76}$$

2.1. *Using rank factorization of the matrix* $p'(N)$ *determine the matrices* $\tilde{p}(N)$, S_{N-1} *of sizes* $m_N \times r_{N-1}, r_{N-1} \times r'_{N-1}$, *respectively, such that*

$$p'(N) = \tilde{p}(N)S_{N-1}, \tag{5.77}$$

$$\operatorname{rank}(\tilde{p}(N)) = \operatorname{rank} S_{N-1} = \operatorname{rank}(p'(N)) = r_{N-1}. \tag{5.78}$$

2.2. *For* $k = N-1, \dots, 2$ *perform the following. Using rank factorization of the matrix* $\begin{pmatrix} p'(k) \\ S_k a'(k) \end{pmatrix}$ *determine the matrices* U_k, S_{k-1} *of sizes* $(m_k + r_k) \times r_{k-1}, r_{k-1} \times r'_{k-1}$, *respectively, such that*

$$\begin{pmatrix} p'(k) \\ S_k a'(k) \end{pmatrix} = U_k S_{k-1}, \tag{5.79}$$

$$\operatorname{rank} U_k = \operatorname{rank} S_{k-1} = \operatorname{rank} \begin{pmatrix} p'(k) \\ S_k a'(k) \end{pmatrix} = r_{k-1}. \tag{5.80}$$

Determine the matrices $\tilde{p}(k), \tilde{a}(k)$ *of sizes* $m_k \times r_{k-1}, r_k \times r_{k-1}$ *respectively, from the partition*

$$U_k = \begin{pmatrix} \tilde{p}(k) \\ \tilde{a}(k) \end{pmatrix}. \tag{5.81}$$

2.3. *For* $k = 1, \dots, N-1$ *compute*

$$\tilde{q}(k) = S_k q'(k). \tag{5.82}$$

Proof. Comparing the corresponding entries in (5.73), (5.75) one gets

$$q(k) = L_k q'(k), \quad a(k)L_{k-1} = L_k a'(k), \quad k = 2, \dots, N-1.$$

Together with the formulas (5.76), (5.71) this implies, by Theorem 5.20, that the elements $p'(i)$ $(i = 2, \dots, N)$, $q'(j)$ $(j = 1, \dots, N-1)$, $a'(k)$ $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the matrix A.

Similarly, comparing the corresponding entries in (5.79), (5.81) one obtains

$$\tilde{a}(k)S_{k-1} = S_k a'(k), \quad p'(k) = \tilde{p}(k)S_{k-1}, \quad k = 2, \dots, N-1.$$

Together with the formulas (5.77), (5.82), by Theorem 5.21, we conclude that the elements $\tilde{p}(i)$ $(i = 2, \dots, N)$, $\tilde{q}(j)$ $(j = 1, \dots, N-1)$, $\tilde{a}(k)$ $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the same matrix A. Let us establish the minimality of these generators. Using the relations (5.78), (5.80) and (5.81) we see that

$$\operatorname{rank} \begin{pmatrix} \tilde{p}(k) \\ \tilde{a}(k) \end{pmatrix} = r_{k-1}, \ k = 2, \dots, N-1; \quad \operatorname{rank}(\tilde{p}(N)) = r_{N-1}. \tag{5.83}$$

It remains to check that

$$\operatorname{rank}(\tilde{q}(1)) = r_1 \tag{5.84}$$

and

$$\operatorname{rank}\begin{pmatrix} \tilde{a}(k) & \tilde{q}(k) \end{pmatrix} = r_k,\ k = 2, \dots, N-1. \tag{5.85}$$

Using the relations (5.72), (5.74), (5.75) one has

$$\operatorname{rank}(q'(1)) = r'_1, \quad \operatorname{rank}\begin{bmatrix} a'(k) & q'(k) \end{bmatrix} = r'_k,\ k = 2, \dots, N-1$$

and using (5.78), (5.80) one obtains

$$\operatorname{rank} S_k = r_k, \quad k = 1, \dots, N-1.$$

This means that all the matrices $q'(1)$, $\begin{pmatrix} a'(k) & q'(k) \end{pmatrix}$, $k = 2, \dots, N-1$; S_k, $k = 1, \dots, N-1$ have full row rank. Now from the equality $\tilde{q}(1) = S_1 q'(1)$ we see that $\operatorname{rank}(\tilde{q}(1)) = \operatorname{rank} S_1$ and hence (5.84) holds. For $k = 2, \dots, N-1$ one has the following. Since the matrix $\begin{pmatrix} S_{k-1} & 0 \\ 0 & I \end{pmatrix}$ has full row rank one obtains

$$\begin{aligned} \operatorname{rank}\begin{pmatrix} \tilde{a}(k) & \tilde{q}(k) \end{pmatrix} &= \operatorname{rank}\left(\begin{pmatrix} \tilde{a}(k) & \tilde{q}(k) \end{pmatrix}\begin{pmatrix} S_{k-1} & 0 \\ 0 & I \end{pmatrix}\right) \\ &= \operatorname{rank}\begin{pmatrix} \tilde{a}(k)S_{k-1} & \tilde{q}(k) \end{pmatrix}. \end{aligned} \tag{5.86}$$

Next, using the equalities $\tilde{a}(k)S_{k-1} = S_k a'(k)$, $\tilde{q}(k) = S_k q'(k)$ one gets

$$\begin{pmatrix} \tilde{a}(k)S_{k-1} & \tilde{q}(k) \end{pmatrix} = S_k \begin{pmatrix} a'(k) & q'(k) \end{pmatrix}$$

which implies

$$\operatorname{rank}\begin{bmatrix} \tilde{a}(k)S_{k-1} & \tilde{q}(k) \end{bmatrix} = \operatorname{rank} S_k = r_k. \tag{5.87}$$

Combining (5.86) and (5.87) together one obtains (5.85). From the relations (5.83)–(5.85) and Theorem 5.18 we conclude that $\tilde{p}(i), \tilde{q}(j), \tilde{a}(k)$ are minimal lower quasiseparable generators of the matrix A. □

§5.8 Normal quasiseparable generators

Now we consider a special case where quasiseparable generators satisfy certain orthonormality conditions.

Definiton 5.27. The lower quasiseparable generators

$$p(i)\ (i = 2, \dots, N),\ q(j)\ (j = 1, \dots, N-1),\ a(k)\ (k = 2, \dots, N-1)$$

of orders r_k $(k = 1, \dots, N-1)$ of a block matrix are said to be in the left normal form if the relations

$$p^*(N)p(N) = I_{r_{N-1}}, \quad a^*(k)a(k) + p^*(k)p(k) = I_{r_{k-1}},\ k = N-1, \dots, 2 \tag{5.88}$$

hold.

The lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r_k $(k = 1, \dots, N-1)$ of a block matrix are said to be in the right normal form if the relations

$$q(1)q^*(1) = I_{r_1}, \quad a(k)a^*(k) + q(k)q^*(k) = I_{r_k}, \ k = 2, \dots, N-1 \tag{5.89}$$

hold.

Let $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ be lower quasiseparable generators of orders r_k $(k = 1, \dots, N-1)$ of a block matrix. These generators determine the matrices Q_k, $k = 1, \dots, N-1$ and P_k, $k = N, \dots, 2$ via the relations (5.1) and (5.2). The conditions (5.88) and (5.89) are equivalent to orthonormality of the columns of the matrices P_k and the rows of the matrices Q_k, respectively.

Lemma 5.28. *Let Q_k, $k = 1, \dots, N-1$ be matrices of the form* (5.1). *Then the relations*

$$Q_k Q_k^* = I_{r_k}, \quad k = 1, \dots, N-1 \tag{5.90}$$

and (5.89) *are equivalent.*

Let P_k, $k = N, \dots, 2$ be matrices of the form (5.2). *Then the relations*

$$P_k^* P_k = I_{r_{k-1}}, \quad k = 2, \dots, N \tag{5.91}$$

and (5.88) *are equivalent.*

Proof. The recursions (5.3) imply

$$q(1)q^*(1) = Q_1 Q_1^*, \quad Q_k Q_k^* = a(k) Q_{k-1} Q_{k-1}^* a^*(k) + q(k)q^*(k), \ k = 2, \dots, N-1. \tag{5.92}$$

Let the conditions (5.90) hold. Using (5.92) one obtains (5.89). Conversely, assuming that (5.89) is valid and using (5.92) one obtains (5.90) by induction.

Next the recursions (5.4) imply

$$P_N^* P_N = p^*(N)p(N), \quad P_k^* P_k = a^*(k) P_{k+1}^* P_{k+1} a(k) + p^*(k)p(k), \ k = N-1, \dots, 2.$$

Hence the equivalence of (5.91) and (5.88) follows. □

Given a set of lower quasiseparable generators of a matrix, one can obtain another set of quasiseparable generators in the left normal form or in the right normal form. For instance, quasiseparable generators in the left normal form are obtained as follows.

Theorem 5.29. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a matrix with block entries A_{ij} of sizes $m_i \times n_j$ and lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r_k $(k = 1, \dots, N-1)$.*

Then a set $\tilde{p}(i)$ $(i = 2, \dots, N)$, $\tilde{q}(j)$ $(j = 1, \dots, N-1)$, $\tilde{a}(k)$ $(k = 2, \dots, N-1)$ of lower quasiseparable generators of the matrix A in the left normal form is determined from the generators $p(i), q(j), a(k)$ via the following algorithm.

1. *Using orthogonal factorization of the matrix* $p(N)$ *determine the matrices* $\tilde{p}(N), S_{N-1}$ *of sizes* $m_N \times r'_{N-1}, r'_{N-1} \times r_{N-1}$*, respectively, such that*

$$p(N) = \tilde{p}(N)S_{N-1}, \tag{5.93}$$

$$\tilde{p}^*(N)\tilde{p}(N) = I_{r'_{N-1}}. \tag{5.94}$$

2. *For* $k = N-1, \dots, 2$ *perform the following. Using orthogonal factorization of the matrix* $\begin{pmatrix} p(k) \\ S_k a(k) \end{pmatrix}$ *determine the matrices* U_k, S_{k-1} *of sizes* $(m_k + r'_k) \times r'_{k-1}, r'_{k-1} \times r_{k-1}$*, respectively, such that*

$$\begin{pmatrix} p(k) \\ S_k a(k) \end{pmatrix} = U_k S_{k-1}, \tag{5.95}$$

$$U_k^* U_k = I_{r'_{k-1}}. \tag{5.96}$$

Determine the matrices $\tilde{p}(k), \tilde{a}(k)$ *of sizes* $m_k \times r'_{k-1}, r'_k \times r'_{k-1}$*, respectively, from the partition*

$$U_k = \begin{pmatrix} \tilde{p}(k) \\ \tilde{a}(k) \end{pmatrix}. \tag{5.97}$$

3. *For* $k = 1, \dots, N-1$ *compute*

$$\tilde{q}(k) = S_k q(k). \tag{5.98}$$

Proof. Comparing the corresponding entries in (5.95), (5.97) one obtains

$$\tilde{a}(k)S_{k-1} = S_k a(k), \quad p(k) = \tilde{p}(k)S_{k-1}, \quad k = 2, \dots, N-1.$$

Together with the formulas (5.93), (5.98) this implies the relations (5.55)–(5.57). Applying Theorem 5.21 we conclude that the elements $\tilde{p}(i)$ $(i = 2, \dots, N)$, $\tilde{q}(j)$ $(j = 1, \dots, N-1)$, $\tilde{a}(k)$ $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the matrix A. Moreover from the relations (5.94) and (5.96), (5.97) we see that these generators are in the left normal form. □

Let A be a block matrix with given entries in the strictly lower triangular part. Using a specification of the algorithm from Theorem 5.9 one obtains a set of minimal quasiseparable generators of A in the left normal form.

Theorem 5.30. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be a matrix with block entries* A_{ij} *of sizes* $m_i \times n_j$.

The following algorithm yields a set $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ *of lower quasiseparable generators in the left normal form of* A.

1. *Set* $A^{(1)} = A(2 : N, 1)$. *Using orthogonal factorization of the matrix* $A^{(1)}$ *determine the matrices* $P_2, q(1)$ *of sizes* $\left(\sum_{i=2}^N m_i\right) \times r_1$ *and* $r_1 \times n_1$*, respectively, such that*

$$A^{(1)} = P_2 q(1), \tag{5.99}$$

with P_2 satisfying the relations

$$P_2^* P_2 = I_{r_1}, \quad r_1 = \operatorname{rank} A^{(1)}. \tag{5.100}$$

2. *For $k = 2, \dots, N-1$ perform the following.*
   Set $A^{(k)} = A(k+1 : N, k)$.
   Determine the matrices $p(k), P_k''$ of sizes $m_k \times r_{k-1}, \left(\sum_{i=k+1}^N m_i\right) \times r_{k-1}$, respectively, from the partition

$$P_k = \begin{pmatrix} p(k) \\ P_k'' \end{pmatrix}. \tag{5.101}$$

   Using orthogonal factorization of the matrix $\begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix}$ determine the matrices P_{k+1}, V_k of sizes $\left(\sum_{i=k+1}^N m_i\right) \times r_k, r_k \times (r_{k-1} + n_k)$, respectively, such that

$$\begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix} = P_{k+1} V_k, \tag{5.102}$$

   with P_{k+1} satisfying the relations

$$P_{k+1}^* P_{k+1} = I_{r_k}, \quad r_k = \operatorname{rank} \begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix}. \tag{5.103}$$

   Determine the matrices $a(k), q(k)$ of sizes $r_k \times r_{k-1}, r_k \times n_k$, respectively, from the partition

$$V_k = \begin{pmatrix} a(k) & q(k) \end{pmatrix}. \tag{5.104}$$

3. *Set*

$$p(N) = P_N. \tag{5.105}$$

Proof. The orthogonal factorization is a particular case of the rank factorization. Hence, by Theorem 5.9, the elements $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ are minimal lower quasiseparable generators of the matrix A. Moreover, using the relations (5.100), (5.103) and the second part of Lemma 5.28 one concludes that these generators are in the left normal form. □

§5.9 Approximation by matrices with quasiseparable representation

Next we consider approximation of a given matrix A by a matrix $\tilde{A}$ with small quasiseparable order. There are two ways to proceed. In the first, the maximal order $\hat{l}$ of quasiseparable generators of the approximation is given. In the second one, the tolerance τ of the approximation is given and in this case the orders of generators are not restricted. For a given block matrix $A = \{A_{ij}\}_{i,j=1}^N$ with block entries A_{ij} of sizes $m_i \times n_j$ based on the algorithm from Theorem 5.30 one obtains the following algorithm to compute lower quasiseparable generators of such an approximation.

Algorithm 5.31.

1. Set $A^{(1)} = A(2:N,1)$. Compute the SVD decomposition of the matrix $A^{(1)}$, i.e., determine the matrices U_1, Σ_1, V_1 of sizes $\left(\sum_{i=2}^{N} m_i\right) \times \rho_1, \rho_1 \times \rho_1, \rho_1 \times n_1$, respectively, such that
$$A^{(1)} = U_1 \Sigma_1 V_1,$$
where
$$\operatorname{rank} A^{(1)} = \operatorname{rank} U_1 = \operatorname{rank} \Sigma_1 = \operatorname{rank} V_1 = \rho_1, \quad U_1^* U_1 = V_1 V_1^* = I_{\rho_1}$$
and Σ_1 is a diagonal matrix with positive diagonal entries. Set
$$r_1 = \min\{\rho_1, \hat{l}\}$$
or take r_1 to be equal to the number of diagonal entries of the matrix Σ_1 greater than the tolerance τ.
Determine the matrices $P_2, q(1)$ via
$$P_2 = U_1(:, 1:r_1), \quad q(1) = \Sigma_1(1:r_1, 1:r_1) V_1(1:r_1, :).$$

2. For $k = 2, \dots, N-1$ perform the following.
Set $A^{(k)} = A(k+1:N, k)$.
Determine the matrices $p(k), P_k''$ of sizes $m_k \times r_{k-1}, \left(\sum_{i=k+1}^{N} m_i\right) \times r_{k-1}$, respectively, from the partition
$$P_k = \begin{pmatrix} p(k) \\ P_k'' \end{pmatrix}.$$
Compute the SVD decomposition of the matrix $\begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix}$, i.e., determine the matrices U_k, Σ_k, V_k of sizes $\left(\sum_{i=k+1}^{N} m_i\right) \times \rho_k, \rho_k \times \rho_k, \rho_k \times (n_k + r_{k-1})$, respectively, such that
$$\begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix} = U_k \Sigma_k V_k,$$
where
$$\operatorname{rank} \begin{pmatrix} P_k'' & A^{(k)} \end{pmatrix} = \operatorname{rank} U_k = \operatorname{rank} \Sigma_k = \operatorname{rank} V_k = \rho_k, \ \ U_k^* U_k = V_k V_k^* = I_{\rho_k}$$
and Σ_k is a diagonal matrix with positive diagonal entries. Set
$$r_k = \min\{\rho_k, \hat{l}\}$$
or take r_k to be equal to the number of diagonal entries of the matrix Σ_k greater than the tolerance τ.

Determine the matrices $P_{k+1}, a(k), q(k)$ via

$$\begin{aligned} P_{k+1} &= U_k(:, 1 : r_k), \\ a(k) &= \Sigma_k(1 : r_k, 1 : r_k)V_k(1 : r_k, 1 : r_{k-1}), \\ q(k) &= \Sigma_k(1 : r_k, 1 : r_k)V_k(1 : r_k, r_{k-1} + 1 : r_{k-1} + n_k). \end{aligned}$$

3. Set $p(N) = P_N$.

Note that this algorithm yields quasiseparable generators in the left normal form.

§5.10 Comments

The auxiliary matrices from Section 1 have been defined and used in the paper [38], the monograph [15] and the paper [20]. An algorithm to compute minimal quasiseparable generators of a matrix has been suggested in [15]. Minimality and similarity conditions for quasiseparable generators were studied in [38] and [15]. The method of reduction to minimal generators was suggested in [15]. The normal quasiseparable generators were used in fact in the monographs [46, 47]. The idea of the approximation method presented in Section 9 is taken from [15] and [6].

The material of this chapter is taken mainly from [27], where the minimality Theorem 5.18 appeared for the first time.

Chapter 6

Rank Numbers of Pairs of Mutually Inverse Matrices, Asplund Theorems

In this chapter we extend the notion of rank numbers introduced in Chapter 4 to wider sets of submatrices. Lower rank numbers for a square matrix relative to the diagonal $i - j$ are introduced as the ranks of the maximal submatrices entirely located under that diagonal, and the upper rank numbers relative to a diagonal are defined correspondingly. If the given matrix is invertible, a strong link exists between these numbers for the matrix and its inverse. In particular, the lower and upper rank numbers relative to the main diagonal are the same for a matrix with square blocks on the main diagonal and for its inverse matrix. This implies that for such a square matrix the lower and upper quasiseparable orders coincide with the ones of the inverse matrix.

A class of square block matrices $A = \{A_{ij}\}$ with square blocks A_{ii} on the main diagonal, and whose inverses are band matrices is thoroughly studied. These are the Green matrices. Namely, it turns out that the lower and upper rank numbers relative to a diagonal are larger than a certain minimal value. A lower Green matrix of order t is a matrix whose lower rank numbers relative to the diagonal $i - j = t$ are the minimal ones. An upper Green matrix of order t is defined accordingly, and a Green matrix of order t is an upper and a lower Green matrix of order t. The first Asplund theorem, which states that A is a Green matrix of order t if and only if A^{-1} is a band matrix of order t, readily follows from the proved relation between the rank numbers relative to a diagonal for the pair (A, A^{-1}).

It is then proved that for a square invertible block matrix A the sum of minimal completion ranks for the lower triangular part of A and plus the corresponding sum for its inverse A^{-1} equals the size of the matrix. This implies the second Asplund theorem. Another corollary of this result gives an equivalent criterion for the equality between the orders of the minimal lower semiseparable generators of A and A^{-1}.

In this chapter we also consider rank-one perturbations of Green matrices and an extension of the results obtained for the inverses of matrices on linear-fractional transformations.

Y. Eidelman et al., *Separable Type Representations of Matrices and Fast Algorithms: Volume 1 Basics. Completion Problems. Multiplication and Inversion Algorithms*, Operator Theory: Advances and Applications 234, DOI 10.1007/978-3-0348-0606-0_6, © Springer Basel 2014

§6.1 Rank numbers of pairs of inverse matrices

Let $A = \{A_{ij}\}_{i,j=1}^N$ be a square matrix with block entries A_{ij} of sizes $m_i \times n_j$, where $\sum_{i=1}^N m_i = \sum_{i=1}^N n_i$, and let t be an integer such that $|t| < N$. Denote $t_0 = \max\{1, -t\}$, $t_N = \min\{N-t-1, N\}$. We will consider maximal submatrices of the matrix A with the indices of entries satisfying $i - j > t$. These submatrices are $A(k+t+1 : N, 1 : k)$, $k = t_0, \dots, t_N$. The ranks $r^L_{k,t}(A)$ of these submatrices are called the *lower rank numbers of the matrix A relative to the diagonal $i-j = t$*, i.e.,

$$r^L_{k,t}(A) = \operatorname{rank} A(k+t+1 : N, 1 : k), \quad k = t_0, \dots, t_N.$$

Similarly we consider the maximal submatrices of A with the indices of entries satisfying $j - i > t$. These submatrices are $A(1 : k, k+t+1 : N)$, $k = t_0, \dots, t_N$. The ranks $r^U_{k,t}$ of these submatrices are called the *upper rank numbers of the matrix A relative to the diagonal $j - i = t$*, i.e.,

$$r^U_{k,t}(A) = \operatorname{rank} A(1 : k, k+t+1 : N), \quad k = t_0, \dots, t_N.$$

We define also $r^L_{k,t}(A)$, $r^U_{k,t}(A)$ for the values of k less than t_0 or greater than t_N setting them to be zero.

We consider here relations between the rank numbers of a matrix and of its inverse.

Theorem 6.1. *Let A be an invertible block matrix with entries of sizes $m_i \times n_j$, $i, j = 1, \dots, N$. Consider the matrix A^{-1} as a block matrix with sizes of entries $n_i \times m_j$, $i, j = 1, \dots, N$.*

The following relations hold:

$$r^L_{k,t}(A) + \sum_{i=k+1}^N n_i = r^L_{k+t,-t}(A^{-1}) + \sum_{i=k+t+1}^N m_i, \quad k = t_0, \dots, t_N, \tag{6.1}$$

$$r^U_{k,t}(A) + \sum_{i=k+1}^N m_i = r^U_{k+t,-t}(A^{-1}) + \sum_{i=k+t+1}^N n_i, \quad k = t_0, \dots, t_N. \tag{6.2}$$

Proof. Let Q be an $N_1 \times N_1$ invertible matrix and l_1, t_1, l_2, t_2 be nonnegative integers such that $l_1 + l_2 = t_1 + t_2 = N_1$. We consider the partitions of the matrices Q, Q^{-1}

$$Q = \begin{pmatrix} U & B \\ C & D \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} U' & B' \\ C' & D' \end{pmatrix},$$

where U, B, C, D are matrices of sizes $l_1 \times t_1$, $l_1 \times t_2$, $l_2 \times t_1$, $l_2 \times t_2$, respectively, and U', B', C', D' are matrices of the corresponding sizes $t_1 \times l_1$, $t_1 \times l_2$, $t_2 \times l_1$, $t_2 \times l_2$.

We prove that the ranks of the matrices C and C' are connected via relations

$$\operatorname{rank} C + t_2 = \operatorname{rank} C' + l_2. \tag{6.3}$$

One can check directly that

$$\begin{pmatrix} C & 0 \\ 0 & I_{t_2} \end{pmatrix} = E \begin{pmatrix} C' & 0 \\ 0 & I_{l_2} \end{pmatrix} F, \tag{6.4}$$

where

$$E = \begin{pmatrix} -D & CB' \\ I_{t_2} & D' \end{pmatrix}, \quad F = Q$$

are invertible matrices with inverses

$$E^{-1} = \begin{pmatrix} -D' & C'B \\ I_{l_2} & D \end{pmatrix}, \quad F^{-1} = Q^{-1}.$$

Indeed, one has

$$E \begin{pmatrix} C' & 0 \\ 0 & I \end{pmatrix} F = \begin{pmatrix} -D & CB' \\ I & D' \end{pmatrix} \begin{pmatrix} C' & 0 \\ 0 & I \end{pmatrix} F = \begin{pmatrix} -DC' & CB' \\ C' & D' \end{pmatrix} \begin{pmatrix} U & B \\ C & D \end{pmatrix}$$
$$= \begin{pmatrix} -DC'U + CB'C & -DC'B + CB'D \\ C'U + D'C & C'B + D'D \end{pmatrix}.$$

It is clear that $C'U + D'C = 0$, $C'B + D'D = I$ and using the equalities $C'U = -D'C$, $DD' + CB' = I$, $C'B + D'D = I$ one obtains

$$-DC'U + CB'C = DD'C + CB'C = C,$$
$$-DC'B + CB'D = -D(I - D'D) + (I - DD')D = 0.$$

Similarly, one obtains

$$E \begin{pmatrix} -D' & C'B \\ I & D \end{pmatrix} = \begin{pmatrix} -D & CB' \\ I & D' \end{pmatrix} \begin{pmatrix} -D' & C'B \\ I & D \end{pmatrix}$$
$$= \begin{pmatrix} DD' + CB' & -DC'B + CB'D \\ 0 & C'B + D'D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

From (6.4) it follows that

$$\operatorname{rank} \begin{pmatrix} C & 0 \\ 0 & I_{t_2} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} C' & 0 \\ 0 & I_{l_2} \end{pmatrix},$$

which implies (6.3).

Now we take $Q = A$, where A is from the statement of the theorem. We set

$$l_2 = \sum_{i=k+t+1}^{N} m_i, \quad t_2 = \sum_{i=k+1}^{N} n_i. \tag{6.5}$$

Then

$$\begin{aligned} C &= A(k+t+1:N, 1:k), \\ C' &= A^{-1}(k+1:N, 1:k+t) \\ &= A^{-1}((k+t)-t+1:N, 1:k+t) \end{aligned}$$

and by the definition of the lower rank numbers relative to some diagonal one obtains

$$\operatorname{rank} C = r^L_{k,t}(A), \quad \operatorname{rank} C' = r^L_{k+t,-t}(A^{-1}). \tag{6.6}$$

Substituting the expressions (6.5), (6.6) in (6.3) one obtains (6.1).

The application of (6.1) to the transpose matrix A^T yields (6.2). □

§6.2 Rank numbers relative to the main diagonal. Quasiseparable orders

Here we consider relations for rank numbers of pairs of mutually inverse matrices relative to the main diagonal, i.e., the case $t = 0$.

Corollary 6.2. *Let A be an invertible block matrix with entries of sizes $m_i \times n_j$, $i, j = 1, \dots, N$. Consider the matrix A^{-1} as a block matrix with sizes of entries $n_i \times m_j$, $i, j = 1, \dots, N$.*

The following relations hold:

$$r^L_{k,0}(A) + \sum_{i=k+1}^{N} n_i = r^L_{k,0}(A^{-1}) + \sum_{i=k+1}^{N} m_i, \quad k = 1, \dots, N-1, \tag{6.7}$$

$$r^U_{k,0}(A) + \sum_{i=k+1}^{N} m_i = r^U_{k,0}(A^{-1}) + \sum_{i=k+1}^{N} n_i, \quad k = 1, \dots, N-1. \tag{6.8}$$

The proof is obtained directly from (6.1), (6.2) by setting $t = 0$.

Now we consider the case of a block matrix with square entries on the main diagonal. Concerning the rank numbers relative to the main diagonal one obtains the following result.

Corollary 6.3. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be an invertible block matrix with entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$.*

Then rank numbers of A relative to the main diagonal $i = j$ coincide with the corresponding rank numbers of A^{-1}.

The proof is obtained directly from (6.7), (6.8) by setting $m_i = n_i$.

Corollary 6.3 implies that for an invertible block matrix A with square entries on the main diagonal, the quasiseparable orders n_L, n_U coincide with quasiseparable orders of the inverse A^{-1}. The same is true for the corresponding rank numbers of A and A^{-1}.

§6.3 Green and band matrices

Here we consider a class of matrices whose inverses, if they exist, are band matrices.

Consider first square scalar matrices $A = \{A_{ij}\}_{i,j=1}^N$. Let t be an integer such that $N > t > 0$. The matrix A is called *an upper Green matrix of order t* if its upper rank numbers relative to the diagonal $j - i = -t$ are not greater than t:

$$r_{k,-t}^U(A) = \operatorname{rank} A(1:k, k-t+1:N) \leq t, \quad k = t, t+1, \dots, N.$$

Notice that if A is an invertible upper Green matrix of order t, then all its upper rank numbers relative to the diagonal $j - i = -t$ equal t. Indeed, assume that for some $k \in \{t, \dots, N\}$ we have $r_{k,-t}^U(A) < t$. We get

$$A(1:k,:) = \begin{pmatrix} A(1:k, 1:k-t) & A(1:k, k-t+1:N) \end{pmatrix}.$$

Here $A(1:k, 1:k-t)$ is a $k \times (k-t)$ matrix and $\operatorname{rank} A(1:k, k-t+1:N) < t$. Hence, $\operatorname{rank} A(1:k,:) < k$ and therefore A is a singular matrix.

A scalar matrix $A = \{A_{ij}\}_{i,j=1}^N$ is called *an upper band matrix of order t* if $A_{ij} = 0$ for $j - i > t$.

Theorem 6.4 (The first Asplund theorem)**.** *An invertible matrix A is an upper band matrix of order t if and only if its inverse is an upper Green matrix of order t.*

The proof follows from the more general Theorem 6.6 obtained below for block matrices.

Next we consider block matrices $A = \{A_{ij}\}_{i,j=1}^N$ with entries of sizes $m_i \times m_j,\ i,j = 1, \dots, N$.

Lemma 6.5. *Let A be an invertible matrix with block entries of sizes $m_i \times m_j,\ i,j = 1, \dots, N$. Then the rank numbers $r_{k,-t}^L(A), r_{k,-t}^U(A)$ $(k = t+1, \dots, N-1)$ of A relative to the diagonals $i - j = -t,\ j - i = -t$, respectively, satisfy the inequalities*

$$r_{k,-t}^L(A),\ r_{k,-t}^U(A) \geq \alpha_k, \quad k = t+1, \dots, N-1,$$

with

$$\alpha_k = \sum_{i=k-t+1}^{k} m_i, \quad k = t+1, \dots, N-1. \tag{6.9}$$

Proof. Assume that for some k with $t + 1 \leq k \leq N - 1$ one has

$$r_{k,-t}^U(A) = \operatorname{rank} A(1:k, k-t+1:N) < \alpha_k. \tag{6.10}$$

The matrix $A(1:k,:)$ contains $m_1 + \dots + m_k$ rows and is obtained from the submatrix $A(1:k, k-t+1:N)$ by the addition of $m_1 + \dots + m_{k-t}$ columns. From (6.10) it follows that the rank of $A(1:k,:)$ is less than the number of its rows. This implies that the matrix A is not invertible.

Using the inequalities $r_{k,-t}^U(A) \geq \alpha_k$ $(k = t+1, \dots, N-1)$ for the transpose matrix A^T one obtains $r_{k,-t}^L(A) \geq \alpha_k$ $(k = t+1, \dots, N-1)$. □

We call the matrix A a *lower Green matrix of order* t if its lower rank numbers relative to the diagonal $i-j=-t$ satisfy the inequalities

$$r^L_{k,-t}(A) \leq \sum_{i=k-t+1}^{k} m_i, \quad k=t+1,\dots,N-1. \tag{6.11}$$

By Lemma 6.5, an invertible lower Green matrix of order t satisfies the equalities

$$r^L_{k,-t}(A) = \sum_{i=k-t+1}^{k} m_i, \quad k=t+1,\dots,N-1. \tag{6.12}$$

The matrix A is called an *upper Green matrix of order* t if its upper rank numbers relative to the diagonal $j-i=-t$ satisfy the inequalities

$$r^U_{k,-t}(A) \leq \sum_{i=k-t+1}^{k} m_i, \quad k=t+1,\dots,N-1. \tag{6.13}$$

By Lemma 6.5, an invertible upper Green matrix of order t satisfies the equalities

$$r^U_{k,-t}(A) = \sum_{i=k-t+1}^{k} m_i, \quad k=t+1,\dots,N-1. \tag{6.14}$$

The matrix A is said to be a *Green matrix of order* t if it is both an upper and a lower Green matrix of order t.

Let A be a Green matrix of order t. Set $n_0 = \max_{t+1\leq k\leq N-1} \alpha_k$ with α_k from (6.9). It is clear that A has quasiseparable order (n_0, n_0) at most. Indeed, each submatrix of A of the form $A(1:k, k+1:N)$ $(k=1,\dots,N-1)$ is a part of the submatrix $\Omega_k = A(1:k, k-t+1:N)$ for $k=t+1,\dots,N-1$ and a part of the submatrix $\Omega_{t+1} = A(1:t+1, 2:N)$ for $k=1,\dots,t$. Since, by assumption, $\operatorname{rank}\Omega_k = \alpha_k$, one obtains

$$\operatorname{rank} A(1:k, k+1:N) \leq n_0, \quad k=1,\dots,N-1.$$

One can check similarly that

$$\operatorname{rank} A(k+1:N, 1:k) \leq n_0, \quad k=1,\dots,N-1.$$

A matrix $A=\{A_{ij}\}_{i,j=1}^N$ is said to be a *lower band matrix of order* t if $A_{ij}=0$ for $i-j>t$, an *upper band matrix of order* t if $A_{ij}=0$ for $j-i>t$, and a *band matrix of order* t if $A_{ij}=0$ for $|i-j|>t$.

Theorem 6.6. *Let $A=\{A_{ij}\}_{i,j=1}^N$ be an invertible matrix with block entries of sizes $m_i\times m_j$, $i,j=1,\dots,N$ and let $t>0$ be an integer.*

Then $A^{-1}=\{B_{ij}\}_{i,j=1}^N$ is a lower band matrix of order t if and only if A is a lower Green matrix of order t.

A^{-1} is an upper band matrix of order t if and only if A is an upper Green matrix of order t.

A^{-1} is a band matrix of order t if and only if A is a Green matrix of order t.

Proof. Setting in (6.1) $n_i = m_i$, changing t to $-t$ and taking into account that $t > 0$, one obtains

$$r^L_{k,-t}(A) = r^L_{k-t,t}(A^{-1}) + \sum_{i=k-t+1}^{k} m_i, \quad k = t+1, \dots, N-1.$$

Hence it follows that (6.12) holds if and only if

$$r^L_{k-t,t}(A^{-1}) = \operatorname{rank} A^{-1}(k+1 : N, 1 : k-t) = 0, \quad k = t+1, \dots, N-1.$$

The last relations are valid if and only if $B_{ij} = 0$ for $i - j > t$.

The application of (6.12) to the transposed matrix A^T yields (6.14). □

Example 6.7. Consider the 7×7 matrix

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

A is a band matrix of order 2, which means that its entries $A_{ij} = 0$, for any $|i-j| > 2$.

Its inverse $B = A^{-1}$ must be a Green matrix of order 2. Indeed, we have

$$B = A^{-1} = \frac{1}{9} \cdot \begin{pmatrix} 7 & -6 & 0 & 4 & -3 & 0 & 1 \\ -6 & 12 & -6 & -3 & 6 & -3 & 0 \\ 0 & -6 & 12 & -6 & -3 & 6 & -3 \\ 4 & -3 & -6 & 13 & -6 & -3 & 4 \\ -3 & 6 & -3 & -6 & 12 & -6 & 0 \\ 0 & -3 & 6 & -3 & -6 & 12 & -6 \\ 1 & 0 & -3 & 4 & 0 & -6 & 7 \end{pmatrix}$$

and one can check easily that

$$\begin{aligned} \operatorname{rank} B(1:7, 1:2) &= \operatorname{rank} B(2:7, 1:3) = \operatorname{rank} B(3:7, 1:4) \\ &= \operatorname{rank} B(4:7, 1:5) = \operatorname{rank} B(5:7, 1:6) \\ &= \operatorname{rank} B(6:7, 1:7) = 2 \end{aligned} \tag{6.15}$$

and

$$\begin{aligned}\operatorname{rank} B(1:2,1:7) &= \operatorname{rank} B(1:3,2:7) = \operatorname{rank} B(1:4,3:7)\\ &= \operatorname{rank} B(1:5,4:7) = \operatorname{rank} B(1:6,5:7)\\ &= \operatorname{rank} B(1:7,6:7) = 2. \end{aligned} \qquad \Diamond$$

§6.4 The inverses of diagonal plus Green of order one matrices

Here we consider invertible block matrices of the form $A = D + G$, where D is a block diagonal matrix and G is a Green matrix of order one. We show that when D is invertible the inverse matrix A^{-1} has the same form.

Theorem 6.8. *Let A be a block invertible matrix with entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, represented in the form $A = D + G$, where $D = \operatorname{diag}(D(1), D(2), \dots, D(N))$ is a block diagonal invertible matrix and G is a block lower Green of order one matrix.*

Then the matrix A^{-1} admits the representation $A^{-1} = D^{-1} + G^{\times}$, where $G^{\times}$ is a block lower Green of order one matrix.

Proof. For $k = 2, \dots, N-1$ we use the partitions of the matrix A in the form

$$A = \begin{pmatrix} A(1:k-1,1:k-1) & A(1:k-1,k) & A(1:k-1,k+1:N) \\ A(k,1:k-1) & A(k,k) & A(k,k+1:N) \\ A(k+1:N,1:k-1) & A(k+1:N,k) & A(k+1:N,k+1:N) \end{pmatrix}. \tag{6.16}$$

Since $A - D$ is a Green of order one matrix, (6.9) yields

$$\operatorname{rank} \begin{pmatrix} A(k,1:k-1) & A(k,k) - D(k) \\ A(k+1:N,1:k-1) & A(k+1:N,k) \end{pmatrix} \le m_k.$$

Consequently,

$$\begin{pmatrix} A(k,1:k-1) & A(k,k) - D(k) \\ A(k+1:N,1:k-1) & A(k+1:N,k) \end{pmatrix} = \begin{pmatrix} p(k) \\ P_{k+1} \end{pmatrix} \begin{pmatrix} Q_{k-1} & q(k) \end{pmatrix} \tag{6.17}$$

with the matrices $p(k), P_{k+1}, Q_{k-1}, q(k)$ of sizes $m_k \times m_k$, $\left(\sum_{i=k+1}^{N} m_i\right) \times m_k$, $m_k \times \left(\sum_{i=1}^{k-1} m_i\right)$, $m_k \times m_k$, respectively.

One can determine the matrices $\tilde{P}_{k-1}$ and $\tilde{Q}_{k+1}$ of sizes $\left(\sum_{i=1}^{k-1} m_i\right) \times m_k$ and $m_k \times \left(\sum_{i=k+1}^{N} m_i\right)$, respectively, such that the matrices

$$B_{k-1} = A(1:k-1,1:k-1) - \tilde{P}_{k-1} Q_{k-1} \tag{6.18}$$

and

$$C_{k+1} = A(k+1:N, k+1:N) - P_{k+1}\tilde{Q}_{k+1} \tag{6.19}$$

are invertible. Indeed, introduce the notations

$$A_{k-1} = A(1:k-1, 1:k-1), \quad P_k = \begin{pmatrix} p(k) \\ P_{k+1} \end{pmatrix}, \quad \nu_k = \sum_{i=1}^{k-1} m_i.$$

From (6.16) and (6.17) it follows that

$$A(:, 1:k-1) = \begin{pmatrix} I & 0 \\ 0 & P_k \end{pmatrix} \begin{pmatrix} A_{k-1} \\ Q_{k-1} \end{pmatrix}.$$

Hence, since A is invertible, we get $\operatorname{rank} \begin{pmatrix} A_{k-1} \\ Q_{k-1} \end{pmatrix} = \nu_k$. Set $\rho_k = \operatorname{rank} A_{k-1}$; it is clear that $\rho_k \le \nu_k$. One can determine an invertible $\nu_k \times \nu_k$ matrix R_k such that

$$R_k A_{k-1} = \begin{pmatrix} A'_{k-1} \\ 0_{(\nu_k - \rho_k)\times \nu_k} \end{pmatrix}, \tag{6.20}$$

with a $\rho_k \times \nu_k$ matrix A'_{k-1} such that $\operatorname{rank} A'_{k-1} = \rho_k$. Let Q'_{k-1} be a $(\nu_k - \rho_k) \times \nu_k$ matrix composed of the rows of the matrix Q_{k-1} which completes the rows of the matrix A'_{k-1} to the row basis of the matrix $\begin{pmatrix} A_{k-1} \\ Q_{k-1} \end{pmatrix}$. One can choose a $\nu_k \times m_k$ matrix Z_k such that

$$Z_k Q_{k-1} = \begin{pmatrix} 0_{\rho_k \times \nu_k} \\ Q'_{k-1} \end{pmatrix}. \tag{6.21}$$

From (6.20) and (6.21) it follows that

$$R_k A_{k-1} + Z_k Q_{k-1} = \begin{pmatrix} A'_{k-1} \\ Q'_{k-1} \end{pmatrix}. \tag{6.22}$$

It is clear that the matrix $\begin{pmatrix} A'_{k-1} \\ Q'_{k-1} \end{pmatrix}$ is invertible. Hence (6.22) implies (6.18) with the invertible matrix

$$B_{k-1} = R_k^{-1} \begin{pmatrix} A'_{k-1} \\ Q'_{k-1} \end{pmatrix}$$

and $\tilde{P}_{k-1} = -R_k^{-1} Z_k$. In a similar way one can obtain the equality (6.19) with an invertible matrix C_{k+1}.

Next set

$$P = \begin{pmatrix} \tilde{P}_{k-1} \\ p(k) \\ P_{k+1} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{k-1} & q(k) & \tilde{Q}_{k+1} \end{pmatrix}.$$

Using the equalities (6.16) and (6.17) we get

$$A - PQ = S,$$

with the block upper triangular invertible matrix S of the form

$$S = \begin{pmatrix} B_{k-1} & * & * \\ 0 & D(k) & * \\ 0 & 0 & C_{k+1} \end{pmatrix}.$$

By Theorem 1.21,

$$A^{-1} = S^{-1} - W, \tag{6.23}$$

with $W = S^{-1}P(I_{m_k} + QS^{-1}P)^{-1}QS^{-1}$. It is clear that

$$S^{-1} = \begin{pmatrix} B_{k-1}^{-1} & * & * \\ 0 & (D(k))^{-1} & * \\ 0 & 0 & C_{k+1}^{-1} \end{pmatrix} \tag{6.24}$$

and

$$\operatorname{rank} W \leq m_k. \tag{6.25}$$

Consider the matrix $G^{\times} = A^{-1} - D^{-1}$. From (6.23) and (6.24) it follows that

$$G^{\times}(k : N, 1 : k) = -W(k : N, 1 : k)$$

and hence, using (6.25), we conclude that

$$\operatorname{rank} G^{\times}(k : N, 1 : k) \leq m_k, \quad k = 2, \dots, N-1.$$

This means that $G^{\times}$ is a block lower Green of order one matrix. □

Corollary 6.9. *Let A be a block invertible matrix with entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, represented in the form $A = D + F$, where $D = \operatorname{diag}(D(1), D(2), \dots, D(N))$ is a block diagonal matrix and F is a block upper Green of order one matrix.*

Then the matrix A^{-1} admits the representation $A^{-1} = D^{-1} + F^{\times}$, where $F^{\times}$ is a block upper Green of order one matrix.

The proof is obtained by applying Theorem 6.8 to the transposed matrix A^T.

Combining Theorem 6.8 and Corollary 6.9 we obtain the corresponding result for diagonal plus Green of order one matrices.

Theorem 6.10. *Let A be a block invertible matrix with entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, represented in the form $A = D + G$, where $D = \operatorname{diag}(D(1), D(2), \dots, D(N))$ is a block diagonal matrix and G is a block Green of order one matrix.*

Then the matrix A^{-1} admits the representation $A^{-1} = D^{-1} + G^{\times}$, where $G^{\times}$ is a block Green of order one matrix.

§6.5 Minimal completion ranks of pairs of mutually inverse matrices. The inverse of an irreducible band matrix

Here we apply Theorem 6.1 to obtain relations between minimal completion ranks of pairs of mutually inverse matrices. As a corollary we obtain a version of the second Asplund theorem.

Theorem 6.11. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be an invertible matrix with block entries* A_{ij} *of size* $m_i \times n_j$*. So* $\sum_{i=1}^N m_i = \sum_{i=1}^N n_i =: N_1$*. The inverse of* A *is partitioned according to the partitioning of* A*:* $A^{-1} := B = \{B_{ij}\}_{i,j=1}^N$*, where* B_{ij} *is of size* $n_i \times m_j$*. Put*

$$\mathcal{A} = \{A_{ij}\}_{1\le j\le i\le N}, \quad \mathcal{B} = \{B_{ij}\}_{1\le j<i\le N}.$$

Then

$$\hat{r}_L(\mathcal{A}) + \hat{r}_L(\mathcal{B}) = N_1, \tag{6.26}$$

where $\hat{r}_L(\mathcal{A})$ *and* $\hat{r}_L(\mathcal{B})$ *are the minimal completion ranks of the parts* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively.*

Proof. Applying formula (2.30) to the matrix A one gets

$$\hat{r}_L(\mathcal{A}) = \sum_{k=1}^N \operatorname{rank} A(k:N, 1:k) - \sum_{k=1}^{N-1} \operatorname{rank} A(k+1:N, 1:k). \tag{6.27}$$

Applying formula (3.3) to the matrix $B = A^{-1}$ one gets

$$\hat{r}_L(\mathcal{B}) = \sum_{k=1}^{N-1} \operatorname{rank} B(k+1:N, 1:k) - \sum_{k=2}^{N-1} \operatorname{rank} B(k+1:N, 1:k-1). \tag{6.28}$$

Using formula (6.1) with $t = 0$ and $t = -1$ one obtains the equalities

$$\begin{aligned} &\operatorname{rank} A(k+1:N, 1:k) + \sum_{i=k+1}^N n_i \\ &\quad = \operatorname{rank} B(k+1:N, 1:k) + \sum_{i=k+1}^N m_i, \quad k = 1, \dots, N-1 \end{aligned} \tag{6.29}$$

and

$$\begin{aligned} &\operatorname{rank} A(k:N, 1:k) + \sum_{i=k+1}^N n_i \\ &\quad = \operatorname{rank} B(k+1:N, 1:k-1) + \sum_{i=k}^N m_i, \quad k = 1, \dots, N. \end{aligned} \tag{6.30}$$

Now (6.29) and (6.30) yield the equalities

$$\begin{aligned}\sum_{k=1}^{N-1} \operatorname{rank} A(k+1:N,1:k) + \sum_{k=1}^{N-1}\sum_{i=k+1}^{N} n_i \\ = \sum_{k=1}^{N-1} \operatorname{rank} B(k+1:N,1:k) + \sum_{k=1}^{N-1}\sum_{i=k+1}^{N} m_i\end{aligned} \tag{6.31}$$

and

$$\begin{aligned}\sum_{k=1}^{N} \operatorname{rank} A(k:N,1:k) + \sum_{k=1}^{N-1}\sum_{i=k+1}^{N} n_i \\ = \sum_{k=1}^{N-1} \operatorname{rank} B(k+1:N,1:k-1) + \sum_{k=1}^{N}\sum_{i=k}^{N} m_i.\end{aligned} \tag{6.32}$$

Subtracting (6.31) from (6.32) and using formulas (6.27), (6.28) one obtains

$$\hat{r}_L(\mathcal{A}) = -\hat{r}_L(\mathcal{B}) + \sum_{k=1}^{N} m_k,$$

which implies (6.26). □

Example 6.12. Consider the 7×7 matrix

$$A = \begin{pmatrix} d & a & a & a & a & a & a \\ a & d & a & a & a & a & a \\ a & a & d & a & a & a & a \\ a & a & a & d & a & a & a \\ a & a & a & a & d & a & a \\ a & a & a & a & a & d & a \\ a & a & a & a & a & a & d \end{pmatrix}.$$

Applying the formula (2.30) to the lower triangular part $\mathcal{A} = \{A_{ij}\}_{1\le j\le i\le 7}$ we obtain the minimal completion rank $\hat{r}_L(\mathcal{A}) = 6$. Theorem 6.11 means that $\hat{r}_L(\mathcal{A}) + \hat{r}_L(\mathcal{B}) = 7$, where $\mathcal{B} = \{B_{ij}\}_{1\le j<i\le 7}$ is the strictly lower triangular part of the matrix $B = A^{-1}$. Notice that we have

$$B = A^{-1} = \frac{1}{(6a+d)(a-d)} \cdot \begin{pmatrix} \alpha & a & a & a & a & a & a \\ a & \alpha & a & a & a & a & a \\ a & a & \alpha & a & a & a & a \\ a & a & a & \alpha & a & a & a \\ a & a & a & a & \alpha & a & a \\ a & a & a & a & a & \alpha & a \\ a & a & a & a & a & a & \alpha \end{pmatrix}$$

where $\alpha = -(5a+d)$ and it follows that the 7×7 matrix of rank 1 which has all its entries equal to a is a minimal rank completion of $\mathcal{B}$. ◊

As a corollary of Theorem 6.11 one obtains the following well-known result.

Theorem 6.13 (The second Asplund theorem). *Let $A = \{A_{ij}\}_{i,j=1}^N$ be an invertible scalar matrix. Let $t \in \{0, 1, \dots, N-1\}$ and let $\mathcal{A}$ denotes the lower triangular part $\mathcal{A} = \{A_{ij}\}_{j<i+t,\ 1\le i,j\le N}$ of the matrix A.*

Then $\hat{r}_L(\mathcal{A}) = t$ if and only if A^{-1} is a lower band of order t matrix with nonzero elements on the diagonal $i - j = t$.

The statement of the theorem follows from a more general Theorem 6.17, which will be proved below.

Example 6.14. Consider the 7×7 matrix B from Example 6.7. This is an invertible matrix. We have (6.15) and also

$$\operatorname{rank} B(2:7,1:2) = \operatorname{rank} B(3:7,1:3) = \operatorname{rank} B(4:7,1:4) = \operatorname{rank} B(5:7,1:5) = \operatorname{rank} B(6:7,1:6) = 2.$$

We treat B as a block $(N-t+1)\times(N-t+1)$ matrix with entries of sizes $m_i \times n_j$, where $m_1 = \cdots = m_t = 1$, $m_{N-t+1} = t$, $n_1 = t$, $n_2 = \cdots = n_{N-t+1} = 1$. $\mathcal{B} = \{B_{ij}\}_{1\le j<i+2\le 7}$ is the lower triangular part of B relative to this partition. By (2.30), we get $\hat{r}_L(\mathcal{B}) = 2$. By the second Asplund theorem, the inverse B^{-1} is a lower band of order 2 irreducible matrix. This is the matrix A from the same example. ♢

Using Theorem 6.13 one can show that there is a matrix A with a given minimal order of diagonal plus semiseparable representation whose inverse A^{-1} has a larger minimal order of diagonal plus semiseparable representation.

Theorem 6.15. *Let A be an invertible $N\times N$ scalar matrix whose part*

$$\mathcal{A} = \{A_{ij}\}_{j<i+t,\ 1\le i,j\le N}$$

has the minimal completion rank t. Then the matrix A admits a diagonal plus semiseparable representation with lower semiseparable generators of order not greater than t and the order of lower semiseparable generators for the matrix A^{-1} is not less than $N-t$.

Proof. The strictly lower triangular part of the matrix A is a part of $\mathcal{A}$. Hence the minimal completion rank of the strictly upper triangular part of A is not greater than t. Therefore, by Theorem 3.2, the matrix A admits a diagonal plus semiseparable representation with lower semiseparable generators of order not greater than t.

Next, by Theorem 6.13, the inverse A^{-1} is a lower band of order t matrix with nonzero elements on the diagonal $i-j=t$. For any completion A_1 of the strictly lower triangular part of the matrix A^{-1}, the submatrix $A_0 = A_1(t+1 : N, 1 : N-t)$ is an upper triangular matrix with nonzero entries on the main diagonal. The matrix A_0 is invertible and therefore $\operatorname{rank} A_0 = N-t$, which implies

$\operatorname{rank} A_1 \geq N - t$. It follows that the minimal completion rank of the strictly lower triangular part of the matrix A is not less than $N - t$. From here, by Theorem 3.2, the minimal order of the lower semiseparable generators of the matrix A^{-1} is greater than or equal to $N - t$. $\square$

Example 6.16. This example illustrates Theorem 6.15 for a specific case of a 7×7 matrix and $t = 2$.

Consider the matrix B which appeared in Example 6.7. This matrix has $N = 7$ and it is invertible, since it has been found in that example as the inverse of the matrix A (up to the constant factor $\frac{1}{9}$, which now will be ignored). For $t = 2$ it follows from Example 6.14 that the part

$$\mathcal{B} = \{B_{ij}\}_{j<i+t,\ 1\leq i,j\leq 7} = \begin{pmatrix} 7 & -6 & * & * & * & * & * \\ -6 & 12 & -6 & * & * & * & * \\ 0 & -6 & 12 & -6 & * & * & * \\ 4 & -3 & -6 & 13 & -6 & * & * \\ -3 & 6 & -3 & -6 & 12 & -6 & * \\ 0 & -3 & 6 & -3 & -6 & 12 & -6 \\ 1 & 0 & -3 & 4 & 0 & -6 & 7 \end{pmatrix}$$

of B has the minimal completion rank t. It follows from Theorem 6.15 that the matrix B admits a diagonal plus semiseparable representation with lower semiseparable generators of order not greater than $t = 2$ and the order of lower semiseparable generators for the matrix $A = B^{-1}$ is not less than $N - t = 7 - 2 = 5$. Also, applying the formula to the matrix A in Example 6.7 we obtain that the lower semiseparable order of this matrix equals 5. $\diamondsuit$

Now we present the proof of a generalization of Theorem 6.13.

Theorem 6.17. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be an invertible block matrix with A_{ij} of size $m \times m$ $(m > 0)$. Let $t \in \{0, 1, \ldots, N-1\}$ and let $\mathcal{A}$ denote the lower triangular part of A, $\mathcal{A} = \{A_{ij}\}_{j<i+t,\ 1\leq i,j\leq N}$.*

Then $\hat{r}_L(\mathcal{A}) = tm$ if and only if A^{-1} is a lower band of order t matrix with invertible entries on the diagonal $i - j = t$.

Proof. First, let $t = 0$, i.e., $\mathcal{A} = \{A_{ij}\}_{1\leq j<i\leq N}$ is the strictly lower triangular part of the matrix A. Then $\hat{r}_L(\mathcal{A}) = 0$ if and only if A is block upper triangular. But since the blocks of A are square, this holds if and only if A^{-1} is upper triangular, and since A^{-1} is invertible its diagonal entries are invertible.

Next, let $t \in \{1, \ldots, N-1\}$. View A as an $(N - t + 1) \times (N - t + 1)$ block matrix where the first t block columns of A and its last t block rows are taken together. In A^{-1} this corresponds to taking together the first t block rows and the last t block columns. Denote $A^{-1} = \{B_{ij}\}_{i,j=1}^N$; in such partition the strictly lower triangular part of A^{-1} has the form

$$\mathcal{B} = \begin{pmatrix} B_{t+1,1} & ? & \dots & ? \\ B_{t+2,1} & B_{t+2,2} & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1} & B_{N2} & \dots & B_{N,N-t} \end{pmatrix}.$$

By Theorem 6.11, $\hat{r}_L(\mathcal{A}) = tm$ if and only if the part $\mathcal{B}$ has the minimal completion rank equal to $Nm - tm$, which is precisely its order. This is equivalent to the fact that the matrix

$$\hat{B} = \begin{pmatrix} B_{t+1,1} & B_{t+1,2} & \dots & B_{t+1,N-t} \\ B_{t+2,1} & B_{t+2,2} & \dots & B_{t+2,N-t} \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1} & B_{N2} & \dots & B_{N,N-t} \end{pmatrix}$$

is invertible and moreover, by Theorem 6.11, the minimal completion rank of the strictly lower triangular part of the matrix $\hat{B}^{-1}$ equals zero. Use now the $t = 0$ case to see that this can happen if and only if $B_{ij} = 0,\ j < i - t$ and $B_{j+t,j},\ j = 1, \dots, N - t$ are invertible matrices. □

Another corollary of Theorem 6.11 concerns the situation when the minimal completion rank of the strictly lower triangular part of a matrix coincides with the minimal completion rank of the strictly lower triangular part of its inverse.

Corollary 6.18. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be an invertible block matrix with* A_{ij} *of size* $m_i \times n_j$*, so that* $\sum_{i=1}^N m_i = \sum_{i=1}^N n_i =: N_1$*. The inverse of* A *is partitioned according to the partitioning of* A*:* $A^{-1} := B = \{B_{ij}\}_{i,j=1}^N$*, where* B_{ij} *is of size* $n_i \times m_j$*. Put*

$$\mathcal{A}_0 = \{A_{ij}\}_{1 \le j < i \le N}, \quad \mathcal{B}_0 = \{B_{ij}\}_{1 \le j < i \le N}.$$

Then

$$\hat{r}_L(\mathcal{A}_0) = \hat{r}_L(\mathcal{B}_0) \tag{6.33}$$

if and only if

$$\sum\nolimits_{k=1}^{N} \operatorname{rank} A(k : N, 1 : k) - \sum\nolimits_{k=2}^{N-1} \operatorname{rank} A(k+1 : N, 1 : k-1) = N_1. \tag{6.34}$$

Proof. Using Theorem 6.11 we see that the condition (6.33) is equivalent to the equality

$$\hat{r}_L(\mathcal{A}_0) = N_1 - \hat{r}_L(\mathcal{A}), \tag{6.35}$$

where $\mathcal{A} = \{A_{ij}\}_{1 \le j \le i \le N}$. Substituting in (6.35) the expression (6.27) and the expression

$$\hat{r}_L(\mathcal{A}_0) = \sum\nolimits_{k=1}^{N-1} \operatorname{rank} A(k+1 : N, 1 : k) - \sum\nolimits_{k=2}^{N-1} \operatorname{rank} A(k+1 : N, 1 : k-1)$$

one obtains (6.34). □

This corollary yields a necessary and sufficient condition for equality of the orders of minimal lower semiseparable generators of a matrix and of its inverse.

Example 6.19. Consider the 7×7 matrix

$$A = \begin{pmatrix} 3 & 2 & 3 & 7 & 1 & 1 & 1 \\ 2 & 5 & 3 & 1 & 1 & 1 & 1 \\ 3 & 3 & 6 & 1 & 1 & 1 & 1 \\ 7 & 1 & 1 & 7 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 8 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 9 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 10 \end{pmatrix}.$$

For the matrix A formula (6.34) is valid. Indeed,

$$\sum_{k=1}^{7} \operatorname{rank} A(k:7,1:k) - \sum_{k=2}^{6} \operatorname{rank} A(k+1:7,1:k-1)$$
$$= (1+2+3+2+2+2+1) - (1+2+1+1+1) = 7. \qquad \Diamond$$

§6.6 Linear-fractional transformations of matrices

Here we extend some results obtained above in this chapter for the inverses to the fractional transformations of matrices.

§6.6.1 The definition and the basic property

A linear-fractional transformation is a function $\Phi : \mathbb{C} \to \mathbb{C}$ defined by

$$\Phi(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{6.36}$$

with some complex numbers $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta - \beta\gamma \neq 0$. Let A be a square matrix such that the matrix $\gamma A + \delta I$ is invertible. Then the matrix $\Phi(A) = (\alpha A + \beta I)(\gamma A + \delta I)^{-1}$ is well defined. We obtain the following generalization of Corollary 6.3.

Theorem 6.20. *Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a block matrix with entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$ and let $\Phi(z)$ be a linear-fractional transformation of the form* (6.36)*. Assume that the matrix $\gamma A + \delta I$ is invertible.*

Then the rank numbers of the matrix A relative to the main diagonal $i = j$ coincide with the corresponding rank numbers of the matrix $\Phi(A)$.

Proof. Assume that $\gamma \neq 0$ and set $b = \delta/\gamma$. We obviously have

$$\Phi(A) = \frac{1}{\gamma}\left(\alpha I + (\beta - \alpha b)(A + bI)^{-1}\right). \tag{6.37}$$

Since $\alpha\delta - \beta\gamma \neq 0$, we have that $\beta - \alpha b \neq 0$. It is clear that the rank numbers of the matrix $\Phi(A)$ relative to the main diagonal coincide with the corresponding rank numbers of the matrix $(A+bI)^{-1}$. By Corollary 6.3, these rank numbers are equal to the ones of the matrix $A+bI$. The rank numbers of the latter matrix relative to the main diagonal are the same as for A.

Now let $\gamma = 0$. Then $\alpha\delta - \beta\gamma \neq 0$ implies $\delta \neq 0, \alpha \neq 0$ and therefore

$$\Phi(A) = \frac{1}{\delta}(\alpha A + \beta I). \tag{6.38}$$

In this case the statement of the theorem is trivial. □

§6.6.2 Linear-fractional transformations of Green and band matrices

Here we extend the results of Section §6.3 on linear-fractional transforms of matrices.

Theorem 6.21. *Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a block matrix with entries of sizes $m_i \times m_j$, $i,j = 1,\dots,N$ and let $t > 0$ be an integer. Let $\Phi(z)$ be a fractional transformation of the form* (6.36). *Assume that $\gamma \neq 0$ and the matrix $\gamma A + \delta I$ is invertible. Set $b = \delta/\gamma$.*

- $\Phi(A)$ *is a lower band matrix of order t if and only if $A+bI$ is a lower Green matrix of order t.*
- $\Phi(A)$ *is an upper band matrix of order t if and only if $A+bI$ is an upper Green matrix of order t.*
- $\Phi(A)$ *is a band matrix of order t if and only if $A+bI$ is a Green matrix of order t.*

Proof. From the formula (6.37) it follows that the matrix $\Phi(A)$ is a lower (upper) band matrix of order t if and only if the matrix $(A+bI)^{-1}$ is. By Theorem 6.6, the latter holds if and only if $A+bI$ is a lower (upper) Green matrix of order t. □

§6.6.3 Unitary Hessenberg and Hermitian matrices

Consider now the linear-fractional transformation (6.36) with the coefficients α, β, γ, δ satisfying the conditions

$$\begin{gathered} \gamma = |\gamma| e^{i\theta_\gamma},\ \delta = |\delta| e^{i\theta_\delta},\ \gamma \neq 0.\ \delta \neq 0,\ e^{2i(\theta_\gamma - \theta_\delta)} \neq 1, \\ \alpha = |\gamma| e^{i\tilde{\theta}},\ \beta = |\delta| e^{i\hat{\theta}},\ \hat{\theta} = \tilde{\theta} + \theta_\gamma - \theta_\delta. \end{gathered} \tag{6.39}$$

It is well known that in this case the transformation $\Phi(z)$ maps the real line onto the unit circle without the point $\hat{z} = \alpha/\gamma$. Assume that A is a Hermitian matrix such that the matrix $\gamma A + \delta I$ is invertible; then the matrix $\Phi(A)$ is well defined and $\Phi(A)$ is unitary if and only if A is a Hermitian matrix. As a direct consequence of Theorem 6.21 we obtain the following statement.

Theorem 6.22. *Let A be a scalar square matrix and let $\Phi(z)$ be a linear-fractional transformation of the form* (6.36) *satisfying the conditions* (6.39)*. Assume that the matrix $\gamma A+\delta I$ is invertible and set $b=\delta/\gamma$.*

Then $\Phi(A)$ is a unitary upper Hessenberg matrix if and only if A is Hermitian and $A+bI$ is a lower Green of order one matrix.

Proof. The matrix A is Hermitian if and only if $\Phi(A)$ is a unitary matrix. By Theorem 6.21 $\Phi(A)$ is a lower band of order one, i.e., an upper Hessenberg matrix, if and only if $A+bI$ is a lower Green of order one matrix. □

§6.6.4 Linear-fractional transformations of diagonal plus Green of order one matrices

Here we derive a generalization of Theorem 6.8 on linear-fractional transformations of matrices.

Theorem 6.23. *Let A be a block matrix with entries of sizes $m_i \times m_j$, $i,j=1,\dots,N$ represented in the form $A=D+G$, where $D=\mathrm{diag}(D(1),D(2),\dots,D(N))$ is a block diagonal invertible matrix and G is a block lower Green of order one matrix. Let $\Phi(z)$ be a fractional transformation of the form* (6.36)*. Assume that $\gamma\neq 0$ and the matrices $\gamma A+\delta I$ and $\gamma D+\delta I$ are invertible.*

Then the matrix $\Phi(A)$ admits the representation

$$\Phi(A)=\Phi(D)+G^{\times}, \tag{6.40}$$

where $G^{\times}$ is a block lower Green of order one matrix.

Proof. Assume that $\gamma\neq 0$. Setting $b=\delta/\gamma$ and using the formula (6.37) we get

$$\Phi(A)=\frac{1}{\gamma}\left(\alpha I+(\beta-\alpha b)[(D+bI)+G]^{-1}\right).$$

Applying Theorem 6.8 to the matrix $(D+bI)+G$ we obtain

$$[(D+bI)+G]^{-1}=(D+bI)^{-1}+G_0,$$

where G_0 is a block lower Green of order one matrix. Hence, it follows that

$$\Phi(A)=\frac{1}{\gamma}\left(\alpha I+(\beta-\alpha b)(D+bI)^{-1}\right)+\frac{1}{\gamma}(\beta-\alpha b)G_0. \tag{6.41}$$

Notice that

$$\frac{1}{\gamma}\left(\alpha I+(\beta-\alpha b)(D+bI)^{-1}\right)=\Phi(D)$$

and $\frac{1}{\gamma}(\beta-\alpha b)G_0$ is a block lower Green of order one matrix. Hence, it follows that (6.41) implies (6.40).

Assume that $\gamma = 0$. Using (6.38) we get

$$\Phi(A) = \frac{1}{\delta}(\alpha D + \alpha G + \beta I) = \frac{1}{\delta}(\alpha D + \beta I) + \frac{\alpha}{\delta} G = \Phi(D) + G^{\times},$$

where $G^{\times} = \frac{\alpha}{\delta} G$ is a block lower Green of order one matrix. □

§6.7 Comments

Symmetric Green matrices, named one-pair (or single-pair) matrices were considered in the monograph by F.R. Gantmacher and M.G. Krein [36]. The classical inversion theorems for Green and band matrices with elements from a non-commutative field were obtained by E. Asplund in the paper [2].

Theorem 6.1 is based on the coupling relations obtained in [3]. The presentation in the first three sections follows the paper [26]. Results similar to ones in Section 4 are presented in the monograph by R. Vandebril, M. Van Barel, and N. Mastronardi [46], but our proofs are different. The results of Section 5 were obtained by H. Woerdeman in [48]. Section 6 is based on the paper by L. Gemignani [33], but our presentation is different.

Chapter 7

Unitary Matrices with Quasiseparable Representations

In this chapter we study in detail the quasiseparable representations of unitary matrices. We show that for unitary matrices the quasiseparable representations are closely connected with factorization representations of a matrix as a product of elementary unitary matrices.

In the first section we present with the proof the well-known results on Givens rotations and QR factorizations of matrices. In the second section we derive relations between rank numbers of unitary matrices. In the third section we study factorization representations of unitary matrices and their connections with quasiseparable representations. In the fourth section we consider a special case of unitary Hessenberg matrices. In the last section we study special quasiseparable representations of matrices for which computations may be performed with a lower complexity.

§7.1 QR and related factorizations of matrices

Let A be an $m \times n$ matrix. Then A may be represented in the form

$$A = Q \cdot R \tag{7.1}$$

with an $m \times m$ unitary matrix Q and an $m \times n$ matrix $R = (R_{ij})$ such that

$$R_{ij} = 0 \quad \text{for} \quad i > j. \tag{7.2}$$

The factorization (7.1) is called *the QR factorization* of the matrix A. To determine the factors Q and R one can proceed as follows.

For a two-dimensional complex vector $x = \begin{pmatrix} a \\ b \end{pmatrix}$ there is a complex Givens

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rotation matrix, i.e., a 2×2 unitary matrix G, such that

$$Gx = \begin{pmatrix} r \\ 0 \end{pmatrix}$$

with some complex number r. The matrix G and the number r may be determined by the formulas

$$G = \begin{pmatrix} c & s \\ -s^* & c^* \end{pmatrix}, \quad r = \sqrt{|a|^2 + |b|^2},$$

where

$$c = \frac{a^*}{r}, \quad s = \frac{b^*}{r}$$

for $x \neq 0$ and $c = 1,\ s = 0$ for $x = 0$.

At first we determine a complex Givens rotation matrix G_{m-1} from the condition

$$G_{m-1}A(m-1:m,1) = \begin{pmatrix} r \\ 0 \end{pmatrix}.$$

Define the $m \times m$ unitary matrix $\tilde{G}_{m-1} = I_{m-2} \oplus G_{m-1}$. Then the matrix $A_1 = \tilde{G}_{m-1}A$ has a zero entry in the $(m,1)$ position. Next we determine a complex Givens rotation matrix G_{m-2} from the condition

$$G_{m-2}A(m-2:m-1,1) = \begin{pmatrix} r' \\ 0 \end{pmatrix}$$

and we define the $m \times m$ unitary matrix $\tilde{G}_{m-2} = I_{m-3} \oplus G_{m-2} \oplus I_1$. The matrix $A_2 = \tilde{G}_{m-2}A_1$ has zero entries in the $(m-1,1)$ and $(m,1)$ positions. We proceed in the same way with the first columns of the matrices $A, A_1, A_2, \dots, A_{m-2}$ and obtain the matrix

$$A^{(1)} := A_{m-1} = \tilde{G}_1 \cdots \tilde{G}_{m-1}A$$

with all the entries zero, except for the first one in the first column:

$$A^{(1)}(2:m,1) = 0.$$

Here $G^{(1)} = \tilde{G}_1 \cdots \tilde{G}_{m-1}$ is an $m \times m$ unitary matrix.

Next we apply the same procedure to the second column of the matrix $A^{(1)}$. We determine a complex Givens rotation matrix $G^{(1)}_{m-1}$ from the condition

$$G^{(1)}_{m-1}A^{(1)}(m-1:m,2) = \begin{pmatrix} r \\ 0 \end{pmatrix}.$$

and define the $m \times m$ unitary matrix $\tilde{G}^{(1)}_{m-1} = I_{m-2} \oplus G^{(1)}_{m-1}$. The matrix $A^{(1)}_1 = \tilde{G}^{(1)}_{m-1}A^{(1)}$ has all the entries zero, except for the first one in the first column, and also a zero entry in the $(m, 2)$ position. We continue in the same way and obtain the matrix

$$A^{(2)} = G^{(2)} G^{(1)} A$$

with zero entries in the two first columns below the main diagonal. Here $G^{(2)}G^{(1)}$ is an $m \times m$ unitary matrix.

We apply this procedure to the third column of the matrix $A^{(2)}$, and so on. Finally we obtain an $m \times m$ unitary matrix Q such that the matrix $R = Q^*A$ satisfies the condition (7.2).

Next we consider some factorizations related to the QR factorization. Let J be the $m \times m$ permutation matrix

$$J = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}.$$

One can rewrite the equality (7.1) in the form

$$A = (QJ)(JR).$$

Setting $QJ = Q',\ JR = R'$ we obtain the representation of the matrix A in the form

$$A = Q' \cdot R', \tag{7.3}$$

with a unitary $m \times m$ matrix Q' and an $m \times n$ matrix $R' = (R'_{ij})$ such that

$$R'_{ij} = 0, \quad \text{for} \quad i + j < m + 1. \tag{7.4}$$

The factorization (7.4) is called *the QR' factorization* of the matrix A.

Applying (7.1) to transposed matrices we obtain the factorization

$$A = LQ, \tag{7.5}$$

where $L = (L_{ij})$ is an $m \times n$ matrix with $L_{ij} = 0$ for $j > i$ and Q is a unitary $n \times n$ matrix. The factorization (7.5) is called *the LQ factorization* of the matrix A. Similarly applying (7.3) to transposed matrices we obtain the factorization

$$A = L'Q', \tag{7.6}$$

where $L' = (L'_{ij})$ is an $m \times n$ matrix with $L'_{ij} = 0$ for $j + i < n + 1$ and Q is a unitary $n \times n$ matrix. The factorization (7.6) is called *the $L'Q$ factorization* of the matrix A.

Assume that $m \geq n$ and consider the QR factorization (7.1) of the matrix A. The condition (7.2) means that the matrix R in (7.1) has the form

$$R = \begin{pmatrix} X \\ 0_{(m-n)\times n} \end{pmatrix},$$

where X is an $n \times n$ upper triangular matrix. Setting $V = Q(:, 1 : n)$ we get

$$A = V \cdot X. \tag{7.7}$$

Here the $m \times n$ matrix V has orthonormal columns, i.e., $V^*V = I_n$, and X is an $n \times n$ upper triangular matrix. The factorization (7.7) is called *the truncated QR factorization* of the matrix A. In the same way one can obtain truncated forms for other factorizations mentioned here.

§7.2 The rank numbers and quasiseparable generators

Here we present relations for lower and upper rank numbers of unitary matrices relative to the main diagonal in a block representation.

Theorem 7.1. *Let V be a unitary block matrix with entries of sizes $m_i \times n_j$, $i, j = 1, \dots, N$, with lower rank numbers ρ_k^L $(k = 1, \dots, N-1)$ and upper rank numbers ρ_k^U $(k = 1, \dots, N-1)$. The following relations hold:*

$$\rho_k^L + \sum_{i=k+1}^{N} n_i = \rho_k^U + \sum_{i=k+1}^{N} m_i, \quad k = 1, \dots, N-1 \tag{7.8}$$

or

$$\begin{aligned} &\operatorname{rank} V(k+1 : N, 1 : k) + \sum_{i=k+1}^{N} n_i \\ &\quad = \operatorname{rank} V(1 : k, k+1 : N) + \sum_{i=k+1}^{N} m_i, \quad k = 1, \dots, N-1. \end{aligned} \tag{7.9}$$

Proof. We consider $V^{-1} = V^*$ as a block matrix with entries of sizes $n_i \times m_j$ $(i, j = 1, \dots, N)$. Using the formula (6.7) from Corollary 6.2 we get

$$r_{k,0}^L(V) + \sum_{i=k+1}^{N} n_i = r_{k,0}^L(V^*) + \sum_{i=k+1}^{N} m_i, \quad k = 1, \dots, N-1.$$

But we have obviously $r_{k,0}^L(V^*) = r_{k,0}^U(V)$ $(k = 1, \dots, N-1)$. Moreover, by the definition of rank numbers we have $r_{k,0}^L(V) = \rho_k^L$, $r_{k,0}^U(V) = \rho_k^U$ and therefore we conclude that the relations (7.8) hold. □

From here one can easily derive a corollary concerning the orders of minimal quasiseparable generators of unitary matrices.

Corollary 7.2. *Let U be a unitary block matrix with entries of sizes $m_i \times n_j$, $i, j = 1, \dots, N$ with lower rank numbers r_k $(k = 1, \dots, N-1)$. Then U has minimal lower quasiseparable generators of orders r_k $(k = 1, \dots, N-1)$ and upper quasiseparable generators of orders $r_k + \mu_k$ $(k = 1, \dots, N-1)$, where $\mu_k = \sum_{i=k+1}^{N}(n_i - m_i)$.*

Proof. By Theorem 7.1, the upper rank numbers of U equal $r_k+\mu_k$ ($k=1,\dots,N-1$). By Theorem 5.9, the matrix U has lower quasiseparable generators of orders r_k ($k=1,\dots,N-1$) and upper quasiseparable generators of orders $r_k+\mu_k$ ($k=1,\dots,N-1$). By Corollary 5.10, these generators are minimal. □

§7.3 Factorization representations

Here we derive factorization representations for matrices with quasiseparable structure.

§7.3.1 Block triangular matrices

We start with representations of block triangular matrices.

Lemma 7.3. *Let W be a block upper triangular unitary matrix with block entries of sizes $\nu_i \times n_j$, $i,j=1,\dots,N$. Set*

$$s_0 = 0, \quad s_k = \sum_{i=1}^{k}(\nu_i - n_i),\ k=1,\dots,N-1. \tag{7.10}$$

Then all the numbers s_k are nonnegative. Moreover, the matrix W admits the factorization

$$W = \tilde{W}_1\tilde{W}_2\cdots\tilde{W}_N \tag{7.11}$$

with

$$\begin{gathered}\tilde{W}_1 = \operatorname{diag}\{W_1, I_{\phi_1}\}; \quad \tilde{W}_k = \operatorname{diag}\{I_{\chi_k}, W_k, I_{\phi_k}\}, \quad k=2,\dots,N-1;\\ \tilde{W}_N = \operatorname{diag}\{I_{\chi_N}, W_N\},\end{gathered} \tag{7.12}$$

where W_k are unitary matrices of sizes $(s_{k-1}+\nu_k)\times(s_{k-1}+\nu_k)$ and

$$\chi_k = \sum_{i=1}^{k-1} n_i, \quad \phi_k = \sum_{i=k+1}^{N} \nu_i.$$

Furthermore one can determine the matrices W_k ($k=1,\dots,N$) using the following algorithm.

1. *Set*
$$\hat{W}_1 = W. \tag{7.13}$$
2. *For $k=1,\dots,N-1$ perform the following. Set*
$$w_k = \hat{W}_k(1:s_{k-1}+\nu_k, 1:n_k) \tag{7.14}$$

and determine the unitary $(s_{k-1}+\nu_k)\times(s_{k-1}+\nu_k)$ matrix W_k from the condition

$$W_k^* w_k = \begin{pmatrix} I_{n_k} \\ 0_{s_k\times n_k} \end{pmatrix}. \tag{7.15}$$

Compute the matrix

$$Z_k = \begin{pmatrix} W_k^* & 0 \\ 0 & I_{\phi_k} \end{pmatrix} \hat{W}_k \tag{7.16}$$

and determine the $\chi_{k+1}\times\chi_{k+1}$ matrix $\hat{W}_{k+1}$ from the partition

$$Z_k = \begin{pmatrix} I_{n_k} & 0 \\ 0 & \hat{W}_{k+1} \end{pmatrix}. \tag{7.17}$$

3. *Set*

$$W_N = \hat{W}_N. \tag{7.18}$$

Proof. At first we show that the condition $s_k \geq 0,\ k=1,\dots,N-1$ holds. Indeed, consider the submatrix $W(:,1:k)$ composed of the first k block columns of the matrix W. We have $W(:,1:k) = \begin{pmatrix} \Omega_k \\ 0 \end{pmatrix}$, where Ω_k is a matrix of size $(\sum_{i=1}^k \nu_i)\times(\sum_{i=1}^k n_i)$. From the invertibility of W it follows that rank $\Omega_k = \sum_{i=1}^k n_i$ and thus $\sum_{i=1}^k n_i \leq \sum_{i=1}^k \nu_i,\ 1\leq k\leq N-1$.

Next we prove by induction that

$$W = \tilde{W}_1\tilde{W}_2\cdots\tilde{W}_k \begin{pmatrix} I_{\chi_{k+1}} & 0 \\ 0 & \hat{W}_{k+1} \end{pmatrix}, \quad k=0,1,\dots,N-1. \tag{7.19}$$

Here $\hat{W}_k$ $(k=1,\dots,N-1)$ are block upper triangular unitary matrices with entries of sizes $\nu_i^{(k)}\times n_j,\ i,j=k,\dots,N,$ where $\nu_k^{(k)} = s_k+n_k,\ \nu_i^{(k)} = \nu_i, i = k+1,\dots,N$.

The relation (7.19) with $k=0$ follows from (7.13) and the definition of the matrix W. Let for some k with $k=1,\dots,N-1$ the representation

$$W = \tilde{W}_1\tilde{W}_2\cdots\tilde{W}_{k-1} \begin{pmatrix} I_{\chi_k} & 0 \\ 0 & \hat{W}_k \end{pmatrix} \tag{7.20}$$

with the matrix $\hat{W}_k$ defined as above hold. The first block column of $\hat{W}_k$ has the form

$$\hat{W}_k(:,1) = \begin{pmatrix} w_k \\ 0_{\phi_k\times n_k} \end{pmatrix},$$

where w_k is an $(s_k+n_k)\times n_k$ matrix such that $w_k^* w_k = I_{n_k}$. We take a unitary $(s_k+n_k)\times(s_k+n_k)$ matrix W_k such that (7.15) holds. Then

$$\begin{pmatrix} W_k^* & 0 \\ 0 & I_{\phi_k} \end{pmatrix} \hat{W}_k(:,1) = \begin{pmatrix} I_{n_k} \\ 0_{(s_k+\phi_k)\times n_k} \end{pmatrix}.$$

Define the matrix Z_k by (7.16). Since Z_k is unitary, the representation (7.17) holds. From (7.16), (7.17) we get

$$\hat{W}_k = \begin{pmatrix} W_k & 0 \\ 0 & I_{\phi_k} \end{pmatrix} \begin{pmatrix} I_{n_k} & 0 \\ 0 & \hat{W}_{k+1} \end{pmatrix}. \tag{7.21}$$

Here $\hat{W}_{k+1}$ is a unitary block upper triangular matrix with entries of sizes $\nu_i^{(k+1)} \times n_j,\ i,j = k{+}1,\dots,N$, where $\nu_i^{(k+1)} = \nu_i, i = k{+}2,\dots,N$ and $\nu_{k+1}^{(k+1)} = \nu_{k+1}{+}s_k = n_{k+1} + s_{k+1}$. Inserting (7.21) in (7.20), we obtain (7.19).

Finally, using (7.19) with $k = N-1$ and (7.18) we obtain the factorization (7.11). □

The reverse statement is the following.

Lemma 7.4. *Let W be a block matrix with block entries of sizes $\nu_i \times n_j,\ i,j = 1,\dots,N$ which admits the factorization* (7.11), (7.12) *with some $(s_{k-1}+\nu_k) \times (s_{k-1}+\nu_k)$ matrices W_k $(k = 1,\dots,N)$, where the numbers $s_k,\ k = 0,\dots,N-1$ are defined in* (7.10).

Then W is a block upper triangular matrix.

Proof. We prove the statement by induction on N. For $N = 2$ the matrix W is a 2×2 block matrix with entries of sizes $\nu_i \times n_j,\ i,j = 1,2$ $(\nu_1+\nu_2 = n_1+n_2)$ and we have

$$W = \tilde{W}_1\tilde{W}_2 = \begin{pmatrix} W_1 & 0 \\ 0 & I_{\nu_2} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ 0 & W_2 \end{pmatrix}.$$

Hence it follows that the left bottom block of size $\nu_2 \times n_1$ in the matrix W equals zero. Let for some $N \geq 2$ the $(N-1)\times(N-1)$ block matrix

$$W' = W_1' \cdots W_{N-1}',$$

with

$$W_1' = \operatorname{diag}\{W_1, I_{\phi_1'}\}, \quad W_k' = \operatorname{diag}\{I_{\chi_k}, W_k, I_{\phi_k'}\},\ k = 2,\dots,N-2,$$
$$W_{N-1}' = \operatorname{diag}\{I_{\chi_{N-1}}, W_{N-1}\},$$

where W_k are matrices of sizes $(s_{k-1}+\nu_k)\times(s_{k-1}+\nu_k)$ and

$$\chi_k = \sum_{i=1}^{k-1} n_i, \quad \phi_k' = \sum_{i=k+1}^{N-1} \nu_i,$$

be upper triangular. The matrix W defined via (7.11), (7.12) may be written in the form

$$W = \tilde{W}_1 \cdots \tilde{W}_{N-1}\tilde{W}_N = \begin{pmatrix} W' & 0 \\ 0 & I_{\nu_N} \end{pmatrix} \begin{pmatrix} I_{\chi_N} & 0 \\ 0 & W_N \end{pmatrix}.$$

Hence using the fact that W' is upper triangular it follows that

$$W(i+1:N,1:i) = \begin{pmatrix} W'(i+1:N-1,1:i) \\ 0 \end{pmatrix} = 0, \quad i=1,\dots,N-1,$$

which completes the proof of the lemma. □

§7.3.2 Factorization of general unitary matrices and compression of generators

Here we consider block unitary matrices with given quasiseparable representations. For such matrices we derive a factorization representation as products of elementary unitary matrices. Together with the factorization of a matrix we obtain another set of its quasiseparable generators. In some cases these generators have smaller orders than the original ones.

Theorem 7.5. *Let $U = \{U_{ij}\}_{i,j=1}^N$ be a block unitary matrix with entries of sizes $m_i \times n_j$, lower quasiseparable generators $p(i)$ $(i=2,\dots,N)$, $q(j)$ $(j=1,\dots,N-1)$, $a(k)$ $(k=2,\dots,N-1)$ of orders r_k^L $(k=1,\dots,N-1)$, upper quasiseparable generators $g(i)$ $(i=1,\dots,N-1)$, $h(j)$ $(j=2,\dots,N)$, $b(k)$ $(k=2,\dots,N-1)$ of orders r_k^U $(k=1,\dots,N-1)$ and diagonal entries $d(k)$ $(k=1,\dots,N)$. Set*

$$\begin{aligned} &\rho_N = 0,\ \rho_{k-1} = \min\{m_k+\rho_k,\ r_{k-1}^L\}, \\ &\nu_k = m_k+\rho_k-\rho_{k-1},\ k=N,\dots,2, \quad \nu_1 = m_1+\rho_1, \\ &s_N = 0, \quad s_{k-1} = n_k+s_k-\nu_k, \quad k=N,\dots,2. \end{aligned} \tag{7.22}$$

Then all the numbers s_k are nonnegative and the matrix U has upper quasiseparable generators of orders s_k $(k=1,\dots,N-1)$ and admits the factorization

$$U = V\cdot F, \tag{7.23}$$

where V is a block lower triangular unitary matrix with block entries of sizes $m_i \times \nu_j$ $(i,j=1,\dots,N)$ and F is a block upper triangular unitary matrix with block entries of sizes $\nu_i \times n_j$ $(i,j=1,\dots,N)$. Moreover, one can choose V in the form

$$V = \tilde{V}_N\tilde{V}_{N-1}\cdots\tilde{V}_2, \tag{7.24}$$

with

$$\tilde{V}_k = \operatorname{diag}\{I_{\eta_k}, V_k, I_{\phi_k}\},\ k=2,\dots,N-1,\ \tilde{V}_N = \operatorname{diag}\{I_{\eta_N}, V_N\}, \tag{7.25}$$

where $\eta_k = \sum_{i=1}^{k-1} m_i$, $\phi_k = \sum_{i=k+1}^N \nu_i$, the V_k $(k=2,\dots,N)$ are $(m_k+\rho_k)\times(m_k+\rho_k)$ unitary matrices and the matrix F has the form

$$F = \tilde{F}_1\tilde{F}_2\cdots\tilde{F}_N, \tag{7.26}$$

with

$$\tilde{F}_1 = \operatorname{diag}\{F_1, I_{\phi_1}\}, \quad \tilde{F}_k = \operatorname{diag}\{I_{\chi_k}, F_k, I_{\phi_k}\}, \quad k = 2, \dots, N-1, \\ \tilde{F}_N = \operatorname{diag}\{I_{\chi_N}, F_N\}, \tag{7.27}$$

where $\chi_k = \sum_{i=1}^{k-1} n_i$, *and* F_k $(k = 1, \dots, N)$ *are* $(s_{k-1} + \nu_k) \times (s_{k-1} + \nu_k)$ *unitary matrices.*

Furthermore, a set of upper quasiseparable generators $g_s(i)$ $(i = 1, \dots, N-1)$, $h_s(j)$ $(j = 2, \dots, N)$, $b_s(k)$ $(k = 2, \dots, N-1)$ *of orders* s_k $(k = 1, \dots, N-1)$ *as well as the unitary matrices* V_k $(k = 2, \dots, N)$, F_k $(k = 1, \dots, N)$ *are obtained using the following algorithm.*

Set $X_{N+1}, Y_{N+1}, z_{N+1}$ *to be the empty* 0×0 *matrices and* $q(N)$, $a(N)$, $g(N)$, $b(N)$, $g_s(N)$ *to be empty matrices of sizes* $0 \times n_N, 0 \times r^L_{N-1}, m_N \times 0, r^U_{N-1} \times 0, m_N \times 0$, *respectively.*

For $k = N, \dots, 2$ *perform the following. Compute the* QR *factorization*

$$\begin{bmatrix} p(k) \\ X_{k+1}a(k) \end{bmatrix} = V_k \begin{pmatrix} X_k \\ 0_{\nu_k \times r^L_{k-1}} \end{pmatrix}, \tag{7.28}$$

where V_k *is a unitary matrix of sizes* $(m_k + \rho_k) \times (m_k + \rho_k)$, X_k *is a matrix of sizes* $\rho_{k-1} \times r^L_{k-1}$. *Compute the* $(m_k + \rho_k) \times (n_k + s_k)$ *matrix*

$$Z_k = V_k^* \begin{pmatrix} d(k) & g_s(k) \\ X_{k+1}q(k) & z_{k+1} \end{pmatrix} \tag{7.29}$$

and determine the matrices $h'_k, h''_k, \Delta_k, \Theta_k$ *of sizes* $\rho_{k-1} \times n_k, \rho_{k-1} \times s_k, \nu_k \times n_k, \nu_k \times s_k$, *respectively, from the partition*

$$Z_k = \begin{bmatrix} h'_k & h''_k \\ \Delta_k & \Theta_k \end{bmatrix}. \tag{7.30}$$

The submatrix $\begin{pmatrix} \Delta_k & \Theta_k \end{pmatrix}$ *has orthonormal rows and one can determine the* $(n_k + s_k) \times (n_k + s_k)$ *unitary matrix* F_k *from the condition*

$$\begin{pmatrix} \Delta_k & \Theta_k \end{pmatrix} F_k^* = \begin{pmatrix} 0_{\nu_k \times s_{k-1}} & I_{\nu_k} \end{pmatrix}. \tag{7.31}$$

Determine the matrices $h_s(k), b_s(k), d_F(k), g_F(k)$ *of sizes* $s_{k-1} \times n_k, s_{k-1} \times s_k, \nu_k \times n_k, \nu_k \times s_k$, *respectively, from the partition*

$$F_k = \begin{bmatrix} h_s(k) & b_s(k) \\ d_F(k) & g_F(k) \end{bmatrix}. \tag{7.32}$$

Compute the matrices Y_k *of the size* $r^U_{k-1} \times s_{k-1}$ *and* z_k *of the size* $\rho_{k-1} \times s_{k-1}$ *by the formulas*

$$Y_k = h(k)h_s^*(k) + b(k)Y_{k+1}b_s^*(k), \quad z_k = h'_k h_s^*(k) + h''_k b_s^*(k). \tag{7.33}$$

Finally, compute

$$g_s(k-1) = g(k-1)Y_k, \tag{7.34}$$

and set

$$F_1 = \begin{bmatrix} d(1) & g(1)Y_2 \\ X_2 q(1) & z_2 \end{bmatrix}. \tag{7.35}$$

Proof. Using the generators $q(k), a(k)$ and $g(k), b(k)$ define the matrices $Q_k,\ k = 1, \dots, N$ of sizes $r_k^L \times \chi_{k+1}$ via relations

$$Q_1 = q(1), \quad Q_k = \begin{pmatrix} a(k)Q_{k-1} & q(k) \end{pmatrix},\ k = 2, \dots, N \tag{7.36}$$

and the matrices $G_k,\ k = 1, \dots, N$ of sizes $\eta_{k+1} \times r_k^U$ via relations

$$G_1 = g(1), \quad G_k = \begin{pmatrix} G_{k-1}b(k) \\ g(k) \end{pmatrix},\ k = 2, \dots, N \tag{7.37}$$

(here we set $r_N^L = r_N^U = 0$). Using the equalities (5.11) and (5.14), we get

$$U(k, 1:k-1) = p(k)Q_{k-1}, \quad k = 2, \dots, N \tag{7.38}$$

and

$$U(1:k-1, k) = G_{k-1}h(k), \quad k = 2, \dots, N. \tag{7.39}$$

We prove by induction that $s_k \geq 0,\ k = N, \dots, 1$ and all the matrices

$$\hat{U}_k = \begin{pmatrix} U(1:k, 1:k) & G_k Y_{k+1} \\ X_{k+1}Q_k & z_{k+1} \end{pmatrix}, \quad k = 1, \dots, N \tag{7.40}$$

are unitary. We prove also that the matrices $\begin{pmatrix} \Delta_k & \Theta_k \end{pmatrix},\ (k = N, \dots, 2)$ have orthonormal rows and the relations

$$G_{k-1}h(k) = G_{k-1}Y_k h_s(k), \quad k = 2, \dots, N \tag{7.41}$$

$$G_{k-1}b(k)Y_{k+1} = G_{k-1}Y_k b_s(k), \quad k = 2, \dots, N \tag{7.42}$$

hold.

For $k = N$ the matrices $G_N Y_{N+1}, X_{N+1}Q_N, z_{N+1}$ are empty and hence $\hat{U}_N = U$ is a unitary matrix. By definition, we have $s_N = 0$. Let for some k with $N \geq k \geq 2$ the matrix $\hat{U}_k$ be unitary. Inserting (7.36), (7.37) and (7.38), (7.39) in (7.40) we get

$$\hat{U}_k = \begin{pmatrix} U(1:k-1, 1:k-1) & G_{k-1}h(k) & G_{k-1}b(k)Y_{k+1} \\ p(k)Q_{k-1} & d(k) & g(k)Y_{k+1} \\ X_{k+1}a(k)Q_{k-1} & X_{k+1}q(k) & z_{k+1} \end{pmatrix}.$$

Using (7.28), (7.29) and (7.30) we obtain

$$\begin{pmatrix} I_{\eta_k} & 0 \\ 0 & V_k^* \end{pmatrix} \hat{U}_k = \begin{pmatrix} U(1:k-1, 1:k-1) & G_{k-1}h(k) & G_{k-1}b(k)Y_{k+1} \\ X_k Q_{k-1} & h'_k & h''_k \\ 0_{\nu_k \times \chi_k} & \Delta_k & \Theta_k \end{pmatrix}. \tag{7.43}$$

Here the matrix $\begin{pmatrix} \Delta_k & \Theta_k \end{pmatrix}$ has the size $\nu_k \times (s_k + n_k)$.

Since the matrix $\begin{pmatrix} I & 0 \\ 0 & V_k^* \end{pmatrix} \hat{U}_k$ is unitary the inequality $s_{k-1} = s_k + n_k - \nu_k \geq 0$ holds and the matrix $\begin{pmatrix} \Delta_k & \Theta_k \end{pmatrix}$ has orthonormal rows. Hence one can determine a unitary $(n_k + s_k) \times (n_k + s_k)$ matrix F_k such that (7.31) holds. Next, using (7.32) we have

$$\begin{pmatrix} h(k) & b(k)Y_{k+1} \\ h'_k & h''_k \end{pmatrix} F_k^* = \begin{pmatrix} Y_k & w'_k \\ z_k & w''_k \end{pmatrix} \tag{7.44}$$

with the matrices Y_k, z_k of sizes $r^U_{k-1} \times s_{k-1}, \rho_{k-1} \times s_{k-1}$ determined via (7.33) and some matrices w'_k, w''_k of sizes $r^U_{k-1} \times \nu_k, \rho_{k-1} \times \nu_k$. Thus using (7.43), (7.31) and (7.44) we get

$$\begin{pmatrix} I_{\eta_k} & 0 \\ 0 & V_k^* \end{pmatrix} \hat{U}_k \begin{pmatrix} I_{\chi_k} & 0 \\ 0 & F_k^* \end{pmatrix} = \begin{pmatrix} U(1:k-1, 1:k-1) & G_{k-1}Y_k & G_{k-1}w'_k \\ X_k Q_{k-1} & z_k & w''_k \\ 0_{\nu_k \times \chi_k} & 0_{\nu_k \times s_{k-1}} & I_{\nu_k} \end{pmatrix}.$$

Since the matrix $\begin{pmatrix} I & 0 \\ 0 & V_k^* \end{pmatrix} \hat{U}_k \begin{pmatrix} I & 0 \\ 0 & F_k^* \end{pmatrix}$ is unitary we conclude that $G_{k-1}w'_k = 0,\ w''_k = 0$ and therefore

$$\begin{aligned} &\begin{pmatrix} I_{\eta_k} & 0 \\ 0 & V_k^* \end{pmatrix} \hat{U}_k \begin{pmatrix} I_{\chi_k} & 0 \\ 0 & F_k^* \end{pmatrix} \\ &= \begin{pmatrix} U(1:k-1, 1:k-1) & G_{k-1}Y_k & 0 \\ X_k Q_{k-1} & z_k & 0 \\ 0 & 0 & I_{\nu_k} \end{pmatrix} = \begin{pmatrix} \hat{U}_{k-1} & 0 \\ 0 & I_{\nu_k} \end{pmatrix}. \end{aligned} \tag{7.45}$$

Hence it follows that the matrix $\hat{U}_{k-1}$ is unitary. Moreover (7.44) implies that

$$\begin{pmatrix} h(k) & b(k)Y_{k+1} \end{pmatrix} F_k^* = \begin{pmatrix} Y_k & w'_k \end{pmatrix}$$

and using the fact that $G_{k-1}w'_k = 0$ we obtain

$$\begin{pmatrix} G_{k-1}h(k) & G_{k-1}b(k)Y_{k+1} \end{pmatrix} = \begin{pmatrix} G_{k-1}Y_k & 0 \end{pmatrix} F_k. \tag{7.46}$$

From here using the partition (7.32) we obtain the equalities (7.41), (7.42).

Thus, by Lemma 5.23, we conclude that $g_s(i)$ $(i = 1, \dots, N-1)$, $h_s(j)$ $(j = 2, \dots, N)$, $b_s(k)$ $(k = 2, \dots, N-1)$ are upper quasiseparable generators of the matrix U.

Next, using (7.45) we get

$$\hat{U}_N = U, \quad \hat{U}_k = \begin{pmatrix} I_{\eta_k} & 0 \\ 0 & V_k \end{pmatrix} \begin{pmatrix} \hat{U}_{k-1} & 0 \\ 0 & I_{\nu_k} \end{pmatrix} \begin{pmatrix} I_{\chi_k} & 0 \\ 0 & F_k \end{pmatrix}, \ k = N, \dots, 2. \tag{7.47}$$

Comparing (7.40) with $k = 1$ and (7.35) we have $\hat{U}_1 = F_1$. Combining this equality with the relations (7.47) we obtain the factorization (7.23)–(7.27). □

Corollary 7.6. *Under the conditions of Theorem* 7.5 *the factorizations*

$$\begin{pmatrix} G_{k-1} & 0 \\ 0 & V_k^* \end{pmatrix} \begin{pmatrix} h(k) & b(k)Y_{k+1} \\ d(k) & g(k)Y_{k+1} \\ X_{k+1}q(k) & z_{k+1} \end{pmatrix} = \begin{pmatrix} G_{k-1} & 0 \\ 0 & I_{\rho_{k-1}+\nu_k} \end{pmatrix} \begin{pmatrix} Y_k & 0 \\ z_k & 0 \\ 0 & I_{\nu_k} \end{pmatrix} F_k,$$
$$k = 2, \dots, N \tag{7.48}$$

hold.

The proof follows directly from the equalities (7.46), (7.30), (7.31) and (7.33).

Next we show that if lower quasiseparable generators of a unitary matrix U are minimal, then Theorem 7.5 may be used for compression of upper quasiseparable generators of U.

Corollary 7.7. *Under the conditions of Theorem* 7.5, *assume that the lower quasiseparable generators* $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ *are minimal.*

Then the upper quasiseparable generators $g_s(i)$ $(i = 1, \dots, N-1)$, $h_s(j)$ $(j = 2, \dots, N)$, $b_s(k)$ $(k = 2, \dots, N-1)$ *obtained in the algorithm are minimal.*

Proof. From (7.22) it follows that

$$s_k = \rho_k + \sum_{i=k+1}^{N} (n_i - m_i), \quad k = 1, \dots, N-1$$

and moreover $\rho_k \le r_k^L$ $(k = 1, \dots, N-1)$. Hence we get

$$s_k \le r_k^L + \sum_{i=k+1}^{N} (n_i - m_i), \quad k = 1, \dots, N-1. \tag{7.49}$$

By Corollary 5.10, the orders r_k^L $(k = 1, \dots, N-1)$ of the generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ equal the corresponding lower rank numbers of the matrix U. Comparing (7.49) with the formula (7.8) we obtain

$$s_k \le \rho_k^U, \quad k = 1, \dots, N-1,$$

where ρ_k^U $(k = 1, \dots, N-1)$ are the upper rank numbers of the matrix U. But s_k $(k = 1, \dots, N-1)$ are the orders of some upper quasiseparable generators of the matrix U and applying Corollary 5.10 again we conclude that these orders are minimal. Hence the upper quasiseparable generators $g_s(i)$ $(i = 1, \dots, N-1)$, $h_s(j)$ $(j = 2, \dots, N)$, $b_s(k)$ $(k = 2, \dots, N-1)$ are minimal. □

§7.3.3 Generators via factorization

Next we prove the converse statement. For a unitary matrix given as a product of elementary matrices we obtain a quasiseparable representation.

Theorem 7.8. *Let U be a block unitary matrix with entries of sizes $m_i \times n_j$, $i, j = 1, \dots, N$ represented as a product*

$$U = V \cdot F, \tag{7.50}$$

where V is a block lower triangular unitary matrix with block entries of sizes $m_i \times \nu_j$ $(i, j = 1, \dots, N)$ and F is a block upper triangular unitary matrix with block entries of sizes $\nu_i \times n_j$ $(i, j = 1, \dots, N)$. Set $\rho_0 = 0$, $s_0 = 0$ and

$$\rho_k = \sum_{i=1}^{k} (\nu_i - m_i), \quad s_k = \sum_{i=1}^{k} (\nu_i - n_i), \quad k = 1, \dots, N. \tag{7.51}$$

By Lemma 7.3, applied to the transposed matrix V^T, the numbers ρ_k are nonnegative and the matrix V admits the factorization

$$V = \tilde{V}_N \tilde{V}_{N-1} \cdots \tilde{V}_2 \tilde{V}_1, \tag{7.52}$$

with

$$\begin{gathered} \tilde{V}_1 = \operatorname{diag}\{V_1, I_{\phi_1}\}, \quad \tilde{V}_k = \operatorname{diag}\{I_{\eta_k}, V_k, I_{\phi_k}\},\ k = 2, \dots, N-1, \\ \tilde{V}_N = \operatorname{diag}\{I_{\eta_N}, V_N\}, \end{gathered} \tag{7.53}$$

where $\eta_k = \sum_{i=1}^{k-1} m_i$, $\phi_k = \sum_{i=k+1}^{N} \nu_i$ and V_k $(k = 2, \dots, N)$ are $(m_k+\rho_k)\times(m_k+\rho_k)$ unitary matrices. By Lemma 7.3, applied to the matrix F, all the numbers s_k are nonnegative and one has the factorization

$$F = \tilde{F}_1 \tilde{F}_2 \cdots \tilde{F}_N, \tag{7.54}$$

with

$$\begin{gathered} \tilde{F}_1 = \operatorname{diag}\{F_1, I_{\phi_1}\},\ \tilde{F}_k = \operatorname{diag}\{I_{\chi_k}, F_k, I_{\phi_k}\},\ k = 2, \dots, N-1, \\ \tilde{F}_N = \operatorname{diag}\{I_{\chi_N}, F_N\}, \end{gathered} \tag{7.55}$$

where $\chi_k = \sum_{i=1}^{k-1} n_i$ and F_k $(k = 1, \dots, N)$ are $(n_k + s_k) \times (n_k + s_k)$ unitary matrices.

Then the matrix U has lower quasiseparable generators of orders ρ_k $(k = 1, \dots, N-1)$ and upper quasiseparable generators of orders s_k $(k = 1, \dots, N-1)$. Moreover, a set of such quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$; $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$; $d(i)$ $(i = 1, \dots, N)$ may be obtained as follows.

Determine the generators $p(i)$ $(i = 2, \dots, N)$, $a(k)$ $(k = 2, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ *from the partitions*

$$V_k = \begin{bmatrix} p(k) & d_V(k) \\ a(k) & q_V(k) \end{bmatrix}, \quad k = 2, \dots, N-1, \tag{7.56}$$

$$V_N = \begin{bmatrix} p(N) & d_V(N) \end{bmatrix}, \tag{7.57}$$

with the matrices $p(k), a(k), d_V(k), q_V(k)$ *of sizes* $m_k \times \rho_{k-1}, \rho_k \times \rho_{k-1}, m_k \times \nu_k, \rho_k \times \nu_k$, *respectively, and*

$$F_k = \begin{bmatrix} h(k) & b(k) \\ d_F(k) & g_F(k) \end{bmatrix}, \ k = 2, \dots, N-1, \tag{7.58}$$

$$F_N = \begin{bmatrix} h(N) \\ d_F(N) \end{bmatrix}, \tag{7.59}$$

with the matrices $h(k), b(k), d_F(k), g_F(k)$ *of sizes* $s_{k-1} \times n_k, s_{k-1} \times s_k, \nu_k \times n_k, \nu_k \times s_k$, *respectively. Next compute the generators* $q(j)$ $(j = 1, \dots, N-1)$, $g(i)$ $(i = 1, \dots, N-1)$, $d(i)$ $(i = 1, \dots, N)$ *via recursion relations as follows.*

1. *Compute the matrix*

$$W_1 = V_1 F_1 \tag{7.60}$$

and determine the matrices $d(1), g(1), q(1), \beta_1$ *of sizes* $m_1 \times n_1, m_1 \times s_1, \rho_1 \times n_1, \rho_1 \times s_1$ *from the partition*

$$W_1 = \begin{bmatrix} d(1) & g(1) \\ q(1) & \beta_1 \end{bmatrix}. \tag{7.61}$$

2. *For* $k = 2, \dots, N-1$ *compute the matrix*

$$W_k = V_k \begin{bmatrix} \beta_{k-1} & 0 \\ 0 & I_{\nu_k} \end{bmatrix} F_k \tag{7.62}$$

and determine the matrices $d(k), g(k), q(k), \beta_k$ *of sizes* $m_k \times n_k, m_k \times s_k, \rho_k \times n_k, \rho_k \times s_k$ *from the partition*

$$W_k = \begin{bmatrix} d(k) & g(k) \\ q(k) & \beta_k \end{bmatrix}, \tag{7.63}$$

with the auxiliary variables β_k, *which are* $\rho_k \times s_k$ *matrices.*

3. *Compute*

$$d(N) = p(N)\beta_{N-1}h(N) + d_V(N)d_F(N). \tag{7.64}$$

Proof. By Lemma 5.3 and Lemma 5.6, one needs to check that the relations

$$U(k, 1 : k-1) = p(k)Q_{k-1}, \ k = 2, \dots, N, \tag{7.65}$$

with the matrices Q_k $(k = 1, \dots, N-1)$ defined in (5.1) and

$$U(1:k-1,k) = G_{k-1}h(k), \quad k = 2, \dots, N, \tag{7.66}$$

with the matrices G_k $(k = 1, \dots, N-1)$ defined in (5.5), and also

$$U(k,k) = d(k), \quad k = 1, \dots, N \tag{7.67}$$

hold.

Define the matrices $\tilde{C}_k$ by

$$\tilde{C}_k = \tilde{V}_k \cdots \tilde{V}_2 \tilde{V}_1 \tilde{F}_1 \cdots \tilde{F}_k, \quad k = 1, \dots, N.$$

Using (7.52), (7.53) and (7.54), (7.55) we get

$$\tilde{C}_k = \begin{pmatrix} C_k & 0 \\ 0 & I_{\phi_k} \end{pmatrix}, \quad k = 1, \dots, N, \tag{7.68}$$

with matrices C_k of sizes $\sigma_k \times \sigma_k$, where

$$\sigma_k = \sum_{i=1}^{k} \nu_i = \eta_{k+1} + \rho_k = \chi_{k+1} + s_k.$$

In particular, $\sigma_0 = 0$. Moreover, using (7.50), (7.52), (7.54) we have

$$U = (\tilde{V}_N \cdots \tilde{V}_{k+1})\tilde{C}_k(\tilde{F}_{k+1} \cdots \tilde{F}_N), \ k = 1, \dots, N-1; \quad U = \tilde{C}_N.$$

Furthermore from (7.53), (7.55) we get

$$\tilde{V}_N \cdots \tilde{V}_{k+1} = \begin{pmatrix} I_{\eta_{k+1}} & 0 \\ 0 & * \end{pmatrix}, \quad \tilde{F}_{k+1} \cdots \tilde{F}_N = \begin{pmatrix} I_{\chi_{k+1}} & 0 \\ 0 & * \end{pmatrix}$$

and therefore

$$U(1:k, 1:k) = C_k(1:\eta_{k+1}, 1:\chi_{k+1}), \quad k = 1, \dots, N. \tag{7.69}$$

Here we treat U as a block matrix with entries of sizes $m_i \times n_j$ and C_k as a scalar $\sigma_k \times \sigma_k$ matrix.

Next we prove by induction that

$$C_k = \begin{pmatrix} C_{k-1}(1:\eta_k, 1:\chi_k) & G_{k-1}h(k) & G_{k-1}b(k) \\ p(k)Q_{k-1} & d(k) & g(k) \\ a(k)Q_{k-1} & q(k) & \beta_k \end{pmatrix}, \quad k = 1, \dots, N-1. \tag{7.70}$$

For $k = 1$, using (7.60), (7.61) we get

$$C_1 = V_1 F_1 = \begin{bmatrix} d(1) & g(1) \\ q(1) & \beta_1 \end{bmatrix}.$$

Let for some k with $1 \le k \le N-2$ the relation (7.70) hold. Using the equalities (5.3) and (5.7) one can rewrite (7.70) in the form

$$C_k = \begin{pmatrix} C_k(1:\eta_{k+1}, 1:\chi_{k+1}) & G_k \\ Q_k & \beta_k \end{pmatrix}.$$

Using the fact that $C_{k+1} = \tilde{V}_{k+1}\tilde{C}_k\tilde{F}_{k+1}$ and (7.68), (7.53) and (7.55) we obtain

$$C_{k+1} = \begin{pmatrix} I_{\eta_{k+1}} & 0 \\ 0 & V_{k+1} \end{pmatrix} \begin{pmatrix} C_k(1:\eta_{k+1}, 1:\chi_{k+1}) & G_k & 0 \\ Q_k & \beta_k & 0 \\ 0 & 0 & I_{\nu_{k+1}} \end{pmatrix} \begin{pmatrix} I_{\chi_{k+1}} & 0 \\ 0 & F_{k+1} \end{pmatrix}. \tag{7.71}$$

From here using the relations (7.56) and (7.58) we get

$$\begin{aligned} C_{k+1}(\eta_{k+1}+1:\eta_{k+2}+\rho_{k+1}, 1:\chi_{k+1}) &= W_{k+1} = \begin{pmatrix} p(k+1)Q_k \\ a(k+1)Q_k \end{pmatrix}, \\ C_{k+1}(1:\eta_{k+1}, \chi_{k+1}+1:\chi_{k+2}+s_{k+1}) &= \begin{pmatrix} G_k h(k+1) & G_k b(k+1) \end{pmatrix}. \end{aligned} \tag{7.72}$$

Next, using (7.62) and (7.63) we obtain

$$\begin{aligned} & C_{k+1}(\eta_{k+1}+1:\eta_{k+2}+\rho_{k+1}, \chi_{k+1}+1:\chi_{k+2}+s_{k+1}) \\ & = V_{k+1}\begin{pmatrix} \beta_k & 0 \\ 0 & I_{\nu_{k+1}} \end{pmatrix} F_{k+1} = \begin{pmatrix} d(k+1) & g(k+1) \\ q(k+1) & \beta_{k+1} \end{pmatrix}. \end{aligned} \tag{7.73}$$

Combining the equalities (7.71), (7.72), (7.73) we obtain

$$C_{k+1} = \begin{pmatrix} C_k(1:\eta_{k+1}, 1:\chi_{k+1}) & G_k h(k+1) & G_k b(k+1) \\ p(k+1)Q_k & d(k+1) & g(k+1) \\ a(k+1)Q_k & q(k+1) & \beta_k \end{pmatrix},$$

which completes the proof of (7.70).

Now combining the relations (7.69) and (7.70) we obtain the equalities (7.65), (7.66) with $k = 2, \dots, N-1$ and (7.67) with $k = 1, \dots, N-1$.

Finally, in the same way as above we obtain the formula (7.71), but with $k = N-1$:

$$C_N = \begin{pmatrix} I_{\eta_N} & 0 \\ 0 & V_N \end{pmatrix} \begin{pmatrix} C_{N-1}(1:\eta_N, 1:\chi_N) & G_{N-1} & 0 \\ Q_{N-1} & \beta_k & 0 \\ 0 & 0 & I_{\nu_N} \end{pmatrix} \begin{pmatrix} I_{\chi_N} & 0 \\ 0 & F_N \end{pmatrix}.$$

From here using the relation $U = \tilde{C}_N = C_N$ and the partitions (7.57) and (7.59) we obtain (7.65) and (7.66) with $k = N$. Using also the equality (7.64) we obtain (7.67) with $k = N$. □

For the case of a block upper triangular matrix we obtain the following statement.

Corollary 7.9. *Let F be a unitary block upper triangular matrix with block entries of sizes $m_i \times n_j$, $i,j = 1,\dots,N$. Set $s_0 = 0$ and*

$$s_k = \sum_{i=1}^{k}(m_i - n_i),\ k = 1,\dots,N. \tag{7.74}$$

By Lemma 7.3, the matrix F admits the factorization (7.54), (7.55) *with the unitary matrices F_1, F_k, $k = 2,\dots,N-1$, F_N of orders m_1, $n_k + s_k$, $k = 2,\dots,N-1$, n_N, respectively.*

Then upper quasiseparable generators $g(i)$ $(i = 1,\dots,N-1)$, $h(j)$ $(j = 2,\dots,N)$, $b(k)$ $(k = 2,\dots,N-1)$ and diagonal entries $d(k)$ $(k = 1,\dots,N)$ of the matrix F are determined from the partitions

$$F_1 = \begin{pmatrix} d(1) & g(1) \end{pmatrix}, \tag{7.75}$$

$$F_k = \begin{pmatrix} h(k) & b(k) \\ d(k) & g(k) \end{pmatrix}, \quad k = 2,\dots,N-1, \tag{7.76}$$

$$F_N = \begin{pmatrix} h(N) \\ d(N) \end{pmatrix}, \tag{7.77}$$

with the matrices $h(k), b(k), d(k), g(k)$ of sizes $s_{k-1} \times n_k, s_{k-1} \times s_k, m_k \times n_k, m_k \times s_k$, respectively.

Proof. We use Theorem 7.8 with $\nu_i = m_i$ $(i = 1,\dots,N)$ and $V = I$. The numbers ρ_k in (7.51) are all zero and the numbers s_k are defined via (7.74). Hence, the partition (7.61) takes the form (7.75). Moreover, from (7.62) and (7.58) it follows that

$$W_k = \begin{pmatrix} 0 & I_{\nu_k} \end{pmatrix} F_k = \begin{pmatrix} d_F(k) & g_F(k) \end{pmatrix}, \quad k = 2,\dots,N-1.$$

From here using (7.63) we get

$$\begin{pmatrix} d(k) & g(k) \end{pmatrix} = \begin{pmatrix} d_F(k) & g_F(k) \end{pmatrix}, \quad k = 2,\dots,N-1.$$

Inserting this in (7.58) we obtain (7.76). Finally, from (7.64) we get $d(N) = d_F(N)$ and combining this with (7.59) we obtain (7.77). □

§7.4 Unitary Hessenberg matrices

Let $U = \{u_{ij}\}_{i,j=1}^{N}$ be an $N \times N$ scalar matrix. The matrix U is called *upper Hessenberg* if its entries below the first subdiagonal are zeros, i.e., $u_{ij} = 0$ for $i > j+1$. We consider here the upper Hessenberg matrices that are also unitary.

One can show that using a similarity transformation with a diagonal unitary matrix one can reduce an upper Hessenberg matrix to an upper Hessenberg one with the real nonnegative subdiagonal entries. Indeed, if

$$U(i+1,i) = \mu_i e^{\varphi_i}, \quad i = 1,\dots,N-1,$$

with $0 \leq \varphi_i < 2\pi$ and $\mu_i \geq 0$, then setting $\psi_i = \sum_{j=i}^{N-1} \varphi_j,\ i = 1, \dots, N-1,\ \psi_N = 0$ and taking $D = \operatorname{diag}\{e^{\psi_i}, \dots, e^{\psi_N}\}$ we obtain the upper Hessenberg matrix $V = D^* U D$ with subdiagonal entries $\mu_k \geq 0,\ k = 1, \dots, N-1$.

Theorem 7.10. *Let U be an $N \times N$ unitary upper Hessenberg matrix. The matrix U admits the factorization*

$$U = \tilde{U}_1 \tilde{U}_2 \cdots \tilde{U}_N, \tag{7.78}$$

with

$$\tilde{U}_k = \operatorname{diag}\{I_{k-1}, U_k, I_{N-k-1}\},\ k = 1, \dots, N-1,\ \tilde{U}_N = \operatorname{diag}\{I_{N-1}, \rho_N\}, \tag{7.79}$$

where U_k are 2×2 unitary matrices and ρ_N is a complex number with $|\rho_N| = 1$.

Moreover, if the subdiagonal entries of the matrix U are nonnegative the matrices $U_k,\ k = 1, \dots, N-1$, may be written in the form

$$U_k = \begin{pmatrix} \rho_k & \mu_k \\ \mu_k & -\rho_k^* \end{pmatrix}, \tag{7.80}$$

with $\mu_k \geq 0,\ |\rho_k|^2 + \mu_k^2 = 1$, and the matrix U has the representation

$$U = \begin{pmatrix} -\rho_1\rho_0^* & -\rho_2\mu_1\rho_0^* & -\rho_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_{N-1}\mu_{N-2}\cdots\mu_1\rho_0^* & -\rho_N\mu_{N-1}\cdots\mu_1\rho_0^* \\ \mu_1 & -\rho_2\rho_1^* & -\rho_3\mu_2\rho_1^* & \cdots & -\rho_{N-1}\mu_{N-2}\cdots\mu_2\rho_1^* & -\rho_N\mu_{N-1}\cdots\mu_2\rho_1^* \\ 0 & \mu_2 & -\rho_3\rho_2^* & \cdots & -\rho_{N-1}\mu_{N-2}\cdots\mu_3\rho_2^* & -\rho_N\mu_{N-1}\cdots\mu_3\rho_2^* \\ \vdots & \ddots & \mu_3 & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{N-1}\rho_{N-2}^* & -\rho_N\mu_{N-1}\rho_{N-2}^* \\ 0 & \cdots & \cdots & 0 & \mu_{N-1} & -\rho_N\rho_{N-1}^* \end{pmatrix}, \tag{7.81}$$

where $\mu_k \geq 0,\ |\rho_k|^2 + \mu_k^2 = 1\ (k = 1, \dots, N-1),\ \rho_0 = -1,\ |\rho_N| = 1$.

Proof. We treat U as an $(N+1) \times (N+1)$ block matrix $U = \{u_{ij}\}_{i,j=0}^N$ with entries of sizes $m_i \times n_j$, where

$$m_0 = m_1 = \cdots = m_{N-1} = 1,\ m_N = 0, \quad n_0 = 0,\ n_1 = n_2 = \cdots = n_N = 1. \tag{7.82}$$

Relative to this partition U is a block upper triangular unitary matrix. By Lemma 7.3, U admits the factorization

$$U = \tilde{U}_0 \tilde{U}_1 \cdots \tilde{U}_{N-1} \tilde{U}_N \tag{7.83}$$

with the matrix $\tilde{U}_0 = \operatorname{diag}\{U_0, I_{N-1}\}$, where U_0 is a complex number with $|U_0| = 1$, and the matrices $\tilde{U}_k,\ k = 1, \dots, N$, defined via (7.79) with 2×2 unitary matrices $U_k,\ k = 1, \dots, N-1$, and a complex number $U_N = \rho_N$ such that $|\rho_N| = 1$.

Moreover, from the proof of Lemma 7.3 it follows that the number U_0 and the 2×2 unitary matrices $\tilde{U}_k,\ k = 1, \dots, N-1$ are determined from the conditions (7.15). Since $n_0 = 0$, the condition (7.15) for U_0 has the form $U_0 w = w'$, where

w, w' are 1×0 empty matrices so that one can take $U_0 = 1$, and therefore $\tilde{U}_0 = I_N$. Inserting $\tilde{U}_0 = I$ in (7.83) we obtain the factorization (7.78).

Next assume that the subdiagonal entries $\mu_k,\ k = 1,\dots,N-1$ of the matrix U are real and nonnegative. The formulas (7.15) mean that the 2×2 matrices $U_k,\ k = 1,\dots,N-1$, are determined from the conditions

$$U_k^* \begin{pmatrix} \rho_k \\ \mu_k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7.84}$$

where $\mu_k \geq 0,\ |\rho_k|^2 + \mu_k^2 = 1$. One can easily check that the matrices U_k defined by (7.80) satisfy (7.84).

Thus, the matrix $U = \{u_{ij}\}_{i,j=0}^N$ with the sizes of the entries defined in (7.82) is represented in the factorized form (7.83) with $\tilde{U}_0 = \mathrm{diag}\{1, I_{N-1}\}$, the matrices $\tilde{U}_k$ as in (7.79) with the matrices U_k of the form (7.80), and $\tilde{U}_N = \mathrm{diag}\{I_{N-1}, \rho_N\}$. Set $\rho_0 = -1$. By Corollary 7.9, upper quasiseparable generators and diagonal entries of the matrix U are given by

$$g(i) = -\rho_i^*,\ i = 0,\dots,N-1; \quad h(j) = \rho_j,\ j = 1,\dots,N;$$
$$b(k) = \mu_k,\ k = 1,\dots,N-1$$

and $d(k) = \mu_k,\ k = 1,\dots,N-1$ (the entries $d(0)$ and $d(N)$ are 1×0 and 0×1 empty matrices). Hence the representation (7.81) follows. □

§7.5 Efficient generators

Here we introduce a special type of quasiseparable generators. Using such generators in algorithms allows in some cases to reduce the complexity of computations. We consider here for simplicity matrices with special partitions into blocks.

Definiton 7.11. Let A be a block matrix with entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, where $m_1 = m_2 = \cdots = m_{N-1} = m, m_N = r$, with lower quasiseparable generators $p(i)$ $(i = 2,\dots,N)$, $q(j)$ $(j = 1,\dots,N-1)$, $a(k)$ $(k = 2,\dots,N-1)$ with orders equal to r. These lower quasiseparable generators are called *efficient* if they are in the left normal form and the $r \times r$ matrices $a(k)$ $(k = 2,\dots,N-1)$ are upper triangular and have upper quasiseparable generators with orders equal m.

We prove here that for any matrix with a special partition with given quasiseparable generators one can obtain a set of efficient quasiseparable generators.

Theorem 7.12. *Let A be a block matrix with entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, where $m_1 = m_2 = \cdots = m_{N-1} = m, m_N = r$ with lower quasiseparable generators $p(i)$ $(i = 2,\dots,N)$, $q(j)$ $(j = 1,\dots,N-1)$, $a(k)$ $(k = 2,\dots,N-1)$ with orders equal to r. A set of efficient lower quasiseparable generators $\tilde{p}(i)$ $(i = 2,\dots,N)$, $\tilde{q}(j)$ $(j = 1,\dots,N-1)$, $\tilde{a}(k)$ $(k = 2,\dots,N-1)$ of the matrix A is determined using the following algorithm.*

1.1. *Compute the* QR *factorization*

$$p(N) = p_V(N)X_N, \tag{7.85}$$

where $p_V(N)$ *is a unitary matrix of size* $r \times r$, X_N *is a matrix of size* $r \times r$.

1.2. *For* $k = N-1, \dots, 2$ *perform the following. Compute the* QR *factorization*

$$\begin{bmatrix} p(k) \\ X_{k+1}a(k) \end{bmatrix} = V_k \begin{pmatrix} X_k \\ 0_{m\times r} \end{pmatrix}, \tag{7.86}$$

where V_k *is a unitary matrix of size* $(m+r)\times(m+r)$, X_k *is a matrix of size* $r \times r$. *Determine the matrices* $p_V(k)$, $a_V(k)$, $d_V(k)$, $q_V(k)$ *of sizes* $m \times r$, $r \times r$, $m \times m$, $r \times m$ *from the partition*

$$V_k = \begin{bmatrix} p_V(k) & d_V(k) \\ a_V(k) & q_V(k) \end{bmatrix}. \tag{7.87}$$

1.3. *For* $k = 1, \dots, N-1$ *compute the matrices*

$$q'(k) = X_{k+1}q(k) \tag{7.88}$$

of sizes $r \times m$.

2.1. *Set* $Q_1 = I_r$ *and for* $k = 2, \dots, N-1$ *compute the* QR *factorizations*

$$a_V(k)Q_{k-1} = Q_k\tilde{a}(k), \tag{7.89}$$

where Q_k *is an* $r\times r$ *unitary matrix and* $\tilde{a}(k)$ *is an* $r \times r$ *upper triangular matrix.*

2.2. *For* $k = 1, \dots, N-1$ *compute the matrices*

$$\tilde{q}(k) = Q_k^* q'(k), \quad \tilde{p}(k+1) = p_V(k+1)Q_k \tag{7.90}$$

of sizes $r \times m$ *and* $m \times r$ *respectively.*

Moreover, for any k *with* $2 \le k \le N-1$ *the upper quasiseparable generators* $g_\alpha^{(k)}(i)$ $(i = 1, \dots, r-1)$, $h_\alpha^{(k)}(j)$ $(j = 2, \dots, r)$, $a_\alpha^{(k)}(t)$ $(t = 2, \dots, r-1)$ *and diagonal entries* $d_\alpha^{(k)}(t)$ $(t = 1, \dots, r)$ *of the matrix* $\tilde{a}(k)$ *are determined using the following algorithm.*

1. *Define the unitary matrix* $W^{(k)}$ *by*

$$W^{(k)} = \begin{pmatrix} I_m & 0_{m\times r} \\ 0_{r\times m} & Q_k^* \end{pmatrix} V_k \begin{pmatrix} Q_{k-1} & 0_{r\times m} \\ 0_{m\times r} & I_m \end{pmatrix} \tag{7.91}$$

and set $\hat{W}_1^{(k)} = W^{(k)}$.

2. *For* $i = 1, \dots, r-1$ *perform the following. Set*

$$w_i^{(k)} = \hat{W}_i^{(k)}(1 : m+1, 1) \tag{7.92}$$

and determine the unitary $(m+1)\times(m+1)$ *matrix* $W_i^{(k)}$ *from the condition*

$$(W_i^{(k)})^* w_i^{(k)} = \begin{pmatrix} 1 \\ 0_{m\times 1} \end{pmatrix}. \tag{7.93}$$

Compute the $(m+r+1-i)\times(m+r+1-i)$ *matrix*

$$Z_i^{(k)} = \begin{pmatrix} (W_i^{(k)})^* & 0 \\ 0 & I_{r-i} \end{pmatrix} \hat{W}_i^{(k)} \tag{7.94}$$

and determine the $(m+r-i)\times(m+r-i)$ *matrix* $\hat{W}_{i+1}^{(k)}$ *from the partition*

$$Z_i^{(k)} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{W}_{i+1}^{(k)} \end{pmatrix}. \tag{7.95}$$

3. *Set*

$$W_r^{(k)} = \hat{W}_r^{(k)} \tag{7.96}$$

which is an $(m+1)\times(m+1)$ *sized matrix.*

4. *Determine the elements* $g_\alpha^{(k)}(i)$ $(i=1,\dots,r-1)$, $h_\alpha^{(k)}(j)$ $(j=2,\dots,r)$, $a_\alpha^{(k)}(t)$ $(t=2,\dots,r-1)$ *and* $d_\alpha^{(k)}(t)$ $(t=1,\dots,r)$ *from the partitions*

$$W_1^{(k)}(m+1,:) = \begin{pmatrix} d_\alpha^{(k)}(1) & g_\alpha^{(k)}(1) \end{pmatrix}, \tag{7.97}$$

$$W_i^{(k)} = \begin{pmatrix} h_\alpha^{(k)}(i) & b_\alpha^{(k)}(i) \\ d_\alpha^{(k)}(i) & g_\alpha^{(k)}(i) \end{pmatrix}, \quad i=2,\dots,r-1, \tag{7.98}$$

$$W_r^{(k)}(:,1) = \begin{pmatrix} h_\alpha^{(k)}(r) \\ d_\alpha^{(k)}(r) \end{pmatrix}, \tag{7.99}$$

with the complex numbers $d_\alpha^{(k)}(i)$ *and the matrices* $g_\alpha^{(k)}(i), h_\alpha^{(k)}(i), b_\alpha^{(k)}(i)$ *of sizes* $1\times m, m\times 1, m\times m$, *respectively.*

Proof. From the formulas (7.85)–(7.88) it follows that

$$p(k) = p_V(k)X_k,\ k=2,\dots,N; \quad X_{k+1}a(k) = a_V(k)X_k,\ k=2,\dots,N-1;$$
$$q'(k) = X_{k+1}q(k),\ k=1,\dots,N-1,$$

with $p_V^*(N)p_V(N) = I_r,\quad a_V^*(k)a_V(k) + p_V^*(k)p_V(k) = I_r,\ k=N-1,\dots,2$. By Theorem 5.29, this means that $p_V(i)$ $(i=2,\dots,N)$, $q'(j)$ $(j=1,\dots,N-1)$, $a_V(k)$ $(k=2,\dots,N-1)$ are lower quasiseparable generators of the matrix A in the left normal form.

Next, formulas (7.89), (7.90) and Theorem 5.20 imply that $\tilde{p}(i)$ $(i=2,\dots,N)$, $\tilde{q}(j)$ $(j=1,\dots,N-1)$, $\tilde{a}(k)$ $(k=2,\dots,N-1)$ are lower quasiseparable generators of the matrix A. Consider the unitary matrices $W^{(k)}$ from (7.91). Using (7.87) we get

$$W^{(k)} = \begin{bmatrix} p_V(k)Q_{k-1} & d_V(k) \\ Q_k^* a_V(k)Q_{k-1} & Q_k^* q_V(k) \end{bmatrix}, \quad k = 2, \dots, N-1.$$

Comparing this with (7.89), (7.90) we see that

$$W^{(k)} = \begin{bmatrix} \tilde{p}(k) & d_V(k) \\ \tilde{a}(k) & Q_k^* q_V(k) \end{bmatrix}, \quad k = 2, \dots, N-1.$$

Hence, that the generators $\tilde{p}(i), \tilde{q}(j), \tilde{a}(k)$ are in the left normal form.

Next, for any k with $2 \le k \le N-1$, since the matrix $\tilde{a}(k)$ is upper triangular, the unitary $(m+r) \times (m+r)$ matrix $W^{(k)}$ may be treated as a block matrix with entries of sizes $\nu_i \times \mu_j,\ i, j = 1, \dots, r$, where

$$\nu_1 = m+1,\ \nu_2 = \dots = \nu_r = 1, \quad \mu_1 = \dots = \mu_{r-1} = 1,\ \mu_r = m+1.$$

By Lemma 7.3, $W^{(k)}$ admits the factorization

$$W^{(k)} = \tilde{W}_1^{(k)} \tilde{W}_2^{(k)} \cdots \tilde{W}_r^{(k)},$$

with

$$\begin{aligned} \tilde{W}_1^{(k)} &= \operatorname{diag}\{W_1^{(k)}, I_{r-1}\}, \\ \tilde{W}_i^{(k)} &= \operatorname{diag}\{I_{i-1}, W_i^{(k)}, I_{r-i}\},\ i = 2, \dots, r-1, \\ \tilde{W}_r^{(k)} &= \operatorname{diag}\{I_{r-1}, W_r^{(k)}\}, \end{aligned}$$

where $W_i^{(k)}$ are unitary matrices of size $(m+1) \times (m+1)$ defined via the relations (7.91)–(7.96). By Corollary 7.9, upper quasiseparable generators $g^{(k)}(i)$ $(i = 1, \dots, r-1)$, $h^{(k)}(j)$ $(j = 2, \dots, r)$, $b^{(k)}(t)$ $(t = 2, \dots, r-1)$ and diagonal entries $d^{(k)}(t)$ $(t = 1, \dots, r)$ of the matrix $W^{(k)}$ are determined from the partitions

$$W_1^{(k)} = \begin{pmatrix} d^{(k)}(1) & g^{(k)}(1) \end{pmatrix},$$

with the elements $d^{(k)}(1), g^{(k)}(1)$ of sizes $(m+1) \times 1, (m+1) \times m$, respectively,

$$W_i^{(k)} = \begin{pmatrix} h^{(k)}(i) & b^{(k)}(i) \\ d^{(k)}(i) & g^{(k)}(i) \end{pmatrix}, \quad i = 2, \dots, r-1,$$

with the complex numbers $d^{(k)}(i)$ and the matrices $g^{(k)}(i), h^{(k)}(i), b^{(k)}(i)$ of sizes $1 \times m, m \times 1, m \times m$ respectively,

$$W_r^{(k)} = \begin{pmatrix} h^{(k)}(r) \\ d^{(k)}(r) \end{pmatrix},$$

with the elements $h^{(k)}(r), d^{(k)}(r)$ of sizes $m \times (m+1), 1 \times (m+1)$, respectively. Using the fact that $\tilde{a}(k) = W^{(k)}(m+1 : m+r, 1 : r)$ we obtain the formulas (7.97)–(7.99) for the upper quasiseparable generators and diagonal entries of the matrix $\tilde{a}(k)$. □

In order to compute the complexity of the algorithm in the previous theorem we proceed as follows. We will not take into account operations of assignment, scalar multiplications by 1 or additions by 0.

In formula (7.91), each of the two matrix multiplications builds an $r \times (m+r)$ matrix where each entry is computed by means of r arithmetical multiplications and $r-1$ arithmetical additions, hence formula (7.91) uses less than $4r^2(m+r)$ arithmetical operations.

Formula (7.93) requires $\rho(m+1)$ operations, where $\rho(k)$ is the cost of inverting a $k \times k$ matrix by a standard method.

In formula (7.94), the matrix multiplication builds an $(m+1) \times (m+r+1-i)$ matrix where each entry is computed by means of $m+1$ arithmetical multiplications and m arithmetical additions, hence formula (7.94) uses less than $2(m+1)^2(m+r)$ arithmetical operations.

Finally, in formula (7.90) each of the two matrix multiplications builds an $r \times m$ matrix where each entry is computed by means of r arithmetical multiplications and $r-1$ arithmetical additions, hence formula (7.90) uses less than $4r^2m$ arithmetical operations.

In total, Step 2.2 of the algorithm performs (7.90) $N-1$ times, while (7.91)–(7.94) are computed $N-2$ times.

If we denote $r_M = \max(r, m)$, then the total complexity c of the algorithm satisfies

$$c < (12r_M(r_M+1)^2 + \rho(r_M+1))(N-2) + 4(r_M)^3(N-1) < 80(r_M)^3N,$$

therefore the algorithm is of complexity $O((r_M)^3N)$. For scalar matrices $r_M = 1$ and there are no additions so that

$$c < 19N - 36.$$

§7.6 Comments

Factorization representations of unitary matrices were used in the monograph [15] and the papers [23], [4]. The representations of unitary matrices via products of Givens matrices were studied by S. Delvaux and M. Van Barel in [14]. The representations of unitary Hessenberg matrices were studied intensively in the literature, see for instance the papers by W.B. Gragg [32] and P.E. Gill, G.H. Golub, W. Murray and M.A. Saunders [34]. Results close to the ones from Section 5 are presented in the paper by S. Delvaux and M. Van Barel [12].

The presentation of the material in such a form as in this chapter appears for the first time.

Part II

Completion of Matrices with Specified Bands

Introduction to Part II

This part is a natural extension of the previous one. It is also of matrix theoretical character, mostly dedicated to completion problems for different classes of partially specified matrices. The main material is contained in the first three chapters, which deal with completion to Green matrices of matrices specified on a band. The novelty is the approach in which the algorithms are described directly in terms of the unspecified entries, without additional intermediate steps (compare with [10, 11, 40]). The direct algorithm of completion to a Green matrix is presented in Chapter 8. The properties of the completions are studied in detail in Chapter 9. Chapter 10 contains applications of the completion method to some special types of specified bands. Chapter 11 is dedicated to application for the completion problem of mutually inverse matrices. Here is considered the problem when the original matrix is specified in its lower (with the diagonal) triangular section and the inverse matrix is specified in its strictly upper triangular section. In the final chapter we consider the problem of completion of a partially specified matrix with a given lower triangular part to a unitary matrix.

The material of this part, except for its last chapter, is not used in the subsequent three parts. The last chapter is used in Part VII. More than that, Part II, except for the last chapters, does not deal directly with semiseparable or quasiseparable representations of matrices. However matrices with small quasiseparable orders appear in examples of Part II. The reader interested in a shortcut to algorithms for semi- and quasi-separable representations may skip this part at the first reading.

Chapter 8

Completion to Green Matrices

In this chapter the problem of completing a given band $\tilde{A} = \{A_{ij}, |i-j| \le n\}_{i,j=1}^{N}$, $n \ge 1$ to a Green matrix of order n is considered. The submatrices of the band $\tilde{A}$,

$$B_k = \tilde{A}(k-n+1 : k, k-n+1 : k), \quad k = n+1, \dots, N-1, \tag{8.1}$$

play a key role.

If all the matrices B_k are invertible then the completion exists and is unique. If some of the matrices B_k are not invertible then it may be that the completion does not exist or is not unique.

The unique completion can be obtained by computing the principal leading submatrices of the given matrix.

§8.1 Auxiliary relations

We start with the following auxiliary result.

Lemma 8.1. *Let Q be a square matrix which has a partition*

$$Q = \begin{pmatrix} B' & X & \Gamma' \\ Y & B & Z \\ \Gamma'' & U & B'' \end{pmatrix},$$

where B', B, B'' are square matrices. Assume that the matrix B is invertible.

Then:

1) *The relations*

$$\operatorname{rank} \begin{pmatrix} X & \Gamma' \\ B & Z \end{pmatrix} = \operatorname{rank} B, \quad \operatorname{rank} \begin{pmatrix} Y & B \\ \Gamma'' & U \end{pmatrix} = \operatorname{rank} B \tag{8.2}$$

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hold if and only if $\Gamma' = XB^{-1}Z$, $\Gamma'' = UB^{-1}Y$. *Moreover, in this case the following factorizations hold:*

$$Q = \begin{pmatrix} I & XB^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \delta & 0 & 0 \\ 0 & B & Z \\ 0 & U & B'' \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ B^{-1}Y & I & 0 \\ 0 & 0 & I \end{pmatrix}, \tag{8.3}$$

where $\delta = B' - XB^{-1}Y$, *and*

$$Q = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & UB^{-1} & I \end{pmatrix} \begin{pmatrix} B' & X & 0 \\ Y & B & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & B^{-1}Z \\ 0 & 0 & I \end{pmatrix}, \tag{8.4}$$

where $\gamma = B'' - UB^{-1}Z$.

2) *Assume that the conditions* (8.2) *are valid. Then the following conditions are equivalent:*

1. *the matrix* Q *is invertible;*
2. *the submatrix* $\begin{pmatrix} B' & X \\ Y & B \end{pmatrix}$ *and the element* γ *are invertible;*
3. *the submatrix* $\begin{pmatrix} B & Z \\ U & B'' \end{pmatrix}$ *and the element* δ *are invertible.*

If these conditions hold, set

$$\begin{pmatrix} B' & X \\ Y & B \end{pmatrix}^{-1} = \begin{pmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{pmatrix}, \quad \begin{pmatrix} B & Z \\ U & B'' \end{pmatrix}^{-1} = \begin{pmatrix} B''_{11} & B''_{12} \\ B''_{21} & B''_{22} \end{pmatrix}.$$

Then the inverse matrix Q^{-1} *is given by the formulas*

$$Q^{-1} = \begin{pmatrix} B'_{11} & B'_{12} & 0 \\ B'_{21} & B'_{22} + B^{-1}Z\gamma^{-1}UB^{-1} & -B^{-1}Z\gamma^{-1} \\ 0 & -\gamma^{-1}UB^{-1} & \gamma^{-1} \end{pmatrix}; \tag{8.5}$$

$$Q^{-1} = \begin{pmatrix} \delta^{-1} & -\delta^{-1}XB^{-1} & 0 \\ -B^{-1}Y\delta^{-1} & B''_{11} + B^{-1}Y\delta^{-1}XB^{-1} & B''_{12} \\ 0 & B''_{21} & B''_{22} \end{pmatrix}. \tag{8.6}$$

Proof. 1) Since the matrix B invertible one obviously gets

$$\operatorname{rank} \begin{pmatrix} X \\ B \end{pmatrix} = \operatorname{rank} \begin{pmatrix} B & Z \end{pmatrix} = \operatorname{rank} B.$$

Hence, by Lemma 2.10, the condition $\operatorname{rank} \begin{pmatrix} X & \Gamma' \\ B & Z \end{pmatrix} = \operatorname{rank} B$ is equivalent to the fact that the matrix $\begin{pmatrix} X & \Gamma' \\ B & Z \end{pmatrix}$ is the unique minimal rank completion of the partially specified matrix $\begin{pmatrix} X & ? \\ B & Z \end{pmatrix}$. By Corollary 2.11, the element Γ' in such completion is determined by the formula $\Gamma' = SZ$ with the matrix S such that $SB = X$, i.e., $S = XB^{-1}$. Consequently, $\Gamma' = XB^{-1}Z$. Applying the same arguments to the matrix transpose to $\begin{pmatrix} Y & B \\ \Gamma'' & U \end{pmatrix}$ one obtains the formula $\Gamma'' = UB^{-1}Y$.

Using the equalities $\Gamma' = XB^{-1}Z$, $\Gamma'' = UB^{-1}Y$ one can directly check the validity of the relations (8.3), (8.4).

2) From the formula (8.4) we conclude that the matrix Q is invertible if and only if the submatrix $\begin{pmatrix} B' & X \\ Y & B \end{pmatrix}$ and the element γ are invertible, in which case

$$Q^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -B^{-1}Z \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} B'_{11} & B'_{12} & 0 \\ B'_{21} & B'_{22} & 0 \\ 0 & 0 & \gamma^{-1} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -UB^{-1} & I \end{pmatrix}.$$

Multiplying the factors in the right-hand side of this equality one obtains (8.5).

3) From the formula (8.3) we conclude that the matrix Q is invertible if and only if the submatrix $\begin{pmatrix} B & Z \\ U & B'' \end{pmatrix}$ and the element δ are invertible, in which case

$$Q^{-1} = \begin{pmatrix} I & 0 & 0 \\ -B^{-1}Y & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \delta^{-1} & 0 & 0 \\ 0 & B''_{11} & B''_{12} \\ 0 & B''_{21} & B''_{22} \end{pmatrix} \begin{pmatrix} I & -XB^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Multiplying the factors in the right-hand side of this equality one obtains (8.6). □

§8.2 Completion formulas

In this section we derive explicit formulas for completion of a given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$, $n \geq 1$ to a Green matrix A of order n.

Theorem 8.2. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$. Assume that all the submatrices of the band $\tilde{A}$ of the form*

$$B_k = \tilde{A}(k-n+1 : k, k-n+1 : k), \quad k = n+1, \dots, N-1 \tag{8.7}$$

are invertible.

Then the band $\tilde{A}$ has a unique completion A to a Green matrix of order n. One obtains this completion by successive computations of its principal leading submatrices

$$A_k := A(1 : k, 1 : k), \quad k = n+1, \dots, N$$

as follows.

In the first step set

$$A_{n+1} = \tilde{A}(1:n+1, 1:n+1). \tag{8.8}$$

Let for some k with $n+1 \leq k \leq N-1$ the matrix A_k be given. The matrix A_{k+1} is obtained via the following operations. Start by partitioning A_k in the form

$$A_k = \begin{bmatrix} B'_k & X_k \\ Y_k & B_k \end{bmatrix} \tag{8.9}$$

with B_k defined in (8.7). *Next, compute the submatrix A_{k+1} by the formula*

$$A_{k+1} = \begin{bmatrix} B'_k & X_k & E_k \\ Y_k & B_k & Z_k \\ F_k & U_k & M_k \end{bmatrix}, \tag{8.10}$$

where M_k, Z_k, U_k are determined from the band via the equalities

$$M_k = \tilde{A}(k+1, k+1), \quad Z_k = \tilde{A}(k-n+1:k, k+1), \quad U_k = \tilde{A}(k+1, k-n+1:k) \tag{8.11}$$

and E_k, F_k are computed by the formulas

$$E_k = X_k B_k^{-1} Z_k, \quad F_k = U_k B_k^{-1} Y_k. \tag{8.12}$$

Finally, set

$$A = A_N. \tag{8.13}$$

Proof. We must prove that the matrix A defined by the relations (8.7)–(8.13) satisfies (6.11) and (6.13). Consider the submatrices of A of the form

$$\Omega_k = A(1:k, k-n+1:N), \quad \Psi_k = A(k-n+1:N, 1:k), \quad k = n+1, \dots, N-1. \tag{8.14}$$

By the definition of rank numbers,

$$r^U_{k,-n}(A) = \operatorname{rank}\Omega_k, \quad r^L_{k,-n}(A) = \operatorname{rank}\Psi_k, \quad k = n+1, \dots, N-1,$$

and thus the conditions (6.11), (6.13) are equivalent to

$$\operatorname{rank}\Omega_k \leq \alpha_k, \quad \operatorname{rank}\Psi_k \leq \alpha_k, \quad k = n+1, \dots, N-1 \tag{8.15}$$

with $\alpha_k = m_{k-n+1} + \cdots + m_k$. Comparing the definitions (8.10) and (8.14), we see that

$$\Omega_k = \begin{pmatrix} X_k & E_k & * \\ B_k & Z_k & * \end{pmatrix}, \quad \Psi_k = \begin{pmatrix} Y_k & B_k \\ F_k & U_k \\ * & * \end{pmatrix}, \quad k = n+1, \dots, N-1.$$

Here each B_k is an invertible matrix of order α_k and hence the conditions (8.15) are equivalent to

$$\operatorname{rank}\Omega_k = \operatorname{rank}\Psi_k = \alpha_k, \quad k = n+1,\dots,N-1. \tag{8.16}$$

Next, using (8.12) one gets

$$\begin{pmatrix} E_j \\ Z_j \end{pmatrix} = \begin{pmatrix} X_j \\ B_j \end{pmatrix} B_j^{-1}Z_j, \quad \begin{pmatrix} F_j & U_j \end{pmatrix} = U_jB_j^{-1}\begin{pmatrix} Y_j & B_j \end{pmatrix}, \quad j = n+1,\dots,N-1.$$

These equalities imply that

$$A(1:j, j+1) = A(1:j, j-n+1:j)(B_j^{-1}Z_j), \quad j = n+1,\dots,N-1 \tag{8.17}$$

and

$$A(j+1, 1:j) = (U_jB_j^{-1})A(j-n+1:j, 1:j), \quad j = n+1,\dots,N-1. \tag{8.18}$$

Let $n+1 \le k \le j \le N-1$. Comparing the first k rows in (8.17) one obtains

$$A(1:k, j+1) = A(1:k, j-n+1:j)(B_j^{-1}Z_j), \quad k = n+1,\dots,N-1,\ k \le j \le N-1. \tag{8.19}$$

This means that every column of the matrix Ω_k, starting with the column with the index α_k+1, is a linear combination of the previous columns of Ω_k. It follows that all the columns of the matrix Ω_k are linear combinations of its first α_k columns, i.e., the columns of the matrix $\begin{bmatrix} X_k \\ B_k \end{bmatrix}$ and consequently

$$\operatorname{rank}\begin{pmatrix} X_k \\ B_k \end{pmatrix} = \operatorname{rank}\Omega_k.$$

In a similar way one gets

$$\operatorname{rank}\begin{pmatrix} Y_k & B_k \end{pmatrix} = \operatorname{rank}\Psi_k.$$

But since B_k is an invertible matrix of order α_k,

$$\operatorname{rank}\Omega_k = \operatorname{rank}\Psi_k = \operatorname{rank}B_k = \alpha_k.$$

To prove the uniqueness, suppose $A = \{A_{ij}\}_{i,j=1}^N$ is a completion of $\tilde{A}$ satisfying (6.11), (6.13), which is equivalent to (8.15) with the matrices Ω_k, Ψ_k defined in (8.14). Since the matrices B_k of the form (8.7) are invertible matrices of sizes $\alpha_k \times \alpha_k$, one gets

$$\operatorname{rank}\Omega_k = \operatorname{rank}\Psi_k = \operatorname{rank}B_k, \quad k = n+1,\dots,N-1. \tag{8.20}$$

Consider the submatrices

$$W_k = A(1:k, k-n+1:k+1),\ C_k = A(k-n+1:k+1, 1:k), \quad k = n+1,\dots,N-1$$

Comparing with (8.10) one gets

$$W_k = \begin{pmatrix} X_k & E_k \\ B_k & Z_k \end{pmatrix}, \; C_k = \begin{pmatrix} Y_k & B_k \\ F_k & U_k \end{pmatrix}.$$

Each W_k contains B_k and is contained in Ω_k and each C_k contains B_k and is contained in Ψ_k. Hence (8.20) implies

$$\operatorname{rank} W_k = \operatorname{rank} C_k = \operatorname{rank} B_k, \quad k = n+1, \dots, N-1,$$

and using the first part of Lemma 8.1 we conclude that

$$E_k = X_k B_k^{-1} Z_k, \quad F_k = U_k B_k^{-1} Y_k, \quad k = n+1, \dots, N-1.$$

This means that all the unspecified entries of $\mathcal{A}$ are uniquely determined by formulas (8.12). □

Example 8.3. This example uses Theorem 8.2 in order to find the Green completion of order 1 of a given scalar band

$$\tilde{A} = \{A_{ij}, |i-j| \leq 1\} = \begin{pmatrix} 0 & \beta_1 & ? & ? & ? & ? \\ \alpha_1 & 1 & \beta_2 & ? & ? & ? \\ ? & \alpha_2 & 1 & \beta_3 & ? & ? \\ ? & ? & \alpha_3 & 1 & \beta_4 & ? \\ ? & ? & ? & \alpha_4 & 1 & \beta_5 \\ ? & ? & ? & ? & \alpha_5 & 0 \end{pmatrix}.$$

For this band $n = 1$ and $N = 6$ and all the submatrices of $\tilde{A}$ of the form (8.7) are scalars, namely

$$B_k = \tilde{A}(k-n+1:k, k-n+1:k) = 1, \quad k = n+1, \dots, N-1$$

and are invertible. Then $\tilde{A}$ has a unique completion A which is a Green matrix of order 1. One obtains this completion by successive computation of its principal leading submatrices

$$A_k := A(1:k, 1:k), \quad k = 2, \dots, 6$$

as follows.

In the first step, set as in (8.8)

$$A_2 = A_{n+1} = \tilde{A}(1:n+1, 1:n+1) = \tilde{A}(1:2, 1:2) = \begin{pmatrix} 0 & \beta_1 \\ \alpha_1 & 1 \end{pmatrix}.$$

Let for $k = 2$ with $2 = n+1 \leq k \leq N-1 = 5$ the matrix A_k be given. The matrix $A_3 = A_{k+1}$ is obtained via the following operations. Start with partitioning

A_2 in the form of (8.9), namely $A_k = \begin{pmatrix} B'_k & X_k \\ Y_k & B_k \end{pmatrix}$, with $B_k = B_2 = 1$ defined in (8.7). It follows that $B'_2 = 0, X_2 = \beta_1, Y_2 = \alpha_1$.

Next compute the submatrix $A_{k+1} = A_3$ by (8.10), namely

$$A_{k+1} = A_3 = \begin{pmatrix} B'_k & X_k & E_k \\ Y_k & B_k & Z_k \\ F_k & U_k & M_k \end{pmatrix},$$

where M_k, U_k, Z_k are determined from the band via (8.11):

$$\begin{aligned} M_k &= M_2 = \tilde{A}(k+1, k+1) = \tilde{A}(3,3) = 1, \\ Z_k &= Z_2 = \tilde{A}(k-n+1 : k, k+1) = \tilde{A}(2,3) = \beta_2, \\ U_k &= U_2 = \tilde{A}(k+1, k-n+1 : k) = \tilde{A}(3,2) = \alpha_2 \end{aligned}$$

and E_k, F_k are computed by (8.12):

$$E_2 = E_k = X_k B_k^{-1} Z_k = \beta_1\beta_2, \quad F_2 = F_k = U_k B_k^{-1} Y_k = \alpha_2\alpha_1.$$

It follows that

$$A_3 = \begin{pmatrix} 0 & \beta_1 & \beta_1\beta_2 \\ \alpha_1 & 1 & \beta_2 \\ \alpha_2\alpha_1 & \alpha_2 & 1 \end{pmatrix}.$$

We continue in the same way and obtain the completion

$$\begin{aligned} A &= A_6 \\ &= \begin{pmatrix} 0 & \beta_1 & \beta_1\beta_2 & \beta_1\beta_2\beta_3 & \beta_1\beta_2\beta_3\beta_4 & \beta_1\beta_2\beta_3\beta_4\beta_5 \\ \alpha_1 & 1 & \beta_2 & \beta_2\beta_3 & \beta_2\beta_3\beta_4 & \beta_2\beta_3\beta_4\beta_5 \\ \alpha_2\alpha_1 & \alpha_2 & 1 & \beta_3 & \beta_3\beta_4 & \beta_3\beta_4\beta_5 \\ \alpha_3\alpha_2\alpha_1 & \alpha_3\alpha_2 & \alpha_3 & 1 & \beta_4 & \beta_4\beta_5 \\ \alpha_4\alpha_3\alpha_2\alpha_1 & \alpha_4\alpha_3\alpha_2 & \alpha_4\alpha_3 & \alpha_4 & 1 & \beta_5 \\ \alpha_5\alpha_4\alpha_3\alpha_2\alpha_1 & \alpha_5\alpha_4\alpha_3\alpha_2 & \alpha_5\alpha_4\alpha_3 & \alpha_5\alpha_4 & \alpha_5 & 0 \end{pmatrix}. \end{aligned}$$

◊

Example 8.4. This example uses Theorem 8.2 in order to find the Green completion of order 2 of a scalar given band

$$\tilde{C} = \{C_{ij}, |i-j| \le 2\} = \begin{pmatrix} 0 & \beta_1 & \beta_1\beta_2 & ? & ? & ? \\ \alpha_1 & 1 & \beta_2 & \beta_2\beta_3 & ? & ? \\ \alpha_2\alpha_1 & \alpha_2 & 1 & \beta_3 & \beta_3\beta_4 & ? \\ ? & \alpha_3\alpha_2 & \alpha_3 & 1 & \beta_4 & \beta_4\beta_5 \\ ? & ? & \alpha_4\alpha_3 & \alpha_4 & 1 & \beta_5 \\ ? & ? & ? & \alpha_5\alpha_4 & \alpha_5 & 0 \end{pmatrix},$$

where $\alpha_k\beta_k \neq 1,\ k = 2,3,4$ and $\alpha_k\beta_k \neq 0,\ k = 1,5$.

For this band $n = 2$ and $N = 6$ and all the submatrices of $\tilde{C}$ of the form (8.7), namely

$$\begin{aligned} B_k &= \tilde{C}(k-n+1:k, k-n+1:k) = C(k-1:k, k-1:k) \\ &= \begin{pmatrix} \gamma_k & \beta_{k-1} \\ \alpha_{k-1} & \delta_k \end{pmatrix}, \quad k = 2, \dots, 6, \end{aligned}$$

are invertible, where $\gamma_k = 1,\ k = 3, \dots, 6$ and $\gamma_2 = 0$ while $\delta_k = 1,\ k = 2, \dots, 5$, $\delta_6 = 0$. Then $\tilde{C}$ has a unique completion C which is a Green matrix of order 2. One obtains this completion by successive computation of its principal leading submatrices

$$C_k := C(1:k, 1:k), \quad k = 3, \dots, 6$$

as follows.

In the first step, set as in (8.8)

$$C_3 = C_{n+1} = \tilde{C}(1:n+1, 1:n+1) = \tilde{C}(1:3, 1:3) = \begin{pmatrix} 0 & \beta_1 & \beta_1\beta_2 \\ \alpha_1 & 1 & \beta_2 \\ \alpha_2\alpha_1 & \alpha_2 & 1 \end{pmatrix}.$$

Note that this matrix is the same as the matrix A_3 in Example 8.3.

Let for $k = 3$ with $3 = n+1 \le k \le N-1 = 5$ the matrix C_k be given. The matrix $C_4 = C_{k+1}$ is obtained via the following operations. Start with partitioning C_3 in the form (8.9), namely $C_k = \begin{pmatrix} B'_k & X_k \\ Y_k & B_k \end{pmatrix}$, with $B_k = B_3 = \begin{pmatrix} 1 & \beta_2 \\ \alpha_2 & 1 \end{pmatrix}$ defined in (8.7). It follows that $B_3^{-1} = \frac{1}{1-\alpha_2\beta_2} \begin{pmatrix} 1 & -\beta_2 \\ -\alpha_2 & 1 \end{pmatrix}$ and $B'_3 = 0$, $X_3 = \begin{pmatrix} \beta_1 & \beta_1\beta_2 \end{pmatrix}$, $Y_3 = \begin{pmatrix} \alpha_1 \\ \alpha_2\alpha_1 \end{pmatrix}$.

Next compute the submatrix $C_{k+1} = C_4$ by the formula (8.10), namely

$$C_{k+1} = C_4 = \begin{pmatrix} B'_k & X_k & E_k \\ Y_k & B_k & Z_k \\ F_k & U_k & M_k \end{pmatrix},$$

where M_k, U_k, Z_k are determined from the band via (8.11):

$$\begin{aligned} M_k &= M_3 = \tilde{C}(k+1, k+1) = \tilde{C}(4,4) = 1, \\ Z_k &= Z_3 = \tilde{C}(k-n+1:k, k+1) = \tilde{C}(2:3, 4) = \begin{pmatrix} \beta_2\beta_3 \\ \beta_3 \end{pmatrix}, \\ U_k &= U_3 = \tilde{C}(k+1, k-n+1:k) = \tilde{C}(4, 2:3) = \begin{pmatrix} \alpha_3\alpha_2 & \alpha_3 \end{pmatrix} \end{aligned}$$

and E_k, F_k are computed by (8.12):

$$E_3 = E_k = X_k B_k^{-1} Z_k = \begin{pmatrix} \beta_1 & \beta_1\beta_2 \end{pmatrix} \frac{1}{1-\alpha_2\beta_2} \begin{pmatrix} 1 & -\beta_2 \\ -\alpha_2 & 1 \end{pmatrix} \begin{pmatrix} \beta_2\beta_3 \\ \beta_3 \end{pmatrix}$$
$$= \begin{pmatrix} \beta_1 & \beta_1\beta_2 \end{pmatrix} \begin{pmatrix} 0 \\ \beta_3 \end{pmatrix} = \beta_1\beta_2\beta_3,$$
$$F_3 = F_k = U_k B_k^{-1} Y_k = \begin{pmatrix} \alpha_3\alpha_2 & \alpha_3 \end{pmatrix} \frac{1}{1-\alpha_2\beta_2} \begin{pmatrix} 1 & -\beta_2 \\ -\alpha_2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2\alpha_1 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_3\alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = \alpha_3\alpha_2\alpha_1.$$

It follows that

$$C_4 = \begin{pmatrix} 0 & \beta_1 & \beta_1\beta_2 & \beta_1\beta_2\beta_3 \\ \alpha_1 & 1 & \beta_2 & \beta_2\beta_3 \\ \alpha_2\alpha_1 & \alpha_2 & 1 & \beta_3 \\ \alpha_3\alpha_2\alpha_1 & \alpha_3\alpha_2 & \alpha_3 & 1 \end{pmatrix}.$$

Continuing this way, we finally get

$$C = C_6$$
$$= \begin{pmatrix} 0 & \beta_1 & \beta_1\beta_2 & \beta_1\beta_2\beta_3 & \beta_1\beta_2\beta_3\beta_4 & \beta_1\beta_2\beta_3\beta_4\beta_5 \\ \alpha_1 & 1 & \beta_2 & \beta_2\beta_3 & \beta_2\beta_3\beta_4 & \beta_2\beta_3\beta_4\beta_5 \\ \alpha_2\alpha_1 & \alpha_2 & 1 & \beta_3 & \beta_3\beta_4 & \beta_3\beta_4\beta_5 \\ \alpha_3\alpha_2\alpha_1 & \alpha_3\alpha_2 & \alpha_3 & 1 & \beta_4 & \beta_4\beta_5 \\ \alpha_4\alpha_3\alpha_2\alpha_1 & \alpha_4\alpha_3\alpha_2 & \alpha_4\alpha_3 & \alpha_4 & 1 & \beta_5 \\ \alpha_5\alpha_4\alpha_3\alpha_2\alpha_1 & \alpha_5\alpha_4\alpha_3\alpha_2 & \alpha_5\alpha_4\alpha_3 & \alpha_5\alpha_4 & \alpha_5 & 0 \end{pmatrix}.$$

Note that this matrix is the same as the matrix $A = A_6$ in Example 8.3. ◊

Example 8.5. This example uses Theorem 8.2 in order to find the Green completion of order 2 of the given scalar band

$$\tilde{A} = \{A_{ij}, |i-j| \le 2\} = \begin{pmatrix} 1 & 2 & 3 & ? & ? & ? \\ 6 & 1 & 2 & 3 & ? & ? \\ 5 & 6 & 1 & 2 & 3 & ? \\ ? & 5 & 6 & 1 & 2 & 3 \\ ? & ? & 5 & 6 & 1 & 2 \\ ? & ? & ? & 5 & 6 & 1 \end{pmatrix}.$$

For this band $n = 2$ and $N = 6$ and all the submatrices of $\tilde{A}$ of the form (8.7), namely

$$B_k = \tilde{A}(k-n+1:k, k-n+1:k) = A(k-1:k, k-1:k) = \begin{pmatrix} 1 & 2 \\ 6 & 1 \end{pmatrix},$$

are invertible. Then $\tilde{A}$ has a unique completion A which is a Green matrix of order 2. One obtains this completion by successive computation of its principal leading submatrices

$$A_k := A(1:k, 1:k), \quad k = 3, \dots, 6,$$

as follows.

In the first step, set as in (8.8)

$$A_3 = A_{n+1} = \tilde{A}(1:n+1, 1:n+1) = \tilde{A}(1:3, 1:3) = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 1 & 2 \\ 5 & 6 & 1 \end{pmatrix}.$$

Let for $k = 3$ with $3 = n+1 \le k \le N-1 = 5$ the matrix A_k be given. The matrix $A_4 = A_{k+1}$ is obtained via the following operations. Start with partitioning A_3 in the form (8.9), namely $A_k = \begin{pmatrix} B'_k & X_k \\ Y_k & B_k \end{pmatrix}$, with $B_k = B_3 = \begin{pmatrix} 1 & 2 \\ 6 & 1 \end{pmatrix}$. It follows that $B_3^{-1} = \frac{1}{11}\begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix}$ and $B'_3 = 1, X_3 = \begin{pmatrix} 2 & 3 \end{pmatrix}, Y_3 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$.

Next compute the submatrix $A_{k+1} = A_4$ by (8.10), namely

$$A_{k+1} = A_4 = \begin{pmatrix} B'_k & X_k & E_k \\ Y_k & B_k & Z_k \\ F_k & U_k & M_k \end{pmatrix},$$

where M_k, U_k, Z_k are determined from the band via (8.11):

$$\begin{aligned} M_k &= M_3 = \tilde{A}(k+1, k+1) = \tilde{A}(4,4) = 1, \\ Z_k &= Z_3 = \tilde{A}(k-n+1:k, k+1) = \tilde{A}(2:3, 4) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \\ U_k &= U_3 = \tilde{A}(k+1, k-n+1:k) = \tilde{A}(4, 2:3) = \begin{pmatrix} 5 & 6 \end{pmatrix} \end{aligned}$$

and E_k, F_k are computed by (8.12):

$$\begin{aligned} E_3 = E_k = X_k B_k^{-1} Z_k &= \begin{pmatrix} 2 & 3 \end{pmatrix} \frac{1}{11} \begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{11} \\ \frac{16}{11} \end{pmatrix} = \frac{50}{11}, \\ F_3 = F_k = U_k B_k^{-1} Y_k &= \begin{pmatrix} 5 & 6 \end{pmatrix} \frac{1}{11} \begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} \frac{4}{11} \\ \frac{31}{11} \end{pmatrix} = \frac{206}{11}. \end{aligned}$$

It follows that

$$A_4 = \begin{pmatrix} 1 & 2 & 3 & \frac{50}{11} \\ 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 \\ \frac{206}{11} & 5 & 6 & 1 \end{pmatrix}.$$

In order to obtain for $k = 4$ the matrix $A_5 = A_{k+1}$ start with partitioning A_4 in the form (8.9), with $B_k = B_4 = \begin{pmatrix} 1 & 2 \\ 6 & 1 \end{pmatrix}$. It follows that $B_4^{-1} = \frac{1}{11}\begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix}$, while

$$B_4' = \begin{pmatrix} 1 & 2 \\ 6 & 1 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 3 & \frac{50}{11} \\ 2 & 3 \end{pmatrix}, \quad \text{and} \quad Y_4 = \begin{pmatrix} 5 & 6 \\ \frac{206}{11} & 5 \end{pmatrix}.$$

Next compute M_4, U_4, Z_4 from the band via (8.11):

$$M_k = M_4 = \tilde{A}(k+1, k+1) = \tilde{A}(5,5) = 1,$$

$$Z_k = Z_4 = \tilde{A}(k-n+1 : k, k+1) = \tilde{A}(3:4, 5) = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

$$U_k = U_4 = \tilde{A}(k+1, k-n+1 : k) = \tilde{A}(5, 3:4) = \begin{pmatrix} 5 & 6 \end{pmatrix}$$

and E_k, F_k are computed by (8.12):

$$\begin{aligned} E_4 = E_k = X_k B_k^{-1} Z_k &= \begin{pmatrix} 3 & \frac{50}{11} \\ 2 & 3 \end{pmatrix} \frac{1}{11} \begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & \frac{50}{11} \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{11} \\ \frac{16}{11} \end{pmatrix} = \begin{pmatrix} \frac{833}{121} \\ \frac{50}{11} \end{pmatrix}, \\ F_4 = F_k = U_k B_k^{-1} Y_k &= \begin{pmatrix} 5 & 6 \end{pmatrix} \frac{1}{11} \begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ \frac{206}{11} & 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{31}{11} & \frac{4}{11} \end{pmatrix} \begin{pmatrix} 5 & 6 \\ \frac{206}{11} & 5 \end{pmatrix} = \begin{pmatrix} \frac{2529}{121} & \frac{206}{11} \end{pmatrix}. \end{aligned}$$

It follows that

$$A_5 = \begin{pmatrix} 1 & 2 & 3 & \frac{50}{11} & \frac{833}{121} \\ 6 & 1 & 2 & 3 & \frac{50}{11} \\ 5 & 6 & 1 & 2 & 3 \\ \frac{206}{11} & 5 & 6 & 1 & 2 \\ \frac{2529}{121} & \frac{206}{11} & 5 & 6 & 1 \end{pmatrix}.$$

Finally, in order to obtain for $k = 5$ the matrix $A_6 = A_{k+1}$ start with partitioning A_5 in the form (8.9), with $B_k = B_5 = \begin{pmatrix} 1 & 2 \\ 6 & 1 \end{pmatrix}$ defined in (8.7). It follows that $B_5^{-1} = \frac{1}{11}\begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix}$ while

$$B_5' = A_3, \quad X_5 = \begin{pmatrix} \frac{50}{11} & \frac{833}{121} \\ 3 & \frac{50}{11} \\ 2 & 3 \end{pmatrix}, \quad \text{and} \quad Y_5 = \begin{pmatrix} \frac{206}{11} & 5 & 6 \\ \frac{2529}{121} & \frac{206}{11} & 5 \end{pmatrix}.$$

Next compute M_5, U_5, Z_5 from the band via (8.11):

$$\begin{aligned} M_k &= M_5 = \tilde{A}(k+1, k+1) = \tilde{A}(6,6) = 1, \\ Z_k &= Z_5 = \tilde{A}(k-n+1 : k, k+1) = \tilde{A}(4:5, 6) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \\ U_k &= U_5 = \tilde{A}(k+1, k-n+1 : k) = \tilde{A}(6, 4:5) = \begin{pmatrix} 5 & 6 \end{pmatrix} \end{aligned}$$

and compute E_k, F_k by (8.12):

$$\begin{aligned} E_5 = E_k = X_k B_k^{-1} Z_k &= \begin{pmatrix} \frac{50}{11} & \frac{833}{121} \\ 3 & \frac{50}{11} \\ 2 & 3 \end{pmatrix} \frac{1}{11} \begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{50}{11} & \frac{833}{121} \\ 3 & \frac{50}{11} \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{11} \\ \frac{16}{11} \end{pmatrix} = \begin{pmatrix} \frac{13878}{1331} \\ \frac{833}{121} \\ \frac{50}{11} \end{pmatrix}, \\ F_5 = F_k = U_k B_k^{-1} Y_k &= \begin{pmatrix} 5 & 6 \end{pmatrix} \frac{1}{11} \begin{pmatrix} -1 & 2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} \frac{206}{11} & 5 & 6 \\ \frac{2529}{121} & \frac{206}{11} & 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{31}{11} & \frac{4}{11} \end{pmatrix} \begin{pmatrix} \frac{206}{11} & 5 & 6 \\ \frac{2529}{121} & \frac{206}{11} & 5 \end{pmatrix} = \begin{pmatrix} \frac{80362}{1331} & \frac{2529}{121} & \frac{206}{11} \end{pmatrix}. \end{aligned}$$

It follows that

$$A = A_6 = \begin{pmatrix} 1 & 2 & 3 & \frac{50}{11} & \frac{833}{121} & \frac{13878}{1331} \\ 6 & 1 & 2 & 3 & \frac{50}{11} & \frac{833}{121} \\ 5 & 6 & 1 & 2 & 3 & \frac{50}{11} \\ \frac{206}{11} & 5 & 6 & 1 & 2 & 3 \\ \frac{2529}{121} & \frac{206}{11} & 5 & 6 & 1 & 2 \\ \frac{80362}{1331} & \frac{2529}{121} & \frac{206}{11} & 5 & 6 & 1 \end{pmatrix}.$$

◊

One can give simple examples showing that if some of the matrices B_k are not invertible, then the desired completion does not exist or is not unique. Indeed consider the partially specified matrix

$$A = \begin{pmatrix} 1 & 1 & x \\ 1 & 0 & 1 \\ x & 1 & 1 \end{pmatrix},$$

where x denotes an unspecified entry. In this case $n = 1$ and a completion which is a Green matrix of order one does not exist. Another example is

$$A = \begin{pmatrix} 1 & 0 & x \\ 0 & 0 & 1 \\ x & 1 & 1 \end{pmatrix}.$$

Here every choice of x defines a completion which is a Green matrix of order one.

Remark. In the conditions of Theorem 8.2 consider the partially specified matrix $\mathcal{A}^t(\mathcal{A}^*)$ with a given band $\tilde{A}^t(\tilde{A}^*)$. One can check easily that the completion of $\tilde{A}^t(\tilde{A}^*)$ to a Green matrix of order n coincides with the matrix $A^t(A^*)$.

§8.3 Comments

The material of the first section may be found in textbooks of linear algebra, for instance the condition (8.2) is contained in the monograph by F.R. Gantmakher [31]. The basic Theorem 8.2 was obtained in [26].

Chapter 9

Completion to Matrices with Band Inverses and with Minimal Ranks

Here we study properties of the unique Green completion obtained in Theorem 8.2 of the previous Chapter 8. In the first section it is shown that this completion is invertible if and only if all the matrices

$$D_k = \tilde{A}(k-n:k, k-n:k), \quad k = n+1, \dots, N \tag{9.1}$$

are invertible. In this case all the principal leading submatrices

$$A^{(j,k)} = A(j:k, j:k),\ 1 \leq j < j+n \leq k \leq N$$

of the completion are also invertible.

In the second section we discuss the properties of the LDU factorization of the completion. It is shown that L and U are lower and respectively upper Green matrices of order n. Also, A^{-1} is a band matrix of order n and in its corresponding factorization L^{-1} is a lower band of order n, while U^{-1} is an upper band of order n and moreover the elements of the matrices L, D, U can be determined explicitly with straightforward formulas.

In the third section we study some remarkable properties of the principal submatrices of the completions to Green matrices. If all the matrices B_k of the form (8.1) are invertible a permanence principle holds true: the Green completion of a matrix $A^{(j,k)}$ coincides with the matrix of the form $G^{(j,k)}$ of the Green completion. If also the matrices D_k of the form (9.1) are invertible, then each element of the inverse of the Green completion of the band coincides with the corresponding element of the inverse of the Green completion of the matrix $A^{(j,k)}$. Moreover, if D_k are invertible the band of order $n+1$ of the Green completion A of order n of a band of order n has a unique completion which is a Green matrix of order $n+1$, and the latter coincides with A. See in this respect Examples 8.3 and 8.4.

In the fourth section we use the permanence principle for the inverse to a Green matrix in order to obtain the inverse explicitly.

Y. Eidelman et al., *Separable Type Representations of Matrices and Fast Algorithms: Volume 1 Basics. Completion Problems. Multiplication and Inversion Algorithms*, Operator Theory: Advances and Applications 234, DOI 10.1007/978-3-0348-0606-0_9,

In the last section we show that the completion obtained in Chapter 8 has minimal rank if and only if all the matrices D_k are of minimal rank.

§9.1 Completion to invertible matrices

At first we get necessary and sufficient conditions for the completion obtained in Theorem 8.2 to be invertible. If these conditions hold we also obtain the invertibility of the principal submatrices of the completion.

Theorem 9.1. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \le n\}$. Assume that all the submatrices of the band $\tilde{A}$ of the form* (8.7) *are invertible. Then by Theorem* 8.2, *the band $\tilde{A}$ has a unique completion A which is a Green matrix of order n.*

The matrix A is invertible if and only if all the submatrices of $\tilde{A}$ of the form

$$D_k = \tilde{A}(k-n:k, k-n:k), \quad k = n+1, \dots, N \tag{9.2}$$

are invertible. Moreover, in this case all the submatrices of A of the form

$$A^{(j,k)} = A(j:k, j:k), \quad 1 \le j, k \le N, \quad k-j \ge n \tag{9.3}$$

are invertible.

Proof. Assume that the matrix A from the statement of the theorem is invertible. Let us prove that this implies invertibility of all the matrices $A^{(j,k)}$ of the form (9.3). From this, setting $j = k-n$, we will obtain invertibility of all the matrices D_k of the form (9.2).

We consider partitions of A in the form

$$A = \begin{pmatrix} B'_k & X_k & \Gamma'_k \\ Y_k & B_k & Z_k \\ \Gamma''_k & U_k & B''_k \end{pmatrix}, \quad k = n+1, \dots, N-1, \tag{9.4}$$

where the matrices B_k are defined in (8.7). Here one has $\begin{pmatrix} B'_k & X_k \\ Y_k & B_k \end{pmatrix} = A^{(1,k)}$. Since A is a Green matrix of order n and the matrices B_k are invertible, one obtains

$$\operatorname{rank} \begin{pmatrix} X_k & \Gamma'_k \\ B_k & Z_k \end{pmatrix} = \operatorname{rank} \begin{pmatrix} Y_k & B_k \\ \Gamma''_k & U_k \end{pmatrix} = \operatorname{rank} B_k, \quad k = n+1, \dots, N-1.$$

From this using the second part of Lemma 8.1 we conclude that all the submatrices $A^{(1,k)}$ with $k = n+1, \dots, N-1$ are invertible.

Next it will be proved that the remaining submatrices $A^{(j,k)}$ of the form (9.3) are also invertible. For any $k \in \{n+2, \dots, N\}$ and $j \in \{2, \dots, k-n\}$ consider the partition of the matrix $A^{(1,k)}$ in the form

$$A^{(1,k)} = \begin{pmatrix} B'_{j_0} & X_{j_0} & \Gamma'_{j_0} \\ Y_{j_0} & B_{j_0} & Z_{j_0} \\ \Gamma''_{j_0} & U_{j_0} & B''_{j_0} \end{pmatrix}, \tag{9.5}$$

where $j_0 = j + n - 1$ and B_{j_0} is defined by (8.7), i.e., $B_{j_0} = A(j : j_0, j : j_0)$. One has $\begin{pmatrix} B_{j_0} & Z_{j_0} \\ U_{j_0} & B''_{j_0} \end{pmatrix} = A^{(j,k)}$. It is easy to see that the invertible submatrix $A^{(1,k)}$, $k \geq n + 2$, of the Green matrix A of order n is also a Green matrix of the same order. Indeed, one obviously has

$$\begin{aligned} \operatorname{rank} A^{(1,k)}(1 : j, j - n + 1 : k) &= \operatorname{rank} A(1 : j, j - n + 1 : k) \\ &\leq \operatorname{rank} A(1 : j, j - n + 1 : N) \\ &= \sum_{i=j-n+1}^{j} m_i := \alpha_j, \quad j = n + 1, \dots, k - 1 \end{aligned}$$

and similarly

$$\operatorname{rank} A^{(1,k)}(j - n + 1 : k, 1 : j) \leq \alpha_j, \quad j = n + 1, \dots, k - 1.$$

Since the numbers α_j are minimal rank numbers for invertible matrices $A^{(1,k)}$ one gets

$$\operatorname{rank} A^{(1,k)}(1 : j, j - n + 1 : k) = \operatorname{rank} A^{(1,k)}(j - n + 1 : k, 1 : j) = \alpha_j, \\ j = n + 1, \dots, k - 1.$$

Thus taking into account that the matrix B_{j_0} is invertible one obtains

$$\operatorname{rank} \begin{pmatrix} X_{j_0} & \Gamma'_{j_0} \\ B_{j_0} & Z_{j_0} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} Y_{j_0} & B_{j_0} \\ \Gamma''_{j_0} & U_{j_0} \end{pmatrix} = \operatorname{rank} B_{j_0}.$$

Hence, since the matrix $A^{(1,k)}$ is invertible, by virtue of the second part of Lemma 8.1 it follows that the submatrix $A^{(j,k)}$ is invertible.

Assume that all the submatrices D_k of the form (9.2) are invertible. It will be proved by induction that this implies that all the submatrices of the matrix A of the form $A^{(1,k)} = A(1 : k, 1 : k)$, $k = n + 1, \dots, N$ are invertible. Taking here $k = N$ we will obtain that the matrix $A = A^{(1,N)}$ is invertible. For $k = n + 1$ one has $A^{(1,n+1)} = D_{n+1}$, which is invertible. Assume that for some $k \geq n + 1$ the matrix $A^{(1,k)}$ is invertible. For the matrix $A^{(1,k+1)}$ consider the partition

$$A^{(1,k+1)} = \begin{pmatrix} B'_k & X_k & \Gamma'_k \\ Y_k & B_k & Z_k \\ \Gamma''_k & U_k & M_k \end{pmatrix},$$

where the matrix B_k is defined in (8.7) and $M_k = A(k+1, k+1)$. Here one has

$$\begin{pmatrix} B'_k & X_k \\ Y_k & B_k \end{pmatrix} = A^{(1,k)}, \quad \begin{pmatrix} B_k & Z_k \\ U_k & M_k \end{pmatrix} = D_{k+1}.$$

Since the matrices B_k and D_{k+1} are invertible, the Schur complement $\gamma_k = M_k - U_k B_k^{-1} Z_k$ is also invertible. The matrix $A^{(1,k)}$ is invertible by the assumption. Using again the second part of Lemma 8.1, we conclude that the matrix $A^{(1,k+1)}$ is invertible. □

Example 9.2. The present example is an illustration of Theorem 9.1. Consider $\mathcal{A}$ in Example 8.3 which is a partially specified scalar matrix with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$, where $n = 1$. Then all the submatrices of the band $\tilde{A}$ of the form (8.7) are scalars equal to 1 and thus invertible. Consider also the completion A of $\tilde{A}$ to a Green matrix of order 1 in the same Example 8.3.

By Theorem 9.1, the matrix A is invertible if and only if all the submatrices of $\tilde{A}$ of the form (9.2), namely

$$D_2 = \begin{pmatrix} 0 & \beta_1 \\ \alpha_1 & 1 \end{pmatrix}, \quad D_k = \begin{pmatrix} 1 & \beta_{k-1} \\ \alpha_{k-1} & 1 \end{pmatrix}, \ k = 3,4,5, \quad D_k = \begin{pmatrix} 1 & \beta_5 \\ \alpha_5 & 0 \end{pmatrix},$$

are invertible. Moreover, in this case all the submatrices of A of the form (9.3), namely $A^{(j,k)} = A(j:k, j:k), \ 1 \leq j,k \leq N, \ k-j \geq n$ are invertible as well.

It is easy to see that all the matrices D_k are invertible if and only if $\alpha_k \beta_k \neq 1, \ k = 3,4,5$ and $\alpha_k \beta_k \neq 0, \ k = 2,6$. In the sequel it will be shown that all the matrices $A(j:k, j:k), \ 1 \leq j < k \leq 6$, are also invertible in this case. ◊

As a direct corollary of Theorem 6.6, Theorem 8.2 and Theorem 9.1 one obtains the following result.

Theorem 9.3. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j, \ i,j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$. Assume that all the submatrices of $\tilde{A}$ of the form* (8.7), (9.2) *are invertible and let A be the completion of $\tilde{A}$ to the Green matrix of order n.*

Then the matrix A is invertible and A^{-1} is a band matrix of order n.

§9.2 The LDU factorization

Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with block entries of sizes $m_i \times m_j$ and invertible principal leading submatrices $\{A_{ij}\}_{i,j=1}^k, \ k = 1, 2, \dots, N$. By Theorem 1.20, A admits the LDU factorization

$$A = L\Delta U, \tag{9.6}$$

where L, U, Δ are block matrices with the same sizes of blocks as A, while L and U are block lower and upper triangular matrices with identities on the main diagonals and Δ is a block diagonal matrix.

Now we proceed with LDU factorizations of Green matrices obtained via the completion procedure.

Theorem 9.4. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \le n\}$. Assume that all the submatrices of the band $\tilde{A}$ of the form*

$$\tilde{A}(k-n+1:k, k-n+1:k), \quad k = n+1, \dots, N-1 \tag{9.7}$$

are invertible. Then by Theorem 8.2, the band $\tilde{A}$ has a unique completion A which is a Green matrix of order n. Assume also that all the submatrices of the band $\tilde{A}$ of the form

$$\tilde{A}(k-n:k, k-n:k), \quad k = n+1, \dots, N, \tag{9.8}$$

and

$$\tilde{A}(1:k, 1:k), \quad k = 1, \dots, n, \tag{9.9}$$

are invertible.

Then all the submatrices of A of the form $A(1:k, 1:k)$, $k = 1, \dots, N$, are invertible and in the factorization (9.6) of A the matrices L and U are, respectively, a lower Green matrix of order n and an upper Green matrix of order n. Moreover, the matrix A^{-1} is a band matrix of order n and in the corresponding factorization

$$A^{-1} = U^{-1}\Delta^{-1}L^{-1} \tag{9.10}$$

L^{-1} is lower band of order n block lower triangular matrix

$$L^{-1}(i,j) = \begin{cases} L_{ij}, & 0 < i-j \le n, \\ I, & i = j, \\ 0, & i < j, i-j > n, \end{cases} \tag{9.11}$$

U^{-1} is upper band of order n block upper triangular matrix

$$U^{-1}(i,j) = \begin{cases} U_{ij}, & 0 < j-i \le n, \\ I, & i = j, \\ 0, & i > j, j-i > n \end{cases} \tag{9.12}$$

and

$$\Delta^{-1} = \mathrm{diag}(\Delta_1, \dots, \Delta_N) \tag{9.13}$$

is a block diagonal matrix.

Furthermore, the elements L_{ij}, U_{ij}, Δ_k are determined explicitly via the formulas

$$\Delta_1 = M_1^{-1}, \quad \Delta_k = (M_k - V_{k-1}B_{k-1}^{-1}Z_{k-1})^{-1}, \quad k = 2, \dots, N, \tag{9.14}$$

$$L_k = -V_{k-1}B_{k-1}^{-1}, \quad k = 2, \dots, N, \tag{9.15}$$

$$U_k = -B_{k-1}^{-1}Z_{k-1}, \quad k = 2, \dots, N, \tag{9.16}$$

where we denote

$$L_k = \begin{pmatrix} L_{k1} & \dots & L_{k,k-1} \end{pmatrix}, \qquad U_k = \begin{pmatrix} U_{1k} \\ \vdots \\ U_{k-1,k} \end{pmatrix}, \quad k = 1, \dots, n,$$

$$L_k = \begin{pmatrix} L_{k,k-n} & \dots & L_{k,k-1} \end{pmatrix}, \quad U_k = \begin{pmatrix} U_{k-n,k} \\ \vdots \\ U_{k-1,k} \end{pmatrix}, \quad k = n+1, \dots, N$$

and also

$$B_k = \tilde{A}(1:k, 1:k),\ Z_k = \tilde{A}(1:k, k+1),\ V_k = \tilde{A}(k+1, 1:k), \quad k = 1, \dots, n,$$
$$B_k = \tilde{A}(k-n+1:k, k-n+1:k), \quad k = n+1, \dots, N-1,$$
$$Z_k = \tilde{A}(k-n+1:k, k+1),\ V_k = \tilde{A}(k+1, k-n+1:k),\ k = n+1, \dots, N-1,$$
$$M_k = \tilde{A}(k,k), \quad k = 1, \dots, N.$$

Proof. The invertibility of the submatrices (9.7) ensures, by Theorem 8.2, that there is a unique completion A of $\tilde{A}$ to a Green matrix of order n. Also, from Theorem 9.1, based on the invertibility of the submatrices (9.8) it follows that all the submatrices of A of the form $A(1:k, 1:k),\ k = n+1, \dots, N$, are invertible. Together with the invertibility of the submatrices (9.9) one obtains the invertibility of all the submatrices $A(1:k, 1:k),\ k = 1, \dots, N$ and therefore, by Theorem 1.20, the factorization (9.6) of the matrix A exists.

Since ΔU is an upper triangular matrix one gets

$$A(k-n+1:N, 1:k) = L(k-n+1:N, 1:k)(\Delta U)(1:k, 1:k), \quad k = n+1, \dots, N$$

and since every matrix $(\Delta U)(1:k, 1:k)$ is invertible one obtains

$$\operatorname{rank} A(k-n+1:N, 1:k) = \operatorname{rank} L(k-n+1:N, 1:k), \quad k = n+1, \dots, N.$$

From here, since A is a Green matrix of order n we conclude that L is a lower Green matrix of the same order. In a similar way one shows that U is an upper Green matrix of order n.

By Theorem 9.3 the matrix A^{-1} is a band matrix of order n. It is clear that L^{-1} and U^{-1} are block lower and upper triangular matrices with identities on the main diagonals. Moreover, by Theorem 6.6, L^{-1} and U^{-1} are lower and upper band matrices of order n and thus the representations (9.11), (9.12) are valid.

Now we derive the formulas (9.14)–(9.16). From (1.51) it follows directly that $\tilde{A}(1,1) = \Delta(1,1)$ and therefore

$$\Delta_1 = M_1^{-1}.$$

Next we set

$$D_k = \tilde{A}(1:k,1:k),\ k=2,\dots,n; \quad D_k = \tilde{A}(k-n:k,k-n:k),\ k=n+1,\dots,N. \tag{9.17}$$

One obviously has the partitions

$$D_k = \begin{pmatrix} B_{k-1} & Z_{k-1} \\ V_{k-1} & M_k \end{pmatrix}, \quad k=2,\dots,N.$$

From here, using (1.54) one obtains the formulas for the inverse matrices

$$D_k^{-1} = \begin{pmatrix} * & -B_{k-1}^{-1}Z_{k-1}\Gamma_k^{-1} \\ -\Gamma_k^{-1}V_{k-1}B_{k-1}^{-1} & \Gamma_k^{-1} \end{pmatrix}, \quad k=2,\dots,N, \tag{9.18}$$

with $\Gamma_k = M_k - V_{k-1}B_{k-1}^{-1}Z_{k-1}$.

From (9.10) one gets

$$\Delta^{-1}L^{-1}A = U.$$

Consider the elements in the lower band:

$$\begin{aligned} (\Delta^{-1}L^{-1}A)(k,1:k) &= U(k,1:k), && k=2,\dots,n, \\ (\Delta^{-1}L^{-1}A)(k,k-n:k) &= U(k,k-n:k), && k=n+1,\dots,N. \end{aligned}$$

Since $\Delta^{-1}L^{-1}$ is a lower triangular lower band of order n matrix,

$$\begin{aligned} (\Delta^{-1}L^{-1})(k,1:k)A(1:k,1:k) &= U(k,1:k), \qquad k=2,\dots,n, \\ (\Delta^{-1}L^{-1})(k,n-k:k)A(k-n:k,k-n:k) &= U(k,k-n:k),\ k=n+1,\dots,N. \end{aligned}$$

Using (9.11), (9.13), (9.17) and the fact that U is upper triangular one gets

$$\Delta_k \begin{pmatrix} L_k & I \end{pmatrix} D_k = \begin{pmatrix} 0 & I \end{pmatrix}, \quad k=2,\dots,N. \tag{9.19}$$

In a similar way, since $AU^{-1}\Delta^{-1} = L$ one gets

$$D_k \begin{pmatrix} U_k \\ I \end{pmatrix} \Delta_k = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad k=2,\dots,N. \tag{9.20}$$

Now from (9.19), (9.20) it follows that

$$\Delta_k \begin{pmatrix} L_k & I \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix} D_k^{-1}, \quad \begin{pmatrix} U_k \\ I \end{pmatrix} \Delta_k = D_k^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad k=2,\dots,N,$$

and using (9.18) one gets

$$\begin{pmatrix} \Delta_k L_k & \Delta_k \end{pmatrix} = \begin{pmatrix} -\Gamma_k^{-1}V_{k-1}B_{k-1}^{-1} & \Gamma_k^{-1} \end{pmatrix}, \quad \begin{pmatrix} U_k\Delta_k \\ \Delta_k \end{pmatrix} = \begin{pmatrix} -B_{k-1}^{-1}Z_{k-1}\Gamma_k^{-1} \\ \Gamma_k^{-1} \end{pmatrix}$$

from which the formulas (9.14) with $k=2,\dots,N$ and the formulas (9.15), (9.16) follow directly. □

Remark. Similar results are valid for the submatrices $A(k:N,k:N)$ and for the UDL factorization of a matrix A which is a completion of a given band to the unique Green matrix.

§9.3 The Permanence Principle

In this section we study some remarkable properties of the principal submatrices of the completions to Green matrices.

In the sequel we use the following notations. For a matrix $A = \{A_{ij}\}_{i,j=1}^{N}$, the symbol $A^{(s,t)}$ $(1 \leq s \leq t \leq N)$ denotes the submatrix of A of the form $A^{(s,t)} = A(s : t, s : t)$ and for a band $\tilde{A} = \{A_{ij}, |i-j| \leq n,\ 1 \leq i,j \leq N\}$ the symbol $\tilde{A}^{(s,t)}$ denotes the band $\tilde{A}^{(s,t)} = \{A_{ij}, |i-j| \leq n,\ s \leq i,j \leq t\}$. For a band $\tilde{A}$ the symbol $G(\tilde{A})$ denotes the completion of $\tilde{A}$ to a Green matrix of order n. We use also the notation $J_{k,l}$ for the set of indices $\{(i,j) :\ k \leq i,j \leq l\}$.

Theorem 9.5 (The Permanence Principle). *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$. Assume that all the submatrices of the band $\tilde{A}$ of the form* (8.7) *are invertible.*

1) *The following relations hold:*

$$(G(\tilde{A}))^{(s,t)} = G(\tilde{A}^{(s,t)}), \quad 1 \leq s,t \leq N,\ t-s \geq n+1. \tag{9.21}$$

2) *Assume additionally that all the submatrices of the band $\tilde{A}$ of the form* (9.2) *are invertible. Then the following relations hold:*

$$[G(\tilde{A})]^{-1}(i,j) = [G(\tilde{A}^{(1,t)})]^{-1}(i,j), \tag{9.22}$$

for $n+1 \leq t \leq N-1,\ (i,j) \in J_{1,t} \setminus J_{t-n+1,t}$*;*

$$[G(\tilde{A})]^{-1}(i,j) = [G(\tilde{A}^{(s,N)})]^{-1}(i,j) \tag{9.23}$$

for $2 \leq s \leq N-n,\ (i,j) \in J_{s,N} \setminus J_{s,s+n-1}$*;*

$$[G(\tilde{A})]^{-1}(i,j) = [G(\tilde{A}^{(s,t)})]^{-1}(i,j) \tag{9.24}$$

for $1 < s < t < N,\ t-s \geq n,\ (i,j) \in J_{s,t} \setminus (J_{s,s+n-1} \cup J_{t-n+1,t})$.

Proof. 1) By Theorem 8.2, the bands $\tilde{A}$, $\tilde{A}^{(s,t)}$ have completions which are Green matrices of order n. Consider the matrix $A = G(\tilde{A})$. Its principal leading submatrices

$$A(s : k, s : k), \quad k = n+s,\dots,t$$

are determined as follows. One obviously has

$$A(s : n+s, s : n+s) = \tilde{A}(s : n+s, s : n+s). \tag{9.25}$$

Next, comparing with (8.9), (8.10) one obtains the formulas

$$A(s : k, s : k) = \begin{pmatrix} B_k^{(s)} & X_k^{(s)} \\ Y_k^{(s)} & B_k \end{pmatrix}, \quad k = n+s,\dots,t-1 \tag{9.26}$$

and

$$A(s:k+1,s:k+1)=\begin{pmatrix} B_k^{(s)} & X_k^{(s)} & E_k^{(s)} \\ Y_k^{(s)} & B_k & Z_k \\ F_k^{(s)} & U_k & M_k \end{pmatrix}, \quad k=n+s,\dots,t-1 \quad (9.27)$$

with B_k and M_k, U_k, Z_k defined in (8.7) and (8.11), and

$$X_k^{(s)} = X_k(s:k-n,:), \quad Y_k^{(s)} = Y_k(:,s:k-n), \quad (9.28)$$

$$E_k^{(s)} = E_k(s:k-n,:), \quad F_k^{(s)} = F_k(:,s:k-n), \quad (9.29)$$

where the matrices X_k, Y_k, E_k, F_k are defined via the partitions (8.9), (8.10). Moreover, using (8.12) one gets

$$E_k^{(s)} = X_k^{(s)} B_k^{-1} Z_k, \quad F_k^{(s)} = U_k B_k^{-1} Y_k^{(s)}. \quad (9.30)$$

Consider the matrix $\hat{G} = G(\tilde{A}^{(s,t)}) = \{G_{ij}\}_{i,j=s}^{t}$. Using the formulas (8.8)–(8.12) one obtains the following expressions for its principal leading submatrices

$$\hat{G}_k = \hat{G}(s:k,s:k), \quad k=n+s,\dots,t.$$

One has

$$\hat{G}_{n+s} = \hat{G}(s:n+s,s:n+s) = \tilde{A}(s:n+s,s:n+s) \quad (9.31)$$

and next for $k=n+s,\dots,t-1$ for the given submatrix $\hat{G}_k$ we use the partition

$$\hat{G}_k = \begin{pmatrix} \hat{B}_k^{(s)} & \hat{X}_k^{(s)} \\ \hat{Y}_k^{(s)} & B_k \end{pmatrix} \quad (9.32)$$

with B_k defined in (8.7) and we compute the submatrix $\hat{G}_{k+1}$ by the recipe

$$\hat{G}_{k+1} = \begin{pmatrix} \hat{B}_k^{(s)} & \hat{X}_k^{(s)} & \hat{E}_k^{(s)} \\ \hat{Y}_k^{(s)} & B_k & Z_k \\ \hat{F}_k^{(s)} & U_k & M_k \end{pmatrix}, \quad (9.33)$$

where M_k, U_k, Z_k are defined in (8.11) and $\hat{E}_k^{(s)}, \hat{F}_k^{(s)}$ are computed by the recipes

$$\hat{E}_k^{(s)} = \hat{X}_k^{(s)} B_k^{-1} Z_k, \quad \hat{F}_k^{(s)} = U_k B_k^{-1} \hat{Y}_k^{(s)}. \quad (9.34)$$

We used here that the specified entries of $\hat{G}$ coincide with the corresponding entries of $A^{(s,t)}$ and hence the matrices B_k, M_k, U_k, Z_k in (9.26), (9.27) and in (9.32), (9.33) respectively, are the same.

Now it is easy to prove by induction that

$$A(s:k,s:k) = \hat{G}(s:k,s:k), \quad k=n+s,\dots,t, \quad (9.35)$$

which implies (9.21). Indeed, comparing (9.25) and (9.31) one gets

$$A(s:n+s,s:n+s)=\hat{G}(s:n+s,s:n+s).$$

Let (9.35) hold for some k with $n+s\le k\le t-1$ the equality. Using (9.26) and (9.32) one gets

$$X_k^{(s)}=\hat{X}_k^{(s)},\quad Y_k^{(s)}=\hat{Y}_k^{(s)}.$$

The formulas (9.27) and (9.34) yield

$$E_k^{(s)}=\hat{E}_k^{(s)},\quad F_k^{(s)}=\hat{F}_k^{(s)}$$

and thus comparing (9.30) and (9.33) one obtains

$$A(s:k+1,s:k+1)=\hat{G}(s:k+1,s:k+1),$$

which completes the proof.

2) By Theorem 9.1, the matrix $A=G(\tilde{A})$ and its submatrices of the form $A^{(s,t)}=A(s:t,s:t)$, $\quad 1\le s,t\le N,\quad s-t\ge n$, are invertible.

At first we consider partitions of the matrix $A=G(\tilde{A})$ in the form

$$A=\begin{pmatrix} B'_t & X_t & \Gamma'_t\\ Y_t & B_t & Z_t\\ \Gamma''_t & U_t & B''_t\end{pmatrix},\quad t=n+1,\dots,N-1, \tag{9.36}$$

where the matrices B_t are defined in (8.7). From here, taking into account (9.21) one obtains $\begin{pmatrix} B'_t & X_t\\ Y_t & B_t\end{pmatrix}=A^{(1,t)}=G(\tilde{A}^{(1,t)})$. Set

$$(A^{(1,t)})^{-1}=\begin{pmatrix} B'_t & X_t\\ Y_t & B_t\end{pmatrix}^{-1}=\begin{pmatrix} (B'_t)_{11} & (B'_t)_{12}\\ (B'_t)_{21} & (B'_t)_{22}\end{pmatrix}.$$

Application of the formula (8.5) to the matrix A partitioned in the form (9.36) yields

$$A^{-1}=\begin{pmatrix} (B'_t)_{11} & (B'_t)_{12} & 0\\ (B'_t)_{21} & * & *\\ 0 & * & *\end{pmatrix}.$$

It follows that the blocks $(B'_t)_{11}$, $(B'_t)_{12}$, $(B'_t)_{21}$ of the matrix $(A^{(1,t)})^{-1}$ coincide with the corresponding blocks of the matrix A^{-1}. This implies that the entries of A^{-1} with the indices $(i,j)\in J_{1,t}\setminus J_{t-n+1,t}$ coincide with the corresponding entries of $(A^{(1,t)})^{-1}$, i.e., (9.22) holds.

Next we consider partitions of A in the form

$$A=\begin{pmatrix} B'_{s_0} & X_{s_0} & \Gamma'_{s_0}\\ Y_{s_0} & B_{s_0} & Z_{s_0}\\ \Gamma''_{s_0} & U_{s_0} & B''_{s_0}\end{pmatrix},\quad s_0=s+n-1,\quad s=2,\dots,N-n, \tag{9.37}$$

where the matrices B_{s_0} are defined in (8.7). From here, taking into account (9.21) one obtains $\begin{pmatrix} B_{s_0} & Z_{s_0} \\ U_{s_0} & B''_{s_0} \end{pmatrix} = A^{(s,N)} = G(\tilde{A}^{(s,N)})$. Set

$$(A^{(s,N)})^{-1} = \begin{pmatrix} B_{s_0} & Z_{s_0} \\ U_{s_0} & B''_{s_0} \end{pmatrix}^{-1} = \begin{pmatrix} (B''_{s_0})_{11} & (B''_{s_0})_{12} \\ (B''_{s_0})_{21} & (B''_{s_0})_{22} \end{pmatrix}.$$

Application of the formula (8.6) to the matrix A partitioned in the form (9.37) yields

$$A^{-1} = \begin{pmatrix} * & * & 0 \\ * & * & (B''_{s_0})_{12} \\ 0 & (B''_{s_0})_{21} & (B''_{s_0})_{22} \end{pmatrix},$$

It follows that the blocks $(B''_{s_0})_{12}$, $(B''_{s_0})_{21}$, $(B''_{s_0})_{22}$ of the matrix $(A^{(s,N)})^{-1}$ coincide with the corresponding blocks of the matrix A^{-1}. This implies that the entries of A^{-1} with the indices $(i,j) \in J_{s,N} \setminus J_{s,s_0}$ coincide with the corresponding entries of $(A^{(s,N)})^{-1}$, i.e., (9.23) holds.

Finally, for $1 < s < t < N$, $t - s \geq n$ we consider the partition of the matrix $A^{(s,N)}$ in the form

$$A^{(s,N)} = \begin{pmatrix} B'_{st} & X_{st} & \Gamma'_{st} \\ Y_{st} & B_t & Z_{st} \\ \Gamma''_{st} & U_{st} & B''_{st} \end{pmatrix}, \tag{9.38}$$

where the matrices B_t are defined in (8.7). From here, taking into account (9.21) one obtains $\begin{pmatrix} B'_{st} & X_{st} \\ Y_{st} & B_t \end{pmatrix} = A^{(s,t)} = G(\tilde{A}^{(s,t)})$. Set

$$(A^{(s,t)})^{-1} = \begin{pmatrix} B'_{st} & X_{st} \\ Y_{s,t} & B_t \end{pmatrix}^{-1} = \begin{pmatrix} (B'_{st})_{11} & (B'_{st})_{12} \\ (B'_{st})_{21} & (B'_{st})_{22} \end{pmatrix}.$$

Application of the formula (8.5) to the matrix $A^{(s,N)}$ partitioned in the form (9.38) yields

$$(A^{(s,N)})^{-1} = \begin{pmatrix} (B'_{st})_{11} & (B'_{st})_{12} & 0 \\ (B'_{st})_{21} & * & * \\ 0 & * & * \end{pmatrix}.$$

It follows that the blocks $(B'_{st})_{11}$, $(B'_{st})_{12}$, $(B'_{st})_{21}$ of the matrix $(A^{(s,t)})^{-1}$ coincide with the corresponding blocks of the matrix $(A^{(s,N)})^{-1}$. This implies that the entries of $(A^{(s,N)})^{-1}$ with indices $(i,j) \in J_{s,t} \setminus J_{t-n+1,t}$ coincide with the corresponding entries of $(A^{(s,t)})^{-1}$, i.e.,

$$[G(\tilde{A}^{(s,t)})]^{-1}(i,j) = [G(\tilde{A}^{(s,N)})]^{-1}(i,j), \quad (i,j) \in J_{s,t} \setminus J_{t-n+1,t}.$$

Comparison of this relation with (9.23) yields (9.24). □

Next we present another result which also has a character of a permanence principle.

Theorem 9.6. *Let A be a block matrix with entries of sizes $m_i \times m_j$, $i, j = 1, \ldots, N$. Assume that A is a Green matrix of order n and that all the submatrices of A of the form (9.2) are invertible. Consider the band $\tilde{G} = \{A_{ij}, |i-j| \leq n+1\}$ of A.*

The band $\tilde{G}$ has a unique completion which is a Green matrix of order $n+1$, moreover this completion coincides with the matrix A.

Proof. Since all the matrices (9.2) are invertible, by Theorem 8.2 the band $\tilde{G}$ has a unique completion G which is a Green matrix of order $n+1$. Let us show that A is a Green matrix of order $n+1$. Then, by the uniqueness stated in Theorem 8.2, we will conclude that $A = G$.

We must prove that

$$\operatorname{rank} A(1:k, k-n:N) = \sum_{s=k-n}^{k} m_s, \quad k = n+2, \ldots, N-1, \tag{9.39}$$

$$\operatorname{rank} A(k-n:N, 1:k) = \sum_{s=k-n}^{k} m_s, \quad k = n+2, \ldots, N-1. \tag{9.40}$$

One has

$$A(1:k, k-n:N) = \begin{pmatrix} A(1:k, k-n) & A(1:k, k-n+1:N) \end{pmatrix}.$$

Here $A(1:k, k-n)$ is a matrix of the size $(m_1 + \cdots + m_k) \times m_{k-n}$. It follows that

$$\operatorname{rank} A(1:k, k-n:N) \leq m_{k-n} + \operatorname{rank} A(1:k, k-n+1:N), \quad k = n+2, \ldots, N-1. \tag{9.41}$$

Since A is a Green matrix of order n,

$$\operatorname{rank} A(1:k, k-n+1:N) = \sum_{s=k-n+1}^{k} m_s, \quad k = n+2, \ldots, N-1. \tag{9.42}$$

From (9.41), (9.42) one obtains

$$\operatorname{rank} A(1:k, k-n:N) \leq m_{k-n} + \sum_{s=k-n+1}^{k} m_s = \sum_{s=k-n}^{k} m_s, \quad k = n+2, \ldots, N-1. \tag{9.43}$$

On the other hand, the matrix $A(1:k, k-n:N)$ contains the submatrix

$$A(k-n:k, k-n:k) = D_k,$$

which is invertible and has order $\sum_{s=k-n}^{k} m_s$. Therefore,

$$\operatorname{rank} A(1:k, k-n:N) \geq \sum_{s=k-n}^{k} m_s, \quad k = n+2, \ldots, N-1. \tag{9.44}$$

Comparing (9.44) and (9.43) one obtains (9.39). The relation (9.40) is obtained similarly. □

Example 9.7. Examples 8.3 and 8.4 illustrate the last of the Permanence Principles. Indeed, consider the band $\tilde{C}$ of order 2 from Example 8.4, which is a band of the order 1 Green completion A obtained in Example 8.3 for the band of order 1 denoted $\tilde{A}$ in that example. $\tilde{C}$ has a unique completion C which is a Green matrix of order 2 and it is obtained in Example 8.4 and this completion satisfies $C = A$, i.e., it is A itself. ◊

§9.4 The inversion formula

Assume that the partially specified block matrix $A = \{A_{ij}\}_{i,j=1}^{N}$ with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$ satisfies the conditions of Theorem 9.3 and let $\tilde{A}$ be completed in such a way that A is a Green matrix of order n. By Theorem 9.3, A is invertible and the matrix A^{-1} is a band of order n. In this section we derive explicit formulas for the entries of A^{-1}.

In this section we use the notation $X(i)$ for the ith entry of a block row or of a block column.

Theorem 9.8. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j$, $i,j = 1,\dots,N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$. Assume that all the submatrices of $\tilde{A}$ of the form*

$$B_k = \tilde{A}(k-n+1:k, k-n+1:k), \quad k = n+1,\dots,N-1$$

are invertible. By Theorem 8.2, the band $\tilde{A}$ has a unique completion A which is a Green matrix of order n. Assume that all the submatrices of $\tilde{A}$ of the form

$$D_k = \tilde{A}(k-n:k, k-n:k), \quad k = n+1,\dots,N,$$

are invertible. For $k = n+1,\dots,N-1$ set

$$\begin{aligned} X_k &= \tilde{A}(k-n, k-n+1:k), & Y_k &= \tilde{A}(k-n+1:k, k-n), \\ V_k &= \tilde{A}(k-n, k-n), & Z_k &= \tilde{A}(k-n+1:k, k+1), \\ U_k &= \tilde{A}(k+1, k-n+1:k), & M_k &= \tilde{A}(k+1, k+1) \end{aligned}$$

and for $k \leq l$ denote by $J_{k,l}$ the set of indices $\{(i,j):\ k \leq i,j \leq l\}$.

Then the entries of the inverse matrix $A^{-1} = \{A'_{ij}\}_{i,j=1}^{N}$ with the indices $|i-j| \leq n$ are given by the formulas:

$$A'_{kk} = \delta_{k+n}^{-1} + \sum_{s=1}^{k-1} \Delta_{s+n}(k-s)\delta_{s+n}^{-1}\Phi_{s+n}(k-s), \tag{9.45}$$

$$A'_{ik} = -\Delta_{k+n}(i-k)\delta_{k+n}^{-1} + \sum_{s=1}^{k-1} \Delta_{s+n}(i-s)\delta_{s+n}^{-1}\Phi_{s+n}(k-s), \tag{9.46}$$

$$A'_{ki} = -\delta_{k+n}^{-1}\Phi_{k+n}(i-k) + \sum_{s=1}^{k-1} \Delta_{s+n}(k-s)\delta_{s+n}^{-1}\Phi_{s+n}(i-s), \tag{9.47}$$

with

$$\Delta_k = B_k^{-1} Y_k, \quad \Phi_k = X_k B_k^{-1}, \quad \delta_k = V_k - X_k B_k^{-1} Y_k$$

for $(i,k),(k,i) \in J_{1,n+1},\ i > k$*;*

$$A'_{kk} = \gamma_{k-1}^{-1} + \sum_{t=k}^{k+n-1} \Lambda_t(k+n-t)\gamma_t^{-1} W_t(k+n-t), \tag{9.48}$$

$$A'_{ik} = -\Lambda_{k-1}(k-i)\gamma_{k-1}^{-1} + \sum_{t=k}^{i+n-1} \Lambda_t(i+n-t)\gamma_t^{-1} W_t(k+n-t), \tag{9.49}$$

$$A'_{ki} = -\gamma_{k-1}^{-1} W_{k-1}(k-i) + \sum_{t=k}^{i+n-1} \Lambda_t(k+n-t)\gamma_t^{-1} W_t(i+n-t), \tag{9.50}$$

with

$$\Lambda_k = B_k^{-1} Z_k, \quad W_k = U_k B_k^{-1}, \quad \gamma_k = M_k - U_k B_k^{-1} Z_k,$$

for $(i,k),(k,i) \in J_{1,N} \setminus (J_{1,n+1} \cup J_{N-n,N}), \quad i = k-n,\dots,k-1$*;*

$$A'_{kk} = \gamma_{k-1}^{-1} + \sum_{t=k}^{N-1} \Lambda_t(k+n-t)\gamma_t^{-1} W_t(k+n-t), \tag{9.51}$$

$$A'_{ik} = -\Lambda_{k-1}(k-i)\gamma_{k-1}^{-1} + \sum_{t=k}^{N-1} \Lambda_t(i+n-t)\gamma_t^{-1} W_t(k+n-t), \tag{9.52}$$

$$A'_{ki} = -\gamma_{k-1}^{-1} W_{k-1}(k-i) + \sum_{t=k}^{N-1} \Lambda_t(k+n-t)\gamma_t^{-1} W_t(i+n-t), \tag{9.53}$$

for $(i,k),(k,i) \in J_{N-n,N},\ i < k$*.*

Proof. To derive (9.45) we take an index k from the range $\{1,\dots,n+1\}$. For any $(i,k),(k,i) \in J_{1,n+1},\ i > k$ one has $(i,k),(k,i) \in J_{1,k+n} \setminus J_{k+1,k+n}$ and hence, by Theorem 9.5, one can obtain the entries A'_{ki}, A'_{ik} from the matrix $(A^{(1,k+n)})^{-1}$ by the formula (9.22), i.e.,

$$A'_{ki} = (A^{(1,k+n)})^{-1}(k,i), \quad A'_{ik} = (A^{(1,k+n)})^{-1}(i,k). \tag{9.54}$$

We set $k+n = t$ and for $s = 1,\dots,k$ consider the matrices $A^{(s,t)}$. For the case $s = k$, i.e., for the matrix $A^{(k,k+n)}$, we use the partition

$$A^{(k,k+n)} = \begin{pmatrix} V_{k+n} & X_{k+n} \\ Y_{k+n} & B_{k+n} \end{pmatrix}.$$

Application of the inversion formula (1.57) yields

$$(A^{(k,k+n)})^{-1} = \begin{pmatrix} \delta_{k+n}^{-1} & -\delta_{k+n}^{-1}\Phi_{k+n} \\ -\Delta_{k+n}\delta_{k+n}^{-1} & * \end{pmatrix}.$$

Hence it follows that

$$(A^{(k,k+n)})^{-1}(k,k) = \delta_{k+n}^{-1}, \qquad k = 1, \ldots, n+1, \tag{9.55}$$

$$(A^{(k,k+n)})^{-1}(k,i) = -\delta_{k+n}^{-1}\Phi_{k+n}(i-k), \quad k = 1, \ldots, n+1,\ i = k+1, \ldots, n+1, \tag{9.56}$$

$$(A^{(k,k+n)})^{-1}(i,k) = -\Delta_{k+n}(i-k)\delta_{k+n}^{-1}, \quad k = 1, \ldots, n+1,\ i = k+1, \ldots, n+1. \tag{9.57}$$

Setting in (9.55)–(9.57) $k = 1$ one obtains the relations (9.45)–(9.47) for the case $k = 1$. Assume now that $k = 2, \ldots, n+1,\ 2 < i \leq n+1,\ i > k$ and for $s = 1, \ldots, k-1$ consider the matrices $A^{(s,t)}$ partitioned in the form

$$A^{(s,t)} = \begin{pmatrix} V_{s+n} & X_{s+n} & \Gamma'_{st} \\ Y_{s+n} & B_{s+n} & Z_{st} \\ \Gamma''_{st} & U_{st} & B''_{st} \end{pmatrix}, \quad s = 1, \ldots, k-1,$$

where $V_{s+n} = A(s,s),\ B_{s+n} = A(s+1 : s+n, s+1 : s+n),\ B''_{st} = A(s+n+1 : t, s+n+1 : t)$. We have $s+1 \leq k \leq n+1 \leq n+s$ and therefore in such a partition the elements with the indices $(i,k), (k,i)$ lie in the middle block in the positions $(i-s, k-s), (k-s, i-s)$, respectively. Note that $\begin{pmatrix} B_{s+n} & Z_{st} \\ U_{st} & B''_{st} \end{pmatrix} = A^{(s+1,t)}$ and set

$$(A^{(s+1,t)})^{-1} = \begin{pmatrix} B_{s+n} & Z_{st} \\ U_{st} & B''_{st} \end{pmatrix}^{-1} = \begin{pmatrix} (A'_{s+1,t})_{11} & (A'_{s+1,t})_{12} \\ (A'_{s+1,t})_{21} & (A'_{s+1,t})_{22} \end{pmatrix}.$$

Application of the formula (8.6) yields

$$(A^{(s,t)})^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (A'_{s+1,t})_{11} & (A'_{s+1,t})_{12} \\ 0 & (A'_{s+1,t})_{21} & (A'_{s+1,t})_{22} \end{pmatrix} + \begin{pmatrix} \delta_{s+n}^{-1} & -\delta_{s+n}^{-1}\Phi_{s+n} & 0 \\ -\Delta_{s+n}\delta_{s+n}^{-1} & \Delta_{s+n}\delta_{s+n}^{-1}\Phi_{s+n} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that

$$(A^{(s,t)})^{-1}(i,k) = (A^{(s+1,t)})^{-1}(i,k) + \Delta_{s+n}(i-s)\delta_{s+n}^{-1}\Phi_{s+n}(k-s), \quad s = 1, \ldots, k-1, \tag{9.58}$$

$$(A^{(s,t)})^{-1}(k,i) = (A^{(s+1,t)})^{-1}(k,i) + \Delta_{s+n}(k-s)\delta_{s+n}^{-1}\Phi_{s+n}(i-s), \quad s = 1, \ldots, k-1. \tag{9.59}$$

Applying the relations (9.58), (9.59) for $s = 1, \ldots, k-1$ successively one obtains

$$(A^{(1,k+n)})^{-1}(i,k) = (A^{(k,k+n)})^{-1}(i,k) + \sum_{s=1}^{k-1}\Delta_{s+n}(i-s)\delta_{s+n}^{-1}\Phi_{s+n}(k-s), \tag{9.60}$$

$$(A^{(1,k+n)})^{-1}(k,i) = (A^{(k,k+n)})^{-1}(k,i) + \sum_{s=1}^{k-1}\Delta_{s+n}(k-s)\delta_{s+n}^{-1}\Phi_{s+n}(i-s). \tag{9.61}$$

Now the relations (9.46), (9.47) follow directly from (9.60), (9.61) and (9.56), (9.57). Setting in (9.60) or in (9.61) $i = k$ and taking into account (9.55) one obtains (9.45).

In order to derive (9.48)–(9.50) we take an index k from the range $\{n+2,\dots,N-n-1\}$. For $i = k-n,\dots,k$ one has $(i,k),(k,i) \in J_{k-n,i+n} \setminus (J_{k-n,k-1} \cup J_{i+1,i+n})$ and hence, by Theorem 9.5, one can obtain the entries A'_{ki}, A'_{ik} from the matrix $(A^{(k-n,i+n)})^{-1}$ by the formula (9.24), i.e.,

$$A'_{ki} = (A^{(k-n,i+n)})^{-1}(k,i), \quad A'_{ik} = (A^{(k-n,i+n)})^{-1}(i,k), \quad i = k-n,\dots,k. \tag{9.62}$$

We set $k-n = s$ and for $t = k,\dots,i+n$ consider the matrices $A^{(s,t)}$. For the case $t = k$, i.e., for the matrix $A^{(k-n,k)}$ one has the partition

$$A^{(k-n,k)} = \begin{pmatrix} B_{k-1} & Z_{k-1} \\ U_{k-1} & M_{k-1} \end{pmatrix}.$$

Application of the inversion formula (1.54) yields

$$(A^{(k-n,k)})^{-1} = \begin{pmatrix} * & -\Lambda_{k-1}\gamma_{k-1}^{-1} \\ -\gamma_{k-1}^{-1}W_{k-1} & \gamma_{k-1}^{-1} \end{pmatrix}.$$

Therefore,

$$(A^{(k-n,k)})^{-1}(k,k) = \gamma_{k-1}^{-1}, \tag{9.63}$$

$$(A^{(k-n,k)})^{-1}(k,i) = -\Lambda_{k-1}(k-i)\gamma_{k-1}^{-1}, \quad i = k-n,\dots,k-1, \tag{9.64}$$

$$(A^{(k-m,k)})^{-1}(i,k) = -\gamma_{k-1}^{-1}W_{k-1}(k-i), \quad i = k-n,\dots,k-1. \tag{9.65}$$

Taking $i = k-n$ in (9.64), (9.65) one obtains the formulas (9.49), (9.50) for the entries $A'_{k-n,k}$, $A'_{k,k-n}$ respectively. Assume that $i = k-n+1,\dots,k$ and for $t = k,\dots,i+n-1$ consider the matrices $A^{(s,t+1)}$ partitioned in the form

$$A^{(s,t+1)} = \begin{pmatrix} B'_{st} & X_{st} & \Gamma'_{st} \\ Y_{st} & B_t & Z_t \\ \Gamma''_{st} & U_t & M_t \end{pmatrix}, \quad t = k,\dots,i+n-1,$$

where $B'_{st} = A(k-n:t-n, k-n:t-n)$, $B_t = A(t-n+1:t, t-n+1:t)$, $M_t = A(t+1,t+1)$. In such a partition the elements with the indices $(i,k),(k,i)$ lie in the middle block in the positions $(i-t+n, k-t+n), (k-t+n, i-t+n)$, respectively. Note that $\begin{pmatrix} B'_{st} & X_{st} \\ Y_{st} & B_t \end{pmatrix} = A^{(s,t)}$ and set

$$(A^{(s,t)})^{-1} = \begin{pmatrix} B'_{st} & X_{st} \\ Y_{s,t} & B_t \end{pmatrix}^{-1} = \begin{pmatrix} (A'_{st})_{11} & (A'_{st})_{12} \\ (A'_{st})_{21} & (A'_{st})_{22} \end{pmatrix}.$$

Application of the formula (8.5) yields

$$(A^{(s,t+1)})^{-1} = \begin{pmatrix} (A'_{st})_{11} & (A'_{st})_{12} & 0 \\ (A'_{st})_{21} & (A'_{st})_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda_t \gamma_t^{-1} W_t & -\Lambda_t \gamma_t^{-1} \\ 0 & -\gamma_t^{-1} W_t & \gamma_t^{-1} \end{pmatrix}.$$

Hence it follows that

$$\begin{aligned}(A^{(s,t+1)})^{-1}(i,k) &= (A^{(s,t)})^{-1}(i,k) + \Lambda_t(i+n-t)\gamma_t^{-1} W_t(k+n-t), \\ t &= k, \dots, i+n-1,\end{aligned} \tag{9.66}$$

$$\begin{aligned}(A^{(s,t+1)})^{-1}(k,i) &= (A^{(s,t)})^{-1}(k,i) + \Lambda_t(k+n-t)\gamma_t^{-1} W_t(i+n-t), \\ t &= k, \dots, i+n-1.\end{aligned} \tag{9.67}$$

Applying the relations (9.66), (9.67) for $t = i+n-1, \dots, k$ successively one obtains

$$(A^{(k-n,k+n)})^{-1}(i,k) = (A^{(k-n,k)})^{-1}(i,k) + \sum_{t=k}^{i+n-1} \Lambda_t(i+n-t)\gamma_t^{-1} W_t(k+n-t), \tag{9.68}$$

$$(A^{(k-n,k+n)})^{-1}(k,i) = (A^{(k-n,k)})^{-1}(k,i) + \sum_{t=k}^{i+n-1} \Lambda_t(k+n-t)\gamma_t^{-1} W_t(i+n-t). \tag{9.69}$$

Now the relations (9.49), (9.50) for $i = k-n+1, \dots, k-1$ follow directly from (9.68), (9.69) and (9.64), (9.65). Setting in (9.68) or in (9.69) $i = k$ and taking into account (9.63) one obtains (9.48).

To derive (9.51)–(9.53) we take an index k from the range $\{N-n, \dots, N\}$. For any $(i,k), (k,i) \in J_{N-n,N}$, $i < k$ one has $(i,k), (k,i) \in J_{k-n,N} \setminus J_{k-n,k-1}$ and hence, by Theorem 9.5, one can obtain the entries A'_{ki}, A'_{ik} from the matrix $(A^{(k-n,N)})^{-1}$ by the formula (9.24), i.e.,

$$A'_{ki} = (A^{(k-n,N)})^{-1}(k,i), \quad A'_{ik} = (A^{(k-n,N)})^{-1}(i,k).$$

Next the relations (9.51)–(9.53) are obtained in the same way as (9.48)–(9.50). □

In the case when a specified band of a block matrix $A = \{A_{ij}\}_{i,j=1}^{N}$ is tridiagonal and hence the corresponding inverse is a tridiagonal matrix, the inversion formulas may be simplified essentially.

Corollary 9.9. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j$, $i,j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq 1\}$. Let all the diagonal entries of $\tilde{A}$,*

$$B_k = A_{kk}, \quad k = 2, \dots, N-1$$

be invertible matrices, by Theorem 8.2 *the band* $\tilde{A}$ *has a unique completion* A *which is a Green matrix of order one. Assume also that all the submatrices of the band* $\tilde{A}$ *of the form*

$$D_k = \tilde{A}(k-1:k, k-1:k), \quad k = 2, \dots, N$$

are invertible. Set

$$\begin{aligned}
&Z_k = A_{k,k+1},\ U_k = A_{k+1,k},\ M_k = A_{k+1,k+1}, \quad k = 2, \dots, N-1,\\
&\Lambda_k = B_k^{-1} Z_k,\ W_k = U_k B_k^{-1},\ \gamma_k = M_k - U_k B_k^{-1} Z_k, \quad k = 2, \dots, N-1,\\
&\Delta_2 = B_2^{-1} A_{21},\ \Phi_2 = A_{12} B_2^{-1},\ \delta_2 = A_{11} - A_{12} B_2^{-1} A_{21},\ \delta_3 = A_{33} - A_{23} B_3^{-1} A_{32}.
\end{aligned}$$

Then the entries of the inverse matrix $A^{-1} = \{A'_{ij}\}_{i,j=1}^N$ *with the indices* $|i-j| \le 1$ *are given by the formulas*

$$\begin{aligned}
A'_{11} &= \delta_2^{-1},\ A'_{22} = \delta_3^{-1} + \Delta_2 \delta_2^{-1} \Phi_2,\ A'_{21} = -\Delta_2 \delta_2^{-1},\ A'_{12} = -\delta_2^{-1} \Phi_2;\\
A'_{kk} &= \gamma_{k-1}^{-1} + \Lambda_k \gamma_k^{-1} W_k, \quad k = 3, \dots, N-1;\ A'_{NN} = \gamma_{N-1}^{-1};\\
A'_{k-1,k} &= -\Lambda_{k-1} \gamma_{k-1}^{-1},\ A'_{k,k-1} = -\gamma_{k-1}^{-1} W_{k-1}, \quad k = 3, \dots, N.
\end{aligned}$$

Example 9.10. In this example the Corollary 9.9 will be used in order to find the inverse matrix A^{-1} of the Green matrix of order one $A = A_6$ which has been obtained in Example 8.3.

The band $\tilde{A}$ from which in Example 8.3 one computes the matrix A, namely

$$\tilde{A} = \{A_{ij}, |i-j| \le 1\} = \begin{pmatrix} 0 & \beta_1 & ? & ? & ? & ? \\ \alpha_1 & 1 & \beta_2 & ? & ? & ? \\ ? & \alpha_2 & 1 & \beta_3 & ? & ? \\ ? & ? & \alpha_3 & 1 & \beta_4 & ? \\ ? & ? & ? & \alpha_4 & 1 & \beta_5 \\ ? & ? & ? & ? & \alpha_5 & 0 \end{pmatrix}$$

has all its submatrices of the form

$$D_k = \tilde{A}(k-1:k, k-1:k), \quad k = 2, \dots, 6$$

invertible if and only if the entries of the given band $\tilde{A}$ satisfy

$$\alpha_k \beta_k \ne 1,\ k = 3, 4, 5, \quad \alpha_k \beta_k \ne 0,\ k = 2, 6, \tag{9.70}$$

as it is easy to see and as it has been also shown in Example 8.4, where the above submatrices have been denoted $B_k,\ k = 2, \dots, 6$. Suppose in the sequel that the conditions (9.70) are satisfied so that the submatrices D_k are invertible.

As Corollary 9.9 asks, set

$$\begin{aligned}
&B_k = 1, \quad k = 2, \dots, 5, \quad Z_k = A_{k,k+1} = \beta_k, \ U_k = A_{k+1,k} = \alpha_k, \quad k = 2, \dots, 5,\\
&M_k = A_{k+1,k+1} = 1, \quad k = 2, \dots, 4, \quad M_5 = A_{5+1,5+1} = 0,\\
&\Lambda_k = B_k^{-1} Z_k = \beta_k, \ W_k = U_k B_k^{-1} = \alpha_k, \quad k = 2, \dots, 5,\\
&\gamma_k = M_k - U_k B_k^{-1} Z_k = 1 - \alpha_k \beta_k, \quad k = 2, \dots, 5,\\
&\Delta_2 = B_2^{-1} A_{21} = \alpha_1, \ \Phi_2 = A_{12} B_2^{-1} = \beta_1,\\
&\delta_2 = A_{11} - A_{12} B_2^{-1} A_{21} = 0 - \alpha_1 \beta_1, \ \delta_3 = A_{33} - A_{23} B_3^{-1} A_{32} = 1 - \alpha_2 \beta_2.
\end{aligned}$$

Then the entries of the inverse matrix $A^{-1} = \{A'_{ij}\}_{i,j=1}^N$ with the indices $|i-j| \le 1$ are given by the formulas

$$\begin{aligned}
A'_{22} &= \delta_3^{-1} + \Delta_2 \delta_2^{-1} \Phi_2 = \frac{\alpha_2 \beta_2}{1 - \alpha_2 \beta_2},\\
A'_{21} &= -\Delta_2 \delta_2^{-1} = \frac{1}{\beta_1}, \quad A'_{12} = -\delta_2^{-1} \Phi_2 = \frac{1}{\alpha_1};\\
A'_{k-1,k} &= -\Lambda_{k-1} \gamma_{k-1}^{-1} = -\frac{\beta_{k-1}}{1 - \alpha_{k-1} \beta_{k-1}}, \quad k = 3, \dots, 6,\\
A'_{k,k-1} &= -\gamma_{k-1}^{-1} W_{k-1} = -\frac{\alpha_{k-1}}{1 - \alpha_{k-1} \beta_{k-1}}, \quad k = 3, \dots, 6,
\end{aligned}$$

and the other diagonal entries are

$$\begin{aligned}
A'_{11} &= \delta_2^{-1} = -\frac{1}{\alpha_1 \beta_1},\\
A'_{kk} &= \gamma_{k-1}^{-1} + \Lambda_k \gamma_k^{-1} W_k = \frac{1}{1 - \alpha_{k-1} \beta_{k-1}} + \frac{\alpha_k \beta_k}{1 - \alpha_k \beta_k}, \quad k = 3, \dots, 5,\\
A'_{66} &= \gamma_{6-1}^{-1} = -\frac{1}{\alpha_5 \beta_5}.
\end{aligned}$$

The inverse of a Green matrix of order one is a tridiagonal matrix, so that all its nonzero entries have already been found. ◊

§9.5 Completion to matrices of minimal ranks

In this section we consider, as above, a partially specified block matrix $\mathcal{A}$ with block entries of sizes $m_i \times m_j, \ i, j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \le n\}$. We assume here that the sizes of the blocks satisfy the condition

$$m_i = m_{n+i}, \quad i = 2, \dots, N - n - 1. \tag{9.71}$$

This implies that all the submatrices of the band $\tilde{A}$ of the form

$$B_k = \tilde{A}(k - n + 1 : k, k - n + 1 : k), \quad k = n + 1, \dots, N - 1 \tag{9.72}$$

have the same size $\nu \times \nu$, where $\nu = \sum_{j=2}^{n+1} m_j$. Assume that all the matrices B_k $(k = n+1, \dots, N-1)$ of the form (9.72) are invertible. By Theorem 8.2, the band $\tilde{A}$ has a unique completion A which is a Green matrix of order n. It is clear that $\operatorname{rank} A \geq \nu$. In this section we obtain necessary and sufficient conditions for the completion A to be of minimal rank ν.

Theorem 9.11. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$ such that the condition* (9.71) *holds. Set $\nu = \sum_{j=2}^{n+1} m_j$. Assume that all the submatrices of the band $\tilde{A}$ of the form* (9.72) *are invertible. By Theorem* 8.2, *the band $\tilde{A}$ has a unique completion A which is a Green matrix of order n.*

Then $\operatorname{rank} A = \nu$ if and only if all the submatrices of $\tilde{A}$ of the form

$$D_k = \tilde{A}(k-n : k, k-n : k), \quad k = n+1, \dots, N \tag{9.73}$$

satisfy the condition

$$\operatorname{rank} D_k = \nu, \quad k = n+1, \dots, N. \tag{9.74}$$

Proof. Assume that $\operatorname{rank} A = \nu$. Since D_k are submatrices of A, we get

$$\operatorname{rank} D_k \leq \nu, \quad k = n+1, \dots, N.$$

At the same time, for $k = n+1, \dots, N-1$ each submatrix D_k contains the submatrix B_k and the submatrix D_N contains the submatrix B_{N-1}. Since the $\nu \times \nu$ matrices B_k $(k = n+1, \dots, N-1)$ are invertible, we get

$$\operatorname{rank} D_k \geq \nu, \quad k = n+1, \dots, N.$$

Hence the equalities (9.74) follow.

Assume that the equalities (9.74) hold. We prove by induction that all the submatrices of A of the form $A^{(1,k)} = A(1:k, 1:k)$, $k = n+1, \dots, N$ satisfy the condition

$$\operatorname{rank} A^{(1,k)} = \nu, \quad k = n+1, \dots, N. \tag{9.75}$$

Taking here $k = N$ we will obtain $\operatorname{rank} A = \operatorname{rank} A^{(1,N)} = \nu$.

For $k = n+1$ we have $A^{(1,n+1)} = D_{n+1}$ and therefore $\operatorname{rank} A^{(1,n+1)} = \operatorname{rank} D_{n+1}$. Assume by induction on k that for some $k \geq n+1$ (9.75) holds. For the matrix $A^{(1,k+1)}$ consider the partition

$$A^{(1,k+1)} = \begin{pmatrix} B'_k & X_k & \Gamma'_k \\ Y_k & B_k & Z_k \\ \Gamma''_k & U_k & M_k \end{pmatrix}, \tag{9.76}$$

where the matrix B_k is defined in (9.72) and $M_k = A(k+1, k+1)$. Here one has

$$\begin{pmatrix} B'_k & X_k \\ Y_k & B_k \end{pmatrix} = A^{(1,k)}, \tag{9.77}$$

$$\begin{pmatrix} B_k & Z_k \\ U_k & M_k \end{pmatrix} = D_{k+1}. \tag{9.78}$$

Applying to the matrix D_{k+1} partitioned in the form (9.78) the factorization (1.52), we get

$$D_{k+1} = \begin{pmatrix} I & 0 \\ U_k B_k^{-1} & I \end{pmatrix} \begin{pmatrix} B_k & 0 \\ 0 & \gamma_k \end{pmatrix} \begin{pmatrix} I & B_k^{-1} Z_k \\ 0 & I \end{pmatrix}$$

with $\gamma_k = M_k - U_k B_k^{-1} Z_k$. From here, since $\operatorname{rank} B_k = \operatorname{rank} D_{k+1}$, we get $\gamma_k = 0$. Now applying the factorization (8.4) to the matrix $A^{(1,k+1)}$ partitioned in the form (9.76) we get

$$A^{(1,k+1)} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & U_k B_k^{-1} & I \end{pmatrix} \begin{pmatrix} B'_k & X_k & 0 \\ Y_k & B_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & B_k^{-1} Z_k \\ 0 & 0 & I \end{pmatrix}.$$

From here, using (9.77) we conclude that $\operatorname{rank} A^{(1,k+1)} = \operatorname{rank} A^{(1,k)} = \nu$. □

With scalar matrices the condition (9.71) is satisfied automatically and we obtain the following.

Corollary 9.12. *Let $\mathcal{A}$ be a partially specified scalar matrix with a given band $\tilde{A} = \{A_{ij}, |i-j| \le n\}$. Assume that all the submatrices of $\tilde{A}$ of the form*

$$B_k = \tilde{A}(k-n+1 : k, k-n+1 : k), \quad k = n+1, \dots, N-1,$$

which are matrices of the size $n \times n$, are invertible. By Theorem 8.2, the band $\tilde{A}$ has a unique completion A which is a Green matrix of order n.

Then the inequality $\operatorname{rank} A \ge n$ holds. Moreover, $\operatorname{rank} A = n$ if and only if the submatrices of $\tilde{A}$ of the form

$$D_k = \tilde{A}(k-n : k, k-n : k), \quad k = n+1, \dots, N$$

satisfy the condition

$$\operatorname{rank} D_k = n, \quad k = n+1, \dots, N.$$

§9.6 Comments

The material of this chapter is taken mostly from the paper [26]. The factorization representation presented in the second section were obtained by H. Dym and I. Gohberg in [10], see also the paper by I. Gohberg, M.A. Kaashoek and H.J. Woerdeman [40]. The first part of Theorem 9.5 generalizes the corresponding result by R.L. Ellis, I. Gohberg and I.D. Lay in [17] obtained for the positive definite case. Theorem 10.3 was obtained in [10] using other methods. Theorem 9.11 appears here for the first time.

Chapter 10

Completion of Special Types of Matrices

Here we consider completions to Green matrices under some conditions on the specified band. If all the submatrices B_k of the form (8.1) are invertible, then the unique completion to a Green matrix is a positive (definite) matrix if and only if all the matrices D_k of the form (9.1) are positive (definite). The positive definite Green completion has the maximal determinant among all the positive definite completions.

The theory takes a nice form in the case of a block Toeplitz band of order n. In this case it is sufficient to ask the invertibility of the principal leading $n \times n$ submatrix. The unique completion to a Green matrix of order n is readily seen to be a Toeplitz matrix.

The example of a tridiagonal band with identities on the main diagonal is treated in detail. Special attention is paid to the case when the tridiagonal band is also Toeplitz.

In the last section we apply the results obtained in this part to special 2×2 block matrices.

For instance, it turns out that such a matrix is positive definite if and only if it is selfadjoint and its entries are strict contractions. The inversion formulas are then applied to the case when the tridiagonal matrix is also Toeplitz.

§10.1 The positive case

In this section we consider the case where the completion of the given band $\tilde{A} = \{A_{ij}, |i-j| \leq n\}$, $n \geq 1$ of a block square matrix $\mathcal{A}$ to a Green matrix will also be a positive or a positive definite matrix. Here a $k \times k$ matrix A is said to be positive if $\langle Ax, x\rangle \geq 0$ for any $x \in \mathbb{C}^k$ and A is said to be positive definite if there exists $\delta > 0$ such that $\langle Ax, x\rangle \geq \delta\langle x, x\rangle$ for any $x \in \mathbb{C}^k$. Here $\langle \cdot, \cdot\rangle$ denotes a scalar product in $\mathbb{C}^k$.

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Theorem 10.1. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i - j| \le n\}$. Assume that all the submatrices of the band $\tilde{A}$ of the form*

$$B_k = \tilde{A}(k - n + 1 : k, k - n + 1 : k), \quad k = n + 1, \dots, N - 1 \tag{10.1}$$

are invertible. Then:

1) *$\tilde{A}$ has a unique completion which is a Green matrix of order n and is a positive matrix if and only if all the submatrices of the band $\tilde{A}$ of the form*

$$D_k = \tilde{A}(k - n : k, k - n : k), \quad k = n + 1, \dots, N \tag{10.2}$$

are positive. In this case the desired completion is determined by successive computation of its principal leading submatrices

$$A_k := A(1 : k, 1 : k), \quad k = n + 1, \dots, N,$$

as follows.
In the first step we set

$$A_{n+1} = \tilde{A}(1 : n + 1, 1 : n + 1). \tag{10.3}$$

Let for some k with $n + 1 \le k \le N - 1$ the matrix A_k be given. The matrix A_{k+1} is obtained via the following operations. We start by partitioning A_k in the form

$$A_k = \begin{pmatrix} B'_k & X_k \\ X_k^* & B_k \end{pmatrix}, \tag{10.4}$$

with B_k defined in (10.1). *Next we compute the submatrix A_{k+1} by the recipe*

$$A_{k+1} = \begin{pmatrix} B'_k & X_k & E_k \\ X_k^* & B_k & Z_k \\ E_k^* & Z_k^* & M_k \end{pmatrix}, \tag{10.5}$$

where M_k, Z_k are determined from the band by the recipes

$$M_k = \tilde{A}(k + 1, k + 1), \quad Z_k = \tilde{A}(k - n + 1 : k, k + 1), \tag{10.6}$$

and E_k are computed by the formula

$$E_k = X_k B_k^{-1} Z_k. \tag{10.7}$$

Finally, we set

$$A = A_N. \tag{10.8}$$

2) *Under the conditions of part* 1), *the completion given by the formulas* (10.3)–(10.8) *is the unique positive definite completion of $\tilde{A}$ such that the inverse matrix A^{-1} is band of order n if and only if all the submatrices D_k, $k = n + 1, \dots, N$ of the form* (10.2) *are positive definite.*

In the proof of the theorem we use the following auxiliary result.

Lemma 10.2. *Let Q be a square matrix which has a partition*

$$Q = \begin{pmatrix} B' & X & \Gamma \\ X^* & B & Z \\ \Gamma^* & Z^* & B'' \end{pmatrix}, \tag{10.9}$$

where B', B, B'' are square matrices. Assume that the matrix B is invertible and the matrices $\begin{pmatrix} B' & X \\ X^* & B \end{pmatrix}$, $\begin{pmatrix} B & Z \\ Z^* & B'' \end{pmatrix}$ are positive (positive definite).

If the condition

$$\operatorname{rank} \begin{pmatrix} X & \Gamma \\ B & Z \end{pmatrix} = \operatorname{rank} B \tag{10.10}$$

is satisfied then the matrix Q is positive (positive definite).

Proof. The matrix Q satisfies the conditions of Lemma 8.1. Application of the equality (8.4) from this lemma yields the factorization

$$Q = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & Z^*B^{-1} & I \end{pmatrix} \begin{pmatrix} B' & X & 0 \\ X^* & B & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & B^{-1}Z \\ 0 & 0 & I \end{pmatrix}, \tag{10.11}$$

where $\gamma = B'' - Z^*B^{-1}Z$. The matrix $\begin{pmatrix} B' & X \\ X^* & B \end{pmatrix}$ is positive (positive definite) by the assumption of the lemma. Since the matrix $\begin{pmatrix} B & Z \\ Z^* & B'' \end{pmatrix}$ is positive (positive definite) the Schur complement γ is also positive (positive definite). Thus the equality (10.11) implies that Q is a positive (positive definite) matrix. □

Proof of the theorem. 1) By Theorem 8.2, the band $\tilde{A}$ has a unique completion which is a Green matrix of order n. Application of the formulas (8.8)–(8.13) to the selfadjoint case yields the formulas (10.3)–(10.8). It remains to prove that this completion A is a positive matrix if and only if all the submatrices D_k, $k = n+1, \dots, N$ of the form (10.2) are positive. The necessity is obvious. To get the sufficiency we shall prove by induction that all the submatrices of the matrix A of the form $A_k = A(1:k, 1:k)$, $k = n+1, \dots, N$, are positive. Taking here $k = N$ we will obtain that the matrix $A = A_N$ is positive. For $k = n+1$ one has $A_{n+1} = D_{n+1}$, which is positive. Assume that for some $k \geq n+1$ the matrix A_k is positive. For the matrix A_{k+1} consider the partition (10.5), where the matrix B_k is defined in (10.1) and $M_k = A(k+1, k+1)$. Here one has

$$\begin{pmatrix} B'_k & X_k \\ X^*_k & B_k \end{pmatrix} = A_k, \quad \begin{pmatrix} B_k & Z_k \\ Z^*_k & M_k \end{pmatrix} = D_{k+1}.$$

The matrix A_k is positive by assumption. The matrix D_{k+1} is positive by the statement of the theorem. Moreover since A_{k+1} is a Green matrix of order n and the matrix B_k is invertible, one has

$$\operatorname{rank}\begin{pmatrix} X_k & E_k \\ B_k & Z_k \end{pmatrix} = \operatorname{rank} B_k.$$

Thus, by Lemma 10.2, we conclude that the matrix A_{k+1} is positive.

2) By Theorem 9.1, the completion A, which is a Green matrix, is invertible if and only if all the submatrices $D_k,\ k = n+1, \dots, N$ of the form (10.2) are invertible. Moreover, in this case the inverse matrix A^{-1} is band of order n. We must prove that the obtained matrix A is positive definite if and only if all the submatrices $D_k,\ k = n+1, \dots, N$ of the form (10.2) are positive definite. The proof is the same as in part 1). □

Next we show that among positive definite completions of a specified band the completion to a Green matrix is the unique completion with the maximal determinant.

Theorem 10.3. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of sizes $m_i \times m_j,\ i,j = 1, \dots, N$, with a given band $\tilde{A} = \{A_{ij}, |i-j| \le n\}$. Assume that $\tilde{A}$ has a positive definite completion A which is a Green matrix of order n. Let H be any other positive definite completion of $\tilde{A}$. Then*

$$\det A \ge \det H$$

with equality if and only if $A = H$.

Proof. First we establish the following auxiliary result. Let Q be a positive definite matrix partitioned in the form (10.9), where the element Γ is unspecified and must be determined in such a way that $\det Q$ will be maximal. It will be proved that the last holds if and only if the condition (10.10) is satisfied.

Multiplying the matrix Q by the matrix $\begin{pmatrix} I & -XB^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ from the left and by the matrix $\begin{pmatrix} I & 0 & 0 \\ -B^{-1}X^* & I & 0 \\ 0 & 0 & I \end{pmatrix}$ from the right, one obtains the matrix

$$Q_1 = \begin{pmatrix} \delta & 0 & \Psi \\ 0 & B & Z \\ \Psi^* & Z^* & B'' \end{pmatrix},$$

where $\delta = B' - XB^{-1}X^*,\ \Psi = \Gamma - XB^{-1}Z$. It is clear that Q_1 and δ are positive definite matrices and $\det Q = \det Q_1$. Next multiplying Q_1 by the matrix

$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -Z^*B^{-1} & I \end{pmatrix}$ from the left and by the matrix $\begin{pmatrix} I & 0 & 0 \\ 0 & I & -B^{-1}Z \\ 0 & 0 & I \end{pmatrix}$ from the right one obtains the matrix

$$Q_2 = \begin{pmatrix} \delta & 0 & \Psi \\ 0 & B & 0 \\ \Psi^* & 0 & \gamma \end{pmatrix},$$

where $\gamma = B'' - Z^*B^{-1}Z$. It is clear that Q_2 is positive definite and $\det Q_1 = \det Q_2$. Multiplying the matrix Q_2 by the matrix $\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\Psi^*\delta^{-1} & 0 & I \end{pmatrix}$ from the left and by the matrix $\begin{pmatrix} I & 0 & -\delta^{-1}\Psi \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ from the right one obtains the matrix

$$Q_3 = \begin{pmatrix} \delta & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \gamma - \Psi^*\delta^{-1}\Psi \end{pmatrix}.$$

The matrix Q_3 is positive definite and moreover one has

$$\det Q = \det Q_3 = \det \delta \cdot \det B \cdot \det(\gamma - \Psi^*\delta^{-1}\Psi).$$

The value $\det Q$ will be maximal if and only if $\det(\gamma - \Psi^*\delta^{-1}\Psi)$ will be maximal. The matrix $\gamma - \Psi^*\delta^{-1}\Psi$ is positive definite and we obviously have $\gamma - \Psi^*\delta^{-1}\Psi \leq \gamma$. This implies that $\det(\gamma - \Psi^*\delta^{-1}\Psi) \leq \det \gamma$ with equality if and only if $\gamma - \Psi^*\delta^{-1}\Psi = \gamma$, i.e., $\Psi = \Gamma - XB^{-1}Z = 0$, which by the first part of Lemma 8.1 is equivalent to (10.10).

Assume now that A is a completion of $\tilde{A}$ such that A is a Green matrix of order n. For any $k \in \{n+1, \ldots, N-1\}$ consider partition of A in the form

$$A = \begin{pmatrix} B'_k & X_k & \Gamma_k \\ X_k^* & B_k & Z_k \\ \Gamma_k^* & Z_k^* & B''_k \end{pmatrix},$$

where the matrix B_k is defined in (10.1). Since A is a Green matrix of order n one has

$$\operatorname{rank} \begin{pmatrix} X_k & \Gamma_k \\ B_k & Z_k \end{pmatrix} = \operatorname{rank} B_k, \quad k = n+1, \ldots, N-1.$$

Let H be any other positive definite completion of $\tilde{A}$. Since, by Theorem 8.2, the completion of $\tilde{A}$ which is a Green matrix of order n is unique, H is not a Green matrix. This implies that for some $k_0 \in \{n+1, \ldots, N-1\}$ the matrix H may be partitioned in the form

$$H = \begin{pmatrix} B'_{k_0} & X_{k_0} & \hat{\Gamma}_{k_0} \\ X^*_{k_0} & B_{k_0} & Z_{k_0} \\ \hat{\Gamma}^*_{k_0} & Z^*_{k_0} & B''_{k_0} \end{pmatrix}$$

with

$$\operatorname{rank} \begin{pmatrix} X_{k_0} & \hat{\Gamma}_{k_0} \\ B_{k_0} & Z_{k_0} \end{pmatrix} > \operatorname{rank} B_{k_0}.$$

It follows that $\det A > \det H$. □

§10.2 The Toeplitz case

Here we consider the special case of a Toeplitz band. We show that the completion of this band to a Green matrix is a Toeplitz matrix.

Theorem 10.4. *Let $\mathcal{A}$ be a partially specified block matrix with block entries of size $m \times m$, with a given Toeplitz band $\tilde{A} = \{A_{i-j}, |i-j| \leq n\}$. Assume that the matrix*

$$B = \begin{pmatrix} A_0 & A_{-1} & \dots & A_{-n+1} \\ A_1 & A_0 & \dots & A_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \dots & A_0 \end{pmatrix} \tag{10.12}$$

is invertible.

Then $\tilde{A}$ has a unique completion which is a Green matrix of order n. Moreover, this completion is a block Toeplitz matrix $A = \{A_{i-j}\}_{i,j=1}^N$ with the unspecified entries determined successively by the relations

$$G_s = \begin{pmatrix} A_{-s+n} & \dots & A_{-s+1} \end{pmatrix}, \; A_{-s} = G_s E, \quad s = n+1, \dots, N-1, \tag{10.13}$$

$$H_s = \begin{pmatrix} A_{s-n} \\ \vdots \\ A_{s-1} \end{pmatrix}, \; A_s = F H_s, \quad s = n+1, \dots, N-1, \tag{10.14}$$

where

$$E = B^{-1} \begin{pmatrix} A_{-n} \\ \vdots \\ A_{-1} \end{pmatrix}, \quad F = \begin{pmatrix} A_n & \dots & A_1 \end{pmatrix} B^{-1}. \tag{10.15}$$

Proof. Since the band $\tilde{A}$ is Toeplitz, all the submatrices of $\tilde{A}$ of the form (8.7) are equal to the matrix B defined by (10.12). By Theorem 8.2, the band $\tilde{A}$ has a unique completion A which is a Green matrix of order n and moreover this completion is determined by the relations (8.8)–(8.13). In the case considered we can rewrite the relations (8.11)–(8.12) as follows. Since $\tilde{A}$ is Toeplitz, the matrices Z_k, U_k from (8.11) have the form

$$Z_k = \begin{pmatrix} A_{-n} \\ \vdots \\ A_{-1} \end{pmatrix}, \quad U_k = \begin{pmatrix} A_n & \dots & A_1 \end{pmatrix}, \quad k = n+1, \dots, N-1.$$

It follows that the matrices $B_k^{-1} Z_k, U_k B_k^{-1}$ do not depend on k, more precisely one has $B_k^{-1} Z_k = E$, $U_k B_k^{-1} = F$, where the matrices E and F are defined by the relations (10.15). Next, one can write down the relations (8.12) in the form

$$A(i, k+1) = A(i, k-n+1:k)E, \quad A(k+1, i) = FA(k-n+1:k, i),$$
$$i = 1, \dots, k-n, \quad k = n+1, \dots, N-1.$$

Changing the index k by $i+s$ one obtains

$$\begin{aligned} A(i, i+s+1) &= A(i, i+s-n+1:i+s)E, \\ A(i+s+1, i) &= FA(i+s-n+1:i+s, i), \\ i = 1, \dots, N-s-1, &\quad s = n, \dots, N-2. \end{aligned} \tag{10.16}$$

Let us prove by induction on s that the elements $A(i, i+s), A(i+s, i)$ do not depend on i and are determined by the relations

$$A(i, i+s) = A_{-s},\ A(i+s, i) = A_s, \quad i = 1, \dots, N-s; \quad s = 1, \dots, N-1, \tag{10.17}$$

where the blocks A_{-s}, A_s are defined in (10.13), (10.14). For $s = 1, \dots, n$ we have

$$A(i, i+s) = \tilde{A}(i, i+s) = A_{-s}, \quad A(i+s, i) = \tilde{A}(i+s, i) = A_s.$$

Assume that for some s with $s \geq n$ the relations

$$A(i, i+\tau) = A_{-\tau},\ A(i+\tau, i) = A_\tau, \quad i = 1, \dots, N-\tau, \quad \tau = 1, \dots, s$$

hold. This implies that

$$A(i, i+s-n+1:i+s) = \begin{pmatrix} A_{-s+n-1} & \dots & A_{-s} \end{pmatrix} = G_{s+1},$$

$$A(i+s-n+1:i+s, i) = \begin{pmatrix} A_{s-n+1} \\ \vdots \\ A_s \end{pmatrix} = H_{s+1}.$$

Substituting these expressions in (10.16) one obtains

$$A(i, i+s+1) = G_{s+1}E = A_{-s-1}, \quad A(i+s+1, i) = FH_{s+1} = A_{s+1}. \qquad \square$$

§10.3 Completion of specified tridiagonal parts with identities on the main diagonal

§10.3.1 The general case

We consider here the case of a partially specified block matrix $\mathcal{A}$ with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given tridiagonal part $\tilde{A} = \{A_{ij}, |i-j| \le 1\}$ such that

$$A_{ii} = I,\ i = 1,\dots,N; \quad A_{i,i+1} = a_i,\ A_{i+1,i} = b_i \quad i = 1,\dots,N-1. \tag{10.18}$$

For such a matrix the matrices B_k from the condition (8.7) of Theorem 8.2 are identities. By Theorem 8.2, $\tilde{A}$ has a unique completion A that is a Green matrix of order one. Moreover using the relations (8.8)–(8.13) with $n = 1$ and the definition (10.18) one gets

$$A(1:2,1:2) = \tilde{A}(1:2,1:2), \tag{10.19}$$

$$A(1:k,1:k) = \begin{bmatrix} B'_k & X_k \\ Y_k & I \end{bmatrix},$$

$$A(1:k+1,1:k+1) = \begin{bmatrix} B'_k & X_k & E_k \\ Y_k & I & Z_k \\ F_k & U_k & I \end{bmatrix}, \tag{10.20}$$

$$k = 2,\dots,N-1$$

with

$$Z_k = a_k, \quad U_k = b_k \tag{10.21}$$

and

$$E_k = X_k Z_k, \quad F_k = U_k Y_k. \tag{10.22}$$

Comparing (10.19), (10.20) one obtains

$$\begin{gathered} A(1:k-1,k) = X_k,\ A(k,1:k-1) = Y_k,\ k = 2,\dots,N; \\ A(k,k) = I,\ k = 1,\dots,N, \end{gathered} \tag{10.23}$$

with

$$X_2 = a_1,\ Y_2 = b_1; \quad X_{k+1} = \begin{pmatrix} E_k \\ Z_k \end{pmatrix}, Y_{k+1} = \begin{pmatrix} F_k & U_k \end{pmatrix},\ k = 2,\dots,N-1. \tag{10.24}$$

Moreover, combining (10.21), (10.22) and (10.24) together one gets

$$\begin{gathered} X_2 = a_1,\ Y_2 = b_1; \quad X_{k+1} = \begin{pmatrix} X_k a_k \\ a_k \end{pmatrix}, \quad Y_{k+1} = \begin{pmatrix} b_k Y_k & b_k \end{pmatrix}, \\ k = 2,\dots,N-1. \end{gathered} \tag{10.25}$$

Thus from (10.23), (10.25) using Lemma 5.6 and Lemma 5.3 we conclude that the completion A is given by the recipe

$$A_{ij} = \begin{cases} a_i a_{i+1} \cdots a_{j-1}, & 1 \le i < j \le N, \\ I, & 1 \le i = j \le N, \\ b_{i-1} \cdots b_{j+1} b_j, & 1 \le j < i \le N. \end{cases} \tag{10.26}$$

Next, by Theorem 9.1, one obtains that the completion A is invertible if and only if all the matrices

$$D_k = \begin{pmatrix} I & a_{k-1} \\ b_{k-1} & I \end{pmatrix}, \ k = 2, \dots, N \tag{10.27}$$

are invertible, which in turn is equivalent to the invertibility of the matrices $I - a_k b_k$ and/or $I - b_k a_k$ $(k = 1, \dots, N-1)$. If these conditions hold, we can compute the inverse matrix $A^{-1} = \{A'_{ij}\}_{i,j=1}^N$ using Corollary 9.9. One has $B_k = I, \ M_k = I, \ Z_k = a_k, \ U_k = b_k, \ \Lambda_k = a_k, \ W_k = b_k, \ \gamma_k = I - b_k a_k, \ \Delta_2 = b_1, \ \Phi_2 = a_1, \ \delta_2 = I - a_1 b_1, \ \delta_3 = I - a_2 b_2$. Furthermore, one obtains

$$\begin{aligned}
A'_{11} &= (I - a_1 b_1)^{-1}, \\
A'_{22} &= (I - a_2 b_2)^{-1} + b_1 (I - a_1 b_1)^{-1} a_1, \\
A'_{21} &= -b_1 (I - a_1 b_1)^{-1}, \\
A'_{12} &= -(I - a_1 b_1)^{-1} a_1; \\
A'_{k-1,k} &= -a_{k-1}(I - b_{k-1} a_{k-1})^{-1} = -(I - a_{k-1} b_{k-1})^{-1} a_{k-1}, \quad k = 3, \dots, N; \\
A'_{k,k-1} &= -(I - b_{k-1} a_{k-1})^{-1} b_{k-1} = -b_{k-1}(I - a_{k-1} b_{k-1})^{-1}, \quad k = 3, \dots, N; \\
A'_{kk} &= (I - b_{k-1} a_{k-1})^{-1} + a_k (I - b_k a_k)^{-1} b_k \\
&= (I - a_k b_k)^{-1} + b_{k-1}(I - a_{k-1} b_{k-1})^{-1} a_{k-1}, \quad k = 3, \dots, N-1; \\
A'_{NN} &= (I - b_{N-1} a_{N-1})^{-1}.
\end{aligned}$$

Thus one obtains the inversion formula

$$\begin{aligned}
A'_{11} = c_1, \quad A'_{kk} &= b_{k-1} c_{k-1} a_{k-1} + c_k, \ k = 2, \dots, N-1, \ A'_{NN} = d; \\
A'_{k,k+1} &= -c_k a_k, \quad A'_{k+1,k} = -b_k c_k, \quad k = 1, \dots, N-1,
\end{aligned} \tag{10.28}$$

where $c_k = (I - a_k b_k)^{-1}$ $(k = 1, \dots, N-1)$, $d = (I - b_{N-1} a_{N-1})^{-1}$.

Next, by Theorem 10.1, the matrix A is positive definite if and only if all the matrices $D_k, \ k = 2, \dots, N$, are positive definite. This is equivalent to the conditions that $b_k = a_k^*$ and the matrices $I - a_k^* a_k$ are positive definite $(k = 1, \dots, N-1)$. The last holds if and only if all the matrices a_k $(k = 1, \dots, N-1)$ are strict contractions.

§10.3.2 The Toeplitz case

Now we consider a particular case where a partially specified block matrix $\mathcal{A}$ with entries of sizes $m \times m$ has a given tridiagonal part $\tilde{A} = \{A_{ij}, |i-j| \leq 1\}$ which is Toeplitz with identities on the main diagonal, i.e.,

$$\mathcal{A} = \begin{pmatrix} I & a & * & \dots & * \\ b & I & a & \dots & * \\ * & b & I & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & I \end{pmatrix}.$$

Here a and b are some $m \times m$ matrices and the asterisks denote the unspecified entries. It was proved above that $\tilde{A}$ has a unique completion which is a Green matrix of order one. Moreover applying the formula (10.26) one obtains that this completion is a Toeplitz matrix of the form

$$A = \begin{pmatrix} I & a & a^2 & \dots & a^{n-1} \\ b & I & a & \dots & a^{n-2} \\ b^2 & b & I & \dots & a^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \dots & I \end{pmatrix}.$$

This fact follows also from Theorem 10.4. Furthermore A is invertible if and only if the matrices $I - ab$ or $I - ba$ are invertible. Formula (10.28) yields the inverse

$$A^{-1} = \begin{pmatrix} c & -ca & 0 & \dots & 0 & 0 & 0 \\ -bc & bca+c & -ca & \dots & 0 & 0 & 0 \\ 0 & -bc & bca+c & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & bca+c & -ca & 0 \\ 0 & 0 & 0 & \dots & -bc & bca+c & -ca \\ 0 & 0 & 0 & \dots & 0 & -bc & d \end{pmatrix}, \qquad (10.29)$$

where $c = (I-ab)^{-1},\ d = (I-ba)^{-1}$.

By Theorem 10.1, the matrix A is positive definite if and only if $b = a^*$ and $\|a\| < 1$.

Also, by Theorem 9.11, $\operatorname{rank} A = m$ if and only if $ab = ba = I$.

§10.4 Completion of special 2 × 2 block matrices

§10.4.1 Completion formulas

Let $m \geq 0$ be an integer and let $\mathcal{A}$ be a partially specified matrix of the form

$$\mathcal{A} = \begin{pmatrix} I & \mathcal{G} \\ \mathcal{H} & I \end{pmatrix},$$

where $\mathcal{G}$ is a partially specified block square matrix with block entries of sizes $\nu_i \times \mu_j,\ i,j = 1,\dots,N$, with a given part $\tilde{G} = \{g_{ij}, j-i \leq m\}$ and $\mathcal{H}$ is a partially specified block square matrix with block entries of sizes $\mu_i \times \nu_j,\ i,j = 1,\dots,N$, with a given part $\tilde{H} = \{h_{ij}, i-j \leq m\}$. This means that $\mathcal{A}$ is a partially specified block square matrix with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq N+m\}$, with

$$\tilde{A}(1:N,1:N) = I, \quad \tilde{A}(N+1:2N, N+1:2N) = I; \tag{10.30}$$

$$\begin{aligned} A_{i,j+N} &= g_{ij}, \quad 1 \leq i,j \leq N,\ j-i \leq m; \\ A_{i+N,j} &= h_{ij}, \quad 1 \leq i,j \leq N,\ i-j \leq m. \end{aligned} \tag{10.31}$$

Lemma 10.5. *Set*

$$G_k^B = \tilde{G}(k+1:N, 1:k+m),\ H_k^B = \tilde{H}(1:k+m, k+1:N), \quad k = 1,\dots,N-m-1$$

and assume that all the matrices $I - G_k^B H_k^B$ *and/or* $\gamma_k = I - H_k^B G_k^B$ $(k = 1,\dots,N-m-1)$ *are invertible.*

Then the band $\tilde{A}$ *has a unique completion* A *which is a Green matrix of order*

$$n = N + m.$$

Moreover, this completion is obtained by successive computation of its principal leading submatrices

$$A_{k+n} := A(1:k+n, 1:k+n), \quad k = 1,\dots,N-m,$$

as follows.

In the first step we set

$$A_{n+1} = \begin{pmatrix} I_N & \tilde{G}(1:N, 1:m+1) \\ \tilde{H}(1:m+1, 1:N) & I_{m+1} \end{pmatrix}. \tag{10.32}$$

Let for some k *with* $1 \leq k \leq N-m-1$ *the matrix* A_{k+n} *be given. The matrix* A_{k+n+1} *is obtained via the following operations. Start by partitioning* A_{k+n} *as*

$$A_{k+n} = \begin{pmatrix} I & 0 & X_k' \\ 0 & I & G_k^B \\ Y_k' & H_k^B & I \end{pmatrix}. \tag{10.33}$$

Next, compute the submatrix A_{k+n+1} by the formula

$$A_{k+n+1} = \begin{pmatrix} I & 0 & X'_k & E'_k \\ 0 & I & G^B_k & Z'_k \\ Y'_k & H^B_k & I & 0 \\ F'_k & U'_k & 0 & I \end{pmatrix}, \tag{10.34}$$

where U'_k, Z'_k are determined from the specified parts $\tilde{G}, \tilde{H}$ via the equalities

$$Z'_k = \tilde{G}(k+1:N, k+m+1), \quad U'_k = \tilde{H}(k+m+1, k+1:N) \tag{10.35}$$

and E'_k, F'_k are computed by the formulas

$$E'_k = -X'_k\gamma_k^{-1}H^B_k Z'_k, \quad F'_k = -U'_k G^B_k \gamma_k^{-1} Y'_k. \tag{10.36}$$

Finally, set

$$A = A_{2N}. \tag{10.37}$$

Proof. In the case considered we represent the matrices from the condition (8.7) of Theorem 8.2 in the form

$$\begin{aligned} B_{k+n} &= \tilde{A}(k+1:k+n, k+1:k+n) \\ &= \begin{pmatrix} \tilde{A}(k+1:N, k+1:N) & \tilde{A}(k+1:N, N+1:k+n) \\ \tilde{A}(N+1:k+n, k+1:N) & \tilde{A}(N+1:k+n, N+1:k+n) \end{pmatrix}, \\ k &= 1, \dots, N-m. \end{aligned}$$

Using the definitions (10.30), (10.31) we get

$$B_{k+n} = \begin{pmatrix} I & \tilde{G}(k+1:N, 1:k+m) \\ \tilde{H}(1:k+m, k+1:N) & I \end{pmatrix},$$

that is

$$B_{k+n} = \begin{pmatrix} I & G^B_k \\ H^B_k & I \end{pmatrix}, \quad k = 1, \dots, N-m-1. \tag{10.38}$$

From the condition of the lemma it follows that all the matrices (10.38) are invertible and hence, by Theorem 8.2, the band $\tilde{A}$ has a unique completion A which is a Green matrix of order $n = N + m$. Moreover, applying the formulas (8.8)–(8.13) one obtains the following.

Applying (8.8) and using the corresponding partition we get

$$\begin{aligned} A_{n+1} &= \tilde{A}(1:N+m+1, 1:N+m+1) \\ &= \begin{pmatrix} \tilde{A}(1:N, 1:N) & \tilde{A}(1:N, N+1:N+m+1) \\ \tilde{A}(N+1:N+m+1, 1:N) & \tilde{A}(N+1:N+m+1, N+1:N+m+1) \end{pmatrix} \end{aligned}$$

and using (10.30), (10.31) we get (10.32).

By (8.9),

$$A_{n+k} = \begin{pmatrix} B'_{n+k} & X_{k+n} \\ Y_{k+n} & B_{k+n} \end{pmatrix}, \quad k = 1, \dots, N - m - 1, \tag{10.39}$$

with

$$B_{k+n} = \tilde{A}(k + 1 : k + m + N, k + 1 : k + m + N).$$

Furthermore using (10.30), (10.31) one obtains

$$\begin{aligned} B'_{k+n} &= A(1 : k, 1 : k) = \tilde{A}(1 : k, 1 : k) = I, \qquad (10.40)\\ X_{k+n} &= A(1 : k, k + 1 : k + N + m) \\ &= \begin{pmatrix} A(1 : k, k + 1 : N) & A(1 : k, N + 1 : k + N + m) \end{pmatrix} \\ &= \begin{pmatrix} 0 & G(1 : k, 1 : k + m) \end{pmatrix}, \end{aligned}$$

i.e.,

$$X_{k+n} = \begin{pmatrix} 0 & X'_k \end{pmatrix}, \tag{10.41}$$

with $X'_k = G(1 : k, 1 : k + m)$, and similarly

$$\begin{aligned} Y_{k+n} &= A(k + 1 : k + N + m, 1 : k) \\ &= \begin{pmatrix} A(k + 1 : N, 1 : k) \\ A(N + 1 : k + N + m, 1 : k) \end{pmatrix} = \begin{pmatrix} 0 \\ H(1 : k + m, 1 : k) \end{pmatrix}, \end{aligned}$$

i.e.,

$$Y_{k+n} = \begin{pmatrix} 0 \\ Y'_k \end{pmatrix}, \tag{10.42}$$

with $Y'_k = H(1 : k + m, 1 : k)$. Combining (10.38)–(10.42) together one obtains (10.33).

The application of (8.10), (8.11) yields

$$A_{k+n+1} = \begin{pmatrix} B'_{n+k} & X_{n+k} & E_{n+k} \\ Y_{n+k} & B_{n+k} & Z_{n+k} \\ F_{n+k} & U_{n+k} & M_{n+k} \end{pmatrix}, \quad k = 1, \dots, N - m, \tag{10.43}$$

with

$$\begin{aligned} M_{n+k} &= \tilde{A}(k + N + m + 1, k + N + m + 1), \\ Z_{k+n} &= \tilde{A}(k + 1 : k + m + N, k + m + N + 1), \\ U_{k+n} &= \tilde{A}(k + m + N + 1, k + 1 : k + m + N). \end{aligned}$$

Using (10.30) one obtains

$$M_{n+k} = \tilde{A}(k + N + m + 1, k + N + m + 1) = I \tag{10.44}$$

and using (10.30), (10.31) one gets

$$Z_{k+n} = \begin{pmatrix} \tilde{A}(k+1:N, k+N+m+1) \\ \tilde{A}(N+1:k+N+m, k+N+m+1) \end{pmatrix} = \begin{pmatrix} \tilde{G}(k+1:N, k+m+1) \\ 0 \end{pmatrix},$$

i.e.,

$$Z_{k+n} = \begin{pmatrix} Z'_k \\ 0 \end{pmatrix}, \tag{10.45}$$

and

$$\begin{aligned} U_{k+n} &= \begin{pmatrix} \tilde{A}(k+N+m+1, k+1:N) & \tilde{A}(k+N+m+1, N+1:k+N+m) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{H}(k+m+1, k+1:N) & 0 \end{pmatrix}, \end{aligned}$$

i.e.,

$$U_{k+n} = \begin{pmatrix} U'_k & 0 \end{pmatrix}, \tag{10.46}$$

with Z'_k, U'_k defined in (10.35). Combining (10.33) and (10.43)–(10.46) one obtains (10.34), (10.35).

Next, by virtue of (1.54) the inverses to the matrices (10.38) are given by the formulas

$$B_{k+n}^{-1} = \begin{pmatrix} I + G_k^B \gamma_k^{-1} H_k^B & -G_k^B \gamma_k^{-1} \\ -\gamma_k^{-1} H_k^B & \gamma_k^{-1} \end{pmatrix}, \quad k = 1, \dots, N-m-1. \tag{10.47}$$

From here, applying (8.12) with the matrices $X_{n+k}, Y_{n+k}, Z_{n+k}, U_{n+k}$ defined in (10.41), (10.42), (10.45), (10.46) one obtains (10.36).

Finally, using the formula (10.37) we obtain the completion A. □

§10.4.2 Completion to invertible and positive matrices

Using Theorem 9.1 one can easily derive the necessary and sufficient conditions for the completion obtained in Lemma 10.5 to be invertible. Indeed, in the case considered 10.30), (10.31) show that the matrices (9.2) of Theorem 9.1 have the form

$$D_{k+n} = \begin{pmatrix} I & G_k^D \\ H_k^D & I \end{pmatrix}, \quad k = 1, \dots, N-m, \tag{10.48}$$

with

$$G_k^D = \tilde{G}(k:N, 1:k+m), \; H_k^D = \tilde{H}(1:k+m, k:N), \quad k = 1, \dots, N-m.$$

By Theorem 9.1, the completion A defined in Lemma 10.5 is an invertible matrix if and only if all the matrices (10.48) are invertible, i.e., if and only if all the matrices $I - G_k^D H_k^D$ and/or $I - H_k^D G_k^D$ $(k = 1, \dots, N-m)$ are invertible. Moreover, by

Theorem 9.3, the matrix $A^{-1} = \{A'_{ij}\}_{i,j=1}^{2N}$ is a band matrix of order $N+m$. Next, using the formula (1.54) for the matrix

$$A = \begin{pmatrix} I & G \\ H & I \end{pmatrix}$$

one can write down the inversion formula

$$A^{-1} = \begin{pmatrix} I + G\gamma^{-1}H & -G\gamma^{-1} \\ -\gamma^{-1}H & \gamma^{-1} \end{pmatrix},$$

where $\gamma = I - HG$. The condition $A'_{ij} = 0$ for $|i-j| > N+m$ means that the completions G and H of the specified parts $\tilde{G}$ and $\tilde{H}$ given in Lemma 10.5 are such that the matrices $G\gamma^{-1}$ and $\gamma^{-1}H$ have zero entries in the parts $j-i > m,\ i-j < m$, respectively. In the case $m=0$ this means that the matrices $G\gamma^{-1}$ and $\gamma^{-1}H$ are lower and upper triangular.

Now we consider the case of a Hermitian partially specified matrix

$$A = \begin{pmatrix} I & G \\ G^* & I \end{pmatrix},$$

where $G = \{g_{ij}\}_{i,j=1}^{N}$ is a partially specified block square matrix with block entries of sizes $\nu_i \times \mu_j,\ i,j = 1,\dots,N$, with a specified part $\tilde{G} = \{g_{ij}, j-i \le m\}$. Set as above

$$G_k^B = \tilde{G}(k+1:N, 1:k+m), \quad k = 1,\dots,N-m-1;$$
$$G_k^D = \tilde{G}(k:N, 1:k+m), \quad k = 1,\dots,N-m.$$

Assume that all the matrices $I - G_k^B(G_k^B)^*$ or $\gamma_k = I - (G_k^B)^*(G_k^B)$ $(k = 1,\dots,N-m-1)$ are invertible. Then by Lemma 10.5 the band $\tilde{A}$ has a unique completion A which is a Green matrix of order $N+m$. Moreover this completion is determined by the formulas (10.32)–(10.37) with Hermitian matrices A_k. By the first part of Theorem 10.1, this completion is a positive matrix if and only if all the matrices

$$\begin{pmatrix} I & G_k^D \\ (G_k^D)^* & I \end{pmatrix}, \quad k = 1,\dots,N-m,$$

are positive. This implies that A is positive if and only if all the matrices $I - (G_k^D)^*(G_k^D),\ k = 1,\dots,N-m$, are positive, i.e., if and only if all the matrices G_k^D are contractions: $\|G_k^D\| \le 1$. Similarly, from the second part of Theorem 10.1 we conclude that the completion A is positive definite if and only if all the matrices G_k^D are strict contractions, i.e., $\|G_k^D\| < 1,\ k = 1,\dots,N-m$.

§10.4.3 Completion to matrices of minimal ranks

Let $\mathcal{A}$ be a partially specified matrix of the form

$$\mathcal{A} = \begin{pmatrix} I & \mathcal{G} \\ \mathcal{H} & I \end{pmatrix},$$

where $\mathcal{G}$ is a partially specified block square matrix with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given part $\tilde{G} = \{g_{ij}, j \leq i\}$ and $\mathcal{H}$ is a partially specified block square matrix with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given part $\tilde{H} = \{h_{ij}, i \leq j\}$. This means that $\mathcal{A}$ is a partially specified block square matrix with a given band $\tilde{A} = \{A_{ij}, |i-j| \leq N\}$, with

$$\tilde{A}(1:N, 1:N) = I, \qquad \tilde{A}(N+1:2N, N+1:2N) = I;$$
$$A_{i,j+N} = g_{ij},\ 1 \leq i,j \leq N,\ j \leq i; \quad A_{i+N,j} = h_{ij},\ 1 \leq i,j \leq N,\ i \leq j.$$

Set $\nu = \sum_{i=1}^{N} m_i$ and assume that all the matrices $I - G_k^B H_k^B$ and/or $I - H_k^B G_k^B$ $(k = 1,\dots,N-1)$, where

$$G_k^B = \tilde{G}(k+1:N, 1:k),\ H_k^B = \tilde{H}(1:k, k+1:N), \quad k = 1,\dots,N-1,$$

are invertible. By Lemma 10.5, the band $\tilde{A}$ has a unique completion A which is a Green matrix of order N.

Theorem 10.6. *Let $\mathcal{A}$ be a partially specified matrix of the form*

$$\mathcal{A} = \begin{pmatrix} I & \mathcal{G} \\ \mathcal{H} & I \end{pmatrix},$$

where $\mathcal{G}$ is a partially specified block square matrix with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given part $\tilde{G} = \{g_{ij}, j \leq i\}$ and $\mathcal{H}$ is a partially specified block square matrix with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given part $\tilde{H} = \{h_{ij}, i \leq j\}$. Set $\nu = \sum_{i=1}^{N} m_i$. Assume that all the matrices $I - G_k^B H_k^B$ and/or $I - H_k^B G_k^B$ $(k = 1,\dots,N-1)$, where

$$G_k^B = \tilde{G}(k+1:N, 1:k),\ H_k^B = \tilde{H}(1:k, k+1:N), \quad k = 1,\dots,N-1,$$

are invertible. By Lemma 10.5, $\mathcal{A}$ has a unique completion A which is a Green matrix of order N. Set also

$$g_k = \tilde{G}(k:N, k), \quad h_k = \tilde{H}(k, k:N), \quad k = 1,\dots,N.$$

Then $\operatorname{rank} A = \nu$ *if and only if*

$$h_1 g_1 = I_{m_1} \tag{10.49}$$

and

$$h_k (I - G_{k-1}^B H_{k-1}^B)^{-1} g_k = I_{m_k}, \quad k = 2,\dots,N. \tag{10.50}$$

Proof. By Theorem 9.11, $\operatorname{rank} A = \nu$ if and only if

$$\operatorname{rank} D_{k+N} = \nu, \quad k = 1,\dots,N. \tag{10.51}$$

For $k=1$ we have the partition

$$D_{N+1} = \begin{pmatrix} I_\nu & g_1 \\ h_1 & I \end{pmatrix}$$

and therefore rank $D_{N+1} = \nu$ if and only if (10.49) holds.

For $k = 2, \dots, N$ we use the partitions

$$D_{k+N} = \begin{pmatrix} B_{k+N-1} & Z_{k+N} \\ U_{k+N} & I \end{pmatrix}.$$

We have

$$B_{k+N-1} = \begin{pmatrix} I & G^B_{k-1} \\ H^B_{k-1} & I \end{pmatrix}, \quad k = 2, \dots, N-1 \tag{10.52}$$

and

$$Z_{k+N} = \begin{pmatrix} g_k \\ 0 \end{pmatrix},\ U_{k+N} = \begin{pmatrix} h_k & 0 \end{pmatrix}, \quad k = 2, \dots, N. \tag{10.53}$$

The conditions (10.51), i.e., rank $D_{k+N} =$ rank B_{k+N-1} hold if and only if

$$I - U_{k+N} B^{-1}_{k+N-1} Z_{k+N} = 0, \quad k = 2, \dots, N. \tag{10.54}$$

Applying the inversion formula (1.57) to the matrix B_{k+N-1} partitioned in the form (10.52) we get

$$B^{-1}_{k+N-1} = \begin{pmatrix} (I - G^B_{k-1} H^B_{k-1})^{-1} & * \\ * & * \end{pmatrix}. \tag{10.55}$$

Using (10.55), (10.54) and (10.53) we obtain (10.50). □

§10.5 Comments

In the presentation of the results of this chapter we follow the paper [26]. The formula (10.29) was obtained by I. Gohberg and G. Heinig in [35]. Theorem 10.6 appears here for the first time.

Chapter 11

Completion of Mutually Inverse Matrices

In this chapter we consider the problem when the original matrix is specified in its lower (with the diagonal) triangular section and the inverse one is specified in its strictly upper triangular section.

§11.1 The statement and preliminaries

Let

$$\mathcal{G} = \begin{pmatrix} g_{11} & ? & \dots & ? \\ g_{21} & g_{22} & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N2} & \dots & g_{NN} \end{pmatrix}$$

be a partially specified block matrix with elements of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given lower triangular part $\tilde{G} = \{g_{ij}\}_{1 \le j \le i \le N}$ and let

$$\mathcal{H} = \begin{pmatrix} ? & h_{12} & \dots & h_{1,N} \\ ? & ? & \dots & h_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \dots & ? \end{pmatrix}$$

be a partially specified block matrix with elements of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given strictly upper triangular part $\tilde{H} = \{h_{ij}\}_{1 \le i < j \le N}$. The problem is to determine an invertible matrix G with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, such that G is a completion of $\mathcal{G}$ and G^{-1} is a completion of $\mathcal{H}$. We study this problem using the previous results on completion of partially specified matrices of the form

$$\mathcal{A} = \begin{pmatrix} I & \mathcal{G} \\ \mathcal{H} & I \end{pmatrix}. \tag{11.1}$$

Y. Eidelman et al., *Separable Type Representations of Matrices and Fast Algorithms: Volume 1 Basics. Completion Problems. Multiplication and Inversion Algorithms*, Operator Theory: Advances and Applications 234, DOI 10.1007/978-3-0348-0606-0_11,

We start with an auxiliary result concerning matrices of the form

$$A = \begin{pmatrix} I_\nu & G \\ H & I_\nu \end{pmatrix} \tag{11.2}$$

with some positive integer ν.

Lemma 11.1. *A matrix A of the form* (11.2) *satisfies the condition* $\operatorname{rank} A = \nu$ *if and only if the matrices G and H are invertible and $H = G^{-1}$.*

Proof. Indeed, the equality $HG = I$ implies

$$A = \begin{pmatrix} I_\nu \\ H \end{pmatrix} \begin{pmatrix} I_\nu & G \end{pmatrix}.$$

Consequently, $\operatorname{rank} A = \nu$.

Now assume that $\operatorname{rank} A = \nu$. Then

$$A = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} \tag{11.3}$$

with $\nu \times \nu$ matrices X_1, X_2, Y_1, Y_2. Comparing (11.2) and (11.3) we get

$$X_1Y_1 = I,\ X_2Y_2 = I,\ H = X_2Y_1,\ G = X_1Y_2.$$

It follows that the matrices X_1 and X_2 are invertible, with $X_1^{-1} = Y_1$ and $X_2^{-1} = Y_2$ and moreover $H = X_2X_1^{-1}$ and $G = X_1X_2^{-1}$. These last equalities mean $GH = HG = I$. □

Corollary 11.2. *A matrix A of the form*

$$A = \begin{pmatrix} I_\nu & G \\ G^* & I_\nu \end{pmatrix}$$

satisfies the condition $\operatorname{rank} A = \nu$ *if and only if the matrix G is unitary.*

As it was mentioned above, in the solution of the completion problem for the partially specified matrices of the form (11.1) a crucial role is played by the submatrices

$$G_k^B = \tilde{G}(k+1:N, 1:k),\ H_k^B = \tilde{H}(1:k, k+1:N), \quad k = 1, \dots, N-1$$

for which all the matrices $I - G_k^B H_k^B$ and/or $I - H_k^B G_k^B$ $k = 1, \dots, N-1$, are invertible. We show that this condition means that the matrices G and H, which are completions of $\mathcal{G}$ and $\mathcal{H}$ such that $H = G^{-1}$, are strongly regular matrices, i.e., all the principal leading submatrices

$$G_k = G(1:k, 1:k), \quad H_k = H(1:k, 1:k), \quad k = 1, \dots, N,$$

are invertible.

Lemma 11.3. *Let* $G = \{g_{ij}\}_{i=1}^{N}$ *and* $H = \{h_{ij}\}_{i=1}^{N}$ *be invertible block matrices with entries of sizes* $m_i \times m_j$ *such that* $H = G^{-1}$. *Set*

$$G_k^B = G(k+1:N, 1:k),\ H_k^B = H(1:k, k+1:N), \quad k = 1, \dots, N-1.$$

All the matrices $I - G_k^B H_k^B$ *and/or* $I - H_k^B G_k^B$ $(k = 1, \dots, N-1)$ *are invertible if and only if all the matrices*

$$G(1:k, 1:k), \qquad H(1:k, 1:k), \qquad k = 1, \dots, N-1$$

and

$$G(k:N, k:N), \quad H(k:N, k:N), \quad k = N, \dots, 2$$

are invertible.

Proof. We obviously have

$$H(1:k, :)G(:, 1:k) = I, \quad G(k:N, :)H(:, k:N) = I, \quad k = 1, \dots, N.$$

Using the partitions

$$H(1:k, :) = \begin{pmatrix} H(1:k, 1:k) & H_k^B \end{pmatrix}, \quad G(:, 1:k) = \begin{pmatrix} G(1:k, 1:k) \\ G_k^B \end{pmatrix}, \\ k = 1, \dots, N-1$$

and

$$G(k:N, :) = \begin{pmatrix} G_{k-1}^B & G(k:N, k:N) \end{pmatrix},$$
$$H(:, k:N) = \begin{pmatrix} H_{k-1}^B \\ H(k:N, k:N) \end{pmatrix}, \quad k = N, \dots, 2,$$

we get

$$H(1:k, 1:k)G(1:k, 1:k) = I - H_k^B G_k^B, \qquad k = 1, \dots, N-1$$

and

$$G(k:N, k:N)H(k:N, k:N) = I - G_{k-1}^B H_{k-1}^B, \quad k = N, \dots, 2.$$

Hence the statement of the lemma follows. □

§11.2 The basic theorem

Here under the assumption that all the matrices $I - G_k^B H_k^B$ and/or $I - H_k^B G_k^B$, $k = 1, \dots, N-1$, are invertible we obtain necessary and sufficient conditions for existence of a solution to the problem of determining an invertible matrix G and the matrix $H = G^{-1}$ such that G is a completion of $\mathcal{G}$ and H is a completion of $\mathcal{H}$.

Theorem 11.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be partially specified block matrices with entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$ such that $\mathcal{G}$ has a given lower triangular part $\tilde{G} = \{g_{ij}\}_{1 \le j \le i \le N}$ and $\mathcal{H}$ has a given strictly upper triangular part $\tilde{H} = \{h_{ij}\}_{1 \le i < j \le N}$. Set*

$$G_k^B = \tilde{G}(k+1:N, 1:k),\ H_k^B = \tilde{H}(1:k, k+1:N), \quad k = 1, \dots, N-1 \quad (11.4)$$

and assume that all the matrices $I - G_k^B H_k^B$ and/or $I - H_k^B G_k^B$ $(k = 1, \dots, N-1)$ are invertible.

For the complete statement of the theorem, set

$$\begin{aligned} g(k) &= \tilde{G}(k,k),\ g_k = \tilde{G}(k:N,k),\ h_k = \tilde{H}(k,k:N), \quad && k = 1, \dots, N, \\ g'_k &= \tilde{G}(k+1:N,k),\ h'_k = \tilde{H}(k,k+1:N), && k = 1, \dots, N-1, \\ G'_k &= \tilde{G}(k, 1:k-1),\ G''_k = \tilde{G}(k+1:N, 1:k-1), && k = 2, \dots, N-1 \end{aligned}$$

and

$$\Delta_k = (I - H_{k-1}^B G_{k-1}^B)^{-1} H_{k-1}^B g_k, \quad k = 2, \dots, N-1.$$

Then there exists an invertible matrix $G = \{g_{ij}\}_{i,j=1}^N$ such that G is a completion of $\mathcal{G}$ and $H = G^{-1} = \{h_{ij}\}_{i,j=1}^N$ is a completion of $\mathcal{H}$ if and only if all the matrices

$$g(1), \quad g(k) + G'_k \Delta_k,\ k = 2, \dots, N-1, \quad g(N) \quad (11.5)$$

are invertible. Such a matrix G is unique.

Furthermore, the matrices G and H are obtained as follows. At first we determine the unspecified diagonal entries h_{ii}, $i = 1, \dots, N$, of $\mathcal{H}$ by the formulas

$$h_{11} = \left(I_{m_1} - H_1^B G_1^B\right)(g(1))^{-1}, \quad (11.6)$$

$$h_{kk} = \left(I_{m_k} - h'_k(g'_k + G''_k \Delta_k)\right)\left(g(k) + G'_k \Delta_k\right)^{-1}, \quad k = 2, \dots, N-1, \quad (11.7)$$

$$h_{NN} = (g(N))^{-1}(I_{m_N} - G_{N-1}^B H_{N-1}^B). \quad (11.8)$$

Next we determine the matrices G and H by successive computation of the submatrices

$$\hat{G}_k := G(1:N, 1:k), \quad \hat{H}_k := H(1:k, 1:N), \quad k = 1, \dots, N,$$

as follows. On the first step we set

$$\hat{G}_1 = \tilde{G}(1:N, 1), \quad \hat{H}_1 = \begin{pmatrix} h_{11} & \tilde{H}(1, 2:N) \end{pmatrix}. \quad (11.9)$$

Let for some k with $1 \le k \le N-1$ the matrices $\hat{G}_k$ and $\hat{H}_k$ be given. We start with partitioning $\hat{G}_k$ and $\hat{H}_k$ in the form

$$\hat{G}_k = \begin{pmatrix} G_k \\ G_k^B \end{pmatrix}, \quad \hat{H}_k = \begin{pmatrix} H_k & H_k^B \end{pmatrix}, \quad (11.10)$$

with

$$G_k = G(1:k, 1:k), \quad H_k = H(1:k, 1:k) \tag{11.11}$$

and G_k^B, H_k^B *defined in* (11.4). *Next we compute the submatrices* $\hat{G}_{k+1}$ *and* $\hat{H}_{k+1}$ *by the formulas*

$$\hat{G}_{k+1} = \begin{pmatrix} G_k & E_k' \\ G_k^B & g_{k+1} \end{pmatrix}, \quad \hat{H}_{k+1} = \begin{pmatrix} H_k & H_k^B \\ F_k' & h_{k+1} \end{pmatrix}, \tag{11.12}$$

where h_{k+1}, E_k', F_k' *are computed by the formulas*

$$\begin{aligned} h_{k+1} &= H(k+1, k+1:N), \\ E_k' &= -G_k(I - H_k^B G_k^B)^{-1} H_k^B g_{k+1}, \\ F_k' &= -h_{k+1} G_k^B (I - H_k^B G_k^B)^{-1} H_k. \end{aligned} \tag{11.13}$$

Finally we get

$$G = \hat{G}_N, \quad H = \hat{H}_N. \tag{11.14}$$

Proof. Assume that all the matrices of the form (11.5) are invertible. We show that the formulas (11.6)–(11.8) yield the diagonal entries $H_{ii},\ i = 1, \dots, N$ of $\mathcal{H}$ such that the conditions (10.49), (10.50) of Theorem 10.6 are satisfied.

Consider the condition (10.49). We use the partitions

$$h_1 := H(1,:) = \begin{pmatrix} h_{11} & \tilde{H}(1, 2:N) \end{pmatrix} = \begin{pmatrix} h_{11} & H_1^B \end{pmatrix}$$

and

$$g_1 = \tilde{G}(1:N, 1) = \begin{pmatrix} g(1) \\ G_1^B \end{pmatrix}$$

and write the equality (10.49) in the form

$$h_{11} g(1) + H_1^B G_1^B = I_{m_1}.$$

Hence it follows that the desired value of h_{11} is given by the formula (11.6).

For $k = 2, \dots, N-1$ we consider the condition (10.50) with the partitions

$$h_k = \begin{pmatrix} h_{kk} & \tilde{H}(k, k+1:N) \end{pmatrix} = \begin{pmatrix} h_{kk} & h_k' \end{pmatrix}$$

and

$$g_k = \tilde{G}(k:N, k) = \begin{pmatrix} g(k) \\ g_k' \end{pmatrix},$$

$$G_{k-1}^B = \tilde{G}(k:N, 1:k-1) = \begin{pmatrix} \tilde{G}(k, 1:k-1) \\ \tilde{G}(k+1:N, 1:k-1) \end{pmatrix} = \begin{pmatrix} G_k' \\ G_k'' \end{pmatrix}.$$

By (1.70), we get

$$(I - G_{k-1}^B H_{k-1}^B)^{-1} = I + G_{k-1}^B (I - H_{k-1}^B G_{k-1}^B)^{-1} H_{k-1}^B$$

and therefore the condition (10.50) may be expressed in the form

$$h_k(g_k + G^B_{k-1}(I - H^B_{k-1}G^B_{k-1})^{-1}H^B_{k-1}g_k) = I_{m_k},$$

i.e.,

$$\begin{pmatrix} h_{kk} & h'_k \end{pmatrix}\left[\begin{pmatrix} g(k) \\ g'_k \end{pmatrix} + \begin{pmatrix} G'_k \\ G''_k \end{pmatrix}\Delta_k\right] = I_{m_k}.$$

Hence it follows that the desired value of h_{kk} is given by the formula (11.7).

Finally, for $k = N$ we use the formulas $h_N = h_{NN}$ and $g_N = g(N)$ and obtain the condition (10.50) in the form

$$h_{NN}(I - G^B_{N-1}H^B_{N-1})^{-1}g(N) = I_{m_N}.$$

Hence it follows that the desired value of h_{NN} is given by the formula (11.8).

Next set $\nu = \sum_{i=1}^N m_i$ and consider the partially specified matrix

$$\mathcal{A} = \begin{pmatrix} I_\nu & \mathcal{G}_1 \\ \mathcal{H}_1 & I_\nu \end{pmatrix}, \tag{11.15}$$

where $\mathcal{G}_1$ is a partially specified block square matrix with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given part $\tilde{G}_1 = \tilde{G} = \{g_{ij}, j \le i\}$ and $\mathcal{H}_1$ is a partially specified block square matrix with block entries of sizes $m_i \times m_j,\ i,j = 1,\dots,N$, with a given part $\tilde{H}_1 = \{h_{ij}, i \le j\}$. By Lemma 10.5, there is a unique completion A of $\mathcal{A}$ such that A is a Green matrix of order N. Moreover, the matrix A is obtained via the formulas (10.32)–(10.37). Notice that the computation of the matrix A via the formulas (10.32)–(10.37) means the computation of the matrices G and H which are the completions of $\mathcal{G}_1$ and $\mathcal{H}_1$ via the formulas (11.9)–(11.14). Next, by Theorem 10.6 the matrix

$$A = \begin{pmatrix} I_\nu & G \\ H & I_\nu \end{pmatrix} \tag{11.16}$$

is of rank ν. By Lemma 11.1, this means that the matrices G and H are invertible and $H = G^{-1}$.

Let G be an invertible matrix such that G is a completion of $\mathcal{G}$ and $H = G^{-1}$ is a completion of $\mathcal{H}$. We prove that this implies that the matrices (11.5) are invertible and that such matrix G is unique. By Lemma 11.3, the matrices $g(1) = G(1,1)$ and $g(N) = G(N,N)$ are invertible. Also, by Lemma 11.3, the matrices $G_k,\ k = 1,\dots,N-1$ are invertible. The matrix $G_N = G$ is invertible by the assumption. For $k = 2,\dots,N$ we use the partitions

$$G_k = \begin{pmatrix} G_{k-1} & E'_{k-1} \\ G'_k & g(k) \end{pmatrix}.$$

Since the matrices G_{k-1} and G_k are invertible, the Schur complements

$$\Gamma_k = g(k) - G'_k G^{-1}_{k-1} E'_{k-1}, \quad k = 2,\dots,N, \tag{11.17}$$

are also invertible. Next consider the matrix A of the form (11.16). By Lemma 11.1, $\operatorname{rank} A = \nu = \sum_{i=1}^{N} m_i$. It follows that

$$\operatorname{rank} A(1 : k, k-N+1 : 2N) \leq \nu, \quad k = N+1, \dots, 2N-1,$$
$$\operatorname{rank} A(k-N+1 : 2N, 1 : k) \leq \nu, \quad k = N+1, \dots, 2N-1,$$

i.e., A is a Green matrix of order N. The matrix A may be treated as a completion of the partially specified matrix (11.15) with the partially specified parts $\mathcal{G}_1$ and $\mathcal{H}_1$ with the given parts $\tilde{G}_1 = \tilde{G} = \{g_{ij}, j \leq i\}$ and $\tilde{H}_1 = \{h_{ij}, i \leq j\}$. By Lemma 10.5, for the given parts $\tilde{G}_1$ and $\tilde{H}_1$ the completion A which is a Green matrix of order N is unique. To get uniqueness of G it remains to check that the specified parts $\tilde{G}_1$ and $\tilde{H}_1$ are uniquely determined. Indeed the part $\tilde{G}_1 = \tilde{G}$ is given. The entries of the upper triangular part $\tilde{H}_1$ coincide for $i < j$ with the given entries h_{ij} of the part $\tilde{H}$. Since $\operatorname{rank} A = \nu$, by Theorem 10.6 the conditions (10.49), (10.50) hold. But as it was shown above the diagonal elements h_{ii} $(i = 1, \dots, N)$ of $\tilde{H}_1$ are uniquely determined from these conditions by the formulas (11.6)–(11.8), which completes the proof of the uniqueness. The matrix A is a completion of the partially specified matrix $\mathcal{A}$ of the form (11.15) and hence, by Lemma 10.5, the matrix A satisfies the equalities (10.32)–(10.37) and therefore the matrix G satisfies the equalities (11.9)–(11.14). Using the first equality in (11.13) we get

$$E'_{k-1} = -G_{k-1}(I - H^B_{k-1} G^B_{k-1})^{-1} H^B_{k-1} g_k, \quad k = 2, \dots, N. \tag{11.18}$$

Inserting (11.18) in (11.17) we obtain

$$\Gamma_k = g(k) + G'_k (I - H^B_{k-1} G^B_{k-1})^{-1} H^B_{k-1} g_k, \quad k = 2, \dots, N-1,$$

and thus the matrices $g(k) + G'_k \Delta_k$ $(k = 2, \dots, N-1)$ are invertible. □

§11.3 The direct method

As it was proved above in Lemma 11.3, invertibility of the matrices $I - G^B_k H^B_k$ and/or $I - H^B_k G^B_k$ $k = 1, \dots, N-1$ implies that the completions G and $H = G^{-1}$ are strongly regular, i.e., all the principal leading submatrices

$$G_k = G(1 : k, 1 : k), \quad H_k = H(1 : k, 1 : k), \quad k = 1, \dots, N,$$

are invertible.

Here we derive some other formulas for the completions G and H using the inverses of the matrices G_k and H_k.

Theorem 11.5. *Under the conditions of Theorem* 11.4 *the matrices* G *and* $H = G^{-1}$ *may be obtained by successive computation of their principal leading submatrices*

$$G_k = G(1 : k, 1 : k), \quad H_k = H(1 : k, 1 : k), \quad k = 1, \dots, N,$$

as follows. In the first step we set

$$G_1 = \tilde{G}(1, 1) \tag{11.19}$$

and compute

$$H_1 = (I_{m_1} - H_1^B G_1^B)(g(1))^{-1}. \tag{11.20}$$

Let for some k with $2 \le k \le N$ the matrices G_{k-1}, H_{k-1} be given. We compute the submatrices G_k, H_k by the formulas

$$G_k = \begin{pmatrix} G_{k-1} & E_k \\ G'_k & g(k) \end{pmatrix} \tag{11.21}$$

and

$$H_k = \begin{pmatrix} W_k \\ F_k \end{pmatrix}, \tag{11.22}$$

with $W_k = \begin{pmatrix} H_{k-1} & H'_k \end{pmatrix}$, where $G'_k, g(k), H'_k$ are determined from the given parts $\tilde{G}$ and $\tilde{H}$ by the recipes

$$G'_k = \tilde{G}(k, 1:k-1),\ g(k) = \tilde{G}(k,k),\ H'_k = \tilde{H}(1:k-1, k)$$

and E_k and F_k are computed by the formulas

$$E_k = -H_{k-1}^{-1} H_{k-1}^B g_k \tag{11.23}$$

and

$$F_k = \begin{pmatrix} -h'_k G''_k & I_{m_k} - h'_k g'_k \end{pmatrix} G_k^{-1}. \tag{11.24}$$

Finally we set

$$G = G_N, \quad H = H_N. \tag{11.25}$$

Proof. By Lemma 11.3, the matrices G_k, H_k $(k = 1, \dots, N)$ are invertible.

The equality (11.19) is obvious. The equality (11.20) follows from (11.6).

For $k = 2, \dots, N$ we proceed as follows. Using the equality

$$H(1:k-1, :)G(:, k) = 0$$

with the partitions

$$H(1:k-1, :) = \begin{pmatrix} H_{k-1} & H_{k-1}^B \end{pmatrix}, \quad G(:, k) = \begin{pmatrix} E_k \\ g_k \end{pmatrix}$$

we obtain

$$H_{k-1}E_k + H_{k-1}^B g_k = 0.$$

This yields (11.23). Moreover, we obtain the matrix G_k by the formula (11.21). Next we use the equality

$$H(k, :)G(:, 1:k) = \begin{pmatrix} 0 & \cdots & 0 & I \end{pmatrix}$$

with the partitions

$$H(k, :) = \begin{pmatrix} F_k & h'_k \end{pmatrix}, \quad G(:, 1:k) = \begin{pmatrix} G_k \\ G_k^B \end{pmatrix}$$

and obtain

$$F_kG_k + h'_kG_k^B = \begin{pmatrix} 0 & \cdots & 0 & I \end{pmatrix}.$$

Hence it follows that

$$F_k = \left[\begin{pmatrix} 0 & \cdots & 0 & I \end{pmatrix} - h'_kG_k^B\right] G_k^{-1}.$$

From here using the partition

$$G_k^B = \begin{pmatrix} G''_k & g'_k \end{pmatrix}$$

we obtain (11.24). Moreover, we obtain the matrix H_k by the formula (11.22). □

§11.4 The factorization

As it was mentioned above, invertibility of the matrices $I - G_k^B H_k^B$ and/or $I - H_k^B G_k^B$ $k = 1, \dots, N-1$ implies that the completions G and $H = G^{-1}$ are strongly regular. By Theorem 1.20, the matrix G admits the factorization $G = LU_1$, where L is a block lower triangular matrix (not necessarily with identities on the main diagonal) and U_1 is an upper triangular matrix with identities on the main diagonal. The inverse $U = U_1^{-1}$ of U_1 is also a block upper triangular matrix with identities on the main diagonal. Next we derive simple formulas for the factors L and U.

Theorem 11.6. *Under the conditions of Theorem* 11.4 *the matrix G admits the factorization*

$$G = LU^{-1}, \tag{11.26}$$

where L is a block lower triangular matrix and U is an upper triangular matrix with identities on the main diagonal. Moreover, the subcolumns of L in the lower triangular part and the subcolumns of U in the strictly upper triangular part are determined by the formulas

$$L(1:N,1) = \tilde{G}(1:N,1), \tag{11.27}$$

$$L(k:N,k) = (I - G_{k-1}^B H_{k-1}^B)^{-1} g_k, \quad k = 2, \dots, N, \tag{11.28}$$

and

$$U(1:k-1,k) = H_{k-1}^B (I - G_{k-1}^B H_{k-1}^B)^{-1} g_k, \quad k = 2, \dots, N, \tag{11.29}$$

where $g_k = \tilde{G}(k:N,k)$.

Proof. By Lemma 11.3, the matrices $G(1:k, 1:k)$ $(k = 1, \dots, N)$ are invertible. By Theorem 1.20, the matrix G admits the factorization $G = LU_1$ with a block lower triangular matrix L and a block upper triangular matrix U_1 with identities on the main diagonal. Set $U = U_1^{-1}$. Then U is also a block upper triangular matrix with identities on the main diagonal. Thus we obtain (11.26), which implies

$$GU = L. \tag{11.30}$$

Moreover, for the matrix $H = G^{-1}$ we get $H = UL^{-1}$, which implies

$$HL = U. \tag{11.31}$$

Comparing the first columns in (11.30) we obtain (11.27).

For $k = 2, \dots, N$ we proceed as follows. Set

$$L_k = L(k : N, k), \quad U_k = U(1 : k-1, k), \quad k = 2, \dots, N.$$

Taking the kth columns and the rows from k to N in (11.30) and using the upper triangular form of U we get

$$G(k : N, 1 : k)U(1 : k, k) = L_k.$$

From here using the partitions

$$G(k : N, 1 : k) = \begin{pmatrix} G^B_{k-1} & g_k \end{pmatrix}, \quad U(1 : k, k) = \begin{pmatrix} U_k \\ I \end{pmatrix}$$

we get $G^B_{k-1}U_k + g_k = L_k$, or

$$L_k - G^B_{k-1}U_k = g_k \tag{11.32}$$

Taking the kth columns and the rows from 1 to $k-1$ in (11.31) and using the lower triangular form of the matrix L we get

$$H(1 : k-1, k : N)L(k : N, k) = U(1 : k-1, k),$$

i.e.,

$$-H^B_{k-1}L_k + U_k = 0. \tag{11.33}$$

Combining (11.32) and (11.33) together we obtain the equation

$$\begin{pmatrix} I & -G^B_{k-1} \\ -H^B_{k-1} & I \end{pmatrix} \begin{pmatrix} L_k \\ U_k \end{pmatrix} = \left[\begin{matrix} g_k \\ 0 \end{matrix} \right).$$

Applying the inversion formula (1.57) we get

$$\begin{pmatrix} L_k \\ U_k \end{pmatrix} = \begin{pmatrix} (I - G^B_{k-1}H^B_{k-1})^{-1} & * \\ H^B_{k-1}(I - G^B_{k-1}H^B_{k-1})^{-1} & * \end{pmatrix} \begin{pmatrix} g_k \\ 0 \end{pmatrix}.$$

Hence the equalities (11.28), (11.29) follow. □

§11.5 Comments

The material of the first three sections appears here for the first time. In the last section we derived the factorization formulas obtained by H. Dym and I. Gohberg in [11].

Chapter 12

Completion to Unitary Matrices

In this chapter we study the problem of completion of a partially specified matrix with a given lower triangular part to a unitary matrix.

§12.1 Auxiliary relations

At first we consider some relations for block positive and positive definite matrices. Recall that a $k \times k$ matrix A is said to be positive, and we write $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for any $x \in \mathbb{C}^k$ and A is said to be positive definite if there exists $\delta > 0$ such that $\langle Ax, x\rangle \geq \delta\langle x, x\rangle$ for any $x \in \mathbb{C}^k$.

Lemma 12.1. *Let*

$$A_0 = \begin{pmatrix} A & Z \\ Z^* & D \end{pmatrix},$$

where A is a square $n \times n$ matrix and D is a square $m \times m$ matrix.

Assume that the matrix A_0 is positive.

Then the equation $AE = Z$ has a solution E and for any such E the matrix $D - Z^*E$ is positive. Moreover, the relation

$$\dim \operatorname{Ker} A_0 = \dim \operatorname{Ker} A + \dim \operatorname{Ker}(D - Z^*E) \tag{12.1}$$

holds.

Proof. Since A_0 is positive, there exists an $(n+m) \times (n+m)$ matrix K such that $A_0 = K^*K$. Consider the partition of the matrix K in the form $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ with submatrices K_1 and K_2 of the sizes $(n+m) \times n$ and $(n+m) \times m$. We have

$$\begin{pmatrix} A & Z \\ Z^* & D \end{pmatrix} = \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} \begin{pmatrix} K_1 & K_2 \end{pmatrix}.$$

The matrix A_0 is positive and, therefore, its principal leading submatrix A is Hermitian. Let us prove that $Z^*x = 0$ for any $x \in \operatorname{Ker} A$, which implies that the

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equation $AE = Z$ has a solution. Indeed, for any $x \in \operatorname{Ker} A$ we have $0 = x^*Ax = x^*K_1^*K_1x = \|K_1x\|^2$, and therefore $K_1x = 0$. Hence $Z^*x = K_2^*K_1x = 0$.

Next let E be a solution of the equation $AE = Z$. We use the factorization

$$\begin{pmatrix} A & Z \\ Z^* & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ E^* & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - Z^*E \end{pmatrix} \begin{pmatrix} I & E \\ 0 & I \end{pmatrix}. \tag{12.2}$$

Since the matrix A_0 is positive this implies that the matrix

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & D - Z^*E \end{pmatrix} \tag{12.3}$$

is positive and hence the matrix $D - Z^*E$ is positive.

The relation (12.1) follows directly from (12.2). □

§12.2 An existence and uniqueness theorem

Let $\mathcal{U}$ be a partially specified block matrix with a given lower triangular part $\tilde{U} = \{u_{ij}, i \geq j\}$. In this chapter we consider the problem of completing the lower triangular part $\tilde{U}$ to a unitary matrix U.

Theorem 12.2. *Let $\mathcal{U}$ be a partially specified block matrix with entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, with a given lower triangular part $\tilde{U} = \{u_{ij}\}_{1 \leq j \leq i \leq N}$.*

The matrix $\mathcal{U}$ has a unitary completion U if and only if the submatrices

$$U_k^D = \tilde{U}(k : N, 1 : k), \quad k = 1, \dots, N$$

satisfy the conditions

$$I - (U_k^D)^* U_k^D \geq 0, \quad \dim \operatorname{Ker}(I - (U_k^D)^* U_k^D) \geq m_k, \quad k = 1, 2, \dots, N. \tag{12.4}$$

Moreover, the unitary completion U is obtained by successive computation of the submatrices

$$\hat{U}_k = U(1 : N, 1 : k),\ k = 1, \dots, N,$$

as follows. In the first step we set

$$\hat{U}_1 = \tilde{U}(1 : N, 1). \tag{12.5}$$

Let for some k with $2 \leq k \leq N$ the matrix $\hat{U}_{k-1}$ be given. We start with partitioning $\hat{U}_{k-1}$ in the form

$$\hat{U}_{k-1} = \begin{pmatrix} U_{k-1} \\ U_{k-1}^B \end{pmatrix}, \tag{12.6}$$

with

$$U_{k-1} = U(1 : k-1, 1 : k-1), \quad U_{k-1}^B = \tilde{U}(k : N, 1 : k-1). \tag{12.7}$$

Next the submatrix $\hat{U}_k$ is obtained by the formulas

$$\hat{U}_k = \begin{pmatrix} U_{k-1} & \Gamma_k \\ U^B_{k-1} & Z_k \end{pmatrix}, \tag{12.8}$$

where Z_k is determined from the specified part $\tilde{U}$ by

$$Z_k = \tilde{U}(k : N, k) \tag{12.9}$$

and Γ_k is defined as

$$\Gamma_k = \Gamma^{(1)}_k + \Gamma^{(2)}_k, \quad k = 2, \dots, N, \tag{12.10}$$

where

$$\Gamma^{(1)}_k = U_{k-1}E_k, \tag{12.11}$$

the matrix E_k being a solution of the equation

$$(I - (U^B_{k-1})^* U^B_{k-1})E_k = -(U^B_{k-1})^* Z_k, \tag{12.12}$$

and the vector $\Gamma^{(2)}_k$ is an arbitrary solution of the equation

$$U^*_{k-1}\Gamma^{(2)}_k = 0 \tag{12.13}$$

satisfying the condition

$$(\Gamma^{(2)}_k)^*\Gamma^{(2)}_k = I - Z^*_k Z_k - (\Gamma^{(1)}_k)^*\Gamma^{(1)}_k. \tag{12.14}$$

The element $\Gamma^{(1)}_k$ is defined uniquely, i.e., it does not depend on the choice of the solution E_k of equation (12.12)*.*

Finally, we get

$$U = \hat{U}_N. \tag{12.15}$$

Furthermore, if the condition (12.4) *holds and additionally*

$$\|U^B_k\| < 1, \quad k = 1, \dots, N-1, \tag{12.16}$$

then the matrices U_k $(k = 1, \dots, N)$ are nonsingular, the unitary completion U is unique and the elements Γ_k $(k = 1, \dots, N)$ are determined by the formulas

$$\Gamma_k = -U_{k-1}(I - (U^B_{k-1})^* U^B_{k-1})^{-1}(U^B_{k-1})^* Z_k, \quad k = 2, \dots, N. \tag{12.17}$$

Proof. First we prove the sufficiency. Assume that the condition of the theorem holds. Determine the first block column $U(:, 1) = \hat{U}_1$ of the matrix U by the formula (12.5). Using (12.4) with $k = 1$ we get $\hat{U}^*_1\hat{U}_1 = I_{m_1}$. Next, for $k = 2, \dots, N$ we determine successively the submatrices $\hat{U}_k = U(:, 1 : k)$ in such a way that the columns of $\hat{U}_k$ are orthonormal. Suppose that for some k, $2 \le k \le N$, the submatrix $\hat{U}_{k-1}$, has been constructed. We show that one can determine the matrix Γ_k of the size $\nu_{k-1} \times m_k$, where $\nu_{k-1} = \sum_{i=1}^{k-1} m_i$, in such a way that the block column $U'_k = \begin{pmatrix} \Gamma_k \\ Z_k \end{pmatrix}$ is orthogonal to the columns of $\hat{U}_{k-1}$ and $(U'_k)^* U'_k = I_{m_k}$. Then we determine the submatrix $\hat{U}_k$ by the formulas (12.6)–(12.9). By using the representation (12.6) we obtain that the block vector column Γ_k is defined by the relations

$$U_{k-1}^* \Gamma_k + (U_{k-1}^B)^* Z_k = 0, \tag{12.18}$$

$$\Gamma_k^* \Gamma_k + Z_k^* Z_k = I_{m_k}. \tag{12.19}$$

We prove that the system of equations (12.18), (12.19) has a solution which is given by the relations (12.10)–(12.14).

Since the columns of the matrix $\hat{U}_{k-1}$ are orthonormal, by using the representation (12.6) we obtain the equality

$$U_{k-1}^* U_{k-1} = I - (U_{k-1}^B)^* U_{k-1}^B. \tag{12.20}$$

Further, from the representation

$$U_k^D = \begin{pmatrix} U_{k-1}^B & Z_k \end{pmatrix}$$

it follows that

$$I - (U_k^D)^* U_k^D = \begin{bmatrix} I - (U_{k-1}^B)^* U_{k-1}^B & -(U_{k-1}^B)^* Z_k \\ -Z_k^* U_{k-1}^B & I - Z_k^* Z_k \end{bmatrix}. \tag{12.21}$$

The first condition in (12.4) means that $I - (U_k^D)^* U_k^D$ is a positive matrix, so by Lemma 12.1 for the matrix $I - (U_k^D)^* U_k^D$ we conclude that the equation (12.12) has a solution E_k. Moreover, one can check easily that the formula (12.11) yields a solution of the equation (12.18). Indeed, using (12.20) we obtain

$$U_{k-1}^* \Gamma_k^{(1)} = U_{k-1}^* U_{k-1} E_k = (I - (U_{k-1}^B)^* U_{k-1}^B) E_k = -(U_{k-1}^B)^* Z_k.$$

Furthermore, the vector $\Gamma_k^{(1)}$ does not depend on the choice of the solution E_k of the equation (12.12). Indeed, for any E such that $(I - (U_{k-1}^B)^* U_{k-1}^B) E = 0$ by using (12.20) we obtain $U_{k-1}^* U_{k-1} E = 0$ and therefore $U_{k-1} E = 0$. Thus the equation (12.18) has a solution $\Gamma_k^{(1)}$ and, moreover, any solution of (12.18) has the form $\Gamma_k = \Gamma_k^{(1)} + \Gamma_k^{(2)}$, where $\Gamma_k^{(2)}$ is a solution of the equation $U_{k-1}^* \Gamma_k^{(2)} = 0$.

Notice that by virtue of (12.11) and (12.13) we have

$$(\Gamma_k^{(1)})^* \Gamma_k^{(2)} = E^* U_{k-1}^* \Gamma_k^{(2)} = 0$$

and therefore

$$\Gamma_k^* \Gamma_k = (\Gamma_k^{(1)})^* \Gamma_k^{(1)} + (\Gamma_k^{(2)})^* \Gamma_k^{(2)}.$$

Hence, in order to satisfy the equation (12.19) we must determine the block column vector $\Gamma_k^{(2)}$ such that $U_{k-1}^*\Gamma_k^{(2)} = 0$ and (12.14) holds. Using (12.11) and (12.12) one can write the right-hand part of the equality (12.14), $\Delta_k := I_{m_k} - Z_k^*Z_k - (\Gamma_k^{(1)})^*\Gamma_k^{(1)}$, in the form

$$\begin{aligned}\Delta_k &= I_{m_k} - Z_k^*Z_k - E_k^*U_{k-1}^*U_{k-1}E_k \\ &= I_{m_k} - Z_k^*Z_k - [(I-(U_{k-1}^B)^*U_{k-1}^B)E_k]^*E_k = I_{m_k} - Z_k^*Z_k + Z_k^*U_{k-1}^BE_k.\end{aligned}$$

Applying Lemma 12.1 to the matrix $I-(U_k^D)^*U_k^D$ represented in the form (12.21) we conclude that the matrix Δ_k is positive. Set $\mu_k = \operatorname{rank}\Delta_k$; clearly, $0 \le \mu_k \le m_k$. If $\mu_k = 0$ one can take $\Gamma_k^{(2)} = 0$. Assume that $0 < \mu_k \le m_k$. Using the condition (12.4) and the formula (12.1) we get

$$\dim\operatorname{Ker}(I-(U_{k-1}^B)^*U_{k-1}^B) + \dim\operatorname{Ker}\Delta_k \ge m_k.$$

Since $\dim\operatorname{Ker}\Delta_k = m_k - \mu_k$, we get

$$\dim\operatorname{Ker}(I-(U_{k-1}^B)^*U_{k-1}^B) \ge \mu_k.$$

But using (12.20) we get

$$\dim\operatorname{Ker}(U_{k-1}^*U_{k-1}) = \dim\operatorname{Ker}U_{k-1}^* \ge \mu_k.$$

It follows that $\operatorname{Ker}U_{k-1}^*$ contains μ_k orthonormal vector columns, i.e., there exists a $\nu_{k-1}\times\mu_k$ matrix Φ_k such that $U_{k-1}^*\Phi_k = 0$ and $\Phi_k^*\Phi_k = I_{\mu_k}$. Since the $m_k\times m_k$ matrix Δ_k is positive and has the rank μ_k, there exists an $m_k\times\mu_k$ matrix Ψ_k such that $\Delta_k = \Psi_k\Psi_k^*$. Set $\Gamma_k^{(2)} = \Phi_k\Psi_k^*$. One can easily check that $U_{k-1}^*\Gamma_k^{(2)} = 0$ and

$$(\Gamma_k^{(2)})^*\Gamma_k^{(2)} = \Psi_k\Phi_k^*\Phi_k\Psi_k^* = \Psi_k\Psi_k^* = \Delta_k.$$

Finally, in the case $\operatorname{Ker}U_{k-1}^* = \{0\}$, equality (12.20) yields

$$\operatorname{Ker}(I-(U_{k-1}^B)^*U_{k-1}^B) = \{0\}.$$

Hence, by Lemma 12.1, using the second condition in (12.4) we obtain $I - Z_k^*Z_k + Z_k^*U_{k-1}^BE_k = 0$.

Now we prove the necessity. Let U be a unitary completion of the lower triangular part $\tilde U$. For $k=1$ we have $\hat U_1 = \tilde U(:,1) = U(:,1)$ and, hence, $\hat U_1^*\hat U_1 = I_{m_1}$. For $k = 2,\dots,N$ we use the representation

$$U_k^D = \begin{pmatrix} U_{k-1}^B & Z_k \end{pmatrix}$$

and the partition (12.8). The orthonormality of the columns of the matrix $\hat U_k = U(:,1:k)$ implies that

$$\begin{aligned} U_{k-1}^*\Gamma_k + (U_{k-1}^B)^*Z_k &= 0, \\ \Gamma_k^*\Gamma_k + Z_k^*Z_k &= I, \quad k=2,\dots,N. \\ U_{k-1}^*U_{k-1} + (U_{k-1}^B)^*U_{k-1}^B &= I.\end{aligned}$$

One can recast these equalities as

$$-(U_{k-1}^B)^* Z_k = U_{k-1}^* \Gamma_k, \qquad -Z_k^* U_{k-1}^B = \Gamma_k^* U_{k-1},$$
$$I - (U_{k-1}^B)^* U_{k-1}^B = U_{k-1}^* U_{k-1}, \quad I - Z_k^* Z_k = \Gamma_k^* \Gamma_k,$$

i.e.,

$$I - (U_k^D)^* U_k^D = \begin{pmatrix} I - (U_{k-1}^B)^* U_{k-1}^B & -(U_{k-1}^B)^* Z_k \\ -Z_k^* U_{k-1}^B & I - Z_k^* Z_k \end{pmatrix}$$
$$= \begin{pmatrix} U_{k-1}^* \\ \Gamma_k^* \end{pmatrix} \begin{pmatrix} U_{k-1} & \Gamma_k \end{pmatrix}, \quad k = 2, \dots, N.$$

This implies $I - (U_k^D)^* U_k^D = M_k^* M_k,\ k = 2, \dots, N$, where $M_k = \begin{pmatrix} U_{k-1} & \Gamma_k \end{pmatrix}$. Hence, the matrices $I - (U_k^D)^* U_k^D$ $(k = 2, \dots, N)$ are positive. Moreover, since the matrices M_k have the sizes $\nu_{k-1} \times (\nu_{k-1} + m_k)$, we conclude that

$$\dim \operatorname{Ker}(I - (U_k^D)^* U_k^D) \geq m_k, \quad k = 2, \dots, N.$$

Finally, we check that if the conditions (12.16) are satisfied, then the unitary completion U is unique and the formulas (12.17) hold. The conditions (12.16) imply that the matrices $I - (U_{k-1}^B)^* U_{k-1}^B,\ k = 2, \dots, N$, are invertible. Moreover, by using the formulas (12.10)–(12.13) we obtain that the unspecified entries of the unitary completion are determined by the relations

$$\Gamma_k = -U_{k-1}(I - (U_{k-1}^B)^* U_{k-1}^B)^{-1} (U_{k-1}^B)^* Z_k + \Gamma_k^{(2)}, \quad k = 2, \dots, N,$$

where $U_{k-1}^* \Gamma_k^{(2)} = 0$. But the equality (12.20) implies that $\operatorname{Ker} U_{k-1}^* = \{0\}$ and, therefore, $\Gamma_k^{(2)} = 0$. We conclude that the unitary completion U is unique and the relations (12.17) hold. □

Remark. By Lemma 11.3, the condition (12.16) means that the unitary completion U is a strongly regular matrix.

Example 12.3. In this example necessary and sufficient conditions for the existence of a unitary completion of a partially specified scalar matrix with a given diagonal and subdiagonal part will be found using Theorem 12.2.

Let

$$\mathcal{U} = \begin{pmatrix} d_1 & * & * & * & * \\ m_1 & d_2 & * & * & * \\ 0 & m_2 & d_3 & * & * \\ 0 & 0 & m_3 & d_4 & * \\ 0 & 0 & 0 & m_4 & d_5 \end{pmatrix}.$$

The partially specified matrix $\mathcal{U}$ has a unitary completion U if and only if the submatrices

$$U_k^D = \tilde{U}(k:5, 1:k), \quad k = 1, \dots, 5$$

satisfy the conditions (12.4). We will now check when these conditions hold.

For $k = 1$ we have

$$I - (U_k^D)^* U_k^D = 1 - \begin{pmatrix} \overline{d_1} & \overline{m_1} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ m_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 - |d_1|^2 - |m_1|^2$$

and therefore (12.4) reads

$$|d_1|^2 + |m_1|^2 = 1. \tag{12.22}$$

For $k = 2$ we have

$$I - (U_2^D)^* U_2^D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \overline{m_1} & 0 & 0 & 0 \\ \overline{d_2} & \overline{m_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} m_1 & d_2 \\ 0 & m_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - |m_1|^2 & -\overline{m_1} d_2 \\ -\overline{d_2} m_1 & 1 - |d_2|^2 - |m_2|^2 \end{pmatrix}.$$

For a 2×2 matrix the condition (12.4) means that the trace of a matrix is non-negative and the determinant is zero. Hence we get

$$|m_1|^2 + |m_2|^2 + |d_2|^2 \le 2, \quad (1 - |m_1|^2)(1 - |m_2|^2 - |d_2|^2) - |m_1|^2 |d_2|^2 = 0,$$

i.e., the trace is positive if and only if $|m_1|^2 \le 1$ and $|d_2|^2 + |m_2|^2 \le 1$ both hold. The first inequality follows readily from the equality which has been deduced for the case $k = 1$, while the second inequality is new. The kernel of the matrix that we obtained is not null if and only if its determinant is zero, which means

$$(1 - |m_1|^2)(1 - |m_2|^2 - |d_2|^2) - |m_1|^2 |d_2|^2 = 0,$$

or

$$1 - |m_1|^2 - |m_2|^2 + |m_1|^2 |m_2|^2 - |d_2|^2 = 0.$$

Therefore, the conditions (12.4) are satisfied for $k = 1$ and $k = 2$ if and only if

$$1 + |m_1|^2 |m_2|^2 = |m_1|^2 + |m_2|^2 + |d_2|^2 \le 2. \tag{12.23}$$

For $k = 3$ we have

$$I - (U_3^D)^* U_3^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \overline{m_2} & 0 & 0 \\ \overline{d_3} & \overline{m_3} & 0 \end{pmatrix} \begin{pmatrix} 0 & m_2 & d_3 \\ 0 & 0 & m_3 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - |m_2|^2 & -\overline{m_2} d_3 \\ 0 & -\overline{d_3} m_2 & 1 - |d_3|^2 - |m_3|^2 \end{pmatrix}.$$

In the same way as above we get

$$1+|m_2|^2|m_3|^2 = |m_2|^2+|m_3|^2+|d_3|^2 \le 2. \tag{12.24}$$

For $k=4$ we have

$$I-(U_4^D)^*U_4^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \overline{m_3} & 0 \\ \overline{d_4} & \overline{m_4} \end{pmatrix} \begin{pmatrix} 0 & 0 & m_3 & d_4 \\ 0 & 0 & 0 & m_4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-|m_3|^2 & -\overline{m_3}d_4 \\ 0 & 0 & -\overline{d_4}m_3 & 1-|d_4|^2-|m_4|^2 \end{pmatrix}$$

and obtain

$$1+|m_3|^2|m_4|^2 = |m_3|^2+|m_4|^2+|d_4|^2 \le 2. \tag{12.25}$$

Finally, for $k=5$ we get

$$I-(U_5^D)^*U_5^D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \overline{m_4} \\ \overline{d_5} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & m_4 & d_5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1-|m_4|^2 & -\overline{m_4}d_5 \\ 0 & 0 & 0 & -\overline{d_5}m_4 & 1-|d_5|^2 \end{pmatrix}.$$

Here we get

$$|m_4|^2+|d_5|^2 \le 2, \quad (1-|m_4|^2)(1-|d_5|^2)-|m_4|^2|d_5|^2 = 0,$$

which means

$$|m_4|^2+|d_5|^2 = 1. \tag{12.26}$$

It follows that the partially specified matrix $\mathcal{U}$ has a unitary completion U if and only if $m_i,\ i=1,\dots,4$ and $d_i,\ i=1,\dots,5$ satisfy the conditions (12.22)–(12.26).

Since the partially specified matrix $\mathcal{U}$ is a Hessenberg matrix it follows that its unitary completions are unitary Hessenberg matrices.

As a particular case of this example take

$$m_1=m_2=m_3=m_4=1, \quad d_1=d_2=d_3=d_4=d_5=0. \tag{12.27}$$

It is easy to check that these values satisfy all the conditions (12.22)–(12.26), therefore unitary completions exist in this particular case. Moreover, any one of them has the form (7.81), where $\rho_0 = -1$, $|\rho_5| = 1$, and from (12.27) it follows that

$$\mu_k = 1,\ \rho_k = 0, \quad k = 1, \dots, 4,$$

therefore the unitary completion is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \rho \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

with $|\rho| = 1$. ◊

§12.3 Unitary completion via quasiseparable representation

Let $\mathcal{U}$ be a partially specified block matrix with a lower triangular part $\tilde{U}$ with given lower quasiseparable generators and diagonal entries. As above, we consider the problem of completing $\tilde{U}$ to a unitary matrix. By Corollary 7.2, such a completion, if it exists, has upper quasiseparable generators with orders equal to the corresponding rank numbers of the given lower part. The problem is to formulate the conditions of Theorem 12.2 in terms of lower quasiseparable generators and diagonal entries of $\tilde{U}$ and to compute a set of upper quasiseparable generators of the completion.

§12.3.1 Existence theorem

Here we present a version of Theorem 12.2 for partially specified matrices with the lower triangular part given in quasiseparable form.

Theorem 12.4. *Let $\mathcal{U}$ be a partially specified block matrix with entries of sizes $m_i \times m_j$, $i, j = 1, \dots, N$, with a lower triangular part $\tilde{U}$ given via lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r_k $(k = 1, \dots, N-1)$ and diagonal entries $d(k)$ $(k = 1, \dots, N)$. Set*

$$\begin{aligned} \rho_N &= 0, \quad \rho_{k-1} = \min\{m_k + \rho_k,\ r_{k-1}\}, \quad k = N, \dots, 2, \\ \tau_0 &= 0, \quad \tau_k = \min\{m_k + \tau_{k-1},\ r_k\}, \quad k = 1, \dots, N-1. \end{aligned} \tag{12.28}$$

Using the given lower generators $p(k), a(k)$ determine via truncated QR *or* QR′ *factorizations* (*see Section* §7.1) *matrices V_k, X_k, $k = N, \dots, 2$ of sizes $(m_k + \rho_k) \times$*

ρ_{k-1} and $\rho_{k-1}\times r_{k-1}$, respectively, such that $V_k^*V_k = I_{\rho_{k-1}}$ and the relations

$$p(N) = V_N X_N, \quad \begin{pmatrix} p(k) \\ X_{k+1}a(k) \end{pmatrix} = V_k X_k,\ k = N-1,\dots,2 \tag{12.29}$$

hold. Using the given lower generators $q(k), a(k)$ determine via truncated LQ *or* L′Q *factorizations (see Section* §7.1*) matrices $Y_k, F_k,\ k = 1,\dots,N-1$ of sizes $r_k \times \tau_k$ and $\tau_k \times (\tau_{k-1}+m_k)$, respectively, such that $F_kF_k^* = I_{\tau_k}$ and the relations*

$$q(1) = Y_1F_1, \quad \begin{pmatrix} a(k)Y_{k-1} & q(k) \end{pmatrix} = Y_kF_k,\ k = 2,\dots,N-1 \tag{12.30}$$

hold.

Define the matrices A_k $(k = 1,\dots,N)$ by

$$A_1 = \begin{pmatrix} d(1) \\ X_2q(1) \end{pmatrix}; \tag{12.31}$$

$$A_k = \begin{pmatrix} p(k)Y_{k-1} & d(k) \\ X_{k+1}a(k)Y_{k-1} & X_{k+1}q(k) \end{pmatrix}, \quad k = 2,\dots,N-1, \tag{12.32}$$

$$A_N = \begin{pmatrix} p(N)Y_{N-1} & d(N) \end{pmatrix}. \tag{12.33}$$

The partially specified matrix $\mathcal{U}$ has a unitary completion U if and only if the conditions

$$I - A_k^*A_k \geq 0, \quad \dim\operatorname{Ker}(I - A_k^*A_k) \geq m_k, \quad k = 1,\dots,N, \tag{12.34}$$

hold.

Furthermore, if the conditions (12.34) *hold and additionally*

$$\|X_{k+1}Y_k\| < 1, \quad k = 1,\dots,N-1, \tag{12.35}$$

then the unitary completion U is unique.

Moreover, if the conditions (12.34) *and* (12.35) *hold, a set of upper quasiseparable generators $g(i)$ $(i = 1,\dots,N-1)$, $h(j)$ $(j = 2,\dots,N)$, $b(k)$ $(k = 2,\dots,N-1)$ of the matrix U is determined by the formulas*

$$g(1) = -d(1)F_1^*, \quad g(i) = -\begin{pmatrix} p(i)Y_{i-1} & d(i) \end{pmatrix} F_i^*,\ i = 2,\dots,N-1, \tag{12.36}$$

$$h(j) = (I - Y_{j-1}^*X_j^*X_jY_{j-1})^{-1}Y_{j-1}^*(p^*(j)d(j) + a^*(j)X_{j+1}^*X_{j+1}q(j)),$$
$$j = 2,\dots,N-1, \tag{12.37}$$
$$h(N) = (I - Y_{N-1}^*X_N^*X_NY_{N-1})^{-1}Y_{N-1}^*p^*(N)d(N),$$

$$b(k) = \begin{pmatrix} I_{\tau_{k-1}} & -h(k) \end{pmatrix} F_k^*,\ k = 2,\dots,N-1. \tag{12.38}$$

Proof. Using the lower quasiseparable generators determine the matrices Q_k $(k = 1,\dots,N-1)$ and P_k $(k = N,\dots,2)$ via equalities (5.1) and (5.2). By Lemma 5.1, we have

$$\tilde{U}(k+1:N, 1:k) = P_{k+1}Q_k, \quad k = 1,\dots,N-1. \tag{12.39}$$

Using also the equalities $Q_1 = q(1),\ P_N = p(N)$ and the recursions (5.3), (5.4) we get

$$\tilde{U}(1:N,1) = \begin{pmatrix} d(1) \\ P_2 q(1) \end{pmatrix}, \tag{12.40}$$

$$\tilde{U}(k:N,1:k) = \begin{pmatrix} p(k)Q_{k-1} & d(k) \\ P_{k+1}a(k)Q_{k-1} & P_{k+1}q(k) \end{pmatrix}, \quad k = 2,\dots,N-1, \tag{12.41}$$

$$\tilde{U}(N,1:N) = \begin{pmatrix} p(N)Q_{N-1} & d(N) \end{pmatrix}. \tag{12.42}$$

Next using the matrices V_k and F_k we determine the matrices $P_k^{(V)}$ and $Q_k^{(F)}$ by means of the recursion relations

$$P_N^{(V)} = V_N, \quad P_k^{(V)} = \begin{pmatrix} I_{m_k} & 0 \\ 0 & P_{k+1}^{(V)} \end{pmatrix} V_k, \quad k = N-1,\dots,2 \tag{12.43}$$

and

$$Q_1^{(F)} = F_1, \quad Q_k^{(F)} = F_k \begin{pmatrix} Q_{k-1}^{(F)} & 0 \\ 0 & I_{m_k} \end{pmatrix}, \quad k = 2,\dots,N-1. \tag{12.44}$$

Using the equalities $V_k^* V_k = I_{\rho_{k-1}}$ and $F_k F_k^* = I_{\tau_k}$ one can easily check that

$$(P_k^{(V)})^* P_k^{(V)} = I_{\rho_{k-1}}, \quad k = 2,\dots,N \tag{12.45}$$

and

$$Q_k^{(F)} (Q_k^{(F)})^* = I_{\tau_k}, \qquad k = 1,\dots,N-1. \tag{12.46}$$

We prove by induction that

$$Q_k = Y_k Q_k^{(F)}, \quad k = 1,\dots,N-1, \tag{12.47}$$

and

$$P_k = P_k^{(V)} X_k, \quad k = N,\dots,2. \tag{12.48}$$

Using the first equalities in (12.30) and (12.44) we have

$$Q_1 = q(1) = Y_1 F_1 = Y_1 Q_1^{(F)}.$$

Let for some k with $2 \le k \le N-2$ the relation $Q_{k-1} = Y_{k-1} Q_{k-1}^{(F)}$ hold. Using the recursion (5.3) we get

$$Q_k = \begin{pmatrix} a(k)Q_{k-1} & q(k) \end{pmatrix} = \begin{pmatrix} a(k)Y_{k-1}Q_{k-1}^{(F)} & q(k) \end{pmatrix}$$
$$= \begin{pmatrix} a(k)Y_{k-1} & q(k) \end{pmatrix} \begin{pmatrix} Q_{k-1}^{(F)} & 0 \\ 0 & I_{m_k} \end{pmatrix}.$$

From here using the factorization (12.30) and the recursion (12.44) we obtain (12.47).

Using the first equalities in (12.29) and (12.43) we have

$$P_N = p(N) = V_N X_N = P_N^{(V)} X_N.$$

Let for some k with $N-1 \geq k \geq 3$ the relation $P_{k+1} = P_{k+1}^{(V)} X_{k+1}$ hold. Using the recursion (5.4) we get

$$P_k = \begin{pmatrix} p(k) \\ P_{k+1}a(k) \end{pmatrix} = \begin{pmatrix} p(k) \\ P_{k+1}^{(V)} X_{k+1} a(k) \end{pmatrix} = \begin{pmatrix} I_{m_k} & 0 \\ 0 & P_{k+1}^{(V)} \end{pmatrix} \begin{pmatrix} p(k) \\ X_{k+1}a(k) \end{pmatrix}.$$

From here using the factorization (12.29) and the recursion (12.43) we obtain (12.47).

Now combining the relations (12.40)–(12.42) and (12.47), (12.48) we obtain the representations

$$\tilde{U}(1:N,1) = \begin{pmatrix} I_{m_1} & 0 \\ 0 & P_2^{(V)} \end{pmatrix} A_1,$$
$$\tilde{U}(k:N,1:k) = \begin{pmatrix} I_{m_k} & 0 \\ 0 & P_{k+1}^{(V)} \end{pmatrix} A_k \begin{pmatrix} Q_{k-1}^{(F)} & 0 \\ 0 & I_{m_k} \end{pmatrix}, \quad k = 2,\dots,N-1,$$
$$\tilde{U}(N,1:N) = A_N \begin{pmatrix} Q_{N-1}^{(F)} & 0 \\ 0 & I_{m_N} \end{pmatrix}$$

with the matrices $A_k,\ k=1,\dots,N$ defined in (12.31)–(12.33). From here using the equalities (12.45), (12.46) we conclude that the conditions (12.4) of Theorem 12.2 and the conditions (12.34) are equivalent. Hence, by Theorem 12.2, the partially specified matrix $\mathcal{U}$ has a unitary completion U if and only if the conditions (12.34) are satisfied.

Combining the relations (12.39) and (12.47), (12.48) we obtain the representations

$$\tilde{U}(k+1:N,1:k) = P_{k+1}^{(V)} X_{k+1} Y_k Q_k^{(F)}, \quad k = 1,\dots,N-1. \tag{12.49}$$

From here using the equalities (12.45), (12.46) we conclude that the conditions (12.16) of Theorem 12.2 and the conditions (12.35) are equivalent. Hence, if (12.35) holds, then by Theorem 12.2 the unitary completion U is unique.

To compute upper quasiseparable generators of the matrix U we apply the formula (12.17). As in Theorem 12.4, we set

$$U_k = U(1:k,1:k), \quad k=1,\dots,N,$$
$$U_k^B = \tilde{U}(k+1:N,1:k),\ k=1,\dots,N-1, \quad Z_k = \tilde{Z}(k:N,k),\ k=2,\dots,N.$$

Using (12.49) and (12.45) we have

$$(U_{k-1}^B)^* U_{k-1}^B = (Q_{k-1}^{(F)})^* Y_{k-1}^* X_k^* X_k Y_{k-1} Q_{k-1}^{(F)}, \quad k=2,\dots,N.$$

Next, the inversion formula (1.70) and the equalities (12.46) yield

$$(I-(U^B_{k-1})^*U^B_{k-1})^{-1}=I+(Q^{(F)}_{k-1})^*Y^*_{k-1}X^*_k(I-X_kY_{k-1}Y^*_{k-1}X^*_k)^{-1}X_kY_{k-1}Q^{(F)}_{k-1},$$
$$k=2,\dots,N,$$

and in conjunction with (12.49) and (12.46) we obtain

$$\begin{aligned}&(I-(U^B_{k-1})^*U^B_{k-1})^{-1}(U^B_{k-1})^*\\ &\quad=(Q^{(F)}_{k-1})^*[I+Y^*_{k-1}X^*_k(I-X_kY_{k-1}Y^*_{k-1}X^*_k)^{-1}X_kY_{k-1}]Y^*_{k-1}X^*_k(P^{(V)}_k)^*,\\ &\qquad k=2,\dots,N.\end{aligned}$$

Using the formula (1.70) we get

$$(I-(U^B_{k-1})^*U^B_{k-1})^{-1}(U^B_{k-1})^*=(Q^{(F)}_{k-1})^*(I-Y^*_{k-1}X^*_kX_kY_{k-1})^{-1}Y^*_{k-1}X^*_k(P^{(V)}_k)^*,$$
$$k=2,\dots,N.$$

Thus, using (12.17) we get

$$U(1:k-1,k)=G_{k-1}h(k),\quad k=2,\dots,N, \tag{12.50}$$

with

$$G_k=-U_k(Q^{(F)}_k)^*,\quad k=1,\dots,N-1, \tag{12.51}$$

and

$$h(k)=(I-Y^*_{k-1}X^*_kX_kY_{k-1})^{-1}Y^*_{k-1}X^*_k(P^{(V)}_k)^*Z_k,\quad k=2,\dots,N. \tag{12.52}$$

Let us check that the elements $h(k)$ are determined by the formulas (12.37). Using the formula (5.10) we have

$$Z_k=\begin{pmatrix} d(k)\\ P_{k+1}q(k)\end{pmatrix},\ k=2,\dots,N-1,\quad Z_N=d(N).$$

Using (12.48) we get

$$Z_k=\begin{pmatrix} I_{m_k} & 0\\ 0 & P^{(V)}_{k+1}\end{pmatrix}\begin{pmatrix} d(k)\\ X_{k+1}q(k)\end{pmatrix},\quad k=2,\dots,N-1,\quad Z_N=d(N),$$

whence, by (12.43) and (12.45),

$$(P^{(V)}_k)^*Z_k=V^*_k\begin{pmatrix} d(k)\\ X_{k+1}q(k)\end{pmatrix},\quad k=2,\dots,N-1,\quad (P^{(V)}_N)^*Z_N=V^*_Nd(N).$$

Next, using the factorizations (12.29) we obtain

$$\begin{aligned}X^*_k(P^{(V)}_k)^*Z_k&=X^*_kV^*_k\begin{pmatrix} d(k)\\ X_{k+1}q(k)\end{pmatrix}\\ &=p^*(k)d(k)+a^*(k)X^*_{k+1}X_{k+1}q(k),\\ &\quad k=2,\dots,N-1,\end{aligned} \tag{12.53}$$

$$X^*_N(P^{(V)}_N)^*Z_N=p^*(N)d(N). \tag{12.54}$$

Inserting (12.53) and (12.54) in (12.52) we obtain the formulas (12.37) for the elements $h(j)$ $(j = 2, \ldots, N)$.

It remains to check that the matrices G_k $(k = 1, \ldots, N-1)$ in (12.51) satisfy the recursion relations

$$G_1 = g(1), \quad G_k = \begin{pmatrix} G_{k-1}b(k) \\ g(k) \end{pmatrix}, \ k = 2, \ldots, N-1, \tag{12.55}$$

with the elements $g(k)$ $(k = 1, \ldots, N-1)$ and $b(k)$ $(k = 2, \ldots, N-1)$ defined in (12.36) and (12.38). By Lemma 5.6, this together with (12.50) implies that $g(i)$ $(i = 1, \ldots, N-1)$, $h(j)$ $(j = 2, \ldots, N)$, $b(k)$ $(k = 2, \ldots, N-1)$ are the upper quasiseparable generators of the matrix U. Using (12.44) with $k = 1$ we have

$$G_1 = -U(1,1)(Q_1^{(F)})^* = -d(1)F_1^*.$$

For $k = 2, \ldots, N-1$ we use the formulas (12.50), (12.51) and (5.11) and obtain the partitions

$$U_k = \begin{pmatrix} U_{k-1} & G_{k-1}h(k) \\ p(k)Q_{k-1} & d(k) \end{pmatrix}.$$

Finally, using (12.44), (12.51), (12.47) and (12.46) we get

$$G_k = \begin{pmatrix} G_{k-1} & -G_{k-1}h(k) \\ -p(k)Y_{k-1} & -d(k) \end{pmatrix} F_k^* = \begin{pmatrix} G_{k-1}b(k) \\ g(k) \end{pmatrix}. \qquad \square$$

§12.3.2 Diagonal correction for scalar matrices

Here for scalar matrices we assume that the conditions (12.35) of Theorem 12.4 are valid and derive other necessary and sufficient conditions for the existence of a unitary completion U of the specified lower triangular part $\tilde{U}$. We suppose that the lower quasiseparable generators of $\tilde{U}$ are fixed and the conditions are imposed on the diagonal entries.

Theorem 12.5. *Let $\mathcal{U}$ be a partially specified scalar matrix with a lower triangular part $\tilde{U}$ given via lower quasiseparable generators $p(i)$ $(i = 2, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N-1)$, $a(k)$ $(k = 2, \ldots, N-1)$ of orders r_k $(k = 1, \ldots, N-1)$ and diagonal entries $d(k)$ $(k = 1, \ldots, N)$. Set*

$$\rho_N = 0, \quad \rho_{k-1} = \min\{1 + \rho_k, \ r_{k-1}\}, \quad k = N-1, \ldots, 2,$$
$$\tau_0 = 0, \qquad \tau_k = \min\{1 + \tau_{k-1}, \ r_k\}, \quad k = 1, \ldots, N-1.$$

Using these lower generators determine the matrices V_k, X_k, $k = N, \ldots, 2$, of sizes $(1+\rho_k) \times \rho_{k-1}$ and $\rho_{k-1} \times r_{k-1}$ such that $V_k^* V_k = I_{\rho_{k-1}}$ via truncated QR *or* QR′ *factorizations (see Section §7.1)*

$$p(N) = V_N X_N, \quad \begin{pmatrix} p(k) \\ X_{k+1}a(k) \end{pmatrix} = V_k X_k, \ k = N-1, \ldots, 2 \tag{12.56}$$

and the matrices Y_k, F_k *of sizes* $r_k \times \tau_k$ *and* $\tau_k \times (\tau_{k-1} + m_k)$ *such that* $F_k F_k^* = I_{\tau_k}$ *via* LQ *or* L′Q *factorizations (see Section* §7.1)

$$q(1) = Y_1 F_1, \quad \begin{pmatrix} a(k)Y_{k-1} & q(k) \end{pmatrix} = Y_k F_k, \; k = 2, \dots, N-1. \tag{12.57}$$

Assume that the conditions

$$\|X_{k+1} Y_k\| < 1, \quad k = 1, \dots, N-1, \tag{12.58}$$

hold.

The partially specified matrix $\mathcal{U}$ *has a unique unitary completion* U *if and only if the conditions*

$$|d(k) + a_k| = b_k, \quad k = 1, \dots, N, \tag{12.59}$$

with

$$a_1 = 0, \quad a_k = \frac{g_k^*(I - \Delta_k)^{-1} f_k}{1 + f_k^*(I - \Delta_k)^{-1} f_k}, \; k = 2, \dots, N-1, \quad a_N = 0, \tag{12.60}$$

$$\begin{gathered} b_1^2 = 1 - h_1^* h_1, \\ b_k^2 = \frac{1 - g_k^*(I - \Delta_k)^{-1} g_k - h_k^* h_k}{1 + f_k^*(I - \Delta_k)^{-1} f_k} + \frac{|g_k^*(I - \Delta_k)^{-1} f_k|^2}{(1 + f_k^*(I - \Delta_k)^{-1} f_k)^2}, \; k = 2, \dots, N-1, \\ b_N^2 = 1 - f_N^* f_N, \end{gathered} \tag{12.61}$$

where

$$\begin{aligned} f_k &= Y_{k-1}^* p^*(k), \; k = 2, \dots, N, \quad h_k = X_{k+1} q(k), \quad k = 1, \dots, N-1, \\ g_k &= Y_{k-1}^* a^*(k) X_{k+1}^* h_k, \quad \Delta_k = Y_{k-1}^* X_k^* X_k Y_{k-1}, \quad k = 2, \dots, N-1, \end{aligned} \tag{12.62}$$

hold.

Proof. By Theorem 12.4, the partially specified matrix $\mathcal{U}$ has a unitary completion if and only if the conditions (12.34) hold, and if this is the case the unitary completion is unique. For a matrix with scalar entries the conditions (12.34) are equivalent to the conditions

$$\|A_k\| = 1, \quad k = 1, \dots, N, \tag{12.63}$$

where the matrices A_k are defined in (12.31)–(12.33). Using (12.31) we obtain the condition (12.63) with $k = 1$ in the form

$$|d(1)|^2 = 1 - h_1^* h_1.$$

Similarly, using (12.33) we obtain the condition (12.63) with $k = N$ in the form

$$|d(N)|^2 = 1 - f_N^* f_N.$$

For $k = 2, \dots, N-1$ we use the fact that the condition (12.63) is equivalent to the condition that the matrix $I - A_k^* A_k$ is positive and singular. Using the formula (12.32) we get

$$A_k^* A_k = \begin{pmatrix} Y_{k-1}^* p^*(k) & Y_{k-1}^* a^*(k) X_{k+1}^* \\ d^*(k) & q^*(k) X_{k+1}^* \end{pmatrix} \begin{pmatrix} p(k) Y_{k-1} & d(k) \\ X_{k+1} a(k) Y_{k-1} & X_{k+1} q(k) \end{pmatrix}.$$

By (12.56), we have

$$p^*(k)p(k) + a^*(k) X_{k+1}^* X_{k+1} a(k) = X_k^* X_k$$

and hence using the notations (12.62) we get

$$I - A_k^* A_k = \begin{pmatrix} I - \Delta_k & -(f_k d(k) + g_k) \\ -(d^*(k) f_k^* + g_k^*) & 1 - (|d(k)|^2 + h_k^* h_k) \end{pmatrix}.$$

By the condition (12.58), the matrix Δ_k is positive definite. Hence, using the equality (12.1) from Lemma 12.1 we conclude that the matrix $I - A_k^* A_k$ is positive and singular if and only if

$$(d^*(k) f_k^* + g_k^*)(I - \Delta_k)^{-1}(f_k d(k) + g_k) + |d(k)|^2 + h_k^* h_k = 1. \tag{12.64}$$

Setting

$$\alpha_k = f_k^*(I - \Delta_k)^{-1} f_k, \quad \beta_k = g_k^*(I - \Delta_k)^{-1} f_k, \quad \gamma_k = g_k^*(I - \Delta_k)^{-1} g_k + h_k^* h_k$$

we rewrite (12.64) in the form

$$(1 + \alpha_k)|d(k)|^2 + 2\mathrm{Re}(d(k)\beta_k) = 1 - \gamma_k,$$

which is equivalent to

$$\left| d(k) + \frac{|\beta_k|}{1 + \alpha_k} \right|^2 = \frac{1 - \gamma_k}{1 + \alpha_k} + \frac{|\beta_k|^2}{(1 + \alpha_k)^2}.$$

This means (12.59) with the numbers a_k, b_k as in (12.60), (12.5). □

§12.4 Comments

In this chapter we extended to block matrices the results obtained in [5] for matrices with scalar entries. Under the conditions (12.4), (12.16) the formula for unitary completion in the factorized form was obtained by H. Dym and I. Gohberg in [11].

Part III

Quasiseparable Representations of Matrices, Descriptor Systems with Boundary Conditions and First Applications

Introduction to Part III

In this part we study the interplay between the quasiseparable and semiseparable representations of matrices and discrete time variant systems with boundary conditions. This part contains the transformation of matrices into descriptor systems with boundary conditions and the deduction from the latter of different fast algorithms. Note that the mentioned transformation allows to represent semi- and quasi-separable representations as a type of forwards and/or backwards recursion relations. As applications we describe the first fast algorithms for inversion of matrices and fast algorithms for multiplication of matrices. The main results are expressed in terms of the appropriate generators.

Chapter 13

Quasiseparable Representations and Descriptor Systems with Boundary Conditions

In this chapter we show that the quasiseparable representation of a matrix is closely connected with the treatment of this matrix as a matrix of the input-output operator of a discrete-time varying linear system with boundary conditions.

§13.1 The algorithm of multiplication by a vector

Here we derive a fast linear complexity algorithm to compute a product of a matrix in the quasiseparable form by a vector. In the subsequent section we show that the relations used in this algorithm form a discrete time descriptor system with homogeneous boundary conditions

Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a matrix of block entries of sizes $m_i \times n_j$ with quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$; $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$; $d(i)$ $(i = 1, \dots, N)$ of orders r_k^L, r_k^U $(k = 1, \dots, N-1)$. This means that the matrix A has the quasiseparable representation

$$A_{ij} = \begin{cases} p(i)a^{>}_{ij}q(j), & 1 \le j < i \le N, \\ d(i), & 1 \le i = j \le N, \\ g(i)b^{<}_{ij}h(j), < quad 1 \le i < j \le N. \end{cases} \tag{13.1}$$

The multiplication of this matrix by a vector may be performed as follows. Let $x = \mathrm{col}(x(i))_{i=1}^{N}$ be a vector with column coordinates $x(i)$ of sizes n_i. The product $y = Ax$ is a vector $y = \mathrm{col}(y(i))_{i=1}^{N}$ with column coordinates $y(i)$ of sizes m_i. The vector y is found as $y = y_L + y_D + y_U$, where $y_L = A_L x$, $y_D = A_D x$,

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$y_U = A_U x$ and $A_L,\ A_D,\ A_U$ are correspondingly the strictly lower triangular, the diagonal and the strictly upper triangular parts of the matrix A.

For y_L one has $y_L(1) = 0$ and for $i \geq 2$, using the first branch of the relations (13.1), one obtains

$$y_L(i) = \sum_{j=1}^{i-1} A_{ij} x(j) = \sum_{j=1}^{i-1} p(i) a^{>}_{ij} q(j) x(j) = p(i)\chi_i,$$

where

$$\chi_i = \sum_{j=1}^{i-1} a^{>}_{i,j} q(j) x(j).$$

One can see that the variable χ_i satisfies the following relations: $\chi_2 = q(1)x(1)$ and for $i = 2, \ldots, N-1$,

$$\chi_{i+1} = \sum_{j=1}^{i} a^{>}_{i+1,j} q(j) x(j) = \sum_{j=1}^{i-1} a^{>}_{i+1,j} q(j) x(j) + a^{>}_{i+1,i} q(i) x(i).$$

Using the equality (4.7) with $i+1$ instead of i and $k = i-1$ one gets $a^{>}_{i+1,j} = a(i) a^{>}_{ij}$. Using also the fact that $a^{>}_{i+1,i} = I_{r^L_i}$ one obtains the recursion

$$\chi_{i+1} = a(i) \sum_{j=1}^{i-1} a^{>}_{ij} q(j) x(j) + q(i) x(i) = a(i)\chi_i + q(i) x(i).$$

For y_U one has $y_U(N) = 0$ and for $i \leq N-1$, using the third branch of the relations (13.1), one obtains

$$y_U(i) = \sum_{j=i+1}^{N} A_{ij} x(j) = \sum_{j=i+1}^{N} g(i) b^{<}_{ij} h(j) x(j) = g(i)\eta_i,$$

where

$$\eta_i = \sum_{j=i+1}^{N} b^{<}_{i,j} h(j) x(j).$$

One has $\eta_{N-1} = h(N)x(N)$ and from the equalities (4.8) and $b^{<}_{i-1,i} = I_{r^U_{i-1}}$ it follows that η_i satisfies the recursion relations

$$\eta_{i-1} = \sum_{j=i}^{N} b^{<}_{i-1,j} h(j) x(j) = b(i) \sum_{j=i+1}^{N} b^{<}_{ij} h(j) x(j) + b^{<}_{i-1,i} h(i) x(i) = b(i)\eta_i + h(i) x(i).$$

For y_D it is obvious that $y_D(i) = d(i)x(i),\ i = 1, \ldots, N.$

From these relations one obtains the following algorithm for computing the product $y = Ax$.

Algorithm 13.1. (Multiplication by a vector)

1. Start with $y_L(1) = 0,\ \chi_2 = q(1)x(1),\ y_L(2) = p(2)\chi_2$ and for $i = 3, \dots, N$ compute recursively

$$\chi_i = a(i-1)\chi_{i-1} + q(i-1)x(i-1), \tag{13.2}$$
$$y_L(i) = p(i)\chi_i. \tag{13.3}$$

2. Compute for $i = 1, \dots, N$

$$y_D(i) = d(i)x(i). \tag{13.4}$$

3. Start with $y_U(N) = 0,\ \eta_{N-1} = h(N)x(N),\ y_U(N-1) = g(N-1)\eta_{N-1}$ and for $i = N-2, \dots, 1$ compute recursively

$$\eta_i = b(i+1)\eta_{i+1} + h(i+1)x(i+1), \tag{13.5}$$
$$y_U(i) = g(i)\eta_i. \tag{13.6}$$

4. Compute the vector y

$$y = y_L + y_D + y_U. \tag{13.7}$$

This is a generalization of Algorithm 1.9.

The complexity of the operations used in Algorithm 13.1 is determined as follows.

1. The formula (13.2): $r^L_{i-1}r^L_{i-2} + r^L_{i-1}n_{i-1}$ multiplications and $(r^L_{i-1} - 1)r^L_{i-2} + r^L_{i-1}(n_{i-1} - 1)$ additions.
2. The formula (13.3): $m_i r^L_{i-1}$ multiplications and $m_i(r^L_{i-1} - 1)$ additions.
3. The formula (13.4): $m_i n_i$ multiplications and $m_i(n_i - 1)$ additions.
4. The formula (13.5): $r^U_i r^U_{i+1} + r^U_i n_{i+1}$ multiplications and $r^U_i(r^U_{i+1} - 1) + r^U_i(n_{i+1} - 1)$ additions.
5. The formula (13.6): $m_i r^U_i$ multiplications and $m_i(r^U_i - 1)$ additions.

Indeed, the operation $a(i-1)\chi_{i-1}$ is a product of an $r^L_{i-2} \times r^L_{i-1}$ matrix $a(i-1)$ by an r^L_{i-1}-dimensional vector χ_{i-1} and hence it requires $r^L_{i-1}r^L_{i-2}$ multiplications and $(r^L_{i-1}-1)r^L_{i-2}$ additions. The operation $q(i-1)x(i-1)$ is a product of an $r^L_{i-1} \times n_{i-1}$ matrix $q(i-1)$ by an n_{i-1}-dimensional vector $x(i-1)$ and hence it requires $r^L_{i-1}n_{i-1}$ multiplications and $r^L_{i-1}(n_{i-1} - 1)$ additions. Thus, the total complexity for computation of the value χ_i is less than $2(r^L_{i-1}r^L_{i-2} + r^L_{i-1}n_{i-1})$ operations. In the same way one obtains complexities for the computation of the other variables of the algorithm. We conclude that the total complexity of Algorithm 13.1 is

$$c < 2\sum_{k=1}^{N}[m_k(r^L_{k-1} + r^U_k + n_k) + n_{k-1}r^L_{k-1} + n_{k+1}r^U_k + r^L_{k-1}r^L_{k-2} + r^U_k r^U_{k+1}]. \tag{13.8}$$

Let the sizes of blocks $m_k,\ n_k$ and the orders of generators $r_k^L,\ r_k^U$ be bounded by the numbers m and r, respectively, i.e., $m_k, n_k \le m, \quad r_k^L, r_k^U \le r$. In this case (13.8) yields the estimate

$$c < 2N(4mr + 2r^2 + m^2).$$

Thus for a matrix with quasiseparable representation the multiplication by a vector costs $O(N)$ operations in contrast to $O(N^2)$ for a matrix in general form.

§13.2 Descriptor systems with homogeneous boundary conditions

Let $A = \{A_{ij}\}_{i,j=1}^N$ be a matrix with quasiseparable generators

$$\begin{array}{lll} p(i)\ (i = 2, \dots, N), & q(j)\ (j = 1, \dots, N-1), & a(k)\ (k = 2, \dots, N-1) \\ g(i)\ (i = 1, \dots, N-1), & h(j)\ (j = 2, \dots, N), & \\ b(k)\ (k = 2, \dots, N-1); & d(i)\ (i = 1, \dots, N) & \end{array}$$

of orders r_k^L, r_k^U $(k = 1, \dots, N-1)$. Let x, y be vectors such that $y = Ax$. Consider in detail the operations used in Algorithm 13.1. Using formulas (13.3), (13.4), (13.6) one has

$$y(k) = p(k)\chi_k + g(k)\eta_k + d(k)x(k), \quad k = 1, \dots, N. \tag{13.9}$$

Here the auxiliary variables χ_k, η_k are determined via the recursion relations

$$\chi_1 = 0, \quad \chi_2 = q(1)x(1), \quad \chi_i = a(i-1)\chi_{i-1} + q(i-1)x(i-1),\ i = 3, \dots, N, \tag{13.10}$$

$$\eta_N = 0, \quad \eta_{N-1} = h(N)x(N), \quad \eta_i = b(i+1)\eta_{i+1} + h(i+1)x(i+1),\ i = N-2, \dots, 1. \tag{13.11}$$

Take r_0^L, r_N^U to be arbitrary nonnegative integers and $p(1), g(N), a(1), b(N)$ to be arbitrary matrices of sizes $m_1 \times r_0^L, m_N \times r_N^U, r_1^L \times r_0^L, r_{N-1}^U \times r_N^U$, respectively. One can rewrite relations (13.10), (13.11) in the form

$$\chi_1 = 0_{r_0^L}, \quad \chi_{k+1} = a(k)\chi_k + q(k)x(k), \quad k = 1, \dots, N-1, \tag{13.12}$$

$$\eta_N = 0_{r_N^U}, \quad \eta_{k-1} = b(k)\eta_k + h(k)x(k), \quad k = N, \dots, 2. \tag{13.13}$$

Relations (13.9), (13.12), (13.13) together form what is called a *discrete-time descriptor system with homogeneous boundary conditions*:

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1, \dots, N-1, \\ \eta_{k-1} = b(k)\eta_k + h(k)x(k), & k = N, \dots, 2, \\ y(k) = p(k)\chi_k + g(k)\eta_k + d(k)x(k), & k = 1, \dots, N, \\ \chi_1 = 0, \quad \eta_N = 0. & \end{cases} \tag{13.14}$$

Here the vectors $x(k)$ $(k=1,\dots,N)$ are called the *input* of the system, the vectors $y(k)$ $(k=1,\dots,N)$ are called the *output*, the vectors χ_k and η_k of sizes r^L_{k-1} and r^U_k are called the *state space variables* of the system. The mapping from $x=(x(k))_{k=1}^N$ to $y=(y(k))_{k=1}^N$ is a linear transformation which transforms the input of the system into the output. This transformation is called the *input-output operator of the system*

For convenience of notation, we define also the matrices $q(N), h(1), a(N), b(1)$ as arbitrary $r^L_N \times n_N, r^U_0 \times n_1, r^L_N \times r^L_{N-1}, r^U_0 \times r^U_1$ matrices respectively. Thus one obtains the following.

Theorem 13.2. *Let A be a block matrix with quasiseparable generators*

$$p(k), q(k), a(k); g(k), h(k), b(k); d(k) \quad k=1,\dots,N.$$

Then A is a matrix of the input-output operator of the system (13.14) *with coefficients equal to the corresponding quasiseparable generators of A.*

The inverse statement is also true.

Theorem 13.3. *Suppose that a system* (13.14) *is given. Then the matrix A with quasiseparable generators $p(k)$, $q(k)$, $a(k)$; $g(k)$, $h(k)$, $b(k)$; $d(k)$ $(k=1,\dots,N)$ which are equal to the corresponding coefficients of the system, is a matrix of the input-output operator of the system* (13.14).

Proof. Let x be an input of the system. One can easily prove by induction that the solution of the first equation in (13.14) is given by

$$\chi_k = \sum_{j=1}^{k-1} a^{>}_{kj} q(j) x(j), \quad k=1,\dots,N. \tag{13.15}$$

Indeed, for $k=1$ the relation (13.15) follows directly from $\chi_1=0$. Let for some k, $k\geq 1$, the relation (13.15) hold. Using the first equation from (13.14) and the equalities $a^{>}_{k+1,k} = I_{r^L_k}$, $a^{>}_{k+1,j} = a(k) a^{>}_{k,j}$, one gets

$$\begin{aligned}\chi_{k+1} &= a(k) \sum_{j=1}^{k-1} a^{>}_{kj} q(j) x(j) + q(k) x(k) \\ &= \sum_{j=1}^{k-1} a^{>}_{k+1,j} q(j) x(j) + a^{>}_{k+1,k} q(k) x(k) = \sum_{j=1}^{k} a^{>}_{k+1,j} q(j) x(j).\end{aligned}$$

Similarly, the solution of the second equation in (13.14) is given by

$$\eta_k = \sum_{j=k+1}^{N} b^{<}_{kj} h(j) x(j), \quad k=N,\dots,1. \tag{13.16}$$

Indeed, for $k = N$ the relation (13.16) follows directly from $\eta_N = 0$. Let for some $k, k \le N$ the relation (13.16) hold. Using the second equation from (13.14) and the equalities $b^<_{k-1,k} = I,\ b^<_{k-1,j} = b(k)b^<_{k,j}$, one gets

$$\begin{aligned}\eta_{k-1} &= b(k) \sum_{j=k+1}^{N} b^<_{kj} h(j)x(j) + h(k)x(k)\\ &= \sum_{j=k+1}^{N} b^<_{k-1,j} h(j)x(j) + b^<_{k-1,k} h(k)x(k) = \sum_{j=k}^{N} b^<_{kj} h(j)x(j).\end{aligned}$$

Thus for the output y one obtains

$$y(k) = p(k)\sum_{j=1}^{k-1} a^>_{kj} q(j)x(j) + d(k)x(k) + g(k)\sum_{j=k+1}^{N} b^<_{kj} h(j)x(j), \quad k = 1, \dots, N.$$

Therefore, one obtains $y = Ax$, with the matrix A given by (13.1). □

Remark 13.4. The system (13.14) may be transformed to the form

$$\begin{cases} A_k u_{k+1} = B_k u_k + C_k x(k), & k = 1, \dots, N,\\ y(k) = E_k u_k + F_k u_{k+1} + d(k)x(k), & k = 1, \dots, N,\\ Pu_1 = 0, \quad Qu_{N+1} = 0, \end{cases} \tag{13.17}$$

with

$$\begin{gathered} z_k = \eta_{k-1}, \qquad u_k = \begin{pmatrix} \chi_k \\ z_k \end{pmatrix},\\ A_k = \begin{pmatrix} I & 0 \\ 0 & -b(k) \end{pmatrix}, \quad B_k = \begin{pmatrix} a(k) & 0 \\ 0 & -I \end{pmatrix}, \quad C_k = \begin{pmatrix} q(k) \\ h(k) \end{pmatrix},\\ E_k = \begin{pmatrix} p(k) & 0 \end{pmatrix}, \quad F_k = \begin{pmatrix} 0 & g(k) \end{pmatrix},\\ P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \end{gathered} \tag{13.18}$$

Conversely, a system (13.17) with E_k, F_k, P, Q as in (13.18) and

$$A_k = \begin{pmatrix} \tilde{a}(k) & 0 \\ 0 & \tilde{b}(k) \end{pmatrix}, \quad B_k = \begin{pmatrix} \tilde{c}(k) & 0 \\ 0 & \tilde{d}(k) \end{pmatrix},$$

where $\tilde{a}(k), \tilde{d}(k)$ are invertible matrices, may be transformed to the form (13.14). The form (13.17) of a descriptor system with boundary conditions is used often in system theory.

Applying Theorem 13.2 and Theorem 13.3 to triangular matrices one obtains the following statements.

Corollary 13.5. *A matrix A is a lower triangular matrix with lower quasiseparable generators $p(k), q(k), a(k)$ $(k = 1, \dots, N)$ and diagonal entries $d(k)$ $(k = 1, \dots, N)$ if and only if A is a matrix of the input-output operator of the system*

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1, \dots, N-1, \\ y(k) = p(k)\chi_k + d(k)x(k), & k = 1, \dots, N, \\ \chi_1 = 0. \end{cases}$$

Corollary 13.6. *A matrix A is an upper triangular matrix with upper quasiseparable generators $g(k), h(k), b(k)$ $(k = 1, \dots, N)$ and diagonal entries $d(k)$ $(k = 1, \dots, N)$ if and only if A is a matrix of the input-output operator of the system*

$$\begin{cases} \eta_{k-1} = b(k)\eta_k + h(k)x(k), & k = N, \dots, 2 \\ y(k) = g(k)\eta_k + d(k)x(k), & k = 1, \dots, N \\ \eta_N = 0. \end{cases}$$

§13.3 Examples

Example 13.7. Consider the $N \times N$ matrix from Example 5.13,

$$A = \begin{pmatrix} d & 1 & 2 & 3 & \cdots & N-2 & N-1 \\ 1 & d & 4 & 6 & \cdots & 2(N-2) & 2(N-1) \\ 2 & 4 & d & 9 & \cdots & 3(N-2) & 3(N-1) \\ 3 & 6 & 9 & d & \cdots & 4(N-2) & 4(N-1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ N-2 & 2(N-2) & 3(N-2) & 4(N-2) & \cdots & d & (N-1)^2 \\ N-1 & 2(N-1) & 3(N-1) & 4(N-1) & \cdots & (N-1)^2 & d \end{pmatrix}.$$

The following lower quasiseparable generators, which have been obtained in the same Example 5.13, and the following diagonal entries can be used:

$$\begin{gathered} p(i) = i-1, i = 2, \dots, N, \quad q(j) = j, j = 1, \dots, N-1, \quad a(k) = 1, k = 2, \dots, N-1, \\ h(i) = i-1, i = 2, \dots, N, \quad g(j) = j, j = 1, \dots, N-1, \quad b(k) = 1, k = 2, \dots, N-1, \\ d(k) = d, k = 1, \dots, N. \end{gathered}$$

Then the descriptor system with boundary conditions (13.14) becomes

$$\begin{aligned} \chi_{k+1} &= a(k)\chi_k + q(k)x(k) = \chi_k + kx(k), & k &= 1, \dots, N-1, \\ \eta_{k-1} &= b(k)\eta_k + h(k)x(k) = \eta_k + (k-1)x(k), & k &= N, \dots, 2, \\ y(k) &= p(k)\chi_k + g(k)\eta_k + d(k)x(k) = (k-1)\chi_k + k\eta_k + dx(k), & k &= 1, \dots, N, \\ \chi_1 &= 0, \quad \eta_N = 0. \end{aligned}$$

◊

Example 13.8. Consider the $N \times N$ matrix

$$A = \begin{pmatrix} \mu & h & h & h & \cdots & h \\ p & \alpha & 0 & 0 & \cdots & 0 \\ p & 0 & \alpha & 0 & \cdots & 0 \\ p & 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p & 0 & 0 & 0 & \cdots & \alpha \end{pmatrix}.$$

For A one can use the following quasiseparable generators

$$\begin{aligned} &p(i) = p,\ i = 2, \dots, N, \quad q(1) = 1,\ \ q(j) = 0,\ j = 2, \dots, N-1, \\ &a(k) = 1,\ k = 2, \dots, N-1, \quad g(1) = 1,\ \ g(j) = 0,\ j = 2, \dots, N-1, \\ &h(i) = h,\ i = 2, \dots, N,\ \ b(k) = 1,\ k = 2, \dots, N-1, \\ &d(1) = \mu,\ d(k) = \alpha,\ k = 2, \dots, N. \end{aligned}$$

Then the descriptor system with boundary conditions (13.14) reads

$$\begin{aligned} \chi_1 &= 0, \quad \eta_N = 0, \quad \chi_2 = a(1)\chi_1 + q(1)x(1) = x(1), \\ \chi_{k+1} &= a(k)\chi_k + q(k)x(k) = 1 \cdot \chi_k + 0 \cdot x(k) = x(1), \quad k = 2, \dots, N-1, \\ \eta_{k-1} &= b(k)\eta_k + h(k)x(k) = \eta_k + hx(k), \quad k = N, \dots, 2, \\ y(k) &= p(k)\chi_k + g(k)\eta_k + d(k)x(k) = px(1) + \eta_k + \alpha x(k), \quad k = 2, \dots, N, \\ y(1) &= p(1)\chi_1 + g(1)\eta_1 + \mu x(1) = \eta_1 + \mu x(1). \end{aligned}$$

◊

Example 13.9. Consider the 5×5 matrix

$$A = \begin{pmatrix} d & 2 & 1 & 1 & 1 \\ 2 & d & 2 & 1 & 1 \\ 1 & 2 & d & 2 & 1 \\ 1 & 1 & 2 & d & 2 \\ 1 & 1 & 1 & 2 & d \end{pmatrix}.$$

from Example 5.16 and use the quasiseparable generators of order two obtained there.

Then the first state space variable of the descriptor system with boundary conditions (13.14) becomes

$$\begin{aligned} \chi_1 &= 0, \quad \chi_2 = a(1)\chi_1 + q(1)x(1) = x(1), \\ \chi_3 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} x(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x(2) = \begin{pmatrix} x(1) \\ x(2) \end{pmatrix}, \\ \chi_4 &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x(3) = \begin{pmatrix} x(1) + x(2) + x(3) \\ x(3) \end{pmatrix}, \\ \chi_5 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x(1) + x(2) + x(3) \\ x(3) \end{pmatrix} + 2x(4) = x(1) + x(2) + x(3) + 2x(4). \end{aligned}$$

The other state space variables of the system becomes

$$\eta_5 = 0, \quad \eta_4 = b(5)\eta_5 + h(5)x(5) = x(5),$$

$$\eta_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x(5) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x(4) = \begin{pmatrix} x(5)+x(4) \\ x(4) \end{pmatrix},$$

$$\eta_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(5)+x(4) \\ x(4) \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} x(3) = \begin{pmatrix} x(5)+x(4)+x(3) \\ x(5)+x(4)+2x(3) \end{pmatrix},$$

$$\eta_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x(5)+x(4)+x(3) \\ x(5)+x(4)+2x(3) \end{pmatrix} + 2x(2) = x(5)+x(4)+x(3)+2x(2).$$

From the form of the state space variables, it follows that the components of the output vector can be obtained as

$$\begin{aligned} y(1) &= p(1)\chi_1 + g(1)\eta_1 + d(1)x(1) = \eta_1 + dx(1) \\ &= dx(1) + 2x(2) + x(3) + x(4) + x(5), \\ y(2) &= p(2)\chi_2 + g(2)\eta_2 + d(2)x(2) = 2x(1) + dx(2) \\ &\quad + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x(5)+x(4)+x(3) \\ x(5)+x(4)+2x(3) \end{pmatrix} \\ &= 2x(1) + dx(2) + 2x(3) + x(4) + x(5), \\ y(3) &= p(3)\chi_3 + g(3)\eta_3 + d(3)x(3) \\ &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x(5)+x(4) \\ x(4) \end{pmatrix} + dx(3) \\ &= x(1) + 2x(2) + dx(3) + 2x(4) + x(5), \\ y(4) &= p(4)\chi_4 + g(4)\eta_4 + d(4)x(4) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x(1)+x(2)+x(3) \\ x(3) \end{pmatrix} + dx(4) + 2x(5) \\ &= x(1) + x(2) + 2x(3) + dx(4) + 2x(5) \end{aligned}$$

and

$$y(5) = x(1) + x(2) + x(3) + 2x(4) + dx(5). \qquad \Diamond$$

§13.4 Inversion of triangular matrices

In the sequel in this chapter we apply the system approach to design of fast algorithms. We start with the simplest case of inversion of triangular matrices.

Theorem 13.10. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be an invertible block lower triangular matrix with entries of sizes $m_i \times m_j$, with lower quasiseparable generators $p(k), q(k), a(k)$ $(k = 1, \dots, N)$ of orders r_k^L and diagonal entries $d(k)$ $(k = 1, \dots, N)$.*

Then the inverse A^{-1} is a block lower triangular matrix with lower quasiseparable generators $-(d(k))^{-1}p(k)$, $q(k)(d(k))^{-1}$, $a(k) - q(k)(d(k))^{-1}p(k)$ $(k = 1, \dots, N)$ of orders r_k^L and diagonal entries $d(k)^{-1}$ $(k = 1, \dots, N)$.

Moreover, the solution of the system of linear algebraic equations $Ax = y$ is obtained via the following algorithm:

Start with

$$x(1) = (d(1))^{-1}y(1),\ \chi_2 = q(1)x(1), \tag{13.19}$$

then for $k = 2, \dots, N-1$ compute recursively

$$x(k) = (d(k))^{-1}(y(k) - p(k)\chi_k), \tag{13.20}$$
$$\chi_{k+1} = a(k)\chi_k + q(k)x(k), \tag{13.21}$$

and finally compute

$$x(N) = (d(N))^{-1}(y(N) - p(N)\chi_N). \tag{13.22}$$

Proof. By Corollary 13.5 A, is a matrix of the input-output operator of the system

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1, \dots, N-1, \\ y(k) = p(k)\chi_k + d(k)x(k), & k = 1, \dots, N, \\ \chi_1 = 0. \end{cases} \tag{13.23}$$

Switching the input and output one obtains the inverse system

$$\begin{cases} \chi_{k+1} = (a(k) - q(k)(d(k))^{-1}p(k))\chi_k + q(k)(d(k))^{-1}y(k), & k = 1, \dots, N-1, \\ x(k) = -(d(k))^{-1}p(k)\chi_k + (d(k))^{-1}y(k), & k = 1, \dots, N, \\ \chi_1 = 0, \end{cases} \tag{13.24}$$

which corresponds to the inverse matrix A^{-1}. Applying Corollary 13.5 again, but in the opposite direction, we conclude that the elements

$$-(d(k))^{-1}p(k), q(k)(d(k))^{-1}, a(k) - q(k)(d(k))^{-1}p(k) \ \ (k = 1, \dots, N)$$

and $d(k)^{-1}$ $(k = 1, \dots, N)$ are lower quasiseparable generators and diagonal entries of the lower triangular matrix A^{-1}. It is clear that the orders of these lower generators are the same as for the original matrix A.

Combining the first and the last equations in (13.23) and the second equation in (13.24) one obtains the formulas (13.19)–(13.22). □

The complexity of the operations used in Theorem 13.10 to compute generators and diagonal entries of the inverse matrix is calculated as follows. The computation of the values $(d(k))^{-1}$ costs $\tilde{\rho}(m_k)$, where $\tilde{\rho}(m)$ denotes complexity of inversion of an $m \times m$ matrix using a standard method. The computation of $(d(k))^{-1}p(k), q(k)(d(k))^{-1}, a(k) - q(k)((d(k))^{-1}p(k))$ costs respectively $m_k^2 r_{k-1}^L, r_k^L m_k^2, r_k^L m_k r_{k-1}^L$ arithmetic multiplications and $m_k(m_k-1)r_{k-1}^L, r_k^L(m_k-$

$1)m_k$, $r_k^L(m_k-1)r_{k-1}^L$ arithmetical additions. Thus the total complexity here is given by the expression

$$c < \sum_{k=1}^{N} (\tilde{\rho}(m_k) + 2m_k^2 r_{k-1}^L + 2r_k^L m_k^2 + 2r_k^L m_k r_{k-1}^L)$$

and setting $m = \max_{1\le k\le N}(m_k),\ r = \max_{0\le k\le N}(r_k^L)$ one obtains the estimate

$$c \le N(\tilde{\rho}(m) + 4m^2 r + 2mr^2). \tag{13.25}$$

The complexity of the solution of the system $Ax = y$ is determined by the number of operations which are used in the formulas (13.20), (13.21) to compute the values x_k, χ_k. The computation by these formulas costs respectively less than $\rho(m_k) + m_k^2 + m_k(m_k-1) + m_k + 2m_k r_{k-1}^L$ and $2r_k^L r_{k-1}^L + 2r_k^L m_k$ arithmetical operations, where $\rho(m)$ is the complexity of solving an $m\times m$ system of linear algebraic equations using a standard method. Thus the total complexity here is

$$c < \sum_{k=1}^{N} (2m_k r_{k-1}^L + 2m_k^2 + \rho(m_k) + 2r_k^L r_{k-1}^L + 2r_k^L m_k)$$

operations and the estimate is

$$c \le (4mr + 2m^2 + 2r^2 + \rho(m))N. \tag{13.26}$$

Note that this way we solve the system without computing the quasiseparable generators of the inverse matrix.

Example 13.11. Consider the $N\times N$ matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

For A one can use the following lower quasiseparable generators of order 1

$$p(i) = 1,\ i = 2,\dots,N, \quad q(j) = 1,\ j = 1,\dots,N-1, \quad a(k) = 1,\ k = 2,\dots,N-1,$$

and the diagonal entries $d(k) = 1,\ k = 1,\dots,N$.

In this example we will apply Theorem 13.10 in order to find the inverse matrix A^{-1} and the solution $x = (x(k))_{k=1,N}$ of the linear system $Ax = y$ for a given vector $y = (y(k))_{k=1,N}$. First, by this theorem the inverse A^{-1} is a block lower triangular matrix with lower quasiseparable generators

$$\begin{aligned} p'(i) &= -(d(i))^{-1}p(i) = -1, && i = 2, \dots, N, \\ q'(j) &= q(j)(d(j))^{-1} = 1, && j = 1, \dots, N-1, \end{aligned}$$

and

$$\begin{aligned} a'(k) &= a(k) - q(k)(d(k))^{-1}p(k) \\ &= 1 - 1 = 0, && k = 2, \dots, N \end{aligned}$$

of orders 1, and diagonal entries $d'(k) = d(k)^{-1} = 1 \quad k = 1, \dots, N$. It follows that

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

Indeed, a direct computation also shows that the matrix A^{-1} above is the inverse of the given matrix A.

Moreover, by Theorem 13.10 the solution of the system of linear algebraic equations $Ax = y$ is obtained via the following algorithm:

Start with (13.19) to compute

$$x(1) = (d(1))^{-1}y(1) = y(1), \ \chi_2 = q(1)x(1) = x(1),$$

then for $k = 2, \dots, N-1$ compute recursively by (13.20)

$$x(2) = x(k) = (d(k))^{-1}(y(k) - p(k)\chi_k) = y(2) - x(1),$$

and by (13.21)

$$\chi_3 = \chi_{k+1} = a(k)\chi_k + q(k)x(k) = \chi_2 + x(2) = x(1) + x(2).$$

Suppose that for a certain k, $2 \le k \le N-2$ we have

$$x(k) = y(k) - x(1) - x(2) - x(3) - \cdots - x(k-1)$$

and

$$\chi_{k+1} = x(1) + x(2) + x(3) + \cdots + x(k)$$

which are true for $k = 2$. We will prove this for $k+1$. We have by (13.20)

$$\begin{aligned} x(k+1) &= (d(k+1))^{-1}(y(k+1) - p(k+1)\chi_{k+1}) \\ &= 1 \cdot y(k+1) - 1 \cdot (x(1) + x(2) + x(3) + \cdots + x(k)) \end{aligned}$$

and by (13.21)

$$\chi_{k+2} = a(k+1)\chi_{k+1} + q(k+1)x(k+1) = 1\cdot(x(1)+x(2)+x(3)+\cdots+x(k)) + 1\cdot x(k+1)$$

so that the induction on k is completed.

Finally, perform the last step of the algorithm, i.e., compute by (13.22)

$$x(N) = (d(N))^{-1}(y(N) - p(N)\chi_N) = y(N) - x(1) - x(2) - x(3) - \ldots - x(N-1). \ \diamondsuit$$

Example 13.12. Consider the $N \times N$ matrix

$$A = \begin{pmatrix} 3 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix}.$$

For A one can use the following lower quasiseparable generators of order 1

$$p(i) = 1, \ i = 2, \ldots, N, \quad q(j) = 1, \ j = 1, \ldots, N-1, \quad a(k) = 0, \ k = 2, \ldots, N-1,$$

and the diagonal entries $d(k) = 3, \ k = 1, \ldots, N$.

In this example we will apply Theorem 13.10 in order to find the inverse matrix A^{-1} and the solution $x = (x(k))_{k=1,N}$ of the linear system $Ax = y$ for a given vector $y = (y(k))_{k=1,N}$. First, by this theorem the inverse A^{-1} is a block lower triangular matrix with lower quasiseparable generators

$$p'(i) = -(d(i))^{-1}p(i) = -\frac{1}{3}, \quad i = 2, \ldots, N,$$
$$q'(j) = q(j)(d(j))^{-1} = \frac{1}{3}, \qquad j = 1, \ldots, N-1,$$

and

$$a'(k) = a(k) - q(k)(d(k))^{-1}p(k) = 0 - \frac{1}{3} = -\frac{1}{3}, \quad k = 2, \ldots, N$$

of orders 1, and diagonal entries $d'(k) = d(k)^{-1} = \frac{1}{3} \quad k = 1, \ldots, N$. It follows that

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{9} & \frac{1}{3} & 0 & \cdots & 0 & 0 \\ \frac{1}{27} & -\frac{1}{9} & \frac{1}{3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^{N-2}}{3^{N-1}} & \frac{(-1)^{N-3}}{3^{N-2}} & \frac{(-1)^{N-4}}{3^{N-3}} & \cdots & \frac{1}{3} & 0 \\ \frac{(-1)^{N-1}}{3^{N}} & \frac{(-1)^{N-2}}{3^{N-1}} & \frac{(-1)^{N-3}}{3^{N-2}} & \cdots & -1 & \frac{1}{3} \end{pmatrix}.$$

$\diamondsuit$

In a similar way as in Theorem 13.10, but using Corollary 13.6. one obtains the corresponding result for upper triangular matrices.

Theorem 13.13. *Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be an invertible block upper triangular matrix with entries of sizes $m_i \times m_j$, with upper quasiseparable generators $g(k), h(k), b(k)$ $(k = 1, \dots, N)$ of orders r_k^U and diagonal entries $d(k)$ $(k = 1, \dots, N)$.*

Then the inverse A^{-1} is a block upper triangular matrix with upper quasiseparable generators $-(d(k))^{-1}g(k), h(k)(d(k))^{-1}, b(k) - h(k)(d(k))^{-1}g(k)$ $(k = 1, \dots, N)$ of orders r_k^U and diagonal entries $d(k)^{-1}$ $(k = 1, \dots, N)$.

Moreover, the solution of the system of linear algebraic equations $Ax = y$ is obtained via the following algorithm:

Start with

$$x(N) = (d(N))^{-1}y(N), \ \eta_{N-1} = h(N)x(N),$$

then for $k = N-1, \dots, 2$ compute recursively

$$x(k) = (d(k))^{-1}(y(k) - g(k)\eta_k),$$
$$\eta_{k-1} = b(k)\eta_k + h(k)x(k)$$

and finally compute

$$x(1) = (d(1))^{-1}(y(1) - g(1)\eta_1).$$

One can easily check that the complexity of the algorithm from Theorem 13.13 satisfies the estimate (13.25).

§13.5 Comments

Systems of the form (13.14) and their input-output matrices were studied and used by I. Gohberg, M.A. Kaashoek and L. Lerer in [38] and by P.M. Dewilde and A.J. van der Veen in [15]. In the paper [22] systems with more general boundary condition were considered.

Chapter 14

The First Inversion Algorithms

Here we consider inversion methods for some classes of matrices and representations. In the first section we apply the idea used in Section §13.4 for a general case of matrices with quasiseparable representations with invertible diagonal entries. In the second section we discuss an inversion method for matrices with lower quasiseparable and upper semisiseparable representations, under some restrictions on generators. This method is based on the representation of the matrix as a sum of an invertible lower triangular matrix and a matrix of a small rank. The same results are obtained in the subsequent Chapter 16 via the system approach.

§14.1 Inversion of matrices in quasiseparable representation with invertible diagonal elements

Consider a matrix $A = \{A_{ij}\}_{i,j=1}^N$ with quasiseparable generators

$$\begin{array}{lll} p(i)\ (i=2,\dots,N), & q(j)\ (j=1,\dots,N-1), & a(k)\ (k=2,\dots,N-1) \\ g(i)\ (i=1,\dots,N-1), & h(j)\ (j=2,\dots,N), & \\ b(k)\ (k=2,\dots,N-1); & d(i)\ (i=1,\dots,N) & \end{array}$$

of orders r_k^L, r_k^U $(k=1,\dots,N-1)$. Let x,y be vectors such that $y = Ax$.

Let r_0^L, r_N^U be arbitrary nonnegative integers and $p(1), g(N), a(1), b(N)$ be arbitrary matrices of sizes $m_1 \times r_0^L, m_N \times r_N^U, r_1^L \times r_0^L, r_{N-1}^U \times r_N^U$, respectively.

Consider also the discrete-time descriptor system with homogeneous boundary conditions (13.14) for which A is the matrix of the input-output operator. Suppose also that all $d(k),\ k=1,\dots,N$ are invertible.

In this case one can obtain the coordinates of the input variable $x(k),\ k = 1,\dots,N$, as an expression in the coordinates of the output variable $y(k),\ k = 1,\dots,N$, the state space variables $\chi_k, \eta_k,\ k=1,\dots,N$ and some of the quasiseparable generators of the matrix A. Indeed, multiplying

$$y(k) = p(k)\chi_k + g(k)\eta_k + d(k)x(k), \quad k=1,\dots,N$$

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on the left by $(d(k))^{-1}$ one obtains

$$x(k) = -(d(k))^{-1}p(k)\chi_k - (d(k))^{-1}g(k)\eta_k + (d(k))^{-1}y(k), \quad k = 1, \dots, N. \quad (14.1)$$

Substituting these coordinates in the recursion relations

$$\chi_{k+1} = a(k)\chi_k + q(k)x(k), \quad k = 1, \dots, N-1,$$
$$\eta_{k-1} = b(k)\eta_k + h(k)x(k), \quad k = N, \dots, 2$$

of the state space variables, it follows that

$$\chi_{k+1} = (a(k) - q(k)(d(k))^{-1}p(k))\chi_k - q(k)(d(k))^{-1}g(k)\eta_k + q(k)(d(k))^{-1}y(k), \quad k = 1, \dots, N-1, \quad (14.2)$$

$$\eta_{k-1} = -h(k)(d(k))^{-1}p(k)\chi_k + (b(k) - h(k)(d(k))^{-1}g(k))\eta_k + h(k)(d(k))^{-1}y(k), \quad k = N, \dots, 2. \quad (14.3)$$

In order to define completely the state space variables only their boundary values

$$\chi_1 = 0, \ \eta_N = 0 \quad (14.4)$$

are needed.

Denote by $a^\times(k), b^\times(k), c^\times(k), e^\times(k), f^\times(k), g^\times(k), \ k = 1, \dots, N$, the matrices

$$a^\times(k) = a(k) - q(k)(d(k))^{-1}p(k), \quad k = 1, \dots, N-1, \quad (14.5)$$
$$b^\times(k) = b(k) - h(k)(d(k))^{-1}g(k), \quad k = N, \dots, 2, \quad (14.6)$$
$$c^\times(k) = q(k)(d(k))^{-1}g(k), \quad k = 1, \dots, N-1, \quad (14.7)$$
$$e^\times(k) = h(k)(d(k))^{-1}p(k), \quad k = N, \dots, 2, \quad (14.8)$$
$$f^\times(k) = h(k)(d(k))^{-1}y(k), \quad k = N, \dots, 2, \quad (14.9)$$
$$g^\times(k) = q(k)(d(k))^{-1}y(k), \quad k = 1, \dots, N-1. \quad (14.10)$$

Then the system (§14.1)–(14.4) becomes

$$\begin{cases} \chi_{k+1} = a^\times(k)\chi_k - c^\times(k)\eta_k + g^\times(k), & k = 1, \dots, N-1, \\ \eta_{k-1} = -e^\times(k)\chi_k + b^\times(k)\eta_k + f^\times(k), & k = N, \dots, 2, \\ \chi_1 = 0, \quad \eta_N = 0. \end{cases} \quad (14.11)$$

If $f^\times(k), \ g^\times(k)$ are taken as the right-hand side the system (14.11) becomes

$$\begin{cases} \chi_{k+1} - a^\times(k)\chi_k + c^\times(k)\eta_k = g^\times(k), & k = 1, \dots, N-1, \\ \eta_{k-1} + e^\times(k)\chi_k - b^\times(k)\eta_k = f^\times(k), & k = N, \dots, 2, \\ \chi_1 = 0, \quad \eta_N = 0. \end{cases}$$

It follows that the system (14.11) can be rewritten as the following system of linear algebraic equations:

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -a^{\times}(1) & c^{\times}(1) & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & e^{\times}(2) & -b^{\times}(2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & -a^{\times}(2) & c^{\times}(2) & I & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & e^{\times}(3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & e^{\times}(N) & -b^{\times}(N) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I \end{pmatrix} \begin{pmatrix} \chi_1 \\ \eta_1 \\ \chi_2 \\ \eta_2 \\ \chi_3 \\ \vdots \\ \chi_N \\ \eta_N \end{pmatrix}$$
$$= \begin{pmatrix} 0 & (g^{\times}(1))^T & (f^{\times}(2))^T & (g^{\times}(2))^T & (f^{\times}(3))^T & \cdots & (f^{\times}(N))^T & 0 \end{pmatrix}^T \tag{14.12}$$

with a block tridiagonal matrix. Thus the inversion of the matrix A has been reduced to the inversion of a block tridiagonal matrix.

Also, let $a(N), b(1)$ be arbitrary matrices of sizes $r_N^L \times r_{N-1}^L$ and $r_0^U \times r_1^U$ respectively.

Theorem 14.1. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with entries of sizes $m_i \times m_j$, with lower quasiseparable generators $p(k), q(k), a(k)$ $(k = 1, \dots, N)$ of orders r_k^L, upper quasiseparable generators $g(k), h(k), b(k)$ $(k = 1, \dots, N)$ of orders r_k^U and invertible diagonal entries $d(k)$ $(k = 1, \dots, N)$. Using these generators define $a^{\times}(k), b^{\times}(k), c^{\times}(k), e^{\times}(k)$ via the relations* (14.5)–(14.8).

The matrix A is invertible if and only if the matrix in (14.12) *is invertible. Moreover, if the last condition holds, then for any block vector $y = (y(k))_{k=1}^N$ the solution $x = (x(k))_{k=1}^N$ of the equation $Ax = y$ is given by the relation* (14.1).

Proof. Let the matrix in (14.12) be invertible. One must prove that A is invertible also. Suppose to this end that $Ax = 0$, and let us show that $x = 0$.

Substituting $y(k) = 0,\ k = 1, \dots, N$ in (14.1), (§14.1), (§14.1), (14.9) and (14.10) one obtains

$$x(k) = -(d(k))^{-1}p(k)\chi_k - (d(k))^{-1}g(k)\eta_k, \quad k = 1, \dots, N, \tag{14.13}$$

with

$$\begin{aligned} {}_{k+1} &= (a(k) - q(k)(d(k))^{-1}p(k))\chi_k - q(k)(d(k))^{-1}g(k)\eta_k, \qquad k = 1, \dots, N-1, \\ \eta_{k-1} &= -h(k)(d(k))^{-1}p(k)\chi_k + (b(k) - h(k)(d(k))^{-1}g(k))\eta_k, \quad k = N, \dots, 2. \end{aligned}$$

In the notations (14.5)–(14.8), it follows that x is given using state space variables χ_k, η_k which satisfy (14.11) with $f^{\times}(k) = 0, g^{\times}(k) = 0$. This means that the state space variables are a solution of the system (14.12) with the right-hand side equal to zero. By the assumption, this means that the state space variables themselves are zero, which by virtue of (14.13) yields $x(k) = 0,\ k = 1, \dots, N$. Therefore, the matrix A is invertible.

Conversely, suppose that the matrix A is invertible. Let χ_k, η_k $k = 1, \dots, N$, be the solution of the system (14.12) with the zero right-hand side. This means

$$\begin{cases} \chi_{k+1} = a^\times(k)\chi_k - c^\times(k)\eta_k, & k = 1, \dots, N-1, \\ \eta_{k-1} = -e^\times(k)\chi_k + b^\times(k)\eta_k, & k = N, \dots, 2, \\ \quad \chi_1 = 0, \quad \eta_N = 0. \end{cases} \tag{14.14}$$

Using the relations (14.5)–(14.8) we get

$$\chi_{k+1} = a(k)\chi_k + q(k)(-(d(k))^{-1}p(k)\chi_k - (d(k))^{-1}g(k)\eta_k), \quad k = 1, \dots, N-1, \tag{14.15}$$

$$\eta_{k-1} = b(k)\eta_k + h(k)(-(d(k))^{-1}p(k)\chi_k - (d(k))^{-1}g(k)\eta_k), \quad k = N, \dots, 2. \tag{14.16}$$

Set $x(k) = -(d(k))^{-1}p(k)\chi_k - (d(k))^{-1}g(k)\eta_k$, which is equivalent to

$$p(k)\chi_k + d(k)x(k) + g(k)\eta_k = 0, \quad k = 1, \dots, N. \tag{14.17}$$

Combining relations (14.15)–(14.17) we get

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1, \dots, N-1, \\ \eta_{k-1} = b(k)\eta_k + h(k)x(k), & k = N, \dots, 2, \\ p(k)\chi_k + d(k)x(k) + g(k)\eta_k = 0, & k = 1, \dots, N, \\ \chi_1 = 0, \quad \eta_N = 0. \end{cases} \tag{14.18}$$

Comparing with (13.14) we conclude that $Ax = 0$, where $x = (x(k))_{k=1}^N$. Since the matrix A is invertible, it follows that $x = 0$. Inserting this in (14.18) we get

$$\chi_1 = 0, \quad \chi_{k+1} = a(k)\chi_k, \quad k = 1, \dots, N-1,$$
$$\eta_N = 0, \quad \eta_{k-1} = b(k)\eta_k, \quad k = N, \dots, 2,$$

whence

$$\chi_k = 0, \qquad \eta_k = 0, \qquad k = 1, \dots, N.$$

Therefore, the matrix in (14.12) is invertible. □

There exist various fast methods for the solution of systems of the form (14.12). We may consider the matrix in (14.12) as a band matrix with the bandwidth $r^L + r^U$, where $r^L = \max_{k=1}^N r_k^L$ and $r^U = \max_{k=1}^N r_k^U$ and use a method described for instance in [43], pp. 202–207. However, we can use the block tridiagonal form of the matrix and the identity entries in the lower subdiagonal in order to simplify the algorithm. Equation (14.12) is equivalent to the conditions (14.4) and the equation

$$\begin{pmatrix} c^\times(1) & I & 0 & 0 & \cdots & 0 & 0 \\ I & e^\times(2) & -b^\times(2) & 0 & \cdots & 0 & 0 \\ 0 & -a^\times(2) & c^\times(2) & I & \cdots & 0 & 0 \\ 0 & 0 & I & e^\times(3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & e^\times(N) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \chi_2 \\ \eta_2 \\ \chi_3 \\ \vdots \\ \chi_N \end{pmatrix} = \begin{pmatrix} g^\times(1) \\ f^\times(2) \\ g^\times(2) \\ f^\times(3) \\ \vdots \\ f^\times(N) \end{pmatrix} \tag{14.19}$$

with the matrix obtained from the matrix in (14.12) by deleting the first and the last columns and rows.

Proceed as in the method of Gaussian elimination. Consider the extended matrix $\tilde{A}$ that corresponds to equation (14.19). In the first stage take the first three rows of $\tilde{A}$,

$$\begin{pmatrix} c^\times(1) & I & 0 & 0 & \cdots & 0 & g^\times(1) \\ I & e^\times(2) & -b^\times(2) & 0 & \cdots & 0 & f^\times(2) \\ 0 & -a^\times(2) & c^\times(2) & I & \cdots & 0 & g^\times(2) \end{pmatrix},$$

and perform the following transformations. Multiply the second row by $c^\times(1)$, subtract the result from the first one and interchange the resulting first two rows. The result is the matrix

$$\begin{pmatrix} I & e^\times(2) & -b^\times(2) & 0 & \cdots & 0 & f^\times(2) \\ 0 & I - c^\times(1)e^\times(2) & c^\times(1)b^\times(2) & 0 & \cdots & 0 & g^\times(1) - c^\times(1)f^\times(2) \\ 0 & -a^\times(2) & c^\times(2) & I & \cdots & 0 & g^\times(2) \end{pmatrix}.$$

Next applying Gaussian elimination with partial pivoting to the last two rows of this matrix we transform it to the upper triangular form

$$\begin{pmatrix} I & e^\times(2) & -b^\times(2) & 0 & 0 & \cdots & 0 & f^\times(2) \\ 0 & \lambda_2 & \rho_2 & w_2 & 0 & \cdots & 0 & \tilde{g}_2 \\ 0 & 0 & \tilde{c}_2 & v_2 & 0 & \cdots & 0 & g_2' \end{pmatrix},$$

with upper triangular $\lambda_2, \tilde{c}_2$. Taking the last row from this matrix and the two next rows from $\tilde{A}$ we obtain the matrix

$$\begin{pmatrix} 0 & 0 & \tilde{c}_2 & v_2 & 0 & 0 & \cdots & 0 & g_2' \\ 0 & 0 & I & e^\times(3) & -b^{\times 3} & 0 & \cdots & 0 & f^\times(3) \\ 0 & 0 & 0 & -a^\times(3) & c^\times(3) & I & \cdots & 0 & g^\times(3) \end{pmatrix}.$$

For such a matrix we may proceed as above, and so on. At every step except the last one we obtain a matrix of the form

$$\begin{pmatrix} 0 & \cdots & 0 & \tilde{c}_{k-1} & v_{k-1} & 0 & 0 & \cdots & 0 & g_{k-1}' \\ 0 & \cdots & 0 & I & e^\times(k) & -b^\times(k) & 0 & \cdots & 0 & f^\times(k) \\ 0 & \cdots & 0 & 0 & -a^\times(k) & c^\times(k) & I & \cdots & 0 & g^\times(k) \end{pmatrix}.$$

Such a matrix is reduced to the form

$$\begin{pmatrix} 0 & \cdots & 0 & I & e^{\times}(k) & -b^{\times}(k) & 0 & 0 & \cdots & 0 & f^{\times}(k) \\ 0 & \cdots & 0 & 0 & \lambda_k & \rho_k & w_k & 0 & \cdots & 0 & \tilde{g}_k \\ 0 & \cdots & 0 & 0 & 0 & \tilde{c}_k & v_k & 0 & \cdots & 0 & g'_k \end{pmatrix}$$

with upper triangular $\lambda_k, \tilde{c}_k$.

In the last step we deal with the last two rows of the transformed matrix, which are

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \tilde{c}_{N-1} & v_{N-1} & g'_{N-1} \\ 0 & 0 & \cdots & 0 & I & e^{\times}(N) & f^{\times}(N) \end{pmatrix}.$$

Multiplying the second row by $\tilde{c}_{N-1}$, subtracting the result from the first row and next changing the first and the second rows we obtain the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & I & e^{\times}(N) & f^{\times}(N) \\ 0 & 0 & \cdots & 0 & \lambda_N & \tilde{g}_N \end{pmatrix},$$

where $\lambda_N = v_{N-1} - \tilde{c}_{N-1}e^{\times}(N)$ and $\tilde{g}_N = g'_{N-1} - \tilde{c}_{N-1}f^{\times}(N)$.

Thus equation (14.19) is reduced to the form

$$\begin{pmatrix} I & e^{\times}(2) & -b^{\times}(2) & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \rho_2 & w_2 & \cdots & 0 & 0 \\ 0 & 0 & I & e^{\times}(3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_N \end{pmatrix} \begin{pmatrix} \eta_1 \\ \chi_2 \\ \eta_2 \\ \chi_3 \\ \vdots \\ \chi_N \end{pmatrix} = \begin{pmatrix} f^{\times}(2) \\ \tilde{g}_2 \\ f^{\times}(3) \\ \tilde{g}_3 \\ \vdots \\ f^{\times}(N) \\ \tilde{g}_N \end{pmatrix}.$$

The last equation can be solved easily.

Thus we obtain the following algorithm.

Algorithm 14.2. Let A be an invertible matrix with quasiseparable generators $p(k), q(k), a(k); g(k), h(k), b(k); d(k)$ $(k = 1, \dots, N)$, such that the diagonal entries $d(k)$ $(k = 1, \dots, N)$ are invertible.

Then the solution x of the equation $Ax = y$ is obtained as follows:

1. For $k = 1, \dots, N$ perform the following operations: find the solutions ψ_k, ϕ_k, μ_k of the equations

$$d(k)\psi_k = p(k), \quad d(k)\phi_k = g(k), \quad d(k)\mu_k = y(k),$$

and compute

$$a^{\times}(k) = a(k) - q(k)\psi_k, \quad c^{\times}(k) = q(k)\phi_k, \quad g^{\times}(k) = q(k)\mu_k,$$
$$b^{\times}(k) = b(k) - h(k)\phi_k, \quad e^{\times}(k) = h(k)\psi_k, \quad f^{\times}(k) = h(k)\mu_k.$$

2. Solve the system (14.12) to obtain the values $\chi_k,\ \eta_k,\ k=1,\dots,N.$ as follows:
   2.1.1. Set $\chi_1=0,\ \eta_N=0,\ v_1=I,\ g'_1=g^\times(1),\ \tilde{c}_1=c^\times(1)$.
   2.1.2. For $k=2,\dots,N-1$, perform the following: compute

$$c'_k=v_{k-1}-\tilde{c}_{k-1}e^\times(k),\quad c''_k=\tilde{c}_{k-1}b^\times(k),\quad g''_k=g'_{k-1}-\tilde{c}_{k-1}f^\times(k),$$

   using Gaussian elimination with partial pivoting transform the matrix

$$\begin{pmatrix} c'_k & c''_k & 0 & g''_k \\ -a^\times(k) & c^\times(k) & I & g^\times(k) \end{pmatrix}$$

   to the form

$$\begin{pmatrix} \lambda_k & \rho_k & w_k & \tilde{g}_k \\ 0 & \tilde{c}_k & v_k & g'_k \end{pmatrix}.$$

   2.1.3. Compute $\lambda_N=v_{N-1}-\tilde{c}_{N-1}e^\times(N),\ \tilde{g}_N=g'_{N-1}-\tilde{c}_{N-1}f^\times(N)$.
   2.2.1. Find the solution χ_N of the equation $\lambda_N\chi_N=\tilde{g}_N$, compute $\eta_{N-1}=f^\times(N)-e^\times(N)\chi_N$.
   2.2.2. For $k=N-1,\dots,2$, find the solution χ_k of the equation $\lambda_k\chi_k=\tilde{g}_k-\rho_k\eta_k-w_k\chi_{k+1}$ and compute $\eta_{k-1}=f^\times(k)-e^\times(k)\chi_k+b^\times(k)\eta_k$.
3. For $k=1,\dots,N.$ find the components of the solution

$$x(k)=-\psi_k\chi_k-\phi_k\eta_k+\mu_k.$$

4. Obtain the solution as $x=\mathrm{col}(x(k))_{k=1}^N$.

Let us calculate the complexity of this algorithm. To this end we denote by

1. $\nu(m)$ the complexity of solving an equation $A_0x=y$ with an unknown block column vector x of size m; if X and Y are matrices with k columns, then the complexity of the equations $A_0X=Y$ is $k\nu(m)$.
2. $\zeta(m,k)$ the complexity of transforming an $m\times k$ submatrix to an upper triangular form using Gaussian elimination.
3. $\upsilon(m)$ the complexity of the solution of a system of linear algebraic equations with an $m\times m$ (upper) triangular matrix.

Denote also $r=\max_{k=0}^N r_k^L$, $s=\max_{k=0}^N r_k^U$, $n=\max_{k=0}^N n_k$. These will be used mainly in upper estimates for the product of two matrices; for example, the product of an $r_k^L\times n_k$ matrix and an $n_k\times r_{k-1}^U$ matrix costs exactly $r_k^L\cdot n_k\cdot r_{k-1}^U$ multiplications and $r_k^L\cdot(n_k-1)\cdot r_{k-1}^U$ additions, hence less that $2rns$ operations.

Step 1 of the algorithm asks solving for each $k=1,\dots,N$ three systems, with an $n_k\times r_{k-1}^L$, an $n_k\times r_k^U$ and an $n_k\times 1$ matrix, which means at most $N\nu(n)(r+s+1)$ operations. For each $k\neq N$, three multiplications of an $r_k^L\times n_k$ matrix with an $n_k\times r_{k-1}^L$, an $n_k\times r_k^U$ and an $n_k\times 1$ matrix are performed, together with a subtraction of two $r_k^L\times r_{k-1}^L$ matrices, which cost in total less than $N(2r^2n+2rns+2rn+r^2)$ operations. For each $k\neq 1$, three multiplications of an $r_{k-1}^U\times n_k$ matrix with an $n_k\times r_k^U$, an $n_k\times r_{k-1}^L$ and an $n_k\times 1$ matrix are performed, together with a subtraction of two $r^U_{k-1} \times r^U_k$ matrices, which cost in total less than $N(2s^2n + 2rns + 2sn + s^2)$ operations.

In Step 2.1.2 of the algorithm, for each $k = 1, \dots, N$ a submatrix of size $(r^L_{k-1} + r^L_k) \times (r^L_{k-1} + r^U_k)$ is brought to the upper triangular form, which costs at most $N\zeta(2r, r+s)$ operations.

The same step as well as Step 2.1.3 performs three matrix multiplications with the $r^L_{k-1} \times r^U_{k-1}$ matrix $\tilde{c}_{k-1}$, respectively $\tilde{c}_{N-1}$, with matrices of sizes $r^U_{k-1} \times r^L_{k-1}$, $r^U_{k-1} \times r^U_k$ and $r^U_{k-1} \times 1$, together with two subtractions of $r^L_{k-1} \times r^L_{k-1}$ and $r^L_{k-1} \times 1$, so that the total cost of these steps is less than $N(2rs(r+s+1)+r^2+r)$.

Steps 2.2.1 and 2.2.2 find the unknown vectors χ_k, $k = 2, \dots, N$, from equations with an upper triangular matrix, which costs $(N-1)\upsilon(r^L_{k-1})$ operations, plus four matrix multiplications with a vector and four subtractions. These have an extra cost of less than $2r^L_{k-1} \cdot r^U_k + 2r^L_{k-1} \cdot r^L_k + 2r^U_{k-1} \cdot r^L_{k-1} + 2r^U_{k-1} \cdot r^U_k + 2r^L_{k-1} + 2r^U_{k-1}$. In total, these two sub-steps cost less than $N(\upsilon(r) + 4rs + 2r^2 + 2s^2 + 2r + 2s)$.

Finally, Step 3 asks for $2N$ multiplications and $2N$ additions and it costs less than $N(2nr + 2ns + 2n)$ operations.

The total complexity of the algorithm is thus less than

$$\big(\nu(n)(r+s+1) + 2(r+s)^2 + (r+s)n + r^2 + s^2 + \zeta(2r, r+s) + 2rs(r+s+1) \\ + r^2 + r + \upsilon(r) + 4rs + 2r^2 + 2s^2 + 2r + 2s + 2nr + 2ns + 2n\big)N.$$

Example 14.3. Consider the $N \times N$ matrix

$$A = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{N-2} & a^{N-1} \\ b & 1 & a & \cdots & a^{N-3} & a^{N-2} \\ b^2 & b & 1 & \cdots & a^{N-4} & a^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b^{N-2} & b^{N-3} & b^{N-4} & \cdots & 1 & a \\ b^{N-1} & b^{N-2} & b^{N-3} & \cdots & b & 1 \end{pmatrix}$$

which has been introduced in Example 5.14. The inverse of this scalar matrix will be found again to be exactly as in (10.29) for the scalar case.

For A a set of quasiseparable generators have been obtained in Example 5.14, namely

$$p(i) = b, \quad i = 2, \dots, N, \quad a(k) = b, \quad k = 2, \dots, N-1,$$
$$h(i) = a, \quad i = 2, \dots, N, \quad b(k) = a, \quad k = 2, \dots, N-1,$$
$$q(j) = 1, \quad g(j) = 1, \quad j = 1, \dots, N-1, \quad d(k) = 1, \quad k = 1, \dots, N.$$

Consider the equation $Ax = y$. Then the above algorithm will find x for a given y in the following way.

Define like in (14.5)–(14.10) for $k = 1, \dots, N-1$

$$a^\times(k) = a(k) - q(k)(d(k))^{-1}p(k) = b - \frac{b}{1} = 0, \quad c^\times(k) = q(k)(d(k))^{-1}g(k) = 1,$$
$$g^\times(k) = q(k)(d(k))^{-1}y(k) = y(k)$$

and for $k = 2, \dots, N$

$$b^\times(k) = b(k) - h(k)(d(k))^{-1}g(k) = a - \frac{a}{1} = 0, \quad e^\times(k) = h(k)(d(k))^{-1}p(k) = ab,$$
$$f^\times(k) = h(k)(d(k))^{-1}y(k) = ay(k).$$

Solve the following linear system of $2N$ equations with $2N$ unknowns with a tridiagonal matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & ab & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & ab & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & ab & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \eta_1 \\ \chi_2 \\ \eta_2 \\ \chi_3 \\ \vdots \\ \chi_N \\ \eta_N \end{pmatrix} = \begin{pmatrix} 0 \\ y(1) \\ ay(2) \\ y(2) \\ ay(3) \\ y(3) \\ \vdots \\ ay(N) \\ 0 \end{pmatrix}.$$

Note that this matrix is invertible under the additional assumption that $ab \neq 1$. From the theorem it follows that the matrix A is also invertible if and only if this condition takes place.

This ends the first step of Algorithm 14.2. In Step 2 solve the system (14.12) to obtain the values $\chi_k, \ \eta_k, \ k = 1, \dots, N$ as follows:

Perform Sub-step 2.1.1, i.e., set

$$\chi_1 = 0, \ \eta_N = 0, \ v_1 = 1, \ g'_1 = g^\times(1) = y(1), \ \tilde{c}_1 = c^\times(1) = 1.$$

Perform Sub-step 2.1.2: first for $k = 2$, i.e., compute

$$c'_k = v_{k-1} - \tilde{c}_{k-1}e^\times(k) = 1 - ab, \quad c''_k = \tilde{c}_{k-1}b^\times(k) = 1 \cdot 0 = 0,$$
$$g''_k = g'_{k-1} - \tilde{c}_{k-1}f^\times(k) = y(1) - ay(2)$$

then form the matrix

$$\begin{pmatrix} c'_k & c''_k & 0 & g''_k \\ -a^\times(k) & c^\times(k) & I & g^\times(k) \end{pmatrix} = \begin{pmatrix} 1-ab & 0 & 0 & y(1)-ay(2) \\ 0 & 1 & 1 & y(2) \end{pmatrix}.$$

No Gaussian elimination with partial pivoting is needed to transform the matrix to the form

$$\begin{pmatrix} \lambda_k & \rho_k & w_k & \tilde{g}_k \\ 0 & \tilde{c}_k & v_k & g'_k \end{pmatrix}.$$

It follows that

$$\lambda_2 = 1 - ab, \quad \rho_2 = 0, \quad w_2 = 0, \quad \tilde{g}_2 = y(1) - ay(2),$$
$$\tilde{c}_2 = 1, \quad v_2 = 1, \quad g'_2 = y(2).$$

Perform again Sub-step 2.1.2, now in turn for $k = 3, \dots, N-1$. Since $a^\times(k)$, $b^\times(k)$, $c^\times(k)$, $e^\times(k)$, $f^\times(k)$ and $g^\times(k)$ are constant, namely equal to $a^\times(2)$, $b^\times(2)$, $c^\times(2)$, $e^\times(2)$, $f^\times(2)$ and $g^\times(2)$, respectively, it follows that for any $k = 3, \dots, N-1$ we have again

$$\lambda_k = 1 - ab, \quad \rho_k = 0, \quad w_k = 0, \quad \tilde{g}_k = y(k-1) - ay(k)$$
$$\tilde{c}_k = 1 \quad v_k = 1, \quad g'_k = y(k).$$

In Sub-step 2.1.3 compute

$$\lambda_N = v_{N-1} - \tilde{c}_{N-1} e^\times(N) = 1 - ab,\ \tilde{g}_N = g'_{N-1} - \tilde{c}_{N-1} f^\times(N) = y(N-1) - ay(N).$$

Sub-step 2.1 is now complete.

In Sub-step 2.2.1 find the solution χ_N of the equation $\lambda_N \chi_N = \tilde{g}_N$, in our case

$$\chi_N = \frac{ay(N) - y(N-1)}{ab - 1},$$

and compute $\eta_{N-1} = f^\times(N) - e^\times(N)\chi_N$. In our case

$$\eta_{N-1} = ay(N) - ab\chi_N = ay(N) - \frac{ab}{ab-1}(ay(N) - y(N-1))$$
$$= \frac{aby(N-1) - ay(N)}{ab - 1}.$$

Perform Sub-step 2.2.2 for $k = N-1$, i.e., find the solution χ_{N-1} of the equation $\lambda_{N-1}\chi_{N-1} = \tilde{g}_{N-1} - \rho_{N-1}\eta_{N-1} - w_{N-1}\chi_N$ and compute $\eta_{N-2} = f^\times(N-1) - e^\times(N-1)\chi_{N-1} - b^\times(N-1)\eta_{N-1}$. In our case $\rho_{N-1} = w_{N-1} = b^\times(N-1) = 0$, so that

$$\chi_{N-1} = \frac{ay(N-1) - y(N-2)}{ab-1}, \quad \eta_{N-2} = \frac{aby(N-2) - ay(N-1)}{ab-1}. \tag{14.20}$$

Perform again Sub-step 2.2.2 for $k = N-2, \dots, 2$, i.e., find the solution χ_k of the equation $\lambda_k \chi_k = \tilde{g}_k - \rho_k \eta_k - w_k \chi_{k+1}$ and compute $\eta_{k-1} = f^\times(k) - e^\times(k)\chi_k - b^\times(k)\eta_k$. Since $\lambda_k, \rho_k, w_k, f^\times(k), e^\times(k), b^\times(k)$ are constant upon k, which means that they are equal to $\lambda_{N-1}, \rho_{N-1}, w_{N-1}, f^\times(N-1), e^\times(N-1), b^\times(N-1)$, respectively, and $\tilde{g}_k = y(k-1) - ay(k),\ k = N-1, \dots, 2$, it follows that, like in (14.20),

$$\chi_k = \frac{ay(k) - y(k-1)}{ab-1}, \quad \eta_{k-1} = \frac{aby(k-1) - ay(k)}{ab-1}.$$

In the third step of Algorithm 14.2, using (14.1) one obtains the solution

$$x(k) = y(k) - \eta_k - b\chi_k, \quad k = 1, \dots, N.$$

For $k = 2, \dots, N-1$

$$\begin{aligned} x(k) &= \frac{(ab-1)y(k) - bay(k) + by(k-1) + ay(k+1) - aby(k)}{ab-1} \\ &= \frac{by(k-1) - (ab+1)y(k) + ay(k+1)}{ab-1}, \end{aligned}$$

while

$$x(1) = y(1) - \eta_1 = \frac{(ab-1)y(1) - aby(1) + ay(2)}{ab-1} = \frac{-y(1) + ay(2)}{ab-1}$$

and

$$x(N) = y(N) - \chi_N = \frac{(ab-1)y(N) - bay(N) + by(N-1)}{ab-1} = \frac{by(N-1-y(N))}{ab-1}.$$

Written in a compact manner, it follows that x as a function of y is as follows

$$\begin{pmatrix} x(1) \\ x(2) \\ x(3) \\ \vdots \\ x(N) \end{pmatrix} = \frac{1}{ab-1} \begin{pmatrix} -1 & a & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & -(ab+1) & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & b & -(ab+1) & a & \cdots & 0 & 0 & 0 \\ 0 & 0 & b & -(ab+1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b & -(ab+1) & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & -1 \end{pmatrix}$$

$$\begin{pmatrix} y(1) & y(2) & y(3) & \cdots & y(N-1) & y(N) \end{pmatrix}^T.$$

Note that the matrix in the above equation is in fact the inverse of the original matrix A. ◊

§14.2 The extension method for matrices with quasiseparable/semiseparable representations

Here we consider matrices with given lower quasiseparable and upper semiseparable generators. For such a matrix we use a representation as a sum of a lower triangular matrix and a matrix of a small rank. Under the conditions on generators such that the lower triangular part is invertible we obtain an explicit inversion formula for the whole matrix. Based on this formula, using a well-known procedure in numerical methods for differential equations, we obtain a stable algorithm for solution of the corresponding system of linear algebraic equations.

§14.2.1 The inversion formula

In this subsection for a matrix with quasiseparable/semiseparable representation an explicit inversion formula is obtained under some restrictions on generators.

Theorem 14.4. *Let A be an invertible block matrix with block entries of sizes $m_i \times m_j$, with lower quasiseparable generators $p(k), q(k), a(k)$ $(k = 1, \dots, N)$ of orders r_k^L $(k = 0, \dots, N)$, upper semiseparable generators $g(k), h(k)$ $(k = 1, \dots, N)$ of order r_U, and diagonal entries $d(k)$ $(k = 1, \dots, N)$. Assume that all the matrices*

$$D(k) = d(k) - g(k)h(k), \quad k = 1, \dots, N, \tag{14.21}$$

are invertible.

Introduce the matrices

$$B(k) = \begin{bmatrix} q(k) \\ -h(k) \end{bmatrix}, \; C(k) = \begin{bmatrix} p(k) & g(k) \end{bmatrix}, \; E(k) = \begin{pmatrix} a(k) & 0 \\ 0 & I_{r_U} \end{pmatrix}, \quad k = 1, \dots, N, \tag{14.22}$$

$$K_j = \begin{pmatrix} 0_{r_j^L \times r_U} \\ I_{r_U} \end{pmatrix}, \quad j = 0, \dots, N, \tag{14.23}$$

and define

$$U(k) = E(k) - B(k)(D(k))^{-1}C(k), \quad U^{\times} = K_N^T U^{>}_{N+1,0} K_0. \tag{14.24}$$

Then the matrix A is invertible if and only if the matrix $U^{\times}$ is invertible, and in this case the inverse matrix A^{-1} is given by the formula

$$A^{-1} = L^{\times} + S^{\times}, \tag{14.25}$$

where

$$L^{\times}(i,j) = \begin{cases} -(D(i))^{-1}C(i)U^{>}_{i,j}B(j)(D(j))^{-1}, & 1 \le j < i \le N, \\ (D(i))^{-1}, & i = j, \\ 0, & i \le j, \end{cases} \tag{14.26}$$

$$S^{\times}(i,j) = (D(i))^{-1}(C(i)U^{>}_{i,0}K_0)(U^{\times})^{-1}(K_N^T U^{>}_{N+1,j}B(j))(D(j))^{-1}, \quad 1 \le i,j \le N. \tag{14.27}$$

Proof. We represent the matrix A in the form

$$A = L + GH,$$

where $G = \mathrm{col}(g(i))_{i=1}^N$, $H = \mathrm{row}(h(i))_{i=1}^N$ and

$$L(i,j) = \begin{cases} 0, & i < j, \\ D(i), & i = j, \\ p(i)a^{>}_{ij}q(j) - g(i)h(j), & 1 \le j < i \le N. \end{cases}$$

For the elements in the strictly lower triangular part of the matrix L one obtains the representation

$$L(i,j) = \begin{pmatrix} p(i) & g(i) \end{pmatrix} \begin{pmatrix} a^{>}_{ij} & 0 \\ 0 & I_{r_U} \end{pmatrix} \begin{pmatrix} q(j) \\ -h(j) \end{pmatrix}, \quad 1 \le j < i \le N,$$

i.e.,

$$L(i,j) = C(i)E^{>}_{ij}B(j), \quad 1 \le j < i \le N,$$

with the matrices $C(i)$, $B(j)$, $E(k)$ defined in (14.22). Hence it follows that L is a block lower triangular matrix with lower quasiseparable generators $C(k)$, $B(k)$, $E(k)$ $(k = 1,\dots,N)$ and diagonal entries $D(k)$ $(k = 1,\dots,N)$. By the formula (1.69),

$$A^{-1} = L^{-1} - L^{-1}G(U^{\times})^{-1}HL^{-1},$$

where

$$U^{\times} = I_{r_U} + HL^{-1}G.$$

Moreover, using (1.72) one gets

$$\det(L+GH) = \det\left(L\left(I + L^{-1}GH\right)\right) = \det L \cdot \det\left(I + HL^{-1}G\right)$$

which implies

$$\det A = \left(\prod_{i=1}^{N} \det D(i)\right) \det U^{\times}. \tag{14.28}$$

It follows that the matrix A is invertible if and only if the matrix $U^{\times}$ is invertible.

Applying Theorem 13.10 to the invertible matrix L one obtains that L^{-1} is a block lower triangular matrix with lower quasiseparable generators

$$-(D(k))^{-1}C(k), B(k)(D(k))^{-1}, U(k) \ \ (k = 1,\dots,N)$$

and diagonal entries $(D(k))^{-1}$ $(k = 1,\dots,N)$. Hence the formula (14.26) follows.

To obtain the representation (14.24) for the matrix $U^{\times}$ and the representations (14.27) for the matrix $S^{\times} = -L^{-1}G(U^{\times})^{-1}HL^{-1}$ we proceed as follows. One obviously has

$$C(i)K_{i-1} = g(i), \ K_i^T B(i) = -h(i), \quad i = 1,\dots,N. \tag{14.29}$$

It will be proved that

$$L^{-1}G = \operatorname{col}\left((D(i))^{-1}C(i)U^{>}_{i,0}K_0\right)_{i=1}^{N}.$$

Indeed, using (14.26) and the first equalities from (14.29) one gets

$$\begin{aligned}(L^{-1}G)(i,:) &= (D(i))^{-1}g(i) - \sum_{m=1}^{i-1}(D(i))^{-1}C(i)U^{>}_{im}B(m)(D(m))^{-1}g(m) \\ &= (D(i))^{-1}C(i)\left[K_{i-1} - \sum_{m=1}^{i-1}U^{>}_{im}B(m)(D(m))^{-1}C(m)K_{m-1}\right].\end{aligned}$$

Furthermore, using the equalities

$$B(m)(D(m))^{-1}C(m) = E(m) - U(m) \tag{14.30}$$

and $E(m)K_{m-1} = K_m,\ U^{>}_{im}U(m) = U^{>}_{i,m-1}$ one gets

$$(L^{-1}G)(i,:) = (D(i))^{-1}C(i)\left[K_{i-1} - \sum_{m=1}^{i-1} U^{>}_{im}(E(m) - U(m))K_{m-1}\right]$$
$$= (D(i))^{-1}C(i)\left[K_{i-1} - \sum_{m=1}^{i-1}(U^{>}_{im}K_m - U^{>}_{i,m-1}K_{m-1})\right] = (D(i))^{-1}C(i)U^{>}_{i,0}K_0.$$

In a similar way we prove that

$$-HL^{-1} = \operatorname{row}\left(K_N^T U^{>}_{N+1,j}B(j)(D(j))^{-1}\right)_{j=1}^{N}.$$

Namely, using (14.26) and the last equality from (14.29) one gets

$$-(HL^{-1})(:,j) = -h(j)(D(j))^{-1} + \sum_{m=j+1}^{N} h(m)(D(m))^{-1}C(m)U^{>}_{mj}B(j)(D(j))^{-1}$$
$$= \left[K_j^T - \sum_{m=j+1}^{N} K_m^T B(m)(D(m))^{-1}C(m)U^{>}_{mj}\right] B(j)(D(j))^{-1}.$$

Furthermore, using (14.30) and $U(m)U^{>}_{mj} = U^{>}_{m+1,j},\ K_m^T E(m) = K_{m-1}^T$ one gets

$$-(HL^{-1})(:,j) = \left[K_j^T + \sum_{m=j+1}^{N}(K_{m-1}^T U^{>}_{mj} - K_m^T U^{>}_{m+1,j})\right] B(j)(D(j))^{-1}$$
$$= K_N^T U^{>}_{N+1,j}B(j)(D(j))^{-1}.$$

Finally,

$$U^{\times} = I_{r_U} + HL^{-1}G = I_{r_U} + \sum_{i=1}^{N} h(i)(D(i))^{-1}C(i)U^{>}_{i,0}K_0$$
$$= I_{r_U} - \sum_{i=1}^{N} K_i^T B(i)(D(i))^{-1}C(i)U^{>}_{i,0}K_0 = I_{r_U} - \sum_{i=1}^{N} K_i^T(E(i) - U(i))U^{>}_{i,0}K_0$$
$$= I_{r_U} - \sum_{i=1}^{N}(K_{i-1}^T U^{>}_{i,0} - K_i^T U^{>}_{i+1,0})K_0 = K_N^T U^{>}_{N+1,0}K_0,$$

which completes the proof. □

In accordance with the formula (14.25), the solution of the system $Ax = y$ is given by the relation

$$x(i) = (D(i))^{-1}y(i) - (D(i))^{-1}C(i)f_i - (D(i))^{-1}C(i)G_i c, \quad 1 \le i \le N, \tag{14.31}$$

where

$$G_i = U^{>}_{i,0}K_0, \quad f_i = \sum_{j=1}^{i-1} U^{>}_{ij}B(j)(D(j))^{-1}y(j), \quad c = -(U^{\times})^{-1}K_N^T f_{N+1}.$$

One can define the elements G_i and f_i recursively:

$$G_1 = K_0, \quad G_{i+1} = U(i)G_i, \qquad 1 \le i \le N; \tag{14.32}$$

$$f_1 = 0, \qquad f_{i+1} = U(i)f_i + B(i)(D(i))^{-1}y(i), \quad 1 \le i \le N \tag{14.33}$$

and rewrite (14.31) as

$$x(i) = (D(i))^{-1}y(i) - (D(i))^{-1}C(i)\chi_i, \tag{14.34}$$

where

$$\chi_i = G_i c + f_i, \quad 1 \le i \le N+1 \tag{14.35}$$

and

$$c = -(K_N^T G_{N+1})^{-1}(K_N^T f_{N+1}). \tag{14.36}$$

§14.2.2 The orthogonalization procedure

The formulas (14.32)–(14.36) give an $O(N)$ algorithm for the solution of the linear system $Ax = y$. However, the direct computation of the solution by this algorithm for large N may lead to considerable errors. This occurs because of large entries in the elements $G_i,\ f_i$ and the fact that the matrix G_i becomes close to a matrix of rank one, which leads to large errors in the inversion of the matrix $K_N^T G_{N+1}$. To improve the performance of the algorithm we apply an orthogonalization procedure which is well known from the shooting method used in the numerical analysis for differential equations (see, for instance [1]). The idea we take from this method is to use on every step, instead of (14.35), another representation

$$\chi_i = \Omega_i c_i + \phi_i, \quad 1 \le i \le N+1, \tag{14.37}$$

where Ω_i are $(r^L_{i-1} + r_U) \times r_U$ matrices with orthonormal columns and ϕ_i are $(r^L_{i-1} + r_U)$-dimensional vectors orthogonal to the columns of Ω_i.

In the first step one has $G_1 = K_0,\ f_1 = 0$, and setting $c_1 = c$ one gets $\chi_1 = G_1 c_1$. Here G_1 is a matrix with orthonormal columns and setting $\Omega_1 = G_1,\ \phi_1 = 0$ one obtains (14.37) with $i = 1$.

Let for some i with $1 \le i \le N-1$ the equality (14.37) hold. Note that from (14.32), (14.33) and (14.35) it follows that

$$\chi_{i+1} = G_{i+1}c + f_i = U(i)G_i c + U(i)f_i + B(i)(D(i))^{-1}y(i),$$

whence

$$\chi_{i+1} = U(i)\chi_i + B(i)(D(i))^{-1}y(i).$$

Using (14.37) one gets

$$\chi_{i+1} = U(i)\Omega_i c_i + U(i)\phi_i + B(i)(D(i))^{-1}y(i).$$

Next we compute the matrix G'_{i+1} and the vector f'_{i+1} by the formulas

$$G'_{i+1} = U(i)\Omega_i, \quad f'_{i+1} = U(i)\phi_i + B(i)(D(i))^{-1}y(i)$$

and obtain

$$\chi_{i+1} = G'_{i+1}c_i + f'_{i+1}.$$

For the matrix G'_{i+1} we compute the QR factorization $G'_{i+1} = \Omega_{i+1}\Lambda_{i+1}$ with the matrix Ω_{i+1} with orthonormal columns and the upper triangular matrix Λ_{i+1}. Hence. one has

$$\chi_{i+1} = \Omega_{i+1}\Lambda_{i+1}c_i + f'_{i+1}. \tag{14.38}$$

Next we compute the vector

$$\phi_{i+1} = f'_{i+1} - \Omega_{i+1}\Omega^*_{i+1}f'_{i+1}. \tag{14.39}$$

Using $\Omega^*_{i+1}\Omega_{i+1} = I$ one gets $\Omega^*_{i+1}\phi_{i+1} = 0$ which implies that ϕ_{i+1} is orthogonal to the columns of Ω_{i+1}. Finally, substituting (14.39) in (14.38) one obtains

$$\chi_{i+1} = \Omega_{i+1}c_{i+1} + \phi_{i+1},$$

with

$$c_{i+1} = \Lambda_{i+1}c_i + \Omega^*_{i+1}f'_{i+1}.$$

Thus, under the assumption that all the matrices Λ_k $(k = 2, \dots, N+1)$ are invertible (this holds for instance if all the matrices $U(k)$ $(k = 1, \dots, N)$ are invertible) one obtains the following algorithm.

Algorithm 14.5.

1. Starting with $\Omega_1 = K_0 = \begin{pmatrix} 0_{r_0^L \times r_U} \\ I_{r_U} \end{pmatrix}$, $\phi_1 = 0_{(r_0^L + r_U)\times 1}$ and for $i = 1, \dots, N$, perform the following operations:

1.1. Compute

$$B(i) = \begin{bmatrix} q(i) \\ -h(i) \end{bmatrix}, \qquad C_i = \begin{bmatrix} p(i) & g(i) \end{bmatrix},$$

$$D(i) = d(i) - g(i)h(i), \quad E(i) = \begin{pmatrix} a(i) & 0 \\ 0 & I_{r_U} \end{pmatrix},$$

$$\tilde{B}_i = B(i)(D(i))^{-1}, \quad U(i) = E(i) - \tilde{B}_i C(i),$$

$$G'_{i+1} = U(i)\Omega_i, \qquad f'_{i+1} = U(i)\phi_i + \tilde{B}_i y(i).$$

1.2. Using a standard orthogonalization procedure determine the matrix Ω_{i+1} with orthonormal columns and the upper triangular matrix Λ_{i+1} such that $G'_{i+1} = \Omega_{i+1}\Lambda_{i+1}$.

1.3. Compute

$$\phi_{i+1} = f'_{i+1} - \Omega_{i+1}\Omega^*_{i+1} f'_{i+1}.$$

2. Compute

$$c_{N+1} = -[K_N^T \Omega_{N+1}]^{-1}(K_N^T \phi_{N+1}).$$

3. For $i = N, \dots, 1$, compute

$$c_i = \Lambda_{i+1}^{-1}(c_{i+1} - \Omega^*_{i+1} f'_{i+1}),$$
$$\chi_i = \Omega_i c_i + \phi_i,$$
$$x(i) = (D(i))^{-1} y(i) - (D(i))^{-1} C(i)\chi_i.$$

Set $m = \max_{1 \le k \le N}(m_k)$, $r = \max\{(\max_{0 \le k \le N-1} r_k^L), r^U)$. The complexity of Algorithm 14.5 is estimated as follows.

1. The matrix $D(i)$: m^2 arithmetical additions and a matrix multiplication which costs $m^2 r$ arithmetical multiplications and $m^2(r-1)$ arithmetical additions.
2. The matrix $\tilde{B}_i$: $r\rho(m)$ arithmetical operations and a matrix multiplication which costs $2rm^2$ arithmetical multiplications and $2r(m-1)m$ arithmetical additions.
3. The matrix $U(i)$: $(2r)^2$ arithmetical additions and a matrix multiplication which costs $(2r)m(2r)$ arithmetical multiplications and $(2r)(m-1)(2r)$ arithmetical additions.
4. The matrix G'_{i+1}: the multiplication of a $2r \times 2r$ matrix with a $2r \times r$ matrix thus less than $8r^3$ arithmetical operations.
5. The computation of f'_{i+1}: two matrix vector multiplications and a vector addition, thus less than $8r^2 + 4rm + 2r$ arithmetical operations.
6. The product $\Omega^*_{i+1} f'_{i+1}$: less than $8r^2$ operations.
7. Step 1.2: $\varphi(r)$ operations.

8. The vector ϕ_{i+1}: less than $4r^2 + 2r$ operations.
9. The vector c_i: another $\varrho(r)$ operations.
10. The vector χ_i: less than $4r^2$ operations
11. The vector $x(i)$: less than $2m^2 + 2m^2 r$ operations.

Here $\varphi(r)$ is the complexity of the QR factorization of a $2r \times r$ matrix, $\rho(r)$ is the complexity of the solution of an $r \times r$ linear system by the standard Gauss method, $\varrho(r)$ is the complexity of the solution of an $r \times r$ linear triangular system by a standard method. Thus, the total complexity of the algorithm does not exceed

$$\left(6m^2 r + \rho(m) + 4m^2 r + 8r^2 m + 8r^3 + 20r^2 + 4r + 2m^2 + \varrho(r) + \varphi(r)\right) N.$$

§14.3 Comments

The diagonal inversion method was suggested by I. Koltracht in [41] for matrices with diagonal plus semiseparable representations. The extension of this method to matrices with quasiseparable representations as well as the treatment via discrete descriptor system appears here for the first time.

Theorem 14.4 in a more general setting was obtained by I. Gohberg and M.A. Kaashoek in [37]. The presentation of the results in §14.2 follows the paper [22] in which results of numerical tests are also presented.

The methods of these chapter were extended in [24], [25] to diagonal plus semiseparable operator matrices; these papers contain also results of numerical tests for block matrices and integral equations.

Chapter 15

Inversion of Matrices in Diagonal Plus Semiseparable Form

Here we study in detail the inversion methods for matrices with diagonal plus semiseparable representations. For scalar matrices we obtain an inversion algorithm without any restrictions.

§15.1 The modified inversion formula

In this section we obtain a specification of the formula (14.25) for matrices with diagonal plus semiseparable representation.

Theorem 15.1. *Let A be a scalar matrix with lower semiseparable generators $p(k), q(k)$, $(k = 1, \dots, N)$ of order r_L, upper semiseparable generators $g(k), h(k)$ of order r_U, and diagonal entries $d(k)$ $(k = 1, \dots, N)$. Assume that the matrices*

$$\delta_k = d(k) - g(k)h(k), \quad l_k = d(k) - p(k)q(k), \quad k = 1, \dots, N \tag{15.1}$$

are invertible.

Introduce the matrices

$$B(k) = \begin{pmatrix} q(k) \\ -h(k) \end{pmatrix}, \qquad C(k) = \begin{pmatrix} p(k) & g(k) \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0_{r_L \times r_U} \\ I_{r_U} \end{pmatrix},$$

$$E(k) = \begin{pmatrix} g(k) & -p(k) \end{pmatrix}, \quad F(k) = \begin{pmatrix} h(k) \\ q(k) \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 0_{r_U \times r_L} \\ I_{r_L} \end{pmatrix}$$

and define

$$U(k) = I_{r_L + r_U} - B(k)\delta_k^{-1}C(k), \tag{15.2}$$

$$U^{\times} = K_0^T U^{>}_{N+1,0} K_0; \tag{15.3}$$

$$V(k) = I_{r_L + r_U} - F(k)l_k^{-1}V(k), \tag{15.4}$$

$$V^{\times} = M_0^T V^{<}_{0,N+1} M_0. \tag{15.5}$$

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Then the matrix A is invertible if and only if the matrix $U^\times$ and/or the matrix $V^\times$ are invertible, and in this case the inverse matrix A^{-1} is given by the formula

$$A^{-1}(i,j) = \begin{cases} (\delta_i)^{-1}(C(i)U^{>}_{i,0}K_0)(U^\times)^{-1}(K_0^T U^{>}_{N+1,j}B(j))(\delta_j)^{-1}, & 1 \le i < j \le N, \\ (\delta_i)^{-1} + (\delta_i)^{-1}(C(i)U^{>}_{i,0}K_0)(U^\times)^{-1}(K_N^T U^{>}_{N+1,i}B(i))(\delta_i)^{-1}, & i = j, \\ (l_i)^{-1}(E(i)V^{<}_{i,N+1}M_0)(V^\times)^{-1}(M_0^T V^{<}_{0,j}F(j))(l_j)^{-1}, & 1 \le j < i \le N. \end{cases} \tag{15.6}$$

Remark. The formula (15.6) means that under the assumption that the matrices (15.1) are invertible, the inverse matrix A^{-1} has semiseparable generators with the same orders as the original matrix. More precisely the elements

$$(\delta_k)^{-1}(C(k)U^{>}_{k,0}K_0)(U^\times)^{-1}, \quad (K_0^T U^{>}_{N+1,k}B(k))(\delta_k)^{-1}, \quad k = 1,\dots,N,$$

are upper semiseparable generators of order r_U and the elements

$$(l_k)^{-1}(E(k)V^{<}_{k,N+1}M_0)(V^\times)^{-1}, \quad (M_0^T V^{<}_{0,k}F(k))(l_k)^{-1}, \quad k = 1,\dots,N,$$

are lower semiseparable generators of order r_L of A^{-1}.

Proof of the theorem. By Theorem 14.4, the matrix A is invertible if and only if $U^\times$ is invertible. Moreover in this case the formulas from (15.6) with $i \le j$ follow directly from (14.25)–(14.27).

In order to obtain the last formula in (15.6) (with $i > j$) consider the matrix A^T. This matrix has lower semiseparable generators $h^T(k), g^T(k)$ of order r_U, upper semiseparable generators $q^T(k), p^T(k)$ of order r_L, and diagonal entries $d^T(k)$. Set

$$B'(k) = \begin{bmatrix} g^T(k) \\ -p^T(k) \end{bmatrix}, \; C'(k) = \begin{bmatrix} h^T(k) & q^T(k) \end{bmatrix}, \quad \delta'_k = d^T(k) - q^T(k)p^T(k),$$

and define

$$U'(k) = I_{r_L+r_U} - B'(k)(\delta'_k)^{-1}C'(k), \quad \tilde{U}^\times = M_0^T (U')^{>}_{N+1,0}M_0.$$

Using (14.28), we get

$$\det A^T = \left(\prod_{i=1}^N \det \delta'_i\right) \det \tilde{U}^\times. \tag{15.7}$$

Moreover, applying the first formula in (15.6) to the matrix A^T we get

$$(A^T)^{-1}(i,j) = (\delta'_i)^{-1}(C'(i)(U')^{>}_{i,0}M_0)(\tilde{U}^\times)^{-1}(M_0^T (U')^{>}_{N+1,j}B'(j))(\delta'_j)^{-1}, \quad 1 \le i < j \le N. \tag{15.8}$$

Using the transposed matrices in (15.7) and the equalities

$$(\delta_k')^T = l_k,\ (B'(k))^T = E(k),\ (C'(k))^T = F(k),\ (U'(k))^T = V(k), \tag{15.9}$$

we get

$$\det A = \left(\prod_{i=1}^{N} \det l_i\right) \det V^{\times}. \tag{15.10}$$

Therefore, the matrix A is invertible if and only if the matrix $V^{\times}$ is invertible. Taking the transposed matrices in (15.8) and using the equalities (15.9) we obtain the last formula in (15.6). □

Example 15.2. Consider the $N \times N$ matrix

$$A = \begin{pmatrix} 0 & a & a & \cdots & a & a \\ -a & 0 & a & \cdots & a & a \\ -a & -a & 0 & \cdots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a & -a & -a & \cdots & 0 & a \\ -a & -a & -a & \cdots & -a & 0 \end{pmatrix},$$

where $a \neq 0$ is a scalar.

For the matrix A one can use the semiseparable generators

$$\begin{aligned} &p(i) = a,\ i = 2, \dots, N, \qquad q(j) = -1, j = 1, \dots, N-1,\ \ d(k) = 0,\ k = 1, \dots, N, \\ &g(j) = 1, j = 1, \dots, N-1,\ \ h(i) = a, \quad i = 2, \dots, N. \end{aligned}$$

Let us use Theorem 15.1 to find the inverse matrix. First,

$$\delta_k = d(k) - g(k)h(k) = 0 - a = -a, \quad l_k = d(k) - p(k)q(k) = 0 - a(-1) = a.$$

Moreover $r_L = r_U = 1$. Introduce the matrices

$$\begin{aligned} &B(k) = \begin{pmatrix} q(k) \\ -h(k) \end{pmatrix} = \begin{pmatrix} -1 \\ -a \end{pmatrix},\ \ C(k) = \begin{pmatrix} p(k) & g(k) \end{pmatrix} = \begin{pmatrix} a & 1 \end{pmatrix},\ \ K_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &E(k) = \begin{pmatrix} 1 & -a \end{pmatrix}, \quad f(k) = \begin{pmatrix} a \\ -1 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned}$$

and define

$$\begin{aligned} U(k) &= I_{r_L + r_U} - \frac{1}{\delta_k} B(k)C(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{-a} \begin{pmatrix} -1 \\ -a \end{pmatrix} \begin{pmatrix} a & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{a} \\ a & 1 \end{pmatrix} = -\begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix}. \end{aligned}$$

Then the product $U(k)U(k+1)$ for two consecutive indices is equal to

$$(-1)^2 \begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

whence

$$U^{>}_{N+1,0} = U(N)U(N-1)\cdot \dots \cdot U(1) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & N = 2r, \\ -\begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix}, & N = 2r-1, \end{cases}$$

and so

$$U^{\times} = K_0^T U^{>}_{N+1,0} K_0 = \begin{cases} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1, & N = 2r, \\ -\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, & N = 2r-1. \end{cases}$$

It follows that A is invertible if and only if N is even. The same result is given by $V^{\times}$. Indeed, one has

$$V(k) = I_{r_L+r_U} - \frac{1}{l_k} F(k)E(k) = U^T(k),$$

therefore

$$V^{<}_{0,N+1} = V(1)V(2)\cdot \dots \cdot V(N) = (-1)^N U^{>}_{N+1,0}$$

and

$$V^{\times} = M_0^T V^{<}_{0,N+1} M_0 = \begin{cases} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1, & N = 2r, \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, & N = 2r-1. \end{cases}$$

Suppose in the sequel that N is even. Then A is invertible according to Theorem 15.1, and by the same theorem one can compute the inverse matrix A^{-1} as follows.

First compute

$$U^{>}_{i,0} = U(i-1)U(i-2)\cdot \dots \cdot U(1) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & i = 2m-1, \\ -\begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix}, & i = 2m, \end{cases}$$

and

$$U^{>}_{N+1,j} = U(N)U(N-1)\cdot \dots \cdot U(j+1) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & j = 2n, \\ -\begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix}, & j = 2n-1, \end{cases}$$

and using them compute

$$\frac{1}{\delta_i}(C(i)U^{>}_{i,0}K_0) = \begin{cases} -\frac{1}{a}\begin{pmatrix} a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{a}, & i = 2m-1, \\ \frac{1}{a}\begin{pmatrix} a & 1 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{a}, & i = 2m, \end{cases}$$

and

$$(K_0^T U^{>}_{N+1,j}B(j))\frac{1}{\delta_j} = \begin{cases} \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -a \end{pmatrix}(-\frac{1}{a}) = 1, & j = 2n, \\ -\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{a} \\ a & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -a \end{pmatrix}(-\frac{1}{a}) = -1, & j = 2n-1. \end{cases}$$

Then if $1 \le i < j \le N$ it follows that

$$A^{-1}(i,j) = \frac{1}{\delta_i}(C(i)U^{>}_{i,0}K_0)(U^{\times})^{-1}(K_0^T U^{>}_{N+1,j}B(j))\frac{1}{\delta_j} = (-1)^{j-i}\frac{1}{a}.$$

Also,

$$A^{-1}(i,i) = \frac{1}{\delta_i} + \frac{1}{\delta_i}(C(i)U^{>}_{i,0}K_0)(U^{\times})^{-1}(K_0^T U^{>}_{N+1,i}B(i))\frac{1}{\delta_i} = -\frac{1}{a} + (-1)^{i-i}\frac{1}{a} = 0.$$

In order to compute the remaining entries of the inverse matrix, first compute

$$V^{<}_{i,N+1} = V(i+1)V(i+2)\cdot\dots\cdot V(N) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & i = 2m, \\ \begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix}, & i = 2m-1, \end{cases}$$

and

$$V^{<}_{0,j} = V(1)V(2)\cdot\dots\cdot V(j-1) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & j = 2n-1, \\ \begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix}, & j = 2n, \end{cases}$$

and using them compute

$$\frac{1}{l_i}(E(i)V^{<}_{i,N+1}M_0) = \begin{cases} \frac{1}{a}\begin{pmatrix} 1 & -a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1, & i = 2m, \\ \frac{1}{a}\begin{pmatrix} 1 & -a \end{pmatrix}\begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1, & i = 2m-1, \end{cases}$$

and

$$(M_0^T V^{<}_{0,j}F(j))\frac{1}{l_j} = \begin{cases} \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ -1 \end{pmatrix}\frac{1}{a} = -\frac{1}{a}, & j = 2n-1, \\ \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix}\begin{pmatrix} a \\ -1 \end{pmatrix}\frac{1}{a} = \frac{1}{a}, & j = 2n. \end{cases}$$

Then if $1 \leq j < i \leq N$ it follows that

$$A^{-1}(i,j) = \frac{1}{l_i}(E(i)V^<_{i,N+1}M_0)(V^\times)^{-1}(M_0^T V^<_{0,j}F(j))\frac{1}{l_j} = (-1)^{i-j+1}\frac{1}{a}.$$

Therefore, the inverse matrix is

$$A^{-1} = \begin{pmatrix} 0 & -\frac{1}{a} & \frac{1}{a} & -\frac{1}{a} & \cdots & \frac{1}{a} & -\frac{1}{a} \\ \frac{1}{a} & 0 & -\frac{1}{a} & \frac{1}{a} & \cdots & \frac{1}{a} & \frac{1}{a} \\ -\frac{1}{a} & \frac{1}{a} & 0 & -\frac{1}{a} & \cdots & \frac{1}{a} & -\frac{1}{a} \\ \frac{1}{a} & -\frac{1}{a} & \frac{1}{a} & 0 & \cdots & \frac{1}{a} & \frac{1}{a} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{a} & \frac{1}{a} & -\frac{1}{a} & \frac{1}{a} & \cdots & 0 & -\frac{1}{a} \\ \frac{1}{a} & -\frac{1}{a} & \frac{1}{a} & -\frac{1}{a} & \cdots & \frac{1}{a} & 0 \end{pmatrix}.$$

◊

§15.2 Scalar matrices with diagonal plus semiseparable representation

The next aim is to derive from Theorem 15.1 an algorithm to compute quasiseparable generators of the inverse to a matrix A with given semiseparable generators, in the case where some of the numbers δ_k or l_k may be zeros.

First we present another version of Theorem 15.1 with some auxiliary matrices.

Lemma 15.3. *Let the conditions of Theorem* 15.1 *hold. Let* Δ_k, Ψ_k $(k = 1, \dots, N)$ *and* Γ_k, Ω_k $(k = 1, \dots, N)$ *be invertible matrices of orders* r_U *and* r_L*, respectively.*

Define the matrices Z_k, S_k, X_k, Y_k *of sizes* $(r_L + r_U) \times r_U, r_U \times (r_L + r_U), r_L \times (r_L + r_U), (r_L + r_U) \times r_L$ *via the forward recursions*

$$Z_1 = K_0, \quad Z_{k+1} = Z_k\Delta_k - B(k)\left((\delta_k)^{-1}C(k)Z_k\Delta_k\right), \quad k = 1, \dots, N, \quad (15.11)$$

$$X_1 = M_0^T, \quad X_{k+1} = \Gamma_k X_k - \left(\Gamma_k X_k F(k)(l_k)^{-1}\right)C(k), \quad k = 1, \dots, N, \quad (15.12)$$

and backward recursions

$$S_{N+1} = K_0^T, \; S_k = \Psi_k S_{k+1} - \left(\Psi_k S_{k+1}B(k)(\delta_k)^{-1}\right)C(k), \; k = N, \dots, 1, \quad (15.13)$$

$$Y_{N+1} = M_0, \; Y_k = Y_{k+1}\Omega_k - F(k)\left((l_k)^{-1}E(k)Y_{k+1}\Omega_k\right), \; k = N, \dots, 1. \quad (15.14)$$

Then the elements of the inverse matrix A^{-1} *can be expressed as*

$$A^{-1}(i,j) = \left((\delta_i)^{-1}C(i)Z_i\Delta_i\right)(S_{i+1}Z_{i+1})^{-1}\Psi^<_{ij}\left(\Psi_j S_{j+1}B(j)(\delta_j)^{-1}\right), \; i < j \quad (15.15)$$

and

$$A^{-1}(i,j) = \left((l_i)^{-1}C(i)Y_{i+1}\Omega_i\right)(X_iY_i)^{-1}\Gamma^>_{ij}\left(\Gamma_j X_j F(j)(l_j)^{-1}\right), \quad i > j. \quad (15.16)$$

Proof. First we consider the elements of (15.6) with $i < j$. The expression $U^{>}_{i,0}K_0$ may be written in the form

$$U^{>}_{i,0}K_0 = Z_i\Delta^{-1}_{i-1}\cdots\Delta^{-1}_{1}, \quad i = 1,\dots,N+1. \tag{15.17}$$

Indeed, for $i = 1$ (15.17) is obvious. Assume that for some i with $1 \le i \le N$ (15.17) has been proven. One has

$$U^{>}_{i+1,0}K_0 = U(i)U^{>}_{i,0}K_0 = U(i)Z_i\Delta^{-1}_{i-1}\cdots\Delta^{-1}_{1}$$

and using (15.2) and (15.11) one gets

$$\begin{aligned} U(i)Z_i &= \left(I - B(i)(\delta_i)^{-1}C(i)\right) Z_i \\ &= \left(Z_i\Delta_i - B(i)\left((\delta_i)^{-1}C(i)Z_i\Delta_i\right)\right)\Delta^{-1}_i = Z_{i+1}\Delta^{-1}_i, \end{aligned}$$

as needed.

Similarly, one gets

$$K_0^T U^{>}_{N+1,j} = \Psi^{-1}_N\cdots\Psi^{-1}_{j+1}S_{j+1}, \quad j = N,\dots,0. \tag{15.18}$$

Indeed, for $j = N$ (15.18) is obvious. Assume that for some j with $N \ge j \ge 1$ (15.18) has been proven. One has

$$K_0^T U^{>}_{N+1,j-1} = K_0^T U^{>}_{N+1,j}U(j) = \Psi^{-1}_N\cdots\Psi^{-1}_{j+1}S_{j+1}U(j)$$

and using (15.2) and (15.13) one gets

$$\begin{aligned} S_{j+1}U(j) &= S_{j+1}\left(I - B(j)(\delta_j)^{-1}C(j)\right) \\ &= \Psi^{-1}_j\left(\Psi_jS_{j+1} - \left(\Psi_jS_{j+1}B(j)(\delta_j)^{-1}\right)C(j)\right) = \Psi^{-1}_jS_j, \end{aligned}$$

as needed.

From (15.18), (15.17) and (15.3) one obtains for $U^{\times}$ for any $k = 0,\dots,N$ the representations

$$U^{\times} = K_0^T U^{>}_{N+1,k}U^{>}_{k+1,0}K_0 = \Psi^{-1}_N\cdots\Psi^{-1}_{k+1}(S_{k+1}Z_{k+1})\Delta^{-1}_k\cdots\Delta^{-1}_1. \tag{15.19}$$

Thus using the first expression in (15.6) and the representations (15.17), (15.18) and (15.19) one gets

$$\begin{aligned} A^{-1}(i,j) = {} & (\delta_i)^{-1}C(i)Z_i\Delta^{-1}_{i-1}\cdots\Delta^{-1}_1\Delta_1\cdots\Delta_{i-1}\Delta_i(S_{i+1}Z_{i+1})^{-1} \\ & \times \Psi_{i+1}\cdots\Psi_{j-1}\Psi_j\Psi_{j+1}\cdots\Psi_N\Psi^{-1}_N\cdots\Psi^{-1}_{j+1}S_{j+1}B(j)(\delta_j)^{-1}, \quad i < j \end{aligned}$$

and hence the representation (15.15) follows.

A similar procedure is applied to the case $i > j$. The expression $M_0^TV^{<}_{0,j}$ may be written in the form

$$M_0^TV^{<}_{0,j} = \Gamma^{-1}_1\cdots\Gamma^{-1}_{j-1}X_j, \quad 1 \le j \le N+1. \tag{15.20}$$

Indeed, for $j = 1$ (15.20) is obvious. Assume that for some j with $1 \leq j \leq N$ (15.20) has been proven. One has

$$M_0^T V_{0,j+1}^{<} = M_0^T V_{0,j}^{<} V(j) = \Gamma_1^{-1} \cdots \Gamma_{j-1}^{-1} X_j V(j)$$

and using (15.4) and (15.12) one gets

$$\begin{aligned} X_j V(j) &= X_j \left(I - F(j)(l_j)^{-1} E(j)\right) \\ &= \Gamma_j^{-1} \left(\Gamma_j X_j - \left(\Gamma_j X_j F(j)(l_j)^{-1}\right) E(j)\right) = \Gamma_j^{-1} X_{j+1}, \end{aligned}$$

as needed.

Similarly, one gets

$$V_{i,N+1}^{<} M_0 = Y_{i+1} \Omega_{i+1}^{-1} \cdots \Omega_N^{-1}, \quad i = N, \dots, 0. \tag{15.21}$$

Indeed, for $i = N$ (15.21) is obvious. Assume that for some i with $N > i \geq 1$ (15.21) has been proven. One has

$$V_{i-1,N+1}^{<} M_0 = V(i) V_{i,N+1}^{<} M_0 = V(i) Y_{i+1} \Omega_{i+1}^{-1} \cdots \Omega_N^{-1}$$

and using (15.4) and (15.14) one gets

$$\begin{aligned} V(i) Y_{i+1} &= \left(I - F(i)(l_i)^{-1} E(i)\right) Y_{i+1} \\ &= \left(Y_{i+1} \Omega_i - F(i) \left((l_i)^{-1} E(i) Y_{i+1} \Omega_i\right)\right) \Omega_i^{-1} = Y_i \Omega_i^{-1}, \end{aligned}$$

as needed.

From (15.21), (15.20) and (15.5) one obtains for $V^{\times}$ for any $k = 0, \dots, N$ the representations

$$V^{\times} = M_0^T V_{0,k}^{<} V_{k-1,N+1}^{<} M_0 = \Gamma_1^{-1} \cdots \Gamma_{i-1}^{-1} (X_i Y_i) \Omega_i^{-1} \cdots \Omega_N^{-1}. \tag{15.22}$$

Thus using the last expression in (15.6) and the representations (15.20), (15.21) and (15.22) one gets

$$\begin{aligned} A^{-1}(i,j) = (l_i)^{-1} E(i) Y_{i+1} \Omega_{i+1}^{-1} \cdots \Omega_N^{-1} \Omega_N \cdots \Omega_{i+1} \Omega_i (X_i Y_i)^{-1} \\ \times \Gamma_{i-1} \cdots \Gamma_{j+1} \Gamma_j \cdots \Gamma_1 \Gamma_1^{-1} \cdots \Gamma_{j-1}^{-1} X(j) F(j) (l_j)^{-1} \end{aligned}$$

and hence the representation (15.16) follows. □

Now assume that A is a scalar matrix. In this case the elements δ_k, l_k are complex numbers. By making the concrete appropriate choice of the matrices $\Delta_k, \Psi_k, \Gamma_k, \Omega_k$ we may eliminate singularities for the values $\delta_k = 0,\ l_k = 0$ in the expressions

$$\frac{1}{\delta_k} C(k) Z_k \Delta_k, \quad \Gamma_k X_k F(k) \frac{1}{l_k}, \quad \Psi_k S_{k+1} B(k) \frac{1}{\delta_k}, \quad \frac{1}{l_k} E(k) Y_{k+1} \Omega_k \tag{15.23}$$

used in Lemma 15.3. Next, let us introduce the following notations. For a vector u, u_{j_0} means the coordinate of u with the maximal absolute value and $\tilde{u}$ is a vector with the same dimension as u with u_{j_0} in the j_0th position and zeros in the others. Now for an n-dimensional column vector u and a number δ we define the $n \times n$ matrix $W(u,\delta)$ by

$$W(u,\delta) = \begin{pmatrix} 1 & 0 & \dots & -\frac{u_1}{u_{j_0}} & \dots & 0 \\ 0 & 1 & \dots & -\frac{u_2}{u_{j_0}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -\frac{u_n}{u_{j_0}} & \dots & 1 \end{pmatrix}$$

for $u \neq 0$, with δ in the (j_0, j_0) position, and $W(u,\delta) = \mathrm{diag}\{\delta, I_{n-1}\}$ for $u = 0$. One obviously has

$$\det W(u,\delta) = \delta \tag{15.24}$$

and furthermore

$$W(u,\delta)u = \begin{pmatrix} 1 & 0 & \dots & -\frac{u_1}{u_{j_0}} & \dots & 0 \\ 0 & 1 & \dots & -\frac{u_2}{u_{j_0}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -\frac{u_n}{u_{j_0}} & \dots & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{j_0} \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \delta u_{j_0} \\ \vdots \\ 0 \end{pmatrix},$$

whence

$$W(u,\delta)u = \delta\tilde{u}. \tag{15.25}$$

Similarly, for an n-dimensional row v one obtains

$$vW^T(v^T,\delta) = \delta\tilde{v}. \tag{15.26}$$

Theorem 15.4. *Let A be a scalar matrix with lower semiseparable generators $p(k), q(k)$ $(k = 1, \dots, N)$ of order r_L, upper semiseparable generators $g(k), h(k)$ of order r_U, and diagonal entries $d(k)$ $(k = 1, \dots, N)$.*

Introduce the matrices and the numbers

$$B(k) = \begin{pmatrix} q(k) \\ -h(k) \end{pmatrix}, \qquad C(k) = \begin{pmatrix} p(k) & g(k) \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0_{r_L \times r_U} \\ I_{r_U} \end{pmatrix},$$

$$E(k) = \begin{pmatrix} g(k) & -p(k) \end{pmatrix}, \quad F(k) = \begin{pmatrix} h(k) \\ q(k) \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 0_{r_U \times r_L} \\ I_{r_L} \end{pmatrix},$$

$$\delta_k = d(k) - g(k)h(k), \qquad l_k = d(k) - p(k)q(k), \qquad 1 \le k \le N,$$

and define forward recursively

$$Z_1 = K_0, \quad v_k = C(k)Z_k,\ Z_{k+1} = Z_k W^T(v_k^T, \delta_k) - B(k)\tilde{v}_k,\ k = 1, \dots, N, \tag{15.27}$$

$$X_1 = M_0^T, \quad f_k = X_k F(k),\ X_{k+1} = W(f_k, l_k)X_k - \tilde{f}_k E(k), \quad k = 1, \dots, N \tag{15.28}$$

and backward recursively

$$S_{N+1} = K_0^T, \quad u_k = S_{k+1}B(k),\ S_k = W(u_k, \delta_k)S_{k+1} - \tilde{u}_k C(k), \quad k = N, \dots, 1, \tag{15.29}$$

$$Y_{N+1} = M_0, \quad w_k = E(k)Y_{k+1},\ Y_k = Y_{k+1}W^T(w_k^T, l_k) - F(k)\tilde{w}_k,\ k = N, \dots, 1. \tag{15.30}$$

Then

$$\det(S_k Z_k) = \det(X_k Y_k) = \det A \neq 0, \quad 1 \leq k \leq N+1, \tag{15.31}$$

and the inverse matrix A^{-1} has the lower quasiseparable generators

$$\tilde{w}_k (X_k Y_k)^{-1},\ \tilde{f}_k,\ W(f_k, l_k), \quad k = 1, \dots, N, \tag{15.32}$$

with orders equal to r_L, upper quasiseparable generators

$$\tilde{v}_k (S_{k+1} Z_{k+1})^{-1},\ \tilde{u}_k,\ W(u_k, \delta_k), \quad k = 1, \dots, N, \tag{15.33}$$

with orders equal to r_U, and diagonal entries

$$\frac{\det(S_{k+1} Z_k)}{\det A}, \quad k = 1, \dots, N. \tag{15.34}$$

Proof. In the expressions (15.23) we put

$$v_k = C(k)Z_k, \quad f_k = X_k F(k), \quad u_k = S_{k+1}B(k), \quad w_k = E(k)Y_{k+1}. \tag{15.35}$$

Next, we take

$$\Delta_k = W^T(v_k^T, \delta_k),\ \Gamma_k = W(f_k, l_k),\ \Psi_k = W(u_k, \delta_k),\ \Omega_k = W^T(w_k^T, l_k) \tag{15.36}$$

and using (15.25) and (15.26) we obtain

$$\begin{aligned} \frac{1}{\delta_k} C(k) Z_k \Delta_k = \tilde{v}_k, \qquad \Gamma_k X_k F(k) \frac{1}{l_k} = \tilde{f}_k, \\ \Psi_k S_{k+1} B(k) \frac{1}{\delta_k} = \tilde{u}_k, \quad \frac{1}{l_k} E(k) Y_{k+1} \Omega_k = \tilde{w}_k. \end{aligned} \tag{15.37}$$

Inserting this in the relations (15.11)–(15.14) we obtain the equalities (15.27)–(15.30), which do not contain δ_k, l_k in the denominators.

Next, substituting $\Delta_k = W^T(v_k^T, \delta_k)$, $\Psi_k = W(u_k, \delta_k)$ in (15.19) we obtain for $U^\times$ for $0 \leq k \leq N$ the representation

$$U^\times = W^{-1}(u_N, \delta_N) \cdots W^{-1}(u_{k+1}, \delta_{k+1})(S_{k+1}Z_{k+1})[W^T(v_k^T, \delta_k)]^{-1} \cdots [W^T(v_1^T, \delta_1)]^{-1}. \tag{15.38}$$

Hence, using (14.28) and (15.24) we conclude that

$$\det(S_k Z_k) = \det A, \quad k = 1, \dots, N+1. \tag{15.39}$$

Similarly, substituting $\Gamma_k = W(f_k, l_k)$, $\Omega_k = W^T(w_k^T, l_k)$ in (15.22) one obtains for $V^\times$ for $k = 0, \dots, N$ the representation

$$V^\times = W^{-1}(f_1, l_1) \cdots W^{-1}(f_{k-1}, l_{k-1})(X_k Y_k)[W^T(w_k^T, l_k)]^{-1} \cdots [W^T(w_N^T, l_N)]^{-1}. \tag{15.40}$$

Hence, using (15.10) and (15.24) we conclude that

$$\det(X_k Y_k) = \det A, \quad k = 1, \dots, N+1. \tag{15.41}$$

Next, substituting (15.37) in (15.15) and (15.16) we obtain

$$A^{-1}(i,j) = \tilde{w}_i (X_i Y_i)^{-1} \Gamma_{ij}^{>} \tilde{f}_j, \quad i > j,$$

with $\Gamma_k = W(f_k, l_k)$, and

$$A^{-1}(i,j) = \tilde{v}_i (S_{i+1} Z_{i+1})^{-1} \Psi_{ij}^{<} \tilde{u}_j, \quad i < j,$$

with $\Psi_k = W(u_k, \delta_k)$. It follows that the elements defined in (15.32) and (15.33) are lower and upper quasiseparable generators of the matrix A^{-1}.

To get the representations of the diagonal entries of the matrix A^{-1} we use the formula

$$A^{-1}(i,i) = \det A'_{ii} / \det A, \tag{15.42}$$

where A'_{ii} is the matrix obtained from A by removing its ith row and ith column. Then applying (15.39) to the matrix A'_{ii} we obtain

$$A^{-1}(i,i) = \det(S_{i+1} Z_i) / \det A. \qquad \square$$

Numerical experiments showed that direct computations by the formulas (15.27)–(15.34) for large N lead to overflow. This occurs because of large entries of the matrices Z_k, S_k, X_k, Y_k. This effect may be overcome by an appropriate scaling. For an $(m+n) \times n$ matrix $F = \mathrm{row}(F(i))_{i=1}^n$ and an $n \times (m+n)$ matrix $F = \mathrm{col}(F(i))_{i=1}^n$, the scaling matrix of size $n \times n$ is defined by

$$\beta(F) = \mathrm{diag}\{1/\|W(i)\|\}_{i=1}^n.$$

Here $\|\cdot\|$ is the Euclidean norm of the vector. In the numerical tests we tried such scaling successfully. Other variants of scaling may be used.

One can use instead of (15.27)–(15.30) other relations containing scaling matrices $\beta(\cdot)$. Set forward recursively

$$Z_1 = K_0, \quad v_k = C(k)Z_k\beta(Z_k), \; Z_{k+1} = Z_k\beta(Z_k)W^T(v_k^T, \delta_k) - B(k)\tilde{v}_k, \quad k = 1, \dots, N,$$
$$X_1 = M_0^T, \quad f_k = \beta(X_k)X_kF(k), \; X_{k+1} = W(f_k, l_k)\beta(X_k)X_k - \tilde{f}_kE(k), \quad k = 1, \dots, N$$

and backward recursively

$$S_{N+1} = K_0^T, \quad u_k = \beta(S_{k+1})S_{k+1}B(k), \; S_k = W(u_k, \delta_k)\beta(S_{k+1})S_{k+1} - \tilde{u}_kC(k), \quad k = N, \dots, 1,$$
$$Y_{N+1} = M_0, \quad w_k = E(k)Y_{k+1}\beta(Y_{k+1}), \; Y_k = Y_{k+1}\beta(Y_{k+1})W^T(w_k^T, l_k) - F(k)\tilde{w}_k, \quad k = N, \dots, 1.$$

This means that instead of (15.35) we use the relations

$$\begin{aligned} v_k &= C(k)Z_k\beta(Z_k), & f_k &= \beta(X_k)X_kF(k), \\ u_k &= \beta(S_{k+1})S_{k+1}B(k), & w_k &= E(k)Y_{k+1}\beta(Y_{k+1}). \end{aligned} \tag{15.43}$$

and instead of (15.36) we take

$$\begin{aligned} \Delta_k &= \beta(Z_k)W^T(v_k^T, \delta_k), & \Gamma_k &= W(f_k, l_k)\beta(X_k), \\ \Psi_k &= W(u_k, \delta_k)\beta(S_{k+1}), & \Omega_k &= \beta(Y_{k+1})W^T(w_k^T, l_k). \end{aligned} \tag{15.44}$$

In the same way as in the proof of Theorem 15.4, we use Lemma 15.3 and conclude that the elements

$$p^{(1)}(k) = \tilde{w}_k(X_kY_k)^{-1}, \qquad q^{(1)}(k) = \tilde{f}_k, \quad a^{(1)}(k) = W(f_k, l_k)\beta(S_{k+1}), \quad k = 1, \dots, N,$$

and

$$g^{(1)}(k) = \tilde{v}_k(S_{k+1}Z_{k+1})^{-1}, \; h^{(1)}(k) = \tilde{u}_k, \quad b^{(1)}(k) = W(u_k, \delta_k)\beta(X_k), \quad k = 1, \dots, N,$$

are lower and upper quasiseparable generators of the matrix A^{-1}.

To obtain representations for diagonal entries we proceed as follows. Similarly to (15.38)

$$\begin{aligned} U^{\times} = \beta^{-1}&(S_{N+1})W^{-1}(u_N, \delta_N) \cdots \beta^{-1}(S_{k+1})W^{-1}(u_k, \delta_k)(S_kZ_k) \\ &\times [W^T(v_{k-1}^T, \delta_{k-1})]^{-1}\beta^{-1}(Z_{k-1}) \cdots [W^T(v_1^T, \delta_1)]^{-1}\beta^{-1}(Z_1). \end{aligned}$$

Using (14.28) and (15.24), this yields

$$\det(S_kZ_k) = \det\beta(S_{N+1}) \cdots \det\beta(S_{k+1}) \det A \det\beta(Z_{k-1}) \cdots \det\beta(Z_1).$$

Applying this formula to the matrix A'_{kk} obtained from A by removing its kth row and kth column one gets

$$\det(S_{k+1}Z_k) = \det\beta(S_{N+1})\cdots\det\beta(S_{k+2})\det A'_{kk}\det\beta(Z_{k-1})\ldots\det\beta(Z_1).$$

Thus, using (15.42) one obtains the representations of diagonal entries of the matrix A^{-1}:

$$d^{(1)}(k) = \det\beta(S_{k+1})\det(S_{k+1}Z_k)/\det(S_kZ_k), \quad k = 1,\ldots,N.$$

Therefore, to compute quasiseparable generators of the matrix A^{-1} one can use the following algorithm.

Algorithm 15.5. 1. Compute for $i = 1,\ldots,N$

$$B(i) = \begin{pmatrix} q(i) \\ -h(i) \end{pmatrix}, \qquad C(i) = \begin{pmatrix} p(i) & g(i) \end{pmatrix}, \qquad \delta_i = d(i) - g(i)h(i),$$

$$l_i = d(i) - p(i)q(i), \quad E(k) = \begin{pmatrix} g(k) & -p(k) \end{pmatrix}, \quad F(k) = \begin{pmatrix} h(k) \\ q(k) \end{pmatrix}.$$

2. Start with $S_{N+1} = \begin{pmatrix} 0_{r_U\times r_L} & I_{r_U} \end{pmatrix}$, and for $i = N,\ldots,1$ perform the following operations:

determine the scaling matrix $\beta(S_{i+1})$ and compute

$$S^0_{i+1} = \beta(S_{i+1})S_{i+1}, \quad u_i = S^0_{i+1}B(i),$$

determine the column vector $\tilde{u}_i$ and the matrix $W(u_i,\delta_i)$, compute

$$S_i = W(u_i,\delta_i)S^0_{i+1} - \tilde{u}_iC(i), \quad h^{(1)}(i) = \tilde{u}_i, \quad b^{(1)}(i) = W(u_i,\delta_i)\beta(S_{i+1}).$$

3. Start with $Z_1 = \begin{pmatrix} 0_{r_L\times r_U} \\ I_{r_U} \end{pmatrix}$ and for $i = 1,\ldots,N$ perform the following operations:

determine the scaling matrix $\beta(Z_i)$ and compute

$$Z^0_i = Z_i\beta(Z_i), \quad v_i = C(i)Z^0_i,$$

determine the row vector $\tilde{v}_i$ and the matrix $W(v_i^T,\delta_i)$, compute

$$Z_{i+1} = Z^0_iW^T(v_i^T,\delta_i) - B(i)\tilde{v}_i, \quad g^{(1)}(i) = \tilde{v}_i(S_{i+1}Z_{i+1})^{-1}.$$

4. Start with $X_1 = \begin{pmatrix} 0_{r_L\times r_U} & I_{r_L} \end{pmatrix}$ and for $i = 1,\ldots,N$ perform the following operations:

determine the scaling matrix $\beta(X_i)$ and compute

$$X^0_i = \beta(X_i)X_i, \quad f_i = X^0_iF(i),$$

determine the column vector $\tilde{f}_i$ and the matrix $W(f_i, l_i)$, compute

$$X_{i+1} = W(f_i, l_i)X_i^0 - \tilde{f}_i E(i), \quad q^{(1)}(i) = \tilde{f}_i, \quad a^{(1)}(i) = W(f_i, l_i)\beta(X_i).$$

5. Start with $Y_{N+1} = \begin{pmatrix} 0_{r_U \times r_L} \\ I_{r_L} \end{pmatrix}$ and for $i = N, \dots, 1$ perform the following operations:

determine the scaling matrix $\beta(Y_{i+1})$ and compute

$$Y_{i+1}^0 = Y_{i+1}\beta(Y_{i+1}), \quad w_i = E(i)Y_{i+1}^0,$$

determine the row vector w_i and the matrix $W(w_i^T, l_i)$, compute

$$Y_i = Y_{i+1}^0 W^T(w_i^T, l_i) - F(i)\tilde{w}_i, \quad p^{(1)}(i) = \tilde{w}_i(X_i Y_i)^{-1}.$$

6. Compute for $i = 1, \dots, N$

$$d^{(1)}(i) = \det\beta(S_{i+1})\det(S_{i+1}Z_i)/\det(S_i Z_i).$$

Set $r = \max\{r^L, r^U\}$. Then the complexity of Algorithm 15.5 is estimated as follows.

1. The numbers δ_i, l_i: $2r$ operations of arithmetical multiplication and $2r$ operations of arithmetical additions.
2. The matrix S_{i+1}^0: $\theta(r)$ operations.
3. The vector u_i: $2r^2$ operations of arithmetical multiplication and $r(2r-1)$ operations of arithmetical additions.
4. The matrix $W(u_i, \delta_i)$: $r-1$ operations.
5. The matrix S_i: less than $4r^3 + 2r^2$ operations of arithmetical multiplication and arithmetical additions.
6. The matrix $b^{(1)}(i)$: less than $2r^3$ operations.
7. The matrix Z_i^0: $\theta(r)$ operations.
8. The vector v_i: less than $4r^2$ operations.
9. The matrix $W(v_i^T, \delta_i)$: $r-1$ operations.
10. The matrix Z_{i+1}: less than $4r^3 + 2r^2$ operations.
11. The product $S_{i+1}Z_{i+1}$: less than $4r^3$ operations.
12. The vector $g^{(1)}(i)$: $\rho(r)$ operations.
13. The matrix X_i^0: $\theta(r)$ operations.
14. The vector f_i: less than $4r^2$ operations.
15. The matrix $W(f_i, l_i)$: $r-1$ operations.
16. The matrix X_{i+1}: less than $4r^3 + 2r$ operations.
17. The matrix $a^{(1)}(i)$: less than $2r^3$ operations.

18. The matrix Y_{i+1}^0: $\theta(r)$ operations.
19. The vector w_i: less than $4r^2$ operations.
20. The matrix $W(w_i^T, l_i)$: $r-1$ operations.
21. The matrix Y_i: less than $4r^3 + 2r^2$ operations.
22. The product X_iY_i: less than $4r^3$ operations.
23. The vector $p^{(1)}(i)$: $\rho(r)$ operations.
24. The product $S_{i+1}Z_i$: less than $4r^3$ operations.
25. The number $d^{(1)}(i)$: $3\zeta(r)$ operations.

Here $\rho(n)$ is the complexity of solving of an $n \times n$ linear system by the standard Gauss method, $\zeta(n)$ is the complexity of computing an $n\times n$ determinant, $\theta(n)$ is the complexity of the scaling operation $\beta(F_1)F_1$ or $F_2\beta(F_2)$ for an $n \times 2n$ matrix F_1 or a $2n \times n$ matrix F_2, respectively. Thus the total complexity of the algorithm does not exceed

$$(2\rho(r) + 3\zeta(r) + 28r^3 + 4\theta(r) + 16r^2 + 10r - 4)N.$$

This number may be reduced by using special forms of the matrices W and β.

§15.3 Comments

The results of this chapter were obtained in the papers [18], [19], which contain also results of numerical tests. In the case of diagonal plus semiseparable of order one representations the algorithm may be simplified essentially (see [18]).

Chapter 16

Quasiseparable/Semiseparable Representations and One-direction Systems

Here we consider matrices with the lower quasiseparable and upper semiseparable representations discussed in Section §14.2. We show that such representations correspond to discrete systems without backward recursions, i.e., to one-direction systems. We study such systems in detail and derive inversion algorithms for matrices of their input-output operators.

§16.1 Systems with diagonal main coefficients and homogeneous boundary conditions

Here we consider systems which correspond to matrices with a given quasiseparable representation for the strictly lower triangular part and a semiseparable representation for the strictly upper triangular part.

Theorem 16.1. *Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a matrix with block entries of sizes $m_i \times n_j$, with lower quasiseparable generators $p(k), q(k), a(k)$ $(k = 1, \dots, N)$ of orders r_k^L $(k = 0, \dots, N)$, upper semiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ of order r_U, and diagonal entries $d(k)$ $(k = 1, \dots, N)$. Define $g(N), h(1)$ to be arbitrary matrices of sizes $m_N \times r_U, r_U \times n_1$ respectively.*

Set

$$\begin{gathered}E(k) = \begin{pmatrix} a(k) & 0 \\ 0 & I_{r_U} \end{pmatrix},\ B(k) = \begin{pmatrix} q(k) \\ -h(k) \end{pmatrix},\ C(k) = \begin{pmatrix} p(k) & g(k) \end{pmatrix}, \\ D(k) = d(k) - g(k)h(k), \quad k = 1, \dots, N, \\ M_1 = \begin{pmatrix} I_{r_0^L} & 0 \\ 0 & 0_{r_U \times r_U} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0_{r_0^L \times r_N^L} & 0 \\ 0 & I_{r_U} \end{pmatrix}.\end{gathered} \tag{16.1}$$

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Then A is a matrix of the input-output operator of the discrete system

$$\begin{cases} f_{k+1} = E(k)f_k + B(k)x(k), & k = 1, \dots, N, \\ y(k) = C(k)f_k + D(k)x(k), & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0, \end{cases} \tag{16.2}$$

with the input $x = (x(k))_{k=1}^N$, the output $y = (y(k))_{k=1}^N$ and the state space variables f_k $(k = 1, \dots, N+1)$.

Proof. The matrix A has quasiseparable generators $p(i)$, $q(j)$, $a(k)$; $g(i)$, $h(j)$, $b(k) = I$; $d(k)$. By Theorem 13.2, A is a matrix of the input-output operator of the system (13.14) with $b(k) = I_{r_U}$:

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1, \dots, N-1, \\ \eta_{k-1} = \eta_k + h(k)x(k), & k = N, \dots, 2, \\ y(k) = p(k)\chi_k + g(k)\eta_k + d(k)x(k), & k = 1, \dots, N. \\ \chi_1 = 0, \quad \eta_N = 0. \end{cases} \tag{16.3}$$

The second recursion in (16.3) may be written in the form

$$\eta_k = \eta_{k-1} - h(k)x(k), \quad k = 2, \dots, N. \tag{16.4}$$

We introduce another state space variable $z_k = \eta_{k-1}$, $k = 2, \dots, N+1$, and obtain

$$z_{k+1} = z_k - h(k)x(k), \quad k = 2, \dots, N. \tag{16.5}$$

Next we set

$$z_1 = z_2 + h(1)x(1). \tag{16.6}$$

Combining (16.5) and (16.6), one gets

$$z_{k+1} = z_k - h(k)x(k), \quad k = 1, \dots, N. \tag{16.7}$$

Moreover using $z_{k+1} = \eta_k$ $(k = 1, \dots, N)$ one has

$$\eta_k = z_k - h(k)x(k), \quad k = 1, \dots, N.$$

Substituting these expressions into the last equations in (16.3) one obtains

$$y(k) = p(k)\chi_k + g(k)(z_k - h(k)x(k)) + d(k)x(k), \quad k = 1, \dots, N,$$

whence

$$y(k) = p(k)\chi_k + g(k)z_k + D(k)x(k), \quad k = 1, \dots, N, \tag{16.8}$$

with the matrices $D(k)$ defined in (16.1). Next we set $\chi_{N+1} = a(N)\chi_N + q(N)x(N)$, which together with the first equations from (16.3) yields

$$\chi_{k+1} = a(k)\chi_k + q(k)x(k), \quad k = 1, \dots, N. \tag{16.9}$$

Finally, using the boundary conditions from (16.3) and $\eta_N = z_{N+1}$ one gets

$$\chi_1 = 0, \quad z_{N+1} = 0. \tag{16.10}$$

Thus combining (16.9), (16.7), (16.8) and (16.10) one obtains the system

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1, \dots, N, \\ z_{k+1} = z_k - h(k)x(k), & k = 1, \dots, N, \\ y(k) = p(k)\chi_k + g(k)z_k + D(k)x(k), & k = 1, \dots, N, \\ \chi_1 = 0, \quad z_{N+1} = 0. \end{cases}$$

If we now introduce the state space variable $f_k = \begin{pmatrix} \chi_k \\ z_k \end{pmatrix}$ $(k = 1, \dots, N+1)$ and use the matrices $E(k), B(k), C(k)$ $(k = 1, \dots, N)$ and M_1, M_2 from (16.1), we obtain the system (16.2). □

Corollary 16.2. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a matrix with block entries of sizes $m_i \times n_j$, with lower semiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ of order r_L, upper semiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ of order r_U, and diagonal entries $d(k)$ $(k = 1, \dots, N)$. Define $p(1)$, $q(N)$, $g(N)$, $h(1)$ to be arbitrary matrices of sizes $m_1 \times r_L, r_L \times n_N, m_N \times r_U, r_U \times n_1$, respectively.*

Set

$$\begin{aligned} B(k) &= \begin{pmatrix} q(k) \\ -h(k) \end{pmatrix}, & C(k) &= \begin{pmatrix} p(k) & g(k) \end{pmatrix}, \\ D(k) &= d(k) - g(k)h(k), & k &= 1, \dots, N; \\ M_1 &= \begin{pmatrix} I_{r_L} & 0 \\ 0 & 0_{r_U \times r_U} \end{pmatrix}, & M_2 &= \begin{pmatrix} 0_{r_L \times r_L} & 0 \\ 0 & I_{r_U} \end{pmatrix}. \end{aligned} \tag{16.11}$$

Then A is a matrix of the input-output operator of the discrete system

$$\begin{cases} f_{k+1} = f_k + B(k)x(k), & k = 1, \dots, N, \\ y(k) = C(k)f_k + D(k)x(k), & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0, \end{cases} \tag{16.12}$$

with the input $x = (x(k))_{k=1}^N$, the output $y = (y(k))_{k=1}^N$, and the state space variables f_k $(k = 1, \dots, N+1)$.

The proof follows directly from Theorem 16.1 by setting $r_k^L = r_L$, $k = 0, \dots, N$ and $a(k) = I_{r_L}$, $k = 1, \dots, N$.

The converse of the statement in Theorem 16.1 reads

Theorem 16.3. *Let there be given a system* (16.2) *with the input vectors $x(k)$, $k = 1, \dots, N$, of sizes n_k, output vectors $y(k)$, $k = 1, \dots, N$, of sizes m_k, and the state* *space variables* $f_k,\ k = 1,\dots,N+1$, *of sizes* $r^L_{k-1} + r_U,\ k = 1,\dots,N+1$. *Let the matrices* $E(k)$ *in* (16.2) *have the form*

$$E(k) = \begin{pmatrix} a(k) & 0 \\ 0 & b(k) \end{pmatrix}, \quad k = 1,\dots,N \tag{16.13}$$

with matrices $a(k)$ *of sizes* $r^L_k \times r^L_{k-1}$ *and invertible matrices* $b(k)$ *of size* $r_U \times r_U$, *and let the matrices* M_1, M_2 *have the form*

$$M_1 = \begin{pmatrix} I_{r^L_1} & 0 \\ 0 & 0_{r_U\times r_U} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0_{r^L_1\times r^L_N} & 0 \\ 0 & I_{r_U} \end{pmatrix}. \tag{16.14}$$

For the matrices $B(k), C(k)$ *of sizes* $(r^L_k + r_U)\times n_k, m_k \times (r^L_{k-1} + r_U)$ *define the partitions*

$$B(k) = \begin{pmatrix} q(k) \\ h(k) \end{pmatrix}, \quad C(k) = \begin{pmatrix} p(k) & g(k) \end{pmatrix} \tag{16.15}$$

with matrices $q(k), h(k), p(k), g(k)$ *of sizes* $r^L_k \times n_k, r_U \times n_k, m_k \times r^L_{k-1}, m_k \times r_U$, *respectively.*

Then the matrix A *with lower quasiseparable generators* $p(k), q(k), a(k)$ $(k = 1,\dots,N)$ *with orders* r^L_k $(k = 0,\dots,N)$, *upper semiseparable generators*

$$\begin{aligned} &\tilde g(1) = g(1)(b(1))^{-1},\ \tilde g(i) = g(i)b^{>}_{i1},\ i = 2,\dots,N-1, \\ &\tilde h(j) = -(b^{>}_{j+1,1})^{-1}h(j), \qquad\qquad j = 2,\dots,N, \end{aligned} \tag{16.16}$$

of order r_U, *and diagonal entries*

$$d(k) = D(k) - g(k)(b(k))^{-1}h(k), \quad k = 1,\dots,N, \tag{16.17}$$

is a matrix of the input-output operator of the system (16.2).

Proof. Set $f_k = \begin{pmatrix} \chi_k \\ z_k \end{pmatrix}$ $(k = 1,\dots,N+1)$ with the vector columns χ_k, z_k of sizes r^L_{k-1}, r_U, respectively. Using the equalities (16.13), (16.14), (16.15) we present the system (16.2) in the form

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1,\dots,N, \\ z_{k+1} = b(k)z_k + h(k)x(k), & k = 1,\dots,N, \\ y(k) = p(k)\chi_k + g(k)z_k + D(k)x(k), & k = 1,\dots,N, \\ \chi_1 = 0, \quad z_{N+1} = 0. \end{cases} \tag{16.18}$$

From the first equations of this system one gets

$$\chi_{k+1} = a(k)\chi_k + q(k)x(k), \quad k = 1,\dots,N-1. \tag{16.19}$$

Using the second equation of the system one gets

$$z_k = (b(k))^{-1} z_{k+1} - (b(k))^{-1} h(k) x(k), \quad k = N, \dots, 1. \tag{16.20}$$

We introduce the variables $\eta_k = z_{k+1},\ k = 1, \dots, N$. Using (16.20) one gets

$$\eta_{k-1} = (b(k))^{-1} \eta_k - (b(k))^{-1} h(k) x(k), \quad k = N, \dots, 1. \tag{16.21}$$

Moreover, (16.20) implies

$$z_k = (b(k))^{-1} \eta_k - (b(k))^{-1} h(k) x(k), \quad k = N, \dots, 1.$$

Substituting these expressions in the third equations from (16.18) one gets

$$y(k) = p(k)\chi_k + g(k)(b(k))^{-1}\eta_k + (D(k) - g(k)(b(k))^{-1} h(k)) x(k), \quad k = 1, \dots, N,$$

whence

$$y(k) = p(k)\chi_k + g(k)(b(k))^{-1}\eta_k + d(k)x(k), \quad k = 1, \dots, N, \tag{16.22}$$

with $d(k)$ defined in (16.17). Using the equality $\eta_N = z_{N+1}$ and the boundary conditions from (16.18) one obtains

$$\chi_1 = 0, \quad \eta_N = 0. \tag{16.23}$$

Combining (16.19), (16.21), (16.22) and (16.23) one obtains the following system equivalent to (16.18):

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1, \dots, N-1, \\ \eta_{k-1} = (b(k))^{-1}\eta_k - (b(k))^{-1} h(k) x(k), & k = N, \dots, 2, \\ y(k) = p(k)\chi_k + g(k)(b(k))^{-1}\eta_k + d(k)x(k), & k = 1, \dots, N, \\ \chi_1 = 0, \quad \eta_N = 0. \end{cases} \tag{16.24}$$

By Theorem 13.3, the matrix A with quasiseparable generators $p(k)$, $q(k)$, $a(k)$; $g(k)(b(k))^{-1}$, $-(b(k))^{-1}h(k)$, $(b(k))^{-1}$; $d(k)$ $(k = 1, \dots, N)$ is a matrix of the input-output operator of the system (16.24).

It remains to check that the matrices $\tilde{g}(i)$ $(i = 1, \dots, N-1)$, $\tilde{h}(j)$ $(j = 2, \dots, N)$ defined in (16.16) are upper semiseparable generators of the matrix A. Set $\tilde{b}(k) = (b(k))^{-1},\ k = 1, \dots, N$. As it was shown above, the elements $g(i)\tilde{b}(i)$ $(i = 1, \dots, N-1)$, $-\tilde{b}(j)h(j)$ $(j = 2, \dots, N)$, $\tilde{b}(k)$ $(k = 2, \dots, N-1)$ are upper quasiseparable generators of A. Since the matrices $\tilde{b}(k)$ are invertible, Theorem 4.3 yields the upper quasiseparable generators of the matrix A

$$\tilde{g}(i) = g(i)\tilde{b}(i)(\tilde{b}^{<}_{1,i+1})^{-1},\ i = 1, \dots, N-1, \quad \tilde{h}(j) = -\tilde{b}^{<}_{1j}\tilde{b}(j)h(j),\ j = 2, \dots, N.$$

Using the equalities

$$\tilde{b}^{<}_{1j}\tilde{b}(j) = \tilde{b}^{<}_{1,j+1}, \quad \tilde{b}^{<}_{1,j+1} = (b^{>}_{j+1,1})^{-1}$$

one obtains $\tilde{h}(j) = -(b^{>}_{j+1,1})^{-1}h(j),\ j = 2, \dots, N$. Using the equality $\tilde{b}^{<}_{1,2} = I$ one gets $\tilde{g}(1) = g(1)(b(1))^{-1}$.

Finally using the equalities

$$\tilde{b}(i)(\tilde{b}^{<}_{1,i+1})^{-1} = (b(i))^{-1}b^{>}_{i+1,1} = b^{>}_{i,1}, \quad i = 2, \dots, N,$$

one gets $\tilde{g}(i) = g(i)b^{>}_{i1}$, $i = 2, \dots, N-1$. □

Corollary 16.4. *Let there be given a system* (16.2) *with the input vectors* $x(k)$, $k = 1, \dots, N$, *of sizes* n_k, *the output vectors* $y(k)$, $k = 1, \dots, N$, *of sizes* m_k, *and the state space variables* f_k, $k = 1, \dots, N+1$, *of sizes* $r_L + r_U$. *For the matrices* $B(k), C(k)$ *of sizes* $(r_L + r_U) \times n_k, m_k \times (r_L + r_U)$ *define the partitions*

$$B(k) = \begin{pmatrix} q(k) \\ h(k) \end{pmatrix}, \quad C(k) = \begin{pmatrix} p(k) & g(k) \end{pmatrix},$$

with matrices $q(k), h(k), p(k), g(k)$ *of sizes* $r_L \times n_k, r_U \times n_k, m_k \times r_L, m_k \times r_U$, *respectively.*

Then the matrix A *with lower semiseparable generators* $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ *of order* r_L, *upper semiseparable generators* $g(i)$ $(i = 1, \dots, N-1)$, $-h(j)$ $(j = 2, \dots, N)$ *of order* r_U, *and diagonal entries*

$$d(k) = D(k) - g(k)h(k), \quad k = 1, \dots, N,$$

is a matrix of the input-output operator of the system (16.12).

The proof follows directly from Theorem 16.3 by setting $r^L_k = r_L$, $k = 0, \dots, N$ and $E(k) = I_{r_L + r_U}$, $k = 1, \dots, N$.

An analog of Theorem 16.1 for matrices with given lower semiseparable and upper quasiseparable generators is the following.

Theorem 16.5. *Let* $A = \{A_{ij}\}^N_{i,j=1}$ *be a matrix with block entries of sizes* $m_i \times n_j$, *with lower semiseparable generators* $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ *of order* r_L, *upper quasiseparable generators* $g(k), h(k), b(k)$ $(k = 1, \dots, N)$ *of orders* r^U_k $(k = 0, \dots, N)$, *and diagonal entries* $d(k)$ $(k = 1, \dots, N)$. *Define* $p(1), q(N)$ *to be arbitrary matrices of sizes* $m_1 \times r_L, r_L \times n_N$, *respectively.*

Set

$$E(k) = \begin{pmatrix} I_{r_L} & 0 \\ 0 & b(k) \end{pmatrix}, \qquad B(k) = \begin{pmatrix} -q(k) \\ h(k) \end{pmatrix}, \quad C(k) = \begin{pmatrix} p(k) & g(k) \end{pmatrix},$$

$$D(k) = d(k) - p(k)q(k), \qquad k = 1, \dots, N,$$

$$M_1 = \begin{pmatrix} I_{r_L} & 0 \\ 0 & 0_{r^U_{N-1} \times r^U_0} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0_{r_L \times r_L} & 0 \\ 0 & I_{r^U_{N-1}} \end{pmatrix}.$$

Then A *is a matrix of the input-output operator of the discrete system*

$$\begin{cases} f_{k-1} = E(k)f_k + B(k)x(k), & k = 1, \dots, N, \\ y(k) = C(k)f_k + D(k)x(k), & k = 1, \dots, N, \\ M_1 f_0 + M_2 f_N = 0, \end{cases}$$

with the input $x = (x(k))_{k=1}^N$, *the output* $y = (y(k))_{k=1}^N$, *and the state space variables* f_k $(k = 0, \dots, N)$.

§16.2 The general one-direction systems

Here we consider discrete systems with boundary conditions which are a generalization of the systems considered in Section §16.1. More precisely we consider systems of the form

$$\begin{cases} f_{k+1} = E(k)f_k + B(k)x(k), & k = 1, \dots, N, \\ y(k) = C(k)f_k + D(k)x(k), & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0. \end{cases} \tag{16.25}$$

Here the input of the system are the n_k-dimensional vectors $x(k)$ $(k = 1, \dots, N)$, the output are the m_k-dimensional vectors $y(k)$ $(k = 1, \dots, N)$, the state space variables are the r_k-dimensional vectors f_k $(k = 1, \dots, N+1)$. The coefficients of the system are the matrices $E(k), B(k), C(k), D(k)$ $(k = 1, \dots, N)$ of sizes $r_{k+1} \times r_k, r_{k+1} \times n_k, m_k \times r_k, m_k \times n_k$, respectively. The boundary conditions are determined by the matrices M_1, M_2 of sizes $r_1 \times r_1, r_1 \times r_{N+1}$, respectively.

The system (16.25) is said to have *well-posed* boundary conditions if the homogeneous system

$$\begin{cases} f_{k+1} = E(k)f_k, & k = 1, \dots, N-1, \\ M_1 f_1 + M_2 f_{N+1} = 0, \end{cases} \tag{16.26}$$

has only the trivial solution. It is easy to see that solution of (16.26) satisfies the relations

$$f_k = E^{>}_{k,0} f_1, \quad k = 1, \dots, N+1. \tag{16.27}$$

In particular, $f_{N+1} = E^{>}_{N+1,0} f_1$ and the boundary conditions yield

$$(M_1 + M_2 E^{>}_{N+1,0}) f_1 = 0. \tag{16.28}$$

It follows that the system (16.25) is well posed if and only if the $r_1 \times r_1$ matrix $M = M_1 + M_2 E^{>}_{N+1,0}$ is invertible. Indeed, if $\det M \neq 0$, then $f_1 = 0$ and by virtue of (16.27) the system (16.26) has only the trivial solution. If (16.26) has only the trivial solution, then (16.28) has only the trivial solution, which implies invertibility of M.

In the case of well-posed boundary conditions the output $y = (y(k))_{k=1}^N$ is uniquely determined by the input $x = (x(k))_{k=1}^N$. Hence a linear operator A such that $y = Ax$ is defined. The operator A is called the *input-output operator*of the system (16.25). We derive explicit formulas for these operators.

Theorem 16.6. *Let there be given a system* (16.25) *with coefficients* $E(k)$, $B(k)$, $C(k)$, $D(k)$ $(k = 1, \dots, N)$ *and* M_1, M_2, *which are matrices of sizes* $r_{k+1} \times r_k, r_{k+1} \times n_k, m_k \times r_k, m_k \times n_k$ *and* $r_1 \times r_1, r_1 \times r_{N+1}$, *respectively, and with well-posed boundary conditions.*

Then the matrix $A = \{A_{ij}\}_{i,j=1}^N$ *of the input-output operator of the system is given by the formula*

$$A_{ij} = \begin{cases} S_{ij} + C(i)E^{>}_{ij}B(j), & 1 \le j < i \le N, \\ D(i) + S_{ii}, & 1 \le i = j \le N, \\ S_{ij}, & 1 \le i < j \le N, \end{cases} \tag{16.29}$$

where

$$S_{ij} = -C(i)E^{>}_{i0}M^{-1}M_2E^{>}_{N+1,j}B(j), \tag{16.30}$$

with

$$M = M_1 + M_2E^{>}_{N+1,0}. \tag{16.31}$$

Moreover, the matrix A *has lower quasiseparable generators* $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ *of orders* r_{k+1} $(k = 0, \dots, N)$ *given by the formulas*

$$\begin{aligned} p(i) &= C(i), & i &= 2, \dots, N, \\ q(j) &= (I - E^{>}_{j+1,0}M^{-1}M_2E^{>}_{N+1,j})B(j), & j &= 1, \dots, N-1, \\ a(k) &= E(k), & k &= 2, \dots, N-1, \end{aligned} \tag{16.32}$$

upper semiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ *of order* r_{N+1} *given by the formulas*

$$g(i) = -C(i)E^{>}_{i0}M^{-1}M_2, \; i = 1, \dots, N-1; \quad h(j) = E^{>}_{N+1,j}B(j), \; j = 2, \dots, N, \tag{16.33}$$

and diagonal entries

$$d(k) = -C(k)E^{>}_{k0}M^{-1}M_2E^{>}_{N+1,k}B(k) + D(k), \quad k = 1, \dots, N. \tag{16.34}$$

Proof. One can check by induction that the solution of the first equation in (16.25) is given by

$$f_k = E^{>}_{k0}f_1 + \sum_{j=1}^{k-1} E^{>}_{kj}B(j)x(j), \quad k = 1, \dots, N+1. \tag{16.35}$$

Indeed, for $k = 1$ the relation (16.35) follows directly from $E^{>}_{1,0} = I$. Let for some $k, k \ge 1$, (16.35) hold. Using the first equation in (16.25) and the equalities

$E(k)E^{>}_{k0} = E^{>}_{k+1,0}$, $E^{>}_{k+1,k} = I$, $E^{>}_{k+1,j} = E(k)E^{>}_{k,j}$ one gets

$$\begin{aligned} f_{k+1} &= E(k)(E^{>}_{k0}f_1 + \sum_{j=1}^{k-1} E^{>}_{kj}B(j)x(j)) + B(k)x(k) \\ &= E^{>}_{k+1,0}f_1 + \sum_{j=1}^{k-1} E^{>}_{k+1,j}B(j)x(j) + E^{>}_{k+1,k}B(k)x(k) \\ &= E^{>}_{k+1,0}f_1 + \sum_{j=1}^{k} E^{>}_{k+1,j}B(j)x(j). \end{aligned}$$

Setting in (16.35) $k = N+1$ and using the boundary conditions of (16.28) and the definition (16.31) one gets

$$f_1 = -M^{-1}M_2\left(\sum_{j=1}^{N} E^{>}_{N+1,j}B(j)x(j)\right).$$

Inserting this formula for f_1 into (16.35) and using the second identity in (16.25) one gets

$$\begin{aligned} y(i) = &- C(i)E^{>}_{i0}M^{-1}M_2\left(\sum_{j=1}^{N} E^{>}_{N+1,j}B(j)x(j)\right) \\ &+ C(i)\sum_{j=1}^{i-1} E^{>}_{ij}B(j)x(j) + D(i)x(i), \quad i = 1,\dots,N. \end{aligned}$$

This means that $y(i) = \sum_{j=1}^{N} A_{ij}x(j),\ j = 1,\dots,N$, where A_{ij} is given by (16.29). Thus the matrix A has the desired representation (16.29).

Now we check that the elements defined in (16.32)–(16.34) are generators of the matrix A.

For $i > j$, using the first formula from (16.29) and the identity $E^{>}_{i0} = E^{>}_{ij}E^{>}_{j+1,0}$ one gets

$$A_{ij} = C(i)E^{>}_{ij}B(j) - C(i)E^{>}_{ij}E^{>}_{j+1,0}M^{-1}M_2E^{>}_{N+1,j}B(j),$$

which implies

$$A_{ij} = C(i)E^{>}_{ij}[(I - E^{>}_{j+1,0}M^{-1}M_2E^{>}_{N+1,j})B(j)], \quad 1 \le j < i \le N.$$

Hence, the matrices defined in (16.32) are lower quasiseparable generators of the matrix A.

For $i < j$, using the third formula in (16.29) one gets

$$A_{ij} = (-C(i)E^{>}_{i0}M^{-1}M_2)(E^{>}_{N+1,j}B(j)), \quad 1 \le i < j \le N.$$

Hence, the matrices defined in (16.33) are upper quasiseparable generators of the matrix A.

The formula (16.34) for diagonal entries follows directly from the second formula in (16.29). □

Corollary 16.7. *Under the conditions of Theorem* 16.6*, let* r_1^L, r_{N+1}^L, r_U *be nonnegative integers such that* $r_1^L + r_U = r_1$*,* $r_{N+1}^L + r_U = r_{N+1}$*, and let the matrices* M_1, M_2 *in the boundary conditions in* (16.25) *have the form*

$$M_1 = \begin{pmatrix} I_{r_1^L} & 0 \\ 0 & 0_{r_U \times r_U} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0_{r_1^L \times r_{N+1}^L} & 0 \\ 0 & I_{r_U} \end{pmatrix}. \tag{16.36}$$

Then the matrices S_{ij} *in* (16.30) *have the form*

$$S_{ij} = g(i)h(j), \quad 1 \le i, j \le N \tag{16.37}$$

with

$$\begin{aligned} g(i) &= -(C(i)E_{i0}^{>})(:, r_1^L + 1 : r_1^L + r_U), & i &= 1, \dots, N-1, \\ h(j) &= E_{22}^{-1}(E_{N+1,j}^{>}B(j))(r_{N+1}^L + 1 : r_{N+1}^L + r_U, :), & j &= 2, \dots, N, \end{aligned} \tag{16.38}$$

where

$$E_{22} = E_{N+1,0}^{>}(r_{N+1}^L + 1 : r_{N+1}^L + r_U, r_1^L + 1 : r_1^L + r_U).$$

Proof. For the $r_{N+1} \times r_1$ matrix $E_{N+1,0}^{>}$ consider the partition

$$E_{N+1,0}^{>} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix},$$

with the matrices $E_{11}, E_{12}, E_{21}, E_{22}$ of sizes $r_{N+1}^L \times r_1^L, r_{N+1}^L \times r_U, r_U \times r_1^L, r_U \times r_U$, respectively. Inserting this expression and the expressions (16.36) in (16.31) one gets

$$M = \begin{pmatrix} I_{r_1^L} & 0 \\ E_{21} & E_{22} \end{pmatrix}$$

and furthermore

$$M^{-1}M_2 = \begin{pmatrix} 0 & 0 \\ 0 & E_{22}^{-1} \end{pmatrix}.$$

Inserting this expression in (16.30) one obtains the representation

$$S_{ij} = -C(i)E_{i0}^{>} \begin{pmatrix} 0 & 0 \\ 0 & E_{22}^{-1} \end{pmatrix} E_{N+1,j}^{>}B(j) = g(i)h(j),$$

with the matrices $g(i), h(j)$ determined in (16.38). □

Remark. The matrices $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ defined in (16.38) are upper semiseparable generators of order r_U of the matrix A. This follows from (16.37) and the last formula in (16.29).

§16.3 Inversion of matrices with quasiseparable/ semiseparable representations via one-direction systems

Here we derive inversion formulas for matrices of input-output operators of one-sided systems.

Theorem 16.8. *Let there be given a system*

$$\begin{cases} f_{k+1} = E(k)f_k + B(k)x(k), & k = 1, \dots, N, \\ y(k) = C(k)f_k + D(k)x(k), & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0, \end{cases} \tag{16.39}$$

with coefficients $E(k), B(k), C(k), D(k)$ $(k = 1, \dots, N)$ *and* M_1, M_2 *which are matrices of sizes* $r_{k+1} \times r_k, r_{k+1} \times m_k, m_k \times r_k, m_k \times m_k$ *and* $r_1 \times r_1, r_1 \times r_{N+1}$*, respectively, and with well-posed boundary conditions. Assume that all the matrices* $D(k)$ $(k = 1, \dots, N)$ *are invertible. Define the matrices*

$$U(k) = E(k) - B(k)(D(k))^{-1}C(k), \quad k = 1, \dots, N.$$

The matrix A *of the input-output operator of the system is invertible if and only if the matrix*

$$M^{\times} = M_1 + M_2 U^{>}_{N+1,0} \tag{16.40}$$

is invertible. Furthermore, if this is the case the inverse matrix A^{-1} *is given by the formula*

$$A^{-1}(i,j) = \begin{cases} S^{\times}_{ij} - (D(i))^{-1}C(i)U^{>}_{ij}B(j)(D(j))^{-1}, & 1 \leq j < i \leq N, \\ (D(i))^{-1} + S^{\times}_{ii}, & 1 \leq i = j \leq N, \\ S^{\times}_{ij}, & 1 \leq i < j \leq N, \end{cases} \tag{16.41}$$

where

$$S^{\times}_{ij} = (D(i))^{-1}C(i)U^{>}_{i0}(M^{\times})^{-1}M_2 U^{>}_{N+1,j}B(j)(D(j))^{-1}. \tag{16.42}$$

Proof. Assume that the matrix $M^{\times}$ is invertible. Let $Ax = 0$. In this case the system (16.39) takes the form

$$\begin{cases} f_{k+1} = E(k)f_k + B(k)x(k), & k = 1, \dots, N, \\ 0 = C(k)f_k + D(k)x(k), & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0. \end{cases} \tag{16.43}$$

From the second equation of this system one gets

$$x(k) = -(D(k))^{-1}C(k)f_k, \quad k = 1, \dots, N. \tag{16.44}$$

Substituting this in the second equation in (16.43) one gets

$$\begin{cases} f_{k+1} = U(k) f_k, & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0. \end{cases}$$

It follows that $(M_1 + M_2 U^{>}_{N+1,0}) f_1 = 0$, and since $M^{\times}$ is invertible, one gets $f_1 = 0$ and therefore $f_k = 0,\ k = 1, \dots, N$. Now using (16.44) one gets $x(k) = 0,\ k = 1, \dots, N$, and therefore the matrix A is invertible.

Assume that the matrix A is invertible. Take f_1 such that $M^{\times} f_1 = 0$. We need to show that $f_1 = 0$. Put $f_k = U^{>}_{k,0} f_1,\ k = 1, \dots, N+1$ and define $x(k),\ k = 1, \dots, N$ via (16.44). One can easily check that the relations (16.43) hold. Hence, $A((x(k))_{k=1}^N) = 0$. Since A is invertible, this implies that $x(k) = 0,\ k = 1, \dots, N$ and so

$$\begin{cases} f_{k+1} = E(k) f_k, & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0. \end{cases} \tag{16.45}$$

But the boundary conditions of (16.39) are well posed. Therefore, (16.45) has only the trivial solution. In particular, $f_1 = 0$.

Now changing the input and the output in the system (16.39) one obtains the system

$$\begin{cases} f_{k+1} = U(k) f_k + B(k)(D(k))^{-1} y(k), & k = 1, \dots, N, \\ x(k) = -(D(k))^{-1} C(k) f_k + (D(k))^{-1} y(k), & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0. \end{cases} \tag{16.46}$$

Let the matrix $M^{\times}$ be invertible. This implies that the system (16.46) has well-posed boundary conditions and hence the input-output operator of this system, which is A^{-1}, is defined. Furthermore, applying the formulas (16.29), (16.30) to the system (16.46) one obtains the formulas (16.41), (16.42). □

Another proof of Theorem 14.4. By Theorem 16.1, A is the matrix of the input-output operator of the discrete system

$$\begin{cases} f_{k+1} = E(k) f_k + B(k) x(k), & k = 1, \dots, N, \\ y(k) = C(k) f_k + D(k) x(k), & k = 1, \dots, N, \\ M_1 f_1 + M_2 f_{N+1} = 0, \end{cases} \tag{16.47}$$

with the coefficients $E(k), B(k), C(k), D(k)$ defined in (14.22), (14.21) and

$$M_1 = \begin{pmatrix} I_{r_0^L} & 0 \\ 0 & 0_{r_U \times r_U} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0_{r_0^L \times r_N^L} & 0 \\ 0 & I_{r_U} \end{pmatrix}.$$

The matrix $M^{\times}$ from (16.40) has the form

$$M^{\times} = \begin{pmatrix} I_{r_0^L} & 0 \\ 0 & 0_{r_U \times r_U} \end{pmatrix} + \begin{pmatrix} 0_{r_0^L \times r_N^L} & 0 \\ 0 & I_{r_U} \end{pmatrix} U^{>}_{N+1,0} = \begin{pmatrix} I_{r_0^L} & 0 \\ * & U^{\times} \end{pmatrix},$$

with the matrix $U^\times$ defined in (14.24). Hence, by Theorem 16.8, the matrix A is invertible if and only if the matrix $U^\times$ is invertible. Furthermore, one gets

$$(M^\times)^{-1}M_2 = K_0(U^\times)^{-1}K_N^T.$$

Inserting this in (16.42) and using also (16.41) one obtains the formulas (14.25)–(14.27). □

§16.4 Comments

The method presented in this chapter was developed by I. Gohberg and M.A. Kaashoek in [37].

Chapter 17

Multiplication of Matrices

This chapter considers the product $A = A_1 A_2$ of block matrices $A_1 = \{A^{(1)}_{ij}\}_{i,j=1}^N$ and $A_2 = \{A^{(2)}_{ij}\}_{i,j=1}^N$ with block entries of compatible sizes $m_i \times \nu_j$ and $\nu_i \times n_j$. One assumes that quasiseparable generators of the factors are given and one derives formulas and algorithms to compute quasiseparable generators of the product.

The product of two matrices is seen as the input-output operator of the cascade system obtained from the two systems whose input-output operator are the two factors. The general case of the product is treated by using the results for the product of lower/upper triangular matrices.

The computational complexity of the product is $O(N)$ and it can further be improved, although keeping the same order, for matrices with diagonal plus semiseparable representations.

§17.1 The rank numbers of the product

We start with the estimate of rank numbers of the product of two matrices via the rank numbers of the factors.

Lemma 17.1. *Let A_1 and A_2 be block matrices with lower and upper rank numbers ρ^L_k, ρ^U_k $(k = 1, \dots, N-1)$ and s^L_k, s^U_k $(k = 1, \dots, N-1)$.*

Then the rank numbers t^L_k, t^U_k $(k = 1, \dots, N-1)$ of the product $A = A_1 A_2$ satisfy the inequalities

$$t^L_k \leq \rho^L_k + s^L_k, \quad t^U_k \leq \rho^U_k + s^U_k, \quad k = 1, \dots, N-1.$$

Proof. One has

$$\begin{aligned} A(k+1:N, 1:k) &= A_1(k+1:N, :)A_2(:, 1:k) \\ &= A_1(k+1:N, 1:k)A_2(1:k, 1:k) \\ &\quad + A_1(k+1:N, k+1:N)A_2(k+1:N, 1:k), \\ &\quad k = 1, \dots, N-1. \end{aligned}$$

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From here and the equalities

$$t_k^L = \operatorname{rank} A(k+1:N, 1:k),$$
$$\rho_k^L = \operatorname{rank} A_1(k+1:N, 1:k),$$
$$s_k^L = \operatorname{rank} A_2(k+1:N, 1:k)$$

it follows that $t_k^L \le \rho_k^L + s_k^L, \quad k = 1, \dots, N-1$.

Applying the obtained result to transposed matrices one obtains the estimates for the upper rank numbers. □

Lemma 17.1 implies that one can obtain generators of the product with orders not greater than the sum of the corresponding orders of the factors.

§17.2 Multiplication of triangular matrices

We start with a detailed study of products of triangular matrices.

Theorem 17.2. *Let A_1 be a block lower triangular matrix with lower quasiseparable generators $p^{(1)}(k), q^{(1)}(k),\ a^{(1)}(k)$ $(k = 1, \dots, N)$ of orders r_k^L $(k = 0, \dots, N)$ and diagonal entries $d^{(1)}(k)$ $(k = 1, \dots, N)$, and let A_2 be a block lower triangular matrix with lower quasiseparable generators $p^{(2)}(k), q^{(2)}(k),\ a^{(2)}(k)$ $(k = 1, \dots, N)$ of orders s_k^L $(k = 0, \dots, N)$ and diagonal entries $d^{(2)}(k)$ $(k = 1, \dots, N)$.*

Then the product $A = A_1 A_2$ is a block lower triangular matrix with lower quasiseparable generators

$$\begin{aligned} p(k) &= \begin{pmatrix} p^{(1)}(k) & d^{(1)}(k)p^{(2)}(k) \end{pmatrix}, \qquad q(k) = \begin{pmatrix} q^{(1)}(k)d^{(2)}(k) \\ q^{(2)}(k) \end{pmatrix}, \\ a(k) &= \begin{pmatrix} a^{(1)}(k) & q^{(1)}(k)p^{(2)}(k) \\ 0 & a^{(2)}(k) \end{pmatrix}, \qquad k = 1, \dots, N \end{aligned} \tag{17.1}$$

of orders $r_k^L + s_k^L$ $(k = 0, \dots, N)$ and diagonal entries

$$d(k) = d^{(1)}(k)d^{(2)}(k), \quad k = 1, \dots, N. \tag{17.2}$$

Proof. By Corollary 13.5, the matrix A_2 is a matrix of the input-output operator of the system

$$\begin{cases} \chi_{k+1}^{(2)} = a^{(2)}(k)\chi_k^{(2)} + q^{(2)}(k)x(k), & k = 1, \dots, N-1, \\ u(k) = p^{(2)}(k)\chi_k^{(2)} + d^{(2)}(k)x(k), & k = 1, \dots, N, \\ \chi_1^{(2)} = 0. \end{cases} \tag{17.3}$$

Here the vector $x = (x(k))_{k=1}^N$ is the input of the system and the vector $u = (u(k))_{k=1}^N$ is the output. Similarly, by Corollary 13.5, A_1 is a matrix of the input-output operator of the system

$$\begin{cases} \chi_{k+1}^{(1)} = a^{(1)}(k)\chi_k^{(1)} + q^{(1)}(k)u(k), & k = 1, \dots, N-1, \\ y(k) = p^{(1)}(k)\chi_k^{(1)} + d^{(1)}(k)u(k), & k = 1, \dots, N, \\ \chi_1^{(1)} = 0. \end{cases} \tag{17.4}$$

Here we take the vector u to be the input of the system (17.4) and the vector $y = (y(k))_{k=1}^N$ is the output.

Substituting the second expression from (17.3) into the system (17.4) one obtains the system

$$\begin{cases} \chi_{k+1}^{(1)} = a^{(1)}(k)\chi_k^{(1)} + q^{(1)}(k)p^{(2)}(k)\chi_k^{(2)} + q^{(1)}(k)d^{(2)}(k)x(k), & k = 1, \dots, N-1, \\ \chi_{k+1}^{(2)} = a^{(2)}(k)\chi_k^{(2)} + q^{(2)}(k)x(k), & k = 1, \dots, N-1, \\ y(k) = p^{(1)}(k)\chi_k^{(1)} + d^{(1)}(k)p^{(2)}(k)\chi_k^{(2)} + d^{(1)}(k)d^{(2)}(k)x(k), & k = 1, \dots, N, \\ \chi_1^{(1)} = 0, \quad \chi_1^{(2)} = 0, \end{cases} \tag{17.5}$$

with the input x and the output y. The product $A = A_1A_2$ is a matrix of the input-output operator of the system (17.5). Introducing the new state space variable

$$\chi_k = \begin{pmatrix} \chi_k^{(1)} \\ \chi_k^{(2)} \end{pmatrix}, \ k = 1, \dots, N$$

and the elements $p(k), q(k), a(k), d(k)$ by the formulas (17.1), (17.2), we represent the system (17.5) in the form

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1, \dots, N-1, \\ y(k) = p(k)\chi_k + d(k)x(k), & k = 1, \dots, N, \\ \chi_1 = 0. \end{cases}$$

By Corollary 13.5, the elements $p(k), q(k), a(k), d(k)$ defined in (17.1), (17.2) are lower quasiseparable generators and the diagonal entries of the lower triangular matrix A. From (17.1) it follows that the orders of these generators equal $r_k^L + r_k^U$. □

Applying Theorem 17.2 to transposed matrices we obtain a similar result for the product of upper triangular matrices.

Theorem 17.3. *Let A_1 be a block upper triangular matrix with upper quasiseparable generators $g^{(1)}(k), h^{(1)}(k),\ b^{(1)}(k)$ $(k = 1, \dots, N)$ of orders r_k^U $(k = 0, \dots, N)$ and diagonal entries $d^{(1)}(k)$ $(k = 1, \dots, N)$ and let A_2 be a block upper triangular matrix with upper quasiseparable generators $g^{(2)}(k), h^{(2)}(k),\ b^{(2)}(k)$ $(k = 1, \dots, N)$ of orders s_k^U $(k = 0, \dots, N)$ and diagonal entries $d^{(2)}(k)$ $(k = 1, \dots, N)$.*

Then the product $A = A_1A_2$ is a block upper triangular matrix with a set of upper quasiseparable generators

$$g(k) = \begin{pmatrix} d^{(1)}(k)g^{(2)}(k) & g^{(1)}(k) \end{pmatrix}, \quad h(k) = \begin{pmatrix} h^{(2)}(k) \\ h^{(1)}(k)d^{(2)}(k) \end{pmatrix},$$
$$b(k) = \begin{pmatrix} b^{(2)}(k) & 0 \\ h^{(1)}(k)g^{(2)}(k) & b^{(1)}(k) \end{pmatrix}, \quad k = 1, \dots, N \tag{17.6}$$

of orders $r_k^U + s_k^U$ $(k = 0, \dots, N)$ and diagonal entries

$$d(k) = d^{(1)}(k)d^{(2)}(k), \quad k = 1, \dots, N. \tag{17.7}$$

Next we obtain quasiseparable generators of a product of a lower triangular matrix and an upper triangular matrix with orders equal to the corresponding orders of the factors.

Theorem 17.4. *Let A_1 be a block lower triangular matrix with lower quasiseparable generators $p^{(1)}(k), q^{(1)}(k), a^{(1)}(k)$ $(k = 1, \dots, N)$ of orders r_k^L $(k = 0, \dots, N)$ and diagonal entries $d^{(1)}(k)$ $(k = 1, \dots, N)$ and let A_2 be a block upper triangular matrix with upper quasiseparable generators $g^{(2)}(k)$, $h^{(2)}(k), b^{(2)}(k)$ $(k = 1, \dots, N)$ of orders r_k^U $(k = 0, \dots, N)$ and diagonal entries $d^{(2)}(k)$ $(k = 1, \dots, N)$.*

Then the product $A = A_1A_2$ is a block matrix with quasiseparable generators $p(k), q(k)$, $a(k); g(k), h(k), b(k); d(k)$ $(k = 1, \dots, N)$ of orders r_k^L, r_k^U $(k = 0, \dots, N)$, where the generators $p(k), a(k), b(k), h(k)$ coincide with the corresponding generators of the factors, i.e.,

$$p(k) = p^{(1)}(k), \;\; h(k) = h^{(2)}(k), \;\; a(k) = a^{(1)}(k), \;\; b(k) = b^{(2)}(k), \quad k = 1, \dots, N, \tag{17.8}$$

and the generators $q(k), g(k), d(k)$ are determined via the recursion relations

$$\beta_0 = 0_{r_0^L \times r_0^U}, \tag{17.9}$$

$$\begin{pmatrix} d(k) & g(k) \\ q(k) & \beta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) & d^{(1)}(k) \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix} \begin{pmatrix} \beta_{k-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} h^{(2)}(k) & b^{(2)}(k) \\ d^{(2)}(k) & g^{(2)}(k) \end{pmatrix},$$
$$k = 1, \dots, N, \tag{17.10}$$

with the auxiliary variables β_k, which are $r_k^L \times r_k^U$ matrices.

Proof. By Corollary 13.6, A_2 is a matrix of the input-output operator of the system

$$\begin{cases} \eta_{k-1} = b^{(2)}(k)\eta_k + h^{(2)}(k)x(k), & k = N, \dots, 2 \\ u(k) = g^{(2)}(k)\eta_k + d^{(2)}(k)x(k), & k = 1, \dots, N \\ \eta_N = 0. \end{cases} \tag{17.11}$$

Here the vector $x = (x(k))_{k=1}^N$ is the input of the system and the vector $u = (u(k))_{k=1}^N$ is the output. By Corollary 13.5, A_1 is a matrix of the input output operator of the system

$$\begin{cases} \chi_{k+1} = a^{(1)}(k)\chi_k + q^{(1)}(k)u(k), & k = 1, \ldots, N-1, \\ y(k) = p^{(1)}(k)\chi_k + d^{(1)}(k)u(k), & k = 1, \ldots, N, \\ \chi_1 = 0. \end{cases} \tag{17.12}$$

Here we take the vector u to be input of the system (17.12) and the vector $y = (y(k))_{k=1}^N$ is the output. We represent the systems (17.11) and (17.12) in the form

$$\eta_N = 0, \quad \begin{pmatrix} \eta_{k-1} \\ u(k) \end{pmatrix} = \begin{pmatrix} h^{(2)}(k) & b^{(2)}(k) \\ d^{(2)}(k) & g^{(2)}(k) \end{pmatrix} \begin{pmatrix} x(k) \\ \eta_k \end{pmatrix}, \; k = N, \ldots, 1, \tag{17.13}$$

and

$$\chi_1 = 0, \quad \begin{pmatrix} y(k) \\ \chi_{k+1} \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) & d^{(1)}(k) \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix} \begin{pmatrix} \chi_k \\ u(k) \end{pmatrix}, \; k = 1, \ldots, N. \tag{17.14}$$

Here by definition $\eta_0 = b^{(2)}(1)\eta_1 + h^{(2)}(1)x(1)$ and $\chi_{N+1} = a^{(1)}(N)\chi_N + q^{(1)}(N)u(N)$. Next we introduce the new state space variable

$$\chi'_k = \chi_k - \beta_{k-1}\eta_{k-1}, \quad k = 1, \ldots, N, \tag{17.15}$$

with β_k defined in (17.9), (17.10). Using (17.9) one obtains the boundary condition

$$\chi'_1 = 0. \tag{17.16}$$

Moreover, substituting the expressions (17.15) in (17.14) one gets

$$\begin{pmatrix} y(k) \\ \chi'_{k+1} + \beta_k\eta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) & d^{(1)}(k) \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix} \begin{pmatrix} \chi'_k + \beta_{k-1}\eta_{k-1} \\ u(k) \end{pmatrix}, \; k = 1, \ldots, N,$$

which implies

$$\begin{pmatrix} y(k) \\ \chi'_{k+1} + \beta_k\eta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) \\ a^{(1)}(k) \end{pmatrix} \chi'_k + \begin{pmatrix} p^{(1)}(k) & d^{(1)}(k) \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix} \begin{pmatrix} \beta_{k-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \eta_{k-1} \\ u(k) \end{pmatrix},$$
$$k = 1, \ldots, N.$$

Using the equalities (17.13) and (17.10) one obtains

$$\begin{pmatrix} y(k) \\ \chi'_{k+1} + \beta_k\eta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) \\ a^{(1)}(k) \end{pmatrix} \chi'_k + \begin{pmatrix} d(k) & g(k) \\ q(k) & \beta_k \end{pmatrix} \begin{pmatrix} x(k) \\ \eta_k \end{pmatrix},$$
$$k = 1, \ldots, N,$$

whence

$$y(k) = p(k)\chi'_k + g(k)\eta_k + d(k)x(k), \quad k = 1, \ldots, N \tag{17.17}$$

and

$$\chi'_{k+1} = a(k)\chi'_k + q(k)x(k), \quad k = 1, \ldots, N-1. \tag{17.18}$$

Combining the relations (17.11), (17.16), (17.17) and (17.18) one obtains the system

$$\begin{cases} \chi'_{k+1} = a(k)\chi'_k + q(k)x(k), & k = 1,\dots,N-1, \\ \eta_{k-1} = b(k)\eta_k + h(k)x(k), & k = N,\dots,2, \\ y(k) = p(k)\chi'_k + g(k)\eta_k + d(k)x(k), & k = 1,\dots,N, \\ \chi'_1 = 0, \quad \eta_N = 0, \end{cases}$$

with the input x and the output y. Moreover, $A = A_1A_2$ is a matrix of the input-output operator of this system. By Theorem 13.3, $p(k)$, $q(k)$, $a(k)$; $g(k)$, $h(k)$, $b(k)$; $d(k)$ $(k = 1,\dots,N)$ are quasiseparable generators of the matrix A. □

A similar result is obtained for the product of upper triangular and lower triangular matrices.

Theorem 17.5. *Let A_1 be a block upper triangular matrix with upper quasiseparable generators $g^{(1)}(k), h^{(1)}(k), b^{(1)}(k)$ $(k = 1,\dots,N)$ of orders r^U_k $(k = 0,\dots,N)$ and diagonal entries $d^{(1)}(k)$ $(k = 1,\dots,N)$ and let A_2 be a block lower triangular matrix with lower quasiseparable generators $p^{(2)}(k)$, $q^{(2)}(k), a^{(2)}(k)$ $(k = 1,\dots,N)$ of orders r^L_k $(k = 0,\dots,N)$ and diagonal entries $d^{(2)}(k)$ $(k = 1,\dots,N)$.*

Then the product $A = A_1A_2$ is a block matrix with quasiseparable generators $p(k), q(k), a(k); g(k), h(k), b(k); d(k)$ $(k = 1,\dots,N)$ of orders r^L_k, r^U_k $(k = 0,\dots,N)$, where the generators $q(k), a(k), b(k), g(k)$ coincide with the corresponding generators of the factors, i.e.,

$$q(k) = q^{(2)}(k), \;\; g(k) = g^{(1)}(k), \;\; a(k) = a^{(2)}(k), \;\; b(k) = b^{(1)}(k), \quad k = 1,\dots,N \tag{17.19}$$

and the generators $p(k), h(k), d(k)$ are determined via the recursion relations

$$\gamma_{N+1} = 0_{r^U_N \times r^L_N}, \tag{17.20}$$

$$\begin{pmatrix} \gamma_k & h(k) \\ p(k) & d(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) & b^{(1)}(k) \\ d^{(1)}(k) & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} p^{(2)}(k) & d^{(2)}(k) \\ a^{(2)}(k) & q^{(2)}(k) \end{pmatrix}, \quad k = N,\dots,1, \tag{17.21}$$

with the auxiliary variables γ_k, which are $r^U_{k-1} \times r^L_{k-1}$ matrices.

Proof. By Corollary 13.5, the matrix A_2 is a matrix of the input output operator of the system

$$\begin{cases} \chi_{k+1} = a^{(2)}(k)\chi_k + q^{(2)}(k)x(k), & k = 1,\dots,N-1, \\ u(k) = p^{(2)}(k)\chi_k + d^{(2)}(k)x(k), & k = 1,\dots,N, \\ \chi_1 = 0. \end{cases} \tag{17.22}$$

Here the vector $x = (x(k))_{k=1}^N$ is the input of the system and the vector $u = (u(k))_{k=1}^N$ is the output. By Corollary 13.6, A_1 is a matrix of the input-output operator of the system

$$\begin{cases} \eta_{k-1} = b^{(1)}(k)\eta_k + h^{(1)}(k)u(k), & k = N, \dots, 2, \\ y(k) = g^{(1)}(k)\eta_k + d^{(1)}(k)u(k), & k = 1, \dots, N, \\ \eta_N = 0. \end{cases} \tag{17.23}$$

Here we take the vector u to be input of the system (17.23) and the vector $y = (y(k))_{k=1}^N$ is the output. We represent the systems (17.22) and (17.23) in the form

$$\chi_1 = 0, \quad \begin{pmatrix} u(k) \\ \chi_{k+1} \end{pmatrix} = \begin{pmatrix} p^{(2)}(k) & d^{(2)}(k) \\ a^{(2)}(k) & q^{(2)}(k) \end{pmatrix} \begin{pmatrix} \chi_k \\ x(k) \end{pmatrix}, \; k = 1, \dots, N, \tag{17.24}$$

and

$$\eta_N = 0, \quad \begin{pmatrix} \eta_{k-1} \\ y(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) & b^{(1)}(k) \\ d^{(1)}(k) & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} u(k) \\ \eta_k \end{pmatrix}, \; k = N, \dots, 2. \tag{17.25}$$

Here $\chi_{N+1} = a^{(2)}(N)\chi_N + q^{(2)}(N)x(N)$ and $\eta_0 = b^{(1)}(1)\eta_1 + h^{(1)}(1)u(1)$, by definition.

Next we introduce the new state space variable

$$\eta'_k = \eta_k - \gamma_{k+1}\chi_{k+1}, \quad k = 1, \dots, N, \tag{17.26}$$

with γ_k defined in (17.20), (17.21). Using (17.20) one obtains the boundary condition

$$\eta'_N = 0. \tag{17.27}$$

Moreover, substituting the expressions (17.26) in (17.25) one gets

$$\begin{pmatrix} \eta'_{k-1} + \gamma_k\chi_k \\ y(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) & b^{(1)}(k) \\ d^{(1)}(k) & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} u(k) \\ \eta'_k + \gamma_{k+1}\chi_{k+1} \end{pmatrix}, \; k = N, \dots, 1,$$

which implies

$$\begin{pmatrix} \eta'_{k-1} + \gamma_k\chi_k \\ y(k) \end{pmatrix} = \begin{pmatrix} b^{(1)}(k) \\ g^{(1)}(k) \end{pmatrix} \eta'_k + \begin{pmatrix} h^{(1)}(k) & b^{(1)}(k) \\ d^{(1)}(k) & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} u(k) \\ \chi_{k+1} \end{pmatrix}, \quad k = N, \dots, 1.$$

Using the equalities (17.24) and (17.21) one obtains

$$\begin{pmatrix} \eta'_{k-1} + \gamma_k\chi_k \\ y(k) \end{pmatrix} = \begin{pmatrix} b^{(1)}(k) \\ g^{(1)}(k) \end{pmatrix} \eta'_k + \begin{pmatrix} \gamma_k & h(k) \\ p(k) & d(k) \end{pmatrix} \begin{pmatrix} \chi_k \\ x(k) \end{pmatrix}, \quad k = N, \dots, 1.$$

It follows that

$$y(k) = p(k)\chi_k + g(k)\eta'_k + d(k)x(k), \quad k = 1, \dots, N, \tag{17.28}$$

and

$$\eta'_{k-1} = b(k)\eta'_k + h(k)x(k), \quad k = N, \dots, 2. \tag{17.29}$$

Combining the relations (17.22), (17.27), (17.28) and (17.29), one obtains the system

$$\begin{cases} \chi_{k+1} = a(k)\chi_k + q(k)x(k), & k = 1,\dots,N-1, \\ \eta'_{k-1} = b(k)\eta'_k + h(k)x(k), & k = N,\dots,2, \\ y(k) = p(k)\chi_k + g(k)\eta'_k + d(k)x(k), & k = 1,\dots,N, \\ \chi_1 = 0, \quad \eta'_N = 0, \end{cases}$$

with the input x and the output y. Moreover, $A = A_1A_2$ is a matrix of the input-output operator of this system. By Theorem 13.3, $p(k)$, $q(k)$, $a(k)$; $g(k)$, $h(k)$, $b(k)$; $d(k)$ $(k = 1,\dots,N)$ are quasiseparable generators of the matrix A. □

§17.3 The general case

Here we derive formulas for quasiseparable generators of a product of two matrices with given quasiseparable representations.

Theorem 17.6. *Let A_1 and A_2 be two block matrices with quasiseparable generators $p^{(1)}(k)$, $q^{(1)}(k)$, $a^{(1)}(k); g^{(1)}(k), h^{(1)}(k), b^{(1)}(k); d^{(1)}(k)$ $(k = 1,\dots,N)$ of orders r^L_k, r^U_k $(k = 0,\dots,N)$ and $p^{(2)}(k)$, $q^{(2)}(k)$, $a^{(2)}(k)$; $g^{(2)}(k)$, $h^{(2)}(k)$, $b^{(2)}(k)$; $d^{(2)}(k)$ $(k = 1,\dots,N)$ of orders s^L_k, s^U_k $(k = 0,\dots,N)$, respectively.*

Then the product $A = A_1A_2$ is a block matrix with quasiseparable generators $p(k), q(k), a(k); g(k), h(k), b(k); d(k)$ $(k = 1,\dots,N)$ of orders $r^L_k + s^L_k, r^U_k + s^U_k$ $(k = 0,\dots,N)$. These generators are determined as follows. We set $\beta_0 = 0_{r^L_0 \times s^U_0}$ and compute recursively

$$\begin{pmatrix} d'(k) & \tilde{g}(k) \\ \tilde{q}(k) & \beta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) & 0 \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix} \begin{pmatrix} \beta_{k-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} h^{(2)}(k) & b^{(2)}(k) \\ 0 & g^{(2)}(k) \end{pmatrix}, \quad k = 1,\dots,N; \tag{17.30}$$

next we set $\gamma_{N+1} = 0_{r^U_N \times s^L_N}$, and compute recursively

$$\begin{pmatrix} \gamma_k & \tilde{h}(k) \\ \tilde{p}(k) & d''(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) & b^{(1)}(k) \\ d^{(1)}(k) & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} p^{(2)}(k) & d^{(2)}(k) \\ a^{(2)}(k) & q^{(2)}(k) \end{pmatrix}, \quad k = N,\dots,1. \tag{17.31}$$

Here β_k, γ_k are auxiliary variables which are matrices of sizes $r^L_k \times s^U_k, r^U_{k-1} \times s^L_{k-1}$. Finally, we set

$$p(k) = \begin{pmatrix} p^{(1)}(k) & \tilde{p}(k) \end{pmatrix}, \quad q(k) = \begin{pmatrix} q^{(1)}(k)d^{(2)}(k) + \tilde{q}(k) \\ q^{(2)}(k) \end{pmatrix}, \tag{17.32}$$

$$a(k) = \begin{pmatrix} a^{(1)}(k) & q^{(1)}(k)p^{(2)}(k) \\ 0 & a^{(2)}(k) \end{pmatrix}, \tag{17.33}$$

$$g(k) = \begin{pmatrix} d^{(1)}(k)g^{(2)}(k) + \tilde{g}(k) & g^{(1)}(k) \end{pmatrix}, \quad h(k) = \begin{pmatrix} h^{(2)}(k) \\ \tilde{h}(k) \end{pmatrix}, \tag{17.34}$$

$$b(k) = \begin{pmatrix} b^{(2)}(k) & 0 \\ h^{(1)}(k)g^{(2)}(k) & b^{(1)}(k) \end{pmatrix}, \tag{17.35}$$

$$d(k) = d'(k) + d''(k), \quad k = 1, \dots, N. \tag{17.36}$$

Proof. For a matrix B we denote by B_L, B_D, B_U the strictly lower triangular, the diagonal and the strictly upper triangular parts extended by zeros. One has $B = B_L + B_D + B_U$. Using these representations for the given matrices A_1, A_2 one gets

$$A = A_1 A_2 = (A_L^{(1)} + A_D^{(1)} + A_U^{(1)})(A_L^{(2)} + A_D^{(2)} + A_U^{(2)}),$$

which implies

$$A = A_L^{(1)}(A_L^{(2)} + A_D^{(2)}) + A_L^{(1)} A_U^{(2)} + (A_D^{(1)} + A_U^{(1)})(A_L^{(2)} + A_D^{(2)}) + (A_D^{(1)} + A_U^{(1)}) A_U^{(2)}.$$

Consequently,

$$A_L = \left(A_L^{(1)}(A_L^{(2)} + A_D^{(2)}) + A_L^{(1)} A_U^{(2)} + (A_D^{(1)} + A_U^{(1)})(A_L^{(2)} + A_D^{(2)}) \right)_L, \tag{17.37}$$

$$A_U = \left(A_L^{(1)} A_U^{(2)} + (A_D^{(1)} + A_U^{(1)})(A_L^{(2)} + A_D^{(2)}) + (A_D^{(1)} + A_U^{(1)}) A_U^{(2)} \right)_U, \tag{17.38}$$

$$A_D = \left(A_L^{(1)} A_U^{(2)} + (A_D^{(1)} + A_U^{(1)})(A_L^{(2)} + A_D^{(2)}) \right)_D. \tag{17.39}$$

$A_L^{(1)}$ is a block lower triangular matrix with lower quasiseparable generators $p^{(1)}(k), q^{(1)}(k),\ a^{(1)}(k)$ $(k = 1, \dots, N)$ and zeros on the diagonal. $A_L^{(2)} + A_D^{(2)}$ is a block lower triangular matrix with lower quasiseparable generators $p^{(2)}(k)$, $q^{(2)}(k)$, $a^{(2)}(k)$ $(k = 1, \dots, N)$ and diagonal entries $d^{(2)}(k)$ $(k = 1, \dots, N)$. By Theorem 17.2, the matrix $A_L^{(1)}(A_L^{(2)} + A_D^{(2)})$ has lower quasiseparable generators $\hat{p}(k), \hat{q}(k), a(k)$ $(k = 1, \dots, N)$, with $\hat{p}(k), \hat{q}(k)$ defined by the formulas

$$\hat{p}(k) = \begin{pmatrix} p^{(1)}(k) & 0 \end{pmatrix}, \quad \hat{q}(k) = \begin{pmatrix} q^{(1)}(k)d^{(2)}(k) \\ q^{(2)}(k) \end{pmatrix}, \tag{17.40}$$

and $a(k)$ defined by the formulas (17.33).

$A_L^{(1)}$ is a block lower triangular matrix with lower quasiseparable generators $p^{(1)}(k), q^{(1)}(k), a^{(1)}(k)$ $(k = 1, \dots, N)$ and zeros on the diagonal, $A_U^{(2)}$ is a block upper triangular matrix with upper quasiseparable generators $g^{(2)}(k)$, $h^{(2)}(k)$, $b^{(2)}(k)$ $(k = 1, \dots, N)$ and zeros on the diagonal. By Theorem 17.4, the matrix $A_L^{(1)} A_U^{(2)}$ has quasiseparable generators $p^{(1)}(k)$, $\tilde{q}(k)$, $a^{(1)}(k)$; $\tilde{g}(k), h^{(2)}(k)$, $b^{(2)}(k); d'(k)$ $(k = 1, \dots, N)$ with $\tilde{q}(k)$, $\tilde{g}(k)$, $d'(k)$ $(k = 1, \dots, N)$ determined via (17.30).

Further, $A_D^{(1)} + A_U^{(1)}$ is a block upper triangular matrix with upper quasiseparable generators $g^{(1)}(k), h^{(1)}(k), b^{(1)}(k)$ $(k = 1, \dots, N)$ and diagonal entries $d^{(1)}(k)$ $(k = 1, \dots, N)$, $A_L^{(2)} + A_D^{(2)}$ is a block lower triangular matrix with lower quasiseparable generators $p^{(2)}(k)$, $q^{(2)}(k), a^{(2)}(k)$ $(k = 1, \dots, N)$ and diagonal entries $d^{(2)}(k)$ $(k = 1, \dots, N)$. By Theorem 17.5, the matrix $(A_D^{(1)} + A_U^{(1)})(A_L^{(2)} + A_D^{(2)})$ has quasiseparable generators $\tilde{p}(k), q^{(2)}(k), a^{(2)}(k); g^{(1)}(k), \tilde{h}(k), b^{(1)}(k); d''(k)$ $(k = 1, \dots, N)$ with $\tilde{p}(k), \tilde{h}(k)$ and $d''(k)$ determined via (17.31).

By the formula (17.39), the diagonal entries of the matrix A are the sums of the diagonal entries of the matrices $A_L^{(1)} A_U^{(2)}$ and $(A_D^{(1)} + A_U^{(1)})(A_L^{(2)} + A_D^{(2)})$ and hence the relations (17.36) hold.

Next, by formula (17.37) one gets

$$A(i,j) = \hat{p}(i) a^{>}_{ij} \hat{q}(j) + \tilde{p}(i)(a^{(2)})^{>}_{ij} q^{(2)}(j) + p^{(1)}(i)(a^{(1)})^{>}_{ij} \tilde{q}(j), \quad 1 \le j < i \le N. \tag{17.41}$$

Consider the elements $p(i) a^{>}_{ij} q(j)$ $(1 \le j < i \le N)$. Using the formulas (17.32) one has

$$p(i) a^{>}_{ij} q(j) = \begin{pmatrix} p^{(1)}(i) & \tilde{p}(i) \end{pmatrix} a^{>}_{ij} \begin{pmatrix} q^{(1)}(j) d^{(2)}(j) + \tilde{q}(j) \\ q^{(2)}(j) \end{pmatrix},$$

which implies

$$p(i) a^{>}_{ij} q(j) = X_1 + X_2 + X_3 + X_4,$$

with

$$X_1 = \begin{pmatrix} p^{(1)}(i) & 0 \end{pmatrix} a^{>}_{ij} \begin{pmatrix} q^{(1)}(j) d^{(2)}(j) \\ q^{(2)}(j) \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & \tilde{p}(i) \end{pmatrix} a^{>}_{ij} \begin{pmatrix} q^{(1)}(j) d^{(2)}(j) \\ q^{(2)}(j) \end{pmatrix},$$
$$X_3 = \begin{pmatrix} p^{(1)}(i) & 0 \end{pmatrix} a^{>}_{ij} \begin{pmatrix} \tilde{q}(j) \\ 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 0 & \tilde{p}(i) \end{pmatrix} a^{>}_{ij} \begin{pmatrix} \tilde{q}(j) \\ 0 \end{pmatrix}.$$

Using (17.40) one obtains

$$X_1 = \hat{p}(i) a^{>}_{ij} \hat{q}(j).$$

Next, note that $a^{>}_{ij}$ is an upper triangular matrix of the form

$$a^{>}_{ij} = \begin{pmatrix} (a^{(1)})^{>}_{ij} & * \\ 0 & (a^{(2)})^{>}_{ij} \end{pmatrix}$$

and therefore one gets

$$X_2 = \tilde{p}(i)(a^{(2)})^{>}_{ij} q^{(2)}(j), \quad X_3 = p^{(1)}(i)(a^{(1)})^{>}_{ij} \tilde{q}(j), \quad X_4 = 0.$$

Hence it follows that the expressions $p(i) a^{>}_{ij} q(j)$ coincide with the right-hand sides of equalities (17.41) and therefore

$$A_{i,j} = p(i) a^{>}_{ij} q(j), \quad 1 \le j < i \le N.$$

This implies that the elements $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ defined in (17.32), (17.33) are lower quasiseparable generators of the matrix A.

In a similar way using the formula (17.38) we derive the formulas (17.34), (17.35) for upper quasiseparable generators of the matrix A. □

Remark 17.7. One can check easily that the relations of Theorem 17.6 may be recast as

$$\beta_0 = 0_{r_0^L \times s_0^U},$$
$$\begin{pmatrix} d'(k) & \tilde{g}(k) \\ \tilde{q}(k) & \beta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) & d^{(1)}(k) \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix} \begin{pmatrix} \beta_{k-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} h^{(2)}(k) & b^{(2)}(k) \\ d^{(2)}(k) & g^{(2)}(k) \end{pmatrix},$$
$$k = 1, \dots, N; \tag{17.42}$$
$$\gamma_{N+1} = 0_{r_N^U \times s_N^L},$$
$$\begin{pmatrix} \gamma_k & \tilde{h}(k) \\ \tilde{p}(k) & d''(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) & b^{(1)}(k) \\ 0 & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} p^{(2)}(k) & 0 \\ a^{(2)}(k) & q^{(2)}(k) \end{pmatrix},$$
$$k = N, \dots, 1. \tag{17.43}$$

Here β_k, γ_k are auxiliary variables, which are matrices of sizes $r_k^L \times s_k^U$, $r_{k-1}^U \times s_{k-1}^L$. Next,

$$p(k) = \begin{pmatrix} p^{(1)}(k) & \tilde{p}(k) + d^{(1)}(k)p^{(2)}(k) \end{pmatrix}, \quad q(k) = \begin{pmatrix} \tilde{q}(k) \\ q^{(2)}(k) \end{pmatrix}, \tag{17.44}$$

$$a(k) = \begin{pmatrix} a^{(1)}(k) & q^{(1)}(k)p^{(2)}(k) \\ 0 & a^{(2)}(k) \end{pmatrix}, \tag{17.45}$$

$$g(k) = \begin{pmatrix} \tilde{g}(k) & g^{(1)}(k) \end{pmatrix}, \quad h(k) = \begin{pmatrix} h^{(2)}(k) \\ \tilde{h}(k) + h^{(1)}(k)d^{(2)}(k) \end{pmatrix}, \tag{17.46}$$

$$b(k) = \begin{pmatrix} b^{(2)}(k) & 0 \\ h^{(1)}(k)g^{(2)}(k) & b^{(1)}(k) \end{pmatrix}, \tag{17.47}$$

$$d(k) = d'(k) + d''(k), \quad k = 1, \dots, N. \tag{17.48}$$

§17.4 Multiplication by triangular matrices

Next we consider particular cases where one of the factors is a triangular matrix.

Corollary 17.8. *Let A_1 be a block upper triangular matrix with upper quasiseparable generators $g^{(1)}(k), h^{(1)}(k), b^{(1)}(k)$ $(k = 1, \dots, N)$ of orders r_k^U $(k = 0, \dots, N)$ and diagonal entries $d^{(1)}(k)$ $(k = 1, \dots, N)$, and A_2 be a block matrix with quasiseparable generators $p^{(2)}(k), q^{(2)}(k), a^{(2)}(k); g^{(2)}(k), h^{(2)}(k), b^{(2)}(k); d^{(2)}(k)$ $(k = 1, \dots, N)$ of orders s_k^L, s_k^U $(k = 0, \dots, N)$ respectively.*

Then the product $A = A_1A_2$ is a block matrix with quasiseparable generators $p(k), q(k),\ a(k); g(k), h(k), b(k); d(k)$ $(k = 1,\dots,N)$ of orders $s_k^L, r_k^U + s_k^U$ $(k = 0,\dots,N)$. These generators are determined as follows. We set $\gamma_{N+1} = 0_{r_N^U \times s_N^L}$, and we compute recursively

$$\begin{pmatrix} \gamma_k & \tilde{h}(k) \\ p(k) & d(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) & b^{(1)}(k) \\ d^{(1)}(k) & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} p^{(2)}(k) & d^{(2)}(k) \\ a^{(2)}(k) & q^{(2)}(k) \end{pmatrix}, \quad k = N,\dots,1. \tag{17.49}$$

Here γ_k are $r_{k-1}^U \times s_{k-1}^L$ matrices. Next we set

$$q(k) = q^{(2)}(k), \qquad a(k) = a^{(2)}(k), \tag{17.50}$$

$$g(k) = \begin{pmatrix} d^{(1)}(k)g^{(2)}(k) & g^{(1)}(k) \end{pmatrix}, \quad h(k) = \begin{pmatrix} h^{(2)}(k) \\ \tilde{h}(k) \end{pmatrix}, \tag{17.51}$$

$$b(k) = \begin{pmatrix} b^{(2)}(k) & 0 \\ h^{(1)}(k)g^{(2)}(k) & b^{(1)}(k) \end{pmatrix}. \tag{17.52}$$

Proof. Since the matrix A_1 is upper triangular one can set

$$p^{(1)}(k) = 0, \quad q^{(1)}(k) = 0, \quad a^{(1)}(k) = 0.$$

Inserting this in (17.30) we get

$$d'(k) = 0, \quad \tilde{q}(k) = 0, \quad \tilde{g}(k) = 0.$$

From here, using (17.31) and (17.34)–(17.36) we obtain the formulas (17.49) and (17.51), (17.52) for upper quasiseparable generators and diagonal entries of the product $A = A_1A_2$. Using (17.32), (17.33) we obtain the formulas

$$\hat{p}(k) = \begin{pmatrix} 0 & p(k) \end{pmatrix}, \ \hat{q}(k) = \begin{pmatrix} 0 \\ q^{(2)}(k) \end{pmatrix}, \ \hat{a}(k) = \begin{pmatrix} 0 & 0 \\ 0 & a^{(2)}(k) \end{pmatrix}$$

for lower quasiseparable generators of the matrix A. One can see easily that such lower quasiseparable generators may be replaced by the other ones $p(k), q(k) = q^{(2)}(k), a(k) = a^{(2)}(k)$ $(k = 1,\dots,N)$. □

Applying Corollary 17.8 to transposed matrices we obtain the following statement.

Corollary 17.9. *Let A_1 be a block matrix with quasiseparable generators $p^{(1)}(k)$, $q^{(1)}(k)$, $a^{(1)}(k)$; $g^{(1)}(k)$, $h^{(1)}(k)$, $b^{(1)}(k)$; $d^{(1)}(k)$ $(k = 1,\dots,N)$ of orders r_k^L, r_k^U $(k = 0,\dots,N)$. Let A_2 be a block lower triangular matrix with lower quasiseparable generators $p^{(2)}(k), q^{(2)}(k)$, $b^{(2)}(k)$ $(k = 1,\dots,N)$ of orders s_k^L $(k = 0,\dots,N)$ and diagonal entries $d^{(2)}(k)$ $(k = 1,\dots,N)$.*

Then the product $A = A_1A_2$ is a block matrix with quasiseparable generators $p(k), q(k),\ a(k); g(k), h(k), b(k); d(k)$ $(k = 1,\dots,N)$ of orders $r_k^L + s_k^L, r_k^U$ $(k =$

$0,\dots,N)$. *These generators are determined as follows. We set* $\gamma_{N+1}=0_{r^U_N\times s^L_N}$, *and we compute recursively*

$$\begin{pmatrix} \gamma_k & h(k) \\ \tilde{p}(k) & d(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) & b^{(1)}(k) \\ d^{(1)}(k) & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} p^{(2)}(k) & d^{(2)}(k) \\ a^{(2)}(k) & q^{(2)}(k) \end{pmatrix}, \quad k=N,\dots,1. \tag{17.53}$$

Here γ_k *are* $r^U_{k-1}\times s^L_{k-1}$ *matrices. Next we set*

$$g(k)=g^{(1)}(k), \qquad b(k)=b^{(1)}(k), \tag{17.54}$$

$$p(k)=\begin{pmatrix} p^{(1)}(k) & \tilde{p}(k) \end{pmatrix}, \qquad q(k)=\begin{pmatrix} q^{(1)}(k)d^{(2)}(k) \\ q^{(2)}(k) \end{pmatrix}, \tag{17.55}$$

$$a(k)=\begin{pmatrix} a^{(1)}(k) & q^{(1)}(k)p^{(2)}(k) \\ 0 & a^{(2)}(k) \end{pmatrix}. \tag{17.56}$$

Corollary 17.10. *Let* A_1 *be a block lower triangular matrix with lower quasiseparable generators* $p^{(1)}(k),q^{(1)}(k),a^{(1)}(k)$ $(k=1,\dots,N)$ *of orders* r^L_k $(k=0,\dots,N)$ *and diagonal entries* $d^{(1)}(k)$ $(k=1,\dots,N)$ *and* A_2 *be a block matrix with quasiseparable generators* $p^{(2)}(k),q^{(2)}(k),a^{(2)}(k);g^{(2)}(k),h^{(2)}(k),b^{(2)}(k);d^{(2)}(k)$ $(k=1,\dots,N)$ *of orders* s^L_k,s^U_k $(k=0,\dots,N)$.

Then the product $A=A_1A_2$ *is a block matrix with quasiseparable generators* $p(k),q(k),\ a(k);g(k),h(k),b(k);d(k)$ $(k=1,\dots,N)$ *of orders* $r^L_k+s^L_k,s^U_k$ $(k=0,\dots,N)$. *These generators are determined as follows. We set* $\beta_0=0_{r^L_0\times s^U_0}$ *and compute recursively*

$$\begin{pmatrix} d(k) & g(k) \\ \tilde{q}(k) & \beta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) & d^{(1)}(k) \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix} \begin{pmatrix} \beta_{k-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} h^{(2)}(k) & b^{(2)}(k) \\ d^{(2)}(k) & g^{(2)}(k) \end{pmatrix}, \quad k=1,\dots,N. \tag{17.57}$$

Here β_k *are* $r^L_k\times s^U_k$ *matrices. Next we set*

$$b(k)=b^{(2)}(k), \qquad h(k)=h^{(2)}(k), \tag{17.58}$$

$$p(k)=\begin{pmatrix} p^{(1)}(k) & d^{(1)}(k)p^{(2)}(k) \end{pmatrix}, \quad q(k)=\begin{pmatrix} \tilde{q}(k) \\ q^{(2)}(k) \end{pmatrix}, \tag{17.59}$$

$$a(k)=\begin{pmatrix} a^{(1)}(k) & q^{(1)}(k)p^{(2)}(k) \\ 0 & a^{(2)}(k) \end{pmatrix}. \tag{17.60}$$

Proof. Since the matrix A_1 is lower triangular, one can set

$$g^{(1)}(k)=0, \quad h^{(1)}(k)=0, \quad b^{(1)}(k)=0.$$

Inserting this in (17.43) we get

$$d''(k)=0, \quad \tilde{h}(k)=0, \quad \tilde{p}(k)=0.$$

From here, using (17.42), (17.44), (17.45) and (17.48), we obtain the formulas (17.57) and (17.59), (17.60) for lower quasiseparable generators and diagonal entries of the product $A = A_1 A_2$. Using (17.46), (17.47) we obtain the formulas

$$\hat{g}(k) = \begin{pmatrix} g(k) & 0 \end{pmatrix},\ \hat{h}(k) = \begin{pmatrix} h^{(2)}(k) \\ 0 \end{pmatrix},\ \hat{b}(k) = \begin{pmatrix} b^{(2)}(k) & 0 \\ 0 & 0 \end{pmatrix}$$

for upper quasiseparable generators of the matrix A. One can see easily that such upper quasiseparable generators may be replaced by the other ones $g(k), h(k) = h^{(2)}(k), b(k) = b^{(2)}(k)$ $(k = 1, \dots, N)$. □

Applying Corollary 17.10 to transposed matrices we obtain the following statement.

Corollary 17.11. *Let A_1 be a block matrix with quasiseparable generators $p^{(1)}(k)$, $q^{(1)}(k), a^{(1)}(k); g^{(1)}(k), h^{(1)}(k), b^{(1)}(k); d^{(1)}(k)$ $(k = 1, \dots, N)$ of orders r_k^L, r_k^U $(k = 0, \dots, N)$. Let A_2 be a block upper triangular matrix with upper quasiseparable generators $g^{(2)}(k), h^{(2)}(k)$, $b^{(2)}(k)$ $(k = 1, \dots, N)$ of orders s_k^U $(k = 0, \dots, N)$ and diagonal entries $d^{(2)}(k)$ $(k = 1, \dots, N)$.*

Then the product $A = A_1 A_2$ is a block matrix with quasiseparable generators $p(k), q(k), a(k); g(k), h(k), b(k); d(k)$ $(k = 1, \dots, N)$ of orders $r_k^L, r_k^U + s_k^U$ $(k = 0, \dots, N)$. These generators are determined as follows. We set $\beta_0 = 0_{r_0^L \times s_0^U}$ and compute recursively

$$\begin{pmatrix} d(k) & \tilde{g}(k) \\ q(k) & \beta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) & d^{(1)}(k) \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix} \begin{pmatrix} \beta_{k-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} h^{(2)}(k) & b^{(2)}(k) \\ d^{(2)}(k) & g^{(2)}(k) \end{pmatrix}, \quad k = 1, \dots, N. \tag{17.61}$$

Here β_k are $r_k^L \times s_k^U$ matrices. Next we set

$$p(k) = p^{(1)}(k), \qquad a(k) = a^{(1)}(k), \tag{17.62}$$

$$g(k) = \begin{pmatrix} \tilde{g}(k) & g^{(1)}(k) \end{pmatrix}, \qquad h(k) = \begin{pmatrix} h^{(2)}(k) \\ h^{(1)}(k) d^{(2)}(k) \end{pmatrix}, \tag{17.63}$$

$$b(k) = \begin{pmatrix} b^{(2)}(k) & 0 \\ h^{(1)}(k) g^{(2)}(k) & b^{(1)}(k) \end{pmatrix}. \tag{17.64}$$

§17.5 Complexity analysis

Here we derive an expression for the complexity of the Algorithm 17.6. Let $A^{(1)} = \{A_{ij}^{(1)}\}_{i,j=1}^N$ and $A^{(2)} = \{A_{ij}^{(2)}\}_{i,j=1}^N$ be matrices with block entries of sizes $m_i \times \nu_j$ and $\nu_i \times n_j$, respectively, and with quasiseparable generators and their orders as in Theorem 17.6.

We start with the formula (17.30). This formula may be written in the form

$$\begin{pmatrix} d'(k) & \tilde{g}(k) \\ \tilde{q}(k) & \beta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) \\ a^{(1)}(k) \end{pmatrix} \beta_{k-1} \begin{pmatrix} h^{(2)}(k) & b^{(2)}(k) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & q^{(1)}(k)g^{(2)}(k) \end{pmatrix}. \tag{17.65}$$

Here the multiplication of the $r^L_{k-1} \times s^U_{k-1}$ matrix β_{k-1} by the $s^U_{k-1} \times (s^U_k + n_k)$ matrix $\begin{pmatrix} h^{(2)}(k) & b^{(2)}(k) \end{pmatrix}$ costs $r^L_{k-1}s^U_{k-1}(s^U_k + n_k)$ multiplications and $r^L_{k-1}(s^U_{k-1} - 1)(s^U_k + n_k)$ additions, thus less than $2r^L_{k-1}s^U_{k-1}(s^U_k + n_k)$ operations. Next, the multiplication of the result by the $(m_k + r^L_k) \times r^L_{k-1}$ matrix $\begin{pmatrix} p^{(1)}(k) \\ a^{(1)}(k) \end{pmatrix}$ costs $(m_k + r^L_k)r^L_{k-1}(s^U_k + n_k)$ operations and $(m_k + r^L_k)(r^L_{k-1} - 1)(s^U_k + n_k)$ additions, thus less than $2(m_k + r^L_k)r^L_{k-1}(s^U_k + n_k)$ operations. Finally, the multiplication of the $r^L_k \times \nu_k$ matrix $q^{(1)}(k)$ by the $\nu_k \times s^U_k$ matrix $g^{(2)}(k)$ costs less than $2r^L_k\nu_k s^U_k$ operations. Thus the total complexity of the formula (17.65) is less than

$$2(r^L_{k-1}s^U_{k-1}(s^U_k + n_k) + (m_k + r^L_k)r^L_{k-1}(s^U_k + n_k) + r^L_k\nu_k s^U_k)$$

operations.

In a similar way we rewrite (17.31) in the form

$$\begin{pmatrix} \gamma_k & \tilde{h}(k) \\ \tilde{p}(k) & d''(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) \\ d^{(1)}(k) \end{pmatrix} \begin{pmatrix} p^{(2)}(k) & d^{(2)}(k) \end{pmatrix} + \begin{pmatrix} b^{(1)}(k) \\ g^{(1)}(k) \end{pmatrix} \gamma_{k+1} \begin{pmatrix} a^{(2)}(k) & q^{(2)}(k) \end{pmatrix}$$

where γ_{k+1} is an $r^U_k s^L_k$ matrix, $\begin{pmatrix} a^{(2)}(k) & q^{(2)}(k) \end{pmatrix}$ is an $s^L_k \times (s^L_{k-1} + n_k)$ matrix, and $\begin{pmatrix} b^{(1)}(k) \\ g^{(1)}(k) \end{pmatrix}$ is an $(r^U_{k-1} + m_k) \times r^U_k$ matrix.

One obtains that the complexity is less than

$$2(r^U_{k-1}\nu_k s^L_{k-1} + r^U_k s^L_k(s^L_{k-1} + n_k) + (r^U_{k-1} + m_k)r^U_k(s^L_{k-1} + n_k)).$$

Next, one can see easily that the computation of the products

$$d^{(1)}(k)p^{(2)}(k), \quad q^{(1)}(k)d^{(2)}(k), \quad q^{(1)}(k)p^{(2)}(k), \quad d^{(1)}(k)g^{(2)}(k),$$
$$h^{(1)}(k)d^{(2)}(k), \quad h^{(1)}(k)g^{(2)}(k), \quad d^{(1)}(k)d^{(2)}(k)$$

by using the formulas (17.32)–(17.36) requires respectively

$$m_k(2\nu_k - 1)s^L_{k-1}, \quad r^L_k(2\nu_k - 1)n_k, \quad r^L_k(2\nu_k - 1)s^L_{k-1}, \quad m_k(2\nu_k - 1)s^U_k,$$
$$r^U_{k-1}(2\nu_k - 1)n_k, \quad r^U_{k-1}(2\nu_k - 1)s^U_k, \quad m_k(2\nu_k - 1)n_k$$

operations.

Thus the total complexity of the algorithm is estimated as follows:

$$c < 2\sum_{k=1}^{N}\Big[r^L_{k-1}(s^U_k+n_k)(s^U_{k-1}+m_k+r^L_k)+r^L_k\nu_k s^U_k \\ +(s^L_k+r^U_{k-1}+m_k)r^U_k(s^L_{k-1}+n_k)+r^U_{k-1}\nu_k s^L_{k-1}+m_k\nu_k s^L_{k-1} \\ +r^L_k\nu_k n_k+r^L_k\nu_k s^L_{k-1}+m_k\nu_k s^U_k+r^U_{k-1}\nu_k n_k+r^U_{k-1}\nu_k s^U_k+m_k\nu_k n_k\Big].$$

Let the sizes of matrices m_k, n_k, ν_k be bounded by the number m and the orders of generators $r^L_k, r^U_k, s^L_k, s^U_k$ be bounded by the numbers r', r'', s', s'':

$$m_k, n_k, \nu_k \le m, \quad r^L_k \le r',\ r^U_k \le r'',\ s^L_k \le s',\ s^U_k \le s''.$$

In this case one obtains the estimate

$$c < 2N\Big[r'(s''+m)(s''+m+r')+r'ms''+(s'+r''+m)r''(s'+m) \\ +r''ms'+r'ms'+m^2(s'+s''+r'+r'')+r'ms''+m^3\Big].$$

If we now set $r=\max\{r', r'', s', s''\}$, then

$$c < 2N(2r(r+m)(2r+m)+4r^2m+4m^2r+m^3),$$

i.e.,

$$c < 2N(4r^3+10r^2m+6m^2r+m^3). \tag{17.66}$$

§17.6 Product of matrices with semiseparable representations

The rules obtained above for quasiseparable representations remain true for semiseparable generators.

Theorem 17.12. *Let A_1 be a block matrix with lower semiseparable generators $p^{(1)}(k)$, $q^{(1)}(k)$ $(k=1,\dots,N)$ of order r_L, upper semiseparable generators $g^{(1)}(k)$, $h^{(1)}(k)$ $(k=1,\dots,N)$ of order r_U, and diagonal entries $d^{(1)}(k)$ $(k=1,\dots,N)$. Let A_2 be a block matrix with lower semiseparable generators $p^{(2)}(k)$, $q^{(2)}(k)$ $(k=1,\dots,N)$ of order s_L, upper semiseparable generators $g^{(2)}(k)$, $h^{(2)}(k)$ $(k=1,\dots,N)$ of order s_U, and diagonal entries $d^{(2)}(k)$ $(k=1,\dots,N)$.*

Then the product $A=A_1A_2$ is a block matrix with lower semiseparable generators $\hat p(i)$ $(i=2,\dots,N)$, $\hat q(j)$ $(j=1,\dots,N-1)$ of order r_L+s_L, upper semiseparable generators $\hat g(i)$ $(i=1,\dots,N-1)$, $\hat h(j)$ $(j=2,\dots,N)$ of order r_U+s_U and diagonal entries $d(k)$ $(k=1,\dots,N)$. These generators are determined as follows. We set $\beta_0=0_{r_L\times s_U}$ and we compute recursively

$$\begin{pmatrix} d'(k) & \tilde g(k) \\ \tilde q(k) & \beta_k \end{pmatrix} = \begin{pmatrix} p^{(1)}(k) & 0 \\ I & q^{(1)}(k) \end{pmatrix}\begin{pmatrix} \beta_{k-1} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} h^{(2)}(k) & I \\ 0 & g^{(2)}(k) \end{pmatrix}, \\ k=1,\dots,N; \tag{17.67}$$

next we set $\gamma_{N+1} = 0_{r_U \times s_L}$, *and compute recursively*

$$\begin{pmatrix} \gamma_k & \tilde{h}(k) \\ \tilde{p}(k) & d''(k) \end{pmatrix} = \begin{pmatrix} h^{(1)}(k) & I \\ d^{(1)}(k) & g^{(1)}(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} p^{(2)}(k) & d^{(2)}(k) \\ I & q^{(2)}(k) \end{pmatrix},$$
$$k = N, \dots, 1. \tag{17.68}$$

Here β_k, γ_k *are auxiliary variables, which are matrices of sizes* $r_L \times s_U, r_U \times s_L$. *Next we set*

$$z_1 = 0, \quad z_i = \sum_{k=2}^{i} q^{(1)}(k) p^{(2)}(k), \; i = 2, \dots, N-1,$$
$$w_1 = 0, \quad w_i = \sum_{k=2}^{i} h^{(1)}(k) g^{(2)}(k), \; i = 2, \dots, N-1.$$

Finally, we determine generators by the formulas

$$\hat{p}(i) = \begin{pmatrix} p^{(1)}(i) & p^{(1)}(i) z_{i-1} + \tilde{p}(i) \end{pmatrix}, \qquad i = 2, \dots, N, \tag{17.69}$$
$$\hat{q}(j) = \begin{pmatrix} q^{(1)}(j) d^{(2)}(j) + \tilde{q}(j) - z_j q^{(2)}(j) \\ q^{(2)}(j) \end{pmatrix}, \qquad j = 1, \dots, N-1; \tag{17.70}$$
$$\hat{g}(i) = \begin{pmatrix} g^{(1)}(i) & d^{(1)}(i) g^{(2)}(i) + \tilde{g}(i) - g^{(1)}(i) w_i \end{pmatrix}, \; i = 1, \dots, N-1, \tag{17.71}$$
$$\hat{h}(j) = \begin{pmatrix} \tilde{h}(j) + w_{j-1} h^{(2)}(j) \\ h^{(2)}(j) \end{pmatrix}, \qquad j = 2, \dots, N, \tag{17.72}$$
$$d(k) = d'(k) + d''(k), \qquad k = 1, \dots, N. \tag{17.73}$$

Proof. We apply Theorem 17.6 with $a^{(1)}(k) = I$, $a^{(2)}(k) = I$, $b^{(1)}(k) = I$, $b^{(2)}(k) = I$. Inserting these values into the formulas (17.30), (17.31) one obtains the formulas (17.67), (17.68) to determine the elements $d'(k), d''(k), \tilde{p}(k), \tilde{q}(k), \tilde{g}(k), \tilde{h}(k)$. Next, using the formulas (17.32)–(17.36) one obtains the formulas (17.73) for the diagonal entries of the matrix $A = A_1 A_2$ and the formulas

$$p(k) = \begin{pmatrix} p^{(1)}(k) & \tilde{p}(k) \end{pmatrix}, \qquad q(k) = \begin{pmatrix} q^{(1)}(k) d^{(2)}(k) + \tilde{q}(k) \\ q^{(2)}(k) \end{pmatrix}, \tag{17.74}$$
$$a(k) = \begin{pmatrix} I & q^{(1)}(k) p^{(2)}(k) \\ 0 & I \end{pmatrix}, \qquad k = 1, \dots, N \tag{17.75}$$

and

$$g(k) = \begin{pmatrix} d^{(1)}(k) g^{(2)}(k) + \tilde{g}(k) & g^{(1)}(k) \end{pmatrix}, \quad h(k) = \begin{pmatrix} h^{(2)}(k) \\ \tilde{h}(k) \end{pmatrix}, \tag{17.76}$$
$$b(k) = \begin{pmatrix} I & 0 \\ h^{(1)}(k) g^{(2)}(k) & I \end{pmatrix}, \qquad k = 1, \dots, N, \tag{17.77}$$

for lower and upper quasiseparable generators of A.

To obtain lower semiseparable generators of the matrix A we apply Theorem 4.2. From (17.75) one obtains easily the equalities

$$a^{>}_{i1} = \begin{pmatrix} I & \sum_{k=2}^{i-1} q^{(1)}(k)p^{(2)}(k) \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & z_{i-1} \\ 0 & I \end{pmatrix}, \quad i = 2, \dots, N \tag{17.78}$$

and

$$(a^{<}_{j+1,1})^{-1} = \begin{pmatrix} I & -\sum_{k=2}^{j} q^{(1)}(k)p^{(2)}(k) \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -z_j \\ 0 & I \end{pmatrix}, \quad j = 1, \dots, N-1. \tag{17.79}$$

Substituting the expressions (17.74), (17.78), (17.79) in the formulas (4.15) one obtains the formulas (17.69), (17.70) for lower semiseparable generators of the matrix A.

Applying the obtained formulas to transposed matrices we obtain the formulas (17.71), (17.72) for upper semiseparable generators of A. □

To get an estimate for the complexity c of the algorithm presented in Theorem 17.12 we set m to be the maximal size of the blocks of the matrices A_1, A_2 and $r = \max\{r_L, r_U, s_L, s_U\}$. Then one can derive the estimate

$$c < 2N(9r^2m + 6m^2r + m^3).$$

which is an improvement over (17.66).

§17.7 Comments

Algorithms for multiplication of matrices via quasiseparable representations can be found for instance in [20]. The treatment via discrete systems appears for the first time in this chapter.

Part IV

Factorization and Inversion

Introduction to Part IV

This part is devoted to the problem of LDU and QR factorizations of matrices in terms of their semiseparable or quasiseparable representations. We also provide fast algorithms for these factorizations in terms of generators of the appropriate representations. The results allow one to design fast algorithms for inversion of matrices via their quasiseparable structure and to deduce via this structure fast solvers for linear algebraic systems. In this part, like in Part III, one can find illustrative examples and the computation of the complexity in detail.

Chapter 18

The LDU Factorization and Inversion

Let A be a block matrix with block entries of sizes $m_i \times m_j$ and with invertible principal leading submatrices $A(1:k, 1:k),\ k = 1, \dots, N$. Such matrix is called strongly regular. By Theorem 1.20, A admits the LDU factorization

$$A = LDU, \tag{18.1}$$

where L, U, D are block matrices with the same sizes of blocks as A and L and U are block lower and upper triangular matrices with identities on the main diagonals, while D is a block diagonal matrix.

It is proved that the lower rank numbers of L are the lower rank numbers of A and the upper rank numbers of U are the upper rank numbers of A. Also, the minimal completion rank of the strictly lower triangular part of L is the minimal completion rank of the strictly lower triangular part of A and the minimal completion rank of the strictly upper triangular part of U is the minimal completion rank of the strictly upper triangular part of A.

It is shown that a part of the quasiseparable generators of the factors L and U are the same as for the original matrix A and the rest of the generators are obtained via an algorithm with linear complexity $O(N)$. This opens the way to solve the linear system $Ax = y$ by factoring $A = LDU$, in fact computing the induced quasiseparable representations for L, D and U and solving $L(D(Ux)) = y$, namely $Lz = y$, $Dw = z$ and then $Ux = w$. Note that linear systems with a triangular matrix have been treated in the preceding chapter.

Since the inverse matrix has the decomposition $A^{-1} = U^{-1}D^{-1}L^{-1}$, we will find quasiseparable generators for A^{-1} via an $O(N)$ algorithm.

§18.1 Rank numbers and minimal completion ranks

Here for a strongly regular matrix we show that the rank numbers and minimal completion ranks of a matrix are equal to the corresponding numbers of the factors in the LDU factorization. This implies that one can obtain quasiseparable and semiseparable representations of the factors with the same orders as for the matrix A.

Y. Eidelman et al., *Separable Type Representations of Matrices and Fast Algorithms: Volume 1 Basics. Completion Problems. Multiplication and Inversion Algorithms*, Operator Theory: Advances and Applications 234, DOI 10.1007/978-3-0348-0606-0_18, © Springer Basel 2014

Lemma 18.1. *Let A be a block matrix with block entries of sizes $m_i \times m_j$ and with invertible principal leading submatrices $A(1:k,1:k)$, $k=1,\dots,N$. Let ρ_k^L $(k=1,\dots,N-1)$ and ρ_k^U $(k=1,\dots,N-1)$ be lower and upper rank numbers of A and $\hat{r}_L$ and $\hat{r}_U$ be minimal completion ranks of the strictly lower triangular and strictly upper triangular parts of A.*

Then in the factorization (18.1) the matrix L is a lower triangular matrix with lower rank numbers ρ_k^L $(k=1,\dots,N-1)$ and U is an upper triangular matrix with upper rank numbers ρ_k^U $(k=1,\dots,N-1)$. Moreover, the minimal completion rank of the strictly lower triangular part of L equals $\hat{r}_L$ and the minimal completion rank of the strictly upper triangular part of U equals $\hat{r}_U$.

Proof. Set

$$L_k = L(1:k,1:k), \quad U_k = (DU)(1:k,1:k), \quad k=1,\dots,N.$$

Formula (18.1) implies

$$A_k = L_k U_k, \quad k=1,\dots,N,$$

and since A_k is invertible the matrices L_k, U_k are also invertible. Using (18.1) and the fact that the matrix U_k is upper triangular one gets

$$A(k+1:N,1:k) = L(k+1:N,1:k)U_k, \qquad k=1,\dots,N-1, \tag{18.2}$$

and

$$A(k+2:N,1:k) = L(k+2:N,1:k)D_kU_k, \quad k=1,\dots,N-2. \tag{18.3}$$

From (18.2), since the matrices U_k are invertible, one obtains

$$\rho_k^L = \operatorname{rank} A(k+1:N,1:k) = \operatorname{rank} L(k+1:N,1:k), \quad k=1,\dots,N-1. \tag{18.4}$$

From (18.3), since the matrices U_k are invertible, one obtains

$$\operatorname{rank} A(k+2:N,1:k) = \operatorname{rank} L(k+2:N,1:k), \quad k=1,\dots,N-2. \tag{18.5}$$

From the formulas (18.4) and (18.5) we see that the expression for the minimal completion rank in the formula (3.3) to be applied to the matrix A coincides with the one for the matrix L.

Using the transposed matrices we obtain the corresponding equalities for the upper rank numbers and minimal completion ranks. □

§18.2 The factorization algorithm

Next we derive an algorithm to compute quasiseparable generators of the factors in the factorization (18.1).

Theorem 18.2. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with block entries of sizes $m_i \times m_j$ and with invertible principal leading block submatrices $A_k = \{A_{ij}\}_{i,j=1}^k$, $k = 1,\dots,N$. Assume that A has lower quasiseparable generators $p(i)$ $(i=2,\dots,N)$,*

$q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r_k^L $(k = 1, \dots, N-1)$, upper quasiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ of orders r_k^U $(k = 1, \dots, N-1)$, and diagonal entries $d(k)$ $(k = 1, \dots, N)$.

Then in the factorization (18.1) the matrix L has lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q^{(1)}(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$, the matrix U has upper quasiseparable generators $g^{(1)}(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$, and the matrix D has the form $D = \mathrm{diag}\{\gamma_1, \dots, \gamma_N\}$. Here the elements $p(i)$, $a(k)$, $h(j)$, $b(k)$ are the same as for the matrix A and the elements $q^{(1)}(j)$, $g^{(1)}(i), \gamma_k$ are determined via the following algorithm.

1. *Compute*

$$\gamma_1 = d(1), \quad q^{(1)}(1) = q(1)\gamma_1^{-1}, \quad g^{(1)}(1) = \gamma_1^{-1} g(1), \tag{18.6}$$

$$f_1 = q^{(1)}(1)\gamma_1 g^{(1)}(1). \tag{18.7}$$

2. *For $k = 2, \dots, N-1$ compute*

$$\gamma_k = d(k) - p(k) f_{k-1} h(k), \tag{18.8}$$

$$q^{(1)}(k) = [q(k) - a(k) f_{k-1} h(k)]\gamma_k^{-1}, \tag{18.9}$$

$$g^{(1)}(k) = \gamma_k^{-1}[g(k) - p(k) f_{k-1} b(k)], \tag{18.10}$$

$$f_k = a(k) f_{k-1} b(k) + q^{(1)}(k)\gamma_k g^{(1)}(k). \tag{18.11}$$

3. *Compute*

$$\gamma_N = d(N) - p(N) f_{N-1} h(N). \tag{18.12}$$

Here f_k $(k = 1, \dots, N-1)$ are auxiliary variables, which are $r_k^L \times r_k^U$ matrices.

Proof. We should check that

$$D = \mathrm{diag}\{\gamma_1, \dots, \gamma_N\}$$

and moreover, by Lemma 5.3 and Lemma 5.6, that the matrices L, U satisfy the relations

$$L(k+1 : N, k) = P_{k+1} q^{(1)}(k), \quad k = 1, \dots, N-1, \tag{18.13}$$

$$U(k, k+1 : N) = g^{(1)}(k) H_{k+1}, \quad k = 1, \dots, N-1, \tag{18.14}$$

with the matrices P_k, H_k defined in (5.2), (5.6) and the elements $\gamma_k, q^{(1)}(k), g^{(1)}(k)$ determined in the algorithm.

Using Lemma 5.1 and Lemma 5.4 one obtains the following partitions of the matrix A:

$$A = \begin{pmatrix} A_{k-1} & G_{k-1} H_k \\ P_k Q_{k-1} & M_k \end{pmatrix}, \quad k = 2, \dots, N,$$

with the matrices P_k, Q_k defined in (5.1), (5.2) and the matrices G_k, H_k defined in (5.5), (5.6). Moreover, using (5.11) and (5.14) one obtains the partitions of the principal leading submatrices $A_k = A(1:k, 1:k)$ in the form

$$A_k = \begin{pmatrix} A_{k-1} & G_{k-1}h(k) \\ p(k)Q_{k-1} & d(k) \end{pmatrix}, \quad k = 2, \dots, N. \tag{18.15}$$

Next we introduce the matrices

$$f_k = Q_k A_k^{-1} G_k, \quad k = 1, \dots, N-1. \tag{18.16}$$

By Theorem 1.20,

$$\gamma_1 = d(1), \quad \gamma_k = d(k) - p(k) f_{k-1} h(k), \quad k = 2, \dots, N \tag{18.17}$$

which mean that the formulas (18.6), (18.8), (18.12) for γ_k $(k = 1, \dots, N)$ hold and moreover

$$L(k:N, k) = \Delta_k(:, 1)\gamma_k^{-1}, \quad k = 1, \dots, N, \tag{18.18}$$

$$U(k, k:N) = \gamma_k^{-1}\Delta_k(1, :), \quad k = 1, \dots, N, \tag{18.19}$$

where

$$\Delta_1 = A, \quad \Delta_k = M_k - P_k f_{k-1} H_k, \quad k = 2, \dots, N. \tag{18.20}$$

Now we will prove the relations (18.13), (18.14). For $k = 1$ one has $\gamma_1 = d(1)$ and, using (5.10), (5.13) one gets

$$\Delta_1(2:N, 1) = A(2:N, 1) = P_2 q(1),$$

$$\Delta_1(1, 2:N) = A(1, 2:N) = g(1)H_2$$

and hence using (18.18), (18.19) one obtains

$$L(2:N, 1) = P_2 q^{(1)}, \quad U(1, 2:N) = g^{(1)}(1)H_2$$

with $q^{(1)}(1)$ and $g^{(1)}(1)$ defined in (18.6).

For $k > 1$ one has the following. Using (5.10) one obtains the representations

$$M_k(:, 1) = A(k:N, k) = \begin{pmatrix} d(k) \\ P_{k+1}q(k) \end{pmatrix}, \quad k = 2, \dots, N-1. \tag{18.21}$$

Similarly, using (5.13) one obtains the representations

$$M_k(1, :) = A(k, k:N) = \begin{pmatrix} d(k) & g(k)H_{k+1} \end{pmatrix}, \quad k = 2, \dots, N-1. \tag{18.22}$$

Using the recursions (5.4) and (5.8) one gets

$$P_k f_{k-1} H_k = \begin{pmatrix} p(k) \\ P_{k+1}a(k) \end{pmatrix} f_{k-1} \begin{pmatrix} h(k) & b(k)H_{k+1} \end{pmatrix}. \tag{18.23}$$

Next, taking the first columns in (18.20), (18.23) and using (18.21) one obtains

$$\Delta_k(:,1) = \begin{pmatrix} d(k) - p(k)f_{k-1}h(k) \\ P_{k+1}(q(k) - a(k)f_{k-1}h(k)) \end{pmatrix}.$$

Similarly, taking the first rows in (18.20), (18.23) and using (18.22) one obtains

$$\Delta_k(1,:) = \begin{pmatrix} d(k) - p(k)f_{k-1}h(k) & (g(k) - p(k)f_{k-1}b(k))H_{k+1} \end{pmatrix}.$$

Thus,

$$\Delta_k(:,1) = \begin{pmatrix} \gamma_k \\ P_{k+1}q'(k) \end{pmatrix}, \qquad k = 2, \dots, N-1,$$

and

$$\Delta_k(1,:) = \begin{pmatrix} \gamma_k & g'(k)H_{k+1} \end{pmatrix}, \quad k = 2, \dots, N-1,$$

with the elements γ_k from (18.8) and

$$q'(k) = q(k) - a(k)f_{k-1}h(k), \quad g'(k) = g(k) - p(k)f_{k-1}b(k).$$

Furthermore, using (18.18), (18.19) one obtains (18.13), (18.14).

It remains to prove the relations (18.7) and (18.11). The equality (18.7) follows directly from the definition (18.16) and the relations

$$Q_1 = q(1), \quad G_1 = g(1), \quad \gamma_1 = d(1).$$

For $k > 1$ applying the factorization (1.52) to the matrices A_k partitioned in the form (18.15) one obtains

$$A_k = \begin{pmatrix} I & 0 \\ p(k)Q_{k-1}A_{k-1}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{k-1} & 0 \\ 0 & \gamma_k \end{pmatrix} \begin{pmatrix} I & A_{k-1}^{-1}G_{k-1}h(k) \\ 0 & I \end{pmatrix},$$

whence

$$A_k^{-1} = \begin{pmatrix} I & -A_{k-1}^{-1}G_{k-1}h(k) \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{k-1}^{-1} & 0 \\ 0 & \gamma_k^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -p(k)Q_{k-1}A_{k-1}^{-1} & I \end{pmatrix}. \tag{18.24}$$

Using the recursion (5.3) and (18.16) one has

$$\begin{aligned} Q_k \begin{pmatrix} I & -A_{k-1}^{-1}G_{k-1}h(k) \\ 0 & I \end{pmatrix} &= \begin{pmatrix} a(k)Q_{k-1} & -a(k)f_{k-1}h(k) + q(k) \end{pmatrix} \\ &= \begin{pmatrix} a(k)Q_{k-1} & q'(k) \end{pmatrix}. \end{aligned} \tag{18.25}$$

Using the recursion (5.7) and (18.16) one has

$$\begin{pmatrix} I & 0 \\ -p(k)Q_{k-1}A_{k-1}^{-1} & I \end{pmatrix} G_k = \begin{pmatrix} G_{k-1}b(k) \\ -p(k)f_{k-1}b(k) + g(k) \end{pmatrix} = \begin{pmatrix} G_{k-1}b(k) \\ g'(k) \end{pmatrix}. \tag{18.26}$$

Now from the definition (18.16) and the relations (18.24)–(18.26) one gets

$$f_k = a(k)f_{k-1}b(k) + q'(k)\gamma_k^{-1}g'(k),$$

which completes the proof. □

Set $m = \max_{1\le k\le N}(m_k)$, $r = \max_{1\le k\le N-1}(r_k^L, r_k^U)$. The complexity of the operations used in the algorithm from Theorem 18.2 is estimated as follows.

1. The formula (18.8): Two matrix multiplications which require mr^2 arithmetical multiplications and $m(r-1)r$ arithmetical additions and m^2r arithmetical multiplications and $m^2(r-1)$ arithmetical additions, respectively, and a matrix addition which costs m^2 arithmetical additions. Thus (18.8) costs less than $2mr^2 + 2m^2r$ arithmetical operations.
2. The formula (18.9): In the parentheses two matrix multiplications which require r^3 arithmetical multiplications and $r^2(r-1)$ arithmetical additions and r^2m arithmetical multiplications and $r(r-1)m$ arithmetical additions, respectively, and a matrix addition which costs rm arithmetical additions.

   Denote by $\rho(m)$ the complexity of the solution of an $m \times m$ system of linear algebraic equations using a standard method. Thus finding γ_k^{-1} costs $\rho(m)$ arithmetical operations. Multiplying by γ_k^{-1} costs rm^2 arithmetical multiplications and $r(m-1)m$ arithmetical additions.

   Thus (18.9) costs in total less than $2r^3+2r^2m+\rho(m)+2rm^2$ arithmetical operations.
3. The formula (18.10) costs the same number of operations, except for $\rho(m)$ arithmetical operations if we keep γ_k^{-1} from before.
4. The formula (18.11): less than $4r^3 + 2m^2r + 2mr^2$ operations.

Thus the complexity is estimated by

$$c < (8r^3 + 8r^2m + 8m^2r + \rho(m))N. \tag{18.27}$$

Example 18.3. Consider the $N \times N$ matrix A from Example 5.14,

$$A = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{N-2} & a^{N-1} \\ b & 1 & a & \cdots & a^{N-3} & a^{N-2} \\ b^2 & b & 1 & \cdots & a^{N-4} & a^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b^{N-2} & b^{N-3} & b^{N-4} & \cdots & 1 & a \\ b^{N-1} & b^{N-2} & b^{N-3} & \cdots & b & 1 \end{pmatrix},$$

with $ab \neq 1$.

For this matrix the quasiseparable generators of the factors in the LDU factorization (18.1) will be computed using the algorithm in Theorem 18.2.

For A one can choose the quasiseparable generators which have been computed in Example 5.14. These are:

$$p(i) = b,\ \ i = 2,\dots,N, \quad a(k) = b,\ \ k = 2,\dots,N-1,$$

which will also be generators for the lower triangular matrix L in the LDU decomposition,

$$h(i) = a,\ \ i = 2,\dots,N, \quad b(k) = a,\ \ k = 2,\dots,N-1,$$

which will also be generators for the upper triangular matrix U in the LDU decomposition, and

$$q(j) = 1,\ \ g(j) = 1,\ \ j = 1,\dots,N-1, \quad d(k) = 1,\ \ k = 1,\dots,N.$$

It remains to compute the entries $\gamma_k,\ \ k = 1,\dots,N$ of the diagonal matrix D, the generators $q^{(1)}(j),\ \ j = 1,\dots,N-1$ of L, the generators $g^{(1)}(j),\ \ j = 1,\dots,N-1$ of U and the auxiliary variables $f_j,\ \ j = 1,\dots,N-1$.

One has

$$\begin{aligned}
\gamma_1 &= d(1) = 1, \quad q^{(1)}(1) = q(1)\gamma_1^{-1} = 1,\\
g^{(1)}(1) &= \gamma_1^{-1} g(1) = 1,\\
f_1 &= q^{(1)}(1)\gamma_1 g^{(1)}(1) = 1,\\
\gamma_2 &= d(2) - p(2) f_1 h(2) = 1 - ab,\\
q^{(1)}(2) &= (q(2) - a(2) f_1 h(2))\gamma_2^{-1} = 1,\\
g^{(1)}(2) &= \gamma_2^{-1}(g(2) - p(2) f_1 b(2)) = 1,\\
f_2 &= a(2) f_1 b(2) + q^{(1)}(2)\gamma_2 g^{(1)}(2) = ab + 1 - ab = 1.
\end{aligned}$$

One can prove by induction that

$$\gamma_k = 1 - ab,\ \ k = 2,\dots,N, \quad q^{(1)}(j) = g^{(1)}(j) = f_j = 1,\ \ j = 1,\dots,N-1.$$

Indeed, suppose that for a certain k these formulas are true. Then for $k+1$ one has

$$\begin{aligned}
\gamma_{k+1} &= d(k+1) - p(k+1) f_k h(k+1) = 1 - a\cdot 1\cdot b = 1 - ab,\\
q^{(1)}(k+1) &= (q(k+1) - a(k+1) f_k h(k+1))\gamma_{k+1}^{-1} = (1 - a\cdot 1\cdot b)/(1-ab) = 1,\\
g^{(1)}(k+1) &= \gamma_{k+1}^{-1}(g(k+1) - p(k+1) f_k b(k+1)) = \frac{1}{1-ab}(1 - a\cdot 1\cdot b) = 1
\end{aligned}$$

and

$$\begin{aligned}
f_{k+1} &= a(k+1) f_k b(k+1) + q^{(1)}(k+1)\gamma_{k+1} g^{(1)}(k+1)\\
&= a\cdot 1\cdot b + 1\cdot(1-ab)\cdot 1 = 1.
\end{aligned}$$

Using the quasiseparable generators which have been found for the factor matrices, one concludes that

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ b & 1 & 0 & \cdots & 0 & 0 \\ b^2 & b & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b^{N-2} & b^{N-3} & b^{N-4} & \cdots & 1 & 0 \\ b^{N-1} & b^{N-2} & b^{N-3} & \cdots & b & 1 \end{pmatrix},$$

$$D = \mathrm{diag}(1, 1-ab, 1-ab, \ldots, 1-ab),$$

and

$$U = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{N-2} & a^{N-1} \\ 0 & 1 & a & \cdots & a^{N-3} & a^{N-2} \\ 0 & 0 & 1 & \cdots & a^{N-4} & a^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

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Example 18.4. Consider the $N \times N$ tridiagonal matrix

$$A = \begin{pmatrix} d & \beta & 0 & \cdots & 0 & 0 \\ \alpha & 0 & \beta & \cdots & 0 & 0 \\ 0 & \alpha & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \beta \\ 0 & 0 & 0 & \cdots & \alpha & 0 \end{pmatrix},$$

with $d \neq 0$. We apply the algorithm from Theorem 18.2 to compute the quasiseparable generators of the factors in the LDU factorization of A.

For A one can choose the quasiseparable generators

$$p(i) = \alpha, \quad i = 2, \ldots, N, \quad a(k) = 0, \quad k = 2, \ldots, N-1,$$

which will also be generators for the lower triangular matrix L in the LDU decomposition,

$$h(i) = 1, \quad i = 2, \ldots, N, \quad b(k) = 0, \quad k = 2, \ldots, N-1,$$

which will also be generators for the upper triangular matrix U in the LDU decomposition and

$$q(j) = 1, \quad g(j) = \beta, \quad j = 1, \ldots, N-1, \quad d(1) = d, \quad d(k) = 0, \quad k = 2, \ldots, N.$$

It remains to compute the entries $\gamma_k, \quad k = 1, \dots, N$, of the diagonal matrix D, the generators $q^{(1)}(j), \quad j = 1, \dots, N-1$, of L, the generators $g^{(1)}(j), \quad j = 1, \dots, N-1$, of U, and the auxiliary variables $f_j, \quad j = 1, \dots, N-1$.

One has

$$\gamma_1 = d(1) = d, \quad q^{(1)}(1) = q(1)\gamma_1^{-1} = \frac{1}{d}, \quad g^{(1)}(1) = \gamma_1^{-1}g(1) = \frac{\beta}{d},$$
$$f_1 = q^{(1)}(1)\gamma_1 g^{(1)}(1) = \frac{\beta}{d}, \quad \gamma_2 = d(2) - p(2)f_1 h(2) = 0 - \alpha f_1 = -\frac{\alpha\beta}{d},$$
$$q^{(1)}(2) = (q(2) - a(2)f_1 h(2))\gamma_2^{-1} = \frac{1}{\gamma_2}, \quad g^{(1)}(2) = \gamma_2^{-1}(g(2) - p(2)f_1 b(2)) = \frac{\beta}{\gamma_2},$$
$$f_2 = a(2)f_1 b(2) + q^{(1)}(2)\gamma_2 g^{(1)}(2) = \frac{\beta}{\gamma_2}.$$

One can prove by induction that

$$\gamma_k = -\frac{\alpha\beta}{\gamma_{k-1}}, \; q^{(1)}(k) = \frac{1}{\gamma_k}, \; f_k = g^{(1)}(k) = \frac{\beta}{\gamma_k}, \quad k = 1, \dots, N-1.$$

Indeed, suppose that for a certain k these formulas are true. Then for $k+1$ one has

$$\gamma_{k+1} = d(k+1) - p(k+1)f_k h(k+1) = -\frac{\alpha\beta}{\gamma_k},$$
$$q^{(1)}(k+1) = (q(k+1) - a(k+1)f_k h(k+1))\gamma_{k+1}^{-1} = \frac{1}{\gamma_{k+1}},$$
$$g^{(1)}(k+1) = \gamma_{k+1}^{-1}(g(k+1) - p(k+1)f_k b(k+1)) = \frac{\beta}{\gamma_{k+1}}$$

and

$$f_{k+1} = a(k+1)f_k b(k+1) + q^{(1)}(k+1)\gamma_{k+1}g^{(1)}(k+1) = \frac{1}{\gamma_{k+1}}\gamma_{k+1}\frac{\beta}{\gamma_{k+1}} = \frac{\beta}{\gamma_{k+1}}.$$

It is clear that the recursion $\gamma_k = \gamma_{k-2}$ holds and therefore

$$\gamma_{2m-1} = d, \; \gamma_{2m} = -\frac{\alpha\beta}{d}, \; m = 1, 2, \dots, \left[\ \tfrac{N+1}{2}\ \right].$$

It follows that

$$D = \operatorname{diag}\left\{d, -\frac{\alpha\beta}{d}, d, -\frac{\alpha\beta}{d}, \dots d\right\}$$

for odd N and

$$D = \operatorname{diag}\left\{d, -\frac{\alpha\beta}{d}, d, -\frac{\alpha\beta}{d}, \dots d, -\frac{\alpha\beta}{d}\right\}$$

for even N.

Using the quasiseparable generators found for the factor matrices, we obtain

$$L=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{\alpha}{d} & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{d}{\beta} & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{\alpha}{d} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{\beta} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{\alpha}{d} & 1 \end{pmatrix}, \quad U=\begin{pmatrix} 1 & \frac{\beta}{d} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\frac{d}{\alpha} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \frac{\beta}{d} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{d}{\alpha} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & \frac{\beta}{d} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

for even N and

$$L=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{\alpha}{d} & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{d}{\beta} & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{\alpha}{d} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{\beta} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{d}{\beta} & 1 \end{pmatrix}, \quad U=\begin{pmatrix} 1 & \frac{\beta}{d} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\frac{d}{\alpha} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \frac{\beta}{d} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{d}{\alpha} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{d}{\alpha} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

for odd N. ◊

Example 18.5. Consider the 5×5 matrix

$$A=\begin{pmatrix} 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 2 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix}.$$

Using Example 5.16 we get quasiseparable generators of A:

$$p(2)=2,\ p(3)=\begin{pmatrix} 1 & 2 \end{pmatrix},\ p(4)=\begin{pmatrix} 1 & 1 \end{pmatrix},\ p(5)=1,$$

$$q(1)=1,\ q(2)=\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ q(3)=\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ q(4)=2,$$

$$a(2)=\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ a(3)=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ a(4)=\begin{pmatrix} 1 & 0 \end{pmatrix};$$

$$g(1)=1,\ g(2)=\begin{pmatrix} 0 & 1 \end{pmatrix},\ g(3)=\begin{pmatrix} 1 & 1 \end{pmatrix},\ g(4)=2,$$

$$h(2)=2,\ h(3)=\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ h(4)=\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ h(5)=1,$$

$$b(2)=\begin{pmatrix} 1 & 0 \end{pmatrix},\ b(3)=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ b(4)=\begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

$$d(k)=3, \quad k=1,\dots,5.$$

From Theorem 18.2 it follows that in the factorization (18.1) for the matrix A the matrix L has lower quasiseparable generators $p(i)$ $(i = 2, \dots, 5)$, $q^{(1)}(j)$ $(j = 1, \dots, 5-1)$, $a(k)$ $(k = 2, \dots, 5-1)$, the matrix U has upper quasiseparable generators $g^{(1)}(i)$ $(i = 1, \dots, 5-1)$, $h(j)$ $(j = 2, \dots, 5)$, $b(k)$ $(k = 2, \dots, 5-1)$, and the matrix D has the form $D = \operatorname{diag}\{\gamma_1, \dots, \gamma_5\}$. Here the elements $p(i)$, $a(k)$, $h(j)$, $b(k)$ are the same as for the matrix A and the elements $q^{(1)}(j)$, $g^{(1)}(i), \gamma_k$ will be determined according to the algorithm in the theorem. Thus we proceed as follows.

Compute

$$\gamma_1 = d(1) = 3, \quad q^{(1)}(1) = q(1)\gamma_1^{-1} = \frac{1}{3}, \quad g^{(1)}(1) = \gamma_1^{-1} g(1) = \frac{1}{3},$$
$$f_1 = q^{(1)}(1)\gamma_1 g^{(1)}(1) = \frac{1}{3}.$$

For $k = 2$ we compute the diagonal entry

$$\gamma_2 = d(2) - p(2) f_{2-1} h(2) = 3 - \frac{4}{3} = \frac{5}{3},$$

the generators

$$q^{(1)}(2) = [q(2) - a(2) f_{2-1} h(2)]\gamma_2^{-1} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{3} \cdot 2 \right) \frac{3}{5} = \frac{1}{5} \begin{pmatrix} -2 \\ 3 \end{pmatrix},$$
$$g^{(1)}(2) = \gamma_2^{-1}[g(2) - p(2) f_{2-1} b(2)] = \frac{3}{5} \left(\begin{pmatrix} 0 & 1 \end{pmatrix} - 2 \cdot \frac{1}{3} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} \right) = \frac{1}{5} \begin{pmatrix} -2 & 3 \end{pmatrix},$$

and the auxiliary matrix

$$\begin{aligned} f_2 &= a(2) f_{2-1} b(2) + q^{(1)}(2)\gamma_2 g^{(1)}(2) \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} \frac{1}{5} \frac{5}{3} \frac{1}{5} \begin{pmatrix} -2 & 3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}. \end{aligned}$$

For $k = 3$ we compute the diagonal entry

$$\gamma_3 = d(3) - p(3) f_2 h(3) = 3 - \begin{pmatrix} 1 & 2 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{8}{5},$$

the generators of L and U, respectively,

$$\begin{aligned} q^{(1)}(3) &= [q(3) - a(3) f_2 h(3)]\gamma_3^{-1} \\ &= \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) \frac{5}{8} = \frac{1}{8} \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \\ g^{(1)}(3) &= \gamma_3^{-1}[g(3) - p(3) f_2 b(3)] \\ &= \frac{5}{8} \left(\begin{pmatrix} 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right) = \frac{1}{8} \begin{pmatrix} 2 & 5 \end{pmatrix}, \end{aligned}$$

and the auxiliary matrix

$$\begin{aligned} f_3 &= a(3)f_2b(3) + q^{(1)}(3)\gamma_3 g^{(1)}(3) \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \frac{8}{5} \frac{1}{8} \begin{pmatrix} 2 & 5 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}. \end{aligned}$$

For $k = 4$ we compute the diagonal entry

$$\gamma_4 = 3 - \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{11}{8},$$

the last generator of L

$$\begin{aligned} q^{(1)}(4) &= [q(4) - a(4)f_3h(4)]\gamma_4^{-1} \\ &= \left(2 - \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \frac{8}{11} = \frac{10}{11}, \end{aligned}$$

and the last generator of U

$$\begin{aligned} g^{(1)}(4) &= \gamma_4^{-1}[g(4) - p(4)f_3b(4)] \\ &= \frac{8}{11} \left(2 - \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{10}{11}. \end{aligned}$$

Also for $k = 4$ we compute f_k, the auxiliary matrix which will be used in the computation of γ_5:

$$\begin{aligned} f_4 &= a(4)f_3b(4) + q^{(1)}(4)\gamma_4 g^{(1)}(4) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{10}{11} \frac{11}{8} \frac{10}{11} = \frac{18}{11}. \end{aligned}$$

Finally, we compute γ_5:

$$\gamma_5 = 3 - 1 \cdot \frac{18}{11} \cdot 1 = \frac{15}{11}.$$

Thus the generators $q^{(1)}(k),\ k = 1, \dots, 4$ of L are

$$q^{(1)}(1) = \frac{1}{3},\ q^{(1)}(2) = \frac{1}{5} \begin{pmatrix} -2 \\ 3 \end{pmatrix},\ q^{(1)}(3) = \frac{1}{8} \begin{pmatrix} 2 \\ 5 \end{pmatrix},\ q^{(1)}(4) = \frac{10}{11},$$

the generators $g^{(1)}(k),\ k = 1, \dots, 4$ of U are

$$g^{(1)}(1) = \frac{1}{3},\ g^{(1)}(2) = \frac{1}{5} \begin{pmatrix} -2 & 3 \end{pmatrix},\ g^{(1)}(3) = \frac{1}{8} \begin{pmatrix} 2 & 5 \end{pmatrix},\ g^{(1)}(4) = \frac{10}{11},$$

and the diagonal entries of D are

$$\gamma_1 = 3,\ \gamma_2 = \frac{5}{3},\ \ g_3 = \frac{8}{5},\ \gamma_4 = \frac{11}{8},\ \gamma_5 = \frac{15}{11}.$$

In the decomposition $A = LDU$ of the matrix A of size $N = 5$, the matrix L has lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q^{(1)}(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$, therefore

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{5} & \frac{7}{8} & 1 & 0 \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{4} & \frac{10}{11} & 1 \end{pmatrix}.$$

The matrix U has upper quasiseparable generators $g^{(1)}(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$, therefore U is equal to L^T:

$$U = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{7}{8} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{10}{11} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix D has the form

$$D = \operatorname{diag}\{\gamma_1, \dots, \gamma_N\} = \operatorname{diag}\left\{3, \frac{5}{3}, \frac{8}{5}, \frac{11}{8}, \frac{15}{11}\right\}. \qquad \Diamond$$

§18.3 Solution of linear systems and analog of Levinson algorithm

Let A be a matrix satisfying the conditions of Theorem 18.2. The solution of the corresponding linear system $Ax = y$ may be calculated as follows.

Algorithm 18.6.

1. Start with

$$\gamma_1 = d(1), \quad \tilde{q}(1) = q(1), \quad g^{(1)}(1) = \gamma_1^{-1} g(1), \tag{18.28}$$

$$f_1 = \tilde{q}(1) g^{(1)}(1), \tag{18.29}$$

then for $k = 2, \dots, N-1$ compute recursively

$$\gamma_k = d(k) - p(k) f_{k-1} h(k), \tag{18.30}$$

$$\tilde{q}(k) = q(k) - a(k) f_{k-1} h(k), \tag{18.31}$$

$$g^{(1)}(k) = \gamma_k^{-1} [g(k) - p(k) f_{k-1} b(k)], \tag{18.32}$$

$$f_k = a(k) f_{k-1} b(k) + \tilde{q}(k) g^{(1)}(k), \tag{18.33}$$

and finally compute

$$\gamma_N = d(N) - p(N)f_{N-1}h(N). \tag{18.34}$$

2. Start with

$$z(1) = \gamma_1^{-1}y(1),\ \chi_2 = \tilde{q}(1)x(1),$$

then for $k = 2, \dots, N-1$ compute recursively

$$z(k) = \gamma_k^{-1}(y(k) - p(k)\chi_k), \tag{18.35}$$

$$\chi_{k+1} = a(k)\chi_k + \tilde{q}(k)z(k), \tag{18.36}$$

and finally compute

$$z(N) = \gamma_N^{-1}(y(N) - p(N)\chi_N). \tag{18.37}$$

3. Start with

$$x(N) = z(N),\ \eta_{N-1} = h(N)x(N),$$

then for $k = N-1, \dots, 2$ compute recursively

$$x(k) = z(k) - g^{(1)}(k)\eta_k, \tag{18.38}$$

$$\eta_{k-1} = b(k)\eta_k + h(k)x(k) \tag{18.39}$$

and finally compute

$$x(1) = z(1) - g^{(1)}(1)\eta_1. \tag{18.40}$$

We justify Algorithm 18.6 as follows. At first we compute the factorization

$$A = \tilde{L}U,$$

where $\tilde{L}$ is a block lower triangular matrix and U is a block upper triangular matrix with identities on the main diagonal. Using the algorithm from Theorem 18.2 one obtains the factorization of A in the form

$$A = LDU,$$

with the block lower and upper triangular matrices L and U with identities on the main diagonals and a block diagonal matrix D. Here L has lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q^{(1)}(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$, U has upper quasiseparable generators $g^{(1)}(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$, and D has the form $D = \text{diag}\{\gamma_1, \dots, \gamma_N\}$, where the elements $p(i)$, $a(k)$, $h(j)$, $b(k)$ are the same as for the matrix A and the elements $q^{(1)}(j)$, $g^{(1)}(i), \gamma_k$ are determined via the formulas (18.6)–(18.12). Set $\tilde{L} = LD$. One can easily see that $\tilde{L}$ is a block lower triangular matrix with lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $\tilde{q}(j) = q^{(1)}(j)\gamma_j$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ and diagonal entries γ_k $(k = 1, \dots, N)$. Thus using the formulas (18.6)–(18.12) we obtain the formulas (18.28)–(18.34) for lower quasiseparable generators and diagonal entries of the matrix $\tilde{L}$ and upper quasiseparable generators of the matrix U.

Applying the algorithm from Theorem 13.10 to the matrix $\tilde{L}$ we obtain Step 2 of the Algorithm 18.6 for determining the solution z of the system $\tilde{L}z = y$. Finally, applying the algorithm from Theorem 13.13 we obtain Step 3 for determining the vector x from the system $Ux = z$.

Set $m = \max_{1\leq k\leq N}(m_k),\ r = \max_{1\leq k\leq N-1}(r_k^L, r_k^U)$. The complexity of the operations used in the first step of Algorithm 18.6 is estimated as follows.

1. The formula (18.30): Like in (18.8) two matrix multiplications which require mr^2 arithmetical multiplications and $m(r-1)r$ arithmetical additions and m^2r arithmetical multiplications and $m^2(r-1)$ arithmetical additions, respectively, and a matrix addition which costs m^2 arithmetical additions. Thus the formula (18.30) costs less than $2mr^2 + 2m^2r$ arithmetical operations.
2. The formula (18.31): Like in the parentheses of the formula (18.9), two matrix multiplications which require r^3 arithmetical multiplications and $r^2(r-1)$ arithmetical additions and r^2m arithmetical multiplications and $r(r-1)m$ arithmetical additions, respectively, and a matrix addition which costs rm arithmetical additions.

   Thus the formula (18.31) costs in total less than $2r^3 + 2r^2m + \rho(m)$ arithmetical operations.
3. The formula (18.32): In the parentheses two matrix multiplications which require r^3 arithmetical multiplications and $r^2(r-1)$ arithmetical additions and r^2m arithmetical multiplications and $r(r-1)m$ arithmetical additions, respectively, and a matrix addition which costs rm arithmetical additions.

   Denote by $\rho(m)$ the complexity of the solution of an $m \times m$ system of linear algebraic equations using a standard method. Thus finding γ_k^{-1} costs $\rho(m)$ arithmetical operations. Multiplying by γ_k^{-1} costs rm^2 arithmetical multiplications and $r(m-1)m$ arithmetical additions.

   Thus the formula (18.32) costs in total less than $2r^3 + 2r^2m + \rho(m) + 2rm^2$ arithmetical operations.
4. The formula (18.33): less than $4r^3 + 2m^2r$ operations.

Thus the total estimate for the complexity of Step 1 is

$$c_1 < (8r^3 + 6r^2m + 6rm^2 + \rho(m))N.$$

Applying the estimate (13.26) to the triangular systems $\tilde{L}z = y$ and $Ux = w$ we obtain the estimates

$$c_2 < (2r^2 + 4mr + 2m^2 + \rho(m))N, \quad c_3 < (2r^2 + 4mr + 2m^2 + \rho(m))N.$$

Thus the total estimate for the complexity of the Algorithm 18.6 is

$$c < (8r^3 + 6r^2m + 6m^2r + 8mr + 4r^2 + 4m^2 + 3\rho(m))N.$$

Next we present another justification of the Algorithm 18.6 based on the successive solution of the equations

$$A_k x^{(k)} = y^{(k)}, \quad k = 1, \dots, N \tag{18.41}$$

with $A_k = A(1:k, 1:k)$, $y^{(k)} = (y(1) \dots y(k))^T$. We use also the auxiliary equations

$$A_k G_k^{(1)} = G_k, \quad k = 1, \dots, N-1 \tag{18.42}$$

with the matrices G_k defined in (5.5). We set $x^{(1)} = z(1)$, $G_1^{(1)} = g^{(1)}(1)$ and for $k = 2, \dots, N$ we use the partitions

$$x^{(k)} = \begin{pmatrix} t_{k-1} \\ z(k) \end{pmatrix}, \quad y^{(k)} = \begin{pmatrix} y^{(k-1)} \\ y(k) \end{pmatrix}, \quad G_k^{(1)} = \begin{pmatrix} G'_k \\ g^{(1)}(k) \end{pmatrix}.$$

Here $z(k)$ and $y(k)$ are the last components of the vectors $x^{(k)}$ and $y^{(k)}$ and $g^{(1)}(k)$ is the last block row of the matrix G_k.

It is clear that $z(1) = (d(1))^{-1} y(1)$ and $g^{(1)}(1) = (d(1))^{-1} g(1)$. For $k = 2, \dots, N$, using the partitions of the submatrices A_k as

$$A_k = \begin{pmatrix} A_{k-1} & G_{k-1} h(k) \\ p(k) Q_{k-1} & d(k) \end{pmatrix}$$

and the recursion (5.7), the system (18.42) may be written in the form

$$\begin{cases} A_{k-1} G'_{k-1} + G_{k-1} h(k) g^{(1)}(k) = G_{k-1} b(k), \\ p(k) Q_{k-1} G'_{k-1} + d(k) g^{(1)}(k) = g(k), \end{cases} \tag{18.43}$$

and the system (18.41) in the form

$$\begin{cases} A_{k-1} t_{k-1} + G_{k-1} h(k) z(k) = y^{(k-1)}, \\ p(k) Q_{k-1} t_{k-1} + d(k) z(k) = y(k). \end{cases} \tag{18.44}$$

From the first equation in (18.43) we get

$$G'_{k-1} = G_{k-1}^{(1)} (b(k) - h(k) g^{(1)}(k)). \tag{18.45}$$

Inserting this in the second equation in (18.43) we get

$$p(k) Q_{k-1} G_{k-1}^{(1)} b(k) + (d(k) - p(k) Q_{k-1} G_{k-1}^{(1)} h(k)) g^{(1)}(k) = g(k). \tag{18.46}$$

Similarly, the first equation in (18.44) may be written in the form

$$t_{k-1} = x^{(k-1)} - G_{k-1}^{(1)} h(k) z(k). \tag{18.47}$$

Inserting this in the second equation in (18.44) we get

$$p(k)Q_{k-1}x^{(k-1)} + (d(k) - p(k)Q_{k-1}G^{(1)}_{k-1}h(k))z(k) = y(k). \tag{18.48}$$

As it was shown in the proof of Theorem 18.2, the variable

$$f_{k-1} = Q_{k-1}A^{-1}_{k-1}G_{k-1} = Q_{k-1}G^{(1)}_{k-1}$$

satisfies the recursion relations (18.29), (18.33). Thus the equalities (18.46), (18.48) may be written in the form

$$\gamma_k g^{(1)}(k) = g(k) - p(k)f_{k-1}b(k) \tag{18.49}$$

and

$$\gamma_k z(k) = y(k) - p(k)\chi_k, \tag{18.50}$$

with γ_k defined in (18.30) and $\chi_k = Q_{k-1}x^{(k-1)}$. From (18.49) and (18.50) we obtain the formulas (18.32) and (18.35), (18.37).

Next we derive recursion relations for χ_k. We obviously have $\chi_2 = q(1)z(1) = \tilde{q}(1)z(1)$ and for $k = 2, \ldots, N-1$, using the recursion (5.3) and the equality (18.47), we get

$$\begin{aligned}\chi_{k+1} = Q_k x^{(k)} &= \begin{pmatrix} a(k)Q_{k-1} & q(k) \end{pmatrix}\begin{pmatrix} t_{k-1} \\ z(k) \end{pmatrix} \\ &= \begin{pmatrix} a(k)Q_{k-1} & q(k) \end{pmatrix}\begin{pmatrix} x^{(k-1)} - G^{(1)}_{k-1}h(k)z(k) \\ z(k) \end{pmatrix} \\ &= a(k)\chi_k + (q(k) - a(k)f_{k-1}h(k))z(k) = a(k)\chi_k + \tilde{q}(k)z(k),\end{aligned}$$

with $\tilde{q}(k)$ defined in (18.31).

It remains to justify Step 3 of Algorithm 18.6. First we prove by induction that

$$t_k = \operatorname{col}(z(i) - g^{(1)}(i)\eta_{ik})_{i=1}^k, \quad k = 1, \ldots, N-1, \tag{18.51}$$

where

$$\eta_{ik} = \sum_{j=i}^{k}(b^{(1)})^{<}_{i,j+1}h(j+1)z(j+1),$$

with

$$b^{(1)}(k) = b(k) - h(k)g^{(1)}(k). \tag{18.52}$$

Using (18.41) with $k = 1, 2$ and the partition $x^{(2)} = \begin{pmatrix} t_1 \\ z(2) \end{pmatrix}$ we get

$$t_1 = x^{(1)} - g^{(1)}(1)h(2)z(2).$$

Let for some k with $2 \leq k \leq N-1$ the relation

$$t_{k-1} = \operatorname{col}(z(i) - g^{(1)}(i)\eta_{i,k-1})_{i=1}^{k-1} \tag{18.53}$$

hold. Using the formula (18.45) and the partitions $G_k^{(1)} = \begin{pmatrix} G'_k \\ g^{(1)}(k) \end{pmatrix}$ one can easily derive that

$$G_k^{(1)} = \mathrm{col}(g^{(1)}(i)(b^{(1)})^{<}_{i,k+1})_{i=1}^{k}, \quad k = 1, \dots, N-1. \tag{18.54}$$

Next, using (18.47), the partition $x^{(k)} = \begin{pmatrix} t_{k-1} \\ z(k) \end{pmatrix}$ and the formulas (18.53), (18.54) we get

$$t_k = \begin{pmatrix} t_{k-1} \\ z(k) \end{pmatrix} - \mathrm{col}(g^{(1)}(i)(b^{(1)})^{<}_{i,k+1}h(k+1)z(k+1))_{i=1}^{k},$$

whence

$$t_k(k) = z(k) - g^{(1)}(k)h(k+1)z(k+1) = z(k) - g^{(1)}(k)\eta_{kk} \tag{18.55}$$

and

$$\begin{aligned} t_k(i) &= t_{k-1}(i) - g^{(1)}(i)(b^{(1)})^{<}_{i,k+1}h(k+1)z(k+1) \\ &= z(i) - g^{(1)}(i)(\eta_{i,k-1} + (b^{(1)})^{<}_{i,k+1}h(k+1)z(k+1))) \\ &= z(i) - g^{(1)}(i)\eta_{ik}, \qquad i = 1, \dots, k-1\,. \end{aligned} \tag{18.56}$$

From (18.55) and (18.56) we obtain (18.51).

Thus we have $x = x^{(N)} = \begin{pmatrix} t_{N-1} \\ x(N) \end{pmatrix}$, and therefore $x(N) = z(N)$ and

$$x(i) = t_{N-1}(i) = z(i) - g^{(1)}(i)\eta_i, \quad i = N-1, \dots, 1, \tag{18.57}$$

where $\eta_i = \eta_{i,N-1}$. We have $\eta_{N-1} = h(N)z(N) = h(N)x(N)$. Next, for $i = N-1, \dots, 2$ we get

$$\begin{aligned} \eta_{i-1} &= \sum_{j=i-1}^{N-1} (b^{(1)})^{<}_{i-1,j+1}h(j+1)z(j+1) \\ &= h(i)z(i) + \sum_{j=i}^{N-1} (b^{(1)})^{<}_{i-1,j+1}h(j+1)z(j+1) \\ &= h(i)z(i) + b^{(1)}(i) \sum_{j=i}^{N-1} (b^{(1)})^{<}_{i,j+1}h(j+1)z(j+1) \\ &= b^{(1)}(i)\eta_i + h(i)z(i), \end{aligned}$$

and so, using (18.57) and (18.52), we get

$$\eta_{i-1} = (b(i) - h(i)g^{(1)}(i))\eta_i + h(i)(x(i) + g^{(1)}(i)) = b(i)\eta_i + h(i)x(i).$$

This recursive algorithm may be viewed as an analog of the famous Levinson algorithm for Toeplitz systems.

§18.4 The inversion formula

Under the assumptions of Theorem 18.2 the generators of the inverse matrix can be obtained via the following algorithm.

Theorem 18.7. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with block entries of sizes $m_i \times m_j$ and with invertible principal leading block submatrices $A_k = \{A_{ij}\}_{i,j=1}^k$, $k = 1, \dots, N$. Assume that A has quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$, $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$, $d(k)$ $(k = 1, \dots, N)$ of orders r_k^L, r_k^U $(k = 1, \dots, N-1)$.*

Then quasiseparable generators $p^{(1)}(i)$ $(i = 2, \dots, N)$, $q^{(1)}(j)$ $(j = 1, \dots, N-1)$, $a^{(1)}(k)$ $(k = 2, \dots, N-1)$, $g^{(1)}(i)$ $(i = 1, \dots, N-1)$, $h^{(1)}(j)$ $(j = 2, \dots, N)$, $b^{(1)}(k)$ $(k = 2, \dots, N-1)$, $d^{(1)}(k)$ $(k = 1, \dots, N)$ of the inverse A^{-1} are determined as follows.

1. Determine the elements $q^{(1)}(j)$ $(j = 1, \dots, N-1)$, $g^{(1)}(i)$ $(i = 1, \dots, N-1)$ and γ_k $(k = 1, \dots, N)$ via the formulas

$$\begin{gathered} \gamma_1 = d(1), \quad q^{(1)}(1) = q(1)\gamma_1^{-1}, \quad g^{(1)}(1) = \gamma_1^{-1} g(1), \\ f_1 = q^{(1)}(1)\gamma_1 g^{(1)}(1); \end{gathered} \tag{18.58}$$

$$\begin{aligned} \gamma_k &= d(k) - p(k) f_{k-1} h(k), \\ q^{(1)}(k) &= [q(k) - a(k) f_{k-1} h(k)]\gamma_k^{-1}, \\ g^{(1)}(k) &= \gamma_k^{-1}[g(k) - p(k) f_{k-1} b(k)], \\ f_k &= a(k) f_{k-1} b(k) + q^{(1)}(k) \cdot \gamma_k \cdot g^{(1)}(k), \\ k &= 2, \dots, N-1; \end{aligned} \tag{18.59}$$

$$\gamma_N = d(N) - p(N) f_{N-1} h(N). \tag{18.60}$$

2. Compute $\lambda_N, h^{(1)}(N), p^{(1)}(N), d^{(1)}(N)$ by

$$\begin{pmatrix} \lambda_N & h^{(1)}(N) \\ p^{(1)}(N) & d^{(1)}(N) \end{pmatrix} = \begin{pmatrix} h(N) \\ I \end{pmatrix} \gamma_N^{-1} \begin{pmatrix} -p(N) & I \end{pmatrix}, \tag{18.61}$$

then for $k = N-1, \dots, 2$ compute $\lambda_k, h^{(1)}(k), p^{(1)}(k), d^{(1)}(k)$ by

$$a^{(1)}(k) = a(k) - q^{(1)}(k) p(k), \quad b^{(1)}(k) = b(k) - h(k) g^{(1)}(k), \tag{18.62}$$

$$\begin{pmatrix} \lambda_k & h^{(1)}(k) \\ p^{(1)}(k) & d^{(1)}(k) \end{pmatrix} = \begin{pmatrix} h(k) & b^{(1)}(k) \\ I & -g^{(1)}(k) \end{pmatrix} \begin{pmatrix} \gamma_k^{-1} & 0 \\ 0 & \lambda_{k+1} \end{pmatrix} \begin{pmatrix} -p(k) & I \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix}, \tag{18.63}$$

and finally compute

$$d^{(1)} = \gamma_1^{-1} - g^{(1)} \lambda_2 q^{(1)}(1). \tag{18.64}$$

Here λ_k $(k = N, \dots, 2)$ are auxiliary variables, which are $r_{k-1}^U \times r_{k-1}^L$ matrices.

Proof. By Theorem 18.2, one obtains the factorization of the matrix A in the form (18.1) with the block lower triangular matrix L which has lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q^{(1)}(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ and identity diagonal entries, the block upper triangular matrix U with upper quasiseparable generators $g^{(1)}(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ and identity diagonal entries, and the block diagonal matrix $D = \operatorname{diag}\{\gamma_1, \dots, \gamma_N\}$. Here, $p(i)$, $a(k)$, $h(j)$, $b(k)$ are the same as for the matrix A and $q^{(1)}(j)$, $g^{(1)}(i), \gamma_k$ are determined via the formulas (18.58)–(18.60).

For the inverse matrix one has

$$A^{-1} = (U^{-1}D^{-1})L^{-1}.$$

By Theorem 13.13, the matrix U^{-1} has upper quasiseparable generators $-g^{(1)}(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b^{(1)}(k)(k = 2, \dots, N-1)$ with $b^{(1)}(k)$ defined in (18.62). One can easily see that the matrix $U^{-1}D^{-1}$ is an upper triangular matrix with upper quasiseparable generators $-g^{(1)}(i)$ $(i = 1, \dots, N-1)$, $h(j)\gamma_j^{-1}$ $(j = 2, \dots, N)$, $b^{(1)}(k)(k = 2, \dots, N-1)$, and diagonal entries γ_k^{-1} $(k = 1, \dots, N)$. Next, by Theorem 13.10, the matrix L^{-1} has lower quasiseparable generators $-p(i)$ $(i = 2, \dots, N)$, $q^{(1)}(j)$ $(j = 1, \dots, N-1)$, $a^{(1)}(k)$ $(k = 2, \dots, N-1)$, with $a^{(1)}(k)$ defined in (18.62). Now by Theorem 17.5 the generators $q^{(1)}(k)$, $a^{(1)}(k)$, $b^{(1)}(k)$, $g^{(1)}(k)$ determined above coincide with the corresponding generators of the matrix A^{-1}, and the rest of the generators $p^{(1)}(k)$, $h^{(1)}(k)$, $d^{(1)}(k)$ of this matrix are determined via recursive relations

$$\lambda_{N+1} = 0_{r_N^U \times r_N^L},$$

$$\begin{pmatrix} \lambda_k & h^{(1)}(k) \\ p^{(1)}(k) & d^{(1)}(k) \end{pmatrix} = \begin{pmatrix} h(k)\gamma_k^{-1} & b^{(1)}(k) \\ \gamma_k^{-1} & -g^{(1)}(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda_{k+1} \end{pmatrix} \begin{pmatrix} -p(k) & I \\ a^{(1)}(k) & q^{(1)}(k) \end{pmatrix},$$

$$k = N, \dots, 1.$$

From here for $k = N$ one obtains the equality (18.61), for $k = N-1, \dots, 2$ the equalities (18.63), and for $k = 1$ the equality (18.64). □

The complexity of the algorithm presented here is estimated as follows. For the computations by the formulas (18.59) one has the estimate (18.27). The use of the formula (18.62) costs at most $2r^2mN$ operations. The formula (18.63) may be written in the form

$$\begin{pmatrix} \lambda_k & h^{(1)}(k) \\ p^{(1)}(k) & d^{(1)}(k) \end{pmatrix} = \begin{pmatrix} h(k) \\ I \end{pmatrix} \gamma_k^{-1} \begin{pmatrix} -p(k) & I \end{pmatrix} + \begin{pmatrix} b^{(1)}(k) \\ -g^{(1)}(k) \end{pmatrix} \lambda_{k+1} \begin{pmatrix} a^{(1)}(k) & q^{(1)}(k) \end{pmatrix}.$$

The cost of computations by this formula does not exceed

$$\tilde{\rho}(m) + 2(m^2(r+m) + (r+m)^2m + r^2(r+m) + r(r+m)^2).$$

Here $\tilde{\rho}(m)$ denotes the complexity of the inversion of an $m \times m$ matrix using a standard method. Thus the total estimate for the complexity is

$$c \le (\tilde{\rho}(m) + 2\rho(m) + 8r^3 + 22r^2m + 12m^2r + 4m^3)N. \tag{18.65}$$

§18.5 The case of a diagonal plus semiseparable representation

In this section we derive a specification of Theorem 18.2 for matrices with diagonal plus semiseparable representations.

Theorem 18.8. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with block entries of sizes $m_i \times m_j$ and with invertible principal leading block submatrices $A_k = \{A_{ij}\}_{i,j=1}^k$, $k = 1, \dots, N$. Assume that A has lower semiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$ of order r_L, upper semiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$ of order r_U, and diagonal entries $d(k)$ $(k = 1, \dots, N)$.*

Then in the factorization (18.1) *the matrix L has lower semiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q^{(1)}(j)$ $(j = 1, \dots, N-1)$, the matrix U has upper semiseparable generators $g^{(1)}(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, and the matrix D has the form $D = \operatorname{diag}\{\gamma_1, \dots, \gamma_N\}$. Here the elements $p(i)$, $h(j)$ are the same as for the matrix A and the elements $q^{(1)}(j)$, $g^{(1)}(i), \gamma_k$ are determined via the following algorithm.*

1. *Compute*

$$\gamma_1 = d(1), \quad q^{(1)}(1) = q(1)\gamma_1^{-1}, \quad g^{(1)}(1) = \gamma_1^{-1} g(1), \tag{18.66}$$

$$f_1 = q^{(1)}(1)\gamma_1 g^{(1)}(1). \tag{18.67}$$

2. *For $k = 2, \dots, N-1$ compute*

$$\gamma_k = d(k) - p(k) f_{k-1} h(k), \tag{18.68}$$

$$q^{(1)}(k) = [q(k) - f_{k-1} h(k)]\gamma_k^{-1}, \tag{18.69}$$

$$g^{(1)}(k) = \gamma_k^{-1}[g(k) - p(k) f_{k-1}], \tag{18.70}$$

$$f_k = f_{k-1} + q^{(1)}(k)\gamma_k g^{(1)}(k). \tag{18.71}$$

3. *Compute*

$$\gamma_N = d(N) - p(N) f_{N-1} h(N). \tag{18.72}$$

Here f_k $(k = 1, \dots, N-1)$ are auxiliary variables, which are $r_L \times r_U$ matrices.

Proof. A may be treated as a matrix with the lower quasiseparable generators

$$p(i),\ i = 2, \dots, N, \quad q(j),\ j = 1, \dots, N-1, \quad a(k) = I_{r_L},\ k = 2, \dots, N-1;$$

and upper quasiseparable generators

$$g(i),\ i = 1, \dots, N-1, \quad h(j),\ j = 2, \dots, N, \quad b(k) = I_{r_U},\ k = 2, \dots, N-1.$$

Hence, by Theorem 18.2, the matrix L has lower quasiseparable generators

$$p(i),\ i = 2, \dots, N, \quad q^{(1)}(j),\ j = 1, \dots, N-1, \quad a(k) = I_{r_L},\ k = 2, \dots, N-1,$$

and the matrix U has upper quasiseparable generators

$$g^{(1)}(i),\ i=1,\dots,N-1,\quad h(j),\ j=2,\dots,N,\quad b(k)=I_{r_U},\ k=2,\dots,N-1.$$

Thus, by Theorem 4.2, the matrix L has lower semiseparable generators $p(i)$ ($i=2,\dots,N$), $q^{(1)}(j)$ ($j=1,\dots,N-1$) and by Theorem 4.3 the elements $g^{(1)}(i)$ ($i=1,\dots,N-1$),; $h(j)$ ($j=2,\dots,N$) are upper semiseparable generators of the matrix U. □

One can easily check that here instead of the estimate for complexity (18.27) one obtains the better one

$$c<(8r^2m+4m^2r+\rho(m))N.$$

In the corresponding algorithm for the solution of the system $Ax=y$ one obtains the estimate

$$c'<(4mr+\rho(m))N$$

instead of (13.25), and the total estimate is

$$c<(8r^2m+4m^2r+8mr+3\rho(m))N.$$

Example 18.9. Consider the matrix A from Example 18.3 with $a\neq 0$, $b\neq 0$, $ab\neq 1$. For this matrix one can use the semiseparable generators

$$p(i)=b^{i-1},\quad h(i)=a^{i-1},\quad i=2,\dots,N,$$
$$q(j)=\frac{1}{a^{j-1}},\quad g(j)=\frac{1}{b^{j-1}},\quad j=1,\dots,N-1,\quad d(k)=1,\quad k=1,\dots,N.$$

One can use Theorem 18.8 to find the LDU factorization of A. It follows that in the factorization (18.1) the matrix L has lower semiseparable generators $p(i)$ ($i=2,\dots,N$), $q^{(1)}(j)$ ($j=1,\dots,N-1$), the matrix U has upper semiseparable generators $g^{(1)}(i)$ ($i=1,\dots,N-1$), $h(j)$ ($j=2,\dots,N$), and the matrix D has the form $D=\operatorname{diag}\{\gamma_1,\dots,\gamma_N\}$. Here the elements $p(i)$, $h(j)$ are the same as for the matrix A and the elements $q^{(1)}(j)$, $g^{(1)}(i),\gamma_k$ are determined via the algorithm in the theorem.

Compute the new semiseparable generators and auxiliary variables with the first step of the algorithm, namely

$$\gamma_1=d(1)=1,\quad q^{(1)}(1)=q(1)\gamma_1^{-1}=1,\quad g^{(1)}(1)=\gamma_1^{-1}g(1)=1,$$
$$f_1=q^{(1)}(1)\gamma_1 g^{(1)}(1)=1,\quad \gamma_2=d(2)-p(2)f_1h(2)=1-ba.$$

One can prove by induction that

$$\gamma_k=1-ab,\quad k=2,\dots,N,$$

and

$$q^{(1)}(k) = \frac{1}{b^{k-1}}, \quad g^{(1)}(k) = \frac{1}{a^{k-1}}, \quad f_k = \frac{1}{a^{k-1}b^{k-1}}, \quad k = 1, \dots, N-1.$$

Indeed, suppose this is true for a certain value of k. Then for $k+1$ one has

$$\gamma_{k+1} = d(k+1) - p(k+1)f_k h(k+1) = 1 - b^k \frac{1}{a^{k-1}b^{k-1}} a^k = 1 - ba,$$

$$q^{(1)}(k+1) = [q(k+1) - f_k h(k+1)]\gamma_{k+1}^{-1} = \left(\frac{1}{b^k} - \frac{1}{a^{k-1}b^{k-1}} a^k\right) \frac{1}{1-ab} = \frac{1}{b^k},$$

$$g^{(1)}(k+1) = \gamma_{k+1}^{-1}[g(k+1) - p(k+1)f_k] = \frac{1}{1-ab}\left(\frac{1}{a^k} - b^k \frac{1}{a^{k-1}b^{k-1}}\right) = \frac{1}{a^k},$$

$$f_{k+1} = f_k + q^{(1)}(k+1)\gamma_{k+1} g^{(1)}(k+1) = \frac{1}{a^{k-1}b^{k-1}} + \frac{1}{b^k}(1-ab)\frac{1}{a^k} = \frac{1}{a^k b^k}$$

and the induction hypothesis is proved.

It is clear that these semiseparable generators for the matrices L, D and U give the same factorization matrices as in Example 18.3. ◊

§18.6 Comments

This chapter is an extension of results obtained by I. Gohberg, T. Kailath and I. Koltracht in [38] for scalar matrices with diagonal plus semiseparable representations. The results of [38] were extended to scalar matrices with quasiseparable representations in [20] and to block matrices in [22]. Further generalizations to matrices with quasiseparable representations out of a band may be found in [28] and to diagonal plus semiseparable operator matrices in [24] and [25].

The recursive method mentioned in Section §18.3 was used by I. Gohberg, T. Kailath and I. Koltracht in [39] for linear algebraic systems with structured matrices. An $O(N^2)$ recursive algorithm for systems with matrices with lower or upper quasiseparable representations was suggested in [16]. The recursive (Levinson-like) fast algorithm for linear algebraic systems with matrices with quasiseparable representations was suggested by R. Vandebril, N. Mastronardi and M. Van Barel in [46, 47].

Chapter 19

Scalar Matrices with Quasiseparable Order One

Generators with orders one are complex numbers. One can then apply the linear complexity inversion algorithm from Theorem 18.2 suggested under the conditions of invertibility of the principal leading submatrices of the matrix. In this chapter an algorithm to determine quasiseparable generators of the inverse without any restrictions on the principal leading submatrices is obtained. In the proofs here, the matrices $\mathrm{adj}A_k$ of principal leading submatrices A_k are used instead of their inverses A_k^{-1}.

The direct use of representations may lead to overflow or underflow in the computing process. That is why an equivalent representation of generators including some scaling coefficients is then used. The particular cases of matrices with diagonal plus semiseparable representations and of tridiagonal matrices are considered separately.

§19.1 Inversion formula

In this section we derive a basic result for this chapter.

Theorem 19.1. *Let A be an invertible scalar matrix with quasiseparable of orders one generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$; $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$; $d(k)$ $(k = 1, \dots, N)$. Using these generators define*

$$l_k = d(k)a(k) - p(k)q(k), \quad \delta_k = d(k)b(k) - g(k)h(k), \quad 2 \le k \le N-1; \tag{19.1}$$

also, define forward recursively $s_1 = q(1)$, $v_1 = g(1)$, $f_1 = q(1)g(1)$, $\gamma_1 = d(1)$

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and for $k = 2, \dots, N-1$

$$\begin{aligned} s_k &= q(k)\gamma_{k-1} - a(k)f_{k-1}h(k), \quad v_k = g(k)\gamma_{k-1} - p(k)f_{k-1}b(k), \\ \gamma_k &= d(k)\gamma_{k-1} - p(k)f_{k-1}h(k), \\ f_k &= a(k)f_{k-1}b(k)d(k) + q(k)v_k + s_k g(k) - \gamma_{k-1}q(k)g(k) \\ \gamma_N &= d(N)\gamma_{N-1} - p(N)f_{N-1}h(N); \end{aligned} \tag{19.2}$$

then define backward recursively $t_N = p(N)$, $u_N = h(N)$, $z_N = h(N)p(N)$, $\theta_N = d(N)$ *and for* $k = N-1, \dots, 2$

$$\begin{aligned} t_k &= p(k)\theta_{k+1} - g(k)z_{k+1}a(k), \quad u_k = h(k)\theta_{k+1} - b(k)z_{k+1}q(k), \\ \theta_k &= d(k)\theta_{k+1} - g(k)z_{k+1}q(k), \\ z_k &= b(k)z_{k+1}a(k)d(k) + h(k)t_k + u_k p(k) - \theta_{k+1}h(k)p(k), \\ \theta_1 &= d(1)\theta_2 - g(1)z_2 q(1) \end{aligned} \tag{19.3}$$

and

$$\begin{aligned} &\rho_k = \gamma_{k-1}\theta_k - f_{k-1}z_k, \ 2 \le k \le N, \quad \rho_1 = \rho_2; \\ &\lambda_1 = \theta_2, \quad \lambda_k = \gamma_{k-1}\theta_{k+1} - a(k)f_{k-1}z_{k+1}b(k), \ 2 \le k \le N-1, \\ &\lambda_N = \gamma_{N-1}, \end{aligned} \tag{19.4}$$

and set $\det A = \rho$.

Then $\rho_k = \rho$, $k = 1, \dots, N$ *and the elements* $-t_i/\rho$ $(i = 2, \dots, N)$, s_j $(j = 1, \dots, N-1)$, l_k $(k = 2, \dots, N-1)$; v_i $(i = 1, \dots, N-1)$, $-u_j/\rho$ $(j = 2, \dots, N)$, δ_k $(k = 2, \dots, N-1)$; λ_k/ρ $(k = 1, \dots, N)$ *are quasiseparable generators of the inverse matrix* A^{-1}.

In the proof of the theorem the following auxiliary result is used essentially.

Lemma 19.2. *Assume that a matrix* A *admits the representation*

$$A = \begin{pmatrix} A_0 & GH \\ PQ & B \end{pmatrix},$$

where A_0, B *are square matrices, and* P, G *and* Q, H *are correspondingly columns and rows with appropriate sizes.*

Then

$$\det A = \det A_0 \cdot \det B - [Q(\operatorname{adj} A_0)G][H(\operatorname{adj} B)P], \tag{19.5}$$

$$\operatorname{adj} A = \begin{pmatrix} A' & -((\operatorname{adj} A_0)G)(H(\operatorname{adj} B)) \\ -((\operatorname{adj} B)P)(Q(\operatorname{adj} A_0)) & B' \end{pmatrix}, \tag{19.6}$$

where A' *and* B' *are matrices of the same sizes as* A_0 *and* B*, respectively.*

Proof. Assume for the moment that the matrices A_0, B, A are invertible. Then the inversion formula (1.54) yields

$$A^{-1} = \begin{pmatrix} A_0^{-1} + (A_0^{-1}G)(H\tilde{B}^{-1}P)(QA_0^{-1}) & -(A_0^{-1}G)(H\tilde{B}^{-1}) \\ -(\tilde{B}^{-1}P)(QA_0^{-1}) & \tilde{B}^{-1} \end{pmatrix}, \tag{19.7}$$

where $\tilde{B} = B - P(QA_0^{-1}G)H$. Moreover, by the formula (1.53) one has $\det A = \det A_0 \cdot \det \tilde{B}$. Next, using (1.72) one gets

$$\det \tilde{B} = \det B \cdot \det \left(I - B^{-1}P(QA_0^{-1}G)H\right) = \det B \left(1 - (QA_0^{-1}G)(HB^{-1}P)\right)$$

and therefore

$$\begin{aligned} \det A &= \det A_0 \cdot \det B \left(1 - \frac{[Q(\operatorname{adj} A_0)G][H(\operatorname{adj} B)P]}{\det A_0 \cdot \det B}\right) \\ &= \det A_0 \cdot \det B - [Q(\operatorname{adj} A_0)G][H(\operatorname{adj} B)P] \end{aligned}$$

and thus (19.5) is valid without any restriction.

Now using the inversion formula (1.69) we represent the matrix $\tilde{B}^{-1}$ in the following form:

$$\begin{aligned} \tilde{B}^{-1} &= B^{-1} + \frac{1}{1 - (HB^{-1}P)(QA_0^{-1}G)}(B^{-1}P)(QA_0^{-1}G)(HB^{-1}) \\ &= \frac{\operatorname{adj} B}{\det B} + \left(1 - \frac{H(\operatorname{adj} B)P}{\det B}\frac{Q(\operatorname{adj} A_0)G}{\det A_0}\right)^{-1} \frac{(\operatorname{adj} B)P}{\det B}\frac{Q(\operatorname{adj} A_0)G}{\det A_0}\frac{H(\operatorname{adj} B)}{\det B} \\ &= \frac{\operatorname{adj} B}{\det B} + \frac{Q(\operatorname{adj} A_0)G}{\det A_0 \cdot \det B - [H(\operatorname{adj} B)P][Q(\operatorname{adj} A_0)G]}\frac{[(\operatorname{adj} B)P][H(\operatorname{adj} B)]}{\det B}. \end{aligned}$$

It follows that

$$\tilde{B}^{-1}P = \frac{(\operatorname{adj} B)P \det A_0}{\det A}, \quad H\tilde{B}^{-1} = \frac{H(\operatorname{adj} B)\det A_0}{\det A}$$

and moreover

$$\begin{aligned} (\tilde{B}^{-1}P)(QA_0^{-1}) &= \frac{[(\operatorname{adj} B)P]([Q(\operatorname{adj} A_0)]}{\det A}, \\ (A_0^{-1}G)(H\tilde{B}^{-1}) &= \frac{[(\operatorname{adj} A_0)G][H(\operatorname{adj} B)]}{\det A}. \end{aligned}$$

Inserting this in (19.7) we obtain (19.6) without any restriction. □

Proof of the theorem. Using Lemma 5.1 and Lemma 5.4 one obtains for every k, $1 \le k \le N-1$, the representation

$$A = \begin{pmatrix} A_k & G_kH_{k+1} \\ P_{k+1}Q_k & B_{k+1} \end{pmatrix}. \tag{19.8}$$

Here A_k is the leading principal submatrix of size $k \times k$ and the column vectors $P_k,\ G_k$ and the row vectors $Q_k,\ H_k$ are given via (5.2), (5.5) and (5.1), (5.6).

We introduce the notations

$$V_k = (\operatorname{adj} A_k)G_k,\ S_k = Q_k(\operatorname{adj} A_k), \quad f_k = Q_k(\operatorname{adj} A_k)G_k,\ \gamma_k = \det A_k; \tag{19.9}$$
$$T_k = (\operatorname{adj} B_k)P_k,\ U_k = H_k(\operatorname{adj} B_k), \quad z_k = H_k(\operatorname{adj} B_k)P_k,\ \theta_k = \det B_k. \tag{19.10}$$

From (19.5) it follows that

$$\begin{aligned}\det A &= \det A_k \cdot \det B_{k+1} - [Q_k(\operatorname{adj} A_k)G_k][H_{k+1}(\operatorname{adj} B_{k+1})P_{k+1}]\\ &= \gamma_k\theta_{k+1} - f_k z_{k+1} = \rho_{k+1}, \quad 1 \le k \le N-1.\end{aligned}$$

Next from (19.6) one obtains

$$\operatorname{adj} A = \begin{pmatrix} A'_k & -V_k U_{k+1} \\ -T_{k+1}S_k & B'_{k+1} \end{pmatrix}, \tag{19.11}$$

where $A'_k,\ B'_{k+1}$ are matrices with the same sizes as $A_k,\ B_{k+1}$.

Consider the elements defined by (19.9). Let $s_k,\ v_k$ be the last entries of the row vector S_k and the column vector V_k, respectively. For $k = 1$ one has

$$S_1 = s_1 = q(1), \quad V_1 = v_1 = g(1), \quad f_1 = q(1)g(1), \quad \gamma_1 = d_1.$$

For $k \ge 2$, the following holds. From the representation

$$A_k = \begin{pmatrix} A_{k-1} & G_{k-1}h(k) \\ p(k)Q_{k-1} & d(k) \end{pmatrix}$$

and from (19.5) it follows that

$$\gamma_k = d(k)\gamma_{k-1} - p(k)f_{k-1}h(k). \tag{19.12}$$

Assume for the moment that A_{k-1} and A_k are invertible and apply the formula (19.7) to the matrix A_k. Then $\tilde{B} = B - P(QA_0^{-1}G)H$ from (19.7) becomes

$$d(k) - p(k)Q_{k-1}\frac{\operatorname{adj} A_{k-1}}{\det A_{k-1}}G_{k-1}h(k) = \frac{\gamma_k}{\det A_{k-1}} = \frac{\gamma_k}{\gamma_{k-1}}.$$

From (19.7) one obtains

$$A_k^{-1} = \begin{pmatrix} A_{k-1}^{-1} + (A_{k-1}^{-1}G_{k-1})(h(k)\frac{\gamma_{k-1}}{\gamma_k}p(k))(Q_{k-1}A_{k-1}^{-1}) & -(A_{k-1}^{-1}G_{k-1})h(k)\frac{\gamma_{k-1}}{\gamma_k} \\ -\frac{\gamma_{k-1}}{\gamma_k}p(k)(Q_{k-1}A_{k-1}^{-1}) & \frac{\gamma_{k-1}}{\gamma_k} \end{pmatrix},$$

whence

$$\operatorname{adj} A_k = \begin{pmatrix} \frac{\gamma_k \operatorname{adj} A_{k-1}}{\gamma_{k-1}} + V_{k-1}(h(k)\frac{1}{\gamma_{k-1}}p(k))S_{k-1} & -V_{k-1}h(k) \\ -p(k)S_{k-1} & \gamma_{k-1} \end{pmatrix}. \tag{19.13}$$

Next, taking into consideration the recursion (5.3) and the equality

$$Q_{k-1}V_{k-1} = f_{k-1},$$

one obtains

$$S_k = Q_k \operatorname{adj} A_k = \begin{pmatrix} a(k)Q_{k-1} & q(k) \end{pmatrix} \operatorname{adj} A_k$$
$$= \Big(\{a(k)\tfrac{1}{\gamma_{k-1}}[\gamma_k + f_{k-1}h(k)p(k)] - q(k)p(k)\}S_{k-1} \quad -a(k)f_{k-1}h(k) + q(k)\gamma_{k-1} \Big).$$

Using (19.12) one obtains

$$\frac{1}{\gamma_{k-1}}[\gamma_k + f_{k-1}h(k)p(k)] = d(k)$$

and thus one concludes that

$$S_k = \begin{pmatrix} (d(k)a(k) - p(k)q(k))S_{k-1} & -a(k)f_{k-1}h(k) + q(k)\gamma_{k-1} \end{pmatrix}, \qquad (19.14)$$

which holds without any restriction.

Similarly, from (5.7) and the equality $S_{k-1}G_{k-1} = f_{k-1}$ one obtains

$$V_k = (\operatorname{adj} A_k)G_k = \operatorname{adj} A_k \begin{pmatrix} G_{k-1}b(k) \\ g(k) \end{pmatrix}$$
$$= \begin{pmatrix} V_{k-1}\{\frac{1}{\gamma_{k-1}}[\gamma_k + h(k)p(k)f_{k-1}]b(k) - h(k)g(k)\} \\ -p(k)f_{k-1}b(k) + \gamma_{k-1}g(k) \end{pmatrix}$$

and then, by virtue of (19.12),

$$V_k = \begin{pmatrix} V_{k-1}(d(k)b(k) - g(k)h(k)) \\ \gamma_{k-1}g(k) - p(k)f_{k-1}b(k) \end{pmatrix}. \qquad (19.15)$$

Next for f_k one gets

$$f_k = Q_k(\operatorname{adj} A_k)G_k = Q_kV_k = \begin{pmatrix} a(k)Q_{k-1} & q(k) \end{pmatrix} \begin{pmatrix} V_{k-1}(d(k)b(k) - g(k)h(k)) \\ \gamma_{k-1}g(k) - p(k)f_{k-1}b(k) \end{pmatrix}$$
$$= a(k)f_{k-1}d(k)b(k) - a(k)f_{k-1}g(k)h(k) + q(k)\gamma_{k-1}g(k) - q(k)p(k)f_{k-1}b(k)$$
$$= a(k)f_{k-1}b(k)d(k) + g(k)[q(k)\gamma_{k-1} - a(k)f_{k-1}h(k)]$$
$$+ q(k)[\gamma_{k-1}g(k) - p(k)f_{k-1}b(k)] - q(k)\gamma_{k-1}g(k). \qquad (19.16)$$

Thus the numbers $s_k,\ v_k,\ f_k, \gamma_k$ satisfy the relations (19.2), and moreover for the vectors $S_k,\ V_k$ one has the recursions

$$S_1 = s_1, \quad S_k = \begin{pmatrix} l_kS_{k-1} & s_k \end{pmatrix}, \quad k = 2, \dots, N-1, \qquad (19.17)$$
$$V_1 = v_1, \quad V_k = \begin{pmatrix} V_{k-1}\delta_k \\ v_k \end{pmatrix}, \quad k = 2, \dots, N-1, \qquad (19.18)$$

where $l_k,\ \delta_k$ are given by (19.1).

Next we consider the elements defined by (19.10). Let $t_k,\ u_k$ be the first entries of the column vector T_k and the row vector U_k, respectively. For $k = N$ one has

$$T_N = t_N = p(N), \quad U_N = u_N = h(N), \quad z_N = h(N)p(N), \quad \theta_N = d(N).$$

For $1 \le k \le N-1$ one has the following. Consider the submatrices $B_k = A(k:N, k:N)$. Using (5.10) and (5.13) one obtains the representations

$$B_k = \begin{pmatrix} d(k) & g(k)H_{k+1} \\ P_{k+1}q(k) & B_{k+1} \end{pmatrix}.$$

The application of (19.5) yields

$$\theta_k = d(k)\theta_{k+1} - q(k)z_{k+1}g(k).$$

Moreover, assuming for the moment that $B_{k+1},\ B_k$ are invertible, we apply the formula (1.57) to the matrix B_k and obtain

$$B_k^{-1} = \begin{pmatrix} \frac{\theta_{k+1}}{\theta_k} & -\frac{\theta_{k+1}}{\theta_k}g(k)H_{k+1}B_{k+1}^{-1} \\ -B_{k+1}^{-1}P_{k+1}q(k)\frac{\theta_{k+1}}{\theta_k} & (B_{k+1}^{-1}P_{k+1})\left(q(k)\frac{\theta_{k+1}}{\theta_k}g(k)\right)(H_{k+1}B_{k+1}^{-1}) + B_{k+1}^{-1} \end{pmatrix}.$$

From here one obtains representations similar to (19.13):

$$\operatorname{adj} B_k = \begin{pmatrix} \theta_{k+1} & -g(k)U_{k+1} \\ -T_{k+1}q(k) & \frac{\theta_k \operatorname{adj} B_{k+1}}{\theta_{k+1}} + T_{k+1}(q(k)\frac{1}{\theta_{k+1}}g(k))U_{k+1} \end{pmatrix}.$$

Then one can proceed as in (19.14)–(19.16), but in the backward direction, to obtain the desired relations (19.3) for $t_k,\ u_k,\ \theta_k,\ z_k$ and to prove that the columns T_k and the rows U_k satisfy the recursions

$$\begin{aligned} &T_N = t_N, \quad T_k = \begin{pmatrix} t_k \\ T_{k+1}l_k \end{pmatrix}, \qquad k = N-1, \dots, 2, \\ &U_N = u_N, \quad U_k = \begin{pmatrix} u_k & \delta_k U_{k+1} \end{pmatrix}, \quad k = N-1 \dots, 2. \end{aligned} \tag{19.19}$$

Indeed, using (5.4) and the equality $U_{k+1}P_{k+1} = z_{k+1}$ one has

$$\begin{aligned} T_k &= (\operatorname{adj} B_k)P_k = (\operatorname{adj} B_k)\begin{pmatrix} p(k) \\ P_{k+1}a(k) \end{pmatrix} \\ &= \begin{pmatrix} \theta_{k+1}p(k) - g(k)z_{k+1}a(k) \\ T_{k+1}\{[(\theta_k + q(k)z_{k+1}g(k))/\theta_{k+1}]a(k) - p(k)q(k)\} \end{pmatrix}. \end{aligned}$$

Since $(\theta_k + g(k)z_{k+1}q(k))/\theta_{k+1} = d(k)$, one obtains

$$T_k = \begin{pmatrix} \theta_{k+1}p(k) - g(k)z_{k+1}a(k) \\ T_{k+1}\{d(k)a(k) - p(k)q(k)\} \end{pmatrix}.$$

Next, using (5.8) and the equality $H_{k+1}T_{k+1} = z_{k+1}$ one has

$$\begin{aligned} U_k &= \begin{pmatrix} h(k) & b(k)H_{k+1} \end{pmatrix} \operatorname{adj} B_k \\ &= \begin{pmatrix} h(k)\theta_{k+1} - b(k)z_{k+1}q(k) & (d(k)b(k) - h(k)g(k))U_{k+1} \end{pmatrix}. \end{aligned}$$

Finally, we conclude that

$$\begin{aligned} z_k &= U_kP_k \\ &= \begin{pmatrix} h(k)\theta_{k+1} - b(k)z_{k+1}q(k) & (d(k)b(k) - h(k)g(k))U_{k+1} \end{pmatrix} \begin{pmatrix} p(k) \\ P_{k+1}a(k) \end{pmatrix} \\ &= h(k)\theta_{k+1}p(k) - b(k)z_{k+1}q(k)p(k) + d(k)b(k)z_{k+1}a(k) - h(k)g(k)z_{k+1}a(k) \\ &= b(k)z_{k+1}a(k)d(k) + h(k)[p(k)\theta_{k+1} - g(k)z_{k+1}a(k)] \\ &\quad + p(k)[h(k)\theta_{k+1} - b(k)z_{k+1}q(k)] - h(k)\theta_{k+1}p(k). \end{aligned}$$

Multiplying (19.19) by $-\rho^{-1}$, $N \geq k \geq 2$ one obtains

$$-T_N/\rho = -t_N/\rho_N, \quad -T_k/\rho = \begin{pmatrix} -t_k/\rho \\ -(T_{k+1}/\rho)l_k \end{pmatrix}, \qquad k = N-1, \dots, 2 \tag{19.20}$$

and

$$-U_N/\rho = -u_N/\rho, \qquad -U_k/\rho = \begin{pmatrix} -u_k/\rho & -\delta_k U_{k+1}/\rho \end{pmatrix}, k = N-1 \dots, 2. \tag{19.21}$$

Let λ_k be the diagonal entries of the matrix $\operatorname{adj} A$. Consider the matrices

$$A'_1 = B_2, \quad A'_k = \begin{pmatrix} A_{k-1} & G_{k-1}b(k)H_{k+1} \\ P_{k+1}a(k)Q_{k-1} & B_{k+1} \end{pmatrix}, \\ k = 2, \dots, N-1; \quad A'_N = A_{N-1}$$

obtained from A by removing its kth row and kth column. One has obviously

$$\lambda_1 = \det B_2 = \theta_2, \quad \lambda_N = \det A_{N-1} = \gamma_{N-1}$$

and using (19.5) one gets

$$\det A'_k = \det A_{k-1} \cdot \det B_{k+1} - a(k)(Q_{k-1} \operatorname{adj} A_{k-1}G_{k-1})(P_{k+1} \operatorname{adj} B_{k+1}H_{k+1})b(k)$$

from which the relations (19.4) for λ_k follow.

Thus for the inverse matrix A^{-1} one has the following. From (19.11) we conclude that

$$A^{-1}(k+1:N, 1:k) = (-T_{k+1}/\rho)S_k, \quad k = 1, \dots, N-1.$$

Moreover, $-T_k/\rho$, S_k satisfy (19.20) and (19.17). Hence, by Lemma 5.3, the elements $-t_i/\rho$ $(i = 2, \dots, N)$, s_j $(j = 1, \dots, N-1)$, l_k $(k = 2, \dots, N-1)$ are lower quasiseparable generators of the inverse matrix A^{-1}. Next, one has

$$A^{-1}(1:k, k+1:N) = V_k(-U_{k+1}/\rho), \quad k = 1, \dots, N-1,$$

where V_k, $(-U_{k+1}/\rho)$ satisfy (19.18) and (19.21). Hence, by Lemma 5.6, the elements v_i $(i = 2, \dots, N)$, $-u_j/\rho$ $(j = 2, \dots, N)$, δ_k $(k = 2, \dots, N-1)$ are upper quasiseparable generators of the inverse matrix A^{-1}. Finally, for the diagonal entries one has

$$A^{-1}(k,k) = \det A'_k / \det A = \lambda_k/\rho.$$

Thus the numbers $-t_i/\rho$ $(i = 2, \dots, N)$, s_j $(j = 1, \dots, N-1)$, l_k $(k = 2, \dots, N-1)$; v_i $(i = 1, \dots, N-1)$, $-u_j/\rho$ $(j = 2, \dots, N)$, δ_k $(k = 2, \dots, N-1)$; λ_k/ρ $(k = 1, \dots, N)$ are quasiseparable generators of the inverse matrix A^{-1}. □

§19.2 Examples

Here Theorem 19.1 is illustrated by concrete examples.

Example 19.3. Consider the matrix

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 & \beta_1 \\ 0 & \alpha_2 & 0 & \beta_2 \\ 0 & 0 & \alpha_3 & \beta_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \alpha_4 \end{pmatrix}.$$

Then one can take the scalar quasiseparable generators

$$\begin{aligned} &g(1) = \beta_1,\ g(2) = \beta_2,\ g(3) = \beta_3,\ h(2) = h(3) = 0,\ h(4) = 1, \\ &p(2) = p(3) = 0,\ p(4) = 1,\ q(1) = \epsilon_1,\ q(2) = \epsilon_2,\ q(3) = \epsilon_3, \\ &d(k) = \alpha_k,\ k = 1, \dots, 4, \quad a(2) = a(3) = b(2) = b(3) = 1. \end{aligned}$$

One can therefore compute

$$\begin{aligned} &l_2 = \alpha_2,\ \delta_2 = \alpha_2,\ l_3 = \alpha_3,\ \delta_3 = \alpha_3, \\ &s_1 = \epsilon_1,\ v_1 = \beta_1,\ f_1 = \epsilon_1\beta_1,\ \gamma_1 = \alpha_1, \\ &s_2 = \epsilon_2\alpha_1,\ v_2 = \beta_2\alpha_1,\ \gamma_2 = \alpha_2\alpha_1, \\ &f_2 = \epsilon_1\beta_1\alpha_2 + \epsilon_2\beta_2\alpha_1 + \epsilon_2\alpha_1\beta_2 - \alpha_1\epsilon_2\beta_2 = \epsilon_1\beta_1\alpha_2 + \epsilon_2\beta_2\alpha_1, \\ &s_3 = \epsilon_3\alpha_2\alpha_1,\ v_3 = \beta_3\alpha_2\alpha_1,\ \gamma_3 = \alpha_3\alpha_2\alpha_1, \\ &f_3 = \epsilon_1\beta_1\alpha_2\alpha_3 + \epsilon_2\beta_2\alpha_1\alpha_3 + \epsilon_3\beta_3\alpha_2\alpha_1 + \epsilon_3\alpha_2\alpha_1\beta_3 - \alpha_2\alpha_1\epsilon_3\beta_3 \\ &\quad = \epsilon_1\beta_1\alpha_2\alpha_3 + \epsilon_2\beta_2\alpha_1\alpha_3 + \epsilon_3\beta_3\alpha_2\alpha_1, \\ &\gamma_4 = \alpha_4\alpha_3\alpha_2\alpha_1 - \epsilon_1\beta_1\alpha_2\alpha_3 - \epsilon_2\beta_2\alpha_1\alpha_3 - \epsilon_3\beta_3\alpha_2\alpha_1. \end{aligned}$$

On the other hand,

$$\begin{aligned} &t_4 = 1,\ u_4 = 1,\ z_4 = 1,\ \theta_4 = \alpha_4,\ t_3 = -\beta_3, \\ &u_3 = -\epsilon_3,\ \theta_3 = \alpha_3\alpha_4 - \beta_3\epsilon_3,\ z_3 = -\alpha_3, \\ &t_2 = -\beta_2\alpha_3,\ u_2 = -\alpha_3\epsilon_3,\ \theta_2 = \alpha_2\alpha_3\alpha_4 - \alpha_2\beta_3\epsilon_3 - \beta_2\alpha_3\epsilon_2,\ z_2 = \alpha_3\alpha_2, \\ &\theta_1 = \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_1\alpha_2\beta_3\epsilon_3 - \alpha_1\beta_2\alpha_3\epsilon_2 - \beta_1\alpha_3\alpha_2\epsilon_1. \end{aligned}$$

Denote $\det A$ by Δ. Then

$$\rho_4 = \alpha_3\alpha_2\alpha_1\alpha_4 - \epsilon_1\beta_1\alpha_2\alpha_3 - \epsilon_2\beta_2\alpha_1\alpha_3 - \epsilon_3\beta_3\alpha_2\alpha_1 = \Delta,$$
$$\rho_3 = \alpha_2\alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_1\beta_3\epsilon_3 - \epsilon_1\beta_1\alpha_2\alpha_3 - \epsilon_2\beta_2\alpha_1\alpha_3 = \Delta,$$
$$\rho_2 = \alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_1\alpha_2\beta_3\epsilon_3 - \alpha_1\beta_2\alpha_3\epsilon_2 - \epsilon_1\beta_1\alpha_3\alpha_2 = \Delta, \quad \rho_1 = \rho_2 = \Delta,$$

and

$$\lambda_1 = \alpha_2\alpha_3\alpha_4 - \alpha_2\beta_3\epsilon_3 - \beta_2\alpha_3\epsilon_2, \quad \lambda_2 = \alpha_1\alpha_3\alpha_4 - \alpha_1\beta_3\epsilon_3 - \epsilon_1\beta_1\alpha_3,$$
$$\lambda_3 = \alpha_2\alpha_1\alpha_4 - \epsilon_1\beta_1\alpha_2 - \epsilon_2\beta_2\alpha_1, \quad \lambda_4 = \gamma_3 = \alpha_3\alpha_2\alpha_1.$$

One defines the following numbers, which will be the quasiseparable generators of the inverse matrix:

$$\tilde{p}(2) = \frac{\beta_2\alpha_3}{\Delta}, \quad \tilde{p}(3) = \frac{\beta_3}{\Delta}, \quad \tilde{p}(4) = \frac{-1}{\Delta},$$
$$\tilde{q}(1) = \epsilon_1, \quad \tilde{q}(2) = \epsilon_2\alpha_1, \quad \tilde{q}(3) = \epsilon_3\alpha_2\alpha_1,$$
$$\tilde{a}(2) = \tilde{b}(2) = \alpha_2, \quad \tilde{a}(3) = \tilde{b}(3) = \alpha_3,$$
$$\tilde{g}(1) = \beta_1, \quad \tilde{g}(2) = \beta_2\alpha_1, \quad \tilde{g}(3) = \beta_3\alpha_2\alpha_1,$$
$$\tilde{h}(2) = \frac{\alpha_3\epsilon_3}{\Delta}, \quad \tilde{h}(3) = \frac{\epsilon_3}{\Delta}, \quad \tilde{h}(4) = \frac{-1}{\Delta},$$

and

$$\tilde{d}(1) = \frac{\lambda_1}{\Delta} = \frac{\alpha_2\alpha_3\alpha_4 - \alpha_2\beta_3\epsilon_3 - \beta_2\alpha_3\epsilon_2}{\Delta},$$
$$\tilde{d}(2) = \frac{\lambda_2}{\Delta} = \frac{\alpha_1\alpha_3\alpha_4 - \alpha_1\beta_3\epsilon_3 - \epsilon_1\beta_1\alpha_3}{\Delta},$$
$$\tilde{d}(3) = \frac{\lambda_3}{\Delta} = \frac{\alpha_2\alpha_1\alpha_4 - \epsilon_1\beta_1\alpha_2 - \epsilon_2\beta_2\alpha_1}{\Delta},$$
$$\tilde{d}(4) = \frac{\alpha_3\alpha_2\alpha_1}{\Delta}.$$

One can build the matrix

$$\begin{pmatrix} \tilde{d}(1) & \tilde{g}(1)\tilde{h}(2) & \tilde{g}(1)\tilde{b}(2)\tilde{h}(3) & \tilde{g}(1)\tilde{b}(2)\tilde{b}(3)\tilde{h}(4) \\ \tilde{p}(2)\tilde{q}(1) & \tilde{d}(2) & \tilde{g}(2)\tilde{h}(3) & \tilde{g}(2)\tilde{b}(3)\tilde{h}(4) \\ \tilde{p}(3)\tilde{a}(2)\tilde{q}(1) & \tilde{p}(3)\tilde{q}(2) & \tilde{d}(3) & \tilde{g}(3)\tilde{h}(4) \\ \tilde{p}(4)\tilde{a}(3)\tilde{a}(2)\tilde{q}(1) & \tilde{p}(4)\tilde{a}(3)\tilde{q}(2) & \tilde{p}(4)\tilde{q}(3) & \tilde{d}(4) \end{pmatrix},$$

namely

$$\frac{1}{\Delta}\begin{pmatrix} \lambda_1 & \beta_1\alpha_3\epsilon_3 & \beta_1\alpha_2\epsilon_3 & -\beta_1\alpha_2\alpha_3 \\ \beta_2\alpha_3\epsilon_1 & \lambda_2 & \beta_2\alpha_1\epsilon_3 & -\beta_2\alpha_1\alpha_3 \\ \beta_3\alpha_2\epsilon_1 & \beta_3\epsilon_2\alpha_1 & \lambda_3 & -\beta_3\alpha_2\alpha_1 \\ -\alpha_3\alpha_2\epsilon_1 & -\alpha_3\epsilon_2\alpha_1 & -\epsilon_3\alpha_2\alpha_1 & \alpha_3\alpha_2\alpha_1 \end{pmatrix},$$

which is indeed A^{-1}. ◊

Example 19.4. Let $a \neq 0$ be a scalar and consider the $(N-1)\times(N-1)$ matrix

$$A = \begin{pmatrix} -(N-1)a & a & a & \cdots & a & a \\ a & -(N-1)a & a & \cdots & a & a \\ a & a & -(N-1)a & \cdots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & a & \cdots & -(N-1)a & a \\ a & a & a & \cdots & a & -(N-1)a \end{pmatrix}.$$

For the matrix A one can use the quasiseparable generators

$$\begin{aligned} &p(i) = a,\ i = 2,\dots,N-1, && q(j) = 1,\ j = 1,\dots,N-2, \\ &a(k) = 1,\ k = 2,\dots,N-2, && g(j) = 1,\ j = 1,\dots,N-2, \\ &h(i) = a,\ i = 2,\dots,N-1, && b(k) = 1,\ k = 2,\dots,N-2, \\ &d(k) = -(N-1)a,\ k = 1,\dots,N-1. \end{aligned}$$

Then

$$l_k = -(N-1)a - a = -Na, \quad \delta_k = -(N-1)a - a = -Na$$

and forwards one can compute

$$s_1 = 1, \quad v_1 = 1, \quad f_1 = 1, \quad \gamma_1 = -(N-1)a,$$

and prove by induction that for $k = 1,\dots,N-1$

$$\gamma_k = (-1)^k N^{k-1} a^k (N-k), \quad f_k = (-1)^{k-1} k N^{k-1} a^{k-1}, \quad v_k = s_k = (-Na)^{k-1}.$$

One can also compute backwards

$$t_{N-1} = a, \quad u_{N-1} = a, \quad z_{N-1} = a^2, \quad \theta_{N-1} = -(N-1)a,$$

and prove by induction that for $k = N-1,\dots,1$

$$\theta_k = \gamma_{N-k} = (-1)^{N-k} k N^{N-k-1} a^{N-k}, \quad u_k = t_k = (-1)^{N-k-1} N^{N-k-1} a^{N-k},$$
$$z_k = (-1)^{N-k-1}(N-k) N^{N-k-1} a^{N-k+1}.$$

It follows that for $k = 1,\dots,N-1$

$$\begin{aligned} \rho_k &= \gamma_{k-1}\theta_k - f_{k-1} z_k \\ &= (-1)^{k-1} N^{k-2} a^{k-1} (N+1-k)(-1)^{N-k} k N^{N-k-1} a^{N-k} \\ &\quad - (-1)^{k-2}(k-1) N^{k-2} a^{k-2} (-1)^{N-k-1}(N-k) N^{N-k-1} a^{N-k+1} \\ &= (-1)^{N-1} N^{N-3} a^{N-1}((N+1-k)k - (k-1)(N-k)) \\ &= (-1)^{N-1} N^{N-2} a^{N-1} = \det A \end{aligned}$$

Also

$$\begin{aligned}\lambda_k &= \gamma_{k-1}\theta_{k+1} - f_{k-1}z_{k+1} \\ &= (-1)^{k-1}N^{k-2}a^{k-1}(N+1-k)(-1)^{N-k-1}(k+1)N^{N-k-2}a^{N-k-1} \\ &\quad - (-1)^{k-2}(k-1)N^{k-2}a^{k-2}(-1)^{N-k-2}(N-k-1)N^{N-k-2}a^{N-k} \\ &= (-1)^N N^{N-4}a^{N-2}((N+1-k)(k+1)-(k-1)(N-k-1)) \\ &= 2(-1)^N N^{N-3}a^{N-2}.\end{aligned}$$

One defines the following numbers, which will be the quasiseparable generators of the inverse matrix

$$\begin{aligned}
&\tilde{p}(i) = -\frac{t_i}{\rho_i} = \frac{(-1)^{N-i}N^{N-i-1}a^{N-i}}{(-1)^{N-1}N^{N-2}a^{N-1}} = (-1)^{i-1}\frac{1}{(Na)^{i-1}}, \quad i=2,\dots,N-1, \\
&\tilde{q}(j) = (-1)^{j-1}(Na)^{j-1}, \;\; j=1,\dots,N-2, \quad \tilde{a}(k) = -Na, \; k=2,\dots,N-2, \\
&\tilde{g}(j) = (-1)^{j-1}(Na)^{j-1} = \tilde{q}(j), \qquad j=1,\dots,N-2, \\
&\tilde{a}(k) = -Na = \tilde{b}(k), \qquad k=2,\dots,N-2, \\
&\tilde{h}(i) = \frac{(-1)^{N-i}N^{N-i-1}a^{N-i}}{(-1)^{N-1}N^{N-2}a^{N-1}} = (-1)^{i-1}\frac{1}{(Na)^{i-1}} = \tilde{p}(i), \quad i=2,\dots,N-1, \\
&\tilde{d}(k) = \frac{\lambda_k}{\rho_k} = \frac{2(-1)^N N^{N-3}a^{N-2}}{(-1)^{N-1}N^{N-2}a^{N-1}} = -\frac{2}{Na}, \qquad k=1,\dots,N-1.
\end{aligned}$$

If $i > j$, then the element $(A^{-1})_{ij}$ of the inverse matrix is

$$\begin{aligned}&\tilde{p}(i)\tilde{q}(j)\tilde{a}(i-1)\tilde{a}(i-2)\cdot \;\cdots\; \cdot\tilde{a}(j+1) \\ &\quad = (-1)^{i-1}\frac{1}{(Na)^{i-1}}(-1)^{j-1}(Na)^{j-1}(-1)^{i-j-1}(Na)^{i-j-1} = -\frac{1}{Na},\end{aligned}$$

and if $i < j$, then the element $(A^{-1})_{ij}$ of the inverse matrix is

$$\tilde{g}(j)\tilde{h}(i)\tilde{b}(j-1)\tilde{b}(j-2)\cdot \;\cdots\; \cdot\tilde{b}(i+1) = -\frac{1}{Na}$$

also. Therefore, the inverse matrix is

$$-\frac{1}{Na}\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix}$$

which is indeed A^{-1}. ◊

Example 19.5. Let $N = 2r+1$ be an odd number and consider the $(N-1)\times(N-1)$ irreducible tridiagonal matrix

$$A = \begin{pmatrix} 0 & \beta_1 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_2 & 0 & \beta_2 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & \alpha_{N-2} & 0 & \beta_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{N-1} & 0 \end{pmatrix},$$

with

$$\alpha_i \neq 0,\ i = 2,\dots,N-1, \quad \beta_j \neq 0,\ j = 1,\dots,N-2.$$

For the matrix A one can use the quasiseparable generators

$$\begin{aligned} p(i) &= \alpha_i,\ i = 2,\dots,N-1, & q(j) &= 1,\quad j = 1,\dots,N-2, \\ a(k) &= 0,\ \ k = 2,\dots,N-2, & g(j) &= \beta_j,\ j = 1,\dots,N-2, \\ h(i) &= 1,\quad i = 2,\dots,N-1, & b(k) &= 0,\ \ k = 2,\dots,N-2, \\ d(k) &= d,\ \ k = 1,\dots,N-1. \end{aligned}$$

Then

$$l_k = -\alpha_k, \quad \delta_k = -\beta_k,\ \ k = 2,\dots,N-2$$

and one can compute forwards

$$\begin{array}{llll} s_1 = 1, & v_1 = \beta_1, & f_1 = \beta_1, & \gamma_1 = 0, \\ s_2 = \gamma_1 = 0, & v_2 = 0, & \gamma_2 = -\alpha_2\beta_1, & f_2 = 0, \\ s_3 = -\alpha_2\beta_1, & v_3 = -\alpha_2\beta_1\beta_3, & \gamma_3 = 0, & f_3 = -\alpha_2\beta_1\beta_3, \\ s_4 = 0, & v_4 = 0, & \gamma_4 = \alpha_2\alpha_4\beta_1\beta_3, & f_4 = 0, \\ s_5 = \alpha_2\alpha_4\beta_1\beta_3, & v_5 = \alpha_2\alpha_4\beta_1\beta_3\beta_5, & \gamma_5 = 0, & f_5 = \alpha_2\alpha_4\beta_1\beta_3\beta_5. \end{array}$$

By induction, one can prove that for $m = 0,\dots,r-1$,

$$\begin{aligned} \gamma_{2m+1} = 0, \quad f_{2m+1} &= (-1)^m \alpha_2\alpha_4 \cdot \cdots \cdot \alpha_{2m}\beta_1\beta_3 \cdot \cdots \cdot \beta_{2m+1}, \\ s_{2m+1} &= (-1)^m \alpha_2\alpha_4 \cdot \cdots \cdot \alpha_{2m}\beta_1\beta_3 \cdot \cdots \cdot \beta_{2m-1}, \\ v_{2m+1} &= \alpha_2\alpha_4 \cdot \cdots \cdot \alpha_{2m}\beta_1\beta_3 \cdot \cdots \cdot \beta_{2m+1}. \end{aligned}$$

Also, for $m = 1,\dots,r$,

$$\gamma_{2m} = (-1)^m \alpha_2\alpha_4 \cdot \cdots \cdot \alpha_{2m}\beta_1\beta_3 \cdot \cdots \cdot \beta_{2m-1}, \quad s_{2m} = v_{2m} = f_{2m} = 0.$$

One can also compute backwards

$$\begin{aligned} &t_{N-1} = \alpha_{N-1}, \quad u_{N-1} = 1, \quad z_{N-1} = \alpha_{N-1}, \quad \theta_{N-1} = 0, \\ &t_{N-2} = 0, \quad u_{N-2} = 0, \quad z_{N-2} = 0, \quad \theta_{N-2} = -\alpha_{N-1}\beta_{N-2}, \end{aligned}$$

$$t_{N-3} = -\alpha_{N-1}\alpha_{N-3}\beta_{N-2}, \quad u_{N-3} = -\alpha_{N-1}\beta_{N-2},$$
$$z_{N-3} = -\alpha_{N-1}\alpha_{N-3}\beta_{N-2}, \quad \theta_{N-3} = 0.$$

By induction, one can prove that for $m = 0, \ldots, r-1$,

$$\theta_{2m+1} = (-1)^{r-m}\alpha_{N-1}\alpha_{N-3}\cdot\cdots\cdot\alpha_{2m+2}\beta_{N-2}\beta_{N-4}\cdot\cdots\cdot\beta_{2m+1},$$
$$z_{2m+1} = t_{2m+1} = u_{2m+1} = 0.$$

Also, for $m = 1, \ldots, r$,

$$z_{2m} = (-1)^{r-m}\alpha_{N-1}\alpha_{N-3}\cdot\cdots\cdot\alpha_{2m}\beta_{N-2}\beta_{N-4}\cdot\cdots\cdot\beta_{2m+1},$$
$$u_{2m} = (-1)^{r-m}\alpha_{N-1}\alpha_{N-3}\cdot\cdots\cdot\alpha_{2m+2}\beta_{N-2}\beta_{N-4}\cdot\cdots\cdot\beta_{2m+1},$$
$$t_{2m} = (-1)^{r-m}\alpha_{N-1}\alpha_{N-3}\cdot\cdots\cdot\alpha_{2m}\beta_{N-2}\beta_{N-4}\cdot\cdots\cdot\beta_{2m+1}, \quad \theta_{2m} = 0.$$

It follows that for $k = 1, \ldots, N-1$

$$\rho_k = (-1)^r\alpha_{N-1}\alpha_{N-3}\cdot\cdots\cdot\alpha_2\beta_{N-2}\beta_{N-4}\cdot\cdots\cdot\beta_1 = \det A$$

and

$$\lambda_1 = \theta_2 = 0, \quad \lambda_k = 0, \;\; k = 2, \ldots, N-2, \quad \lambda_{N-1} = 0.$$

One defines the following numbers, which will be the quasiseparable generators of the inverse matrix

$$\tilde{p}(2m) = (-1)^{m+1}\frac{1}{\alpha_2\alpha_4\cdot\cdots\cdot\alpha_{2m-2}\beta_1\beta_3\cdot\cdots\cdot\beta_{2m-1}}, \quad \tilde{p}(2m+1) = 0,$$
$$\tilde{h}(2m) = (-1)^{m+1}\frac{1}{\alpha_2\alpha_4\cdot\cdots\cdot\alpha_{2m}\beta_1\beta_3\cdot\cdots\cdot\beta_{2m-1}}, \quad \tilde{h}(2m+1) = 0,$$
$$\tilde{d}(k) = 0, \quad \tilde{q}(j) = s_j, \quad \tilde{g}(j) = v_j,$$
$$\tilde{a}(k) = l_k = -\alpha_k, \quad \tilde{b}(k) = \delta_k = -\beta_k.$$

With the above generators one can write the inverse matrix A^{-1}. For instance, the last column L of the inverse matrix has the even components $L(2m) = 0, \;\; m = 1, \ldots, r-1$, since $s_{2m} = 0$ and $L(2r) = 0$, since $\tilde{d}(2r) = 0$. The odd components are given by

$$\tilde{h}(2r) = (-1)^{r+1}\frac{1}{\alpha_2\alpha_4\cdot\cdots\cdot\alpha_{2r}\beta_1\beta_3\cdot\cdots\cdot\beta_{2r-1}}$$

and it follows that

$$L(2m+1) = (-1)^{r-m+1}\frac{1}{\alpha_{N-1}}\frac{\beta_{N-3}}{\alpha_{N-3}}\cdot\cdots\cdot\frac{\beta_{2(m+1)}}{\alpha_{2(m+1)}}. \tag{19.22}$$

Similarly one can find the remaining part of the matrix A^{-1}. ◊

§19.3 Inversion algorithm with scaling

We present here an algorithm to compute quasiseparable generators of the inverse matrix A^{-1}. The direct use of representations derived in Theorem 19.1 may lead to an overflow or underflow in the computing process. That is why we use an equivalent representation of generators including some scaling coefficients.

Algorithm 19.6. Let A be an invertible scalar matrix with quasiseparable of orders one generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$; $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$; $d(k)$ $(k = 1, \dots, N)$.

Then quasiseparable of orders one generators $\tilde{t}_i$ $(i = 2, \dots, N)$, $\tilde{s}_j$ $(j = 1, \dots, N-1)$, $\tilde{l}_k$ $(k = 2, \dots, N-1)$; $\tilde{v}_i$ $(i = 1, \dots, N-1)$, $\tilde{u}_j$ $(j = 2, \dots, N)$, $\tilde{\delta}_k$ $(k = 2, \dots, N-1)$; $\tilde{\lambda}_k$ $(k = 1, \dots, N)$ of the matrix A^{-1} are given as follows.

1.1.1. Set $s'_1 = q(1)$, $v'_1 = g(1)$, $\gamma'_1 = d(1)$ and compute $f'_1 = q(1)g(1)$.

1.1.2. Introduce the scaling coefficient β^I_1 as $\beta^I_1 = 1/\max(|\gamma'_1|, |f'_1|)$ or using another method.

1.1.3. Compute

$$\tilde{\gamma}_1 = \gamma'_1 \beta^I_1, \quad \tilde{f}_1 = f'_1 \beta^I_1, \quad \tilde{s}_1 = \beta^I_1 s'_1, \quad \tilde{v}_1 = v'_1 \beta^I_1.$$

1.2. For $k = 2, \dots, N-1$, perform the following operations:

1.2.1. Compute

$$\begin{aligned}
l_k &= d(k)a(k) - q(k)p(k), & \delta_k &= d(k)b(k) - h(k)g(k), \\
s'_k &= q(k)\tilde{\gamma}_{k-1} - a(k)\tilde{f}_{k-1}h(k), & v'_k &= g(k)\tilde{\gamma}_{k-1} - p(k)\tilde{f}_{k-1}b(k), \\
\gamma'_k &= d(k)\tilde{\gamma}_{k-1} - p(k)\tilde{f}_{k-1}h(k), \\
f'_k &= a(k)\tilde{f}_{k-1}b(k)d(k) + q(k)v'_k + s'_k g(k) - \tilde{\gamma}_{k-1}q(k)g(k).
\end{aligned}$$

1.2.2. Introduce the scaling coefficient β^I_k as $\beta^I_k = 1/\max(|\gamma'_k|, |f'_k|)$ or by using another method.

1.2.3. Compute

$$\begin{aligned}
\tilde{\gamma}_k &= \gamma'_k \beta^I_k, \quad \tilde{f}_k = f'_k \beta^I_k, \quad \tilde{s}_k = \beta^I_k s'_k, \\
\tilde{v}_k &= v'_k \beta^I_k, \quad \tilde{l}_k = \beta^I_k l_k, \quad \tilde{\delta}_k = \beta^I_k \delta_k.
\end{aligned}$$

2.1.1. Set $t'_N = p_N$, $u'_N = h_N$, $\theta'_N = d_N$ and compute $z'_N = h_N p_N$.

2.1.2. Introduce the scaling coefficient β^U_N as $\beta^U_N = 1/\max(|\theta'_N|, |z'_N|)$ or by using another method.

2.1.3. Compute

$$\begin{aligned}
&\tilde{\theta}_N = \theta'_N \beta^U_N, \quad \tilde{z}_N = z'_N \beta^U_N, \quad t''_N = \beta^U_N t'_N, \quad u''_N = u'_N \beta^U_N, \\
&\tilde{\rho}_N = \tilde{\gamma}_{N-1}\tilde{\theta}_N - \tilde{f}_{N-1}\tilde{z}_N, \quad \tilde{t}_N = -t''_N/\tilde{\rho}_N, \quad \tilde{u}_N = -u''_N/\tilde{\rho}_N.
\end{aligned}$$

2.2. For $k = N-1, \dots, 2$, perform the following operations:

2.2.1. Compute

$$t'_k = p(k)\tilde{\theta}_{k+1} - g(k)\tilde{z}_{k+1}a(k), \quad u'_k = h(k)\tilde{\theta}_{k+1} - b(k)\tilde{z}_{k+1}q(k),$$
$$\theta'_k = d_k\tilde{\theta}_{k+1} - g(k)\tilde{z}_{k+1}q(k),$$
$$z'_k = b(k)\tilde{z}_{k+1}a(k)d(k) + h(k)t'_k + u'_k p(k) - \tilde{\theta}_{k+1}h(k)p(k).$$

2.2.2. Introduce the scaling coefficient β^U_k as $\beta^U_k = 1/\max(|\theta'_k|, |z'_k|)$ or by using another method.

2.2.3. Compute

$$\tilde{\theta}_k = \theta'_k\beta^U_k, \quad \tilde{z}_k = z'_k\beta^U_k, \quad t''_k = \beta^U_k t'_k, \qquad u''_k = u'_k\beta^U_k,$$
$$\tilde{\rho}_k = \tilde{\gamma}_{k-1}\tilde{\theta}_k - \tilde{f}_{k-1}\tilde{z}_k, \qquad \tilde{t}_k = -t''_k/\tilde{\rho}_k, \quad \tilde{u}_k = -u''_k/\tilde{\rho}_k.$$

3.1. Compute $\lambda_1 = \tilde{\theta}_2\beta^I_1/\tilde{\rho}_2$.

3.2. For $k = 2, \dots, N-1$, compute

$$\lambda'_k = \tilde{\gamma}_{k-1}\tilde{\theta}_{k+1} - a(k)\tilde{f}_{k-1}\tilde{z}_{k+1}b(k), \quad \tilde{\lambda}_k = \lambda'_k\beta^U_k/\tilde{\rho}_k.$$

3.3. Compute $\tilde{\lambda}_N = \tilde{\gamma}_{N-1}\beta^U_N/\tilde{\rho}_N$.

To justify this algorithm note the following. It is easy to prove by induction that the elements $\tilde{s}_k$, $\tilde{v}_k$, $\tilde{\gamma}_k$, $\tilde{f}_k$ and t''_k, u''_k, $\tilde{\theta}_k$, $\tilde{z}_k$ are connected with the corresponding elements of (19.2), (19.3) via the relations

$$\tilde{s}_k = s_k\prod_{i=1}^k \beta^I_i, \;\; \tilde{v}_k = v_k\prod_{i=1}^k \beta^I_i, \; \tilde{\gamma}_k = \gamma_k\prod_{i=1}^k \beta^I_i, \; \tilde{f}_k = f_k\prod_{i=1}^k \beta^I_i, \quad 1 \le k \le N-1,$$

and

$$u''_k = u_k\prod_{i=k}^N \beta^U_i, \; t''_k = t_k\prod_{i=k}^N \beta^U_i, \; \tilde{\theta}_k = \theta_k\prod_{i=k}^N \beta^U_i, \; \tilde{z}_k = z_k\prod_{i=k}^N \beta^U_i, \quad N \ge k \ge 2.$$

It follows that

$$\tilde{\rho}_k = \left(\prod_{i=1}^{k-1} \beta^I_i\right)\rho\left(\prod_{i=k}^{N} \beta^U_i\right), \qquad 2 \le k \le N$$

and

$$\lambda'_k = \left(\prod_{i=1}^{k-1} \beta^I_i\right)\lambda_k\left(\prod_{i=k+1}^{N} \beta^U_i\right), \quad 2 \le k \le N-1.$$

Notice also that

$$\tilde{l}^>_{ij} = l^>_{ij}(\beta^I_{ij})^>, \; i > j; \quad \tilde{\delta}^<_{ij} = \delta^<_{ij}(\beta^I_{ij})^<, \; i < j.$$

Thus one obtains

$$\begin{aligned}
A_{ij} &= -\frac{t_i l^{>}_{ij} s_j}{\rho} \\
&= -\frac{t''_i (\prod_{m=i}^{N} \beta^U_m)^{-1} \tilde{l}^{>}_{ij} ((\beta^I_{ij})^{>})^{-1} \tilde{s}_j (\prod_{m=1}^{j} \beta^I_m)^{-1}}{(\prod_{m=1}^{i-1} \beta^I_m)^{-1} \tilde{\rho}_i (\prod_{m=i}^{N} \beta^U_m)^{-1}} \\
&= -\frac{t''_i \tilde{l}^{>}_{ij} \tilde{s}_j}{\tilde{\rho}_i} = \tilde{t}_i \tilde{l}^{>}_{ij} \tilde{s}_j, \quad i > j,
\end{aligned}$$

and

$$\begin{aligned}
A_{ij} &= -\frac{v_i \delta^{<}_{ij} u_j}{\rho} \\
&= -\frac{\tilde{v}_i (\prod_{m=1}^{i} \beta^I_m)^{-1} \tilde{\delta}^{<}_{ij} ((\beta^I_{ij})^{<})^{-1} u''_j (\prod_{m=j}^{N} \beta^U_m)^{-1}}{(\prod_{m=1}^{j-1} \beta^I_m)^{-1} \tilde{\rho}_j (\prod_{m=j}^{N} \beta^U_m)^{-1}} \\
&= -\frac{\tilde{v}_i \tilde{\delta}^{<}_{ij} u''_j}{\tilde{\rho}_j} = \tilde{v}_i \tilde{\delta}^{<}_{ij} \tilde{u}_j, \quad i < j.
\end{aligned}$$

Hence the elements $\tilde{t}_i$ $(i = 2, \dots, N)$, $\tilde{s}_j$ $(j = 1, \dots, N-1)$, $\tilde{l}_k$ $(k = 2, \dots, N-1)$ and $\tilde{v}_i$ $(i = 1, \dots, N-1)$, $\tilde{u}_j$ $(j = 2, \dots, N)$, $\tilde{\delta}_k$ $(k = 2, \dots, N-1)$ given by Algorithm 19.6 are correspondingly lower and upper quasiseparable generators of the matrix A. The diagonal entries of A may be expressed as follows:

$$\begin{aligned}
\tilde{\lambda}_1 = \theta_2/\rho &= \frac{\tilde{\theta}_2 (\prod_{i=2}^{N} \beta^U_i)^{-1}}{(\beta^I_1)^{-1} \tilde{\rho}_2 (\prod_{i=2}^{N} \beta^U_i)^{-1}} = \frac{\tilde{\theta}_2 \beta^I_1}{\tilde{\rho}_2}, \\
\tilde{\lambda}_k = \lambda_k/\rho &= \frac{(\prod_{i=1}^{k-1} \beta^I_i)^{-1} \lambda'_k (\prod_{i=k+1}^{N} \beta^U_i)^{-1}}{(\prod_{i=1}^{k-1} \beta^I_i)^{-1} \tilde{\rho}_k (\prod_{i=k}^{N} \beta^U_i)^{-1}} = \frac{\lambda'_k \beta^U_k}{\tilde{\rho}_k}, \quad 2 \le k \le N-1, \\
\tilde{\lambda}_N = \gamma_{N-1}/\rho &= \frac{\tilde{\gamma}_{N-1} (\prod_{i=1}^{N-1} \beta^I_i)^{-1}}{(\prod_{i=1}^{N-1} \beta^I_i)^{-1} \tilde{\rho}_N (\beta^U_N)^{-1}} = \frac{\tilde{\gamma}_{N-1} \beta^U_N}{\tilde{\rho}_N},
\end{aligned}$$

which corresponds to Steps 3.1–3.3 in Algorithm 19.6.

An easy calculation shows that Algorithm 19.6 requires $58(N-2) + 20$ operations. At the same time, substituting $m = 1$, $r = 1$ in (18.65) one obtains the estimate $c \le 52N$ for the complexity of the algorithm from Theorem 18.7, to be applied to scalar matrices with quasiseparable order one. The last estimate is better than that for Algorithm 19.6, but the algorithm from Theorem 18.7 is obtained under some restrictions.

It is clear that using consequently Algorithm 19.6 for quasiseparable generators of the matrix A^{-1} and then applying Algorithm 13.1 to the product $x = A^{-1}y$ one obtains an algorithm of linear complexity for the solution of the linear equation $Ax = y$.

§19.4 The case of diagonal plus semiseparable representation

Let A be a scalar matrix with lower and upper semiseparable generators $p(i)$ $(i = 2,\dots,N)$, $q(j)$ $(j = 1,\dots,N-1)$ and $g(i)$ $(i = 1,\dots,N-1)$, $h(j)$ $(j = 2,\dots,N)$ of order one, and diagonal entries $d(k)$ $(k = 1,\dots,N)$. Such a matrix has quasiseparable generators $p(i)$ $(i = 2,\dots,N)$, $q(j)$ $(j = 1,\dots,N-1)$, $a(k) = 1$, $(k = 2,\dots,N-1)$; $g(i)$ $(i = 1,\dots,N-1)$, $h(j)$ $(j = 2,\dots,N)$, $b(k) = 1$ $(k = 2,\dots,N-1)$; $d(k)$ $(k = 1,\dots,N)$. Hence one can set $a_k = b_k = 1$ in Algorithm 19.6 and obtain the following method.

Algorithm 19.7. Let A be a scalar matrix with lower and upper semiseparable generators $p(i)$ $(i = 2,\dots,N)$, $q(j)$ $(j = 1,\dots,N-1)$ and $g(i)$ $(i = 1,\dots,N-1)$, $h(j)$ $(j = 2,\dots,N)$ of order one and diagonal entries $d(k)$ $(k = 1,\dots,N)$. Then the quasiseparable of orders one generators $-\tilde{t}_i$ $(i = 2,\dots,N)$, $\tilde{s}_j$ $(j = 1,\dots,N-1)$, l_k $(k = 2,\dots,N-1)$; $\tilde{v}_i$ $(i = 1,\dots,N-1)$, $-\tilde{u}_j$ $(j = 2,\dots,N)$, $\tilde{\delta}_k$ $(k = 2,\dots,N-1)$; $\tilde{\lambda}_k$ $(k = 1,\dots,N)$ of the matrix A^{-1} are obtained as follows.

1.1.1. Set $s'_1 = q(1)$, $v'_1 = g(1)$, $\gamma'_1 = d(1)$ and compute $f'_1 = q(1)g(1)$.

1.1.2. Introduce the scaling coefficient β^I_1 as $\beta^I_1 = 1/\max(|\gamma'_1|, |f'_1|)$ or by using another method.

1.1.3. Compute

$$\tilde{\gamma}_1 = \gamma'_1\beta^I_1, \quad \tilde{f}_1 = f'_1\beta^I_1, \quad \tilde{s}_1 = \beta^I_1 s'_1, \quad \tilde{v}_1 = v'_1\beta^I_1.$$

1.2. For $k = 2,\dots,N-1$, perform the following operations:

1.2.1. Compute

$$\begin{aligned} l_k &= d(k) - q(k)p(k), & \delta_k &= d(k) - h(k)g(k),\\ s'_k &= q(k)\tilde{\gamma}_{k-1} - \tilde{f}_{k-1}h(k), & v'_k &= g(k)\tilde{\gamma}_{k-1} - p(k)\tilde{f}_{k-1},\\ \gamma'_k &= d(k)\tilde{\gamma}_{k-1} - p(k)\tilde{f}_{k-1}h(k), && \\ f'_k &= \tilde{f}_{k-1}d(k) + q(k)v'_k + s'_k g(k) - \tilde{\gamma}_{k-1}q(k)g(k). && \end{aligned}$$

1.2.2. Introduce the scaling coefficient β^I_k as $\beta^I_k = 1/\max(|\gamma'_k|, |f'_k|)$ or by using another method.

1.2.3. Compute

$$\begin{aligned} \tilde{\gamma}_k &= \gamma'_k\beta^I_k, & \tilde{f}_k &= f'_k\beta^I_k, & \tilde{s}_k &= \beta^I_k s'_k,\\ \tilde{v}_k &= v'_k\beta^I_k, & \tilde{l}_k &= \beta^I_k l_k, & \tilde{\delta}_k &= \beta^I_k\delta_k. \end{aligned}$$

2.1.1. Set $t'_N = p(N)$, $u'_N = h(N)$, $\theta'_N = d(N)$ and compute $z'_N = h(N)p(N)$.

2.1.2. Introduce the scaling coefficient β^U_N as $\beta^U_N = 1/\max(|\theta'_N|, |z'_N|)$ or by using another method.

2.1.3. Compute

$$\tilde{\theta}_N = \theta'_N \beta^U_N, \quad \tilde{z}_N = z'_N \beta^U_N, \quad t''_N = \beta^U_N t'_N, \qquad u''_N = u'_N \beta^U_N,$$
$$\tilde{\rho}_N = \tilde{\gamma}_{N-1}\tilde{\theta}_N - \tilde{f}_{N-1}\tilde{z}_N, \qquad \tilde{t}_N = -t''_N/\rho_N, \quad \tilde{u}_N = -u''_N/\rho_N.$$

2.2. For $k = N-1, \dots, 2$, perform the following operations:

2.2.1. Compute

$$t'_k = p(k)\tilde{\theta}_{k+1} - g(k)\tilde{z}_{k+1}, \quad u'_k = h(k)\tilde{\theta}_{k+1} - \tilde{z}_{k+1}q(k),$$
$$\theta'_k = d(k)\tilde{\theta}_{k+1} - g(k)\tilde{z}_{k+1}q(k),$$
$$z'_k = \tilde{z}_{k+1}d(k) + h(k)t'_k + u'_k p(k) - \tilde{\theta}_{k+1}h(k)p(k).$$

2.2.2. Introduce the scaling coefficient β^U_N as $\beta^U_k = 1/\max(|\theta'_k|, |z'_k|)$ or by using another method.

2.2.3. Compute

$$\tilde{\theta}_k = \theta'_k \beta^U_k, \quad \tilde{z}_k = z'_k \beta^U_k, \quad t''_k = \beta^U_k t'_k, \qquad u''_k = u'_k \beta^U_k,$$
$$\tilde{\rho}_k = \tilde{\gamma}_{k-1}\tilde{\theta}_k - \tilde{f}_{k-1}\tilde{z}_k, \qquad \tilde{t}_k = -t''_k/\tilde{\rho}_k, \quad \tilde{u}_k = -u''_k/\tilde{\rho}_k.$$

3.1. Compute $\lambda_1 = \tilde{\theta}_2 \beta^I_1/\tilde{\rho}_2$.

3.2. For $k = 2, \dots, N-1$, compute

$$\lambda'_k = \tilde{\gamma}_{k-1}\tilde{\theta}_{k+1} - \tilde{f}_{k-1}\tilde{z}_{k+1}, \quad \tilde{\lambda}_k = \lambda'_k \beta^U_k/\tilde{\rho}_k.$$

3.3. Compute $\tilde{\lambda}_N = \tilde{\gamma}_{N-1}\beta^U_N/\tilde{\rho}_N$.

The amount of operations for this algorithm is $42(N-2)+20$.

§19.5 The case of a tridiagonal matrix

Consider a tridiagonal scalar matrix $A = \{A_{ij}\}_{i,j=1}^N$, i.e., $A_{ij} = 0$ for $|i-j| > 1$. Quasiseparable of orders one generators of the matrix A may be defined (see §4.11) via

$$\begin{aligned}
&p(i) = 1, && i = 2, \dots, N, && q(j) = A_{j+1,j},\ j = 1, \dots, N-1,\\
&a(k) = b(k) = 0,\ k = 2, \dots, N-1; &&&&\\
&g(i) = A_{i,i+1}, && i = 1, \dots, N-1, && h(j) = 1, \qquad j = 2, \dots, N;\\
&d(k) = A_{kk}, && k = 1, \dots N. &&
\end{aligned}$$

As a direct consequence of Theorem 19.1 one obtains the following inversion formula that is valid for any invertible tridiagonal matrix.

Theorem 19.8. *Let* $A = \{A_{ij}\}_{i,j=1}^{N}$ *be an invertible tridiagonal matrix. Let us define:*

$$l_k = -A_{k+1,k}, \quad \delta_k = -A_{k,k+1}, \quad 2 \le k \le N-1;$$

forward recursively $s_1 = A_{2,1},\ v_1 = A_{1,2},\ f_1 = s_1 v_1,\ \gamma_1 = A_{1,1}$ *and for* $k = 2, \dots, N-1$

$$\begin{aligned} s_k &= A_{k+1,k}\gamma_{k-1}, \quad v_k = A_{k,k+1}\gamma_{k-1}, \\ \gamma_k &= A_{k,k}\gamma_{k-1} - f_{k-1}, \\ f_k &= \gamma_{k-1}A_{k+1,k}A_{k,k+1}, \end{aligned}$$

$$\gamma_N = A_{N,N}\gamma_{N-1} - f_{N-1},$$

backward recursively $t_N = 1,\ u_N = 1,\ z_N = 1,\ \theta_N = A_{N,N}$ *and for* $k = N-1, \dots, 2$

$$\begin{aligned} t_k &= \theta_{k+1}, \quad u_k = \theta_{k+1}, \quad z_k = \theta_{k+1}, \\ \theta_k &= A_{k,k}\theta_{k+1} - A_{k,k+1}z_{k+1}A_{k+1,k}, \\ \theta_1 &= A_{1,1}\theta_2 - A_{2,1}z_2A_{1,2}, \end{aligned}$$

and

$$\begin{aligned} &\rho_k = \gamma_{k-1}\theta_k - f_{k-1}z_k, \quad 2 \le k \le N, \qquad \rho_1 = \rho_2, \\ &\lambda_1 = \theta_2, \ \ \lambda_k = \gamma_{k-1}\theta_{k+1}, 2 \le k \le N-1, \quad \lambda_N = \gamma_{N-1}. \end{aligned}$$

Set $\det A = \rho$.

Then $\rho_k = \rho,\ k = 1, \dots, N$, *and the elements* $-t_i/\rho_i$ $(i = 2, \dots, N)$, s_j $(j = 1, \dots, N-1)$, l_k $(k = 2, \dots, N-1)$; v_i $(i = 1, \dots, N-1)$, $-u_j/\rho_j$ $(j = 2, \dots, N)$, δ_k $(k = 2, \dots, N-1)$; λ_k/ρ_k $(k = 1, \dots, N)$ *are quasiseparable of orders one generators of the matrix* A^{-1}.

Applying the scaling method described in Subsection §19.3 one obtains in the case of a tridiagonal matrix the following method.

Algorithm 19.9. Let $A = \{A_{ij}\}_{i,j=1}^{N}$ be a tridiagonal matrix. Then the quasiseparable generators $-\tilde{t}_i$ $(i = 2, \dots, N)$, $\tilde{s}_j$ $(j = 1, \dots, N-1)$, $\tilde{l}_k$ $(k = 2, \dots, N-1)$; $\tilde{v}_i$ $(i = 1, \dots, N-1)$, $-\tilde{u}_j$ $(j = 2, \dots, N)$, $\tilde{\delta}_k$ $(k = 2, \dots, N-1)$; $\tilde{\lambda}_k$ $(k = 1, \dots, N)$ of orders one of the matrix A^{-1} are obtained as follows.

1.1.1. Set $s_1' = A_{2,1},\ v_1' = A_{1,2},\ \gamma_1' = A_{1,1}$ and compute $f_1' = s_1' v_1'$.

1.1.2. Introduce the scaling coefficient β_1^I as $\beta_1^I = 1/\max(|\gamma_1'|, |f_1'|)$ or by using another method.

1.1.3. Compute

$$\tilde{\gamma}_1 = \gamma_1'\beta_1^I, \quad \tilde{f}_1 = f_1'\beta_1^I, \quad \tilde{s}_1 = \beta_1^I s_1', \quad \tilde{v}_1 = v_1'\beta_1^I.$$

1.2. For $k = 2, \dots, N-1$, perform the following operations:

1.2.1. Compute

$$l_k = -A_{k+1,k}, \qquad \delta_k = -A_{k,k+1};$$
$$s'_k = A_{k+1,k}\tilde{\gamma}_{k-1}, \qquad v'_k = A_{k,k+1}\tilde{\gamma}_{k-1},$$
$$\gamma'_k = A_{kk}\tilde{\gamma}_{k-1} - \tilde{f}_{k-1}, \quad f'_k = A_{k+1,k}v'_k.$$

1.2.2. Introduce the scaling coefficient β^I_k as $\beta^I_k = 1/\max(|\gamma'_k|, |f'_k|)$ or by using another method.

1.2.3. Compute

$$\tilde{\gamma}_k = \gamma'_k\beta^I_k, \quad \tilde{f}_k = f'_k\beta^I_k, \quad \tilde{s}_k = \beta^I_k s'_k,$$
$$\tilde{v}_k = v'_k\beta^I_k, \quad \tilde{l}_k = \beta^I_k l_k, \quad \tilde{\delta}_k = \beta^I_k\delta_k.$$

2.1.1. Set $\theta'_N = r_{N,N},\ z'_N = 1$.

2.1.2. Introduce the scaling coefficient β^U_N as $\beta^U_N = 1/\max(|\theta'_N|, |z'_N|)$ or by using another method.

2.1.3. Compute

$$\tilde{\theta}_N = \theta'_N\beta^U_N, \quad \tilde{z}_N = z'_N\beta^U_N,$$
$$\tilde{\rho}_N = \tilde{\gamma}_{N-1}\tilde{\theta}_N - \tilde{f}_{N-1}\tilde{z}_N, \quad \tilde{t}_N = -\tilde{z}_N/\rho_N, \quad \tilde{u}_N = \tilde{t}_N.$$

2.2. For $k = N-1, \ldots, 2$, perform the following operations:

2.2.1. Set $z'_k = \tilde{\theta}_{k+1}$ and compute

$$\theta'_k = A_{k,k}\tilde{\theta}_{k+1} - A_{k,k+1}\tilde{z}_{k+1}A_{k+1,k}.$$

2.2.2. Introduce the scaling coefficient β^U_k as $\beta^U_k = 1/\max(|\theta'_k|, |z'_k|)$ or by using another method.

2.2.3. Compute

$$\tilde{\theta}_k = \theta'_k\beta^U_k, \quad \tilde{z}_k = z'_k\beta^U_k,$$
$$\tilde{\rho}_k = \tilde{\gamma}_{k-1}\tilde{\theta}_k - \tilde{f}_{k-1}\tilde{z}_k, \quad \tilde{t}_k = -\tilde{z}_k/\tilde{\rho}_k, \quad \tilde{u}_k = \tilde{t}_k.$$

3.1. Compute $\lambda_1 = \tilde{\theta}_2\beta^I_1/\tilde{\rho}_2$.

3.2. For $k = 2, \ldots, N-1$, compute

$$\tilde{\lambda}_k = \tilde{\gamma}_{k-1}\tilde{\theta}_{k+1}\beta^U_k/\tilde{\rho}_k.$$

3.3. Compute $\lambda_N = \tilde{\gamma}_{N-1}\beta^U_N/\tilde{\rho}_N$.

This algorithm requires $23(N-2)+14$ operations.

§19.6 Comments

The material of this chapter is taken mostly from the paper [21].

Chapter 20

The QR-Factorization Based Method

In this chapter we present a method for the inversion of block matrices with given quasiseparable representations without any restriction on the matrix except its invertibility. It is based on a special representation of a block invertible matrix A in the form

$$A = VUR, \tag{20.1}$$

where V is a block lower triangular unitary matrix and U is a block upper triangular unitary matrix, with nonsquare blocks, and R is a block upper triangular matrix with square invertible blocks on the main diagonal. This is a form of the QR factorization of the matrix A in which the unitary Q-factor is written in a special form.

The matrices V, U, R are given by their quasiseparable generators, which are computed via quasiseparable generators of the original matrix A. Using this representation we find the solution of the system of linear algebraic equations $Ax = y$ as $x = R^{-1}U^*V^*y$. As a result, we obtain a linear complexity algorithm to find the solution x.

In the first step of the method we compute the factorization $A = VT$, where V is a block lower triangular unitary matrix and T is a block upper triangular matrix. In general, these matrices have rectangular blocks on the main diagonal. In order to obtain matrices which are convenient for inversion, we compute for the matrix T the factorization $T = UR$, where U is a block upper triangular unitary matrix and R is a block upper triangular matrix with square invertible blocks on the main diagonal. Below we present the description of both steps with the detailed justification.

§20.1 Factorization of triangular matrices

We derive here factorizations which are valid for any block triangular matrices with given quasiseparable generators.

Y. Eidelman et al., *Separable Type Representations of Matrices and Fast Algorithms: Volume 1 Basics. Completion Problems. Multiplication and Inversion Algorithms*, Operator Theory: Advances and Applications 234, DOI 10.1007/978-3-0348-0606-0_20,

Lemma 20.1. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be a block lower triangular matrix with entries of sizes* $m_i \times n_j$ *and lower quasiseparable generators* $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ *and diagonal entries* $d(k)$ $(k = 1, \dots, N)$. *Using these generators and the diagonal entries define matrices*

$$A_1 = \begin{pmatrix} d(1) \\ q(1) \end{pmatrix}, \quad A_k = \begin{pmatrix} p(k) & d(k) \\ a(k) & q(k) \end{pmatrix}, \; k = 2, \dots, N-1, \quad A_N = \begin{pmatrix} p(N) & d(N) \end{pmatrix} \tag{20.2}$$

and then set

$$\tilde{A}_1 = \operatorname{diag}\{A_1, I_{\gamma_1}\}, \; \tilde{A}_k = \operatorname{diag}\{I_{\eta_k}, A_k, I_{\gamma_k}\}, \; k = 2, \dots, N-1, \\ \tilde{A}_N = \operatorname{diag}\{I_{\eta_N}, A_N\}, \tag{20.3}$$

where $\eta_k = \sum_{i=1}^{k-1} m_i, \; \gamma_k = \sum_{i=k+1}^{N} n_i$.

Then

$$A = \tilde{A}_N \tilde{A}_{N-1} \cdots \tilde{A}_1. \tag{20.4}$$

Proof. Let us prove by induction the validity of the relations

$$\tilde{A}_k \cdots \tilde{A}_1 = \begin{pmatrix} A(1:k, 1:k) & 0 \\ Q_k & 0 \\ 0 & I_{\gamma_k} \end{pmatrix}, \quad k = 1, \dots, N-1, \tag{20.5}$$

where the matrices Q_k are given by (5.1).

For $k = 1$ (20.5) is obvious. Suppose (20.5) holds for k with $1 \leq k \leq N-2$. Then

$$\begin{aligned} &\tilde{A}_{k+1} \tilde{A}_k \cdots \tilde{A}_1 \\ &= \begin{pmatrix} I_{\eta_{k+1}} & 0 & 0 & 0 \\ 0 & p(k+1) & d(k+1) & 0 \\ 0 & a(k+1) & q(k+1) & 0 \\ 0 & 0 & 0 & I_{\gamma_{k+1}} \end{pmatrix} \begin{pmatrix} A(1:k, 1:k) & 0 & 0 \\ Q_k & 0 & 0 \\ 0 & I_{n_{k+1}} & 0 \\ 0 & 0 & I_{\gamma_{k+1}} \end{pmatrix} \\ &= \begin{pmatrix} A(1:k, 1:k) & 0 & 0 \\ p(k+1)Q_k & d(k+1) & 0 \\ a(k+1)Q_k & q(k+1) & 0 \\ 0 & 0 & I_{\gamma_{k+1}} \end{pmatrix}. \end{aligned}$$

Using the equality (5.11) we get

$$\begin{pmatrix} A(1:k, 1:k) & 0 \\ p(k+1)Q_k & d(k+1) \end{pmatrix} = A(1:k+1, 1:k+1)$$

and thus using (5.3) we conclude that

$$\tilde{A}_{k+1} \tilde{A}_k \cdots \tilde{A}_1 = \begin{pmatrix} A(1:k+1, 1:k+1) & 0 \\ Q_{k+1} & 0 \\ 0 & I_{\gamma_{k+1}} \end{pmatrix}.$$

The relation (20.5) with $k = N-1$ and the relation (5.11) yield

$$\tilde{A}_N \cdots \tilde{A}_1 = \begin{pmatrix} I_{\eta_N} & 0 & 0 \\ 0 & p(N) & d(N) \end{pmatrix} \begin{pmatrix} A(1:N-1, 1:N-1) & 0 \\ Q_{N-1} & 0 \\ 0 & I_{n_N} \end{pmatrix}$$
$$= \begin{pmatrix} A(1:N-1, 1:N-1) & 0 \\ p(N)Q_{N-1} & d(N) \end{pmatrix} = A(1:N, 1:N) = A. \qquad \square$$

The inverse statement is also valid.

Lemma 20.2. *Let A be a block matrix with entries of sizes $m_i \times n_j,\ i,j = 1,\dots,N$ and let be given the factorization*

$$A = \tilde{A}_N \tilde{A}_{N-1} \cdots \tilde{A}_1 \tag{20.6}$$

with

$$\tilde{A}_1 = \operatorname{diag}\{A_1, I_{\gamma_1}\},\ \tilde{A}_k = \operatorname{diag}\{I_{\eta_k}, A_k, I_{\gamma_k}\},\ k = 2,\dots,N-1,$$
$$\tilde{A}_N = \operatorname{diag}\{I_{\eta_N}, A_N\},$$

where $\eta_k = \sum_{i=1}^{k-1} m_i,\ \gamma_k = \sum_{i=k+1}^{N} n_i$, and with matrices $A_1,\ A_k\ (k = 2,\dots,N-1),\ A_N$ of sizes $(m_1 + r_1^L) \times n_1,\ (m_k + r_k^L) \times (n_k + r_{k-1}^L)\ (k = 2,\dots,N-1),\ m_N \times (n_N + r_{N-1}^L)$, respectively. Assume that the matrices $A_k\ (k = 1,\dots,N)$ are partitioned in the form

$$A_1 = \begin{pmatrix} d(1) \\ q(1) \end{pmatrix}; \qquad A_k = \begin{pmatrix} p(k) & d(k) \\ a(k) & q(k) \end{pmatrix},$$
$$k = 2,\dots,N-1; \quad A_N = \begin{pmatrix} p(N) & d(N) \end{pmatrix}, \tag{20.7}$$

with submatrices $d(k)\ (k = 1,\dots,N),\ p(i)\ (i = 2,\dots,N),\ q(j)\ (j = 1,\dots,N-1),\ a(k)\ (k = 2,\dots,N-1)$ of sizes $m_k \times n_k,\ m_i \times r_{i-1}^L,\ r_j^L \times n_j,\ r_k^L \times r_{k-1}^L$, respectively.

Then A is a block lower triangular matrix with lower quasiseparable generators $p(i)\ (i = 2,\dots,N),\ q(j)\ (j = 1,\dots,N-1),\ a(k)\ (k = 2,\dots,N-1)$ of orders $r_k^L\ (k = 1,\dots,N-1)$ and diagonal entries $d(k)\ (k = 1,\dots,N)$.

Proof. Using the elements $d(k), p(k), q(k), a(k)$ from the partitions (20.7), define the block lower triangular matrix $A' = \{A'_{ij}\}_{i,j=1}^N$ by

$$A'_{ij} = \begin{cases} p(i)a^{>}_{ij}q(j), & 1 \le j < i \le N, \\ d(i), & 1 \le i = j \le N, \\ 0, & 1 \le i < j \le N. \end{cases}$$

This means that $p(i)\ (i = 2,\dots,N),\ q(j)\ (j = 1,\dots,N-1),\ a(k)\ (k = 2,\dots,N-1)$ are lower quasiseparable generators of orders $r_k^L\ (k = 1,\dots,N-1)$ of the matrix A' and $d(k)\ (k = 1,\dots,N)$ are diagonal entries of A'. Using Lemma 20.1

and the formula (20.6), we conclude that $A = A'$ and therefore A is a block lower triangular matrix with these lower quasiseparable generators and diagonal entries $d(k)$ $(k = 1, \dots, N)$. $\square$

Similar results are valid for block upper triangular matrices and are obtained by passing to transposed matrices.

Lemma 20.3. *Let A be a block upper triangular matrix with entries of sizes $m_i \times n_j$ and upper quasiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ and diagonal entries $d(k)$ $(k = 1, \dots, N)$. Using these quasiseparable generators and diagonal entries define matrices*

$$A_1 = \begin{pmatrix} d(1) & g(1) \end{pmatrix}, \quad A_k = \begin{pmatrix} h(k) & b(k) \\ d(k) & g(k) \end{pmatrix}, \qquad k = 2, \dots, N-1, \quad A_N = \begin{pmatrix} h(N) \\ d(N) \end{pmatrix}, \tag{20.8}$$

and then set

$$\tilde{A}_1 = \operatorname{diag}\{A_1, I_{\phi_1}\}; \quad \tilde{A}_k = \operatorname{diag}\{I_{\chi_k}, A_k, I_{\phi_k}\}, \qquad k = 2, \dots, N-1; \quad \tilde{A}_N = \operatorname{diag}\{I_{\chi_N}, A_N\}, \tag{20.9}$$

where $\chi_k = \sum_{i=1}^{k-1} n_i$, $\phi_k = \sum_{i=k+1}^{N} m_i$.

Then

$$A = \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_N. \tag{20.10}$$

Lemma 20.4. *Let A be a block matrix with entries of sizes $m_i \times n_j$, $i, j = 1, \dots, N$, and let be given the factorization*

$$A = \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_N,$$

with

$$\tilde{A}_1 = \operatorname{diag}\{A_1, I_{\phi_1}\}, \quad \tilde{A}_k = \operatorname{diag}\{I_{\chi_k}, A_k, I_{\phi_k}\}, \qquad k = 2, \dots, N-1, \quad \tilde{A}_N = \operatorname{diag}\{I_{\chi_N}, A_N\},$$

where $\chi_k = \sum_{i=1}^{k-1} n_i$, $\phi_k = \sum_{i=k+1}^{N} m_i$, and with matrices A_1, A_k $(k = 2, \dots, N-1)$, A_N of sizes $m_1 \times (n_1 + r_1^U)$, $(m_k + r_{k-1}^U) \times (n_k + r_k^U)$ $(k = 2, \dots, N-1)$, $(m_N + r_{N-1}^U) \times n_N$ respectively. Assume that the matrices A_k $(k = 1, \dots, N)$ are partitioned in the form

$$A_1 = \begin{pmatrix} d(1) & g(1) \end{pmatrix}, \quad A_k = \begin{pmatrix} h(k) & b(k) \\ d(k) & g(k) \end{pmatrix}, \qquad k = 2, \dots, N-1, \quad A_N = \begin{pmatrix} h(N) \\ d(N) \end{pmatrix},$$

with submatrices $d(k)$ $(k = 1, \dots, N)$, $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ of sizes $m_k \times n_k$, $m_i \times r_i^U$, $r_{j-1}^U \times n_j$, $r_{k-1}^U \times r_k^U$, respectively.

Then A is a block upper triangular matrix with upper quasiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ of orders r^U_k $(k = 1, \dots, N-1)$ and diagonal entries $d(k)$ $(k = 1, \dots, N)$.

§20.2 The first factorization theorem

Let A be a block matrix with given quasiseparable generators. We present here an algorithm for computing generators and diagonal entries of a unitary block lower triangular matrix V and a block upper triangular matrix T such that $A = VT$.

Theorem 20.5. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with entries of sizes $m_i \times n_j$, lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r^L_k $(k = 1, \dots, N-1)$, upper quasiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ of orders r^U_k $(k = 1, \dots, N-1)$, and diagonal entries $d(k)$ $(k = 1, \dots, N)$. Set*

$$\begin{aligned} &\rho_N = 0, \;\; \rho_{k-1} = \min\{m_k + \rho_k, \; r^L_{k-1}\}, \quad k = N, \dots, 2, \; \rho_0 = 0, \\ &\nu_k = m_k + \rho_k - \rho_{k-1}, \; \rho'_k = r^U_k + \rho_k, \qquad k = 1, \dots, N \end{aligned} \tag{20.11}$$

and

$$\eta_k = \sum_{i=1}^{k-1} m_i, \quad \phi_k = \sum_{i=k+1}^{N} \nu_i.$$

Then the matrix A admits the factorization

$$A = VT, \tag{20.12}$$

where V is a block lower triangular unitary matrix with block entries of sizes $m_i \times \nu_j$ $(i, j = 1, \dots, N)$ and T is a block upper triangular matrix with block entries of sizes $\nu_i \times n_j$ $(i, j = 1, \dots, N)$. The matrix V is determined via the relations

$$V = \tilde{V}_N \tilde{V}_{N-1} \cdots \tilde{V}_1, \tag{20.13}$$

where

$$\begin{aligned} &\tilde{V}_1 = \mathrm{diag}\{V_1, I_{\phi_1}\}, \; \tilde{V}_k = \mathrm{diag}\{I_{\eta_k}, V_k, I_{\phi_k}\}, \\ &k = 2, \dots, N-1, \; \tilde{V}_N = \mathrm{diag}\{I_{\eta_N}, V_N\}, \end{aligned}$$

with $(m_k + \rho_k) \times (m_k + \rho_k)$ unitary matrices obtained by means of the following algorithm.

1. *Using* QR *factorization or another method, compute the factorization*

$$p(N) = V_N \begin{pmatrix} X_N \\ 0_{\nu_N \times r^L_{N-1}} \end{pmatrix}, \tag{20.14}$$

where V_N is a unitary matrix of size $m_N \times m_N$ and X_N is a matrix of size $\rho_{N-1} \times r^L_{N-1}$.

2. *For $k = N-1, \dots, 2$ using* QR *factorization or another method, compute the factorization*

$$\begin{pmatrix} p(k) \\ X_{k+1}a(k) \end{pmatrix} = V_k \begin{pmatrix} X_k \\ 0_{\nu_k \times r^L_{k-1}} \end{pmatrix}, \tag{20.15}$$

where V_k is a unitary matrix of size $(m_k + \rho_k) \times (m_k + \rho_k)$ and X_k is a matrix of size $\rho_{k-1} \times r^L_{k-1}$.

3. *Set V_1 to be a $\nu_1 \times \nu_1$ unitary matrix.*

Moreover, lower quasiseparable generators $p_V(i)$ $(i = 2, \dots, N)$, $q_V(j)$ $(j = 1, \dots, N-1)$, $a_V(k)$ $(k = 2, \dots, N-1)$ of orders ρ_k $(k = 1, \dots, N-1)$, and diagonal entries $d_V(k)$ $(k = 1, \dots, N)$ of the matrix V, which are matrices of sizes $m_i \times \rho_{i-1}, \rho_j \times \nu_j, \rho_k \times \rho_{k-1}$ and $m_k \times \nu_k$, respectively, are determined from the partitions

$$V_N = \begin{pmatrix} p_V(N) & d_V(N) \end{pmatrix}, \tag{20.16}$$

$$V_k = \begin{pmatrix} p_V(k) & d_V(k) \\ a_V(k) & q_V(k) \end{pmatrix}, \quad k = N-1, \dots, 2, \tag{20.17}$$

$$V_1 = \begin{pmatrix} d_V(1) \\ q_V(1) \end{pmatrix}. \tag{20.18}$$

Furthermore, upper quasiseparable generators $g_T(i)$ $(i = 1, \dots, N-1)$, $h_T(j)$ $(j = 2, \dots, N)$, $b_T(k)$ $(k = 2, \dots, N-1)$ of orders ρ'_k $(k = 1, \dots, N-1)$ are determined by the formulas

$$\tilde{h}(N) = p^*_V(N)d(N), \quad h_T(N) = \begin{pmatrix} h(N) \\ \tilde{h}(N) \end{pmatrix}, \quad d_T(N) = d^*_V(N)d(N), \tag{20.19}$$

$$\begin{gathered} g_T(k) = \begin{pmatrix} d^*_V(k)g(k) & q^*_V(k) \end{pmatrix}, \ d_T(k) = d^*_V(k)d(k) + q^*_V(k)X_{k+1}q(k), \\ \tilde{h}(k) = p^*_V(k)d(k) + a^*_V(k)X_{k+1}q(k), \quad h_T(k) = \begin{pmatrix} h(k) \\ \tilde{h}(k) \end{pmatrix}, \\ b_T(k) = \begin{pmatrix} b(k) & 0 \\ p^*_V(k)g(k) & a^*_V(k) \end{pmatrix}, \quad k = N-1, \dots, 2, \end{gathered} \tag{20.20}$$

$$d_T(1) = d^*_V(1)d(1) + q^*_V(1)X_2q(1), \quad g_T(1) = \begin{pmatrix} d^*_V(1)g(1) & q^*_V(1) \end{pmatrix}. \tag{20.21}$$

Proof. By Lemma 20.2, (20.13) implies that V is a block lower triangular matrix with lower quasiseparable generators $p_V(i)$ $(i = 2, \dots, N)$, $q_V(j)$ $(j = 1, \dots, N-1)$, $a_V(k)$ $(k = 2, \dots, N-1)$, and diagonal entries $d_V(k)$ $(k = 1, \dots, N)$ defined in (20.16), (20.17), (20.18). Hence, V^* is a block upper triangular matrix with entries of sizes $\nu_i \times m_j$ $(i, j = 1, \dots, N)$, with upper quasiseparable generators $q^*_V(i)$ $(i = 1, \dots, N-1)$, $p^*_V(j)$ $(j = 2, \dots, N)$, $a^*_V(k)$ $(k = 2, \dots, N-1)$ and diagonal entries $d^*_V(k)$ $(k = 1, \dots, N)$. Consider the matrix $T = V^*A$. Applying

Corollary 17.8 with

$$\begin{aligned}
&g^{(1)}(k) = q_V^*(k),\ h^{(1)}(k) = p_V^*(k),\ b^{(1)}(k) = a_V^*(k), \quad d^{(1)}(k) = d_V^*(k),\\
&p^{(2)}(k) = p(k), \quad q^{(2)}(k) = q(k), \quad a^{(2)}(k) = a(k),\\
&g^{(2)}(k) = g(k), \quad h^{(2)}(k) = h(k), \quad b^{(2)}(k) = b(k), \qquad d^{(2)}(k) = d(k)
\end{aligned}$$

and using the notation X_k instead of γ_k for the auxiliary variables, we obtain the formulas in (20.20) and (20.21) for upper generators $b_T(k)$ and $g_T(k)$, the formula (20.19) and the relations

$$X_N = p_V^*(N)p(N), \quad p_T(N) = d_V^*(N)p(N), \tag{20.22}$$

$$\begin{pmatrix} X_k & \tilde{h}(k) \\ p_T(k) & d_T(k) \end{pmatrix} = \begin{pmatrix} p_V^*(k) & a_V^*(k) \\ d_V^*(k) & q_V^*(k) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & X_{k+1} \end{pmatrix} \begin{pmatrix} p(k) & d(k) \\ a(k) & q(k) \end{pmatrix},$$
$$k = N-1, \dots, 1, \tag{20.23}$$

$$h_T(k) = \begin{pmatrix} h(k) \\ \tilde{h}(k) \end{pmatrix}, \quad k = 2, \dots, N \tag{20.24}$$

for the upper generators $h_T(k)$, diagonal entries $d_T(k)$ and lower generators $p_T(k)$. Here γ_k are auxiliary variables.

From (20.23), (20.24) we obtain the formulas (20.20) and (20.21) for $h_T(k)$ and $d_T(k)$ and the relations

$$\begin{aligned}
X_k &= p_V^*(k)p(k) + a_V^*(k)X_{k+1}a(k),\\
p_T(k) &= d_V^*(k)p(k) + q_V^*(k)X_{k+1}a(k), \quad k = N-1, \dots, 2.
\end{aligned} \tag{20.25}$$

It remains to check that T is a block lower triangular matrix. Combining (20.14), (20.15) and (20.16), (20.17) with (20.22), (20.25), we conclude that $p_T(k) = 0,\ k = 2, \dots, N$. Hence, T is lower triangular. □

Corollary 20.6. *Under the conditions of Theorem* 20.5*, the orders of lower quasiseparable generators of the matrix V are not greater than the corresponding orders of lower quasiseparable generators of the matrix A and the orders of upper quasiseparable generators of the matrix T are not greater than the sum of the corresponding orders of lower and upper quasiseparable generators of A:*

$$\rho_k \le r_k^L, \quad \rho'_k \le r_k^L + r_k^U, \quad k = 1, \dots, N-1. \tag{20.26}$$

Furthermore, if A has quasiseparable order (ρ_L, ρ_U), the matrix V has the lower quasiseparable order ρ_L at most and the matrix T has the upper quasiseparable order $\rho_L + \rho_U$ at most.

Proof. The inequalities (20.26) follow directly from the relations

$$\rho_{N-1} = \min\{m_N,\ r_{N-1}^L\}, \quad \rho_{k-1} = \min\{m_k + \rho_k, r_{k-1}^L\},\ k = N-1, \dots, 2,$$
$$\rho'_k = r_k^U + \rho_k, \qquad k = 1, \dots, N-1.$$

Taking quasiseparable generators of A with minimal orders, Corollary 5.11 yields

$$r_k^L \leq \rho_L, \quad r_k^U \leq \rho_U, \qquad k = 1, \dots, N-1,$$

and therefore

$$\rho_k \leq \rho_L, \quad \rho'_k \leq \rho_L + \rho_U, \quad k = 1, \dots, N-1.$$

Hence, the lower quasiseparable order of V does not exceed ρ_L and the upper quasiseparable order of T does not exceed $\rho_L + \rho_U$. □

§20.3 The second factorization theorem

In this section we consider the factorization of a block upper triangular matrix T with given generators in the form $T = UR$ with a block upper triangular unitary matrix U and a block upper triangular matrix R with square blocks on the main diagonal.

Theorem 20.7. *Let $T = \{T_{ij}\}_{i,j=1}^N$ be a block upper triangular matrix with entries of sizes $\nu_i \times n_j$ such that*

$$s_k := \sum_{i=1}^{k} (\nu_i - n_i) \geq 0, \ k = 1, \dots, N-1, \quad s_N = \sum_{i=1}^{N} (\nu_i - n_i) = 0, \tag{20.27}$$

and with upper quasiseparable generators $g_T(i)$ $(i = 1, \dots, N-1)$, $h_T(j)$ $(j = 2, \dots, N)$, $b_T(k)$ $(k = 2, \dots, N-1)$ of orders ρ'_k $(k = 1, \dots, N-1)$ and diagonal entries $d_T(k)$ $(k = 1, \dots, N)$.

Then T admits the factorization

$$T = UR, \tag{20.28}$$

where U is a block upper triangular unitary matrix with block entries of sizes $\nu_i \times n_j$ $(i, j = 1, \dots, N)$ with upper quasiseparable generators $g_U(i)$ $(i = 1, \dots, N-1)$, $h_U(j)$ $(j = 2, \dots, N)$, $b_U(k)$ $(k = 2, \dots, N-1)$ of orders $s_k = \sum_{i=1}^{k} (\nu_i - n_i)$ $(k = 1, \dots, N-1)$ and diagonal entries $d_U(k)$ $(k = 1, \dots, N)$, and R is a block upper triangular matrix with block entries of sizes $n_i \times n_j$ $(i, j = 1, \dots, N)$, upper quasiseparable generators $g_R(i)$ $(i = 1, \dots, N-1)$, $h_R(j)$ $(j = 2, \dots, N)$, $b_R(k)$ $(k = 2, \dots, N-1)$ of orders ρ'_k $(k = 1, \dots, N-1)$ and square upper triangular diagonal entries $d_R(k)$ $(k = 1, \dots, N)$.

Quasiseparable generators and the diagonal entries of the matrices U, R are determined using the following algorithm.

1. *Compute $s_1 = \nu_1 - n_1$. Compute the* QR *factorization*

$$d_T(1) = U_1 \begin{pmatrix} d_R(1) \\ 0_{s_1 \times n_1} \end{pmatrix}, \tag{20.29}$$

where U_1 *is a* $\nu_1 \times \nu_1$ *unitary matrix and* $d_R(1)$ *is an upper triangular* $n_1 \times n_1$ *matrix.*

Determine the matrices $d_U(1)$, $g_U(1)$ *of sizes* $\nu_1 \times n_1$, $\nu_1 \times s_1$ *from the partition*

$$U_1 = \begin{pmatrix} d_U(1) & g_U(1) \end{pmatrix}. \tag{20.30}$$

Compute

$$g_R(1) = d_U^*(1) g_T(1), \quad Y_1 = g_U^*(1) g_T(1). \tag{20.31}$$

2. *For* $k = 2, \dots, N-1$, *perform the following. Compute* $s_k = s_{k-1} + \nu_k - n_k$. *Compute the* QR *factorization*

$$\begin{pmatrix} Y_{k-1} h_T(k) \\ d_T(k) \end{pmatrix} = U_k \begin{pmatrix} d_R(k) \\ 0_{s_k \times n_k} \end{pmatrix}, \tag{20.32}$$

where U_k *is an* $(n_k + s_k) \times (n_k + s_k)$ *unitary matrix and* $d_R(k)$ *is an* $n_k \times n_k$ *upper triangular matrix.*

Determine the matrices $d_U(k)$, $g_U(k)$, $h_U(k)$, $b_U(k)$ *of sizes* $\nu_k \times n_k$, $\nu_k \times s_k$, $s_{k-1} \times n_k$, $s_{k-1} \times s_k$ *from the partition*

$$U_k = \begin{pmatrix} h_U(k) & b_U(k) \\ d_U(k) & g_U(k) \end{pmatrix}. \tag{20.33}$$

Compute

$$g_R(k) = h_U^*(k) Y_{k-1} b_T(k) + d_U^*(k) g_T(k), \quad Y_k = b_U^*(k) Y_{k-1} b_T(k) + g_U^*(k) g_T(k), \tag{20.34}$$

where U_k *is an* $(n_k + s_k) \times (n_k + s_k)$ *unitary matrix,* $d_R(k)$ *is an* $n_k \times n_k$ *upper triangular matrix, and* $g_R(k)$ *and* Y_k *are matrices of sizes* $n_k \times \rho'_k$ *and* $s_k \times \rho'_k$, *respectively.*

Set

$$h_R(k) = h_T(k), \quad b_R(k) = b_T(k). \tag{20.35}$$

3. *Compute the* QR *factorization*

$$\begin{pmatrix} Y_{N-1} h_T(N) \\ d_T(N) \end{pmatrix} = U_N d_R(N), \tag{20.36}$$

where U_N *is a unitary matrix of sizes* $(\nu_N + s_{N-1}) \times (\nu_N + s_{N-1})$ *and* $d_R(N)$ *is an upper triangular matrix of sizes* $n_N \times n_N$.

Set

$$h_R(N) = h_T(N). \tag{20.37}$$

Determine the matrices $d_U(N)$, $h_U(N)$ *of sizes* $\nu_N \times n_N$, $s_{N-1} \times n_N$ *from the partition*

$$U_N = \begin{pmatrix} h_U(N) \\ d_U(N) \end{pmatrix}. \tag{20.38}$$

Proof. From (20.27) it follows that all the numbers s_k are nonnegative and

$$\nu_1 = s_1 + n_1, \quad \nu_k + s_{k-1} = n_k + s_k,\ k = 2, \dots, N-1, \quad \nu_N + s_{N-1} = n_N.$$

Let $U_k,\ k = 1, \dots, N$, be the unitary matrices defined in (20.29), (20.32), (20.36). Set

$$\tilde{U}_1 = \operatorname{diag}\{U_1, I_{\phi_1}\}, \quad \tilde{U}_k = \operatorname{diag}\{I_{\chi_k}, U_k, I_{\phi_k}\},$$
$$k = 2, \dots, N-1, \quad \tilde{U}_N = \operatorname{diag}\{I_{\chi_N}, U_N\},$$

with $\chi_k = \sum_{i=1}^{k-1} n_i, \quad \phi_k = \sum_{i=k+1}^{N} \nu_i$, and then set

$$U = \tilde{U}_1 \tilde{U}_2 \cdots \tilde{U}_N. \tag{20.39}$$

Since the matrices U_k are unitary, all matrices $\tilde{U}_k$ are also unitary and hence so is the matrix U. Moreover, from Lemma 20.4 it follows that U is a block upper triangular matrix with upper quasiseparable generators $g_U(i)$ $(i = 1, \dots, N-1)$, $h_U(j)$ $(j = 2, \dots, N)$, $b_U(k)$ $(k = 2, \dots, N-1)$ and diagonal entries $d_U(k)$ $(k = 1, \dots, N)$ defined in (20.30), (20.33), (20.38). Hence U^* is a block lower triangular unitary matrix with entries of sizes $n_i \times \nu_j$ $(i, j = 1, \dots, N)$ and with lower quasiseparable generators $h_U^*(i)$ $(i = 2, \dots, N)$, $g_U^*(j)$ $(j = 1, \dots, N-1)$, $b_U^*(k)$ $(k = 2, \dots, N-1)$ and diagonal entries $d_U^*(k)$ $(k = 1, \dots, N)$.

We apply Theorem 17.4 to determine quasiseparable generators of the product $R = U^*T$. Using (17.8) we obtain the formulas (20.35), (20.37) for the generators $h_R(k), b_R(k)$. Next, using the recursion relations (17.9), (17.10) we get

$$\begin{pmatrix} d_R(1) & g_R(1) \\ q_R(1) & Y_1 \end{pmatrix} = U_1^* \begin{pmatrix} d_T(1) & g_T(1) \end{pmatrix}, \tag{20.40}$$

$$\begin{pmatrix} d_R(k) & g_R(k) \\ q_R(k) & Y_k \end{pmatrix} = U_k^* \begin{pmatrix} Y_{k-1} h_T(k) & Y_{k-1} b_T(k) \\ d_T(k) & g_T(k) \end{pmatrix},$$
$$k = 2, \dots, N-1, \tag{20.41}$$

$$d_R(N) = U_N^* \begin{bmatrix} Y_{N-1} h_T(N) \\ d_T(N) \end{bmatrix}, \tag{20.42}$$

with the auxiliary variables Y_k, which are $s_k \times \rho'_k$ matrices. From (20.40) and (20.41) we obtain the formulas

$$\begin{pmatrix} d_R(1) \\ q_R(1) \end{pmatrix} = U_1^* d_T(1), \quad \begin{pmatrix} d_R(k) \\ q_R(k) \end{pmatrix} = U_k^* \begin{pmatrix} Y_{k-1} h_T(k) \\ d_T(k) \end{pmatrix},\ k = 2, \dots, N-1 \tag{20.43}$$

for the diagonal entries $d_R(k)$ and the lower quasiseparable generators $q_R(k)$, and

$$\begin{pmatrix} g_R(1) \\ Y_1 \end{pmatrix} = U_1^* g_T(1), \quad \begin{pmatrix} g_R(k) \\ Y_k \end{pmatrix} = U_k^* \begin{pmatrix} Y_{k-1} b_T(k) \\ g_T(k) \end{pmatrix},\ k = 2, \dots, N-1, \tag{20.44}$$

for determining upper quasiseparable generators $g_T(k)$ and auxiliary variables Y_k. Comparing (20.43), (20.42) with (20.29), (20.32), (20.38), we conclude that $q_R(k) = 0,\ k = 1, \dots, N-1$, and $d_R(k),\ k = 1, \dots, N$ are upper triangular matrices. Hence, R is an upper triangular matrix. From (20.44) using the partitions (20.30), (20.33) we obtain the formulas (20.31), (20.34).

From the equalities $b_R(j) = b_T(j), j = 2, \dots, N-1$ it follows that the orders of the upper quasiseparable generators of the matrix R are equal to the corresponding orders ρ'_k $(k = 1, \dots, N-1)$ of upper quasiseparable generators of the matrix T. □

Remark. The condition (20.27) of Theorem 20.7 holds if the matrix T is invertible. Indeed, consider the submatrix $T(:, 1:k)$ composed of the first k block columns of T. We have $T(:, 1:k) = \begin{pmatrix} T_k \\ 0 \end{pmatrix}$, where T_k is a matrix of size $(\sum_{i=1}^k \nu_i) \times (\sum_{i=1}^k n_i)$. From the invertibility of the matrix T it follows that $\operatorname{rank} T_k = \sum_{i=1}^k n_i$ and thus $\sum_{i=1}^k n_i \le \sum_{i=1}^k \nu_i,\ 1 \le k \le N-1$. Next, since T is invertible, it is a square matrix and hence $\sum_{i=1}^N \nu_i = \sum_{i=1}^N n_i$. Moreover one can see easily that in the case where T is invertible the diagonal blocks $d_R(k)$ $(k = 1, \dots, N)$ are invertible matrices.

Below we will show that the condition (20.27) is satisfied if the matrix T is obtained via the factorization (20.12), where the original matrix A has the sizes of blocks $m_i \times m_j,\ i, j = 1, \dots, N$. Moreover, in this case the numbers $s_k = \sum_{i=1}^k (\nu_i - n_i)$ $(k = 1, \dots, N-1)$ are equal to the corresponding orders ρ_k of lower quasiseparable generators of the matrix V.

Corollary 20.8. *Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block matrix with entries of sizes $m_i \times m_j$ (with square blocks on the main diagonal). Let V, T be the matrices obtained in Theorem 20.5 and U, R be the matrices obtained in Theorem 20.7.*

Then the orders of upper quasiseparable generators of the matrix U are equal to the corresponding orders of lower quasiseparable generators of the matrix V:

$$s_k = \rho_k, \quad k = 1, \dots, N. \tag{20.45}$$

Furthermore, if A has quasiseparable order (ρ_L, ρ_U), then the matrix U has the lower quasiseparable order ρ_L at most and the matrix R has the upper quasiseparable order $\rho_L + \rho_U$ at most.

Proof. The orders s_k $(k = 1, \dots, N-1)$ of upper quasiseparable generators of the matrix U are obtained via the relations

$$s_k = \sum_{i=1}^k (\nu_i - n_i), \quad k = 1, \dots, N-1,$$

with

$$\nu_1 = m_1 + \rho_1, \quad \nu_k = m_k + \rho_k - \rho_{k-1},\ k = 2, \dots, N-1,$$

where ρ_k $(k = 1, \ldots, N-1)$ are the orders of lower quasiseparable generators of the matrix V. By the condition of the corollary, $m_i = n_i$ $(i = 1, \ldots, N)$ which implies (20.45). Furthermore, using the fact that the orders of upper quasiseparable generators of the matrices T and R coincide and the second part of Corollary 20.6 we conclude that the matrix U has the lower quasiseparable order ρ_L at most and the matrix R has the upper quasiseparable order $\rho_L + \rho_U$ at most. □

§20.4 Solution of linear systems

Let us now consider the system $Ax = y$ of linear algebraic equations with block invertible matrix A with given quasiseparable generators. Using Theorems 20.5, 20.7, Algorithm 13.1 and the algorithm from Theorem 13.13 we obtain the following algorithm.

Algorithm 20.9. Let $A = \{A_{ij}\}_{i,j=1}^N$ be a block invertible matrix with entries of sizes $m_i \times n_j$, lower quasiseparable generators $p(i)$ $(i = 2, \ldots, N)$, $q(j)$ $(j = 1, \ldots, N-1)$, $a(k)$ $(k = 2, \ldots, N-1)$ of orders r_k^L $(k = 1, \ldots, N-1)$, upper quasiseparable generators $g(i)$ $(i = 1, \ldots, N-1)$, $h(j)$ $(j = 2, \ldots, N)$, $b(k)$ $(k = 2, \ldots, N-1)$ of orders r_k^U $(k = 1, \ldots, N-1)$ and diagonal entries $d(k)$ $(k = 1, \ldots, N)$. Then solution of the system $Ax = y$ is computed as follows.

1. Using the algorithm from Theorem 20.5, compute quasiseparable generators $p_V(i)$ $(i = 2, \ldots, N)$, $q_V(j)$ $(j = 1, \ldots, N-1)$, $a_V(k)$ $(k = 2, \ldots, N-1)$, $g_T(i)$ $(i = 1, \ldots, N-1)$, $h_T(j)$ $(j = 2, \ldots, N)$, $b_T(k)$ $(k = 2, \ldots, N-1)$ and diagonal entries $d_V(k)$, $d_T(k)$ $(k = 1, \ldots, N)$ of the block lower triangular unitary matrix V and the block upper triangular matrix T such that $A = VT$.
2. Using the algorithm from Theorem 20.7, compute quasiseparable generators $g_U(i)$ $(i = 1, \ldots, N-1)$, $h_U(j)$ $(j = 2, \ldots, N)$, $b_U(k)$ $(k = 2, \ldots, N-1)$, $g_R(i)$ $(i = 1, \ldots, N-1)$, $h_R(j)$ $(j = 2, \ldots, N)$, $b_R(k)$ $(k = 2, \ldots, N-1)$ and diagonal entries $d_U(k)$, $d_R(k)$ $(k = 1, \ldots, N)$ of the block upper triangular unitary matrix U and the block upper triangular matrix R with invertible diagonal entries such that $T = UR$.
3. Compute the product $\tilde{x} = V^* y$ as follows: start with $\tilde{x}(N) = d_V^*(N)y(N)$, $w_{N-1} = p_V^*(N)y(N)$, $\tilde{x}(N-1) = q_V^*(N-1)w_{N-1} + d_V^*(N-1)y(N-1)$, and for $i = N-2, \ldots, 1$ compute recursively
$$w_i = a_V^*(i+1)w_{i+1} + p_V^*(i+1)y(i+1), \quad \tilde{x}(i) = q_V^*(i)w_i + d_V^*(i)y(i).$$
4. Compute the product $\hat{x} = U^* \tilde{x}$ as follows: start with $\hat{x}(1) = d_U^*(1)\tilde{x}(1)$, $z_2 = g_U^*(1)\tilde{x}(1)$, $\hat{x}(2) = h_U^*(2)z_2 + d_U^*(2)\tilde{x}(2)$, and for $i = 3, \ldots, N$ compute recursively
$$z_i = b_U^*(i-1)z_{i-1} + g_U^*(i-1)\tilde{x}(i-1), \quad \hat{x}(i) = d_U^*(i)\tilde{x}(i) + h_U^*(i)z_i.$$
5. Compute the solution x of the equation $Rx = \hat{x}$ as follows: start with $x(N) = (d_R(N))^{-1}\hat{x}(N)$, $\eta_{N-1} = h_R(N)\hat{x}(N)$, and for $i = N-1, \ldots, 2$ compute

recursively

$$\eta_{i-1} = b_R(i)\eta_i + h_R(i)x(i), \quad x(i) = (d_R(i))^{-1}(\hat{x}(i) - g_R(i)\eta_i),$$

and finally compute $x(1) = (d_R(1))^{-1}(\hat{x}(1) - g_R(1)\eta_1)$.

Here in Steps 3 and 4 we used Algorithm 13.1 for the upper triangular matrix V^* with upper quasiseparable generators $q_V^*(i),\ (i = 1,\dots,N-1),\ p_V^*(j),\ (j = 2,\dots,N),\ a_V^*(k)\ (k = 2,\dots,N-1)$ and diagonal entries $d_V^*(k)\ (k = 1,\dots,N)$ and for the lower triangular matrix U^* with lower quasiseparable generators $h_U^*(i)\ (i = 2,\dots,N),\ g_U^*(j)\ (j = 1,\dots,N-1),\ b_U^*(k)\ (k = 2,\dots,N-1)$ and diagonal entries $d_U^*(k)\ (k = 1,\dots,N)$. Computations in Steps 3 and 4 may be performed also based on the relation (20.13) for the matrix V and the relation (20.39) for the matrix U. In Step 5 we apply the algorithm from Theorem 13.13 to the upper triangular matrix R.

§20.5 Complexity

Let us estimate the costs of computations in the Algorithm 20.9 presented above. In Step 1, i.e., in the algorithm from Theorem 20.5, costs are determined by the relations (20.15) and (20.20). In (20.15), computing of the product $X_{k+1}a(k)$ requires $\rho_k r_k^L r_{k-1}^L$ operations of arithmetical multiplication and less operations of arithmetical addition, and the QR factorization costs $\vartheta(m_k+\rho_k, r_{k-1}^L)$ operations. Here $\vartheta(m,n)$ means the complexity of QR factorization for a matrix of size $m \times n$. In (20.20), the computation of the products $p_V^*(k)d(k),\ a_V^*(k)X_{k+1}q(k),\ p_V^*(k)g(k),\ d_V^*(k)g(k),\ d_V^*(k)d(k),\ q_V^*(k)X_{k+1}q(k)$ cost respectively $\rho_{k-1}m_kn_k,\ \rho_{k-1}\rho_kr_k^Ln_k,\ \rho_{k-1}m_kr_k^U,\ \nu_km_kr_k^U,\ \nu_km_kn_k,\ \nu_k\rho_kr_k^Ln_k$ operations of arithmetical multiplication and less operations of arithmetical addition.

Thus the total complexity of Step 1 is

$$c_1 < \sum_{k=1}^{N} \big[\vartheta(m_k+\rho_k, r_{k-1}^L) + 2(\rho_{k-1}m_kn_k + \rho_{k-1}\rho_kr_k^Ln_k + \rho_{k-1}m_kr_k^U + \nu_km_kr_k^U + \nu_km_kn_k + \nu_k\rho_kr_k^Ln_k)\big]$$

operations. In Step 2, i.e., in the algorithm from Theorem 20.7, costs are determined by the relations (20.32). Computation of the products $Y_{k-1}h_T(k)$ and $Y_{k-1}b_T(k)$ costs less than $2s_{k-1}\rho_{k-1}^Ln_k$ and $2s_{k-1}\rho_{k-1}^L\rho_k^L$ arithmetical operations, respectively, the computation of the QR factorization costs $\vartheta(s_{k-1}+\nu_k, n_k+\rho_k^L)$ operations. Thus the total complexity of Step 2 is

$$c_2 < \sum_{k=1}^{N} [\vartheta(s_{k-1}+\nu_k, n_k+\rho_k^L) + 2s_{k-1}\rho_{k-1}^L(n_k+\rho_k^L)]$$

operations. In Step 3, we apply to the upper triangular matrix V^* the relation (13.8) with $m_k = \nu_k,\ n_k = m_k,\ r_k^U = \rho_k,\ r_k^L = 0$ and obtain the complexity

$$c_3 = \sum_{k=1}^{N} [\nu_k \rho_k + m_{k+1}\rho_k + \rho_k \rho_{k+1} + \nu_k m_k].$$

In Step 4, we apply to the lower triangular matrix U^* the relation (13.8) with $m_k = n_k,\ n_k = \nu_k,\ r_k^U = 0,\ r_k^L = s_k$ and obtain the complexity

$$c_4 = \sum_{k=1}^{N} [n_k s_{k-1} + \nu_{k-1} s_{k-1} + s_{k-1} s_{k-2} + n_k \nu_k].$$

And finally the complexity of Step 5 is given by

$$c_5 = \sum_{k=1}^{N} [n_k \rho'_k + n_{k+1}\rho'_k + \rho'_k \rho'_{k+1} + \zeta(n_k)],$$

where $\tilde{\zeta}(n)$ is the complexity of solving an $n \times n$ linear triangular system by the standard method. The total complexity of Algorithm 20.9 is the sum $c = c_1 + c_2 + c_3 + c_4 + c_5$.

Assume that the sizes of blocks $m_k,\ n_k$, the orders of quasiseparable generators $r_k^L,\ r_k^U$ of the matrix A and the values $\sum_{i=1}^{k}(m_i - n_i)$ are bounded by the numbers $m,\ r,\ s_0$, respectively, i.e.,

$$m_k,\ n_k \le m,\ k = 1, \ldots, N, \quad r_k^L,\ r_k^U \le r,\ \sum_{i=1}^{k}(m_i - n_i) \le s_0,\ k = 1, \ldots, N-1.$$

Then the following estimates are obtained. From the relation $\rho_{k-1} = \min\{m_k + \rho_k,\ r_{k-1}^L\}$ it follows that $\rho_k \le r$ and from the equality $\rho'_k = r_k^U + \rho_k$ we conclude that $\rho'_k \le 2r$. Next, we have

$$\sum_{k=1}^{N} \nu_k = \sum_{k=1}^{N} m_k \le mN$$

and from $\nu_k = m_k + \rho_k - \rho_{k-1}$ we conclude that

$$s_k = \sum_{i=1}^{k}(\nu_i - n_i) = \sum_{i=1}^{k}(m_i + \rho_i - \rho_{i-1} - n_i) = \rho_k + \sum_{i=1}^{k}(m_i - n_i) \le r + s_0,$$

$$\nu_k + s_{k-1} = m_k + \rho_k + \sum_{i=1}^{k-1}(m_i - n_i) \le m + r + s_0.$$

Using these relations the complexities $c_1,\ c_2,\ c_3,\ c_4,\ c_5$ are estimated as follows:

$$\begin{aligned}
c_1 &< (\vartheta(m+r,r) + 4rm^2 + 2r^3m + 2r^2m + 2m^3 + 2r^2m^2)N,\\
c_2 &< (\vartheta(m+r+s_0, n+2r) + 4(rm+2r^2)s_0 + 4r^2m + 8r^3)N,\\
c_3 &< 2(2mr + r^2 + m^2)N,\\
c_4 &\le N(2mr + r^2 + s_0(2mr + 2r + s_0) + m^2),\\
c_5 &\le N(4mr + 4r^2 + \tilde{\zeta}(m)).
\end{aligned}$$

Thus, the total complexity of Algorithm 20.9 is estimated as

$$\begin{aligned}
c < \big(&\vartheta(m+r,r) + \vartheta(r+m+s_0, m+r) + \tilde{\zeta}(m) + 4rm^2 + 2r^3m + 6r^2m\\
&+ 2r^2m^2 + 2m^3 + 8r^3 + 8mr + 6r^2 + 2m^2 + s_0(4mr + 4r^2 + 2r + s_0)\big)N.
\end{aligned}$$

Therefore, in this case Algorithm 20.9 has a linear $O(N)$ complexity.

Assume that the sizes of the blocks of the matrix A satisfy $m_k = n_k,\ k = 1,\dots,N$. Then since $s_0 = 0$ we conclude that

$$\begin{aligned}
c < \big(&\vartheta(m+r,r) + \vartheta(r+m, m+r) + \tilde{\zeta}(m) + 4rm^2 + 2r^3m + 6r^2m\\
&+ 2r^2m^2 + 2m^3 + 8r^3 + 8mr + 6r^2 + 2m^2\big)N.
\end{aligned} \tag{20.46}$$

§20.6 The case of scalar matrices

We consider here the case of a matrix $A = \{A_{ij}\}_{i,j=1}^N$ with scalar entries, i.e., $m_k = n_k = 1$. Let r_k^L $(k = 1,\dots,N-1)$ be the orders of lower quasiseparable generators of A. In the factorization $A = VUR$ the matrix R is a scalar upper triangular matrix and $V,\ U$ are unitary matrices. Thus we have here a special form of the QR factorization in which the unitary factor is represented as the product VU. The matrix $V = \{v_{ij}\}_{i,j=1}^N$ with scalar entries v_{ij} may be treated, by Theorem 20.5, as a block lower triangular matrix with blocks of sizes $1 \times \nu_k$. Here $\nu_k = 1 + \rho_k - \rho_{k-1}$ $(k = 1,\dots,N)$, where ρ_k are the orders of lower quasiseparable generators of the block matrix V which are defined by the relations $\rho_N = 0,\ \rho_{k-1} = \min\{1+\rho_k,\ r_{k-1}^L\}$ $(k = N-1,\dots,1)$. The fact that V is a block lower triangular matrix means that $v_{ij} = 0$ for $j > \sum_{k=1}^{i} \nu_k = i + \rho_i$. Similarly, for the unitary matrix $U = \{u_{ij}\}_{i,j=1}^N$, one has that it follows from Theorem 20.7 that $u_{ij} = 0$ for $i > j + \rho_j$. Moreover, by Corollary 20.8, the orders of upper quasiseparable generators of U equal ρ_k. If for some r holds $r_k^L \le r,\ k = 1,\dots,N-1$, we obtain $\rho_k \le r$ and hence the matrices V and U satisfy the relations $v_{ij} = 0$ for $j > i + r$ and $u_{ij} = 0$ for $i > j + r$.

In the case of scalar matrices we obtain the following specification of the factorization Theorems 20.5 and 20.7.

Theorem 20.10. *Let* $A = \{A_{ij}\}_{i,j=1}^N$ *be a scalar matrix with lower quasiseparable generators* $p(i)$ $(i = 2,\dots,N)$, $q(j)$ $(j = 1,\dots,N-1)$, $a(k)$ $(k = 2,\dots,N-1)$ *of* *orders* r_k^L $(k = 1, \dots, N-1)$, *upper quasiseparable generators* $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ *of orders* r_k^U $(k = 1, \dots, N-1)$, *and diagonal entries* $d(k)$ $(k = 1, \dots, N)$. *Let us define the numbers* ρ_k *via the recursion relations* $\rho_N = 0,\ \rho_0 = 0,\ \rho_{k-1} = \min\{1 + \rho_k,\ r_{k-1}^L\},\ k = N, \dots, 2$ *and set* $m_k = 1, n_k = 1, \nu_k = 1 + \rho_k - \rho_{k-1},\ \rho'_k = r_k^U + \rho_k,\ k = 1, \dots, N$.

The matrix A *admits the factorization*

$$A = VUR,$$

where V *is a unitary matrix represented in the block lower triangular form with blocks of sizes* $m_i \times \nu_j$ $(i, j = 1, \dots, N)$, *lower generators* $p_V(i)$ $(i = 2, \dots, N)$, $q_V(j)$ $(j = 1, \dots, N-1)$, $a_V(k)$ $(k = 2, \dots, N-1)$ *of orders* ρ_k $(k = 1, \dots, N-1)$, *and diagonal entries* $d_V(k)$ $(k = 1, \dots, N)$, U *is a unitary matrix represented in the block upper triangular form with blocks of sizes* $\nu_i \times n_j$ $(i, j = 1, \dots, N)$, *upper quasiseparable generators* $g_U(i)$ $(i = 1, \dots, N-1)$, $h_U(j)$ $(j = 2, \dots, N)$, $b_U(k)$ $(k = 2, \dots, N-1)$ *of orders* ρ_k $(k = 1, \dots, N-1)$, *and diagonal entries* $d_U(k)$ $(k = 1, \dots, N)$, *and* R *is an upper triangular scalar matrix with upper generators* $g_R(i)$ $(i = 1, \dots, N-1)$, $h_R(j)$ $(j = 2, \dots, N)$, $b_R(k)$ $(k = 2, \dots, N-1)$ *of orders* ρ'_k $(k = 1, \dots, N-1)$, *and diagonal entries* $d_R(k)$ $(k = 1, \dots, N)$.

The generators and the diagonal entries of the matrices V, U, R *are determined using the following algorithm.*

1.1. *Set* $V_N = 1$. *If* $r_{N-1}^L > 0$, *set* $X_N = p(N)$, $p_V(N) = 1$, $h_R(N) = \begin{pmatrix} h(N) \\ d(N) \end{pmatrix}$ *and* $d_V(N), d_T(N)$ *to be* $1, \times 0$ *and* 0×1 *empty matrices; else set* $X_N, p_V(N)$ *to be the* 0×0 *and* 1×0 *empty matrices,* $d_V(N) = 1$, $h_R(N) = h(N)$, $d_T(N) = d(N)$.

1.2. *For* $k = N-1, \dots, 2$ *perform the following. Compute the* QR *factorization*

$$\begin{pmatrix} p(k) \\ X_{k+1}a(k) \end{pmatrix} = V_k \begin{pmatrix} X_k \\ 0_{\nu_k \times r_{k-1}^L} \end{pmatrix},$$

where V_k *is a unitary matrix of sizes* $(1+\rho_k) \times (1+\rho_k)$ *and* X_k *is a matrix of sizes* $\rho_{k-1} \times r_{k-1}^L$. *Determine matrices* $p_V(k)$, $a_V(k)$, $d_V(k)$, $q_V(k)$ *of sizes* $1 \times \rho_{k-1},\ \rho_k \times \rho_{k-1},\ 1 \times (1 + \rho_k - \rho_{k-1}),\ \rho_k \times (1 + \rho_k - \rho_{k-1})$ *from the partition*

$$V_k = \begin{pmatrix} p_V(k) & d_V(k) \\ a_V(k) & q_V(k) \end{pmatrix}.$$

Compute

$$h'_k = p_V^*(k)d(k) + a_V^*(k)X_{k+1}q(k), \quad h_R(k) = \begin{pmatrix} h(k) \\ h'_k \end{pmatrix},$$

$$b_R(k) = \begin{pmatrix} b(k) & 0 \\ p_V^*(k)g(k) & a_V^*(k) \end{pmatrix},$$

$$g_T(k) = \begin{pmatrix} d_V^*(k)g(k) & q_V^*(k) \end{pmatrix}, \quad d_T(k) = d_V^*(k)d(k) + q_V^*(k)X_{k+1}q(k).$$

1.3. *Choose a unitary matrix* V_1 *of sizes* $\nu_1 \times \nu_1$. *Determine the matrices* $d_V(1)$, $q_V(1)$ *of sizes* $1 \times \nu_1$, $\rho_1 \times \nu_1$ *from the partition*

$$V_1 = \begin{pmatrix} d_V(1) \\ q_V(1) \end{pmatrix}.$$

Compute

$$d_T(1) = d_V^*(1)d(1) + q_V^*(1)X_2q(1), \quad g_T(1) = \begin{pmatrix} d_V^*(1)g(1) & q_V^*(1) \end{pmatrix}.$$

Thus we have computed the matrices V_k *and quasiseparable generators* $b_R(k)$, $h_R(k)$ *of the matrix* R.

2.1. *Compute the* QR *factorization*

$$d_T(1) = U_1 \begin{pmatrix} d_R(1) \\ 0_{\rho_1 \times 1} \end{pmatrix},$$

where U_1 *is a* $\nu_1 \times \nu_1$ *unitary matrix and* $d_R(1)$ *is a number.*
Determine the matrices $d_U(1)$, $g_U(1)$ *of sizes* $\nu_1 \times 1$, $\nu_1 \times \rho_1$ *from the partition*

$$U_1 = \begin{pmatrix} d_U(1) & g_U(1) \end{pmatrix}.$$

Compute

$$g_R(1) = d_U^*(1)g_T(1), \quad Y_1 = g_U^*(1)g_T(1).$$

2.2. *Compute the* QR *factorization*

$$\begin{pmatrix} Y_{k-1}h_R(k) \\ d_T(k) \end{pmatrix} = U_k \begin{pmatrix} d_R(k) \\ 0_{\rho_k \times 1} \end{pmatrix},$$

where U_k *is a* $(1+\rho_k) \times (1+\rho_k)$ *unitary matrix and* $d_R(k)$ *is a number. Determine the matrices* $d_U(k)$, $g_U(k)$, $h_U(k)$, $b_U(k)$ *of sizes* $\nu_k \times 1$, $\nu_k \times \rho_k$, $\rho_{k-1} \times 1$, $\rho_{k-1} \times \rho_k$ *from the partition*

$$U_k = \begin{pmatrix} h_U(k) & b_U(k) \\ d_U(k) & g_U(k) \end{pmatrix}.$$

Compute

$$g_R(k) = h_U^*(k)Y_{k-1}b_R(k) + d_U^*(k)g_T(k), \quad Y_k = b_U^*(k)Y_{k-1}b_R(k) + g_U^*(k)g_T(k).$$

2.3. *Set* $U_N = 1$. *If* $r_{N-1}^L > 0$ *set* $h_U(N) = 1$ *and* $d_U(N)$ *to be* 0×1 *empty matrix; else set* $d_U(N) = 1$ *and* $h_U(N)$ *to be* 0×1 *empty matrix.*
Compute $d_R(N) = \begin{pmatrix} Y_{N-1}h_R(N) \\ d_T(N) \end{pmatrix}$.
Thus we have computed quasiseparable generators $g_R(k)$ *and diagonal entries* $d_R(k)$ *of the matrix* R.

For a matrix with scalar entries we obtain the following algorithm for solving a system of linear algebraic equations.

Algorithm 20.11. Let $A = \{A_{ij}\}_{i,j=1}^N$ be an invertible matrix with scalar entries and with lower quasiseparable generators $p(i)$ $(i = 2, \dots, N)$, $q(j)$ $(j = 1, \dots, N-1)$, $a(k)$ $(k = 2, \dots, N-1)$ of orders r_k^L $(k = 1, \dots, N-1)$, upper quasiseparable generators $g(i)$ $(i = 1, \dots, N-1)$, $h(j)$ $(j = 2, \dots, N)$, $b(k)$ $(k = 2, \dots, N-1)$ of orders r_k^U $(k = 1, \dots, N-1)$, and diagonal entries $d(k)$ $(k = 1, \dots, N)$. Then the solution of the system $Ax = y$ is given as follows.

1. Using the algorithm from Theorem 20.10, compute lower quasiseparable generators $p_V(i)$ $(i = 2, \dots, N)$, $q_V(j)$ $(j = 1, \dots, N-1)$, $a_V(k)$ $(k = 2, \dots, N-1)$ and diagonal entries $d_V(k)$ $(i = 1, \dots, N)$ of the unitary block lower triangular matrix V, upper quasiseparable generators $g_U(i)$ $(i = 1, \dots, N-1)$, $h_U(j)$ $(j = 2, \dots, N)$, $b_U(k)$ $(k = 2, \dots, N-1)$ and diagonal entries $d_U(k)$ $(k = 1, \dots, N)$ of the unitary block upper triangular matrix U, and upper quasiseparable generators $g_R(i)$ $(i = 1, \dots, N-1)$, $h_R(j)$ $(j = 2, \dots, N)$, $b_R(k)$ $(k = 2, \dots, N-1)$ and diagonal entries $d_R(k)$ $(k = 1, \dots, N)$ of the upper triangular matrix R, so that $A = VUR$.
2. Compute the product $\tilde{x} = V^* y$ as follows: start with $\tilde{x}(N) = d_V^*(N)y(N)$, $w_{N-1} = p_V^*(N)y(N)$, $\tilde{x}(N-1) = q_V^*(N-1)w_{N-1} + d_V^*(N-1)y(N-1)$ and for $i = N-2, \dots, 1$ compute recursively
$$w_i = a_V^*(i+1)w_{i+1} + p_V^*(i+1)y(i+1), \quad \tilde{x}(i) = q_V^*(i)w_i + d_V^*(i)y(i).$$
3. Compute the product $\hat{x} = U^*\tilde{x}$ as follows: start with $\hat{x}(1) = d_U^*(1)\tilde{x}(1)$, $z_2 = g_U^*(1)\tilde{x}(1)$, $\hat{x}(2) = h_U^*(2)z_2 + d_U^*(2)\tilde{x}(2)$ and for $i = 3, \dots, N$ compute recursively
$$z_i = b_U^*(i-1)z_{i-1} + g_U^*(i-1)\tilde{x}(i-1), \quad \hat{x}(i) = d_U^*(i)\tilde{x}(i) + h_U^*(i)z_i.$$
4. Compute the solution x of the equation $Rx = \hat{x}$ as follows: start with $x(N) = (d_R(N))^{-1}\hat{x}(N)$, $\eta_{N-1} = h_R(N)\hat{x}(N)$ and for $i = N-1, \dots, 2$ compute recursively
$$\eta_{i-1} = b_R(i)\eta_i + h_R(i)x(i), \quad x(i) = (d_R(i))^{-1}(\hat{x}(i) - g_R(i)\eta_i),$$
and finally compute $x(1) = (d_R(1))^{-1}(\hat{x}(1) - g_R(1)\eta_1)$.

The inequality (20.46) for a scalar matrix yields the following estimate for the complexity of Algorithm 20.11:
$$c \le N(\vartheta(1+r, r) + \vartheta(r+1, r+1) + 5r^3 + 10r^2 + 10r + 4).$$

§20.7 Comments

The idea of the method used in this chapter was suggested by P.M. Dewilde and A.J. van der Veen in the monograph [15] for infinite matrices. The theorems and algorithms of this chapter suitable for finite block matrices were obtained in the paper [23], but the proofs presented here are essentially simpler. A similar method, but using Givens representations instead of a part of quasiseparable generators, was suggested by M. Van Barel and S. Delvaux in [13].

Instead of the factorization (20.1) one can use in a similar way the representation of the matrix A in the form $A = ULV$ with unitary block triangular matrices U, V and a triangular matrix L. Such an approach was used by N. Mastronardi, S. Chandrasekaran, S. Van Huffel, E. Van Camp and M. Van Barel for matrices with diagonal plus semiseparable of order one representations in [42, 9], and by S. Chandrasekaran and M. Gu for matrices with banded plus semiseparable of order one representations in [7], by S. Chandrasekaran, P.M. Dewilde, M. Gu, T. Pals, X. Sun and A.J. van der Veen in [6, 8] for solving some inversion and least squares problems for matrices with quasiseparable representations. This method was extended to matrices with hierarchically semiseparable representations via reduction to quasiseparable ones, see the paper [49] by J. Xia, S. Chandrasekaran, M. Gu and X.S. Li and the literature cited therein.

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