UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

AUXILIARY
LIBRARY ROUTINE V 11 - 294

TITLE: Modified Bessel Functions (SADOI Only)

TYPE: Closed

NUMBER OF WORDS: 300

TEMPORARY STORAGE: 0 - 10 inclusive

ACCURACY: At least 7 decimal places

DURATION: Approximately 190 m.sec.

DESCRIPTION: This routine calculates the modified Bessel Functions which are defined as:

$$I_v(z) = \sum_{m=0}^{\infty} \frac{z^{v+2m}}{2^{v+2m} m! \, \Gamma(v + m + 1)}$$

$$K_v(z) = \frac{\pi}{2 \sin v\pi} [I_{-v}(z) - I_v(z)]$$

These are solutions of the modified Bessel equation of order v which is

$$z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} - (z^2 + v^2) w = 0.$$

The accuracy of the routine depends upon the accuracy of the argument. For this reason, double precision argument may be used. In cases where the argument, scaled by a suitable scaling factor, has sufficient accuracy in single precision, a single precision argument may be used. The true value of the argument must be greater than .000001. Values smaller than this will give incorrect results. To use a double precision argument, where the argument is scaled by 2^{-M}, store the MSP in location OF and LSP in the accumulator and enter with

p	50 MF
	50 pF
p+1	26 ( )

where M $\geq$ 0, but M $\leq$ 39.

If a single precision argument is to be used, the accumulator must be clear upon entry. One form for entry may be

p	50 MF
	51 pF
p + 1	26 ( )

Again the MSP must be in location OF. The routine will leave the following in the locations shown:

in location	7F	$1/16\ e^{-x} I_1(x)$	
" "	8F	$1/16\ e^{x} K_0(x)$	
" "	9F	$1/16\ e^{-x} I_0(x)$	
" "	10F	$1/16\ xe^{x} K_1(x)$	for $x < 2$
" "	10F	$1/16\ e^{x} K_1(x)$	for $x \geq 2$

NOTE: Notice the two forms for $K_1(x)$ depending on the size of the argument.

The aim of the double precision argument is to allow the use of a wide range of arguments, all of which may be scaled by the same scaling factor. This means, for example, that if a range of values for the argument is from .00005 to 2^{12}, they may all be scaled by 2^{13}. The routine rescales each value down to the minimum value needed for each. For small arguments scaled by large factors, a double precision argument is needed to insure the accuracy of the routine.

This routine contains the subroutines; (R 1) Square Root, (S 3) Natural Logarithm, (R 5) Reciprocal, and (S 4) Exponential. These may be used by other parts of the program and may be entered in the usual manner since they are referred to in this routine as symbolic addresses.

The method used to obtain these functions may be found in "Mathematical Tables and Aids to Computation" 1956, Vol. 10, pp. 162-163.

RT: 1/10/61

DATE May 16, 1960

PROGRAMMED BY Marvin Harding

APPROVED BY J. N. Snyder

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LOCATION	ORDER	NOTES	PAGE 1	V 11
	00K(V11)			
0	40 1F			
	K5 170L	Plant Link		
1	42 108L			
	10 20F			
2	42 188L	Store M		
	41 189L			
3	26 237L			
	50 1F	Place argument		
4	L5 F	in AQ		
	00 1F			
5	36 6L			
	26 9L	Rescale arg.		
6	40 F			
	F5 189L	by new factor M'		
7	40 189L			
	L0 188L			
8	36 9L			
	26 4L			
9	L5 F			
	40 5F	$X \cdot 2^{-M'}$		
10	L5 188L			
	L0 189L			
11	40 6F	Store M'		
	L3 6F	Test size of		
12	32 30L			
	40 F	M' and transfer		
13	F5 F			
	36 29L	to proper portion		
14	40 F			
	F5 F	of routine		
15	36 16L			
	26 109L			
16	L5 169L			
	L0 5F	Test for X = 2		

LOCATION	ORDER	NOTES	PAGE 2	V 11
17	40 F			
	L3 F			
18	32 19L	X = 2		
	26 22L	X > 2		
19	50 158L			
	L5 19L			
20	46 94L			
	L5 86L			
21	42 72L			
	26 48L			
22	L5 190L			
	L0 5F	Test for		
23	40 F	X = 3.75		
	L3 F			
24	36 26L	X = 3.75		
	L5 F			
25	36 47L	X < 3.75		
	26 109L	X > 3.75		
26	L5 202L			
	40 9F			
27	L5 179L			
	40 7F			
28	22 138L			
	50 193L			
29	L5 5F			
	10 1F	Form X/4		
30	22 31L			
	L5 5F			
31	10 2F			
	40 5F			
32	L5 5F			
	00 1F			
33	50 77L			
	50 33L			
34	26 (S3)			
	10 6F	Form 1/32 ln X/2		

LOCATION	ORDER	NOTES PAGE 3 V 11
35	75 187L	
	00 2F	
36	40 176L	
	L1 5F	
37	50 69L	
	50 37L	
38	26 (S4)	
	40 F	Form e^{-x}
39	50 F	
	7J F	
40	40 F	
	50 F	Form e^{-x}
41	7J F	
	40 177L	
42	50 95L	
	50 42L	
43	26 (R5)	
	10 3F	Form $1/8\ e^{x}$
44	S5 99L	
	50 F	
45	40 178L	
	L5 61L	
46	42 72L	
	26 53L	
47	L5 68L	
	42 72L	
48	L1 5F	
	50 48L	
49	26 (S4)	
	40 F	Form e^{-x}
50	50 F	
	7J F	
51	40 F	
	50 F	
52	7J F	
	40 177L	

LOCATION	ORDER	NOTES PAGE 4 V 11
53	L5 28L	
	42 62L	
54	50 191L	
	L5 5F	Form X/3.75
55	66 190L	
	85 F	
56	40 F	
	L5 37L	
57	46 65L	
	L5 191L	
58	40 199L	
	L5 200L	
59	40 201L	
	50 F	
60	7J F	
	40 1F	
61	40 2F	
	50 73L	
62	50 2F	
	7J F	Form Summation
63	L4 199L	
	40 199L	of terms
64	F5 201L	
	40 201L	for small
65	36 F	
	50 1F	arguments
66	7J 2F	
	40 2F	
67	F5 62L	
	42 62L	
68	26 62L	
	50 85L	
69	L5 199L	
	L4 192L	Form $1/16\ e^{-x}\ I_0(x)$

LOCATION	ORDER	NOTES PAGE 5 V 11
70	40 3F	
	50 181L	for small arg.
71	50 177L	
	7J 3F	
72	40 7F	
	26 F	
73	L5 70L	
	42 62L	
74	L5 33L	
	46 65L	
75	L5 5F	
	00 1F	Form 1/16 $e^x K_0(x)$
76	40 F	
	22 57L	for small arg.
77	L5 199L	
	L0 180L	
78	40 8F	
	50 176L	
79	75 3F	
	00 4F	
80	40 F	
	L5 8F	
81	10 5F	
	L0 F	
82	40 8F	
	50 178L	
83	75 8F	
	00 4F	
84	40 8F	
	26 87L	
85	L5 86L	
	46 94L	
86	50 138L	
	26 88L	
87	L5 42L	
	46 94L	

LOCATION	ORDER	NOTES PAGE 6 V 11
88	L5 94L	
	42 62L	
89	L5 7F	Place 1/16 e^{-x} $I_0(x)$
	40 9F	in 9F
90	L5 91L	
	42 72L	
91	26 54L	
	50 92L	1/16 e^{-x} $I_1(x)$ for
92	50 5F	
	75 7F	small arg.
93	00 1F	
	40 7F	
94	26 F	
	50 163L	
95	L5 L	
	42 62L	
96	L5 44L	
	46 65L	
97	L5 5F	
	00 1F	
98	40 F	
	22 57L	
99	L5 199L	
	L4 169L	
100	40 10F	
	50 176L	
101	75 3F	
	00 3F	
102	40 F	
	50 F	1/16x e^{x} $K_1(x)$ for
103	75 1F	
	00 3F	small arg.
104	40 F	
	L5 10F	
105	10 3F	
	L4 F	

LOCATION	ORDER	NOTES PAGE 7 V 11
106	40 F	
	50 F	
107	75 178L	
	00 3F	
108	40 10F	
	22 F	
109	L5 6F	
	42 111L	
110	50 136L	
	50 191L	
112	L5 190L	
	10 F	Form 3.75/X
112	66 5F	
	00 2F	
113	S5 132L	
	40 3F	
114	7J 203L	
	50 114L	
115	26 (R1)	Form $1/\sqrt{X}$
	40 4F	
116	L5 119L	
	42 121L	
117	L5 113L	
	46 131L	
118	41 199L	
	L5 161L	
119	40 162L	
	50 219L	
120	L5 3F	
	40 2F	
121	50 2F	
	7J F	
122	L4 199L	
	40 199L	Terms for
123	F5 162L	
	40 162L	large arg.

LOCATION	ORDER	NOTES PAGE 8 V 11
124	32 127L	
	50 3F	
125	7J 2F	
	40 2F	
126	F5 121L	
	42 121L	
127	26 121L	
	L5 199L	
128	L4 218L	
	40 F	for large arg.
129	50 F	
	75 4F	
130	10 4F	$1/16\ e^{-x}\ I_0(x)$
	40 7F	
131	26 F	$1/16\ e^{-x}\ I_1(x)$
	50 227L	
132	L5 131L	
	42 121L	
133	L5 7F	
	40 9F	
134	L5 110L	
	46 131L	
135	26 118L	
	50 206L	
136	50 3F	
	75 204L	
137	40 3F	2/X
	26 143L	
138	50 F	
	50 191L	
139	L5 169L	
	66 5F	
140	S5 150L	
	40 3F	2/X
141	10 1F	
	50 141L	

LOCATION	ORDER	NOTES
142	26 (R1)	
	40 4F	$1/\sqrt{x}$
143	L5 135L	
	42 62L	
144	L5 149L	
	42 153L	
145	L5 140L	
	46 65L	
146	L5 3F	
	40 1F	
147	40 2F	
	41 199L	
148	L5 200L	$1/16\ e^x K_1(x)$
	40 201L	
149	26 62L	$1/16\ e^x K_0(x)$
.	50 154L	
150	L5 199L	
	L4 205L	for large arg.
151	40 F	
	50 F	
152	75 4F	
	10 3F	
153	40 10F	
	26 F	
154	L5 160L	
	42 62L	
155	L5 10F	
	40 8F	
156	L5 157L	
	42 153L	
157	26 145L	
	50 108L	
158	L5 235L	
	40 8F	
159	L5 236L	
	40 10F	

LOCATION	ORDER	NOTES PAGE 10 V 11
160	22 108L	Return to master
	50 212L	waste
161	LL 4095F	
	LL 4088F	-8
162	00 F	
	00 F	
163	00 F	
	00 1098 6324 2500 J	
164	00 F	
	00 6437 3586 250 J	Coeff. for
165	00 F	
	00 1885 6167 500 J	$I_1(x)$
166	00 F	small arg.
	00 3323 4162 250 J	
167	00 F	
	00 3769 15000 J	
168	00 F	
	00 4051 3750 J	
169	00 F	
	00 5000 0000 0000 J	
170	00 F	
	00 7721 5720 000 J	
171	80 F	
	00 6636 0710 5000 J	coeff. for
172	80 F	
	00 9092 1551 0000 J	$K_1(x)$
173	80 F	
	00 9904 0299 0000 J	Small arg.
174	80 F	
	00 9994 4798 0000 J	
175	80 F	
	00 9999 7657 0000 J	
176	00 F	
	00 F	

LOCATION	ORDER	NOTES PAGE 11 V 11
177	00 F	
	00 F	
178	00 F	
	00 F	
179	00 F	
	00 1143 5526 312 J	
180	00 F	
	00 5772 1566 0000 J	
181	00 F	
	00 4227 8420 0000 J	
182	00 F	
	00 2306 9756 0000 J	
183	00 F	
	00 3488 5900 000 J	
184	00 F	Coeff. for
	00 2626 980 000 J	$K_0(x)$
185	00 F	small arg.
	00 1075 00000 J	
186	00 F	
	00 7400 000 J	
187	00 F	
	00 3465 7359 0280 J	
188	00 F	
	00 F	
189	00 F	
	00 F	
190	00 F	
	00 9375 0000 0000 J	
191	00 F	
	00 F	
192	00 F	
	00 6250 0000 000 J	
193	00 F	
	00 2197 2643 1200 J	

LOCATION	ORDER	NOTES PAGE 12 V 11
194	00 F	
	00 1931 2140 0000 J	Coeff.
195	00 F	
	00 7542 1825 000 J	for $I_0(x)$
196	00 F	
	00 1662 3325 000 J	small arg.
197	00 F	
	00 2254 8000 00 J	
198	00 F	
	00 2863 3125 0 J	
199	00 F	
	00 F	
200	LL 4095F	-6
	LL 4090F	
201	00 F	
	00 F	
202	00 F	
	00 1340 3565 603 J	
203	00 F	
	00 2666 6666 6667 J	
204	00 F	
	00 5333 3333 3334 J	
205	00 F	
	00 6266 5707 0000 J	
206	80 F	
	00 9608 3821 0000 J	Coeff. for
207	00 F	
	00 1094 7840 000 J	$K_0(x)$
208	80 F	
	00 9946 8777 0000 J	large arg.
209	00 F	
	00 2939 3600 00 J	
210	80 F	
	00 9987 4230 0000 J	

LOCATION	ORDER	NOTES
211	00 F	
	00 2660 4000 0 J	
212	00 F	
	00 1174 9309 5000 J	
213	80 F	
	00 9817 2190 0000 J	Coeff. for
214	00 F	
	00 7521 3400 00 J	$K_1(x)$
215	80 F	
	00 9960 9823 5000 J	large arg.
216	00 F	
	00 1628 0700 00 J	
217	80 F	
	00 9996 5877 5000 J	
218	00 F	
	00 3989 4228 0000 J	
219	00 F	
	00 1328 5917 000 J	
220	00 F	
	00 2253 1870 00 J	
221	80 F	
	00 9984 2435 1000 J	
222	00 F	
	00 9162 8080 00 J	Coeff. for
223	80 F	
	00 9794 2293 7000 J	I_0
224	00 F	
	00 2635 5372 000 J	large arg.
225	80 F	
	00 9835 2367 1000 J	
226	00 F	
	00 3923 7670 00 J	
227	80 F	
	00 9601 1975 8000 J	

LOCATION	ORDER	NOTES
228	80 F	
	00 9963 7981 7000 J	
229	00 F	
	00 1638 0140 00 J	
230	80 F	
	00 9896 8445 0000 J	
231	00 F	
	00 2282 9673 000 J	Coeff. for
232	80 F	
	00 9710 4687 9000 J	$I_1(x)$
233	00 F	
	00 1787 6535 000 J	large arg.
234	80 F	
	00 9957 9941 3000 J	
235	00 F	
	00 5259 8013 292 J	
236	00 F	
	00 6459 2302 791 J	
237	L3 188L	
	36 9L	
238	22 3L	
	50 F	
	(R1) 00K	239-247
	(S3) 00K	248-261
	(R5) 00K	262-278
	(S4) 00K	279-299