

## UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

Aux.

LIBRARY ROUTINE V 8 - 219

**TITLE** Ordinary Bessel Functions (DOI OR SADOI)  
**TYPE** Closed with program parameter  
**NO. OF WORDS** 257  
**TEMPORARY STORAGE** 0, 1, 2, 3; 6, 7, 8, 9  
**ACCURACY** About 4 decimal places (see below)  
**DURATION** At most 100 ms. (see below)  
**USE** Enter with  $x \cdot 2^{-M}$  in A,  $M \geq 0$

p	XY MF	(this routine at g)
	50 pF	
p+1	26 gF	

X = 5 → Functions of first kind ( $J_N$ ) only will be computed

X = J → Functions of both first and second kinds ( $Y_n$  often written  $N_n$ )  
will be computed

Y = 4 → Functions of zero order ( $J_0, Y_0$ ) only will be computed

Y = 2 → Functions of first order ( $J_1, Y_1$ ) only will be computed

Y = 0 → Functions of both zero and first orders will be computed

More explicitly:

XY	Ins. computed
54	$J_0(x)$
52	$J_1(x)$
50	$J_0(x), J_1(x)$
J4	$J_0(x), Y_0(x)$
J2	$J_1(x), Y_1(x)$
J0	$J_0(x), J_1(x), Y_0(x), Y_1(x)$

The results are scaled by  $2^{19}$  and are left in locations 6 - 9 as follows:

$J_0(x) \cdot 2^{-19}$  in loc 6

$J_1(x) \cdot 2^{-19}$  in loc 7

$Y_0(x) \cdot 2^{-19}$  in loc 8  
 $Y_1(x) \cdot 2^{-19}$  in loc 9

Illiac will stop if  $x \leq 0$ , and upon starting again will transfer control to the right hand order at the location specified by the preset parameter S3.

NOTE The integer M need not be the smallest integer such that  $X \cdot 2^{-M} < 1$ .

DESCRIPTION If  $x \geq 16$ , the following asymptotic expressions are used:

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \left[ \sin \left( x + \frac{\pi}{4} \right) + \frac{1}{8x} \sin \left( x - \frac{\pi}{4} \right) \right]$$

$$J_1(x) \approx \sqrt{\frac{2}{\pi x}} \left[ \sin \left( x - \frac{\pi}{4} \right) + \frac{3}{8x} \sin \left( x + \frac{\pi}{4} \right) \right]$$

$$Y_0(x) \approx \sqrt{\frac{2}{\pi x}} \left[ \sin \left( x - \frac{\pi}{4} \right) - \frac{1}{8x} \sin \left( x + \frac{\pi}{4} \right) \right]$$

$$Y_1(x) \approx \sqrt{\frac{2}{\pi x}} \left[ -\sin \left( x + \frac{\pi}{4} \right) + \frac{3}{8x} \sin \left( x - \frac{\pi}{4} \right) \right]$$

No matter what entry is used, if  $x \geq 16$ , then all four functions are computed. The time is about 25 ms. At  $x = 16$ , where these formulae give the worst results, the error is less than .0001.

If  $0 < x < 16$ , the following expansions are used

$$J_0(x) = \sum_{n=0}^N (-1)^n \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}$$

$$J_1(x) = \frac{x}{2} \sum_{n=0}^N (-1)^n \frac{\left(\frac{x}{2}\right)^{2n}}{n! (n+1)!}$$

$$Y_0(x) = \frac{2}{\pi} \left[ \left(\gamma + \ln \frac{x}{2}\right) J_0(x) - \sum_{n=1}^N \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{(n!)^2} \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) \right]$$

$$Y_1(x) = \frac{2}{\pi} \left[ \left(\gamma + \ln \frac{x}{2}\right) J_1(x) - \frac{1}{x} - \frac{1}{\pi} \sum_{n=0}^N \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{n! (n+1)!} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} + 1 + \frac{1}{2} + \dots + \frac{1}{n+1}\right) \right]$$

where  $N$  is large enough in each case so that the last term is less than  $2^{-16}$ . The time to compute  $J_0(x)$  or  $J_1(x)$  is at most 40 ms. depending on the size of  $x$  (less time for smaller  $x$ ). The time to compute  $J_0(x)$  and  $Y_0(x)$  or  $J_1(x)$  and  $Y_1(x)$  is at most 100 ms. The error is less than  $2^{-16}$ . If  $J_n(x)$  or  $Y_n(x)$  with  $n \neq 0, 1$  are desired, use the recurrence relations:

$$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$$

$$\frac{2n}{x} Y_n(x) = Y_{n+1}(x) + Y_{n-1}(x)$$

For  $x \leq 0$ ;

$$J_0(0) = 1$$
$$J_1(0) = 0$$
$$Y_0(0) = -\infty$$
$$Y_1(0) = -\infty$$

We also have

$$J_0(-x) = J_0(x)$$

$$J_1(-x) = -J_1(x)$$

$$Y_0(-x) = Y_0(x) + (2n+1) \pi i J_0(x)$$

$$Y_1(-x) = Y_1(x) - (2n+1) \pi i J_1(x)$$

The branch point at the origin produces the multiple values of  $Y_0(-x)$  and  $Y_1(-x)$

Library routines S1 (Natural Logarithm) and T5 (Sine - Cosine) are included in this routine and may be used by other parts of the program. S1 starts at word  $g + 191$  and T5 starts at word  $g + 236$ , where  $g$  is the location of the first word of this routine.

RT: 4/3/61

DATE	July 24, 1956
CODED BY	<i>Blankfield</i>
APPROVED BY	<i>J. P. Nash</i>

Ordinary Bessel Functions

LOCATION	ORDER OOK (V8)	NOTES	PAGE 1
0	40 185L K5 F		
1	42 168L 40 F	Plant Link	
2	46 169L 10 20F	OO M OO M	
3	42 169L L1 185L		
4	30 S3L 51 69L	Stop if $X \leq 0$	
5	L0 170L 32 26L		
6	L5 169L L0 171L	$M \leq 4$	
7	32 10L L5 171L		
8	L0 169L 42 9L		
9	L5 185L 10 F	Adjust M to 4	Bookkeeping
10	40 185L L5 F		
11	36 19L 00 5F		
12	36 14L L5 175L		
13	42 88L 26 61L		
14	00 1F 32 16L	Functions of 1st and 2nd kinds	
15	L5 175L 42 122L		
16	26 89L L5 174L		

LOCATION	ORDER	NOTES	PAGE 2
17	42 88L L5 175L		
18	42 122L 26 61L		
19	00 5F 32 21L		
20	L5 175L 42 47L		
21	26 35L 00 1F		
22	36 24L L5 175L	Functions of 1st kind only	
23	42 60L 26 48L		
24	L5 176L 42 47L		
25	L5 175L 42 60L		
26	26 35L L4 28L		
27	46 28L L5 185L	$M \geq 5$	Bookkeeping
28	00 1F 32 29L		
29	26 127L 40 1F		
30	L5 28L 46 32L	Is $X < 16$ ?	
31	32 31L L5 1F		
32	10 F L0 185L		
33	36 34L 26 127L		
34	L5 1F 26 10L		

LOCATION	ORDER	NOTES	PAGE 3
35	50 185L		
	7J 185L		
36	40 F11		
	41 188L		
37	L1 177L		
	40 187L	Preset	
38	L5 177L		
	40 190L		
39	40 6F		
	L5 187L		
40	L4 178L		
	40 187L	Count	
41	L4 188L		$J_0(x) \ x < 16$
	40 188L		
42	50 F		
	71 190L		
43	10 13F		
	66 188L		
44	S5 F		
	40 190L	Compute	
45	L4 6F		
	40 6F		
46	L7 190L		
	L0 179L	Test	
47	32 39L		
	26 F		
48	50 185L		
	7J 185L		
49	40 F		
	L5 185L		$J_1(x) \ x < 16$
50	10 16F		
	40 190L	Preset	
51	40 7F		
	41 187L		
52	41 188L		
	L5 187L		












LOCATION	ORDER	NOTES	PAGE 4	v8
53	L4 178L			
	40 187L	Count		
54	L4 188L			
	40 188L			
55	50 F			
	71 190L			
56	10 13F			
	66 188L			
57	S5 F		$J_1(x) x < 16$	
	40 190L	Compute		
58	L4 7F			
	40 7F			
59	L7 190L			
	L0 179L	Test		
60	32 52L			
	26 F			
61	50 185L			
	7J 185L			
62	40 F			
	41 187L			
63	41 188L			
	41 189L	Preset		
64	41 8F			
	L5 180L			
65	10 9F			
	40 190L		$Y_0(x) x < 16$	
66	40 6F			
	50 179L			
67	L5 187L			
	L4 180L			
68	40 187L			
	00 1F			
69	L0 180L			
	L4 188L	Count		
70	40 188L			
	L5 180L			

LOCATION	ORDER	NOTES	PAGE 5
71	10 5F 66 187L		
72	S5 F L4 189L		
73	40 189L 50 F		
74	71 190L 10 4F		
75	66 188L S5 F	Compute	
76	40 190L L4 6F		
77	40 6F 50 190L		
78	75 189L 00 5F		
79	L4 8F 40 8F		$y_0(x) \ x < 16$
80	L7 190L L0 179L	Test	
81	32 66L L5 185L		
82	22 82L 50 82L		
83	26 191L L4 182L	Logarithm	
84	40 F 50 F		
85	75 6F 00 5F		
86	L0 8F 40 F		
87	50 F 7J 183L		
88	40 8F 26 F		

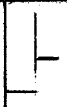

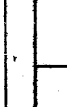
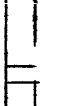
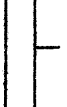















LOCATION	ORDER	NOTES	PAGE 6
89	11 17F 19 22F		
90	L0 185L 36 123L	Is it o.k. to divide?	
91	L4 185L 66 185L		
92	S5 F 40 186L		
93	50 185L 7J 185L		
94	40 F 41 187L		
95	41 188L 50 179L		
96	L5 181L 00 4F		
97	40 189L L5 185L	Preset	$Y_1(x) \ x < 16$
98	10 16F 40 190L		
99	40 7F 10 1F		
100	40 9F L5 187L		
101	L4 181L 40 187L	Count	
102	L4 188L 40 188L		
103	L5 180L L4 187L		
104	50 179L 10 5F	Compute	
105	66 188L S5 F		
106	L4 189L 40 189L		

LOCATION	ORDER	NOTES	PAGE 7	v8
107	50 F 71 190L			
108	10 4F 66 188L			
109	S5 F 40 190L	Compute		
110	L4 7F 40 7F			
111	50 190L 75 189L			
112	00 4F L4 9F			
113	40 9F L7 190L			
114	L0 179L 32 100L	Test		
115	L5 185L 50 115L		$-Y_1(x) \quad x < 16$	
116	26 191L L4 182L	Logarithm		
117	40 F 50 F			
118	75 7F 00 5F			
119	L0 186L L0 9F			
120	40 F 50 F			
121	7J 183L 40 9F			
122	22 122L 26 F			
123	41 F 09 1F			
124	F0 F 40 9F	$x \leq 2^{-19}$		

LOCATION	ORDER	NOTES	PAGE 8
125	L5 122L 42 60L	 $x \leq 2^{-19}$	 $-Y_1(x) \quad x < 16$
126	26 48L 00 F		
127	L5 169L 42 134L		
128	46 138L L4 172L		
129	42 133L 46 137L		
130	L4 172L 42 150L		
131	L0 173L 46 141L		
132	50 184L 7J 185L		
133	40 186L 19 F		$-x \geq 16$
134	L4 186L 00 F		
135	50 135L 50 135L		
136	26 236L 40 187L		
137	19 F L0 186L		
138	00 F 50 138L		
139	26 236L 40 186L		
140	50 179L L5 184L		
141	10 F 66 185L		
142	41 F S5 F		

LOCATION	ORDER	NOTES	PAGE 9
143	40 1F 51 1F		
144	10 1F SJ F		
145	40 188L 50 F		
146	L5 1F 66 188L	Square Root	
147	S5 F L0 188L		
148	10 1F 36 150L		
149	L4 188L 26 145L		
150	50 179L 19 F		
151	66 185L S5 F		x ≥ 16
152	40 189L 00 1F		
153	L4 189L 40 190L		
154	50 188L 7J 187L		
155	40 187L 50 188L		
156	79 186L 40 186L		
157	50 189L 7J 186L		
158	L4 187L 10 19F	J <sub>0</sub>	
159	40 6F 50 190L		
160	7J 187L L4 186L	J <sub>1</sub>	

LOCATION	ORDER	NOTES	PAGE 10
161	10 19F 40 7F		
162	50 189L 79 187L		
163	14 186L 10 19F		
164	40 8F 50 190L		x ≥ 16
165	7J 186L L0 187L		
166	10 19F 40 9F		
167	L5 14L 46 28L		Reset Location 28 relative
168	22 168L 22 F		Link
169	00 F 00 F		
170	00 5F 00 5F		
171	00 4F 00 4F		
172	00 1F 00 1F		
173	00 5F 00 F		Constants and Storage
174	00 F 00 89L		
175	00 F 00 167L		
176	00 F 00 48L		
177	00 1F 00 F		
178	00 2F 00 F		

LOCATION	ORDER	NOTES	Page 11
179	00 F 00 4F		
180	00 512F 00 F		
181	00 1024F 00 F		
182	00 F 00 83020537705J		
183	40 F 00 136619772368J		
184	00 F 00 318309886184J		
185	00 F 00 F		Constants and Storage
186	00 F 00 F		
187	00 F 00 F		
188	00 F 00 F		
189	00 F 00 F		
190	00 F 00 F		
191	40 2F S5 1F		
192	L4 191L 42 226L		
193	41 F 50 F		Natural Logarithm S1
194	L5 2F L4 235L		
195	32 199L L1 228L		
196	L4 F 30 S3		

LOCATION	ORDER	NOTES	PAGE 12
197	40 F		
	L5 2F		
198	00 6F		
	40 2F		
199	22 194L		
	L1 227L		
200	22 201L		
	40 2F		
201	L1 227L		
	00 1F		
202	L4 F		
	40 F		
203	L5 2F		
	00 1F		
204	32 200L		
	L5 227L		
205	00 1F		
	L4 F		
206	40 F		
	L5 2F		
207	10 1F		
	40 1F		
208	L5 229L		
	10 1F		
209	L4 1F		
	40 1F		
210	41 3F		
	50 3F		
211	L5 2F		
	L0 229L		
212	66 1F		
	S5 F		
213	40 1F		
	50 1F		
214	7J 1F		
	40 2F		

Natural  
Logarithm S1

LOCATION	ORDER	NOTES	PAGE 13
215	50 230L 7J 2F		
216	L4 231L 40 3F		
217	50 3F 7J 2F		
218	L4 232L 40 3F		
219	50 3F 7J 2F		
220	L4 233L 40 3F		
221	50 3F 7J 1F		
222	40 3F 50 3F		
223	7J 2F L4 1F		Natural Logarithm S1
224	L4 234L 10 5F		
225	L0 227L L4 F		
226	40 F 22 F		
227	00 F 00 10830424696J		
228	00 F 00 129965096355J		
229	40 F 00 207106781187J		
230	00 F 00 464044457J		
231	00 F 00 2228558603J		
232	00 F 00 12500185911J		



LOCATION	ORDER	NOTES	PAGE 14
233	00 F 00 83333329444J		
234	00 F 00 16F		Natural Logarithm Sl
235	LF F 00 F		
236	40 F K5 250L		
237	42 248L L9 F		
238	40 F LS F		
239	40 F 50 F		
240	71 F 00 2F		
241	40 1F L5 236L		Sine-Cosine T5
242	42 243L 50 249L		
243	79 1F L4 255L		
244	40 2F 50 2F		
245	F5 243L 42 243L		
246	L0 256L 36 243L		
247	LJ 245L 74 F		
248	00 1F 22 F		
249	00 F 00 27232J		
250	NO F 00 499998202409J		

LOCATION	ORDER	NOTES	PAGE 15
251	00 F 00 80219199J		
252	NO F 00 497659123501J		
253	00 F 00 39846313004J		
254	NO F 00 177017951258J		-Sine-Cosine T5
255	40 F 00 285398163397J		
256	L9 1F L4 256L		