UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

ILLINOIS CODE G1 - 78

TITLE	Laplace's Equation - Liebmann Method (DOI Only)
TYPE	Open
NUMBER OF WORDS	5 through 46. Grid points and key words may fill memory
TEMPORARY STORAGE	0 - 8
ACCURACY	Arbitrarily set at 10^{-5}, but may be changed by user.
DURATION	3PQ milliseconds, where P = number of interior points Q = number of times over grid For rectangular regions and 5 - place accuracy Q is roughly approximated by $\left\| \frac{5}{\log_{10} K^*} \right\|$,

where $K^* = [1 - \pi^2/4 ([N-1]^{-2} + [M-1]^{-2})]^2$,

N = number of columns,

M = number of rows.

DESCRIPTION

This code gives the solution V(x,y), corresponding to given boundary values, of the difference equation $V_{xx} + V_{yy} = 0$, which approximates Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The solution is found for all interior grid points of an arbitrary closed region. The scheme used is that of Liebmann, which requires an initial approximation for each interior point and then improves the approximation by taking for a new value at a point the average of the previous function values at the four neighboring points of the grid. When successive approximations at every point of the grid fail to change by more than a predetermined amount, the present values of V are taken as the solution. For an initial approximation the interior points are taken as linear interpolations between the boundary points marking each horizontal sequence of interior points.

The maximum number of boundary and interior points depends on the complexity of the boundary but is limited to about 950 for very simple boundaries. First, the boundary should be enclosed in a rectangular grid of M rows and N columns, preferably with $N \geq M$. Known boundary points, of course, should correspond to points of the grid. If a given interior point has a sequence position i in the M x N grid (reading from left to right and top to bottom) then its neighbors have the positions i - N, i - 1, i + 1, i + N. Since the exterior points will not be stored it is necessary to find the actual addresses of the points in sequential positions. i, i - N, and i + N. This is done as follows: in preparing the problem a count is made of the number of points in each sequence of interior, boundary, and exterior points, with the sequences arranged in the same order as they are on the grid. These are called temporary key words and are placed on the tape after 00 65K as 20 nF 00 F, 30 nF 00 F, 40 nF 00 F for interior, boundary, exterior sequences, respectively, where n is the number of points in a sequence. A tally should be kept of the total number of boundary and interior points, for this number plus the number of sequences of interior points may not exceed 950. The tape of temporary key words must be terminated with 00 F 00 F. It is placed in the memory in locations (65 ff) later occupied by the function values at the grid points. Temporary key words are operated on to form the addresses of locations i, i - N, i + N, for each i which starts a sequence of interior points, and to place them into one word. These words are called the <u>key words</u> and are put into the memory in sequence starting at any location following the space required for the grid points. A suggested position is at address j = [998 minus number of sequences of interior points]. Key words are of the form $[m \times 2^{-9} + p \times 2^{-19} + q \times 2^{-39}]$, where N(m), N(p), N(q) are the (i - N)th, i - th, (i + N) - th numbers of the grid, respectively.

Without loss of generality all boundary point values are required to lie in the range $0 \leq V < 1/4$. Then the interior points are also within this range and the boundary point values may be marked by entering them as negative values. Requiring a maximum of 1/4 permits adding the four points to be averaged without chance of overflow. The boundary point

values are written as S followed by up to 11 decimal digits and are placed on the tape in sequential order except that after the first boundary point following a sequence of n interior points the number K (n + 1), where (n + 1) must be a 3 - decimal digit number, is inserted. This list is terminated by the character N. Input of this tape places in order the boundary point values and fills in all sequences of interior point values by linear interpolations between the boundary values marking each sequence. Boundary point values are stored as their negatives; interior points are unchanged.

Parameters required are (1) N, the number of columns, to be placed in location 3 in the form $N \times 2^{-19}$, and (2) j, the address for the start of the key words, to be placed in location 4 in the form $j \times 2^{-39}$. The print parameters are (1) N (3) = $k \times 2^{-39}$, where k is the number of decimal places to be printed, and (2) N (4) = $s \times 2^{-39}$, where s is the address of the last point to be printed.

The complete routine for solving the problem requires the following tapes:

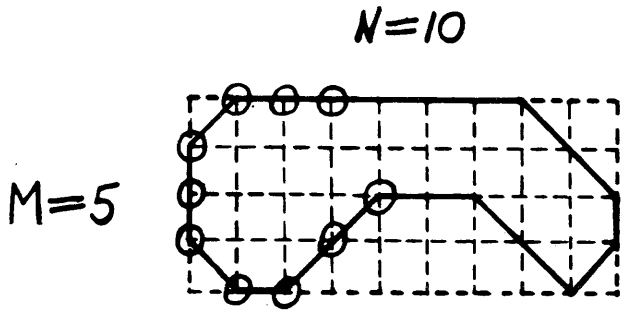
1. Decimal Order Input (X1 - 18)
2. Temporary key words (in 65 ff) and parameters
3. Routine to form key words (in 10 - 64), followed by 26 10N
4. Decimal number sequence input (N3 - 23) and modification (in 26 - 46). N3 is obsolete, but it has been copied into G1 and no copy is required by the programmer.
5. Boundary input routine (in 5 - 25), followed by 26 28N
6. Boundary points, followed by N
7. Parameters again and routine evaluating interior points (in 10 - 38), followed by 24 10N
8. Print parameters, print routine, and Single Column print (P6 - 25) (in 10 - 28), followed by 24 10N

The order listed must be adhered to although only 2, 6, and parameters need to be constructed for each problem. Codes 4 and 5 overwrite 3, 7 overwrites 4 and 5, and 8 overwrites 7.

Relative location 28 in the code for evaluating interior points (fixed location 38) contains the constant which determines the accuracy and thus the stopping point of the routine. This has been set arbitrarily

at 5×10^{-6} to give five-place accuracy, but may be varied at the discretion of the user by setting $N(38F) = 5 \times 10^{-(n+1)}$, (where n is the number of decimal places of accuracy desired) after input of the routine for evaluating interior points, but before its execution.

An example follows:



Suppose the encircled boundary points have the value 10 and the others the value (-2). First add 2 to the function to get it non-negative throughout the region. Then scale by a factor of 50 so that the function values V are all in the range ($0 \leq V \leq .24$). The temporary keys are 40 1F 00 F, 30 7F 00 F, 40 2F 00 F, 30 1F 00 F, 20 7F 00 F, 30 1F 00 F, 40 1F 00 F, 30 1F 00 F, 20 3F 00 F, 30 3F 00 F, 20 2F 00 F, 30 2F 00 F, 20 2F 00 F, 30 1F 00 F, 40 3F 00 F, 30 1F 00 F, 20 1F 00 F, 30 1F 00 F, 40 1F 00 F, 30 2F 00 F, 40 5F 00 F, 30 1F 00 F, 00 F 00 F. Parameters are $N(3) = 10 \times 2^{-19}$ and $N(4) = [998 - 5] \times 2^{-39} = 993 \times 2^{-39}$. The list of boundary points is S24, S24, S24, S, S, S, S, S24, S, K008, S24, S24, K004, S, S, S, K003, S24, S24, K003, S, S, K002, S24, S24, S, N.

lgr

DATE October 19, 1954 Rev. 3/19/59
CODED BY Richard King
APPROVED BY J. H. Nash

WORDS

ATION	ORDER		NOTES PAGE 1 G 1
	00 10K		
0	41 7F		N(7) = N(8) = 0
	40 8F		
1	50 (65)F	From 46'	Q = current temporary key
	01 3F		
2	L0 52L		Test for end of temporary keys
	36 4L	No	
3	41 ( )F	By 41	Insert 0 at end of keys
	24 999F		Read in "boundary input"
4	L0 52L	From 2'	Test for interior sequence
	L0 52L		
5	36 47L	No	
	41 5F	Yes	N(5) = N(6) = 0
6	40 6F		
	L5 7F		N(0) = [number of interior parts
7	L0 8F		+ number of boundary points] 2^{-19}
	26 8L		
8	40 F		
	L5 1L		Plant address of current temporary key
9	46 26L		
	46 42L		
10	L0 51L		plant address of previous temporary key
	46 11L		
11	50 ( )F	By 10' 17	
	01 3F	From 17'	Test for exterior sequence
12	L0 53L		
	36 18L	Yes	
13	11 3F	No	
	L5 5F	From 20	N(5) = [number of points we have backed
14	S4 F		up] . 2^{-19}
	40 5F		
15	L0 3F		Have we backed up a row?
	32 20L	Yes	
16	L5 11L	No	Plant address of previous temporary key
	L0 51L		

ORDER		NOTES
46 11L		
26 11L		
11 3F	From 12'	
L5 6F		N(6) = [number of exterior points
S4 F		we have backed up] . 2^{-19}
40 6F		
22 13L		
L5 54L	From 15'	A = [address of first mesh word] . 2^{-19}
L4 F		
40 1F		
L0 3F		N(1) = m . 2^{-9} + p . 2^{-19}
L4 6F		
00 10F		
L4 1F		
40 1F		
41 5F		
41 6F		N(5) = N(6) = 0
26 26L		
50 ( )F	By 9,	
01 3F	From 32'	Test for exterior sequence
L0 53L	Yes	
36 33L		
11 3F	No	N(5) = [number of points we have
L5 5F	From 35	progressed . 2^{-19}
S4 F		
40 5F		
L0 3F		Have we progressed a row?
32 35L	Yes	
L5 26L	No	
L4 51L		Plant address of next temporary key
46 26L		
26 26L		
11 3F	From 27'	
L5 6F		N(6) = [number of exterior parts we
S4 F		have progressed] . 2^{-19}
40 6F		

LOCATION	ORDER		NOTES	PAGE 3 G 1
35	22 28L			
	L5 1F	From 30'		
36	L4 3F		$N(1) = m \cdot 2^{-9} + p \cdot 2^{-19} + q \cdot 2^{-39}$	
	L0 6F			
37	10 20F			
	42 1F		= key word	
38	L5 1F			
	26 39L		Store key word	
39	40 ( )S4			
	L5 39L			
40	L4 51L			
	46 39L		Plant addresses for next key word	
41	46 3L			
	26 42L			
42	50 ( )F	By 9'	Q = [number of parts in present interior	
	00 3F		sequence] . 2^{-16}	
43	11 3F	From 48	N(7) = [number of interior parts +	
	L5 7F	From 50'	number of exterior points + number of	
44	S4 F		boundary points] . 2^{-19}	
	40 7F			
45	L5 1L			
	L4 51L		Set address for next temporary key	
46	46 1L			
	26 1L			
47	L0 52L	From 5	Test for boundary sequence	
	32 48L	No		
48	26 43L	Yes		
	11 3F	From 47'		
49	L5 8F		N(8) = [number of exterior points] x	
	S4 F			
50	40 8F			
	22 43L			
51	00 1F			
	00 F		2^{-19}	
52	00 F			
	00 1F		2^{-39}	

LOCATION	ORDER		NOTES	PAGE 4 G 1
53	00 F			
	00 4F		4×2^{-39}	
54	00 47F			
	00 F		47×2^{-19}	
	26 10N			
BOUNDARY INPUT				
	00 26K			
	DNSI Code N3		26 - 46F	
	00 36K			
36	66 1F			
	26 5F			
37	40 (47)F			
	L5 37F		Modification to DNSI code (N3)	
38	L4 32F			
	46 37F			
39	L5 2F			
	L0 45F			
40	40 1F			
	34 999F		Read in "Evaluate interior points" routine	
	00 5K			
0	S5 1F			
	32 1L		interior points or boundary points	
1	26 37F			
	40 1F		$N(1) = 10^{-3}(n + 1)$	
2	L5 37F		Get address of interior points	
	L0 20L			
3	46 7L			
	46 10L		Plant first interior point address	
4	46 14L			
	L0 20L			
5	46 6L		Plant left hand boundary point address	
	46 13L			
6	L5 ( )F	By 5		
	50 21L			
7	L0 ( )F	By 3	Find functional increment per interior	
	40 4F		point	

LOCATION	ORDER		NOTES PAGE 5 G 1
8	75 4F		
	66 1F		
9	S5 F		
	40 4F		
10	L5 ( )F	By 3'	N(3) = right hand boundary point
	40 3F		
11	L5 21L		
	L2 1F		Count to n and test
12	40 1F		
	36 18L		
13	L7 ( )F	By 5',15	Form initial values for interior points
	L4 4F		
14	40 ( )F	By 4, 16	
	L5 14L		
15	46 13L		
	L4 20L		
16	46 14L		Set addresses in loop
	00 63F		
17	00 63F		
	26 11L		
18	L5 14L		
	46 37L		Prepare to store right hand boundary
19	L5 3F		point
	26 37F		
20	00 1F		$2^{-19} + 2^{-39}$
	// 00 1F		
21	00F 00001		
	// 00000 1000J		$10^{-3} + 10^{-9}$
	// 26 28N		
	EVALUATE INTERIOR POINTS.		
	00 10K		
0	41 6F		N(6) = 0
	L3 ( )S4	By 23	Test for end of grid
1	32 19L		
	L5 ( )S4	By 23'	N(2) = current key word

LOCATION	ORDER		NOTES
2	40 2F		
	46 7L		
3	46 14L		Plant p, q
	42 11L		
4	L4 25L		
	46 13L		Plant p + 1
5	L0 26L		
	46 12L		Plant p - 1
6	10 30F		
	42 12L		Plant m
7	L5 (p)F	By 2'	N(p) = N(4)
	40 4F		
8	36 11L		Test for boundary point
	L5 L	Yes	
9	L4 25L		Plant key work addresses
	42 L		
10	42 1L		
	22 L		End of row
11	19 37F	From 8	
	L7 (q)F	By 3'	
12	L6 (p-1)F	By 5'	$N_{i+1}(p) = 1/4\ [N_{i+1}(m) + N_i(q) + N_{i+1}$
	L6 (m)F	By 6'	$(p-1) + N_i(p+1)]$
13	L6 (p+1)F	By 4'	
	10 2F		
14	40 (p)F		
	L0 4F		
15	40 4F		
	L7 4F		$N(6) = \Sigma\ \lvert N_{i+1}(p) - N_i(p) \rvert$
16	L4 6F		
	40 6F		
17	36 18L		Test for overflow
	40 27L	Yes	N(27L) < 0
18	L5 2F		
	L4 25L		Step m, p, q by 1
19	26 2L		
	L1 27L	From 1	

LOCATION	ORDER		NOTES
20	36 22L		Test for overflow
	L5 28L	No	
21	L0 6F		Test for end
	34 999F		Read in print routines
22	49 27L	From 20	Reset N(27L) $\geq$ 0
	L5 24L		
23	42 L		
	42 1L		Reset addresses and start over on grid
24	26 L		
	0F S4		Address used
25	00 1025F		$2^{-9} + 2^{-19} + 2^{-29} + 2^{-39}$
	00 1025F		
26	00 1026F		
	00 1025F		$2^{-9} + 2 \cdot 2^{-19} + 2^{-29} + 2^{-39}$
27	40 F		
	00 F		1/2
28	00F 00000		
	005 000000J		$5 \cdot 10^{-6}$
	24 10N		

PRINT ROUTINE

	00 10K		
0	L5 (47)F		Argument to "single column p
	50 L		
1	26 16F		
	19 18F		
2	L4 L		Step address for new argument
	46 L		
3	L0 5L		Test for end
	32 4L		
4	26 L		Print next point
	00 F		final stop
5	L5 1S4		End constant
	50 L		
	00 16K		
	Single Column Print (P 6)		
	24 10N		