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SURFACES FOR COMPUTER-AIDED DESIGN OF SPACE FORMS

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by

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Project MAC

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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ABSTRACT

The design of airplanes, ships, automobiles, and so-called "sculptured parts" involves the design, delineation, and mathematical description of bounding surfaces. A method is described which makes possible the description of free-form doubly curved surfaces of a very general kind. An extension of these ideas to hyper-surfaces in higher dimensional spaces is also indicated.

This surface technique has been specifically devised for use in the Computer-Aided Design Project at M.I.T., and has already been successfully implemented here and elsewhere.

ACKNOWLEDGEMENTS

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SECTION I

INTRODUCTION

The purpose of this work is to present the mathematics of a certain class of surfaces which are suitable for the design and description of arbitrary shapes. In the past, the subject of surface mathematics has been investigated, in analytical geometry and in differential geometry, from the standpoint of the analysis of geometric properties of surfaces that already exist, but very little literature has been produced on the subject of the creation of such surfaces. As a typical example, the design of the hull of a racing yacht requires the description of a surface of considerable subtlety and complexity, and the process is traditionally carried out by purely graphical procedures which are exceedingly laborious, since they entail a large amount of trial and error iteration in order to assure that the surface is completely described, and is smooth and "fair." The design of automobile bodies and airplane fuselages is similarly tedious and time consuming, although mathematical techniques have been applied to aircraft design for a number of years.

A few papers have been written on the subject of fitting existing ship hull shapes by means of various types of polynominals, with the two-fold purpose of smoothing and interpolating the information contained in preliminary graphically derived hull lines, and of replacing this graphical information with formulas and equations that will permit further analytical techniques to be applied, such as structural analysis and the discipline of fluid mechanics. But these mathematical techniques are applicable only when the surface has already been designed to some degree of completeness, so as to furnish enough information to make the mathematics work.

The mathematical structure of the surfaces to be described in the following discussion has been devised to implement the surface design process itself, so as to make it, from the designer's standpoint, extremely natural and easy. The designer himself need not know or care about these internal mathematical details, any more than he needs to know the specific composition of the pencils with which he writes or the mechanics of the splines with which he now draws curves. The mathematics is relatively simple, but it is nevertheless too complicated for hand calculation, and is designed for use on a computer.

In the design of a three-dimensional object, whether it be an airplane fuselage, an automobile body, a ship's hull, or a single sculptured part of a machine, the designer requires a system which will permit him to define a surface with a minimum of input information, and then to modify this surface, if he feels so inclined, either by changing the original input, or by adding more design constraints to the system.

As a specific example, suppose a designer wishes to design an airplane fuselage, using the SKETCHPAD system.[1,2,3] He would like to be able to draw the outline of the airplane as seen from the side, the outline of the airplane as seen from above, and some arbitrarily selected section midships. With these three arbitrary curves designed, he would like to have the computer automatically and immediately generate a "fair" surface and display this surface to him in sufficient detail so that he could make appropriate judgments. If the surface so generated does not satisfy him, he would perhaps like to modify his original design curves, or else he might perhaps like to add other new sections and have the computer automatically and instantly re-fair the surface to fit this additional information.

The following sections describe a very simple, flexible and general class of surfaces which are able to fulfill these requirements. It will be shown that a single algorithmic structure and essentially only two symbol types serve to provide the following features:

1. Smooth, fair surfaces can be defined by a minimum number of curves, and then adjacent surfaces can be designed to match position, slope, curvature, and indeed any desired order of derivative along the adjoining boundaries.
2. The design curves that define the surface can be of any kind whatsoever, including circles, second-degree curves, polynominals, transcendentals, and also sketched curves with no known mathematical formula whatsoever.
3. Some classic surfaces are not necessarily members of the family of surfaces to be described; nevertheless, these classic surfaces can be matched along their boundaries to any order of derivative desired.

4. The arithmetic involved in constructing these surfaces is extremely simple and, we have found, easy to implement on a digital computer. It also lends itself to special-purpose computing hardware, such as digital or analog differential analysers. In addition, by virtue of the form of the algorithm, the parameters that define the shapes are extremely easy to compute. (In some cases they may require no computation at all.)

We intend to develop a method to construct complex arbitrary surfaces by piecing together surface "patches." Each such patch will be defined by four boundary curves, in principle, although it is harmless for one of the boundary curves to be degenerate, and to appear as a point instead of a curve segment. In the design of a surface, it is intended that the designer begin with a single surface patch, or a very small number of patches, and then subdivide these regions with additional design curves defining boundaries of smaller patches only when the internal surface needs modification. This is somewhat at variance with the customary procedure for mathematical curve fitting and surface fitting of existing curves and surfaces, in which a relatively large number of surface points already defined by some other procedure are used to obtain mathematical expressions for a surface that best fits them. Instead, the system to be described is intended to be used by the designer at the outset, in the process of designing the surface, rather than later on as a means for making it mathematical.

This is not to say that the surface-patch technique cannot be used to formulate patch-wise mathematical expressions for existing surfaces, but rather to indicate that the primary purpose of this surface technique is to facilitate the initial design process itself.

When the design process is completed, the surface will be completely mathematically defined, since this definition occurs automatically and concurrently with design.

Ordinarily a ship's hull or an airplane fuselage is described by certain important curves such as, in the case of the hull, a keel curve, a midships section, and a curve representing the sheer or deck line; these curves are sufficient to determine a surface, since they form the boundaries of a surface patch. However, ordinarily this primary surface will not have certain desired characteristics, and it will have to be modified by introducing additional information, such as for instance one or two other section curves. When these additional curves are introduced, the surface algorithm permits the computer to "re-fair" the original hull form to contain these curves.

Similarly, an airplane fuselage can be designed by drawing a profile curve, a maximum half-breadth curve, and a mid-section of the fuselage. Again these curves suffice to define a primary surface, which in most cases will require modification by the addition of a few more curves to make more explicit the designer's wishes. As these additional curves are introduced, the original surface will be sub-divided into patches, but the algorithm will automatically insure continuity of surface slope and curvature (if desired) and will incorporate these additional curves into the surface automatically. This should make initial surface design virtually painless, and is intended to remove the tedious process of surface fairing as it is now practiced in naval architecture. Airplane fuselages are usually somewhat simpler shapes than, say, yacht hulls, and for a number of years second-degree curves have been successfully used for fuselage design; on the other hand, naval architects have steadily resisted the use of such methods in their work, since the complexity of yacht shapes makes it necessary to pay attention to the irksome details of the geometry involved, and second-degree curves prove to be cumbersome in such applications.

The system that is described in this report is intended to furnish the flexibility that second-degree curve techniques lack, and to remove almost entirely the need for the designer to be an analytical geometer. With this system implemented on a computer, there is reason to believe that the computer can take over all of the geometrical and mathematical burden of the design process, and leave the user free to be a sculptor assisted by an exquisitely skillful mechanical slave.

Ultimately, when a graphical input-output hardware for a computer is available in the engineering design office, these methods will permit designers to delineate complex shapes with great ease, by simply drawing the salient curves that define and describe them. Already experiments along these lines are in progress in a few isolated laboratories both in universities and in industry. Very soon the two severe handicaps that have inhibited the wider use of such graphical devices will be removed. These inhibiting factors have been high cost for the terminal hardware and small size of the working area. Rapid strides are being made on both these fronts, and within a few years it will be possible not only to draw on a virtually unlimited drawing surface, but to draw objects directly in three-dimensional space, and to view these constructed objects as one would view an actual physical thing.

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SECTION II

NOTATION

We shall in what follows relate the x, y, and z coordinates of points on a surface to two independent variables u and w, so that we could write

$$x = f(u, w)$$
$$y = g(u, w)$$
$$z = h(u, w).$$

If the functions f, g and h were specified, then for a pair of values of u and w, a point in space would be defined. If we held one of the independent variables fixed, say w, then by allowing u to vary, the point in space would trace out a curve. If subsequently we set w to a new fixed value and again allowed u to vary, we would trace out another curve, and so on. Clearly by stepping the values of w by small increments and allowing u to vary after each such step, we could produce a family of space curves that would lie on the surface and define it. All that is needed is some convenient and systematic way of arriving at the functions f, g, and h.

It will turn out that the form of all of these three functions is the same; only certain internal numerical values are different. In vector notation we can write

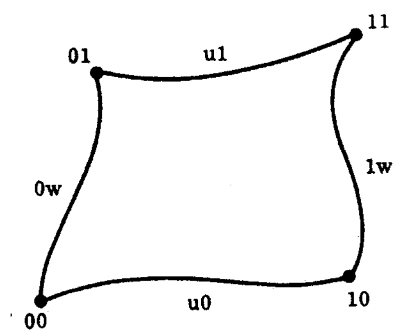
$$\left[x \;\; y \;\; z\right] = \left[f(u,w) \quad g(u,w) \quad h(u,w)\right]$$

Since $V = \left[x \;\; y \;\; z\right]$ is a suitable conventional abbreviation for the vector on the left, we introduce a similar abbreviation for the right hand side:

$$(uw) = \left[f(u,w) \quad g(u,w) \quad h(u,w)\right]$$

Here, in the abbreviated symbol on the left, we shall omit the comma between the two letters. Later on, when no ambiguity can arise, we shall omit the parentheses as well, and write simply uw to stand for the vector. It is to be remembered that uw does not stand for the ordinary product of the two quantities, but is merely a bi-literal symbol standing for a vector whose components are functions of the two variables.

We plan to build up surfaces by adjoining surface "patches," in an analogy of the piecewise fitting of complicated curves by curve segments suitably joined together. Accordingly, we shall at the beginning focus our attention on one such surface patch. To simplify arithmetic, we shall stipulate that the independent variables, or **parameters**, u and w can take on only values between 0 and 1. Then a surface patch can be considered to be a surface segment bounded by four space curves, (0 w), (1 w), (u 0) and (u 1).



Here, typically, the symbol (0w) stands for the vector describing the x, y, and z coordinates of points along the curve generated by allowing w to vary continuously from 0 to 1, while u is held fixed and equal to 0.

We shall introduce two scalar functions, F_0 and F_1 each a function of a single variable. These will be referred to as "blending functions" for reasons that will become clear.

In order to compress the surface equation, and the proofs that we wish to demonstrate, we shall use a kind of indicial notation; we introduce the indices i and j, which can assume only the values 0 and 1, and we invoke the customary summation convention for terms with repeated indices. This convention in our case simply means that when an index is repeated in a term, we write out all the possible terms that the actual indicial values generate, and then add them.

SECTION III

THE SURFACE EQUATION

With these conventions and notational peculiarities in mind, we write

$$(uw) = (iw)F_i(u) + (uj)F_j(w) - (ij)F_i(u)F_j(w).$$

(Typically, the first term on the right expands as follows:

$$(iw)F_i(u) = (0w)F_0(u) + (1w)F_1(u).$$

Thus the complete expansion would consist of eight terms, if carried out.) We shall proceed to demonstrate that this surface equation represents a surface that contains the four boundary curves, and is thus defined by them.

We must make a stipulation, a weak one, on the nature of the blending functions F_0 and F_1:

$$F_0(0) = 1 \qquad\qquad F_0(1) = 0$$

$$F_1(1) = 1 \qquad\qquad F_1(0) = 0$$

A further stipulation is that F_0 and F_1 be continuous and monotonic over the interval.

Now set u = a, where a can only be either 0 or 1. Then, substituting in the surface equation,

$$(aw) = (iw)F_i(a) + (aj)F_j(w) - (ij)F_1(a)F_j(w).$$

Consider $F_i(a)$ which occurs twice in the equation. By the stipulation, if i = a, $F_i(a) = 1$.

Otherwise, if $i \neq a$, $F_i(a) = 0$.

Hence all terms in the expansion that contain $i \neq a$ vanish; we can set i = a and what remains is

$$\begin{aligned}(aw) &= (aw)F_a(a) + (aj)F_j(w) = (aj)F_a(a)F_j(w)\\ &= (aw) \qquad + (aj)F_j(w) - (aj)F_j(w)\\ &= (aw).\end{aligned}$$

This shows that for a = 0 or 1, and hence (aw) = (0w) or (1w), the surface equation reduces to an identity. This implies that the surface contains its boundaries. An entirely parallel argument would show that the equation also reduces to an identity for the other two boundaries (u0) and (u1).

Provided a pair of functions F_0 and F_1 are chosen once and for all that satisfy the stipulations, the surface equation may be constructed immediately and uniquely for any set of boundary curves (u0) (u1) (0w) and (1w). It is to be observed that no restrictions have been placed on the form of the boundary curves; there is perhaps the restriction that they form a closed boundary, at least at the corners (ij) = (00), (01), (10), and (11) otherwise there will be multiple values within the surface segment; similarly they should be continuous functions, but apart from these rather obvious restrictions, they can be of any shape whatever, including curves that can only be represented by tables of values.

We can gain intuitive insight into the nature of such a surface if we look at one of the terms, say $(uj)F_j(w)$.

We have the expansion

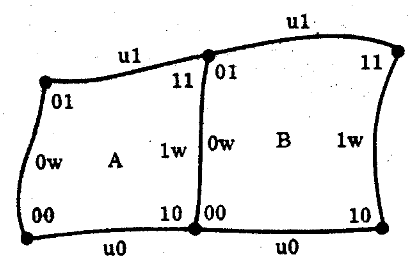
$$(uj)F_j(w) = (u0)F_0(w) + (u1)F_1(w).$$

This represents a weighted average of the quantities (u0) and (u1). When w = 0, $F_0(0) = 1$ and $F_1(0) = 0$, and the expression becomes simply (u0). As w increases, the weight of $F_0(w)$ decreases, while that of $F_1(w)$ increases, so that the surface partakes of the nature of both boundary curves. As w approaches the value 1, the influence of (u0) on the shape of the surface gradually disappears, while the influence of (u1) gradually becomes dominant. Finally, at w = 1, the curve (u1) represents the shape of the surface. We can say that the surface is generated by a gradual transition from (u0) to (u1), and that these two curve shapes are "blended" together by virtue of the blending functions F_0 and F_1. This discussion is somewhat oversimplified, since we have omitted the term $(iw)F_j(u)$ and it too plays a part in determining the shape of the internal surface, as does of course the term involving the corner coordinates, $(ij)F_i(u)F_j(w)$.

The entire surface equation is seen to be symmetric in u and w, and by virtue of this and a secondary symmetry in the functions F_0 and F_1, we can abbreviate proofs about the behavior of the surface along all boundaries by exhibiting a typical proof for any one boundary.

3.1 BOUNDARY SLOPE CONTINUITY

It is our aim to design and delineate complicated surfaces by adjoining surface patches, in a piecewise fashion. Consider two such patches A and B,



with a common boundary. For patch A the boundary is (1w); for patch B it is (0w), and the vectors of coordinates are equal,

$$A\ (1w) = B\ (0w).$$

Then the two patches will be continuous across their common boundary. They will however in general be discontinuous <u>in slope</u> across the boundary, and we wish to investigate this and make some amendments that will correct this discontinuity of slope.

We take the partial derivative with respect to u: Our symbolism for this partial derivative is $(uw)_u = \frac{\partial\,(uw)}{\partial\,u}$, and when we substitute, say, u = 0, we can write $(0w)_u$ to mean the value of the partial derivative so obtained. Then

$$(uw)_u = (iw)\,F_i'(u) + (uj)_u F_j(w) - (ij)\,F_i'(u)\,F_j(w).$$

Now substitute u = a = 0 or 1, as before.

$$(aw)_u = (iw)\,F_i'(a) + (aj)_u F_j(w) - (ij)\,F_i'(a)\,F_j(w).$$

If we now place additional constraints on the blending functions, that their first derivatives

$$F_i'(a) = 0 \qquad (a = \text{either } 0 \text{ or } 1)$$

we obtain the result

$$(aw)_u = (aj)_u F_j(w),$$

all other terms vanishing.

This implies, for example, that when a = 0,

$$(0w)_u = (00)_u F_0(w) + (01)_u F_1(w),$$

or, the derivative anywhere along the boundary in the u direction (across the boundary) depends only upon the derivatives at the end-points of the boundary; it is entirely independent of the shapes of the four boundary curves, including the boundary (0w) itself.

Thus for the two patches A and B, if

$$A(10)_u = B(00)_u$$

and

$$A(11)_u = B(01)_u$$

i.e., if the boundary curves are continuous in slope in the u direction at the ends of the contiguous boundary between patches, we are guaranteed to have

$A(1w)_u = B(0w)_u$ everywhere along the boundary regardless of the shapes of the boundary curves of A and B. This is a remarkably powerful and useful property, achieved at the slight expense of extending the stipulations on the F_i.

Similarly, the second derivative with respect to u is

$$(uw)_{uu} = (iw)\, F_i''(u) + (uj)_{uu} F_j(w) - (ij)\, F_i''(u) F_j(w)$$

and if we further stipulate that $F_i''(a) = 0$ we obtain

$$(aw)_{uu} = (aj)_{uu} F_j(w).$$

This establishes second derivative (or curvature) continuity as an automatic and inherent property of adjacent patches, provided their boundary curves have this kind of continuity at the end-points of the boundary. It is easy to see that we may escalate in this way to any level of derivative continuity we wish along contiguous boundaries.

3.2 SLOPE CORRECTION SURFACE

The surface equation already described is very general, in the sense that it can contain virtually any boundary curve we wish, and it has certain benign properties of derivative matching along boundaries; nevertheless it is not a universal formula for all surfaces, and there are many that do not belong to its family. We have already seen that surfaces generated by the surface equation have a definite intrinsic slope along boundaries, whose variation is rigidly prescribed by a simple formula. Obviously surfaces exist whose boundary slopes do not match this intrinsic slope, except at the end-points of boundaries. Nevertheless, we wish to be able to patch together such other surfaces with our special surfaces, so as to have slope continuity (or continuity of any level of derivative).

To do so, we introduce a new surface equation, describing a slope-correction surface, which when added to the first surface equation has the property of leaving the boundaries unchanged, but causing the derivatives across boundaries to vary in any arbitrary way we wish, as we move along the boundary.

The equation resembles the first form very strongly. It is

$$(uw) = (iw)_u G_i(u) + (uj)_w G_j(w) - (ij)_{uw} G_i(u) G_j(w).$$

Here, typically, $(iw)_u$ is a function of w only, and describes the arbitrary variation of the derivative with respect to u as w varies, along the curve (iw), and similarly for the other boundaries. The vector $(ij)_{uw}$ represents the cross derivatives of the four corners. Typically,

$$(00)_{uw} = \frac{\partial^2 (uw)}{\partial u \partial w} \qquad \begin{matrix} u = 0 \\ w = 0 \end{matrix}$$

The functions G_0 and G_1 are again blending functions or weighting functions, but they have properties different from the functions F_0 and F_1. We stipulate

$$G_0(0) = G_1(0) = G_0(1) = G_1(1) = 0$$

$$G'_0(0) = 1 \qquad G'_1(1) = 1$$

$$G'_0(1) = 0 \qquad G'_1(0) = 0$$

or in the indicial notation used earlier,

$$G_i(a) = 0, \text{ a and i} = 0 \text{ or } 1.$$

$$G'_i(a) = 0, \ a \neq i.$$

$$G'_i(a) = 1, \ a = i.$$

We need to ensure that the vectors describing the boundaries vanish identically, and that the vectors describing the slope variation along boundaries are indeed given by the equation. The proof proceeds along precisely the same lines we used before. First, substitute u = a. The equation becomes

$$(aw) = (iw)_u G_i(a) + (aj)_w G_j(w) - (ij)_{uw} G_i(a) G_j(w).$$

$$= (aj)_w G_j(w).$$

Consider $(aj)_w$. We wish to have the correction surface leave the original boundary vectors unchanged, and hence the boundary vectors of the correction surface must vanish; ie,

$$(iw) = 0$$

$$(uj) = 0$$

Then the derivatives of these boundaries must also vanish; in particular,

$(iw)_w = 0$ and then $(ij)_w = 0$, when w = j. Hence $(aj)_w = 0$.

Thus (aw) = 0 indicates the desired behavior of the correction surface along a boundary. Similarly (ua) = 0.

To examine the slope variation along a boundary, differentiate the equation with respect to u:

$$(uw)_u = (iw)_u G_i'(u) + (uj)_{uw} G_j(w)$$
$$- (ij)_{uw} G_i'(u) G_j(w).$$

Now substitute u = a

$$(aw)_u = (iw)_u G_i'(a) + (aj)_{uw} G_j(w) - (ij)_{uw} G_i'(a) G_j(w)$$

As before, $G_i'(a) = 1$ if and only if $a = i$,
so we get

$$(aw)_u = (aw)_u G_a'(a) + (aj)_{uw} G_j(w) - (aj)_{uw} G_a'(a) G_j(w)$$
$$= (aw)_u + (aj)_{uw} G_j(w) - (aj)_{uw} G_j(w)$$
$$= (aw)_u.$$

This demonstrates that the surface has the slope variation along the boundary as required. To make use of this slope correction surface, we must first determine what the intrinsic slope of the surface to be corrected is, and then we must subtract this slope from the desired boundary slope, to yield the correction slopes that enter into the equation. Thus if $(0w)_u$ is the desired slope, and

$I\ (0w)_u$ is the intrinsic slope, then

$C\ (0w)_u$ will be the correction slope,

$C\ (0w)_u = (0w)_u - I\ (0w)_u.$

The correction slopes $C\ (iw)_u$ and $C\ (uj)_w$ are the four functions that enter into the slope correction surface. The desired surface is obtained by adding the correction surface to the first surface:

$$(uw) = I\ (uw) + C\ (uw)$$

where we use the symbol I (uw) to represent the surface whose boundary slope is being modified.

3.3 HIGHER-ORDER CORRECTION SURFACES

Analogous forms may be obtained for correction of higher derivatives along boundaries. For second derivative correction, the surface equation is

$$(uw) = (iw)_{uu}H_i(u) + (uj)_{ww}H_j(w) - (ij)_{uuww}H_i(u)H_j(w).$$

In this equation, the blending functions H_i have the stipulations that, for a = 0 or 1 as before,

$$H_i(a) = 0$$

$$H_i'(a) = 0$$

$$H_i''(a) = 0, \ i \neq a$$

$$H_i''(i) = 1, \ i = a.$$

With these constraints on the H_i, it is easy to arrange matters so that this second-order correction surface is zero everywhere on the boundary, has zero slopes across boundaries, and has second derivatives across boundaries specified by $(iw)_{uu}$ and $(uj)_{ww}$ whatever these functions may be. The addition of this surface vector to a given surface vector will then provide a means for boundary second-derivative correction without disturbing either the boundary shapes or boundary slopes.

Although we have already carried out a similar proof for slope correction, it might be well to exhibit once again the course of the argument.

First, to show that the boundary vectors are zero, substitute u = a: (a = 0 or 1.)

$$(aw) = (iw)_{uu}H_i(a) + (aj)_{ww}H_i(w) - (ij)_{uuww}H_i(a)H_j(w)$$

$$= (aj)_{ww}H_i(w).$$

The term $(aj)_{ww}$ refers to the second derivative in the w sense at each of the four corners, such as, typically, (00). As in the case of slope correction, we must have

$$(iw) = 0 \text{ along boundaries.}$$

Then $(iw)_{ww} = 0$, $(ij)_{ww} = 0$, and in particular $(aj)_{ww} = 0$, so that the equation satisfies the boundary condition.

For boundary slope vectors, differentiate with respect to u:

$$(uw)_u = (iw)_{uu} H_i'(u) + (uj)_{wwu} H_j(w) - (ij)_{uuww} H_i'(u) H_j(w)$$

Set u = a:

$$(aw)_u = (aj)_{wwu} H_j(w).$$

We wish to have the slope vectors vanish along boundaries, so typically

$(iw)_u = 0$ for all w.

But then $(iw)_{uw} = 0$ and $(iw)_{uww} = 0$ by taking derivatives. The order of differentiation is immaterial, so

$(iw)_{uww} = (iw)_{wwu}$, and finally we can conclude that

$(aj)_{wwu} = 0$; again the right and left hand sides of the equation are in agreement.

Finally, we differentiate again with respect to u:

$$(uw)_{uu} = (iw)_{uu} H_i''(u) + (uj)_{wwuu} H_j(w) - (ij)_{wwuu} H_i''(u) H_j(w).$$

Set u = a; only terms in which a = i remain:

$$\begin{aligned}(aw)_{uu} &= (aw)_{uu} H_a''(a) + (aj)_{wwuu} H_j(w) - (aj)_{wwuu} H_a''(a) H_j(w)\\ &= (aw)_{uu}\end{aligned}$$

Again we have demonstrated an identity. The escalation to any level of boundary derivative correction vector is obvious.

3.4 MATRIX FORM

The surface equation

$$(uw) = (iw) F_i(u) + (uj) F_j(w) - (ij) F_i(u) F_j(w)$$

may be expanded directly into matrices, to yield:

$$(uw) = \begin{bmatrix} u0 & u1 \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \end{bmatrix} + \begin{bmatrix} F_0 u & F_1 u \end{bmatrix} \begin{bmatrix} 0w \\ 1w \end{bmatrix}$$

$$- \begin{bmatrix} F_0 u & F_1 u \end{bmatrix} \begin{bmatrix} 00 & 01 \\ 10 & 11 \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \end{bmatrix}.$$

In this we have treated the indicial form term by term in a straightforward way. We shall in what follows omit parentheses, since no misunderstanding can arise. Thus typically $F_0 u$ is written in place of $F_0(u)$ as a matter of convenience and economy. Similarly, typically 00 is written instead of (00); the reader should be reminded that this is merely a compact way of exhibiting the x, y, z coordinates at point (00).

It means:

$$00 = \begin{bmatrix} x(00), y(00), z(00) \end{bmatrix} \quad \text{when written out completely.}$$

The three vector (matrix) products are equivalent to the following three products:

$$\begin{bmatrix} 1 & F_0 u & F_1 u \end{bmatrix} \begin{bmatrix} 0 & u0 & u1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ F_0 w \\ F_1 w \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & F_0 u & F_1 u \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0w & 0 & 0 \\ 1w & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ F_0 w \\ F_1 w \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & F_0 u & F_1 u \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -00 & -01 \\ 0 & -10 & -11 \end{bmatrix} \begin{bmatrix} 1 \\ F_0 w \\ F_1 w \end{bmatrix}$$

and in this form we can perform the addition, obtaining

$$(uw) = \begin{bmatrix} 1 & F_0 u & F_1 u \end{bmatrix} \begin{bmatrix} 0 & u0 & u1 \\ 0w & -00 & -01 \\ 1w & -10 & -11 \end{bmatrix} \begin{bmatrix} 1 \\ F_0 w \\ F_1 w \end{bmatrix}.$$

It is slightly more convenient to rewrite this in the equivalent form

$$(uw) = - \begin{bmatrix} -1 & F_0u & F_1u \end{bmatrix} \begin{bmatrix} 0 & u0 & u1 \\ 0w & 00 & 01 \\ 1w & 10 & 11 \end{bmatrix} \begin{bmatrix} -1 \\ F_0w \\ F_1w \end{bmatrix}$$

so as to avoid the awkward minus signs in the 3 x 3 matrix.

Two facts should be noted. The leading row vector in front of the matrix and the trailing column vector following the matrix are transposes of one another, but with different arguments; the matrix represents the boundary conditions of a patch. The partition $\begin{bmatrix} 00 & 01 \\ 10 & 11 \end{bmatrix}$ is redundant, since its elements must agree with uj and iw for u and w equal to 0 or 1.

We have already suggested that we can maintain slope continuity across boundaries by suitable stipulations on F_i, and we have also already suggested that when desired we can adjust slopes across boundaries by a second additive vector with suitable stipulations on its G_i. We shall now investigate the combined form of the surface equation. To do so we shall prefix a symbol to the vector uw to indicate whether we are talking about the first surface equation, or the correction surface equation, and we shall omit the prefix symbol when we are talking about the combined form. Thus

uw = suw + cuw, with
suw = the primary surface
cuw = the correction surface
uw = the combination.

Accordingly, using this notational convention, we will take derivatives, with respect to u, of the surface equation suw in order to determine its slope vector in the u direction.

$$suw_u = - \begin{bmatrix} 0 & F_0{}'u & F_1{}'u \end{bmatrix} \begin{bmatrix} 0 & su0 & su1 \\ s0w & s00 & s01 \\ s1w & s10 & s11 \end{bmatrix} \begin{bmatrix} -1 \\ F_0w \\ F_1w \end{bmatrix}$$

$$- \begin{bmatrix} -1 & F_0u & F_1u \end{bmatrix} \begin{bmatrix} 0 & su0_u & su1_u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ F_0w \\ F_1w \end{bmatrix}$$

We substitute u = 0, and obtain

$$s0w_u = \begin{bmatrix} s00_u & s01_0 \end{bmatrix} \begin{bmatrix} F_0w \\ F_1w \end{bmatrix} .$$

Now consider, for example, $s00_u$ and the desired 00_u. The symbol $s00_u$ refers to the slope vector at a corner; we have already seen that at corners the correction surface $c00_u = 0$, and so $00_u = s00_u$. This is bourne out intuitively by the reflection that at (ij) corners, the two crossing boundary curves completely define the slopes there; since this is so, no correction of slope need or can be applied.

Hence we should write

$$s0w_u = \begin{bmatrix} 00_u & 01_u \end{bmatrix} \begin{bmatrix} F_0w \\ F_1w \end{bmatrix} .$$

By analogy and symmetry we can write the remaining three statements:

$$s1w_u = \begin{bmatrix} 10_u & 11_u \end{bmatrix} \begin{bmatrix} F_0w \\ F_1w \end{bmatrix}$$

$$su0_w = \begin{bmatrix} F_0u & F_1u \end{bmatrix} \begin{bmatrix} 00_w \\ 10_w \end{bmatrix}$$

$$su1_w = \begin{bmatrix} F_0u & F_1u \end{bmatrix} \begin{bmatrix} 01_w \\ 11_w \end{bmatrix} .$$

In order to obtain a desired slope vector along any of the boundaries, we add the correction surface, whose equation is

$$cuw = -\begin{bmatrix}-1 & G_0u & G_1u\end{bmatrix}\begin{bmatrix}0 & cuo_w & cu1_w \\ c0w_u & c00_{uw} & c01_{uw} \\ c1w_u & c10_{uw} & c11_{uw}\end{bmatrix}\begin{bmatrix}-1 \\ G_0w \\ G_1w\end{bmatrix}$$

As we have already remarked, the desired surface uw is the sum of the vectors suw and cuw. Hence the correction slope vector, such as, typically, $cu0_w$, is

$$cu0_w = u0_w - \begin{bmatrix}F_0u & F_1u\end{bmatrix}\begin{bmatrix}00_w \\ 10_w\end{bmatrix}$$

This is an entry in the correction surface matrix.

Now we introduce a new fact: the corner cross derivatives of the primary surface equation are all zero. To show this, differentiate the indicial expression first with respect to u, then with respect to w, and finally set u = a, w = b, where a and b are as usual either 0 or 1. We have

$$uw_u = (iw)F_i'(u) + (uj)_u F_j(w) - (ij)F_i'(u)F_j(w), \text{ and}$$

$$uw_{uw} = (iw)_w F_i'(u) + (uj)_u F_j'(w) - (ij)F_i'(u)F_j'(w).$$

Evidently this expression vanishes for (uw) = (a b). This shows that the corner "twists", or cross derivatives, of the original surface all vanish; it is a peculiarity of the first fundamental surface equation.

Hence we can assert that

$cij_{uw} = ij_{uw}$; This says that the desired twists at corners are identical with the correction surface twists, since the fundaments surface has <u>no</u> twist. We shall use this result to replace the partition $\begin{bmatrix}c00_{uw} & c01_{uw} \\ c10_{uw} & c11_{uw}\end{bmatrix}$

with

$$\begin{bmatrix}00_{uw} & 01_{uw} \\ 10_{uw} & 11_{uw}\end{bmatrix}$$

We can rewrite the expression for cuo_w as follows:

$$cuo_w = \begin{bmatrix} 1 & -F_0u & -F_1u \end{bmatrix} \begin{bmatrix} u0_w \\ 00_w \\ 10_w \end{bmatrix}$$

and

$$cu1_w = \begin{bmatrix} 1 & -F_0u & -F_1u \end{bmatrix} \begin{bmatrix} u1_w \\ 01_w \\ 11_w \end{bmatrix}$$

and of course, perhaps trivially,

$$0 = \begin{bmatrix} 1 & -F_0u & -F_1u \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Each of these matric products represents an element of the top row of the correction surface matrix.

Since the row matrix $\begin{bmatrix} 1 & -F_0u & -F_1u \end{bmatrix}$ is common to these three products, it can be factored out and introduced into the matrix $\begin{bmatrix} -1 & G_0u & G_1u \end{bmatrix}$ to yield

$$\begin{bmatrix} - \begin{bmatrix} 1 & -F_0u & -F_1u \end{bmatrix} & G_0u & G_1u \end{bmatrix}$$

which is the same as the vector

$$\begin{bmatrix} -1 & F_0u & F_1u & G_0u & G_1u \end{bmatrix}$$

We replace the elements of the top row of the correction surface matrix by the three matrices $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} u0_w \\ 00_w \\ 10_w \end{bmatrix}$ and $\begin{bmatrix} u1_w \\ 01_w \\ 11_w \end{bmatrix}$

This causes it to become a 5 x 3 matrix, and we now have the intermediate result,

$$cuw = -\begin{bmatrix} -1 & F_0u & F_1u & G_0u & G_1u \end{bmatrix} \left[\begin{array}{c|c|c} 0 & u0_w & u1_w \\ 0 & 00_w & 01_w \\ 0 & 10_w & 11_w \\ \hline c0w_u & 00_{uw} & 01_{uw} \\ c1w_u & 10_{uw} & 11_{uw} \end{array}\right] \begin{bmatrix} -1 \\ G_0w \\ G_1w \end{bmatrix}$$

By similar procedures, we can write for the elements of the first column of the correction surface matrix,

$$c0w_u = \begin{bmatrix} 0w_u & 00_u & 01_u \end{bmatrix} \begin{bmatrix} 1 \\ -F_0w \\ -F_1w \end{bmatrix}$$

$$c1w_u = \begin{bmatrix} 1w_u & 10_u & 11_u \end{bmatrix} \begin{bmatrix} 1 \\ -F_0w \\ -F_1w \end{bmatrix}$$

and again trivially, perhaps,

$$0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -F_0w \\ -F_1w \end{bmatrix}$$

When we factor out the common column matrix as before, and replace each entry of the column matrix $\begin{bmatrix} 0 \\ 0 \\ 0 \\ c0w_u \\ c1w_u \end{bmatrix}$ by the above expressions, we obtain the complete matric expression for the correction surface:

$$cuw = -\begin{bmatrix}-1 & F_0u & F_1u & G_0u & G_1u\end{bmatrix}\left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & u0_w & u1_w \\ 0 & 0 & 0 & 00_w & 01_w \\ 0 & 0 & 0 & 10_w & 11_w \\ \hline 0w_u & 00_u & 01_u & 00_{uw} & 01_{uw} \\ 1w_u & 10_u & 11_u & 10_{uw} & 11_{uw} \end{array}\right]\begin{bmatrix}-1 \\ F_0w \\ F_1w \\ G_0w \\ G_1w\end{bmatrix}$$

If now we border the original surface equation matrix, it can be written,

$$suw = -\begin{bmatrix}-1 & F_0u & F_1u & G_0u & G_1u\end{bmatrix}\left[\begin{array}{c|c|c|c|c} 0 & u0 & u1 & 0 & 0 \\ 0w & 00 & 01 & 0 & 0 \\ 1w & 10 & 11 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{bmatrix}-1 \\ F_0w \\ F_1w \\ G_0w \\ G_1w\end{bmatrix}$$

In this bordering process, the value of the matric product is unchanged.

Since the pre- and post-multiplicative matrices in this equation are the same as those of the correction surface equation, we can add the two 5 x 5 matrices and pre- and post-multiply by the two vectors. We shall perform, in fact,

uw = suw + cuw, and obtain

$$uw = -\begin{bmatrix}-1 & F_0u & F_1u & G_0u & G_1u\end{bmatrix}\left[\begin{array}{c|c|c|c|c} 0 & u0 & u1 & u0_w & u1_w \\ \hline 0w & 00 & 01 & 00_w & 01_w \\ 1w & 10 & 11 & 10_w & 11_w \\ \hline 0w_u & 00_u & 01_u & 00_{uw} & 01_{uw} \\ 1w_u & 10_u & 11_u & 10_{uw} & 11_{uw} \end{array}\right]\begin{bmatrix}-1 \\ F_0w \\ F_1w \\ G_0w \\ G_1w\end{bmatrix}$$

This is a general expression for a slope-matching, slope continuous surface patch with entirely arbitrary boundaries and entirely arbitrary slopes across these boundaries. There are no stipulations whatever on the nature of the boundary slope function. The stipulation on the F and G functions have already been discussed.

Now that we have constructively arrived at a general expression for surfaces that have a prescribed boundary vector and a prescribed boundary slope vector, it might be interesting to apply a proof to a conjectured higher order surface equation in which not only boundaries, boundary slopes, but also boundary second derivatives are vector quantities under control.

We postulate, therefore, that by analogy the surface equation is

$$uw = - \begin{bmatrix} -1 & F_0u & F_1u & G_0u & G_1u & H_0u & H_1u \end{bmatrix}$$

$$\times \left[\begin{array}{c|c|c|c|c|c|c} 0 & u0 & u1 & u0_{w} & u1_{w} & u0_{ww} & u1_{ww} \\ \hline 0w & 00 & 01 & 00_{w} & 01_{w} & 00_{uww} & 01_{ww} \\ 1w & 10 & 11 & 10_{w} & 11_{w} & 10_{ww} & 11_{ww} \\ \hline 0w_{u} & 00_{u} & 01_{u} & 00_{uw} & 01_{uw} & 00_{uww} & 01_{uww} \\ 1w_{u} & 10_{u} & 11_{u} & 10_{uw} & 11_{uw} & 10_{uww} & 11_{uww} \\ \hline 0w_{uu} & 00_{uu} & 01_{uu} & 00_{uuw} & 01_{uuw} & 00_{uuww} & 01_{uuww} \\ 1w_{uu} & 10_{uu} & 11_{uu} & 10_{uuw} & 11_{uuw} & 10_{uuww} & 11_{uuww} \end{array}\right] \begin{bmatrix} -1 \\ F_0w \\ F_1w \\ G_0w \\ G_1w \\ H_0w \\ H_1w \end{bmatrix}$$

It represents a surface patch whose vectors of coordinates, slope, and curvature as well, are everywhere arbitrary along its boundaries. The first column and first row of the 7 x 7 matrix represent these boundary conditions; the remainder of the matrix is redundant, since the quantities this partition contains must all come from the column and row by differentiation.

We can test this equation by seeing whether it contains a boundary curve. To this end, set $u = 0$, so that we check whether it contains the boundary (0w). We obtain, invoking the stipulations on the F, G, and H functions,

$$0w = - \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X \begin{bmatrix} 0 & 00 & 01 & 00_w & 01_w & 00_{ww} & 01_{ww} \\ 0w & 00 & 01 & 00_w & 01_w & 00_{ww} & 01_{ww} \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{bmatrix} \begin{bmatrix} -1 \\ F_0w \\ F_1w \\ G_0w \\ G_1w \\ H_0w \\ H_1w \end{bmatrix}$$

In the boundary matrix we have omitted irrelevant terms, because of the zero's in the pre-multiplying vector. We obtain, by performing the multiplication,

$$0w = - \begin{bmatrix} 0w & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ F_0w \\ F_1w \\ G_0w \\ G_1w \\ H_0w \\ H_1w \end{bmatrix}$$

$= 0w$, which is the hoped-for identity.

We can next try to see whether the equation also conforms to the boundary second derivative conditions. It will be more convenient in what follows to introduce some abbreviated notation.

Set

$$\begin{bmatrix} fu \end{bmatrix} = \begin{bmatrix} -1 & F_0u & F_1u & G_0u & G_1u & H_0u & H_1u \end{bmatrix}$$

and a similar expression for $\begin{bmatrix} fw \end{bmatrix}$.

Set the 7 x 7 boundary condition matrix equal to $[B]$.

With these abbreviations, the surface equation is

$$uw = - [fu] \; [B] \; [fw]^T .$$

We differentiate with respect to u:

$$uw_u = - \left([f'u] \; [B] + [fu] \; [B_u] \right) [fw]^T$$

and again:

$$uw_{uu} = - \left([f''u] \; [B] + 2 \; [f'u] \; [B_u] + [fu] \; [B_{uu}] \right) [fw]^T$$

We wish to investigate the right hand side of this equation for u = 0, that is, for $uw_{uu} = 0w_{uu}$. The blending function vector and its derivatives become

$$[f''0] = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]$$
$$[f'0] = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]$$
$$[f0] = [-1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

As for the first and second partial derivatives of the $[B]$ matrix, all elements of $[B_u]$ and $[B_{uu}]$ vanish except for those in the top row.

Then

$$[f''0] \; [B] = [0w_{uu} \quad 00_{uu} \quad 01_{uu} \quad 00_{uuw} \quad 01_{uuw} \quad 00_{uuww} \quad 01_{uuww}]$$

$$2 \; [f'0] \; [B_u] = [\text{ the null vector}]$$

$$[f0] \; [B_{uu}] = [\; 0 \quad -00_{uu} \quad -01_{uu} \quad -00_{uuw} \quad -01_{uuw} \quad -00_{uuww} \quad -01_{uuww}]$$

The sum of these vectors is evidently

$$[0w_{uu} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

Finally,

$$0w_{uu} = - [0w_{uu} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \; [fw]^T$$

$$= 0w_{uu} \text{, as expected.}$$

We have shown that the extended surface equation satisfies the second derivative boundary conditions. In a similar way it can be shown to satisfy the first derivative boundary conditions, but this was skipped in favor of the proof for the higher derivative, since the procedure exhibits a few interesting points.

By analogy we could construct matrix products to represent surfaces which satisfy even higher derivative conditions across boundaries.

3.5 BOUNDARY CURVES

It is often convenient to use particular boundary curve functions defined by the curve end-points and end-point tangent vectors. We can use the blending functions themselves to define such curves. For example, the u0 boundary curve can be described by the equation

$$u0 = \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \begin{bmatrix} 00 \\ 10 \\ 00_u \\ 10_u \end{bmatrix}$$

where the column vector contains the end-point information. We observe that the row vector becomes $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ when u = 0; it becomes $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, when u = 1. Again, if we take derivatives of this row vector with respect to u we obtain $\begin{bmatrix} F_0'u & F_1'u & G_0'u & G_1'u \end{bmatrix}$ and this becomes $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ for u = 0, and it becomes $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$ for u = 1.

With this behavior of the row vector, it is easy to see that the equation does indeed represent a curve satisfying the end-point conditions.

The matrix form of the surface equation has been shown to be

$$uw = - \begin{bmatrix} -1 & F_0u & F_1u & G_0u & G_1u \end{bmatrix} \begin{bmatrix} 0 & u0 & u1 & u0_w & u1_w \\ 0w & 00 & 01 & 00_w & 01_w \\ 0w_u & 00_u & 01_u & 00_{uw} & 01_{uw} \\ 1w_u & 10_u & 11_u & 10_{uw} & 11_{uw} \end{bmatrix} \begin{bmatrix} -1 \\ F_0w \\ G_0w \\ G_1w \end{bmatrix}$$

Now when, in computing uw, we perform the matrix multiplication from the left, we have for the second column

$$\begin{bmatrix} -1 & F_0u & F_1u & G_0u & G_1u \end{bmatrix} \begin{bmatrix} u0 \\ 00 \\ 10 \\ 00_u \\ 10_u \end{bmatrix}$$

$$= -u0 + \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \begin{bmatrix} 00 \\ 10 \\ 00_u \\ 10_u \end{bmatrix} = -u0 + u0 = 0.$$

Now if similarly u1, $u0_w$ and $u1_w$ are functions of the same kind, their corresponding column products vanish just as in the case of u0. Accordingly, the resulting product of the three matrices has the form

$$uw = -\begin{bmatrix} P & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix} = P,$$

where P is the product of the row vector and the first column of the matrix, or

$$P = \begin{bmatrix} -1 & F_0u & F_1u & G_0u & G_1u \end{bmatrix} \begin{bmatrix} w \\ 0w \\ 1w \\ 0w_u \\ 1w_u \end{bmatrix}$$

$$= \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \begin{bmatrix} 0w \\ 1w \\ 0w_u \\ 1w_u \end{bmatrix}$$

If again the elements of the column vector, 0w, 1w, $0w_u$ and $1w_u$ are likewise functions described as outlined, we can write typically

$$0w = [\,00 \quad 01 \quad 00_w \quad 01_w\,] \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}$$

For the complete column vector we have

$$\begin{bmatrix} 0w \\ 1w \\ 0w_u \\ 1w_u \end{bmatrix} = \begin{bmatrix} 00 & 01 & 00_w & 01_w \\ 10 & 11 & 10_w & 11_w \\ 00_u & 01_u & 00_{uw} & 01_{uw} \\ 10_u & 11_u & 10_{uw} & 11_{uw} \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}$$

When we substitute this result for the column vector, we obtain the surface equation

$$uw = [\,F_0 u \quad F_1 u \quad G_0 u \quad G_1 u\,] \left[\begin{array}{cc|cc} 00 & 01 & 00_w & 01_w \\ 10 & 11 & 10_w & 11_w \\ \hline 00_u & 01_u & 00_{uw} & 01_{uw} \\ 10_u & 11_u & 10_{uw} & 11_{uw} \end{array}\right] \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}$$

This is a particularly convenient form for computation. The 4 x 4 matrix contains nothing but information about the corner coordinates, corner slopes, and corner twists; all entries are constants, and the partitions of the matrix systematically group these quantities. The leading row vector and the trailing column vector are transposes of one another, (but with different arguments, of course.)

We shall refer to the 4 x 4 matrix as the "boundary condition" matrix, and shall assign to it the symbol B, so that the matric equation for the surface could be written

$$uw = [\, F_0u \;\; F_1u \;\; G_0u \;\; G_1u \,] \;\; B \;\; \begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}$$

It must be remembered that each of the entries in B is a three-vector, whose components are x, y, and z coordinates and slopes and twists. This means that B is really a tensor.

3.6 BLENDING FUNCTIONS

We can relate the blending function vector to a so-called basis vector in the following way. Let $[\, u_1 \; u_2 \; u_3 \; u_4 \,]$ be a vector whose elements are a set of linearly independent functions of the variable u. Then we can postulate the existence of a matrix M such that

$$[\, F_0u \;\; F_1u \;\; G_0u \;\; G_1u \,] = [\, u_1 \;\; u_2 \;\; u_3 \;\; u_4 \,] \;\; M.$$

To evaluate the M matrix, we substitute u = 0, u = 1 on both left and right hand sides of the equation. Then we take derivatives of both sides, and again substitute u = 0 and u = 1. There results

$$\begin{bmatrix} F_0 0 & F_1 0 & G_0 0 & G_1 0 \\ F_0 1 & F_1 1 & G_0 1 & G_1 1 \\ F'_0 0 & F'_1 0 & G'_0 0 & G'_1 0 \\ F'_0 1 & F'_1 1 & G'_0 1 & G'_1 1 \end{bmatrix} = \begin{bmatrix} 0_1 & 0_2 & 0_3 & 0_4 \\ 1_1 & 1_2 & 1_3 & 1_4 \\ 0'_1 & 0'_2 & 0'_3 & 0'_4 \\ 1'_1 & 1'_2 & 1'_3 & 1'_4 \end{bmatrix} \;\; M.$$

The matrix on the left is the identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ by virtue of the stipulations on the blending functions and their derivatives. From this, we conclude that

$$M^{-1} = \begin{bmatrix} 0_1 & 0_2 & 0_3 & 0_4 \\ 1_1 & 1_2 & 1_3 & 1_4 \\ 0'_1 & 0'_2 & 0'_3 & 0'_4 \\ 1'_1 & 1'_2 & 1'_3 & 1'_4 \end{bmatrix}$$

and we need only find the inverse of this matrix (if possible) to obtain M.

(In the matrix, the notation $1'_2$ means $\left. \frac{d(u_2)}{du} \right|_{u=1}$ typically.)

In the next section we shall for the first time be specific about the basis vector $[u_1 \ u_2 \ u_3 \ u_4]$, but it is interesting and important to realize that so far in the discussion nothing has been said to diminish the generality of the mathematical structure. It is hoped that the reader will not lose sight of the fact that the surface equations in their several forms can be implemented in many ways. We propose to develop one such implementation in detail, but it is only one of many.

3.7 CUBIC BASIS VECTOR

Let the basis vector be

$$[u_1 \ u_2 \ u_3 \ u_4] = [u^3 \ u^2 \ u \ 1].$$

The vector on the right contains four specifically chosen linearly independent functions of u, the powers of u, and when multiplied by a coefficient vector yields cubic polynomials:

$$[u^3 \ u^2 \ u \ 1] \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = Au^3 + Bu^2 + Cu + D.$$

By the reasoning of the last section, we have for this vector basis,

$$M^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

from which we can obtain the desired inverse

$$M = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Now we can write

$$[F_0u \quad F_1u \quad G_0u \quad G_1u] = [u^3 \quad u^2 \quad u \quad 1] \; M.$$

We shall abbreviate the notation for the basis vector in what follows. We shall write

$$[u^3 \quad u^2 \quad u \quad 1] = U$$
$$\text{and } [w^3 \quad w^2 \quad w \quad 1] = W.$$

The matrice surface equation

$$uw = [F_0u \quad F_1u \quad G_0u \quad G_1u] \quad B \quad \begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}$$

now becomes, simply and compactly,

$$uw = U\,M\,B\,M^t\,W^t. \qquad \text{(Superscript t means transpose.)}$$

If U and W are cubic basis vectors, then the surface patch is the so-called bi-cubic surface. Such surfaces are very easy to compute, particularly since the basis vector is so easy to evaluate. In passing it is important to remark that the above compact surface equation is not limited to cubics;

U and W are not restricted to cubic basis vectors, and M is simply the matrix that generates the appropriate set of blending functions. Among other possibilities, U and W might be higher order polynomial basis vectors; or they might be any set of linearly independent functions. Provided the associated M^{-1} matrix has an inverse, these basis vectors are acceptable.

We can write, for w held fixed, an expression for a u - varying curve on the surface:

$$uw = U(M B M^t W^t) = U A$$

where A is a column vector of constant coefficients. We can write a similar expression for u held fixed and w varying. The matrix product MBM^t is the same in either case. This suggests that for any surface patch this product should be evaluated first; thereafter, we can either obtain u-varying, w-constant curves or w-varying, u-constant curves in an obvious way.

We shall investigate another basis vector that is composed of another set of linearly independent functions (not powers of u) in a later article.

3.8 DIFFERENCE EQUATIONS

If the basis vectors are polynomial bases, we can invoke the techniques of finite differences to calculate points on the surface patch.

Consider the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Then

$$L \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ a+b \\ a+b+c \\ a+b+c+d \end{bmatrix}$$

If a, b, c, d are respectively third, second, first and zero-order differences of the cubic n^3, then the column matrix on the right of the equation represents the corresponding differences for the cubic $(n + 1)^3$. The differences for $(n + k)^3$ are given by

$$L^k \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$ where L^k means k successive

multiplications by the L matrix.

When n = 0, we can easily find that

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 1 \\ 0 \end{bmatrix}$$ for cubics.

Using this,

$$k^3 = [\,0001\,]\; L^k \begin{bmatrix} 6 \\ -6 \\ 1 \\ 0 \end{bmatrix}$$

In this expression, the vector [0 0 0 1] serves to select the bottom element of the resulting column vector after k multiplications by L.

By extension, we can write the more complete statement

$$[\,k^3 \;\; k^2 \;\; k \;\; 1\,] = [\,0 \;\; 0 \;\; 0 \;\; 1\,]\; L^k \begin{bmatrix} 6 & 0 & 0 & 0 \\ -6 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We shall call the [0 0 0 1] vector l; and we shall call the 4 x 4 matrix N, so that

$$[\,k^3 \;\; k^2 \;\; k \;\; 1\,] = l\, L^k\, N.$$

Now let the usual parametric variable u be represented by $u = k\delta$, where $k = 0, 1, 2 \ldots$ and where δ is an increment size. Then

$$[u^3 \; u^2 \; u \; 1] = [k^3 \; k^2 \; k \; 1] \begin{bmatrix} \delta^3 & & & \\ & \delta^2 & & \\ & & \delta & \\ & & & 1 \end{bmatrix}$$

Call this last square matrix Δ. Then

$$U = 1 L^k N \Delta .$$

This expression states that we may step along the u parameter, in δ increments, by successive multiplications by the L matrix, and thus evaluate the U vector at these steps. In order for u to go from 0 to 1, k must start at 0 and go to $\frac{1}{\delta}$, since $k\delta = 1$.

We can also write, for the W parameter

$$W = 1 L^k N \Delta$$

The surface equation in difference form can now be written out in full:

$$uw = 1 L^j N\Delta \; M \; B \; M^t \; \Delta^t \; N^t \; L^{tk} \; 1^t.$$

Call the partial product $N\Delta \; M \; B \; M^t \; \Delta^t \; N^t = S_{00}$, a square 4 x 4 matrix.

Then $L \, S_{00} = S_{10}$ a new square matrix,

and $L^j \, S_{00} = S_{j0}$, after j multiplications.

We remember that for any column of S_{00} the multiplication by L is a process of cumulative addition, as shown by

$$L \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ a+b \\ a+b+c \\ a+b+c+d \end{bmatrix}$$

We can write in general that

$$L\, S_{jk}\, L^t = S_{j+1,\, k+1}$$

where the new square matrix is obtained from the old by cumulative addition of column elements, followed by cumulative addition of row elements. These operations are furthermore commutative, which means that we obtain the same result if we first add row elements and then afterward add column elements:

We have, finally, that at $u = 0$, $w = 0$
the surface equation is

$00 = 1\, S_{00}\ 1^t$ and in general

$uw = 1\, S_{jk}\ 1^t$ where $u = j\delta$, $w = k\delta$, and S_{jk} has been formed from S_{00} by j column additions and k row additions.

This obviously furnishes an extremely simple way to generate discrete points on a surface patch. The pre-multiplier 1 has the effect of selecting the bottom row of S_{jk} and similarly the post-multiplier 1^t has the effect of selecting the last column of S_{jk}. The bottom right hand corner element of S_{jk} is the value of the coordinate for a point on the surface, at $u = j\delta$, $w = k\delta$.

Consider $1\, S_{j0}$. This represents the row vector obtained after j cumulative addition operations have been performed on the columns of S_{00}. The right hand element of this vector is the value of the coordinate at $u = j\delta$, $w = 0$. We can hold u fixed and step out successive values of the coordinate for w - varying, simply by cumulative addition on this row vector alone. In this case, the resulting right hand element is the marching coordinate value.

An analogous remark can be made for the product $S_{0k}\ 1^T$. This is a column vector, and successive cumulative additions of its elements marches out values of the surface coordinate for $w = k\delta$ fixed, and u varying.

Although the arithmetic of the foregoing difference method is very attractive, it possesses certain drawbacks that must be made explicit. The coordinate values are precise if and only if no truncation error whatever is allowed in the arithmetic. Error is cumulative, and the least departure from a precise number at the start will rapidly propagate. In the Δ matrix there are numbers of δ^3 magnitude; for 100 calculated points along a curve, this calls for 6 decimal digits, all exact, or 18 binary bits. For display purposes, 100 points along a curve are adequate, but for engineering purposes this information is too sparse. To calculate intermediate points, the Δ matrices need to be changed and the S matrix recalculated. Furthermore, it is clear that we soon reach an upper limit on the number of available bits in the computer word, becasue of the rapid growth of δ^3.

We can of course use the difference technique to calculate coefficients for a cubic and then calculate points using them. For consider

$$uw = U\ M\ B\ M^T\ \ \Delta^T\ N^T\ L^{Tk}\ \ 1^T.$$

In this, the partial product $M\ B\ M^T\ \ \Delta^T\ N^T\ L^{Tk}$ represents a square matrix whose last column consists of the coefficients for the cubic at fixed $w = k\delta$, with u varying: If S_k is this square matrix,

$$uw = U\ S_k\ 1^T = U \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

The cubic can be generated by digital methods for any u, or it could be generated quite easily by analog differential analyses hardware. The integrators of the analog differential analyses are loaded from the values of the column coefficient vector, and it is then unclamped and allowed to generate the curve. Meanwhile the digital machine can perform a cumulative addition on the rows of S_k to yield S_{k+1}. The last column of this new matrix then contains the new coefficients for the curve of $w = (k + 1)\delta$, ready to be loaded into the integrators.

Recent developments in hardware, particularly a hybrid digital-analog multiplying device, may make it possible to generate surfaces for display directly from the indicial forms of the equation, or from the matrix form

$$uw = U\ M\ B\ M^T\ W^W$$

without the necessity for recourse to this last difference technique. It is however interesting to know that the method exists.

SECTION IV

HYPERSURFACES — HIGHER DIMENSIONS

We can readily extend the surface equation to describe hyper-surfaces immersed in hyper-space. For this purpose we shall introduce a slight variant on our notation. We shall write typically

$$\underset{i}{u} \equiv F_i u. \quad i = 0 \text{ or } 1.$$

This will be a standard replacement for the blending function notation. The stipulations on the F_i are as before, so that if $u = a$, $a = 0$ or 1, we can write

$$\underset{i}{a} = 0 \text{ when } a \neq i$$

$$\underset{i}{a} = 1 \text{ when } a = i.$$

For slope continuity across boundaries, typically

$\underset{i}{a'} = 0$ where this symbol means the first derivative of the biending function with the argument $= a$.

For higher order continuity across boundaries, the additional stipulations on the blending fucntions are the same as for ordinary surfaces with two degrees of freedom and have already been discussed.

The general surface equation for hyper-space is, in indicial form,

$$\begin{aligned}(uvw\ldots) &= (ujk\ldots)\underset{j}{v}\,\underset{k}{w}\ldots \\ &+ (ivk\ldots)\underset{i}{u}\,\underset{k}{w}\ldots \\ &+ (ijw\ldots)\underset{i}{u}\,\underset{j}{v}\ldots \\ &+ \ldots. \\ &- (N-1)\,(ijk\ldots)\underset{i}{u}\,\underset{j}{v}\,\underset{k}{w}\ldots\end{aligned}$$

In this equation, N is the number of independent parameters in (u v w. . .); it is the number of degrees of freedom of a point on the hyper-surface. The indices i, j, k etc. can take on only the values 0 or 1.

Let us proceed to prove that this surface contains a boundary, say for example the boundary (u 0 0 . . .). We hope that the following equality holds:

$$
\begin{aligned}
(u\ 0\ 0\ .\ .\ .) &= (u\ j\ k\ .\ .\ .)^{0\ 0}_{j\ k}\ .\ .\ . \\
&+ (i\ 0\ k\ .\ .\ .)^{u\ 0}_{i\ k}\ .\ .\ . \\
&+ (i\ j\ 0\ .\ .\ .)^{u\ 0}_{i\ j}\ .\ .\ . \\
&+\ .\ .\ .\ . \\
&- (N-1)\,(i\ j\ k\ .\ .\ .)^{u\ 0\ 0}_{i\ j\ k}\ .\ .\ .
\end{aligned}
$$

The last term in this expression is non-vanishing if and only if all indices other than i are zero, i. e., $j = 0$, $k = 0$, etc. We can accordingly rewrite this term as

$$(N-1)\quad (i\ 0\ 0.\ .\ .)^{u}_{i}\ .$$

Next consider the second term on the right:

$$(i\ 0\ k\ .\ .\ .)^{u\ 0}_{i\ k}\ .\ .\ .$$

It is non-vanishing if and only if $k = 0$, etc.

We can accordingly rewrite it as

$$(\,i\ 0\ 0\ .\ .\ .)^{u}_{i}\ .$$

A similar consideration applies to

$(i\ j\ 0\ .\ .\ .)^{u\ 0}_{i\ j}\ .\ .\ .$ which also becomes

$$(i\ 0\ 0\ .\ .\ .)^{u}_{i}\ .$$

There are evidently N-1 such terms, all identical, and they are removed by the last term. All that is left is the term

$$(u\ j\ k\ .\ .\ .)\,{}^{0\ 0}_{j\ k}\ .\ .\ .$$

but since j = 0 and k = 0, we finally have (u 0 0 . . .) on the right. This establishes the identify, and the surface equation has thus been shown to contain this boundary curve. It is trivial to show that the surface contains all boundaries, and is defined by them.

We can also show that the hyper-surface contains boundary <u>surfaces</u> of lower order. We shall content ourselves with the case for N = 3, and show that it contains surfaces for N = 2 which are identical with our ordinary surfaces. We have

$$\begin{aligned}(u\ v\ w) &= (u\ j\ k)\,{}^{v\ w}_{j\ k}\\ &+ (i\ v\ k)\,{}^{u\ w}_{i\ k}\\ &+ (i\ j\ w)\,{}^{u\ v}_{i\ j}\\ &- 2\,(i\ j\ k)\,{}^{u\ v\ w}_{i\ j\ k}\ .\end{aligned}$$

Set v = 0. Then substituting, and retaining only non-vanishing terms, (which means that j must be replaced by 0 whenever it occurs, and ${}^{v}_{j} = {}^{0}_{0} = 1$)

$$\begin{aligned}(u\ 0\ w) &= (u\ 0\ 1)\,{}^{w}_{k}\\ &+ (i\ 0\ k)\,{}^{u\ w}_{i\ k}\\ &+ (i\ 0\ w)\,{}^{u}_{i}\\ &-2\,(i\ 0\ k)\,{}^{u\ w}_{i\ k}\end{aligned}$$

or

$$(u\;0\;w) = (u\;0\;k)^{w}_{k}$$

$$+ (i\;0\;w)^{u}_{i}$$

$$- (i\;0\;k)^{u\;w}_{i\;k} .$$

This is the two-degree-of-freedom surface

$$(uw) = (uk)^{w}_{k} + (iw)^{u}_{i} - (ik)^{u\;w}_{i\;k} .$$

We shall next consider the slope vector of such a hyper-surface. We take partial derivatives with respect to one of the variables, say u, and get

$$(u\;v\;w\;.\;.\;.)_{u} = (u\;j\;k\;.\;.\;.)^{\;\;v\;w}_{u\;j\;k}\;.\;.\;.$$

$$+ (i\;v\;k\;.\;.\;.)^{u'\;w}_{i\;\;k}\;.\;.\;.$$

$$+ (i\;j\;w\;.\;.\;.)^{u'\;v}_{i\;\;j}\;.\;.\;.$$

$$+\;.\;.\;.\;.$$

$$- (N-1)\;(i\;j\;k\;.\;.\;.)^{u'\;v\;w}_{i\;\;j\;k}\;.\;.\;.$$

In this, set u = 0.

$$(0\;v\;w\;.\;.\;.)_{u} = (0\;j\;k\;.\;.\;.)^{\;\;v\;w}_{u\;j\;k}\;.\;.\;.$$

$$+ (i\;v\;k\;.\;.\;.)^{0'\;w}_{i\;\;k}\;.\;.\;.$$

$$+ (i\;j\;w\;.\;.\;.)^{0'\;v}_{i\;\;k}\;.\;.\;.$$

$$- (N-1)\;(i\;j\;k\;.\;.\;.)^{0'\;v\;w}_{i\;\;j\;\;k}\;.\;.\;.$$

or

$$(0\ v\ w\ .\ .\ .)_u = (0\ j\ k\ .\ .\ .)\genfrac{}{}{0pt}{}{\ \ v\,w}{u\,j\,k}\ .\ .\ .\ ,$$

since all other terms vanish by virtue of $\genfrac{}{}{0pt}{}{0'}{i} = 0.$

This result is analogous to the one obtained for boundaries of ordinary surfaces; it says that the slope anywhere on a boundary is a function only of the slopes at the "ends" of the boundary, and are otherwise independent of the boundary shapes. Slope continuity across boundaries is a consequence.

The hyper-surface equation just developed is defined by ordinary curves, or single-degree-of freedom boundaries; we can also write a hyper-surface equation for N degrees of freedom, defined by boundaries with N-1 degrees of freedom. We shall exhibit the result for N = 3:

$$\begin{aligned}(u\ v\ w) &= (i\ v\ w)\genfrac{}{}{0pt}{}{u}{i}\\ &+ (u\ j\ w)\genfrac{}{}{0pt}{}{v}{j}\\ &+ (u\ v\ k)\genfrac{}{}{0pt}{}{w}{k}\\ &- (i\ j\ w)\genfrac{}{}{0pt}{}{u\ v}{i\ j}\\ &- (i\ v\ k)\genfrac{}{}{0pt}{}{u\ w}{i\ k}\\ &- (u\ j\ k)\genfrac{}{}{0pt}{}{v\ w}{j\ k}\\ &+ (i\ j\ k)\genfrac{}{}{0pt}{}{u\ v\ w}{i\ j\ k}\ .\end{aligned}$$

The proof that this space contains, for example, the boundary subspace (0 v w) follows the preceding proofs in principle and will not be carried out.

SECTION V

SURFACE NORMAL VECTORS

5.1 GENERAL SLOPE CONTINUITY CRITERIA

The surface normal vector furnishes a convenient mechanism for the investigation of general criteria for continuity of surface slope across boundaries between surface patches. It will be seen that the continuity conditions already established are much stronger than are necessary, but that they are expedient.

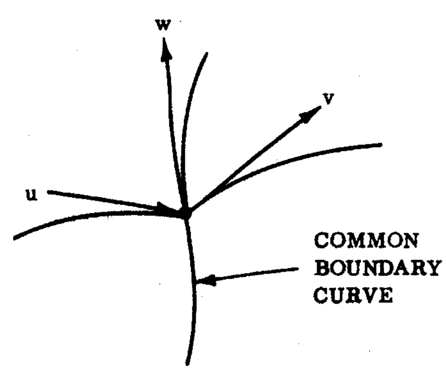
Put

$$U = \left[x_u \; y_u \; z_u\right]$$

$$W = \left[x_w \; y_w \; z_w\right]$$

for the tangent vectors of a surface patch at some point. Let us assume that another adjacent surface has a common boundary curve along u = constant, w varying, so that W is common to both patches. Let the parameter for this second patch be v, and for its tangent vector put

$$V = \left[x_v \; y_v \; z_v\right] .$$



The two surfaces will be continuous in slope across the boundary at the point in question in case the three vectors U, W, and V are coplanar there.

The surface normal vector for one surface is

$$N = U \times W.$$

If V is perpendicular to N,

$$V \cdot N = 0.$$

But then in this case, the three vectors U, W and V are coplanar since they are all perpendicular to N.

In detail, this gives

$$N = U \times W = \begin{bmatrix} J_x & J_y & J_z \end{bmatrix}$$

a vector whose components are the familiar Jacobians, and

$$V \cdot N = VN^T$$

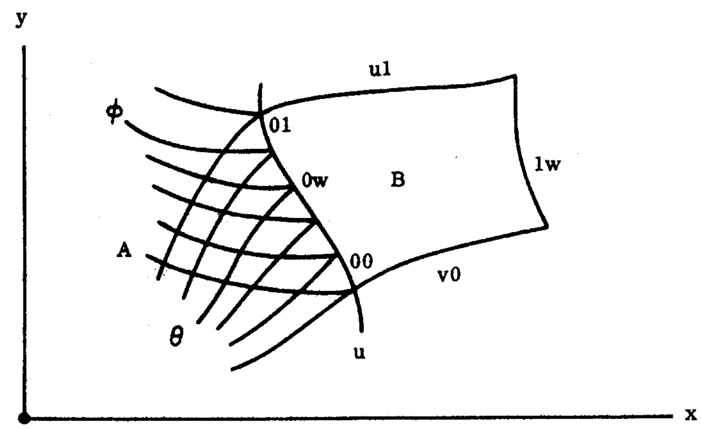
$$= \begin{bmatrix} x_v & y_v & z_v \end{bmatrix} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} x_v & y_v & z_v \\ x_u & y_u & z_u \\ x_w & y_w & z_w \end{bmatrix} = 0$$

(This is the so-called "scalar triple product" of the vectors.) Thus the vanishing of the determinant of the matrix of the three tangent vectors is the general condition for slope continuity between two patches, at any point on their common boundary.

This equation also shows that we may have slope continuity of surfaces even though the curvilinear coordinates of the two surfaces are <u>not</u> slope - continuous across the boundary.

If the tangent vectors U and V are equal everywhere along the boundary curve, the determinant is sure to vanish; similarly if the tangent vectors U and V are scalar multiples of one another, even when the scalar multiplier is a variable quantity.

GENERAL CONSTRUCTION - TANGENT SURFACES



Suppose that a surface A already exists, defined by the parametric vector equation

$$A = \left[x(\phi\ \theta)\ \ y(\phi\ \theta)\ \ z(\phi\ \theta).\right]$$

Let a be a curve on the surface; it is always possible to write the vector equation for a in either of two forms:

$$a(\phi) = \left[x(\phi)\ \ y(\phi)\ \ z(\phi)\right]$$

or

$$a(\theta) = \left[x(\theta)\ \ y(\theta)\ \ z(\theta)\right]$$

Suppose we wish to attach a surface B to surface A, in such a way as to make curve a common to both surfaces, and suppose furthermore that we wish to maintain slope continuity across this mutual boundary.

We shall consider curve a to be the boundary (0w) of the B surface. We are at liberty to design, arbitrarily, a projection of the other three boundaries, (u0), (u1), and (1w). Say for example that we design these curves in the xy projection. Then the curves represent the x and y components of their coordinate vectors.

We are now ready to obtain the missing z component of the tangent vector across (0w). We first compute the surface normal to A along curve a. For this purpose we can use any one of the expressions

$$(\phi\ \theta)_{\phi} \ \times\ (\phi\ \theta)_{\theta}$$

$$\text{or}\quad (\phi\ \theta)_{\phi} \ \times\ a(\theta)_{\theta}$$

$$\text{or}\quad a(\phi)_{\phi} \ \times\ (\phi\theta)_{\theta}$$

Each expression yields a surface normal N; the three results are identical. We can evaluate this surface normal vector at any point on (0w) since we have a correspondence between w and the variables ϕ and θ .

We also have the equation

$(0w)_u\ N^T = 0$. This is the familiar condition for surface slope continuity.

Let $N = \begin{bmatrix} a & b & c \end{bmatrix}$ after evaluation at w.

Then the equation becomes

$$\begin{bmatrix} x(0w)_u & y(0w)_u & z(0w)_u \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

We already have the x and y components of this equation, and can solve for the z component:

$$z(0w)_u = \frac{a\ x(0w)_u + b\ y(0w)_u}{-c} .$$

This z component has a magnitude that ensures that the complete vector $(0w)_u$ is coplanar with surface A at w. Hence $(0w)_u$ is the desired tangent vector of surface B across (0w).

We next find the z tangent vector components $z(00)_u$ and $z(01)_u$ from $(0w)_u$, and use them in the equation

$$z(0w)^1_u = z(00)_u F_0 w + z(01)_u F_1 w.$$

Here the 1 superscript indicates that this is an intermediate result; it is the intrinsic boundary tangent vector for the F-type surface, and does not yet match the $z(0w)_u$ vector function obtained from the A surface.

Accordingly, we must add to the F-type surface a G-type tangent vector correction surface, so as to make the combination have the desired slope along (0w).

This G-type correction surface is, as we have already shown,

$$(uw)^c = \begin{bmatrix} 1 & G_0 u & G_1 u \end{bmatrix} \begin{bmatrix} 0 & u0^c_w & u1^c_w \\ 0w^c_u & -00^c_{uw} & -01^c_{uw} \\ 1w^c_u & -10^c_{uw} & -11^c_{uw} \end{bmatrix} \begin{bmatrix} 1 \\ G_0 w \\ G_1 w \end{bmatrix} .$$

The superscript c indicates that this is a correction surface.

Slope correction is necessary only along the boundary (0w); we can enter the value for $0w^c_u$ in the matrix, but the other entries must be looked at in detail.

We have, for the slope correction across (0w),

$$0w^c_u = 0w_u - 0w^1_u .$$

These latter two quantities have already been found for the z component, and so $0w^c_u$ is known.

Consider $1w^c_u$. This is at a free boundary, (1w), remote from (0w), and we can set it equal to 0. Then 10^c_u and 11^c_u are both zero also. On the other hand $u0^c_w$ and $u1^c_w$ are connected to (0w) at (00) and (01) and so we must specify them in such a way as to satisfy the conditions at these points. Elsewhere, they too are arbitrary.

We write the G function expression:

$$(uw) = \begin{bmatrix} 1 & G_0u & G_1u \end{bmatrix} \begin{bmatrix} 0 & u0_w & u1_w \\ 0w_u & -00_{uw} & -01_{uw} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ G_0w \\ G_1w \end{bmatrix}$$

(we omit the c super-script temporarily.)

Performing the first multiplication, we have

$$(uw) = \begin{bmatrix} 0w_u G_0u & | & u0_w - 00_{uw} G_0u & | & u1_w - 01_{uw} G_0u \end{bmatrix} \begin{bmatrix} 1 \\ G_0w \\ G_1w \end{bmatrix}$$

Consider the element $u0_w - 00_{uw} G_o u$ in the row vector. Since $u0_w$ is arbitrary, it would be convenient to choose it so as to make the entire vector element vanish. We therefore write

$$u0_w = 00_{uw} G_0u.$$

Then $00_w = 0$ as it should, and

since $u0_{wu} = 00_{uw} G_0'u,$

$$00_{wu} = 00_{uw} \text{ as it should.}$$

Similarly, we may set

$$u1_w = 01_{uw} G_0u.$$

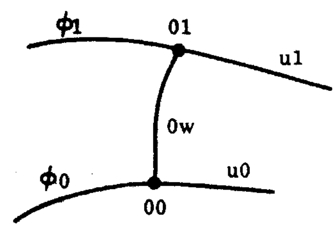
The result of these choices of $u0_w$ and $u1_w$ is to reduce the G equation to

$$uw^c = 0w_u^c \; G_0u.$$

This represents the correction surface z component which must be added to the z component of the F surface in order to obtain slope continuity along (0w) between the given A surface and the designed B surface.

5.2 ADJACENT-PATCH SLOPE CONTINUITY

It is sometimes desirable to define the boundary curves for two adjacent patches so that at the junction between the curves the tangent vectors have the same direction but are of different magnitudes. This is particularly useful when the boundary curves are parametric cubics, because then the magnitudes of the tangent-vectors at the end points control the behavior of the curve segment.



As a specific case, consider the boundary (0w) common to two patches; let the tangent vectors at (00) and (01) for the first patch be 00_ϕ and 01_ϕ, and let the tangent vectors for the next patch be 00_u and 01_u.

If the tangent vectors have the same direction, they are scalar multiples of one another,

or $$00_u = m\ 00_\phi$$

and $$01_u = n\ 01_\phi\ .$$

Suppose that the patch (ϕ w) already exists. We need to obtain an appropriate expression for (uw) so as to match surface slopes across (0w). By the results of the preceeding article, we can accomplish this in very general ways, but in our present case let us make a special requirement on the tangent vectors: let us assume that everywhere across (0w) the tangent vectors have the same direction, and differ only in magnitude. Then for any w,

$$0w_u = \lambda\ 0w_\phi$$

where λ is a scalar. We know that λ takes the value m at w = 0, and takes the value n at w = 1, and we conclude that λ therefore must be a scalar function of w.

With this relationship between $(0w_u)$ and $(0w_\phi)$ the vector cross product is always the null vector:

$$(0w)_u \times (0w)_\phi = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Hence the scalar triple product of $(0w)_u$, $(0w)_\phi$, and $(0w)_w$ vanishes for any $(0w)_w$. This ensures that the two surfaces will be continuous in slope across (0w) for <u>any</u> shape of (0w) and for <u>any</u> $\lambda = \lambda(w)$ that has the proper behavior at w = 0 and w = 1.

We could for example take

$$\lambda(w) = m(1-w) + nw.$$

This is a linear variation of λ with respect to w. It has one disadvantage, however, in that it introduces un-wanted cross derivatives or twists at (00) and (01). In order to avoid this, we might use

$$\lambda(w) = m\, F_0 w + n\, F_1 w.$$

Then the required slope function across (0w) for the (uw) patch is

$$0w_u = (m\, F_0 w + n\, F_1 w)\; 0w_\phi.$$

We can check to find the cross derivatives introduced by this relationship. The cross-derivative is obtained by differentiating with respect to w, and yields

$$0w_{uw} = (m\, F_0' w + n\, F_1' w)\; 0w_\phi$$
$$+ (m\, F_0 w + n\, F_1 w)\; 0w_{\phi w}.$$

At (00), $00_{uw} = m\; 00_{\phi w}$

and at (01), $01_{uw} = n\; 0w_{\phi w}$.

This shows that the λ function does not introduce additional twists at the corners of the patch, beyond, of course, those already inherent in the (ϕw) surface.

If the (uw) surface already exists, defined perhaps by an F-type equation, its intrinsic tangent vector across (0w) is a known function of (0w), say $0w_u^1$, where the 1 superscript indicates that it is an intermediate result. Then, as before, the correction of slope is

$$0w_u^c = 0w_u - 0w_u^1. \quad \text{The correction surface}$$

is $$(uw)^c = 0w_u^c G_0 u.$$

When this correction surface is added to the original surface, the combination will be continuous in slope with the (ϕw) surface across (0w). The u and ϕ curvilinear coordinates of the two patches will be continuous in slope across (0w), but their tangent vectors will be different in magnitude.

5.3 APPLICATIONS

Let dU represent a differential vector, so that

$$dU = \left[dx\ dy\ dz \right], \text{ in which}$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial w} dw.$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial w} dw.$$

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial w} dw.$$

If dU is tangent to a u-varying, w-fixed curve, these become

$$dx = \frac{\partial x}{\partial u} du = x_u\, du$$

$$dy = y_u\, du.$$

$$dz = z_u\, du, \text{ since } dw = 0.$$

Hence $dU = \left[x_u\ y_u\ z_u \right] du.$

Similarly, if dW is a differential vector tangent to a w-varying, u-fixed curve,

$$dW = \begin{bmatrix} x_w & y_w & z_w \end{bmatrix} dw.$$

The normal differential vector at a point of the surface will be given by the vector cross product:

$$dn = dU \times dW$$

$$= \begin{bmatrix} \begin{vmatrix} y_u & z_u \\ y_w & z_w \end{vmatrix} & \begin{vmatrix} z_u & x_u \\ z_w & x_w \end{vmatrix} & \begin{vmatrix} x_u & y_u \\ x_w & y_w \end{vmatrix} \end{bmatrix} du\, dw.$$

The determinants that comprise the elements of the vector are the jabobians J_x, J_y and J_z so that we may write

$$dn = \begin{bmatrix} J_x & J_y & J_z \end{bmatrix} du\, dw.$$

The magnitude of dn is equal to the differential area of the elemental parallelogram described by dU and dW. This magnitude is

$$|dn| = \sqrt{dn \cdot dn} \quad (\text{or} \sqrt{[dn]\,[dn]^T})$$

$$= du\, dw \sqrt{J_x^2 + J_y^2 + J_z^2}\,.$$

From this, we can construct an algorithm for finding surface areas of patches; we simply perform numerical integration of the expression

$$A = \int_0^1 \int_0^1 \sqrt{J_x^2 + J_y^2 + J_z^2}\; du\, dw.$$

Again, if N is the unit normal vector to the surface at a point, then

$$dn = N\,|dn|, \text{ from which}$$

$$N = \frac{dn}{|dn|}$$

$$= \frac{J_x}{S} \quad \frac{J_y}{S} \quad \frac{J_z}{S}$$

where $$S = \sqrt{J_x^2 + J_y^2 + J_z^2}\,.$$

The quantities $\frac{J_x}{S}$, $\frac{J_y}{S}$, $\frac{J_z}{S}$ are the coordinate components a, b and c of the surface normal.

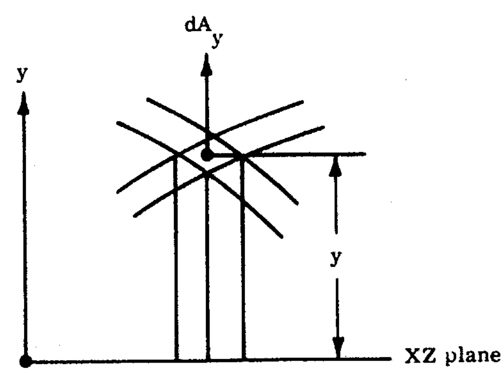
If the surface is to be manufactured by milling with a ball-end cutter of radius R, then the cutter-center vector $\left[x_c \; y_c \; z_c\right]$ is related to the surface vector $\left[x \; y \; z\right]$ by the simple expression

$$\left[x_c \; y_c \; z_c\right] = \left[x \; y \; z\right] + R \left[a \; b \; c\right].$$

This describes a "parallel" surface spaced a distance R away from the designed surface.

The normal vector can also be used to calculate volumes enclosed by surface patches and planes, as follows.

Suppose we wish to calculate the volume contained between a surface patch and the xz plane.



We can imagine the volume broken up into a number of slender prisms whose axes are all parallel to the y axis (and perpendicular to the xz plane.) The area of the base of one of these prisms is the projection of the small element of surface area, or

$$dA_y = J_y \, du \, dw.$$

The volume of this prism is

$$dV = y\,dA_y = yJ_y\,du\,dw$$

$$V = \int_u \int_w yJ_y\,du\,dw.$$

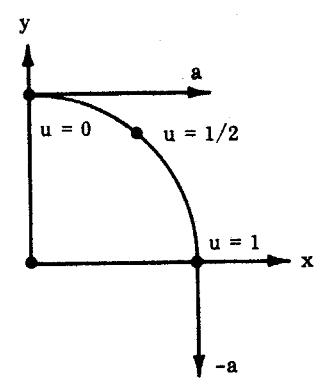
SECTION VI

CORNER TWIST VECTORS

6.1 THE QUASI-SPHERE

It is possible to choose a parametric cubic that very nearly approximates a circle for one quadrant. We shall go into detail about this shortly; intuition suggests that similarly we ought to be able to construct an approximation of an octant of a sphere by means of a bicubic surface, bounded by these approximations to circles.

For the circle approximation, let us assume that we will be content to make the quasi-circle pass through a point on the true circle at $u = 1/2$. (This is not the best possible approximation, but it yields quite good results and the arithmetic is simple.)



We shall assume a circle of unit radius, centered at the origin, with end-point values of the parameter u as shown. The tangent vectors are symmetric, but have yet undefined magnitudes.

We have

$$x = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \; M \begin{bmatrix} 0 \\ 1 \\ a \\ 0 \end{bmatrix}$$

Where the column vector on the right represents the end conditions for the curve.

When u = 1/2, this becomes

$$x = \frac{1}{8} \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ a \\ 0 \end{bmatrix}$$

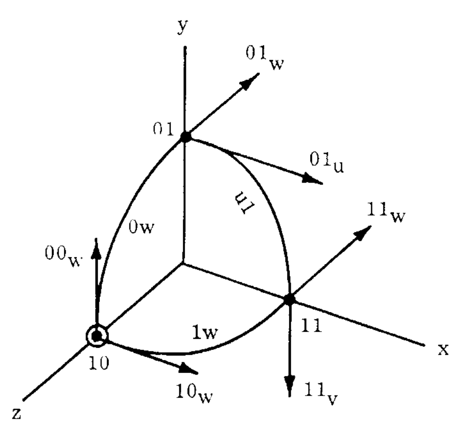
$$= \frac{1}{8} \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} -2 + a \\ 3 - 2a \\ a \\ 0 \end{bmatrix}$$

$$= \frac{1}{8} (- 2 + a + 6 - 4a + 4a) = \frac{1}{8} (4 + a)$$

$a = 8x - 4$. But at $u = 1/2$, $x = \frac{\sqrt{2}}{2}$ since it is a point on the circle (by symmetry, at $\frac{\pi}{4}$).

Hence $a = 4(\sqrt{2} - 1)$. This is the required magnitude of the two tangent vectors at $u = 0$ and $u = 1$. (Calculation reveals that the quasi-circle has a radius of about 1.00016 at $u = 1/3$ (at $\frac{\pi}{6}$ or 30°) so it is a good approximation.)

We now establish a coordinate system for the sphere, and show its boundary curves.



u0 = degenerate curve boundary

The boundary curves 0w, 1w, and u1 are all unit circles; the boundary u0 however is a degenerate circle, and appears as a point.

We shall first investigate the z component of the uw surface vector

$$z(uw) = \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \left[\begin{array}{cc|cc} 00 & 01 & 00_w & 01_w \\ 10 & 11 & 10_w & 11_w \\ \hline 00_u & 01_u & 00_{uw} & 01_{uw} \\ 10_u & 11_u & 10_{uw} & 11_{uw} \end{array}\right] \begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}$$

$$= \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \left[\begin{array}{cc|cc} 1 & 0 & 0 & -a \\ 1 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}.$$

When we perform the first multiplication, we obtain

$$z(uw) = \left[\begin{array}{c|c|c|c} F_0u + F_1u & 0 & 0 & -a(F_0u + F_1u) \end{array}\right] \begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}.$$

But $F_0u + F_1u = 1$, by virtue of the definition of the F functions.

Hence

$$z(uw) = \begin{bmatrix} 1 & 0 & 0 & -a \end{bmatrix} \begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}$$

$$= F_0w - a\ G_1w.$$

The curvilinear coordinates for w constant thus yield constant z; this implies that z is independent of the other variable u, and the w curves are plane curves. They must of course be quasi-circles.

We have obtained the value of the number a by investigating a unit circle. For a circle of radius r, the tangent vector magnitudes must be proportional to r, or equal to ra. We can find these radii for various values of w from either the quasi-circle y(0w) or x(1w).

We have

$$x(1w) = \begin{bmatrix} 10 & 11 & 10_w & 11_w \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}$$

and

$$y(0w) = \begin{bmatrix} 00 & 01 & 00_w & 01_w \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}$$

in either case,

$$r(0w) = \begin{bmatrix} 0 & 1 & a & 0 \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}.$$

For w fixed, the x and y coordinates of a quasi-circle are given by

$$x = \begin{bmatrix} F_0 u & F_1 u & G_0 u & G_1 u \end{bmatrix} \begin{bmatrix} 0 \\ r \\ ar \\ 0 \end{bmatrix}$$

and

$$y = \begin{bmatrix} F_0 u & F_1 u & G_0 u & G_1 u \end{bmatrix} \begin{bmatrix} r \\ 0 \\ 0 \\ -ar \end{bmatrix}$$

where r is a function of w, shown above.

But

$$\begin{bmatrix} 0 \\ r \\ ar \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & a & a^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}$$

and

$$\begin{bmatrix} r \\ 0 \\ 0 \\ -ar \end{bmatrix} = \begin{bmatrix} 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -a & -a^2 & 0 \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix} .$$

We obtain these last results by simply writing rows in the 4 x 4 matrix that correspond to elements in the vector of the left side. When we combine results, we have

$$x(uw) = \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ \hline 0 & a & a^2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}$$

$$y(uw) = \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \left[\begin{array}{cc|cc} 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -a & -a^2 & 0 \end{array}\right] \begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}$$

The equation for z(uw) has already been shown.

The striking thing about the B matrix as it appears in these equations is that is has non-zero entries in the bottom right partition,

$$\begin{bmatrix} 00_{uw} & 01_{uw} \\ 10_{uw} & 11_{uw} \end{bmatrix}.$$ By comparison, we see

that $x(00_{uw}) = a^2$

and $y(10_{uw}) = -a^2$.

These are the cross-derivatives at the corners 00 and 10. All other cross derivatives vanish. We shall refer to these cross-derivatives as "twists" of the surface; uw_{uw} is the twist vector at a generalized point on the surface.

6.2 THE EXACT SPHERE

A bi-cubic surface cannot fit a sphere exactly, and it would be interesting to see whether by an appropriate choice of F and G functions other than cubics, an exact equation can be constructed.

The functions

$$F_0 u = \cos^2 \frac{\pi}{2} u$$

and

$$F_1 u = \sin^2 \frac{\pi}{2} u$$

resemble the cubic functions in shape; they satisfy the conditions

$$F_0' 0 = F_0' 1 = 0$$

$$F_1' 0 = F_1' 1 = 0$$

as well as

$$F_0 0 = 1 \qquad F_0 1 = 0$$

$$F_1 0 = 0 \qquad F_1 1 = 1.$$

We shall choose these functions, and determine appropriate G functions so that the equation

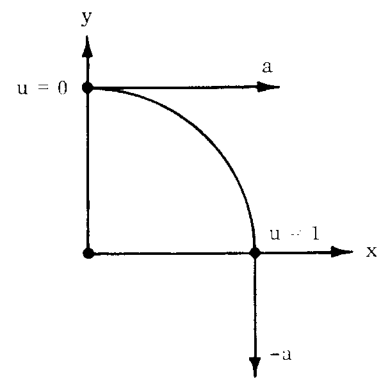
$$(u) = \begin{bmatrix} F_0 u & F_1 u & G_0 u & G_1 u \end{bmatrix} \begin{bmatrix} (0) \\ (1) \\ (0)_u \\ (1)_u \end{bmatrix}$$

represents an exact circle, and not the approximation of the last section.

We already have the well-known parametric equations for a circle;

$$x = \sin au$$

$$y = \cos au \qquad \text{where } a = \frac{\pi}{2}$$



We can compare the x equation with

$$x(u) = \begin{bmatrix} \cos^2 au & \sin^2 au & G_0 u & G_1 u \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ a \\ 0 \end{bmatrix}$$

$$= \sin^2 au + a\, G_0 u$$

Then $\sin au = \sin^2 au + a\, G_0 u$, whence

$$G_0 u = \frac{1}{a} (\sin au - \sin^2 au).$$

Again,

$$y = \cos au = \begin{bmatrix} \cos^2 au & \sin^2 au & G_0 u & G_1 u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -a \end{bmatrix}$$

$$\cos au = \cos^2 au - a\, G_1 u$$

whence

$$G_1 u = \frac{1}{a} (\cos^2 au - \cos au).$$

We can easily verify that these G functions satisfy the same stipulations as the cubic G functions:

$$G_0 0 = G_0 1 = 0$$

$$G_1 0 = G_1 1 = 0$$

$$G_0' 0 = 1 \qquad G_0' 1 = 0$$

$$G_1' 0 = 0 \qquad G_1' 1 = 1 .$$

If we now use the same boundary value matrices as were used in the previous case of the quasi-sphere, but with $a = \frac{\pi}{2}$ throughout, we obtain for the z component of the surface vector

$$z(uw) = F_0 w - a G_1 w .$$

This is, with the new F and G functions

$$\begin{aligned} z(uw) &= \cos^2 aw - (\cos^2 aw - \cos aw) \\ &= \cos aw . \end{aligned}$$

As before, this shows that the z coordinate of the surface is independent of u: the w curves are plane curves, and they are indeed circles. Their radii are given by

$$x(1w) = \begin{bmatrix} 10 & 11 & 10_w & 11_w \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & a & 0 \end{bmatrix} \begin{bmatrix} F_0 w \\ F_1 w \\ G_0 w \\ G_1 w \end{bmatrix}$$

$$= F_1 w - a G_0 w$$

$$= \sin^2 aw - (\sin aw - \sin^2 aw)$$

or $\quad r = \sin aw .$

Then

$$x(uw) = \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \begin{bmatrix} 0 \\ r \\ ar \\ 0 \end{bmatrix}$$

$$= r(F_1u + aG_0u) = \sin aw \sin au,$$

and

$$y(uw) = \begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} \begin{bmatrix} r \\ 0 \\ 0 \\ -ar \end{bmatrix}$$

$$= r(F_0u - aG_1u)$$

$$= \sin aw (\cos^2 au - \cos^2 au - \cos au)$$

$$= \sin aw \cos au.$$

The resulting parametric equations, when collected, are

$$x = \sin aw \sin au$$

$$y = \sin aw \cos au$$

$$z = \cos aw$$

and these are well-known.

This demonstrates that the sphere is a special case of the general surface equation, provided the blending functions are suitably chosen.

The F and G functions are by inspection, seen to be linear combinations of the linearly independent functions of u,

$$\begin{bmatrix} \cos^2 au & \sin^2 au & \cos au & \sin au \end{bmatrix},$$

and this may be taken as an appropriate basis vector. Then

$$\begin{bmatrix} F_0u & F_1u & G_0u & G_1u \end{bmatrix} = \begin{bmatrix} \cos^2 au & \sin^2 au & \cos au & \sin au \end{bmatrix} \begin{bmatrix} M \end{bmatrix}$$

where the M matrix is, in this case,

$$M = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{a} \\ 0 & 1 & -\frac{1}{a} & 0 \\ 0 & 0 & 0 & -\frac{1}{a} \\ 0 & 0 & \frac{1}{a} & 0 \end{bmatrix} .$$

Incidentally, its inverse is

$$M^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{bmatrix} .$$

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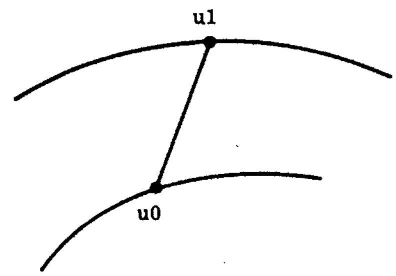
SECTION VII

RULED SURFACES

The locus of straight lines connecting corresponding points on two curves (u0) and (u1) is a surface. The lines are called "rulings" of the surface.



The equation for a generalized line of the surface is also the equation for the surface:

$$(uw) = (u1)w - (u0)w + (u0).$$

This is equivalent to

$$(uw) = (u1)w + (u0)(1-w).$$

From this equation we obtain the derivatives

$$(uw)_u = (u1)_u w + (u0)_u (1-w)$$

$$(uw)_w = (u1) - (u0)$$

$$(uw)_{uw} = (u1)_u - (u0)_u.$$

7.1 DEVELOPABLE SURFACES

A special case of such ruled surfaces is of importance and interest. If the ruled surface has the property of being tangent, along the rulings, to a moving plane which rolls around the surface, then it may be deformed by simple bending and flattened out into a plane. Such a process is called "development" of the surface. We call such surfaces "developables" or "wrapped surfaces". (A sheet of paper can be wrapped around the two curves (u0) and (u1) to form the surface. These surfaces are also known as "convolutes".)

The tangent-plane condition can be established by showing that the tangent vector at a point on (u0), the tangent vector at a corresponding point on (u1), and the tangent vectors along the line joining these points, are all coplanar. We need to form the scalar triple product of these vectors, and show that it vanishes.

The tangent vectors in question are $(u0)_u$, $(u1)_u$, $(u0)_w$ and $(u1)_w$.

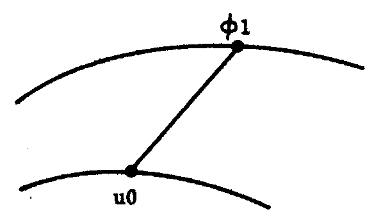
First observe that, for a ruled surface,

$(uw)_w = (u1) - (u0)$. This tangent vector is independent of w, so that $(u0)_w = (u1)_w$. Moreover, the vector is simply the line segment joining the two points, as might be expected. We can write, for the scalar triple product,

$$\begin{vmatrix} (u0)_u \\ (u1)_u \\ (u1) - (u0) \end{vmatrix} = 0$$

where the notation represents the determinant of the matrix of the three (row) vectors. If the determinant vanishes for all values of u, the surface is developable.

The preceding describes an analytical test to ensure that a ruled surface is developable. We shall now describe a construction that will enable us to define a ruled surface by means of two space curves. Suppose that the two space curves are defined by vector functions of two <u>different</u> parameters, u and ϕ



The scalar triple product is

$$\begin{vmatrix} (u0)_u \\ (\phi 1)_\phi \\ (\phi 1) - (u0) \end{vmatrix} = 0$$

If we consider u the independent variable, the equation enables us to find ϕ for any value of u; this value of ϕ determines the point on (ϕ1) which corresponds to a point on (u0), so that the line joining these points is coplanar with the tangent to (ϕ1) and the tangent to (u0). We allow u to vary, and obtain related ϕ values; these values of ϕ enable us to evaluate the components of the vector (ϕ1). These components are the same as the components of the desired (u1) vector.

Provided we remember that the (unspecified) functions of u and ϕ are different, the symbolism

$$(u1) = (\phi 1), \phi = \phi (u)$$

represents the statement that the $\begin{bmatrix} x & y & z \end{bmatrix}$ vector is the same for both.

With the correspondence established between points on the two curves, we can write the equation for the developable surface,

$$(uw) = (u1)w + (u0)(1 - w).$$

This is the ruled surface equation, but with a special relationship between curves (u0) and (u1).

7.2 PLANE/SURFACE INTERSECTIONS

The general surface equation can be cast in the form

$uw = UBW^T$ where U and W are vector functions of u and w respectively, and where B is a square matrix describing the boundary curves. For example, we might be dealing with the first F-type surface equation,

$$uw = - \begin{bmatrix} -1 & F_0u & F_1u \end{bmatrix} \begin{bmatrix} 0 & u0 & u1 \\ 0w & 00 & 01 \\ 1w & 10 & 11 \end{bmatrix} \begin{bmatrix} -1 \\ F_0w \\ F_1w \end{bmatrix}$$

in which these vectors and the matrix are explicit:

Since uw is in reality a vector consisting of an x, a y and a z component, there are three B matrices which we can call B_x B_y B_z.

We wish to find the intersection of this surface with the plane

$$ax + by + cz + d = 0.$$

We can substitute $x = UB_xW^T$, $y = UB_yW^T$ and $z = UB_zW^T$ into this equation, and write the result in the form

$$U \left[aB_x + bB_y + cB_z\right] W^T + d = 0.$$

It is permissible to interchange the order of multiplication from aU, bU, cU to Ua, Ub, Uc because a b and c are scalars. In this form, the sum $\left[aB_x + bB_y + cB_z\right] = S$, a square matrix function of u and w, and

$USW^T = -d$ is an equation in the two variables u and w. If w is assigned a fixed value, there results an equation in u which when solved will yield a point on the intersection curve of the surface with the plane, (provided of course such a point exists for the chosen value of w.)

If the surface in question is a bi-cubic, the matrix S is no longer a function of the variables u and w, but consists of constant elements. In this case the above procedure reduces to the solution of a series of cubic equations in u, where the coefficients of the cubics are determined by successive fixed values of w.

In any case, if the spacing of the w values is close, the old value of u just previously determined for a particular choice of w can appropriately be used as a first trial solution for the new value of w. Algorithms for the improvement of this initial trial value of u are not difficult to construct, and will not be discussed in detail.

If the plane is given by, say, the equation

$$x + d = 0$$

the solution procedure is unaltered. Not much simplification results.

SECTION VIII

RATIONAL POLYNOMIAL FUNCTIONS

8.1 BOUNDARY CURVES AND BLENDING FUNCTIONS

Two kinds of curves have for many years been traditionally used in airplane lines design — cubic polynomials, and conics. Unfortunately, each of these curve forms for itself has certain drawbacks. In the parametric form, for ordinary cubics, the entire shape of a curve segment is governed by end tangent vectors. Sometimes these end tangent vectors lead to unwanted hooks and bulges in the curve segments. On the other hand, conics, although more benignly behaved, cannot by their very nature yield curves with points of inflection. Yet such curves very often exist in aircraft shapes — as for instance in the case of wing fillets.

Because of these short-comings, a new curve type has been developed. It is based upon rational polynomial functions. It contains both conics and ordinary cubics as special cases, and provides a great degree of generality and flexibility.

We start by establishing the form of this function.

Let v be a vector, so that for example $v = [x\ y\ z\ 1]$ or $v = [x\ y\ 1]$ or $v = [x\ 1]$. The first of these can be thought of as the vector (or matrix) of coordinates on a space curve; the second is the vector of coordinates for a plane curve, and the last is the vector of a single coordinate. Since this last vector yields the most general case, we shall begin with it, and show how one might evaluate a set of numbers in a matrix to define each of the parametric coordinates of a curve.

The product of v and a variable scalar w is $wv = [wx\ \ w]$. Here both wx and w are cubic functions of a parameter, v, and obviously

$$x = \frac{wx}{w}.$$

This is the ratio of two cubic polynomials (hence the term "rational function"). We can represent the two cubic polynomials by the matric equation

$$wv = [u^3\ u^2\ u\ 1]\ A.$$

Since $wv = [wx\ \ w]$, the matrix A must consist of four rows and two columns of constant coefficients. We now proceed to show how these numbers may be found so as to define a coordinate of a curve.

We shall be interested in the end-point coordinates of the curve at $v = 0$ and $v = 1$. These coordinates are $v_0 = [x_0\ \ 1]$ and $v_1 = [x_1\ \ 1]$ respectively. A tangent vector anywhere on the curve is clearly

$$v' = [x'\ \ 0]$$

where the prime mark means differentiation with respect to the parameter v. The tangent vectors at $v = 0$ and $v = 1$ are therefore $v_o' = [x_o'\ \ 0]$ and $v_1' = [x_1'\ \ 0]$ respectively.

Now differentiate both sides of

$$wv = [u^3\ u^2\ u\ 1]\ A \qquad \text{and obtain}$$

$$(wv)' = [3u^2\ 2u\ 1\ 0]\ A.$$

Substitution of $u = 0$ and $u = 1$ into these two expressions yields

$$\begin{bmatrix} w_0\ v_0 \\ w_1\ v_1 \\ (w_0\ v_0)' \\ (w_1\ v_1)' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} A.$$

The 4 x 4 matrix on the right has an inverse, and we may write

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} w_0 v_0 \\ w_1 v_1 \\ (w_0 v_0)' \\ (w_1 v_1)' \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_0 v_0 \\ w_1 v_1 \\ w_0' v_0 + w_0 v_0' \\ w_1' v_1 + w_1 v_1' \end{bmatrix} .$$

The square matrix inverse is constant and always the same, and reappears in the algebra so often that we shall henceforth call it the matrix M.

The matrix equation can be factored and rewritten in the form

$$A = M \begin{bmatrix} w_0 & 0 & 0 & 0 \\ 0 & w_1 & 0 & 0 \\ w_0' & 0 & w_0 & 0 \\ 0 & w_1' & 0 & w_1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix} .$$

The right hand matrix of v's represents the desired end conditions on the curve. In our present case, it is of course a 4 x 2 matrix.

The middle matrix is 4 x 4 and contains the four numbers $[w_0\ w_1\ w_0'\ w_1']$. Any arbitrary set of four numbers inserted into this matrix will serve to define a unique pair of cubic functions of the parameter u, from which x can be found, by using, as we have said, the ratio $x = \frac{wx}{w}$.

Instead of picking these four numbers arbitrarily, however, we shall impose further conditions on the curve until enough conditions are imposed to define $[w_0\ w_1\ w_0'\ w_1']$ uniquely.

We begin by introducing desired second derivative vectors at the endpoints; these vectors are clearly

$$v_0'' = [x_0'' \;\; 0] \quad \text{and}$$

$$v_1'' = [x_1'' \;\; 0].$$

(Incidentally, in the case of vectors $v' = [x' \; y' \; 0]$, $v'' = [x'' \; y'' \; 0]$ if the determinant of the matrix $\begin{bmatrix} x' & y' \\ x'' & y'' \end{bmatrix}$ vanishes, the curve will have a point of inflection at v. If the determinant is positive, the center of curvature will lie on the left as one proceeds along the curve; if the determinant is negative, the center of curvature lies on the right. If two curve segments have equal v' and v'' at a junction, they are continuous both in slope and curvature at such a junction.)

When we take second derivatives of both sides of

$wv = [u^3 \; u^2 \; u \; 1] \; A$, we obtain

$$(wv)'' = [6u \; 2 \; 0 \; 0] \; A.$$

At $u = 0$, this is

$$(w_0 v_0)'' = [0 \;\; 2 \;\; 0 \;\; 0] \; M \begin{bmatrix} w_0 v_0 \\ w_1 v_1 \\ w_0' v_0 + w_0 v_0' \\ w_1' v_1 + w_1 v_1' \end{bmatrix}.$$

But $(w_0 v_0)'' = w_0'' v_0 + 2 w'_0 v'_0 + w_0 v_0''$ and solving for $w_0 v_0''$,

$$w_0 v_0'' = (w_0 v_0)'' - w_0'' v_0 - 2 w_0' v_0'.$$

Now $(w_0 v_0)'' = [-6 \;\; 6 \;\; -4 \;\; -2] \begin{bmatrix} w_0 v_0 \\ w_1 v_1 \\ w_0' v_0 + w_0 v_0' \\ w_1' v_1 + w_1 v_1' \end{bmatrix}.$

Furthermore, since in general

$$(wv)'' = [(wx)'' \quad w''],$$

the quantity w'' is the second component of the vector of (wv)'' and therefore is associated with the last column of the matrix $\begin{bmatrix} w_0 v_0 \\ w_1 v_1 \\ w_0' v_0 + w_0 v_0' \\ w_1' v_1 + w_1 v_1' \end{bmatrix}$.

But the last components of v_0 and v_1 are both 1, and the last components of v_0' and v_1' are both zero. Hence

$$w_0'' = [-6 \quad 6 \quad -4 \quad -2] \begin{bmatrix} w_0 \\ w_1 \\ w_0' \\ w_1' \end{bmatrix}$$

We now can write, by combining results:

$$w_0 v_0'' = [-6 \quad 6 \quad -4 \quad -2] \begin{bmatrix} w_0 (v_0 - v_0) \\ w_1 (v_1 - v_0) \\ w_0' (v_0 - v_0) + w_0 v_0' \\ w_1' (v_1 - v_0) + w_1 v_1' \end{bmatrix} -2 w_0' v_0'$$

$$= 6 w_1 (v_1 - v_0) - 4 w_0 v_0' - 2 w_1' (v_1 - v_0) - 2 w_1 v_1' - 2 w_0' v_0$$

Collecting,

$$w_0 v_0'' = w_0 (-4 v_0') + w_1 (6 (v_1 - v_0) - 2 v_1')$$

$$+ w_0' (-2 v_0') + w_1' (-2 (v_1 - v_0)).$$

We now restore this last expression to matrix form:

$$w_0 v_0'' = [w_0\ w_1\ w_0'\ w_1'] \begin{bmatrix} -4\ v_0' \\ 6\ (v_1 - v_0) - 2\ v_1' \\ -2\ v_0' \\ -2\ (v_1 - v_0) \end{bmatrix}.$$

In our present case, the matrix on the right consists of a column of numbers and a column of zeros. Hence the column of zeros can be discarded, and the result is a 4 x 1 matrix. On the left, $w_0\ v_0''$ is a scalar.

Similarly, we can find by analogous algebraic procedures that

$$w_1 v_1'' = [w_0\ w_1\ w_0'\ w_1'] \begin{bmatrix} 6\ (v_0 - v_1) + 2\ v_0' \\ 4\ v_1' \\ 2\ (v_0 - v_1) \\ -\ 2\ v_1' \end{bmatrix}.$$

Then, writing a matrix equation, we have, so far,

$$[w_0\ v_0''\ \ w_1 v_1''] = [w_0\ w_1\ w_0'\ w_1']\ \ [P\ |\ Q\ ]$$

where $[P|Q]$ represents a 4 x 2 matrix consisting of the separate 4 x 1 matrices for $w_0\ v_0''$ and $w_1\ v_1''$, written side by side as columns.

We now introduce another condition. Let it be required that the curve pass through the point $v_c = [x_c\ \ 1]$ when $u = \frac{1}{2}$. (This value of u is of course arbitrary.)

This condition leads to

$$v_c = \frac{1}{8}\ [1\ 2\ 4\ 8]\ \ M \begin{bmatrix} w_0\ v_0 \\ w_1\ v_1 \\ w_0'\ v_0 + w_0\ v_0' \\ w_1'\ v_1 + w_1\ v_1' \end{bmatrix}.$$

By algebraic manipulations similar to the preceding, we can rearrange the equation to read

$$8\,v_c = [w_0\, w_1\, w_0'\, w_1'] \begin{bmatrix} 4\,v_0 + v_0' \\ 4\,v_1 - v_1' \\ v_0 \\ -v_1 \end{bmatrix}$$

This is an equation in which v_c has two components, x_c and 1. The right hand matrix is a 4 x 2. Call it the R matrix. Then we can adjoin these matrices, to obtain

$$[w_0\, v_0''\;\; w_1\, v_1''\;\; 8\,v_c] = [w_0\, w_1\, w_0'\, w_1']\;\; [\,P\, Q\, R\,].$$

Now [P Q R] represents a 4 x 4 matrix; P and Q are each 4 x 1 matrices, but R is a 4 x 2 matrix. We next transfer $w_0\, v_0''$ and $w_1\, v_1''$ to the right hand side, obtaining

$$[0\;\; 0\;\; 8\,v_c] = [w_0\;\; w_1\;\; w_0'\;\; w_1'] \left([\,P\,Q\,R\,] - \left[\begin{array}{cc|cc} v_0'' & 0 & 0 & 0 \\ 0 & v_1'' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \right)$$

The right hand matrix is now a 4 x 4. Provided its determinant does not vanish, it has an inverse, and

$$[\,w_0\;\; w_1\;\; w_0'\;\; w_1'\,] = [\,0 \mid 0 \mid 8\,v_c\,]\;\; S$$

where S is the 4 x 4 inverse of the matrix.

Now that [w_0 w_1 w_0' and w_1'] have been evaluated, the curve is completely defined, since the rational function $\frac{wv}{w}$ is completely defined.

8.2 PLANE CURVES

With some loss of generality and flexibility, we can have $v = [x \ y \ 1]$, a plane curve. We shall next show that for an appropriate choice of $[w_0 \ w_1 \ w_0'$ and $w_1']$, the curve reduces to a conic.

We have the equation

$$wv = [wx \ wy \ w]$$
$$= [u^3 \ u^2 \ u \ 1] \ A.$$

In this case, A is a 4 x 3 matrix. Now if the top row of this matrix is $[0 \ 0 \ 0]$, the equation reduces to

$$wv = [u^2 \ u \ 1] \ A$$

when the top row of A has been omitted. A is now a 3 x 3 matrix, and it is possible to show that this equation is a parametric form for the general conic, expressed as a quadratic rational function.

For the top row of A, we have the vector equation

$$[2 \ -2 \ 1 \ 1] \begin{bmatrix} w_0 v_0 \\ w_1 v_1 \\ w_0' v_0 + w_0 v_0' \\ w_1' v_1 + w_1 v_1' \end{bmatrix} = [0 \ 0 \ 0]$$

Expanding:

$$2 w_0 v_0 - 2 w_1 v_1 + w_0' v_0 + w_0 v_0' + w_1' v_1 + w_1 v_1' = [0 \ 0 \ 0]$$

Collecting:

$$w_0 (2 v_0 + v_0') + w_1 (-2 v_1 + v_1') + w_0' (v_0) + w_1' (v_1) = [0 \ 0 \ 0]$$

In matrix form,

$$[0 \ 0 \ 0] = [w_0 \ w_1 \ w_0' \ w_1'] \begin{bmatrix} 2 v_0 + v_0' \\ -2 v_1 + v_1' \\ v_0 \\ v_1 \end{bmatrix}$$

The matrix is a 4 x 3.

Now we can adjoin a column to the matrix and an element to the vector, and write

$$[0 \;\; 0 \;\; 0 \;\; w_0] = [w_0 \;\; w_1 \;\; w_0' \;\; w_1'] \begin{bmatrix} 2\,v_0 + v_0' & 1 \\ -2\,v_1 + v_1 & 0 \\ v_0 & 0 \\ v_1 & 0 \end{bmatrix}$$

then

$$[w_0 \;\; w_1 \;\; w_0' \;\; w_1'] = [0 \;\; 0 \;\; 0 \;\; w_0] \begin{bmatrix} 2\,v_0 + v_0' & 1 \\ -2\,v_1 + v_1' & 0 \\ v_0 & 0 \\ v_1 & 0 \end{bmatrix}^{-1} .$$

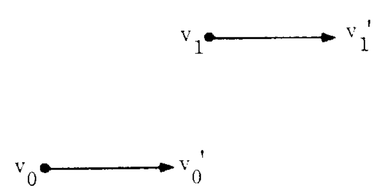
If the indicated inverse exists, then a solution can be obtained in terms of w_0. Furthermore, w_0 can be set equal to 1 arbitrarily.

The matrix has an inverse in case the determinant

$$\begin{vmatrix} -2\,v_1 + v_1' \\ v_0 \\ v_1 \end{vmatrix} \neq 0.$$

As a test, construct a conic with end conditions

$$\begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 2\,v_0 + v_0' \\ -2\,v_1 + v_1' \\ v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} [\,0\;0\;2\,] + [\,1\;0\;0\,] \\ [\,-2\;-2\;-2\,] + [\,1\;0\;0\,] \\ 0\;0\;1 \\ 1\;1\;1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -2 & -2 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

We first test to see whether the determinant vanishes:

$$\begin{vmatrix} -1 & -2 & -2 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} = -1.$$ Hence the augmented matrix

will have an inverse.

The matrix is $\begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & -2 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ and its inverse is $\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & -2 & -2 \end{bmatrix}$

then

$$[\,1\;\; w_1\;\; w_0'\;\; w_1'\,] = [\,0\;\;0\;\;0\;\;1\,] \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & -2 & -2 \end{bmatrix}$$

whence $[\,1\;\; w_1\;\; w_0'\;\; w_1'\,] = [\,1\;\;-1\;\;-2\;\;-2\,]$.

The conic equation is

$$wv = [\,u^3\;u^2\;u\;1\,] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= [u^3\ u^2\ u\ 1] \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & -2 & 1 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= [u^3\ u^2\ u\ 1] \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} = [u^2\ u\ 1] \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[wx\ \ wy\ \ w] = \left[-2u^2 + u \;\middle|\; -u^2 \;\middle|\; -2u + 1\right]$$

$$x = u$$

$y = \dfrac{-u^2}{1 - 2u} = \dfrac{-x^2}{1 - 2x}$. The curve is hyperbolic, with an asymptote at $1 - 2x = 0$, $x = \dfrac{1}{2}$.

It is always possible in all of the foregoing to set $w_0 = 1$. This is because all equations are homogeneous. It is never possible for $w_0 = 0$, since this leads to certain degenerate cases.

We remark in passing that when

$$[w_0\ \ w_1\ \ w_0'\ \ w_1'] = [1\ \ 1\ \ 0\ \ 0]$$

the equation reduces to the ordinary parametric cubic, given by

$$v = [u^3\ u^2\ u\ 1]\ M \begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix}$$

and w is constant and equal to 1. Hence the rational polynomial functions contain as special cases all conics, ordinary cubics, and of course therefore straight lines and circles.

Their use as boundary curves for surface patches is obvious. They maintain tangent vector continuity between adjacent patches; indeed, if the F_0 and F_1 functions are constructed as rational functions, we can establish the F_1 function

$$F_1(u) = \frac{u_3}{3u^2 - 3u + 1}.$$

This function has the end conditions

$$\begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \\ v_0'' \\ v_1'' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $v_0'' = \lambda v_0' = 0[1 \ 0 \ 0]$ and $v_1'' = \lambda v_1' = 0[1 \ 0 \ 0]$, the curve has a point of inflection at $u = 0$ and $u = 1$. Hence its use insures curvature continuity across boundaries between patches, provided of course the boundary curves have similar curvature continuity at patch corners.

The cure is symmetric. Furthermore, we can put

$$F_0(u) = 1 - F_1(u)$$

and obtain directly the F_0 function, another cubic rational function, with similar properties to F_1.

8.3 AN EXAMPLE

We shall work out the equation for the $F_1(u)$ blending function with the customary stipulations that

$$F_1(1) = 1, \ F_1(0) = F_1'(0) = F_1'(1) = 0,$$

and with the two additional stipulations that $F_1''(0) = F_1''(1) = 0$ as well.

This blending function will give <u>both</u> slope and <u>curvature</u> continuity across the common boundary between two contiguous patches. The end-conditions are, for $v = [\, F_0(u) \;\; 1 \,]$,

$$\begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \\ v_0'' \\ v_1'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix $[\, P \;\; Q \;\; R \,] - \begin{bmatrix} v_0'' & 0 & 0 & 0 \\ 0 & v_1'' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -6 & 0 & 4 \\ 6 & 0 & 4 & 4 \\ 0 & -2 & 0 & 1 \\ -2 & 0 & -1 & -1 \end{bmatrix}$$

obtained by direct substitution in the given form.

Its inverse is $\begin{bmatrix} 0 & -1 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ -2 & 2 & 6 & 6 \\ 2 & 0 & -6 & 0 \end{bmatrix} = S$

Now set $v_c = \left[\frac{1}{2} \quad 1\right]$ for symmetry. Then, since $u = \frac{1}{2}$,

$$[w_0 \ w_1 \ w_0' \ w_1'] = [0 \ 0 \ 8 \ v_c] \ S$$

$$= [0 \ 0 \ 4 \ 8] \begin{bmatrix} 0 & -1 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ -2 & 2 & 6 & 6 \\ 2 & 0 & -6 & 0 \end{bmatrix}$$

$$= [8 \ 8 \ -24 \ 24].$$

Now if $w_0 = 1$, instead of 8, the equation becomes

$$[w_0 \ w_1 \ w_0' \ w_1'] = [1 \ 1 \ -3 \ 3]$$

we have

$$A = M \begin{bmatrix} w_0 v_0 \\ w_1 v_1 \\ w_0' v_0 + w_0 v_0' \\ w_1' v_1 + w_1 v_1' \end{bmatrix}$$

and substituting the values of $[w_0 \ w_1 \ w_0' \ w_1']$:

$$A = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & -3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Finally, $wv = [wx \ w] = [u^3 \ u^2 \ u \ 1] \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & -3 \\ 0 & 1 \end{bmatrix}$

$$[wx \ w] = [u^3 \mid 3u^2 - 3u + 1]$$

Hence $x = \frac{wx}{w} = \frac{u^3}{3u^2 - 3u + 1} = F_1(u)$ as required. The other F_0 function is

$$F_0(u) = 1 - F_1(u) = 1 - \frac{u^3}{3u^2 - 3u + 1}$$

$$= \frac{-u^3 + 3u^2 - 3u + 1}{3u^2 - 3u + 1} .$$

This is seen to be a rational cubic function also.

8.4 PLANE CURVE THROUGH A POINT

The plane curve vector is $v = [x \;\; y \;\; 1]$, $wv = [wx \;\; wy \;\; w]$. Here the polynomial denominators in $x = \frac{wx}{w}$ and $y = \frac{wy}{w}$ are both the same.

As before, the end conditions on the curve are contained in the matrix $\begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix}$.

We wish to cause the curve to pass through some arbitrary point v_c (commonly called a "shoulder point") and it will turn out that we shall also be free to choose some arbitrary slope at this point. It is important to distinguish between the term "slope" and "tangent vector". The slope of a curve implies that the direction of the tangent vector is known, but the magnitude of the vector is not under our control.

We begin by assuming some value of the parameter u to correspond to v_c. For purposes of illustration, let $u = \frac{1}{2}$ at this point.

Then, from wv = $[u^3\ u^2\ u\ 1]$ A, we write

$$v_c = \frac{1}{8}[1\ 2\ 4\ 8]\ M \begin{bmatrix} w_0 v_0 \\ w_1 v_1 \\ w_0' v_0 + w_0 v_0' \\ w_1' v_1 + w_1 v_1' \end{bmatrix}$$

Observe that we have arbitrarily set $w_c = 1$. This is harmless, since the equation is, as we have observed, homogeneous.

By multiplying the matrices, combining, and collecting terms as we have done before, we achieve the result

$$8\,v_c = [w_0\ w_1\ w_0'\ w_1'] \begin{bmatrix} 4\,v_0 + v_0' \\ 4\,v_1 - v_1' \\ v_0 \\ -v_1 \end{bmatrix}$$

The matrix on the right is a 4 x 3 matrix; to make it square, so that it can have an inverse, we need an additional column. This column can be provided by a scaler equation, and the slope relationship will furnish this equation.

We first find an expression for the tangent vector at v_c. Differentiating, we obtain as usual,

$$(wv)' = [3u^2\ 2u\ 1\ 0]\ A$$

$$w' = [3u^2\ 2u\ 1\ 0]\ M \begin{bmatrix} w_0 \\ w_1 \\ w_0' \\ w_1' \end{bmatrix}$$

This last equation comes from the equation for (wv)' by an argument that we have used before; that since $wv = [\,wx \;\; wy \;\; w\,]$, $(wv)' = [\,(wx)' \; (wy)' \; w'\,]$.

Hence w' corresponds to the last column of $\begin{bmatrix} w_o v_o \\ w_1 v_1 \\ (w_0 v_0)' \\ (w_1 v_1)' \end{bmatrix}$

which is $\begin{bmatrix} w_0 \\ w_1 \\ w_0' \\ w_1' \end{bmatrix}$.

Now at v_c, we have already set $u = \frac{1}{2}$. We make this substitution, and obtain, from

$$wv' = (wv)' - w'v$$

$$v_c' = \frac{1}{4} \; [\,-6 \;\; 6 \;\; -1 \;\; 1\,] \begin{bmatrix} w_0 (v_o - v_c) \\ w_1 (v_1 - v_c) \\ w_0' (v_0 - v_c) + w_0 v_0' \\ w_1' (v_1 - v_c) + w_1 v_1' \end{bmatrix} .$$

When we perform the indicated multiplications, and then collect results and restore to matrix form, we have

$$4 v_c' = [\,w_0 \;\; w_1 \;\; w_0' \;\; w_1'\,] \begin{bmatrix} -6 (v_0 - v_c) - v_0' \\ 6 (v_1 - v_c) - v_1' \\ v_c - v_0 \\ v_c - v_1 \end{bmatrix} .$$

We are now ready to introduce the slope condition. We could write $\frac{dy}{dx} = \frac{y'}{x'}$, but this would yield awkward results when the slope became very great and approached an infinite slope. Instead, we choose two numbers a and b so that $ax' = by'$. It is obvious that these numbers can very appropriately be $a = \sin\theta$, $b = \cos\theta$ where θ is the slope angle. Then, for v_c,

$$0 = by'_c - ax'_c$$

This is a scalar equation.

Now x'_c is the first component of v'_c, and y'_c is the second component of v'_c. Hence these quantities correspond to the first and second columns of the matrix in the equation for v'_c, respectively. We can write this out in detail:

$$0 = [w_0 \;\; w_1 \;\; w_0' \;\; w_1'] \left(b \begin{bmatrix} -6(y_0 - y_c) - y_0' \\ 6(y_1 - y_c) - y_1' \\ y_c - y_0 \\ y_c - y_1 \end{bmatrix} - a \begin{bmatrix} -6(x_0 - x_c) - x_0' \\ 6(x_1 - x_c) - x_1' \\ x_c - x_0 \\ x_c - x_1 \end{bmatrix} \right) = [w_0 \;\; w_1 \;\; w_0' \;\; w_1'] \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$$

where the brackets on the right enclose the resulting 4 x 1 matrix (or column vector).

The factor 4 (of 4 v_c') obviously drops out of the equation.

We now adjoin this to the equation for v_c. Again we can drop the 8 (from $8\ v_c$), and obtain

$$[v_c \ \ 0] = [w_0 \ \ w_1 \ \ w_0' \ \ w_1'] \begin{bmatrix} 4v_0 + v_0' & p \\ 4v_1 - v_1 & q \\ v_0 & r \\ -v_1 & s \end{bmatrix}.$$

The vector on the left consists of four components: $[x_c \ \ y_c \ \ 1 \ \ 0]$; the matrix on the right is a 4 x 4 matrix. If it has an inverse, S, then we can solve for $[w_0 \ \ w_1 \ \ w_0' \ \ w_1']$ by the equation

$$[w_0 \ \ w_1 \ \ w_0' \ \ w_1'] = [v_c \ \ 0] \ S.$$

These values of the w vector cause the curve to satisfy the desired conditions.

8.5 SECOND DERIVATIVE VECTORS

We have already discussed rational functions for

$v = [x \ \ 1]$ (and of course $[y \ \ 1]$ and $[z \ \ 1]$.)

In particular, we showed that curves based upon these functions can usually have arbitrary first and second derivative vectors at the end-points, and in addition can be caused to pass through some shoulder point v_c, also arbitrarily chosen.

When the vector $v = [x \ \ y \ \ 1]$, the complete generality of the resulting curves is somewhat curtailed. We shall investigate the conditions under which such a plane curve can satisfy end conditions including second derivatives.

We have already obtained an expression for the vector

$$[w_0 v_0'' \ \ w_1 v_1''] = [w_0 \ \ w_1 \ \ w_0' \ \ w_1'] \ \ [P \mid Q].$$

Before, the vectors $w_0\ v_0''$ and $w_1\ v_1''$ were actually scalars, since they came from $v'' = [x'' \ 0]$. But since $v'' = [x'' \ \ y'' \ \ 0]$, they are each 2-component vectors, and their combination makes a 4-component vector. Similarly P and Q are now each a 4 x 2 matrix, and their combination is a 4 x 4 matrix.

We carry $w_0 v_0''$ and $w_1 v_1''$ across the equal sign, and obtain on the left the null vector:

$$[0\ 0\ 0\ 0] = [w_0\ w_1\ w_0'\ w_1'] \left(\left[P \mid Q \right] - \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{v_0''} & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{v_1''} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \right)$$

The matrix on the right is 4 x 4. Now the condition that must hold, in order for there to be a solution for $[w_0\ w_1\ w_0'\ w_1']$, is that this matrix must be singular; the determinant of this matrix must vanish.

This last remark tells us that v_0'' and v_1'' <u>cannot</u> be chosen entirely arbitrarily. However, it is always possible to make the determinant of the matrix vanish by the adjustment of any one of the four components of v_0'' and v_1''. Thus if one of the four components is the number a, we can expand the determinant in such a way as to obtain the equation.

$k_1 a + k_2 = 0$, from which a can be found.

Suppose the matrix is, or has been caused to be, singular. Then, if

$$[0\ 0\ 0\ 0] = [w_0\ w_1\ w_0'\ w_1']\ S,$$

we make it non-singular by an appropriate modification. In some cases, this might consist in adding 1 to an element in the top row of S. As an illustration, we might have

$$[0\ 0\ 0\ w_0] = [w_0\ w_1\ w_0'\ w_1'] \left(S + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

It can be seen that this modification is still a valid equation; if the modified S matrix now has an inverse, we can immediately obtain it and solve for the w vector.

As an illustration, consider the end conditions

$$\begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \\ v_0'' \\ v_1'' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ a & 0 & 0 \end{bmatrix}$$

We plan to adjust a in v_1'' until the matrix is singular,

We require first the matrix [P | Q] , which is given by

$$\left[\begin{array}{l|l} -4\,v_0' & 6\,(v_0 - v_1) + 2\,v_0' \\ 6\,(v_1 - v_0) - 2\,v_1' & 4\,v_1' \\ -2\,v_c' & 2\,(v_0 - v_1) \\ -2\,(v_1 - v_0) & -2\,v_1' \end{array}\right] .$$

Substitution of the end conditions gives the 4 x 4 matrix:

$$\left[\begin{array}{l|c} -4 \quad 0 & [-6 \ \ -6] + [2 \ \ 0] \\ [6 \ \ 6] - [2 \ \ 0] & 4 \quad 0 \\ -2 \quad 0 & -2 \quad -2 \\ -2 \quad -2 & -2 \quad 0 \end{array}\right] = \begin{bmatrix} -4 & 0 & -4 & -6 \\ 4 & 6 & 4 & 0 \\ -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & 0 \end{bmatrix}$$

In passing, we note that if our end conditions had been $v_0'' = [0 \ 0 \ 0]$ and if also $v_1'' = [0 \ 0 \ 0]$, the resulting matrix would be singular, because the first and third columns of [P | Q] are identical. However, this is not our present case.

We now subtract the matrix

$$\left[\begin{array}{cc|cc} v_0'' & & 0 & 0 \\ 0 & 0 & v_1'' & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

from [P | Q] and obtain

$$\begin{bmatrix} -5 & 0 & -4 & -6 \\ 4 & 6 & (4-a) & 0 \\ -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & 0 \end{bmatrix} .$$

By a series of reductions accomplished by multiplying rows of the matrix and additions (or subtractions) of rows to remove elements, we can obtain the determinant

$$\begin{vmatrix} (a-1) & -1 \\ 1 & -1 \end{vmatrix} = 0$$

This implies

$$a = 2.$$

This is the value of a that makes the matrix singular.

The singular matrix is $\begin{bmatrix} -5 & 0 & -4 & -6 \\ 4 & 6 & 2 & 0 \\ -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & 0 \end{bmatrix}$

We make it non-singular by adding 1 to the top left element. The matrix is now

$$\begin{bmatrix} -4 & 0 & -4 & -6 \\ 4 & 6 & 2 & 0 \\ -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & 0 \end{bmatrix}$$

and corresponds to the vector $[\, w_0 \;\; 0 \;\; 0 \;\; 0 \,]$ on the left of the equation.

The inverse of this matrix is

$$\frac{1}{2}\begin{vmatrix} 2 & 1 & -6 & 3 \\ -1 & 0 & 3 & -1 \\ -1 & -1 & 3 & -3 \\ -1 & 0 & 2 & 0 \end{vmatrix} = R.$$

Finally, $[\, w_0 \;\; w_1 \;\; w_0' \;\; w_1' \,] = [\, w_0 \;\; 0 \;\; 0 \;\; 0 \,] \;\; R.$

If we arbitrarily set $w_0 = 2$, then the required solution is just the top row of R, or

$$[\, w_0 \;\; w_1 \;\; w_0' \;\; w_1' \,] = [\, 2 \;\; 1 \;\; -6 \;\; 3 \,].$$

With these numbers known, the curve equation is completely defined. We obtain it by substition in the canonical form:

$$wv = [\, u^3 \; u^2 \; u \; 1 \,] \;\; M \begin{bmatrix} w_0 \, v_0 \\ w_1 \, v_1 \\ w_0' \, v_0 + w_0 \, v_0' \\ w_1' \, v_1 + w_1 \, v_1' \end{bmatrix}$$

$$= [\, u^3 \; u^2 \; u \; 1 \,] \;\; M \begin{bmatrix} 0 \;\; 0 \;\; 2 \\ 1 \;\; 1 \;\; 1 \\ [\, 0 \;\; 0 \;\; -6 \,] + [\, 2 \;\; 0 \;\; 0 \,] \\ [\, 3 \;\; 3 \;\; 3 \,] + [\, 1 \;\; 0 \;\; 0 \,] \end{bmatrix}$$

$$= [\, u^3 \; u^2 \; u \; 1 \,] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & -6 \\ 4 & 3 & 3 \end{bmatrix}$$

$$= [u^3\ u^2\ u\ 1] \begin{bmatrix} 4 & 1 & -1 \\ -5 & 0 & 6 \\ 2 & 0 & -6 \\ 0 & 0 & 2 \end{bmatrix} .$$

The separate equations for the x and y coordinates of the curve can be exhibited:

$$x = \frac{4u^3 - 5u^2 + 2u}{-u^3 + 6u^2 - 6u + 2}$$

$$y = \frac{u^3}{-u^3 + 6u^2 - 6u + 2} .$$

In the foregoing, certain matrices have occurred. These matrices are significant ones, and can be written as transformations of the common matrix $\begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix}$

as follows:

For the conic condition matrix,

$$\begin{bmatrix} 2\ v_0 + v_0' \\ -2\ v_1 + v_1' \\ v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix} .$$

For the $u = \frac{1}{2}$, v_c shoulder point condition,

$$\begin{bmatrix} 4\,v_0 + v_0' \\ 4\,v_1 - v_1' \\ v_0 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix} .$$

For the P and Q matrices associated with $w_0\,v_0''$ and $w_1\,v_1''$,

$$\begin{bmatrix} -4\,v_0' \\ 6\,v_1 - 6\,v_0 - 2\,v_1' \\ -2\,v_0' \\ -2\,v_1 + 2\,v_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 & 0 \\ -6 & 6 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 2 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix}$$

$$\begin{bmatrix} 6\,v_0 - 6\,v_1 + 2\,v_0' \\ 4\,v_1' \\ 2\,v_0 - 2\,v_1 \\ -2\,v_1' \end{bmatrix} = \begin{bmatrix} 6 & -6 & 2 & 0 \\ 0 & 0 & 0 & 4 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_0' \\ v_1' \end{bmatrix} .$$

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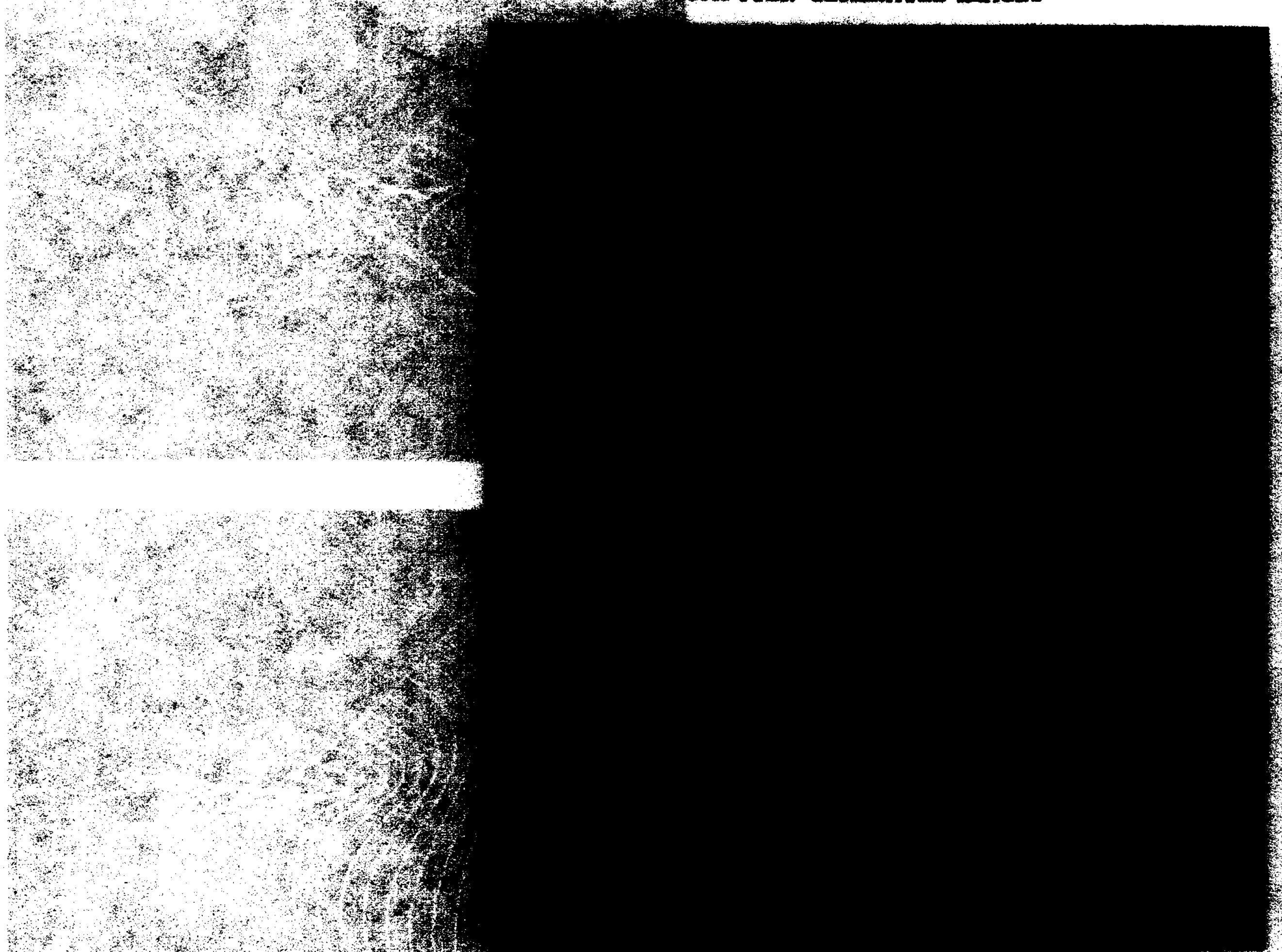
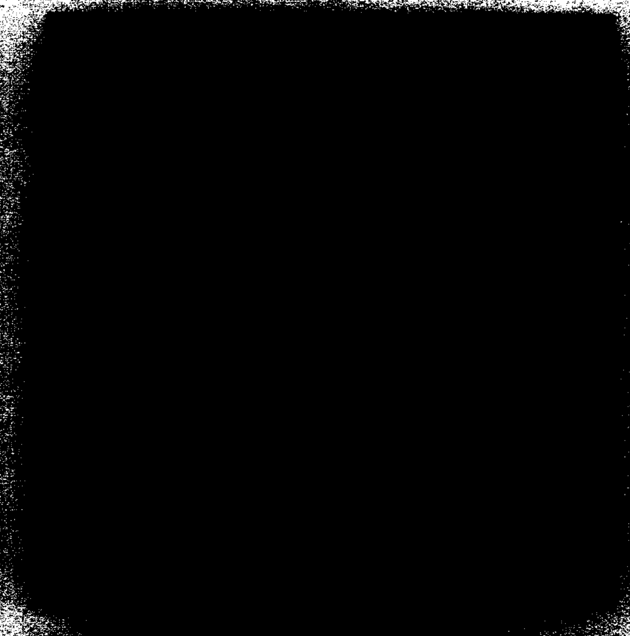
APPENDIX

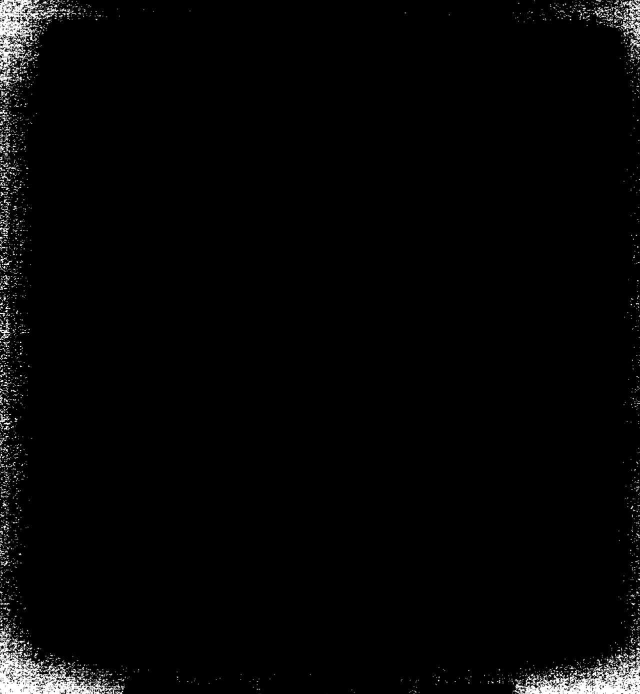
COMPUTER-GENERATED IMAGES



[illegible] views and a perspective view (upper right) of a single surface. [illegible] was generated by the computer from the three orthog- [illegible] When registered on a cathode-ray tube a change in any one of the views will automatically cause a change to be made in the other views.

*These computer-generated images were provided by Prof. B. Herzog of the University of Michigan, and are shown through the courtesy of Ford Motor Company.





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13. ABSTRACT

The design of airplanes, ships, automobiles, and so-called "sculptured parts" involves the design, delineation, and mathematical description of bounding surfaces. A method is described which makes possible the description of free-form doubly curved surfaces of a very general kind. An extension of these ideas to hyper-surfaces in higher dimensional spaces is also indicated.

This surface technique has been specifically devised for use in the Comp ter-Aided Design Project at M.I.T., and has already been successfully implemented here and elsewhere.

14. KEY WORDS

Arbitrary-surface design	Multiple-access computers	Surface "patches"
Computer-aided design	On-line computers	Time-shared computers
Machine-aided cognition	Real-time computers	Time-sharing

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