16. Abstracts

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17. Key Words and Document Analysis. 17a. Descriptors

Group Theory

Symbol Manipulation

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group

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MAC TECHNICAL MEMORANDUM 35

AN INTERACTIVE IMPLEMENTATION
OF THE
TODD-COXETER ALGORITHM

Richard J. Bonneau

December 1973

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

PROJECT   MAC

CAMBRIDGE   MASSACHUSETTS 02139

An Interactive Implementation

of the

Todd-Coxeter Algorithm

Abstract

The Todd-Coxeter algorithm provides a systematic approach
to the enumeration of cosets of a finitely presented group.
This memo describes an interactive implementation of algorithm,
including a manual on its use, examples, and methods of
accessing the program. Applications of this algorithm are
also discussed.

TABLE OF CONTENTS

1. Introduction

This memo presents a user's manual for the operation of an interactive version of the Todd-Coxeter algorithm for the computation of the index of a subgroup with respect to a given group. The bulk of the FORTRAN program was derived from the program written by Cannon, et. al [1], but has been re-organized to provide for interactive use.

The Todd-Coxeter Algorithm, as described in [2], [3] and [4], provides an efficient method for determining the order of a finite group which is described in terms of generators and relations. The algorithm also can be used to compute the index of any subgroup provided the subgroup is presented by generators which are words in the generators of the group.

This algorithm has been used widely by group theorists and mathematicians for a number of applications. These applications include construction of a presentation of a known group by generators and relations [5], computation of part of the Schur multiplier of a small finite Lie group (PSU(3,4)) [5], calculating double cosets H x K given the single coset table form of the subgroups H and K [6], and determination of a set of coset representatives for a subgroup [6]. The Todd-Coxeter algorithm has been acknowledged as a powerful and necessary tool for any automated group theory project.

The current program is written in the FORTRAN IV programming language and runs under a simulated PDP-10/50 operating system. The actual configuration is the Project MAC, MATHLAB PDP-10 computer at M.I.T.

This memo presents instructions, and examples, for the use of this program, along with an annotated listing of the program. Any comments, criticisms or suggestions will be graciously accepted and gratefully acknowledged and should be addressed to the author.

The implementation of the Todd-Coxeter algorithm represents   the first step in an effort to develop a sophisticated collection of programs to aid in computations with groups. It is our hope to be releasing future memos detailing improvements and additions to this project.

2. USER'S MANUAL FOR THE TODD-COXETER ALGORITHM

1. Find a free console on the MATHLAB system.
   Type "control-Z" (ctrl-Z). (Non-Printing)

2. The system will type several lines relating to
   the number of users, operating system version,
   and other information. When it has been printed
   out, type

   :LOGIN GROUP followed by a carriage return (CR).

3. TEN50 will be awaiting commands when it types a
   new line with only a period on it (.) .

4. To run the Todd-Coxter program type:

   R   TC   (CR)

5. At this point, the user has a number of options relating
   to the input to the program: input may either
   be interactive via the teleypeor automatic via
   files on the disk. Similary, the output can be
   printed either on the teleype or onto a file on
   disk. The remainder of the description will assume
   teletype (TTY) for both input and output.

6. The program now prints out the following question:

   INPUT ON DSK (20) OR TTY (05) ?

7. Since we desire TTY as input source, we type

   05   (CR)

8. The question is now printed:

   OUTPUT TO DSK (21) OR TTY (05) ?

9. Again, for TTY output we type

   05   (CR)

10. The program now types out the following message:

    TODD-COXETER ALGORITHM running on file: TTY.DAT
                       putting output on file: TTY.DAT

11. The program now prints out

    NEW PROBLEM INPUT

12. At this point, the program is expecting a control number to direct its actions. To begin a new problem (i.e., a group and a subgroup), type

    | -1 (CR) |

13. The program will now respond:

    ****

    PROBLEM   n
    GROUP RELATORS

    where n is the current problem number.

14. At this point, the group relators are input, using the following scheme:

    For each relator, one line of numbers is input.
    Each relator is numerically represented by a
         sequence of two digit numbers in the form:

    $e_1 x_1 x_2 \ldots x_n$ (CR)

    where e_1 is a positive two digit number signifying the multiplicity of the relator
         x_i is non-zero integer representing either a generator of the group (x_i positive) or the inverse of a generator of the group (x_i negative).

    E.g., if a relator is

    $(aba^{-1}b^{-2})^3 = 1$

    then it is entered as the following string:

    030102-1-2-2   (CR)

15. After each line is entered the program will verify the input by printing (as in the example above) a

    EXP 3   1   2 -1 -2 -2

16. Enter as many lines as there are group relators. To inform the program that all the relators have been input, enter

    | -2 (CR) |

17. The following line will now be typed:

SUBGROUP GENERATORS

18. You now enter, in the same format as described above,
    the generators of the desired subgroup; again,
    one line of input per generator.

19. All the generators have been input, enter a

    -3 (CR)

    This will inform the program that all the input information
    has been given, and also that program execution should
    start. (Note: if the identity subgroup is the desired input
    you need only type -3 when subgroup generators are asked
    for).

20. The Todd-Coxeter algorithm is now being applied to the
    input data. The program will now terminate in one
    of two ways:

    Successful Termination: In this case, the index
          has been determined and the following is
          printed:

    INDEX= i   MAX= m   TOTAL= t

          Here, i will be the index, m will denote
          the maximum number of cosets existing at any
          one time during the execution, and t is the
          total number of cosets created.

    Unsuccessful Termination: In this case, the core
          storage was not sufficient to accomodate the
          computation necessary to complete the problem.

    LOOKAHEAD UNABLE TO CONTINUE

21. In both cases, the program again types out

    NEW PROBLEM INPUT

22. At this point, the user may wish to proceed in one of
    four ways:

    1) Input a new problem: enter -1 (cr) and proceed
        to step 13.

    2) Terminate the session: enter -9 (CR) and proceed to
        step 23.

3) Maintain the same group as just previously input
but input a new subgroup: enter -4 (CR)
and proceed to step 17.

4) Maintain the session, but with new I/O configuration:
enter -8 (CR)
and proceed to step 6

23. At the end of the session, the system types out certain
run-time statistics, then types

EXIT
↑C

24. To log off the system, type (cntrl-Z) and then
:logout (CR)

3. Sample Session Number 1

```
^C

.^Z
10556)    .SLEEP 17,    $$u
ML ITS 811  CONSOLE 7 FREE. 16:46:22

ML ITS.811. DDT.516.
11. USERS
EXPERIMENTAL SYSTEM WITH DYNAMOD MAGTAPE ROUTINES,ETC
USE CAUTION, AND DONT RELOAD OLD SYSTEM WITHOUT SEEING ME.   —rg
:login group
     INIT
*:TCR50
.R TC


INPUT ON DSK (20) OR TTY (05) ?
05


OUTPUT TO DSK (21) OR TTY (05) ?
05



TODD-COXETER ALGORITHM RUNNING ON FILE: TTY  .DAT
              PUTTING OUTPUT ON FILE: TTY  .DAT



NEW PROBLEM INPUT
-1

**********************************
PROBLEM   1


GROUP RELATORS
 9 1                              $a^9 = 1$



EXP  9       1
 8 2                              $b^8 = 1$



EXP  8       2
 1 1 2-1-2                        $aba^{-1}b^{-1} = 1$
```

```
EXP   1        1  2 -1 -2

-2


SUBGROUP GENERATORS
-3



INDEX=    72      MAX=   121     TOTAL=   448

NEW PROBLEM INPUT
-9


EXECUTION TIME:        4.86 SEC.
TOTAL ELAPSED TIME:    55.00 SEC.
NO EXECUTION ERRORS DETECTED

EXIT
^C


  .
```

4. Disk Files for Input and/or Output

The current implementation of the Todd-Coxeter algorithm provides a number of input-output media. In particular, the teletype (TTY) has been used for both input and output in the sample session. This section will be concerned with the use of disk files for both input and output.

Naming Conventions

Disk files used for the FORTRAN I/O must agree with the following rules:

1) Each file name is of the form

    Name-ext

2) Name must be a 5 character word

3) Ext must be the 3 characters DAT

4) For standard (i.e. default) names are:

    INPUT: FØR20.DAT

    OUTPUT: FØR21.DAT

The user is permitted to use standard file names or provide specific I/O file names.

Disk Files for Input

The structure of these files mirror exactly the inputs as given in the algorithm.

A sample session with disk input from file NEWCN. DAT and output to TTY is given below, along with listing of NEWCN. DAT.

5.   Sample Session Number 2

.START

INPUT ON DSK (20) OR TTY (05) ?
20

STANDARD FILE NAME (20) OR NOT (22) ?
22

ENTER 5 LETTER FILE NAME
NEWCA\A\N

OUTPUT TO DSK (21) OR TTY (05) ?
05

TODD-COXETER ALGORITHM RUNNING ON FILE: NEWCN.DAT
                PUTTING OUTPUT ON FILE: TTY .DAT

****
PROBLEM   2

GROUP RELATORS

EXP 8   1

EXP 7   2

EXP 2   1 2

EXP 3   -1 2
SUBGROUP GENERATORS

EXP 2   1

EXP 1   -1 2

INDEX=   448   MAX= 2176   TOTAL= 2626
****

PROBLEM   3

GROUP RELATORS

EXP 4   3

EXP 5   2

EXP 1   2   2 -3 -2   3

EXP 2   2   3   3

EXP 1   1   1   1   1 -2 -1   2

EXP 3   1   2   3

EXP 11   1

EXP 3   1   1   1   1   3   3
SUBGROUP GENERATORS

EXP 1   1

OVERFLOW   MAX= 2740   TOTAL= 3995
   720 COSETS REMAIN AFTER LOOKAHEAD

INDEX=   720   MAX= 2740   TOTAL= 3995
****
PROBLEM   4

GROUP RELATORS

EXP 3   1

EXP 2   1   2

EXP 2   2   3

EXP 2   3   1

EXP 2   1   2   3

EXP 7   2

EXP 13   3
SUBGROUP GENERATORS

```
OVERFLOW              MAX=  2744       TOTAL=  3710
  2065 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW              MAX=  2744       TOTAL=  4477
  2527 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW              MAX=  2744       TOTAL=  4704
  2675 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW              MAX=  2744       TOTAL=  4776
  1092 COSETS REMAIN AFTER LOOKAHEAD

  INDEX=  1092      MAX=  2744       TOTAL=  4776
*********************************
PROBLEM   5


GROUP RELATORS


EXP  3          1

EXP  3          2

EXP  3          3

EXP  3          2  3

EXP  4          1  3

EXP  4          1 -3

EXP  4         -2  3

EXP  1          1 -2  1  2 -3  1  3  1 -3

EXP  5          1  2

EXP  5         -1  2
SUBGROUP GENERATORS


EXP  1          1

EXP  1          3

OVERFLOW              MAX=  2739       TOTAL=  3394
  2287 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW              MAX=  2739       TOTAL=  3848
  2539 COSETS REMAIN AFTER LOOKAHEAD
```

```
OVERFLOW               MAX=  2739        TOTAL=  4052
  2702 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW               MAX=  2739        TOTAL=  4089
  2726 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW               MAX=  2739        TOTAL=  4102
  2722 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW               MAX=  2739        TOTAL=  4119
  2738 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW               MAX=  2739        TOTAL=  4120
  2739 COSETS REMAIN AFTER LOOKAHEAD


LOOKAHEAD UNABLE TO CONTINUE
```

.TYPE HERON.DAT

```
-1
 0  1
 7  2
 2  1  2
 3 -1  2
-2
 2  1
 1 -1  2
-3
-1
 4  3
 5  2
 1  2  2 -3 -2  3
 2  2  3  3
 1  1  1  1  1 -2 -1  2
 3  1  2  3
11  1
 3  1  1  1  1  3  3
-2
 1  1
-3
-1
 2  1
 2  1  2
 2  2  3
 2  3  1
 2  1  2  3
 7  2
13  3
-3
-3
-1
 3  1
 3  2
 3  2
 4  1  3
 4  1 -3
 4 -2  3
 1  1 -2  1  2 -3  1  3  1 -3
 5  1  2
 5 -1  2
-2
 1  3
 1  2
-3
-8
```

.TYPE

*

Disk Files for Output

The structure of these files mirrors exactly the output as would be obtained using the TTY for output. In this made, however, summary information is output to the TTY. A sample session with input from file NEW CN.DAT and output to file OUTCN.DAT is given below, as well as the type - out of OUT CN.DAT.

6.   Sample Session Number 3

4237)   .IOT 1,16 $$u
ML ITS 811   CONSOLE 7 FREE. 17:01:31

ML ITS.811. DDT.516.
12. USERS
THE 110 BAUD NUMBER IS 258-7894.   SORRY ABOUT THE MIX UP. —PJ
:login group
    INIT
*:TEN
.R TC

INPUT ON DSK (20) OR TTY (05) ?
20

STANDARD FILE NAME (20) OR NOT (22) ?
22

ENTER 5 LETTER FILE NAME
NEWCN

OUTPUT TO DSK (21) OR TTY (05) ?
21

STANDARD FILE NAME (21) OR NOT (23) ?
23

ENTER 5 LETTER FILE NAME
OUTCN

TODD-COXETER ALGORITHM RUNNING ON FILE: NEWCN.DAT
                PUTTING OUTPUT ON FILE: OUTCN.DAT

****
PROBLEM   1

INDEX=   448   MAX=   2176   TOTAL=   2626
****
PROBLEM   2

INDEX=   720   MAX=   2740   TOTAL=   3995
****
PROBLEM   3

INDEX=   1092   MAX=   2744   TOTAL=   4776
****

PROBLEM   4

LOOKAHEAD UNABLE TO CONTINUE

INPUT ON DSK (20) OR TTY (05) ?

.TY OUTCN.DAT

****
PROBLEM   1

GROUP RELATORS

EXP 8   1

EXP 7   2

EXP 2   1   2

EXP 3   -1   2
SUBGROUP GENERATORS

EXP 2   1

EXP 1   -1   2

INDEX=   448   MAX= 2176   TOTAL= 2626
****
PROBLEM   2

GROUP RELATORS

EXP 4   3

EXP 5   2

EXP 1   2   2 -3 -2   3

EXP 2   2   3   3

EXP 1   1   1   1   1 -2 -1   2

EXP 3   1   2   3

EXP 11   1

i

SUBGROUP GENERATORS 1 1 3 3

EXP 1   1

OVERFLOW   MAX= 2740   TOTAL= 3995
   720 COSETS REMAIN AFTER LOOKAHEAD

INDEX=   720   MAX= 2740   TOTAL= 3995
****
PROBLEM   3

GROUP RELATORS

EXP 3   1

EXP 2   1 2

EXP 2   2 3

EXP 2   3 1

EXP 2   1 2 3

EXP 7   2

EXP 13   3
SUBGROUP GENERATORS

OVERFLOW   MAX= 2744   TOTAL= 3710
   2065 COSETS REMAIN AFTER LOOKAHEAD

i

OVERFLOW   MAX= 2744   TOTAL= 4477
   2527 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW   MAX= 2744   TOTAL= 4704
   2675 COSETS REMAIN AFTER LOOKAHEAD

OVERFLOW   MAX= 2744   TOTAL= 4776
   1092 COSETS REMAIN AFTER LOOKAHEAD

INDEX= 1092   MAX= 2744   TOTAL= 4776
****

PROBLEM   4

GROUP RELATORS

EXP 3   1
EXP 3   2
EXP 3   3
EXP 3   2 3
EXP 4   1
EXP 3   1
EXP 3   -2 3

EXP 3   1 -2 1   3 1 -3
EXP 3   1 2
EXP 3   -1 2
SUBGROUP GENERATORS

EXP 1   1
EXP 1   2

OVERFLOW   MAX= 2725   TOTAL= 4104

i 2722 COSMIC REMAIN AFTER LOOKAHEAD

  OVERFLOW   MAX= 2729   TOTAL= 4119
   1730 COSMIC REMAIN AFTER LOOKAHEAD

  OVERFLOW   MAX= 2731   TOTAL= 4120
   1729 COSMIC REMAIN AFTER LOOKAHEAD

  LOOKAHEAD UNABLE TO CONTINUE

7. MISCELLANEOUS INFORMATION

Typing Errors

1. Before carriage return

2. Use "RUBOUT" or "DELETE" Keys

3. Will echo the deleted character

4. May be used in succession to delete several char.

Program Interrupt:

1. Type control -C ($\uparrow$ c) once

2. Until EXIT
   $\uparrow C$
   .
   appears

To continue after $\uparrow$ C interrupt:

1. Type CONT

2. Back to same spot!

To Restart

1. Type START at monitor level
2. Continue as normal

Current Limitations

1. Number of generators $\leqq 9$.
2. Number of terms in generators or relation words $\leqq 40$.

8.   Getting onto the MATHLAB Computer

In order to run the Todd-Coxeter Algorithm on a given problem, one must first gain access to the MATHLAB computer located at M.I.T. There are three ways of doing so:

1)   If you are at M.I.T. and have access to a free ML console, simply follow the instructions in Section 2 of this memo.

2)   If you are not at M.I.T., but have access to a computer on the ARPA network of computers, then you can use the MATHLAB computer by accessing it over the network. Consult your local expert on the ARPA network to find out how to do this.

3)   You can access the MATHLAB computer using a terminal (teletype or Memorex) by telephone, if you have an acoustic coupler or a Dataphone.   The telephone numbers are:available from Dr. Vera Pless or Richard Bonneau, telephone number (617) 253-6026.

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AN INTERACTIVE IMPLEMENTATION

OF THE

TODD - COXETER ALGORITHM

Richard J. Bonneau

December 1973

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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AD 770-565

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