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A Dexterity Measure for the Kinematic Control of Robot Manipulators with Redundancy

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Abstract: We have derived a new performance measure, the product of the minors of the Jacobian matrix, that tells us how far a kinematically redundant manipulator is from singularity. It has been demonstrated that previously used performance measures, namely the condition number and the manipulability measure, allow changes in the type of arm configuration (or aspect) along a trajectory, causing repeatability problems and discontinuities in motion. The new measure, on the other hand, ensures that the arm remains in the same type of configuration, thus effectively preventing these problems.

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1 Introduction.

The benefits of using a quantitative measure in engineering systems are well known. More specifically, a quantitative measure provides one with a rational basis upon which one can, without having to rely on experience and intuition alone, analyze, design, and control the systems as follows:

- one can evaluate the performance of a given system and analyze the system, by estimating this measure; or
- one can design a system that achieves the performance to a certain degree, by maximizing (or minimizing) this measure; or
- one can control, in an on-line way, a given system to achieve it, by maximizing the measure at each moment.

In robotic systems, also, various performance measures have been used to quantify desired performance features such as obstacle avoidance (Yoshikawa, 1984; Maciejewski, 1985; Espiau, 1985; Khatib, 1986), torque minimization (Hollerbach, 1985), kinetic energy minimization (Whitney, 1972), and constraining the joint variables within their physical limits (Liegeois, 1977).

Another desire, in addition to optimizing these performance features, is the achievement of *dexterous manipulation*. This performance characteristic, however, is, in fact, quite ambiguous unless the concept of *dexterity* is made more precise. One concept of dexterity was specified as the volume of that part of the workspace, within which the end effector can achieve any orientation (Vijaykumar, Tsai, and Waldron, 1985): the larger the volume, the more dexterous the manipulator.

Another concept of dexterity, suggested by Yoshikawa (1985a), is the *ease* of changing the position and orientation of the end effector. This ease is a function of the dynamic characteristics of the manipulator. Another concept of dexterity proposed by Klein (1984, 1987) appeared to mean: (1) the quality of the linear system of differential relationships, as indicated by either the determinant or condition number of the Jacobian matrix; or (2) the naturalness of the appearance, as measured by how evenly distributed the joint variables are. Naturalness would be achieved by minimizing the sum of the squares of the deviation of actuators' displacements from their midpoints.

According to these definitions, a manipulator would be the least dexterous at a singular point; for at a singularity the conditioning of the linear system is at its worst, and creates awkward appearances due to links lining up or folding. In this sense, therefore, dexterity may be viewed as a measure of the *distance* from a singularity. In this paper, the meaning of dexterity is explicitly specified as the distance from singularity.

1.1 Existing Performance Measures.

To quantitatively represent the distance, several measures have been proposed (Yoshikawa, 1985a, 1985b; Uchiyama, 1985; Maciejewski, 1985; Salisbury, 1982): (a) the determinant of the Jacobian matrix, (b) its condition number, and (c) combinations of its singular values. It is not surprising that all of these are based on the Jacobian matrix, because the instantaneous end effector velocity in Cartesian space is related to the velocities in joint variable space through this matrix.

1.1.1 The Determinant of the Jacobian Matrix.

In linear algebra, the determinant of a matrix is an important measure used to test the invertibility of the matrix and its distance from singularity. Accordingly, the determinant of the Jacobian matrix has been tried as a dexterity measure for both nonredundant and redundant manipulators. For nonredundant manipulators, for instance, the determinant has been used as a measure of degeneracy for the analysis of wrist configurations (Paul and Stevenson, 1983). For redundant manipulators, on the other hand, Yoshikawa (1984) has proposed a measure called *manipulability*, defined as the square root of the determinant of $\mathbf{JJ}^T$. This measure is often viewed as a generalization of the concept of the determinant, because of the following:

- the manipulability reduces to the regular determinant in the nonredundant case.
- the manipulability becomes zero when the workspace rank is reduced at a singularity, just as the regular determinant of a square Jacobian matrix does.
- since the singular values of $\mathbf{JJ}^T$ are the squares of the singular values of $\mathbf{J}$, the determinant of $\mathbf{JJ}^T$ may be regarded as if it were the square of the regular determinant of a square Jacobian matrix.

1.1.2 The Condition Number.

Meanwhile, since the condition number of the Jacobian matrix is another important measure, whose inverse also indicates how far a matrix is from singularity, it too has been proposed as a measure of dexterity (Salisbury, 1982). It is noteworthy that this measure was initially used to determine the kinematic solution that minimizes the propagation from joint actuator torque error to end-effector force error—equivalently, the velocity error propagation from joint space to workspace—for nonredundant manipulators.

1.1.3 Singular Values and Their Combinations.

The determinant and the condition number of the Jacobian matrix can be expressed in terms of the singular values of the matrix: the determinant is the product of all the singular values, while the condition number is the ratio of the largest to the smallest singular value. Since the smallest singular value becomes zero when the matrix becomes singular, and roughly controls the behaviour of the two measures near a singularity, the smallest singular value itself has been suggested as a measure of distance from singularity (Klein, 1985). In addition to its simple expression, this measure has a relatively clear physical meaning: it may be interpreted in terms of the minimum responsiveness of the manipulator. That is, the worst ratio of the magnitude of the end effector velocity to the magnitude of the joint velocity (Klein, 1985), at a particular point in the workspace.

Other combinations of singular values that have been suggested as dexterity measures are the geometric mean and harmonic mean (Yoshikawa, 1985b). These may be viewed essentially as variations of the aforementioned measures.

1.1.4 Some Common Features.

The features common to all of these measures are the following:

- They indicate the presence of singularity: when singular, the value of these measures become zero (except for the condition number, the value of which becomes infinity).
- Their absolute values (inverse of the absolute value in the case of the condition number) appear to represent, in one way or another, the distance from singularity. That is, the larger the value, the farther the manipulator is from singularity.

In the case of redundant manipulators, however, these measures do not explicitly indicate the successive changes in the available degrees of freedom as long as the workspace rank is preserved. For instance, suppose we have a five degree of freedom manipulator that is to move in a three-dimensional workspace, hence having two degrees of redundancy. These measures do not necessarily become zero when such a manipulator has lost one (or even two) degrees of freedom.

Because the degree of redundancy is an important constituent of the distance from a singularity, there is therefore an obvious shortcoming of these distance measures. Furthermore, it is also undesirable for a measure to be insensitive to relative differences in the distance from singularity for a particular degree of redundancy. Therefore, we feel that a satisfactory dexterity measure should indicate *both* changes in the degrees of redundancy, also well as relative differences within a particular degree of redundancy.

Losing degrees of freedom may not in itself be a serious drawback, as long as the workspace rank is fully preserved so that the desired location of the end effector can be achieved by suitable selection of the joint variables. What may be of more concern are potential problems that can be expected to arise—by analogy with the nonredundant case—when degrees of freedom are lost. More specifically, in the nonredundant case, the collection of points where degrees of freedom are lost—at the singularities—constitutes the boundaries between regions of different types of joint configuration or *aspects* (for example, elbow up versus elbow down) (Uchiyama, 1979).

When such a boundary is crossed, the manipulator switches from one distinct type of configuration to another. A trajectory control scheme that allows the manipulator to cross over such boundaries often leads to *repeatability problems*, since it will use a different set of joint variables to reach a given point in the work space depending on the past history of the motion. Besides, near the points on the trajectory where switching occurs, there are often accompanying discontinuities in motion, resulting in large joint variable velocities. The same problems are expected in the redundant case, since in this case too there exist multiple solutions involving distinct types of configurations (Borrel, 1986).

Now the boundaries between regions corresponding to different types of configurations are the points where the degrees of freedom are reduced. It appears, however, that these (nontrivial) problems tend to be veiled because of the fact that, owing to the redundancy, the switching can happen without causing the more serious (and obvious) problems that occur at true singularities. To our knowledge, there has not appeared any analysis of these problems for redundant manipulators, and any performance measures that is designed to prevent them.

2 Preview.

The objectives of this paper are as follows:

- to analyze the aforementioned relative distance ideas for redundant manipulators and to derive a new distance concept;
- to derive from this concept a new performance measure that represents the dexterity of manipulators including relative distance from singularity;
- to examine whether the new performance measure helps in the task of avoiding the repeatability problems and discontinuous motions due to switching between different types of configurations.

This performance measure is intended to be used either with on-line kinematic control methods, or for off-line design purposes.

In order to better understand the degree of redundancy and the relative distance differences, we first review in Section 2 the concept of singularity, and some basic information about the degree of redundancy. Then we will derive a new concept of the distance from singularity for the kinematically redundant case. This concept is obtained by observing the structure of the Jacobian matrix of a redundant manipulator. Then, from this concept, a new performance measure will be developed. The special properties of this measure will be discussed in Section 3. In addition, the new performance measure will be qualitatively compared with two existing performance measures: the manipulability measure and the (inverse of the) condition number. In Section 4, the results of numerical simulations will be shown with redundant manipulators to compare the effectiveness of each measure in achieving dexterous movements. In the comparison, both the repeatability problems, as well as the ability to preserve the type of joint configuration are shown. Finally some concluding remarks will be made in Section 5.

3 New Distance Concept and Performance Measure.

This subsection presents reviews of two basic concepts, singularity and kinematic redundancy, to provide a better understanding of the distance from singularity in redundant manipulators. Then the new distance concept and its corresponding performance measure will be derived.

3.1 Review of Singularity and Redundancy.

In this subsection we review the concept of kinematic singularity and compare it with the concept of singularity of a matrix. We then review the concept of kinematic redundancy and compare and contrast this with the concept of redundancy of a matrix.

3.1.1 Kinematic Singularity.

Singularities can be easily discovered by examining the differential relationship[1], between changes in the joint variables and changes in the end effector location:

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\boldsymbol{\theta}}, \tag{1}$$

where $\boldsymbol{\theta}$ is a n-dimensional vector representing the joint variables, while $\mathbf{x}$ is a m-dimensional vector representing the end effector location and J is the Jacobian matrix of the transformation from joint variables to end effector location.

[1]Much of the discussion here is based on unpublished notes on singularities by B.K.P. Horn.

For the nonredundant case ($n = m$), the Jacobian matrix $\mathbf{J}$ is square. When the Jacobian matrix becomes singular, a manipulator is said to be at a *singular* point. Hence at a singular point, the determinant of $\mathbf{J}$, that is, $\det(\mathbf{J})$ equals zero. This simple fact, together with the fact that the determinant is a continuous function of the joint variables, provides some important insights:

1. When $\det(\mathbf{J}) = 0$, the rank of $\mathbf{J}$ is reduced, and the manipulator loses some of its degrees of freedom. This is because the rank of the column space is reduced. The result is an inability to move in a certain direction by any combination of small changes in the joint variables.

2. Furthermore, as a manipulator comes close to this point, small movements in some direction requires very large changes in the joint variables.

3. The sign of the determinant changes as one passes through a singular point. Since the determinant is the ratio of the differential volume in work space coordinates to the differential volume in joint coordinates, the sign change in the determinant indicates a change from one type of configuration to another. In fact, just as a change of sign in a continuous function cannot occur without passing through zero, so the manipulator cannot change from one type of configuration to another without passing through a singularity. This property has also been discussed by Uchiyama (1979).

4. At a singularity, two different kinds of solutions collapse into one; hence, the number of types of configurations is reduced.

5. Items 3 and 4 can be explained in terms of Riemann sheets: multiple solutions correspond to multiple sheets, each of which represents a mapping from Cartesian coordinates to joint variables; singular points lie on the *folds* of these sheets.

The first two items may explain why keeping far from a singularity is closely related to dexterity.

From the fact that the determinant becomes zero at a singularity, it functions in a sense, as an indicator of the presence of a singular point. A geometrical interpretation of the absolute value of the determinant is the *volume* of a parallelepiped made of n column vectors (or row vectors) of the Jacobian matrix. This interpretation, together with the fact that at a singularity the parallelepiped collapses and the volume becomes zero, is in fact the basis of the idea that the determinant is a *measure of distance* from singularity.

One disadvantage of using the determinant, however, as an indicator of singularity is that, once the rank of $\mathbf{J}$ is reduced, the determinant is zero, and there is no way to distinguish between two different degress of singularity, even though the ranks of the corresponding Jacobian matrices may be different. A more accurate indicator for this purpose probably is the remaining rank, or degree of freedom, itself.

For redundant manipulators, the measure equivalent to det($\mathbf{J}$) is the manipulability measure, namely

$$\sqrt{\det(\mathbf{J}\mathbf{J}^T)}.$$

Yet this measure, as mentioned in Section 1, cannot indicate a change in the degree of redundancy, something that will be reviewed in the following subsection.

3.1.2 Kinematic Redundancy.

The degree of redundancy r is formally defined as

$$r = n - m, \tag{2}$$

where n is the number of degrees of freedom, and m the rank of workspace. In terms of linear algebra (Strang, 1980), the workspace rank corresponds to the dimension of the row space of the Jacobian matrix, and the number of degrees of freedom to the dimension of its column space, while the degree of redundancy corresponds to the dimension of the null space.

In other words, the degree of redundancy is the maximum number of linearly independent vectors in the null space, $\mathbf{e}_i$, defined by the equation

$$\mathbf{J}\mathbf{e}_i = \mathbf{0}. \tag{3}$$

But, we find that this definition is not appropriate for describing the concept of *kinematic redundancy.*

For instance, consider the following Jacobian matrix, consisting of three two-dimensional column vectors, J^1, J^2, and J^3, representing a three degree of freedom, planar, redundant manipulator,

$$\mathbf{J} = (J^1 \; J^2 \; J^3).$$

If the second and the third links line up, we find that $J^3 = cJ^2$ (with c a nonzero constant), and we know that the redundancy has been eliminated. However, the null space vector that satisfies (3) is equal to

$$\mathbf{e} = (0, c, -1)^T,$$

which is a nonzero vector. Thus, according to the above definition, the degree of redundancy is one, whereas the observation indicates there is no redundancy. This discrepancy can be resolved if we modify the meaning of n in (2), to be equal to the *available* degrees of freedom. But, as mentioned in Section 1, it turns out that this modified definition is still inadequate to describe the relative differences in the distance for a given degree of redundancy.

3.2 New Concept of Distance from Singularity in the Redundant Case.

In this subsection, we will consider some Jacobian matrices of kinematically redundant manipulators, and identify relative differences in the distance from singularity. On the basis of the observations, we will propose a new definition of the distance from singularity.

Again, the kinematic equation of a kinematically redundant manipulator is generally given as follows:

$$\mathbf{x} = \mathbf{f}(\boldsymbol{\theta}),$$

where $\mathbf{x} \in \Re^m$, and $\boldsymbol{\theta} \in \Re^n$ with with $m < n$. Then, the Jacobian matrix for the transformation from $\boldsymbol{\theta}$ to $\mathbf{x}$, namely $\mathbf{J} \in \Re^{m \times n}$, may be denoted in general as,

$$\mathbf{J} = (J^1 \; J^2 \; \ldots \; J^m \; J^{m+1} \; \ldots \; J^n),$$

where J^k is the k-th column vector. If m linearly independent vectors are chosen, without loss of generality, as the first m column vectors of $\mathbf{J}$, then, from linear algebra, the remaining $n - m$ vectors $J^{m+1}, \ldots, J^n$ are linear combinations of the first m vectors $J^1, \ldots, J^m$ (Strang, 1980).

Observations indicate that how many of these first m vectors are included in the linear combination for each of the remaining $n - m$ vectors determines *how far* from a singularity the manipulator is.

To illustrate this point, let us select a manipulator with just one degree of redundancy, i.e., $n = m + 1$, where

$$\mathbf{J} = (J^1 \; J^2 \; \ldots \; J^m \; J^{m+1}).$$

Consider the following three cases of linear combinations for J^{m+1}:

1. $J^{m+1} = a_1 J^1$,

2. $J^{m+1} = a_1 J^1 + a_2 J^2$,

3. $J^{m+1} = a_1 J^1 + a_2 J^2 + \cdots + a_m J^m$,

where the a_i's, ($i = 1, 2, \ldots, m$) are arbitrary nonzero constants. What are the differences between these cases?

According to the formal definition in (2), the degree of redundancy for each of the three cases is one. Alternatively, if the modified definition is used, then the manipulator in case 1 has no redundancy, whereas it has one degree of redundancy in both cases 2 and 3. However, careful observation reveals that there still exists another difference in the distance from a singularity between cases 2 and 3. The differences between the three cases may be explained as follows:

1. In case 1, the manipulator gets into a singularity, reducing its rank (to less than m), if *any* two of the first m column vectors happen to line up.

2. In case 2, a singularity arises if *any* two, except for J^1 and J^2, of the m column vectors line up.

3. In case 3, the Jacobian matrix preserves its rank (m), even when *any* two of the column vectors happen to line up.

In other words, the chance for the manipulator to get into a singularity decreases by degrees, as the number of linearly independent vectors to be included in the combination increases. These differences in the possibility of getting into singularity determine the *relative differences*, for systems with the same degrees of redundancy, in the distance from singularity.

Meanwhile, the number of the J^1, J^2, ..., J^m that appear in each of the J^{m+1}, ..., J^n uniquely determines the number of distinct combinations of m linearly independent column vectors, or the number of distinct submatrices of rank m in the Jacobian matrix. Hence, the number of nonsingular square submatrices also represents the margin from singularity; as the number increases, the system is less likely to become singular. Of course, this number is reduced as the number of column vectors that line up increases. While these two measures are equivalent, determining the number of nonsingular submatrices is easier than selecting redundant vectors in the set of m vectors. Note, at the same time, that these observations are not confined to this particular example of a one degree of redundancy case, but are evidently valid for the general cases, where the degree of redundancy is more than one.

As an example, consider the following five jointed robot having a three dimensional workspace and thus two degrees of redundancy, where the Jacobian matrix is given as,

$$\mathbf{J} = (J^1\ J^2\ J^3\ J^4\ J^5),$$

where the J^i's are the three dimensional column vectors. If J^1, J^2, and J^3 are selected as linearly independent vectors, then J^4 and J^5 can, in general, be represented as

$$J^4 = c_1 J^1 + c_2 J^2 + c_3 J^3,$$
$$J^5 = d_1 J^1 + d_2 J^2 + d_3 J^3.$$

Depending on *how many* and *which* of the coefficients c_i's and d_i's are zero, we have different numbers and combinations of linearly independent vectors appearing in J^4 and J^5. At the same time, this number and the particular combinations of vectors determine the number of submatrices of rank 3 in the Jacobian matrix. Table 1 shows the relationship between the number of submatrices and the number (and the combination) of linearly independent vectors.

Table 1: The relationship between the number of linearly independent vectors used in representing the remaining vectors, and the number of nonsingular submatrices in the Jacobian matrix.

c_1 c_2 c_3	* * *	* * *	0 0 *	0 * *	0 0 *	0 * *
d_1 d_2 d_3	* * *	0 * *	* * *	* * 0	* * 0	0 * *
No. of sub-matrices	10 ($_5C_3$)	9	8	8	7	6
Notes:	A	B	C	D	E	F
c_1 c_2 c_3	0 0 *	0 0 *	0 0 0	0 * 0	0 0 0	0 0 0
d_1 d_2 d_3	0 * *	* 0 0	* * *	0 * 0	0 * 0	0 0 0
No. of sub-matrices	5	4	4	3	2	1
Notes:	G	H	I	J	K	L

* represent any nonzero value

<u>Notes:</u>

A All of the c_i's and the d_i's are nonzero.

B Only one of the c_i's and the d_i's is zero.

C Any two of either the c_i's or the d_i's is zero.

D One of the c_i's and one of the d_j's are zero, with $i \neq j$.

E Two of either the c_i's or the d_i's are zero and one of the other coefficients, d_j's or c_j's, is zero with $i \neq j$.

F One of the c_i's and one of the d_i's are zero with $i = j$.

G Two of either the c_i's or the d_i's are zero and one of the other coefficients, d_j's or c_j's, is zero with $i = j$.

H Two of both the c_i's and the d_j's are zero, with one overlapping $i = j$.

I All of either the c_i's or the d_j's are zero.

J One of both the c_i's and the d_j's are nonzero, with $i = j$.

K Only one of either the c_i's or the d_j's is nonzero.

L All of the c_i's and the d_j's are zero.

It is noteworthy that the number of nonsingular submatrices successively drops from the maximum, 10, to the minimum, 1, depending on the number and combination of linearly independent vectors. Again, even within a particular degree of redundancy, there are differences in the number of nonsingular submatrices. Clearly the number of nonsingular submatrices gives an indication of the relative distance from singularity.

In addition, note that the absolute value of the determinant of each submatrix, called a *minor*, represents the distance from its own singular or degenerate state. Therefore, a measure of the *overall* distance from singularity should take into account the value of each minor of the Jacobian matrix. In other words, in addition to the *number* of submatrices of rank m, one should take into account their determinants. The further *each submatrix* is from a singularity, as indicated by a larger absolute value of the minor, the smaller the chance of singularity.

The above observations directly leads to a definition of *distance* from singularity as follows:

Definition:

> The distance from singularity is represented by *the number* of distinct nonsingular submatrices of rank m and the *magnitude* of the determinant of each submatrix, that is, the magnitude of each minor of the Jacobian matrix.

In using the concept of distance from singularity, for example, in finding kinematic solutions, it is unwieldy to bring this detailed definition to bear in full. What is needed is a single number that summarizes the salient aspects of the concept above. This leads us to the new performance measure to be introduced in the next section.

3.3 Derivation of A New Performance Measure.

We will derive a performance measure for the purposes of kinematic control and manipulator design, based on the distance concept developed above. More specifically, the following objectives are simultaneously to be met in order to achieve the desired performance:

> - To keep *the number* of distinct nonsingular submatrices of rank m as large as possible;
> - To make *the magnitude* of each minor as large as possible.

As an index that explicitly represents these objectives, we propose the following measure:

$$H = \left| \prod_{i=1}^{p} \Delta_i \right|^{1/p}, \tag{4}$$

where the Δ_i's for $i = 1, 2, \ldots, p$, with $p = {}_nC_m$, are minors of rank m of the Jacobian matrix. Clearly, this measure contains in its expression the two elements of the distance, the *number* and the *magnitude* of distinct minors, in such a way that both objectives are automatically achieved as it increases. To be more specific, since the measure has nonzero values only if all of the minors are nonzero, keeping it greater than zero guarantees the maximum *number* of distinct nonsingular submatrices. At the same time, since the measure cannot have a large value unless each minor is large, increasing the measure tends to increase the *magnitude* of each. Furthermore, since the measure is a product, it becomes smaller if the minors have uneven values (for a given total value). Therefore, this tends to prevent any minor from being particularly large at the expense of forcing others to be too small.

In (4), the exponent $1/p$ is primarily used so that, when $n = m$, the measure reduces to the absolute value of the determinant. We find a similar treatment, in (Yoshikawa, 1984), where the manipulability measure is defined as the square root of $\det(\mathbf{J}\mathbf{J}^T)$. Of course, if exponents are used, then the physical interpretations of the measure changes. This will be considered in the next section. Also note that the use of an exponent, when the measure is used in the null space of the resolved motion method, results in a different time response of convergence toward the optimal kinematic solution. Except for these differences, the essential characteristics are not changed by introducing the exponent.

4 Properties of the New Measure and Relationship to other Measures.

In this section we first explore the properties of the new performance measure and then compare it with the existing performance measures.

4.1 Properties of the New Measure.

Examining the new measure, we find the following important properties:

- When $m = n$, i.e., for nonredundant manipulators, the measure reduces to

  $$H = |\det(\mathbf{J})|,$$

  which is the same as that proposed by (Paul and Stevenson, 1983). This measure may be conceptually interpreted as the *volume* of a parallelepiped in

m-dimensional space, the edges of which come from the columns (or equivalently, the rows) of the Jacobian matrix, $\mathbf{J}$.

- When $n > m$, the measure represents the *geometric mean* of the volumes of p parallelepipeds made of each of the possible combinations of m column vectors out of the n.

- The points where $\Delta_i = 0$ are the boundaries between one *type* of joint configuration and another. These points are also the points where some of the column vectors in Δ_i are linear combinations of the remaining ones, thereby causing the corresponding minor to become zero.

Note that the last property may be considered an extension of the nonredundant case in Section 2 to the redundant case, where the points satisfying $\det(\mathbf{J}) = 0$ determine the boundaries. This property, in fact, was used by Borrel and Liegeois (Borrel, 1986) to determine the boundaries of different types of joint configurations. These boundaries divide the joint space into subsets, called *aspects*, each of which consists of one type of joint configuration.

In addition to determining aspects, we can use this property to *force* the joint configuration to stay within a preferred aspect. More specifically, by keeping all of the Δ_i's nonzero, we preserve the type of joint configuration of a redundant manipulator.

Why do we think that it is often useful to force the joint configuration to stay within a particular aspect? The reasons, already discussed in Section 1, can be summarized as follows:

- switching between aspects can cause a certain type of repeatability problem, and

- discontinuities in motion and awkward configurations may accompany the switching.

Clearly, maximizing the new performance measure directly prevents all of the Δ_i's from becoming zero, so it immediately addresses these problems. In other words, by virtue of this property, the new performance measure is expected to help solve these problems.

4.2 Relationship to other Measures.

In this subsection, we investigate the relationship between the proposed measure and the two others: first the manipulability measure and then the condition number.

4.2.1 Relationship to the Manipulability Measure.

The new measure and manipulability are loosely related, since both measures represent the distance from a singularity. But what is the precise relationship between the two? The following theorem answers this question:

Theorem 1 *For any matrix,* $\mathbf{J} \in \Re^{m \times n}$, *with* $m < n$,

$$\det(\mathbf{J}\mathbf{J}^T) = \sum_{i=1}^{p} \Delta_i^2,$$

where the Δ_i*'s,* $i = 1, 2, \ldots, p$, *with* $p = {}_nC_m$, *are the minors of rank* m *of the matrix* $\mathbf{J}$.

The proof of this theorem is given in Appendix 1. Since the manipulability measure, H_1, is defined by

$$H_1 = \sqrt{\det(\mathbf{J}\mathbf{J}^T)},$$

it can be expressed in terms of the minors as follows:

$$H_1 = \sqrt{\sum_{i=1}^{p} \Delta_i^2}, \tag{5}$$

whereas the new measure, expressed in (4) is,

$$H = \left| \prod_{i=1}^{p} \Delta_i \right|^{1/p}.$$

Comparing the two measures, we note the following differences:

1. Geometrically, the manipulability measure may be interpreted as the Euclidean norm of the vector representing the present state in the minor coordinate system, with components $\Delta_1, \Delta_2, \ldots, \Delta_p$. In contrast, the new measure is proportional to the radius of a sphere whose volume is equal to that of a hyper-cube with edges of length $\Delta_1, \Delta_2, \ldots, \Delta_p$.

2. As mentioned in Section 3, the new measure tends not to have a large value if the values of the Δ_i's are uneven; whereas the other measure can still have a large value as long as *some* of the dominant minors have large absolute values. The manipulability measure can have, in the extreme, some zero minors, as long as the workspace rank is preserved.

   Hence the new measure tends to give more balanced minors than the manipulability measure, in addition to preventing minors from becoming equal to zero, thus directly controlling the switching between aspects. In contrast, the manipulability measure does not have an immediate effect on the switching between aspects.

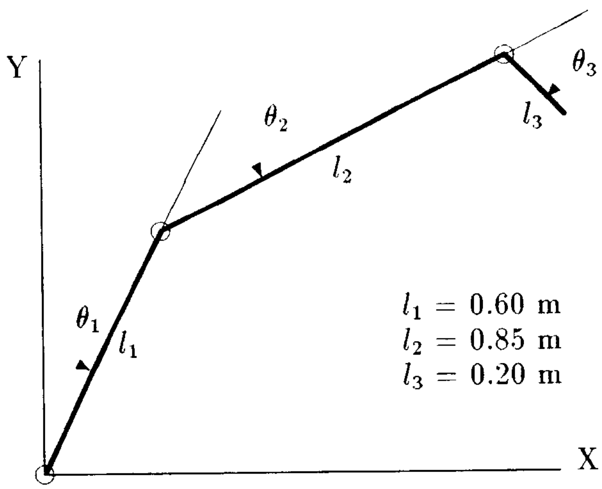


Figure 1: Schematic diagram of a planar redundant manipulator with three links.

3. Note that the manipulability can be also expressed as (Yoshikawa, 1985)

$$H_1 = \prod_{k=1}^{m} \sigma_k,$$

   where σ_k is the k-th singular value of $\mathbf{JJ}^T$.

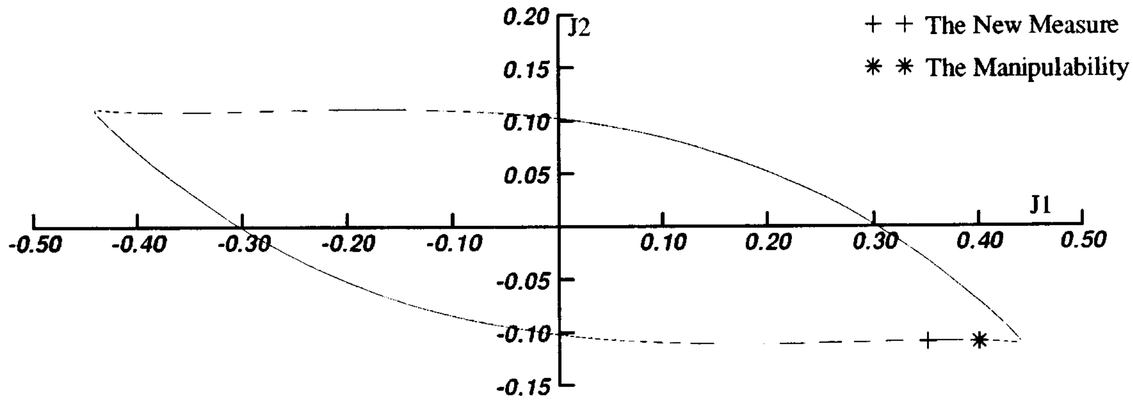
   This expression shows that the measure has a similar form to the new measure in that it is a product; the difference is that the manipulability measure is the product of the singular values representing the workspace, whereas the product of minors in the new measure represents the joint space.

   This difference implies, in a sense, that the former concentrates on preserving the workspace rank while the latter concentrates on degrees of freedom of joint space. Since keeping as many degrees of freedom as possible in joint space automatically preserves the workspace rank, the latter imposes stricter constraints.

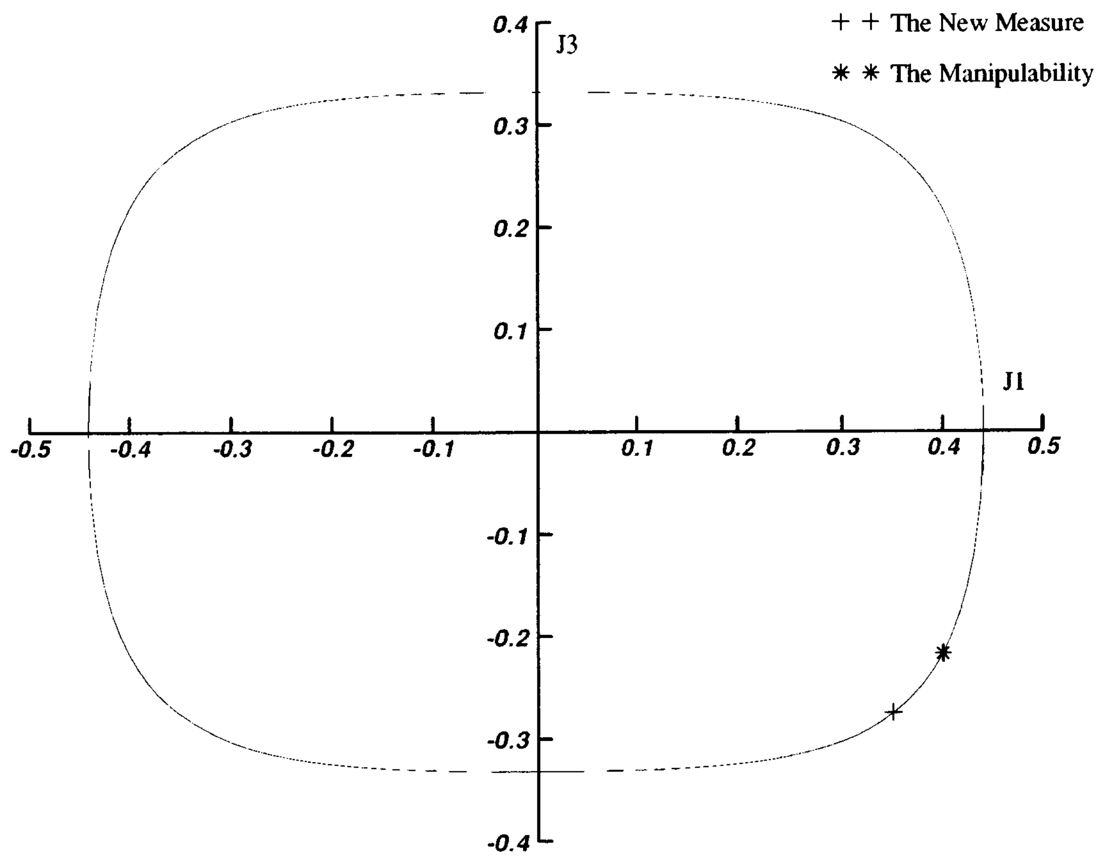
To illustrate the second difference, let us consider a three degree of freedom, redundant manipulator as shown in Figure 1, which is to locate the end effector at a certain (x, y) position.

Each end effector location within the workspace can be achieved by an infinite number of joint variable combinations. Each such kinematic solution is represented by a distinct Jacobian matrix and thus a distinct set of minors.

It is useful to consider a space where the coordinates are the values of the minors, Δ_1, Δ_2, and Δ_3. For a given end effector location, the infinite set of solutions determines a curve in this space, as shown in Figure 2. Particular points



Cross Plot of Minor Trajectory: J1 vs. J2



Cross Plot of Minor Trajectory: J1 vs. J3

Figure 2: The locus of the minors $J1$, $J2$, and $J3$, for all possible kinematic solutions, when the end effector is at the point $x = 0.2$ m, $y = 0$ m. The three dimensional curve is here represented by projection onto the $J3 = 0$ and $J2 = 0$ planes.

in this curve will maximize specific performance measures. Applying the inverse kinematic method presented in (Chang, 1986), for example, when the end effector is located at $x = 0.2$ m, $y = 0$ m, we obtain two particular sets of joint values, each of which maximizes one of the two measures of performance. The corresponding sets of minors are plotted in the same curve in Figure 2. These plots confirm a feature predicted above: the new performance measure gives somewhat more balanced minor values than the manipulability measure. The balance of the magnitudes of the minors changes with the end effector location; this improvement is more noticeable as the end effector moves toward the outer or inner workspace limits (which is where it matters most).

4.3 Relationship to the Condition Number.

The relationship of the new measure with the condition number is not so clear as that with the manipulability measure, because of the difficulty in deriving such a pair of simple expressions as (4) and (5). As is well known, the condition number H_2 is defined by

$$H_2 = \frac{\sigma_{\max}}{\sigma_{\min}}, \tag{6}$$

where $\sigma_{\min}$ and $\sigma_{\max}$ are minimum and maximum values of the singular values, respectively. The singular values correspond to the workspace rank: a nonzero value of $\sigma_{\min}$ guarantees full workspace rank. Therefore, minimizing the condition number, in effect, results in maximizing $\sigma_{\min}$. This measure thus tends to weight preservation of rank without weighting what happens within the redundant degrees of freedom.

5 Numerical Simulations.

In this section, the new measure is quantitatively compared with the two other measures. To this end, some numerical experiments were carried out for the case of a three degrees of freedom planar manipulator. These experiments allow us to examine:

1. whether the new measure can help achieve the desired performance, namely avoiding singularities, if used for kinematic control;

2. whether the measure can preserve *the aspect* (that is, the type of joint configurations) and how this alleviates the repeatability problem;

3. what other effects are induced by transitions between different aspects.

To examine the first point (the ability to overcome singularity), two kinds of simulations were performed: (a) where the manipulator is initially in a nearly singular configuration, and (b) when the end effector touches the base. To examine the second and third points, the end effector is made to radially reciprocate between the base and the outer limits of its workspace.

Either the fixed inverse kinematic mapping method (Chang, 1986) or the resolved motion method (Liegeois, 1977) can be used for the kinematic control in these experiments. The fixed inverse kinematic mapping method obtains the joint variables, $\boldsymbol{\theta}$, by numerically solving the following system of nonlinear equations:

$$\left\{ \begin{array}{ll} \mathbf{x} & = \mathbf{f}(\boldsymbol{\theta}), \\ \mathbf{Zh} & = \mathbf{0}, \end{array} \right\} \tag{7}$$

where the upper equations are the kinematic equations, and the lower equations the optimizing equations. Here $\mathbf{Z}$ is the null space matrix defined by

$$\mathbf{Z} = \left(\mathbf{J}_{n-m}\, \mathbf{J}_m^{-1} \ : \ -\mathbf{I}_{n-m} \right), \tag{8}$$

while $\mathbf{h}$ is the gradient of the performance measure functions defined by

$$\begin{array}{ll} \mathbf{h} & = (h_1, h_2, \ldots, h_n)^T, \\ h_i & = \frac{\partial H}{\partial \theta_i}, \quad i = 1, 2, \ldots, n. \end{array} \tag{9}$$

The resolved motion method, on the other hand, uses the equation

$$\dot{\boldsymbol{\theta}} = \mathbf{J}^+ \dot{\mathbf{x}} + \alpha (\mathbf{I} - \mathbf{J}^+ \mathbf{J}) \mathbf{h}, \tag{10}$$

where α is a gain constant, $\mathbf{I}$ the n-dimensional identity matrix, and $\mathbf{J}^+$ the matrix known as the Moore-Penrose pseudoinverse defined by

$$\mathbf{J}^+ = \mathbf{J}^T (\mathbf{J}\mathbf{J}^T)^{-1}. \tag{11}$$

5.1 Overcoming Singularity.

The two experiments examining the ability of the kinematic solution method to overcome singularity are made with a manipulator that has three revolute joints with equal lengths of 0.55 m, as shown in Figure 3.

5.1.1 Escaping from a Nearly Singular Configuration.

In the first experiment, we start from a configuration specified by the joint variables $\boldsymbol{\theta} = (-90°, 179.5°, 0°)^T$. This configuration is nearly singular. The manipulator is then commanded to use self-motion to escape from this configuration, using each of the three performance measures in turn.

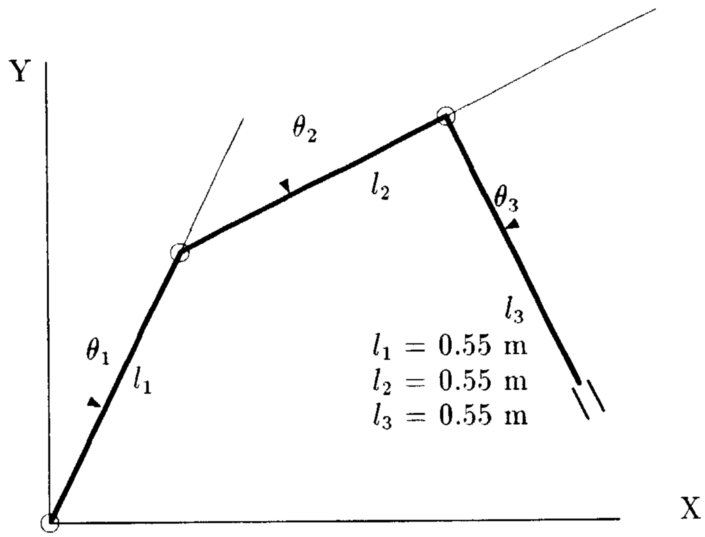
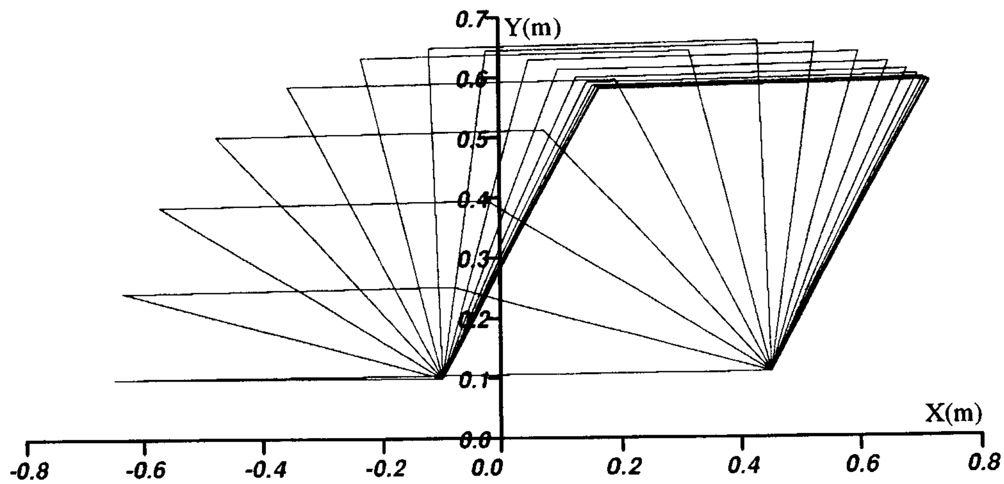


Figure 3: Schematic diagram of a planar redundant manipulator with three links of equal length.

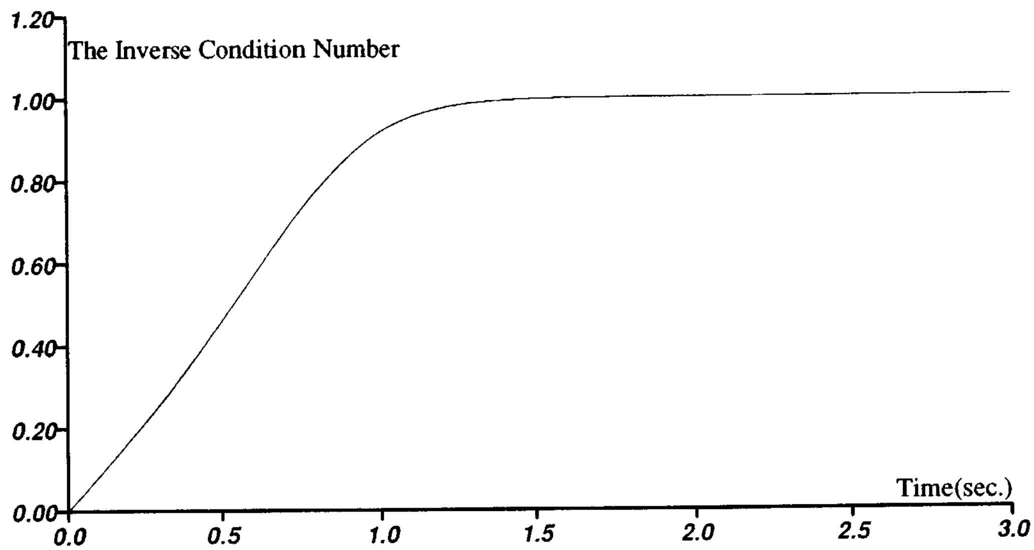
The resolved motion method was used, with $\alpha = 10$ in equation (10), to examine the change of joint variable configuration. In order to avoid a large value for the null space term, the condition number was minimized by maximizing its inverse.

The results of the experiments are given in Figures 4, 5, 6, where the change of the configuration and the time response is given for each performance measure. It should be clear from these figures that each measure, if included in the null space term, makes the manipulator escape from the singular configuration, driving joint values toward the state where the measure has the maximum value for the chosen end effector location. The effect of including the performance measures is significant because there is no self-motion *without* their use. The speed of convergence in the case of the condition number case is noticeably slower than that in the case of the other two measures, which are about equally fast.

Whereas the condition number leads to a considerably different steady state configuration, the new measure and the manipulability measure have steady state configurations that look surprisingly similar. Yet a close inspection shows that they are in fact slightly different. The reason for the similarity is as follows: the optimizing equation, $\mathbf{Zh} = \mathbf{0}$ in (7), for a particular measure is in general quite different from that for another measure. Even for the manipulator in Figure 3, with its symmetric geometry, joint solutions for the different measures are typically different. However, the above optimal conditions for each measure turn out to be satisfied only at end effector locations (x, y) satisfying $x^2 + y^2 = l^2$, with a particular joint values of $\theta_2 = \theta_3 = 90°$ or $\theta_2 = \theta_3 = -90°$. Here l is the length of each of the three links of the given manipulator.

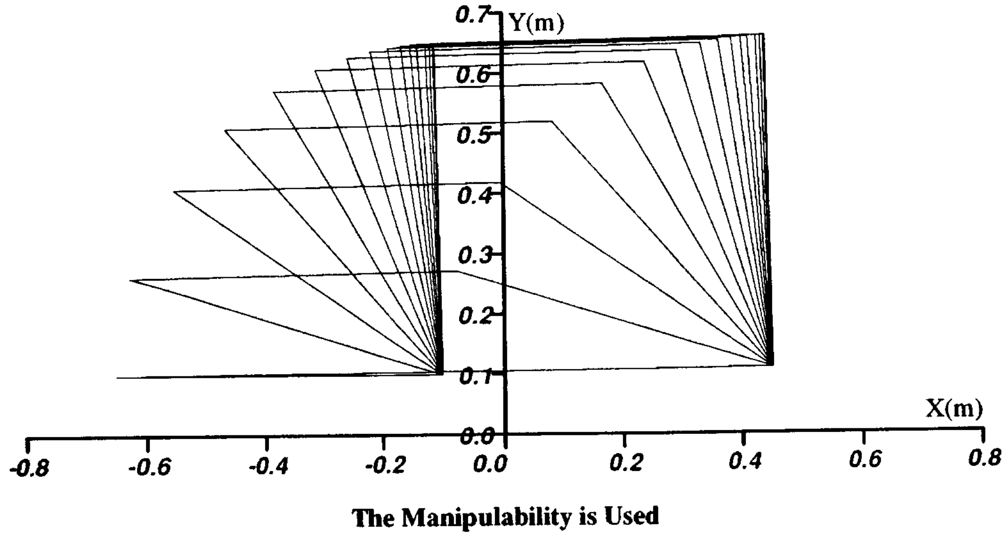


The Condition Number Is Used



Time Response of The Inverse Condition Number

Figure 4: Illustration of the ability to escape from near a singularity when the condition number is used: self-motion brings the arm closer to the optimal configuration as time goes on. The link lengths are $l_1 = l_2 = l_3 = 0.55$ m.



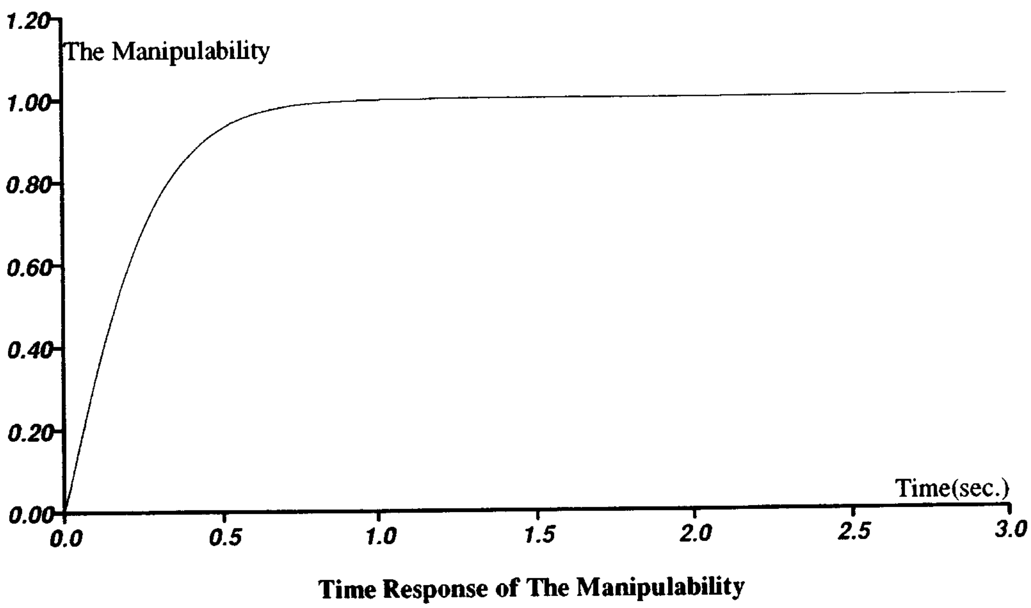
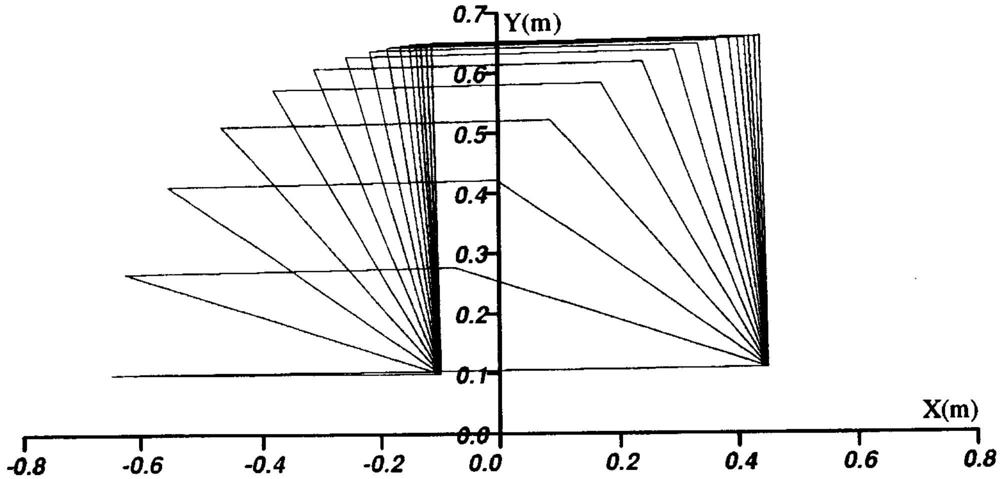
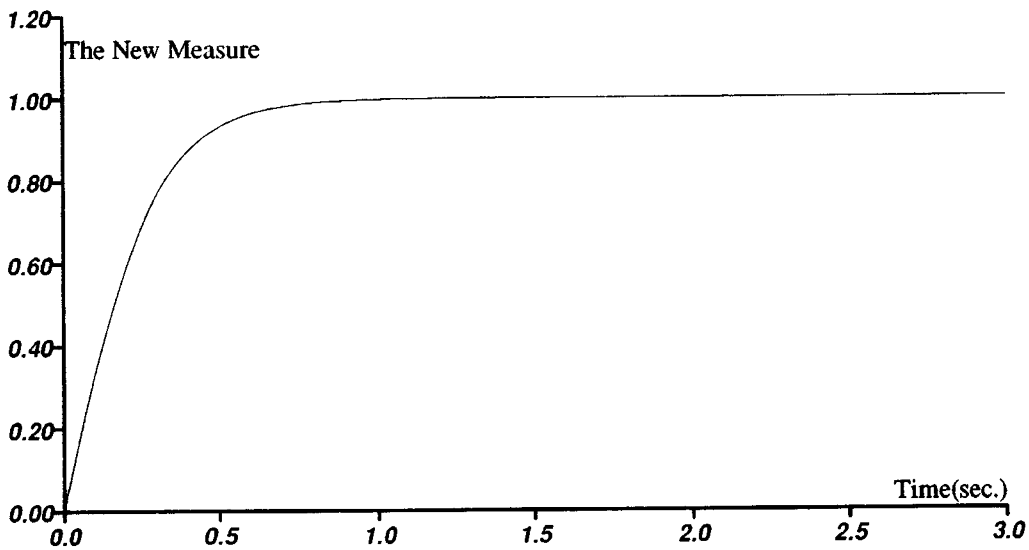


Figure 5: Illustration of the ability to escape from near a singularity when the manipulability measure is used: self-motion brings the arm closer to the optimal configuration as time goes on. The link lengths are $l_1 = l_2 = l_3 = 0.55$ m.



The New Measure Is Used



Time Response of The New Measure

Figure 6: Illustration of the ability to escape from near a singularity when the new measure is used: self-motion brings the arm closer to the optimal configuration as time goes on. The link lengths are $l_1 = l_2 = l_3 = 0.55$ m.

5.1.2 Passing the Base.

Whereas the previous experiment examines the ability to escape from near a singularity, the present one examines the behavior of the manipulator when the tip touches the base, forming a closed kinematic chain—a triangle. This case is of interest because we see intuitively that self-motion is not possible, except for rigid body rotation of the whole triangle with respect to the base. This intuition can be easily confirmed if the projection matrix, $\mathbf{I} - \mathbf{J}^+\mathbf{J}$, in the homogeneous solution term of (10), is symbolically derived. To be specific, when the tip is located at the base, the Jacobian matrix in (1) has the following degenerate form:

$$\mathbf{J} = \begin{pmatrix} 0 & j_{12} & j_{13} \\ 0 & j_{22} & j_{23} \end{pmatrix}, \qquad (12)$$

from which the projection matrix can be derived as,

$$\mathbf{I} - \mathbf{J}^+\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (13)$$

By looking at the projection matrix, we see that only θ_1 is affected by the gradient of the performance measure, $\mathbf{h}$, resulting in a rigid body rotation of the triangle with respect to the base, without making any other changes in the joint variable configuration.

The question then is, is it impossible for the manipulator to get into and out of this special configuration? In other words, can we develop an inverse kinematic solution that resolves the motion when the tip is passing the base? The answer to this question is that, although the homogeneous term becomes ineffective with the tip at the base, it is still possible for the manipulator to get into and out of the point. The reasons for this may be analyzed as follows:

- When the tip is *approaching* the base, the homogeneous solution term, although diminishing, still exists, continuing the effort to achieve the optimal configuration, until the tip touches the base.

- When the tip is at the base, the projection matrix is given as in (13). The homogeneous term does now not contribute to overcoming the closed chain configuration. Yet, since the rank is still preserved, the pseudoinverse $\mathbf{J}^+$ is available, which can be derived from (11) as

$$\mathbf{J}^+ = \begin{pmatrix} 0 & 0 \\ +j_{23} & -j_{13} \\ -j_{22} & +j_{12} \end{pmatrix}. \qquad (14)$$

Since, in this matrix, the second and the third row vectors are linearly independent, we may have a differential tip displacement in *any* direction we like in the workspace.

- Finally, once the tip has moved away from the base—no matter how small a distance—the homogeneous term immediately begins to restore its effectiveness.

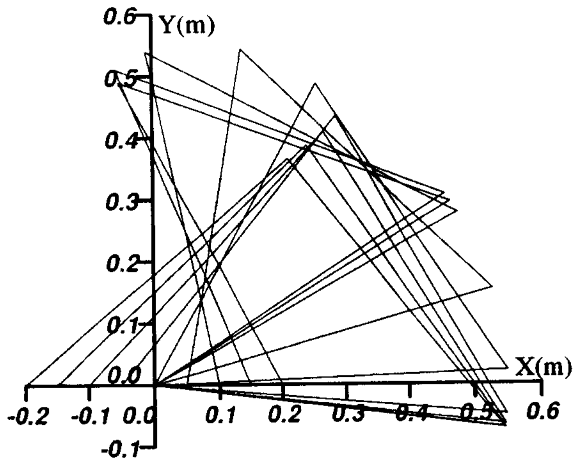
To sum up, at the base, where the particular solution term is still well defined, this term drives the tip away, while the homogeneous term is momentarily ineffective; in the remaining region of the workspace, both terms are effective. Furthermore, the transitions between the two regions are smooth, without discontinuities.

One may suspect that the ability to overcome the singularity at the base in the numerical experiments could have arisen from some small error in tip location—the tip could be some small distance away from the base—due to the linearization characteristics of (10). Similarly, it might be thought that an inexact Jacobian matrix could have made it possible for the tip to get away from the base, which could perhaps be impossible with the exact Jacobian matrix. But these considerations do not apply, because both the Jacobian matrix in (12) and the pseudoinverse in (14) are exact expressions defined at an exact point (the base). Rather, the ability to escape from the singularity comes from an intrinsic feature of the kinematic redundancy.

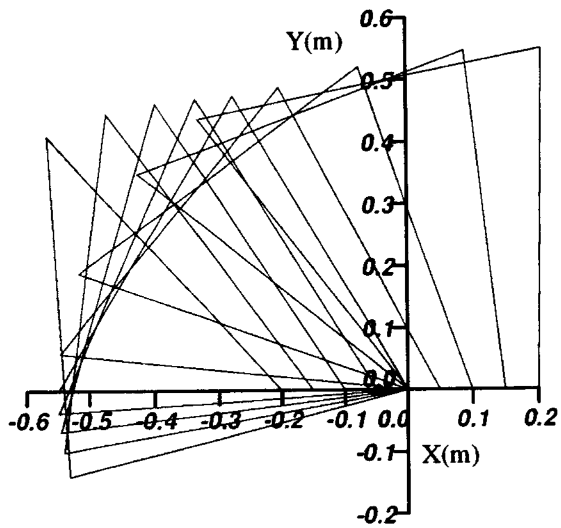
The aforementioned analysis is confirmed by the following experiment: the tip is made to move along the straight line starting from $\mathbf{x} = (0.2, 0)^T$ to $(-0.2, 0)^T$, passing through the base, $(0, 0)^T$ (the measurements are in meters). Together with the desired end effector motion, the three performance measures are given to the resolved motion method, which provides the exact equilibrium solution after a sufficiently large number of iterations (Chang, 1986), thus cancelling out the effects of possible inaccuracies in both the tip location and the Jacobian matrix.

From the result shown in Figures 7, we see that with any one of the three measures the manipulator has no difficulty in getting into and out of the special point at the origin. We may conclude that the use of a redundant manipulator seals the hole in the workspace at the origin, where, without the kinematic redundancy, a singularity is unavoidable.

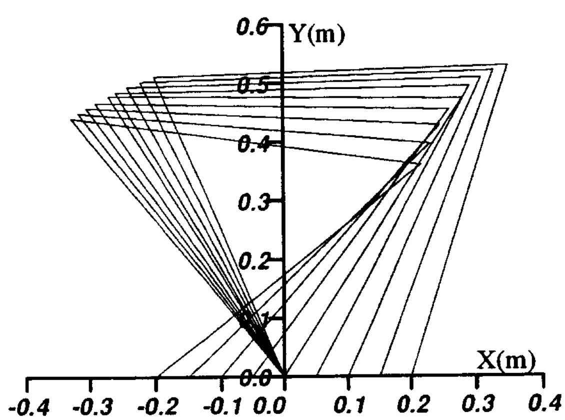
In Figure 7, one may note the smoothness of motion when the new measure is used, as compared to motions with the other two measures: when the other measures are used, the motions approaching the base from $\mathbf{x} = (0.2, 0)^T$ are abrupt in θ_1. The reason for the smoothness when the new measure is used is not entirely clear at this point; but we may conjecture that keeping the minors balanced prevents abrupt changes in the joint variables.



The Condition Number Is Used



The Manipulability is Used



The New Measure Is Used

Figure 7: Use of the three performance measures, when the tip is passing the base from x = 0.2 m to x = -0.2 m along the x-axis. Here the link lengths are equal, $l_1 = l_2 = l_3 = 0.55$ m.

5.2 Preserving the Aspect and its Effect on Repeatability.

In the following experiment, we examine whether the manipulator, with the new performance measure, can preserve the aspect and compare the behaviour to that in the case of the other measures. The manipulator to be used for this purpose is a three degree of freedom planar manipulator with revolute joints, with $l_1 = 0.6$ m, $l_2 = 0.85$ m, and $l_3 = 0.2$ m.

In the experiment, the end effector is made to reciprocate radially between the base and the outer limit of the workspace, where the manipulator is fully extended. The radial motion itself is not of primary concern; it is chosen because it is a way of scanning the workspace to examine the ability to preserve the type of configuration or aspect. Because of rotational symmetry, a series of configurations corresponding to the tip reciprocating in one radial direction represent configurations in all of the other directions, thus covering the whole workspace. The rotational symmetry comes, of course, from the fact that the new performance measure, as well as the other measures, depends on θ_2 and θ_3 only, and is independent of θ_1—hence, one optimal configuration for a fixed tip location is symmetrical to any other location that is the same distance from the base.

Of the two kinematic control methods, the fixed inverse kinematic method is the more convenient one for obtaining the equilibrium states. Therefore it is used here with the three measures. When applying the method to solve for successive joint configurations as the tip moves, the present joint values are used as the initial guesses for the next tip location. The very first joint variable values, corresponding to the starting point of the tip, by the way, are determined by obtaining the global minimum of the performance measure. To do this, we first determine all of the local minima, by starting from points in a dense sampling of the joint variable space, and then solving the non-linear equations (7). In parallel with this, all the local maxima at each tip location are obtained with the proposed method, in order to determine whether successive generations of joint values are indeed correct. In addition to the joint configurations, corresponding minor values are obtained to examine the correlation between joint configurations and minor values.

In Figures 8, 10, 12, the optimal joint configurations based on each of the performance measures and the value of each measure are plotted. Corresponding minor values are plotted in Figures 9, 11, 13.

As shown in the figures, each performance measure has two distinct types of configurations, either one of which, depending on the tip location, can give the global optimum. Hence, each of the two types of configurations has its own corresponding performance measure curve: the solid curve corresponds mostly to the "scaffold"-shaped configuration (configuration A); and the broken curve corresponds mostly to "N"-shaped configurations (configuration B).

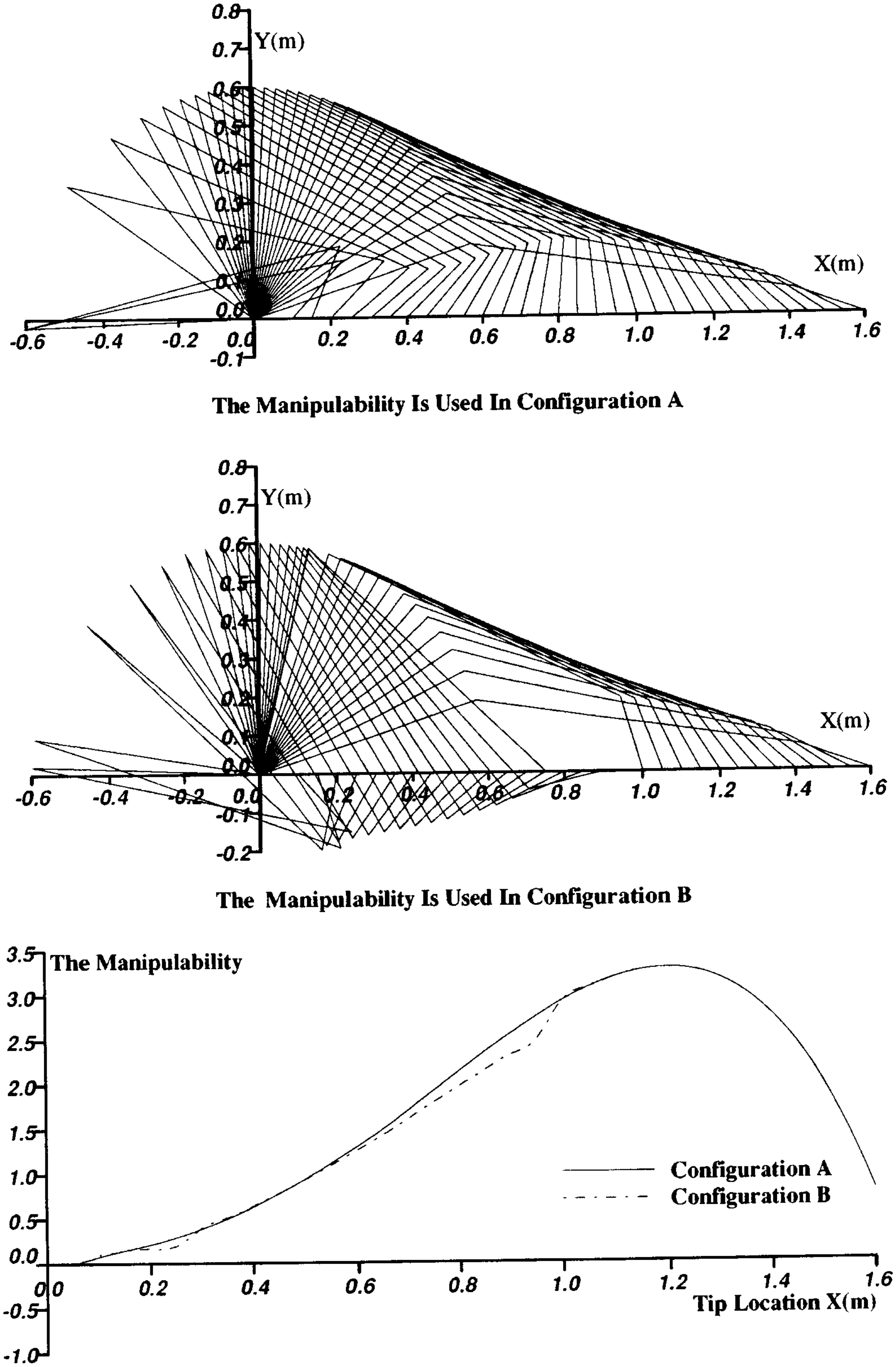
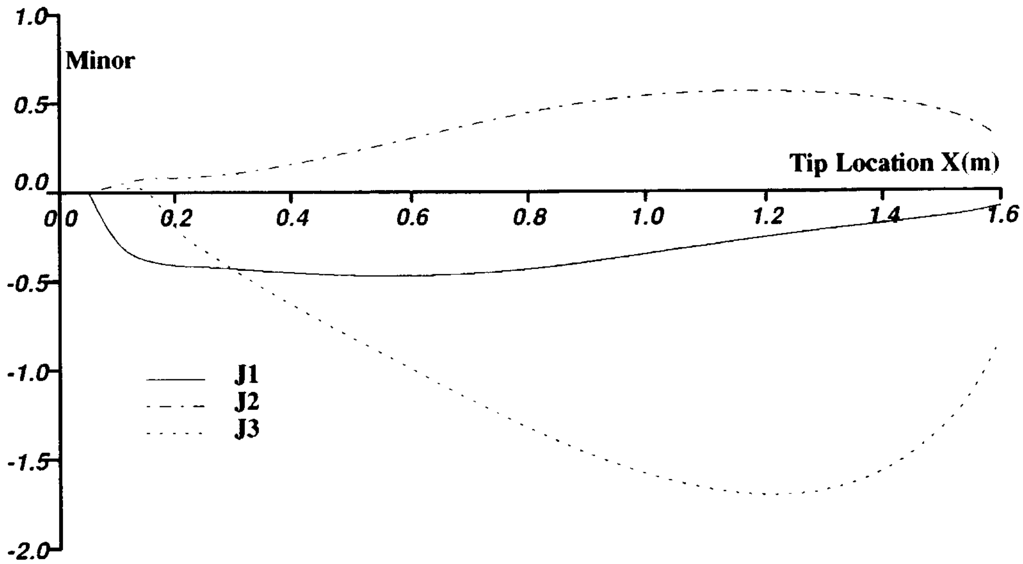
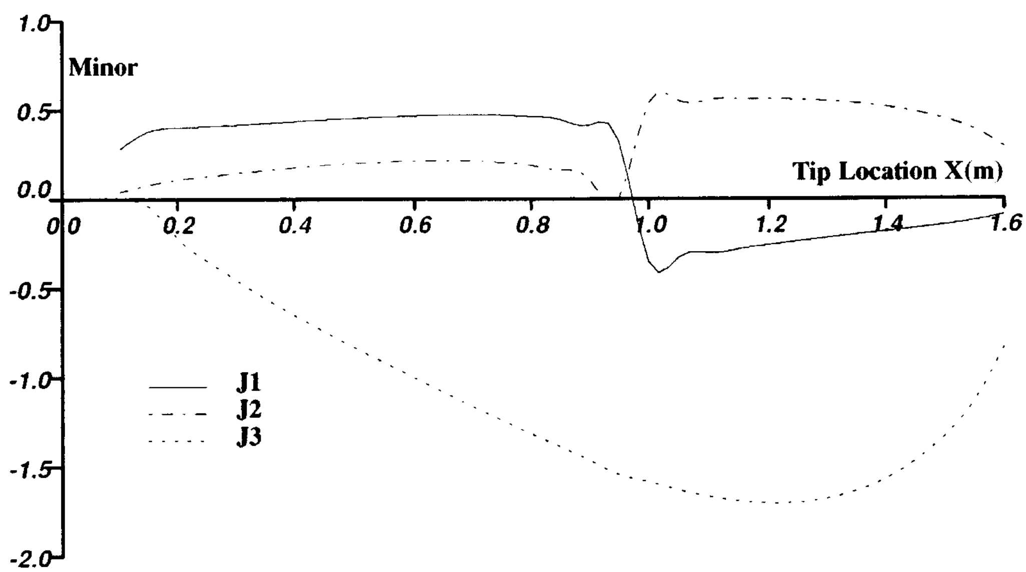


Figure 8: Configurations A and B, when the manipulability is used as the performance measure, and corresponding values of the performance measure, where A and B refer to the tip-motion with different initial configurations.



Minors For Configuration A



Minors For Configuration B

Figure 9: The minor values in configurations A and B, when the manipulability is used as the performance measure, where A and B refer to the tip-motion with different initial configuration.

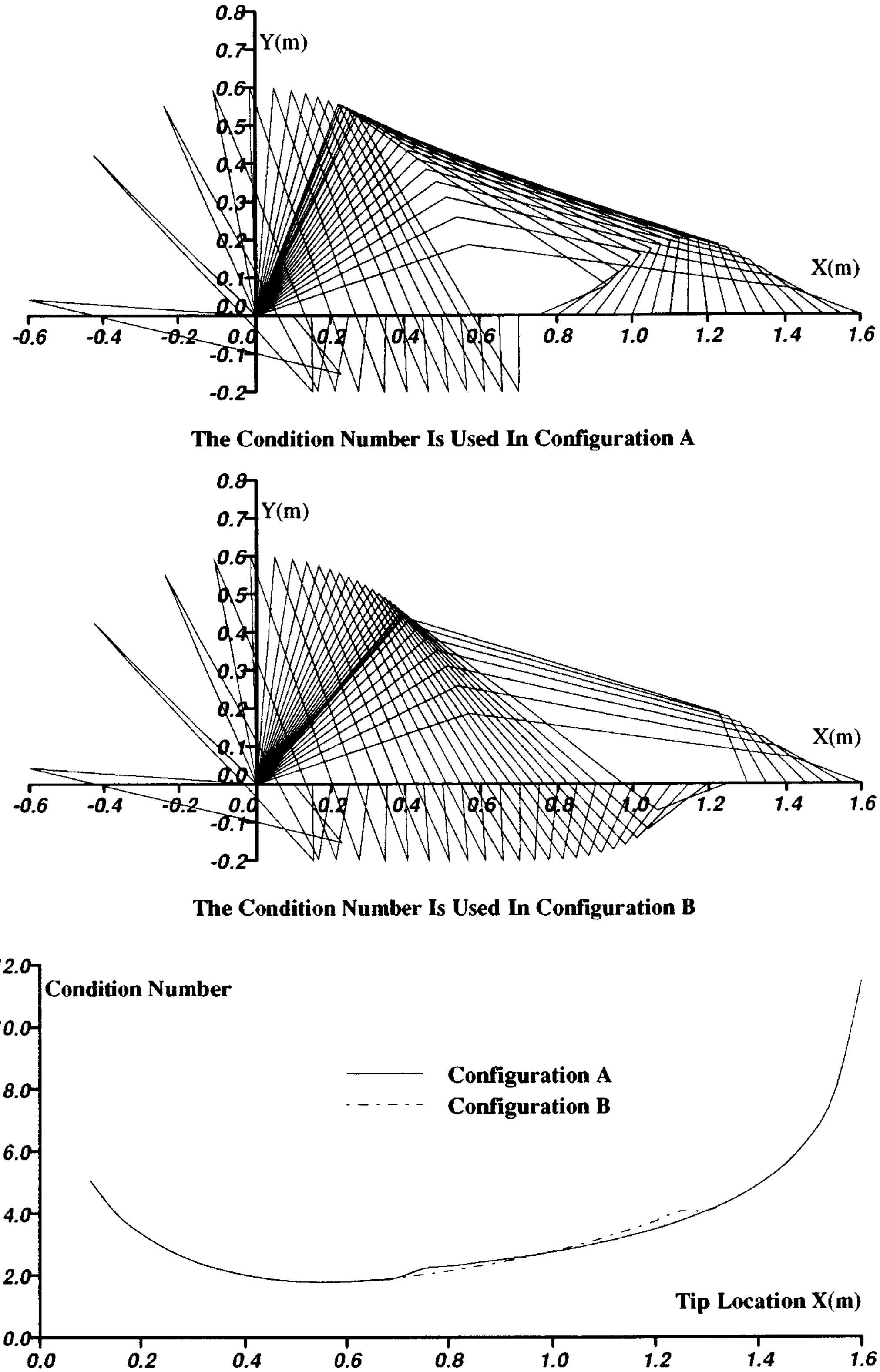
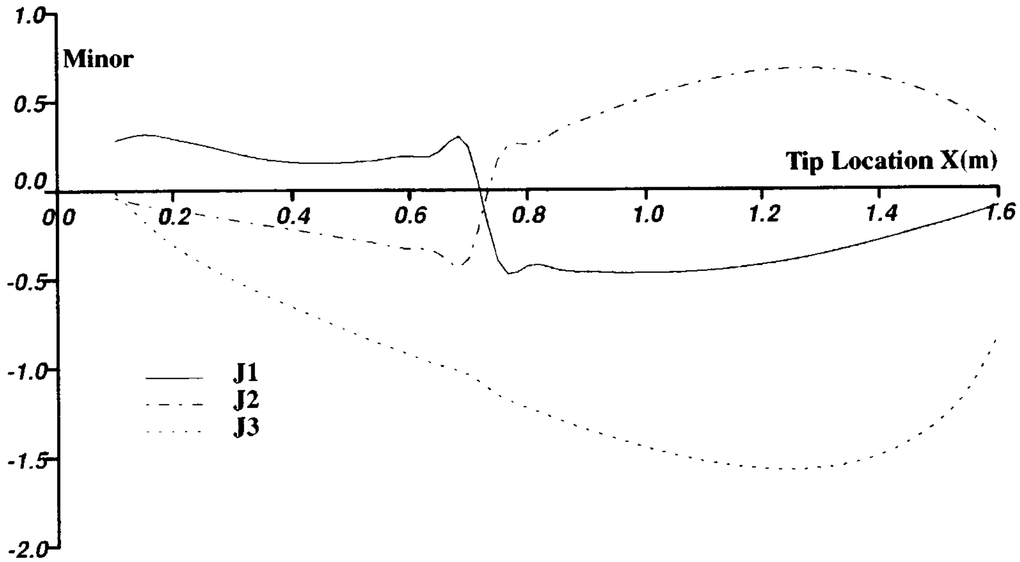
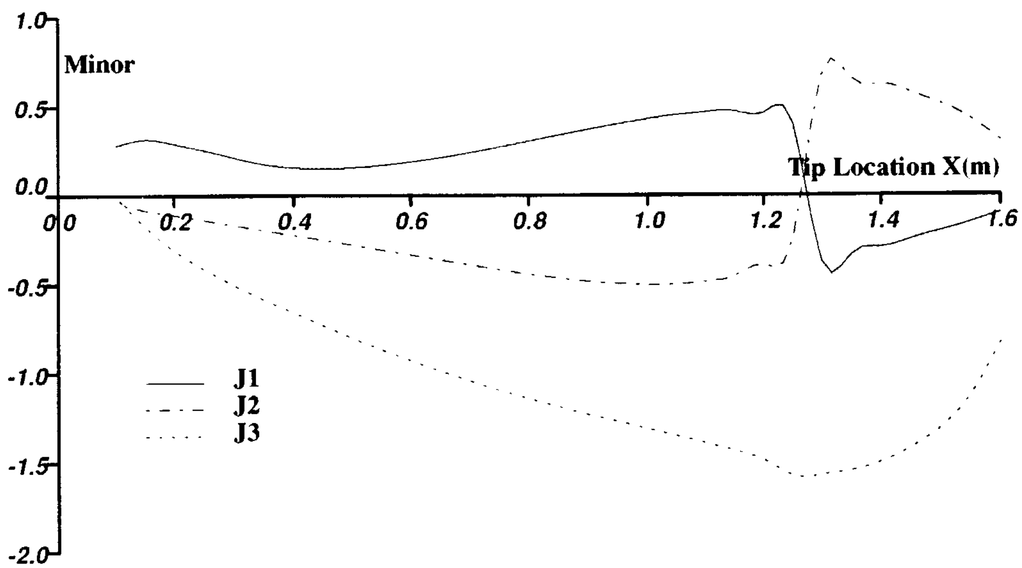


Figure 10: Configurations A and B, when the condition number is used as the performance measure, and corresponding values of the performance measure, where A refers to the tip-motion from $x = 1.6$ m to 0.1 m and B refers to the tip-motion from $x = 0.1$ m to 1.6 m.



Minors For Configuration A



Minors For Configuration B

Figure 11: The minor values in configurations A and B, when the condition number is used as the performance measure, where A and B refer to the tip-motion with different initial tip location.

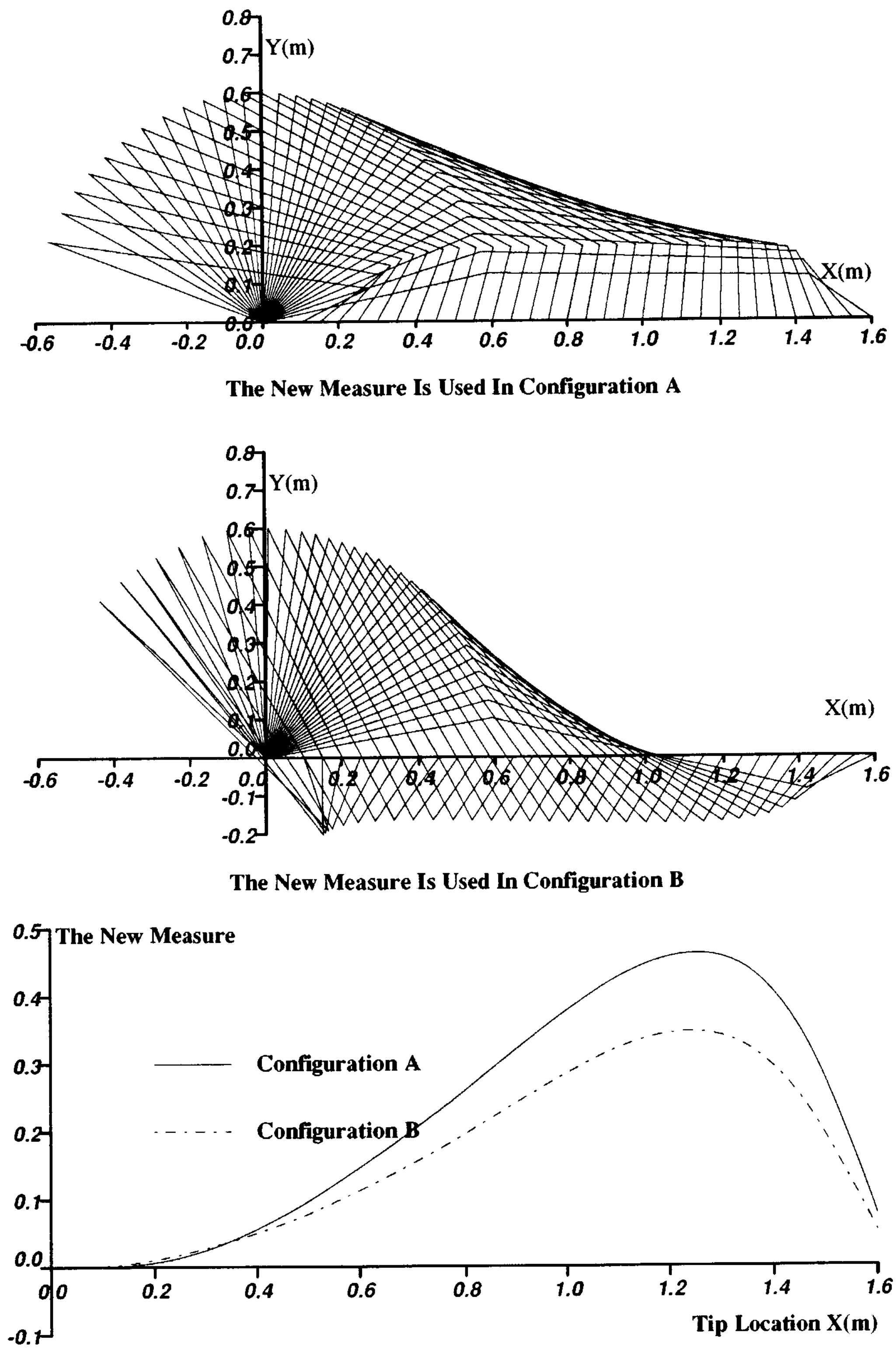
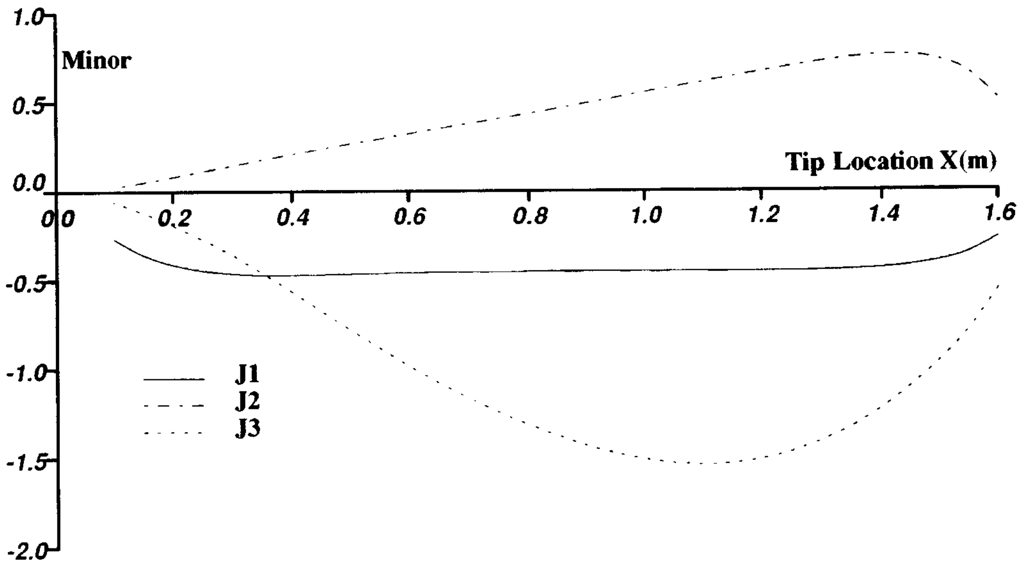
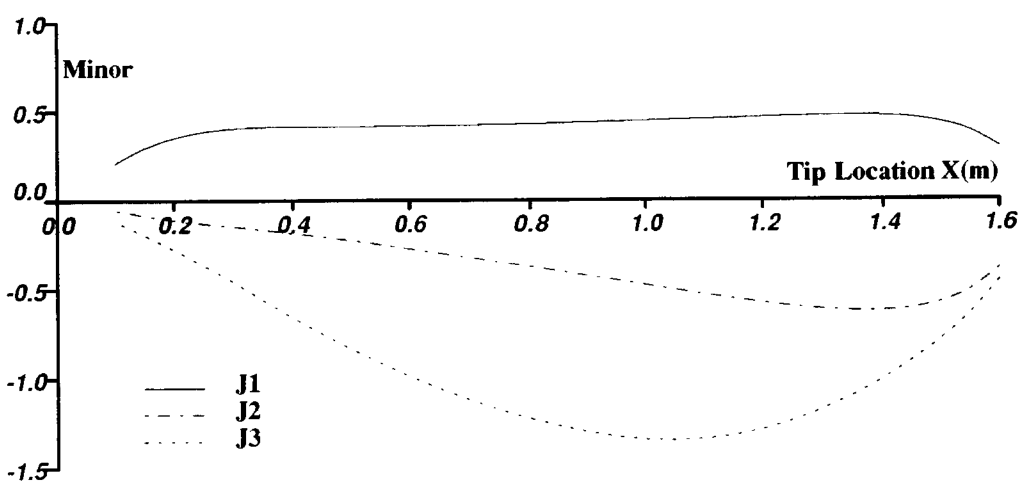


Figure 12: Configurations A and B, when the new measure is used as the performance measure, and corresponding values of the measure, where A and B refer to the tip-motions with different initial configurations.



Minors For Configuration A



Minors For Configuration B

Figure 13: The minor values in configurations A and B, when the new measure is used as the performance measure, where A and B refer to the tip-motion with different initial configuration.

Note also that the mirror-image sets of configurations A and B with respect to the x-axis are not included in the figures, since they have the same values of a given performance measure. Of course, there are still additional sets of configurations—corresponding to local maxima instead of global maxima—that are not shown in the figures either. What do we learn from the resulting sets of configurations? Let us examine the configurations generated by each performance measure in turn.

5.2.1 The Manipulability Measure.

Figure 8 shows the two types of configurations and the corresponding values of the performance measure using the manipulability measure. As shown in the figure, depending on the tip location, either one of the two types of configurations may yield the larger performance measure values. In the region between $x = 1.1$ m and $x = 1.6$ m, however, the two configurations become identical, having the same values for the performance measure. The question then is, what happens to the two types of configurations, when the tip is coming out of this region? To answer this question, we need more careful observations as follows:

In configuration A, the initial type of configuration is preserved in almost the entire workspace, excluding only the region between the base and x=0.1 m. That is, except for this region, the type of configuration is independent of the tip location and the direction of the tip motion—toward or away from the base.

In configuration B, on the other hand, where the tip starts near the base, the initial type of configuration is preserved only if the tip is located within a certain distance from the base (about 1 m). Outside of this range, the configuration *shifts* to, or *merges* with, configuration A. And once merged, configurations corresponding to subsequent tip motions stay within configuration A, never returning to configuration B.

Here, we observe that the manipulator has switched the type of joint configuration or the aspect. Moreover, the type of joint configuration, once switched from one aspect to another, does not return to the initial type of configuration: this is the source of the repeatability problem. An important question then is, what happens to the minors when this switching of aspect occurred? Do they change their sign, passing through zero values? Figure 9 clearly shows that is the case: the tip location where two configurations merge is also the place where one of the minors changes its sign.

5.2.2 The Condition Number.

In the case of the condition number, the situation is even more complicated. In this case, configuration A consists of successive joint configurations, where the tip starts from close to the outer limit of the workspace and moves toward the base, whereas configuration B represents the movements in the opposite direction from the base. In distinction to what happens in the case where the manipulability is used as the measure, here *both* configurations A and B, merge into the same type of configuration. Furthermore, the locations where merging occurs are different: about $x = 0.7$ m for configuration A; and $x = 1.3$ m for configuration B. And when $x < 0.7$ m or $x > 1.3$ m, the two types of configurations are identical, giving the same value for the performance measure.

Hence, the repeatability problem occurs, between $x = 0.7$ m and $x = 1.3$ m, when the tip reverses its direction after experiencing a merging of types of configurations. Again, the curve of the minor values in Figure 11 shows that, when the switchings occur, the signs of the minors change.

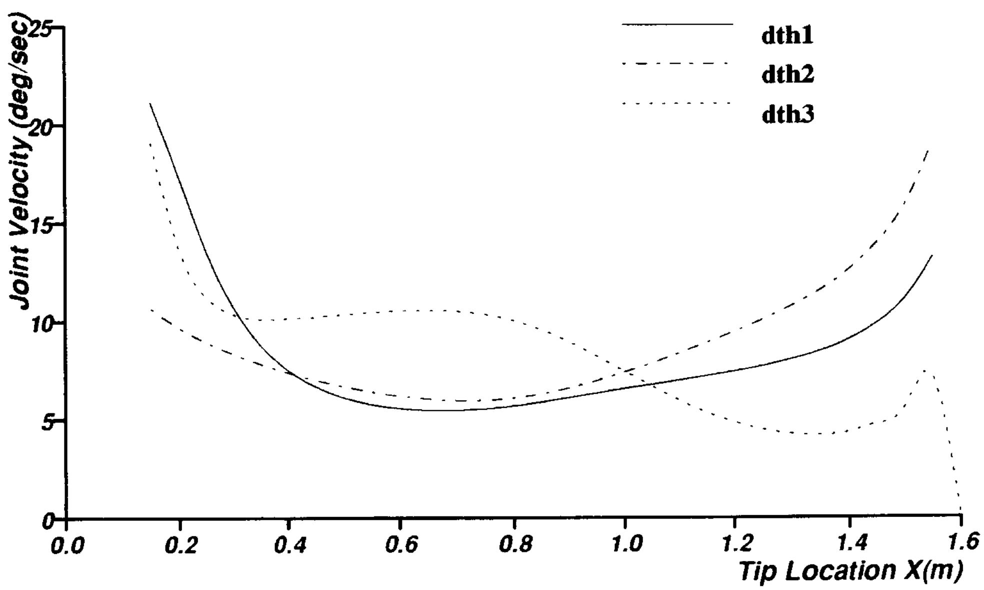
5.2.3 The New Measure.

When the new performance measure is used, there still exist two distinct types of configurations. One thing particularly noticeable, however, is that there is no switching for either type of configuration. Consequently, there is no repeatability problem at all. Since there is no merging of different types of configurations, the initial types of configurations are preserved, showing also distinct curves for the performance measures. As expected, the curve of the value of the minors in Figure 13 clearly shows that there is no sign change at all for the three minors. We conclude from this an obvious consequence of using a measure that has direct control over each minor value. In addition, perusal of the results shows that the new measure leads to smoother motions near the base.

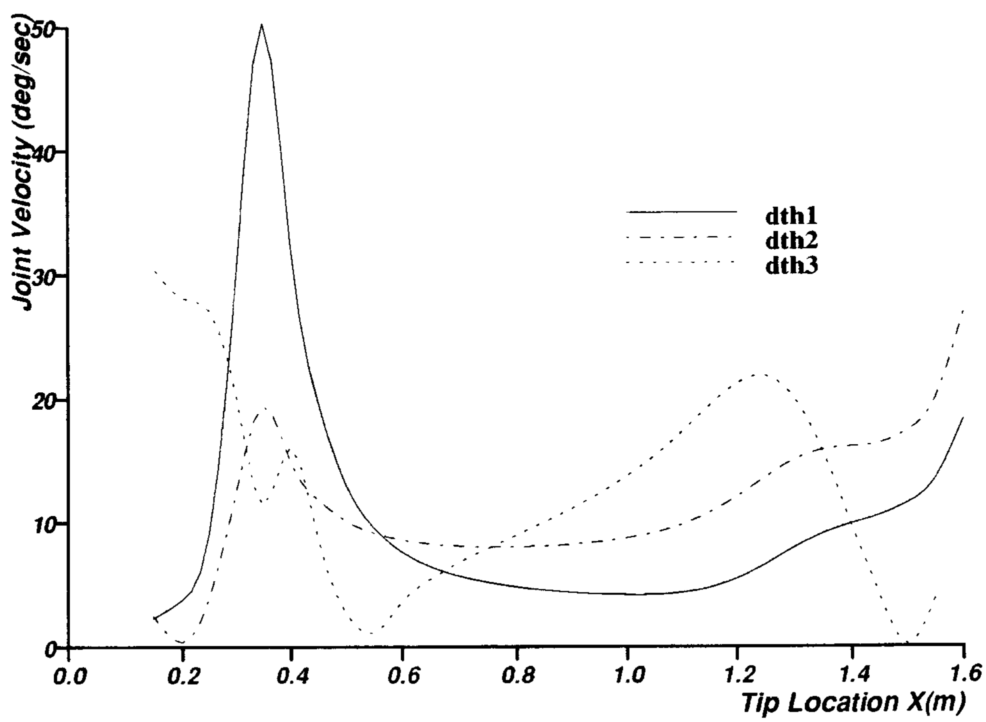
5.3 Discontinuity effects.

As mentioned in Section 4, when merging of different types of configurations or switching of aspects occurs, discontinuous joint motion can be expected. To test this prediction, we computed joint velocities for both types of configurations for each of the performance measures. In this experiment the tip was made to move with a constant velocity of 0.1 (m/sec). Here, to resolve the velocity, we used the resolved motion method.

Figures 14, 15, 16 show the resulting velocity curve corresponding to the configurations obtained in the previous subsection. As expected, simulations in which there is no switching (both in the case of simulations using the new measure and simulations involving configuration A of the manipulability measure), show quite smooth joint velocity curves, as plotted in Figures 14 and 16. Simulations where there is switching, on the other hand, show rugged velocity curves. (The solution method employed here tended to smooth out the transitions in the graphs shown).

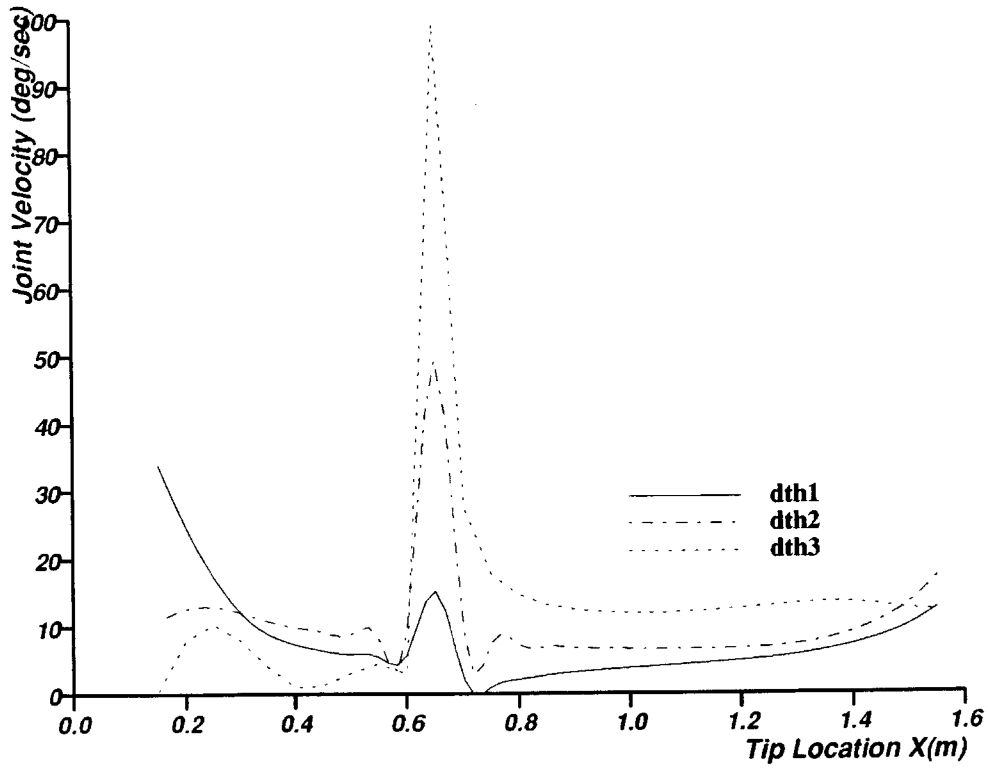


Joint Velocities With The Manipulability: Configuration A

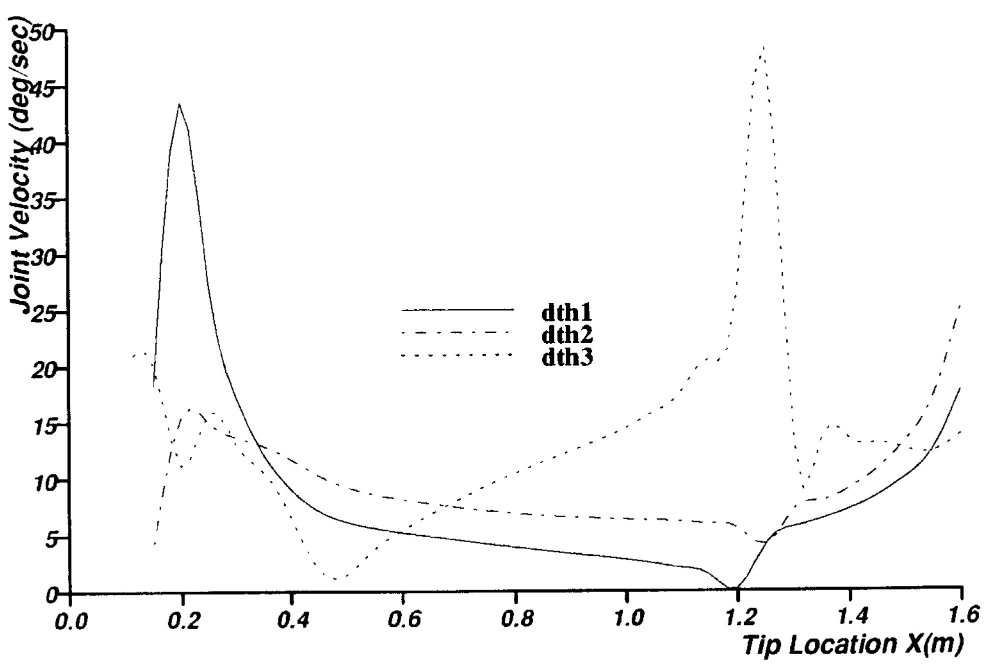


Joint Velocities With The Manipulability: Configuration B

Figure 14: The joint velocities in configurations A and B, when the manipulability is used as the performance measure, where A and B refer to the tip-motion with different initial configurations.

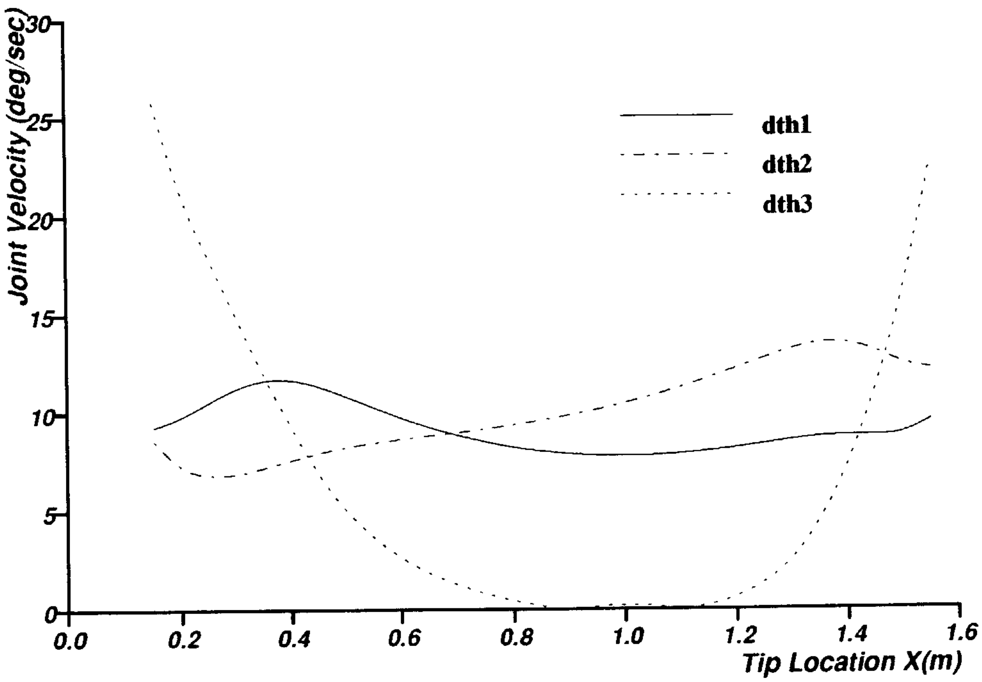


Joint Velocities With The Condition Number: Configuration A

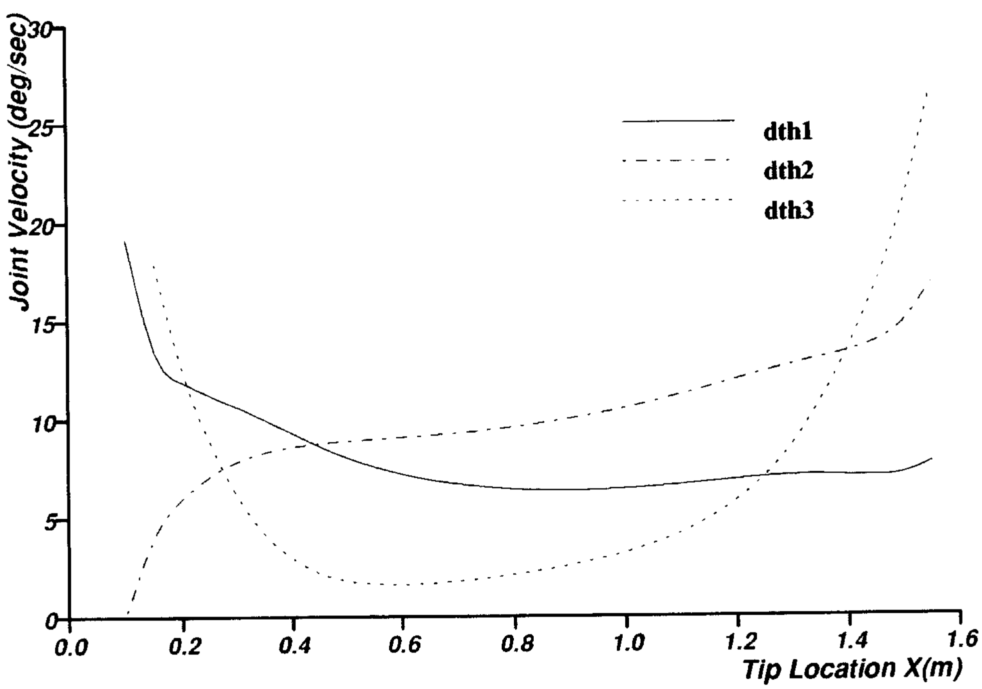


Joint Velocities With The Condition Number: Configuration B

Figure 15: The joint velocities in configurations A and B, when the condition number is used as the performance measure, where A and B refer to the tip-motion with different initial configurations.



Joint Velocities With The New Measure: Configuration A



Joint Velocities With The New Measure: Configuration B

Figure 16: The joint velocities in configurations A and B, when the new measure is used as the performance measure, where A and B refer to the tip-motion with different initial configurations.

For the remaining cases, where switching does occur, the velocity trajectory is generally rugged, confirming our prediction. Yet, the degree of ruggedness of the curves corresponding to the two different performance measures is different. Careful observation reveals the following: for the manipulability measure, the merging happens through some intermediate configurations, reducing the degree of discontinuity; whereas, for the condition number measure, the switching happens instantaneously with few intermediate states, resulting in much larger values of joint velocity.

It is not clear at this point why there is this difference. We will probably be able to get some clue, when the gradients of the two measure are first expressed in symbolic form and then each term is examined to pinpoint the cause of the abrupt switching. Determining the symbolic expressions needed for this inspection appears to be possible, although quite complicated, even for this simple manipulator.

5.4 Conclusions from Experimental Results.

Summing up, we have compared the new measure of distance from singularity with two existing measures, both qualitatively and quantitatively. Analysis shows that the qualitative relationship agrees well with the experimental results.

To summarize these results, all the measures showed the ability to overcome singularity by successfully treating two cases: (a) a location corresponding to a configuration of the kinematic chain where the links are almost in a straight line and (b) the situation where the tip is at the base or origin. The essential difference between the new measure and the other two is its ability to explicitly prevent the minors from becoming zero. This ability, in effect, prevents the merging of different types of configurations or switching of aspects, which in turn removes the repeatability problems and discontinuous motions. In addition, balancing the values of the minors appears to contribute to noticeably smoother movements near the base.

6 Conclusion.

In this paper, we have defined the concept of dexterity as distance from singularity. Then we reviewed the concepts of singularity and redundancy to enable further investigation of the distance concept. We have illustrated that there are different degrees of distance from singularity within a particular degree of redundancy, showing that the conventional concept of redundancy is not sufficient to describe this distance. The new distance concept we derived was the number of nonzero minors along with the values of the minors. On the basis of the new distance concept, a new performance measure was derived, namely the product of the minors of the Jacobian matrix. Then we have related the new performance measure to the manipulability measure and the condition number. Having investigated the qualitative relationship, we pointed out that the other measures do not have the ability to explicitly prevent minors from becoming zero. Through another series of numerical experiments, the advantages of the ability to prevent minors from becoming zero was clearly confirmed. Whereas the two other measures showed repeatability problems and discontinuous motions, the new measure consistently did not.

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Appendix 1: Proof of Theorem 1

In this appendix, we provide a proof of Theorem 1. If the Jacobian matrix is expressed in the form:

$$\mathbf{J} = \begin{pmatrix} j_{11} & \cdots & j_{1n} \\ & \cdots & \\ j_{m1} & \cdots & j_{mn} \end{pmatrix},$$

then

$$\mathbf{J}\mathbf{J}^T = \begin{pmatrix} \sum_{k=1}^{n} {j_{1k}}^2 & \cdots & \sum_{k=1}^{n} j_{1k} j_{mk} \\ & \cdots & \\ \sum_{k=1}^{n} j_{mk} j_{1k} & \cdots & \sum_{k=1}^{n} {j_{mk}}^2 \end{pmatrix}.$$

In general, the determinant of an $m \times m$ matrix $\mathbf{A}$ can be written explicitly as:

$$\det(\mathbf{A}) = \sum_{\boldsymbol{\sigma}} (a_{1\alpha} a_{2\beta} \cdots a_{m\nu}) \det(P_{\boldsymbol{\sigma}}), \tag{1}$$

where a_{ij} is the element of $\mathbf{A}$ in i-th row and the j-th column, while $\boldsymbol{\sigma} = (\alpha, \beta, \cdots, \nu)$ is a permutation of the integers from 1 to m, and $P_{\boldsymbol{\sigma}}$ is the corresponding permutation matrix (Thus $\det(P_{\boldsymbol{\sigma}}) = +1$ when $\boldsymbol{\sigma}$ is an even permutation, and $\det(P_{\boldsymbol{\sigma}}) = -1$ when $\boldsymbol{\sigma}$ is an odd permutation). The sum is to be taken over all $m!$ permutations of $\boldsymbol{\sigma}$. Hence, the determinant of $\mathbf{J}\mathbf{J}^T$ is

$$\det(\mathbf{J}\mathbf{J}^T) = \sum_{\boldsymbol{\sigma}} (\sum_{k=1}^{n} j_{1k} j_{\alpha k})(\sum_{k=1}^{n} j_{2k} j_{\beta k}) \cdots (\sum_{k=1}^{n} j_{mk} j_{\nu k}) \det(P_{\boldsymbol{\sigma}}).$$

Expanding this, we have

$$\det(\mathbf{J}\mathbf{J}^T) = \sum_{\boldsymbol{\sigma}} \left(\sum_{k_1, \cdots, k_m = 1}^{n} j_{1k_1} j_{\alpha k_1} j_{2k_2} j_{\beta k_2} \cdots j_{mk_m} j_{\nu k_m} \right) \det(P_{\boldsymbol{\sigma}}). \tag{2}$$

Here we make use of the fact that the determinant of the permutation matrix is zero when $\boldsymbol{\sigma}$ is not a permutation (that is, when the integers are not distinct). As a result, terms that have non-distinct k_i's are multiplied by zero. For instance, if $k_2 = k_1 = 1$, then the term in $j_{11} j_{\alpha 1} j_{21} j_{\beta 1} \cdots j_{mm} j_{\nu m}$ disappears when $\alpha = 1$ and $\beta = 2$. Thus, in Equation 2, summation is effectively applied only to the terms with distinct k_i's. Note also that the number p of different sets of distinct k_i's is

$$p = {}_nC_m.$$

Changing the order of summation in Equation 2, we have

$$\det(\mathbf{J}\mathbf{J}^T) = \sum_{k_1, \cdots, k_m = 1}^{n} \left(\sum_{\boldsymbol{\sigma}} j_{1k_1} j_{2k_2} \cdots j_{mk_k} \det(P_{\boldsymbol{\sigma}}) \right) . (j_{\alpha k_\alpha} j_{\beta k_\beta} \cdots j_{\nu k_\nu})$$

Here the summation over $k_1, \cdots, k_m$ may be rearranged into summation over all permutations $\boldsymbol{\sigma}_i$, as follows:

$$\sum_{k_1,\cdots,k_m=1}^{n} (.) = \sum_{i=1}^{p} \sum_{\boldsymbol{\sigma}_i} (.),$$

where $\boldsymbol{\sigma}_i = (k_1, k_2, \cdots, k_m)$ is the i-th permutation. Therefore Equation 2 becomes

$$\det(\mathbf{J}\mathbf{J}^T) = \sum_{i=1}^{p} \sum_{\boldsymbol{\sigma}_i} \left(\sum_{\boldsymbol{\sigma}} j_{\alpha k_1} j_{\beta k_2} \cdots j_{\nu k_m} \det(P_{\boldsymbol{\sigma}}) \right) j_{1k_1} j_{2k_2} \cdots j_{mk_m}. \tag{3}$$

Let us use the notation

$$\Delta_i = \sum_{\boldsymbol{\sigma}} j_{\alpha k_1} j_{\beta k_2}, \cdots, j_{\nu k_m} \det(P_{\boldsymbol{\sigma}}).$$

Then we note that Equation 1 shows that Δ_i is the determinant of the transpose of the submatrix made of k_i's column vectors as

$$\Delta_i = \det\Big((J^{k_1}\ J^{k_2} \cdots J^{k_m})^T \Big),$$

where J^{k_i} is the k_i-th column vector of the Jacobian matrix. Once a set of k_i's is chosen, the absolute value of Δ_i is fixed; only its sign changes as k_i's make permutations. If we let the absolute value be $|\Delta_i|$, then Equation 3 becomes

$$\det(\mathbf{J}\mathbf{J}^T) = \sum_{i=1}^{p} (\pm) |\Delta_i| \left(\sum_{\boldsymbol{\sigma}_i} j_{1k_1} j_{2k_2} \cdots j_{mk_m} \det(P_{\boldsymbol{\sigma}_i}) \right).$$

The fact that the determinant of a square matrix is equal to that of the transpose of the matrix, and the fact that $\mathbf{J}\mathbf{J}^T$ is positive definite imply that

$$\det(\mathbf{J}\mathbf{J}^T) = \sum_{i=1}^{p} {\Delta_i}^2.$$

Q.E.D.

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Abstract. We have derived a new performance measure, product of minors of the Jacobian matrix, that tells how far kinematically redundant manipulators are from singularity. It was demonstrated that previously used performance measures, namely condition number and manipulability measure allowed to change configurations, causing repeatability problems and discontinuity effects. The new measure, on the other hand, assures that the arm solution remains in the same configuration, thus effectively preventing these problems.

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