Numerical Recipes in Fortran 90
Second Edition

Volume 2 of
Fortran Numerical Recipes
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Preface to Volume 2

Fortran 90 is not just the long-awaited updating of the Fortran language to modern computing practices. It is also the vanguard of a much larger revolution in computing, that of multiprocessor computers and widespread parallel programming. Parallel computing has been a feature of the largest supercomputers for quite some time. Now, however, it is rapidly moving towards the desktop.

As we watched the gestation and birth of Fortran 90 by its governing “X3J3 Committee” (a process interestingly described by a leading committee member, Michael Metcalf, in the Foreword that follows), it became clear to us that the right moment for moving Numerical Recipes from Fortran 77 to Fortran 90 was sooner, rather than later.

Fortran 90 compilers are now widely available. Microsoft’s Fortran PowerStation for Windows 95 brings that firm’s undeniable marketing force to PC desktop; we have tested this compiler thoroughly on our code and found it excellent in compatibility and performance. In the UNIX world, we have similarly tested, and had generally fine experiences with, DEC’s Fortran 90 for Alpha AXP and IBM’s xlf for RS/6000 and similar machines. NAG’s Fortran 90 compiler also brings excellent Fortran 90 compatibility to a variety of UNIX platforms. There are no doubt other excellent compilers, both available and on the way. Fortran 90 is completely backwards compatible with Fortran 77, by the way, so you don’t have to throw away your legacy code, or keep an old compiler around.

There have been previous special versions of Fortran for parallel supercomputers, but always specific to a particular hardware. Fortran 90, by contrast, is designed to provide a general, architecture-independent framework for parallel computation. Equally importantly, it is an international standard, agreed upon by a large group of computer hardware and software manufacturers and international standards bodies.

With the Fortran 90 language as a tool, we want this volume to be your complete guide for learning how to “think parallel.” The language itself is very general in this regard, and applicable to many present and future computers, or even to other parallel computing languages as they come along. Our treatment emphasizes general principles, but we are also not shy about pointing out parallelization “tricks” that have frequent applicability. These are not only discussed in this volume’s principal text chapters (Chapters 21–23), but are also sprinkled throughout the chapters of Fortran 90 code, called out by a special “parallel hint” logo (left, above). Also scattered throughout the code chapters are specific “Fortran 90 tips,” with their own distinct graphic call-out (left). After you read the text chapters, you might want simply to browse among these hints and tips.

A special note to C programmers: Right now, there is no effort at producing a parallel version of C that is comparable to Fortran 90 in maturity, acceptance, and stability. We think, therefore, that C programmers will be well served by using this volume for an educational excursion into Fortran 90, its parallel programming constructions, and the numerical algorithms that capitalize on them. C and C++ programming have not been far from our minds as we have written this volume, and we think that you will find that time spent in absorbing its principal lessons (in Chapters 21–23) will be amply repaid in the future, as C and C++ eventually develop standard parallel extensions.
A final word of truth in packaging: **Don’t buy this volume unless you also buy (or already have) Volume 1** (now retitled *Numerical Recipes in Fortran 77*). Volume 2 does not repeat any of the discussion of what individual programs actually do, or of the mathematical methods they utilize, or how to use them. While our Fortran 90 code is thoroughly commented, and includes a header comment for each routine that describes its input and output quantities, these comments are not supposed to be a complete description of the programs; the complete descriptions are in Volume 1, which we reference frequently. But here’s a money-saving hint to our previous readers: If you already own a Second Edition version whose title is *Numerical Recipes in FORTRAN* (which doesn’t indicate either “Volume 1” or “Volume 2” on its title page) then take a marking pen and write in the words “Volume 1.” There! (Differences between the previous reprintings and the newest reprinting, the one labeled “Volume 1,” are minor.)

**Acknowledgments**

We continue to be in the debt of many colleagues who give us the benefit of their numerical and computational experience. Many, though not all, of these are listed by name in the preface to the second edition, in Volume 1. To that list we must now certainly add George Marsaglia, whose ideas have greatly influenced our new discussion of random numbers in this volume (Chapter B7).

With this volume, we must acknowledge our additional gratitude and debt to a number of people who generously provided advice, expertise, and time (a great deal of time, in some cases) in the areas of parallel programming and Fortran 90. The original inspiration for this volume came from Mike Metcalf, whose clear lectures on Fortran 90 (in this case, overlooking the beautiful Adriatic at Trieste) convinced us that Fortran 90 could serve as the vehicle for a book with the larger scope of parallel programming generally, and whose continuing advice throughout the project has been indispensable. Gyan Bhanot also played a vital early role in the development of this book; his first translations of our Fortran 77 programs taught us a lot about parallel programming. We are also grateful to Greg Lindhorst, Charles Van Loan, Amos Yahil, Keith Kimball, Malcolm Cohen, Barry Caplin, Loren Meissner, Mitsu Sakamoto, and George Schnurer for helpful correspondence and/or discussion of Fortran 90’s subtler aspects.

We once again express in the strongest terms our gratitude to programming consultant Seth Finkelstein, whose contribution to both the coding and the thorough testing of all the routines in this book (against multiple compilers and in sometimes-buggy, and always challenging, early versions) cannot be overstated.

WHP and SAT acknowledge the continued support of the U.S. National Science Foundation for their research on computational methods.

*February 1996*

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Foreword

by Michael Metcalf

Sipping coffee on a sunbaked terrace can be surprisingly productive. One of the Numerical Recipes authors and I were each lecturing at the International Center for Theoretical Physics in Trieste, Italy, he on numerical analysis and I on Fortran 90. The numerical analysis community had made important contributions to the development of the new Fortran standard, and so, unsurprisingly, it became quickly apparent that the algorithms for which Numerical Recipes had become renowned could, to great advantage, be recast in a new mold. These algorithms had, hitherto, been expressed in serial form, first in Fortran 77 and then in C, Pascal, and Basic. Now, nested iterations could be replaced by array operations and assignments, and the other features of a rich array language could be exploited. Thus was the idea of a “Numerical Recipes in Fortran 90” first conceived and, after three years’ gestation, it is a delight to assist at the birth.

But what is Fortran 90? How did it begin, what shaped it, and how, after nearly foundering, did its driving forces finally steer it to a successful conclusion?

The Birth of a Standard

Back in 1966, the version of Fortran now known as Fortran 66 was the first language ever to be standardized, by the predecessor of the present American National Standards Institute (ANSI). It was an all-American affair. Fortran had first been developed by John Backus of IBM in New York, and it was the dominant scientific programming language in North America. Many Europeans preferred Algol (in which Backus had also had a hand). Eventually, however, the mathematicians who favored Algol for its precisely expressible syntax began to defer to the scientists and engineers who appreciated Fortran’s pragmatic, even natural, style. In 1978, the upgraded Fortran 77 was standardized by the ANSI technical committee, X3J3, and subsequently endorsed by other national bodies and by ISO in Geneva, Switzerland. Its dominance in all fields of scientific and numerical computing grew as new, highly optimizing compilers came onto the market. Although newer languages, particularly Pascal, Basic, PL/1, and later Ada attracted their own adherents, scientific users throughout the 1980s remained true to Fortran. Only towards the end of that decade did C draw increasing support from scientific programmers who had discovered the power of structures and pointers.

During all this time, X3J3 kept functioning, developing the successor version to Fortran 77. It was to be a decade of strife and contention. The early plans, in the late 1970s, were mainly to add to Fortran 77 features that had had to be left out of that standard. Among these were dynamic storage and an array language, enabling it to map directly onto the architecture of supercomputers, then coming onto the market. The intention was to have this new version ready within five years, in 1982. But two new factors became significant at that time. The first was the decision that the next standard should not just codify existing practice, as had largely been the case in 1966 and 1978, but also extend the functionality of the language through
innovative additions (even though, for the array language, there was significant borrowing from John Iverson’s APL and from DAP Fortran). The second factor was that X3J3 no longer operated under only American auspices. In the course of the 1980s, the standardization of programming languages came increasingly under the authority of the international body, ISO. Initially this was in an advisory role, but now ISO is the body that, through its technical committee WG5 (in full, ISO/IEC JTC1/SC22/WG5), is responsible for determining the course of the language. WG5 also steers the work of the development body, then as now, the highly skilled and competent X3J3. As we shall see, this shift in authority was crucial at the most difficult moment of Fortran 90’s development.

The internationalization of the standards effort was reflected in the welcome given by X3J3 to six or seven European members; they, and about one-third of X3J3’s U.S. members, provided the overlapping core of membership of X3J3 and WG5 that was vital in the final years in bringing the work to a successful conclusion. X3J3 membership, which peaked at about 45, is restricted to one voting member per organization, and significant decisions require a majority of two-thirds of those voting. Nationality plays no role, except in determining the U.S. position on an international issue. Members, who are drawn mainly from the vendors, large research laboratories, and academia, must be present or represented at two-thirds of all meetings in order to retain voting rights.

In 1980, X3J3 reported on its plans to the forerunner of WG5 in Amsterdam, Holland. Fortran 8x, as it was dubbed, was to have a basic array language, new looping constructs, a bit data type, data structures, a free source form, a mechanism to “group” procedures, and another to manage the global name space. Old features, including COMMON, EQUIVALENCE, and the arithmetic-IF, were to be consigned to a so-called obsolete module, destined to disappear in a subsequent revision. This was part of the “core plus modules” architecture, for adding new features and retiring old ones, an aid to backwards compatibility. Even though Fortran 77 compilers were barely available, the work seemed well advanced and the mood was optimistic. Publication was intended to take place in 1985. It was not to be.

One problem was the sheer number of new features that were proposed as additions to the language, most of them worthwhile in themselves but with the totality being too large. This became a recurrent theme throughout the development of the standard. One example was the suggestion of Lawrie Schonfelder (Liverpool University), at a WG5 meeting in Vienna, Austria, in 1982, that certain features already proposed as additions could be combined to provide a full-blown derived data type facility, thus providing Fortran with abstract data types. This idea was taken up by X3J3 and has since come to be recognized, along with the array language, as one of the two main advances brought about by what became Fortran 90. However, the ramifications go very deep: all the technical details of how to handle arrays of objects of derived types that in turn have array components that have the pointer attribute, and so forth, have to be precisely defined and rigorously specified.

**Conflict**

The meetings of X3J3 were often full of drama. Most compiler vendors were represented as a matter of course but, for many, their main objective appeared to be to maintain the status quo and to ensure that Fortran 90 never saw the light of
day. One vendor’s extended (and much-copied) version of Fortran 77 had virtually become an industry standard, and it saw as its mission the maintenance of this lead. A new standard would cost it its perceived precious advantage. Other large vendors had similar points of view, although those marketing supercomputers were clearly keen on the array language. Most users, on the other hand, were hardly prepared to invest large amounts of their employers’ and their own resources in simply settling for a trivial set of improvements to the existing standard. However, as long as X3J3 worked under a simple-majority voting rule, at least some apparent progress could be made, although the underlying differences often surfaced. These were even sometimes between users — those who wanted Fortran to become a truly modern language and those wanting to maintain indefinite backwards compatibility for their billions of lines of existing code.

At a watershed meeting, in Scranton, Pennsylvania, in 1986, held in an atmosphere that sometimes verged on despair, a fragile compromise was reached as a basis for further work. One breakthrough was to weaken the procedures for removing outdated features from the language, particularly by removing no features whatsoever from the next standard and by striking storage association (i.e., COMMON and EQUIVALENCE) from the list of features to be designated as obsolescent (as they are now known). A series of votes definitively removed from the language all plans to add: arrays of arrays, exception handling, nesting of internal procedures, the FORALL statement (now in Fortran 95), and a means to access skew array sections. There were other features on this list that, although removed, were reinstated at later meetings: user-defined operators, operator overloading, array and structure constructors, and vector-valued subscripts. After many more travails, the committee voted, a year later, by 26 votes to 9, to forward the document for what was to become the first of three periods of public comment.

While the document was going through the formal standards bureaucracy and being placed before the public, X3J3 polished it further. X3J3 also prepared procedures for processing the comments it anticipated receiving from the public, and to each of which, under the rules, it would have to reply individually. It was just as well. Roughly 400 replies flooded in, many of them very detailed and, disappointingly for those of us wanting a new standard quickly, unquestionably negative towards our work. For many it was too radical, but many others pleaded for yet more modern features, such as pointers.

Now the committee was deadlocked. Given that a document had already been published, any further change required not a simple but a two-thirds majority. The conservatives and the radicals could each block a move to modify the draft standard, or to accept a revised one for public review — and just that happened, in Champagne-Urbana, Illinois, in 1988. Any change, be it on the one hand to modify the list of obsolescent features, to add the pointers or bit data type wanted by the public, to add multi-byte characters to support Kanji and other non-European languages or, on the other hand, to emasculate the language by removing modules or operator overloading, and hence abstract data types, to name but some suggestions, none of these could be done individually or collectively in a way that would achieve consensus. I wrote:

“In my opinion, no standard can now emerge without either a huge concession by the users to the vendors (MODULE / USE) and/or a major change in the composition of the committee. I do not see how members who have worked for up to a decade
or more, devoting time and intellectual energy far beyond the call of duty, can be expected to make yet more personal sacrifices if no end to the work is in sight, or if that end is nothing but a travesty of what had been designed and intended as a modern scientific programming language. . . . I think the August meeting will be a watershed — if no progress is achieved there will be dramatic resignations, and ISO could even remove the work from ANSI, which is failing conspicuously in its task.”

(However, the same notes began with a quotation from The Taming of the Shrew: “And do as adversaries do in law, / Strive mightily, but eat and drink / as friend.” That we always did, copiously.)

Resolution

The “August meeting” was, unexpectedly, imbued with a spirit of compromise that had been so sadly lacking at the previous one. Nevertheless, after a week of discussing four separate plans to rescue the standard, no agreement was reached. Now the question seriously arose: Was X3J3 incapable of producing a new Fortran standard for the international community, doomed to eternal deadlock, a victim of ANSI procedures?

Breakthrough was achieved at a traumatic meeting of WG5 in Paris, France, a month later. The committee spent several extraordinary days drawing up a detailed list of what it wanted to be in Fortran 8x. Finally, it set X3J3 an ultimatum that was unprecedented in the standards world: The ANSI committee was to produce a new draft document, corresponding to WG5’s wishes, within five months! Failing that, WG5 would assume responsibility and produce the new standard itself.

This decision was backed by the senior U.S. committee, X3, which effectively directed X3J3 to carry out WG5’s wishes. And it did! The following November, it implemented most of the technical changes, adding pointers, bit manipulation intrinsic procedures, and vector-valued subscripts, and removing user-defined elemental functions (now in Fortran 95). The actual list of changes was much longer. X3J3 and WG5, now collaborating closely, often in gruelling six-day meetings, spent the next 18 months and two more periods of (positive) public comment putting the finishing touches to what was now called Fortran 90, and it was finally adopted, after some cliff-hanging votes, for forwarding as a U.S. and international standard on April 11, 1991, in Minneapolis, Minnesota.

Among the remaining issues that were decided along the way were whether pointers should be a data type or be defined in terms of an attribute of a variable, implying strong typing (the latter was chosen), whether the new standard should coexist alongside the old one rather than definitively replace it (it coexisted for a while in the U.S., but was a replacement elsewhere, under ISO rules), and whether, in the new free source form, blanks should be significant (fortunately, they are).

Fortran 90

The main new features of Fortran 90 are, first and foremost, the array language and abstract data types. The first is built on whole array operations and assignments, array sections, intrinsic procedures for arrays, and dynamic storage. It was designed with optimization in mind. The second is built on modules and module procedures, derived data types, operator overloading and generic interfaces, together with
pointers. Also important are the new facilities for numerical computation including a set of numeric inquiry functions, the parametrization of the intrinsic types, new control constructs — SELECT CASE and new forms of DO, internal and recursive procedures and optional and keyword arguments, improved I/O facilities, and many new intrinsic procedures. Last but not least are the new free source form, an improved style of attribute-oriented specifications, the IMPLICIT NONE statement, and a mechanism for identifying redundant features for subsequent removal from the language. The requirement on compilers to be able to identify, for example, syntax extensions, and to report why a program has been rejected, are also significant. The resulting language is not only a far more powerful tool than its successor, but a safer and more reliable one too. Storage association, with its attendant dangers, is not abolished, but rendered unnecessary. Indeed, experience shows that compilers detect errors far more frequently than before, resulting in a faster development cycle. The array syntax and recursion also allow quite compact code to be written, a further aid to safe programming.

No programming language can succeed if it consists simply of a definition (witness Algol 68). Also required are robust compilers from a wide variety of vendors, documentation at various levels, and a body of experience. The first Fortran 90 compiler appeared surprisingly quickly, in 1991, especially in view of the widely touted opinion that it would be very difficult to write one. Even more remarkable was that it was written by one person, Malcolm Cohen of NAG, in Oxford, U.K. There was a gap before other compilers appeared, but now they exist as native implementations for almost all leading computers, from the largest to PCs. For the most part, they produce very efficient object code; where, for certain new features, this is not the case, work is in progress to improve them.

The first book, *Fortran 90 Explained*, was published by John Reid and me shortly before the standard itself was published. Others followed in quick succession, including excellent texts aimed at the college market. At the time of writing there are at least 19 books in English and 22 in various other languages: Chinese, Dutch, French, Japanese, Russian, and Swedish. Thus, the documentation condition is fulfilled.

The body of experience, on the other hand, has yet to be built up to a critical size. Teaching of the language at college level has only just begun. However, I am certain that this present volume will contribute decisively to a significant breakthrough, as it provides models not only of the numerical algorithms for which previous editions are already famed, but also of an excellent Fortran 90 style, something that can develop only with time. Redundant features are abjured. It shows that, if we abandon these features and use new ones in their place, the appearance of code can initially seem unfamiliar, but, in fact, the advantages become rapidly apparent. This new edition of *Numerical Recipes* stands as a landmark in this regard.

*Fortran Evolution*

The formal procedures under which languages are standardized require them either to evolve or to die. A standard that has not been revised for some years must either be revised and approved anew, or be withdrawn. This matches the technical pressure on the language developers to accommodate the increasing complexity both of the problems to be tackled in scientific computation and of the underlying hardware
on which programs run. Increasing problem complexity requires more powerful features and syntax; new hardware needs language features that map onto it well.

Thus it was that X3J3 and WG5, having finished Fortran 90, began a new round of improvement. They decided very quickly on new procedures that would avoid the disputes that bedevilled the previous work: WG5 would decide on a plan for future standards, and X3J3 would act as the so-called development body that would actually produce them. This would be done to a strict timetable, such that any feature that could not be completed on time would have to wait for the next round. It was further decided that the next major revision should appear a decade after Fortran 90 but, given the somewhat discomforting number of requests for interpretation that had arrived, about 200, that a minor revision should be prepared for mid-term, in 1995. This should contain only “corrections, clarifications and interpretations” and a very limited number (some thought none) of minor improvements.

At the same time, scientific programmers were becoming increasingly concerned at the variety of methods that were necessary to gain efficient performance from the ever-more widely used parallel architectures. Each vendor provided a different set of parallel extensions for Fortran, and some academic researchers had developed yet others. On the initiative of Ken Kennedy of Rice University, a High-Performance Fortran Forum was established. A coalition of vendors and users, its aim was to produce an ad hoc set of extensions to Fortran that would become an informal but widely accepted standard for portable code. It set itself the daunting task of achieving that in just one year, and succeeded. Melding existing dialects like Fortran D, CM Fortran, and Vienna Fortran, and adopting the new Fortran 90 as a base, because of its array syntax, High-Performance Fortran (HPF) was published in 1993 and has since become widely implemented. However, although HPF was designed for data parallel codes and mainly implemented in the form of directives that appear to non-HPF processors as comment lines, an adequate functionality could not be achieved without extending the Fortran syntax. This was done in the form of the PURE attribute for functions — an assertion that they contain no side effects — and the FORALL construct — a form of array assignment expressed with the help of indices.

The dangers of having diverging or competing forms of Fortran 90 were immediately apparent, and the standards committees wisely decided to incorporate these two syntactic changes also into Fortran 95. But they didn’t stop there. Two further extensions, useful not only for their expressive power but also to access parallel hardware, were added: elemental functions, ones written in terms of scalars but that accept array arguments of any permitted shape or size, and an extension to allow nesting of WHERE constructs, Fortran’s form of masked assignment. To readers of Numerical Recipes, perhaps the most relevant of the minor improvements that Fortran 95 brings are the ability to distinguish between a negative and a positive real zero, automatic deallocation of allocatable arrays, and a means to initialize the values of components of objects of derived data types and to initialize pointers to null.

The medium-term objective of a relatively minor upgrade has been achieved on schedule. But what does the future hold? Developments in the underlying principles of procedural programming languages have not ceased. Early Fortran introduced the concepts of expression abstraction (X=Y+Z) and later control expression (e.g., the DO loop). Fortran 77 continued this with the if-then-else, and Fortran 90 with the DO and SELECT CASE constructs. Fortran 90 has a still higher level of expression abstraction (array assignments and expressions) as well as data structures and even
full-blown abstract data types. However, during the 1980s the concept of objects
came to the fore, with methods bound to the objects on which they operate. Here,
one particular language, C++, has come to dominate the field. Fortran 90 lacks
a means to point to functions, but otherwise has most of the necessary features in
place, and the standards committees are now faced with the dilemma of deciding
whether to make the planned Fortran 2000 a fully object-oriented language. This
could possibly jeopardize its powerful, and efficient, numerical capabilities by too
great an increase in language complexity, so should they simply batten down the
hatches and not defer to what might be only a passing storm? At the time of writing,
this is an open issue. One issue that is not open is Fortran’s lack of in-built exception
handling. It is virtually certain that such a facility, much requested by the numerical
community, and guided by John Reid, will be part of the next major revision. The
list of other requirements is long but speculative, but some at the top of the list
are conditional compilation, command line argument handling, I/O for objects of
derived type, and asynchronous I/O (which is also planned for the next release
of HPF). In the meantime, some particularly pressing needs have been identified,
for the handling of floating-point exceptions, interoperability with C, and allowing
allocatable arrays as structure components, dummy arguments, and function results.
These have led WG5 to begin processing these three items using a special form of
fast track, so that they might become optional but standard extensions well before

Conclusion

Writing a book is always something of a gamble. Unlike a novel that stands
or falls on its own, a book devoted to a programming language is dependent on
the success of others, and so the risk is greater still. However, this new Numerical
Recipes in Fortran 90 volume is no ordinary book, since it comes as the continuation
of a highly successful series, and so great is its significance that it can, in fact,
influence the outcome in its own favor. I am entirely confident that its publication
will be seen as an important event in the story of Fortran 90, and congratulate its
authors on having performed a great service to the field of numerical computing.

Geneva, Switzerland
January 1996

Michael Metcalf
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Chapter 21. Introduction to Fortran 90

Language Features

21.0 Introduction

Fortran 90 is in many respects a backwards-compatible modernization of the long-used (and much abused) Fortran 77 language, but it is also, in other respects, a new language for parallel programming on present and future multiprocessor machines. These twin design goals of the language sometimes add confusion to the process of becoming fluent in Fortran 90 programming.

In a certain trivial sense, Fortran 90 is strictly backwards-compatible with Fortran 77. That is, any Fortran 90 compiler is supposed to be able to compile any legacy Fortran 77 code without error. The reason for terming this compatibility trivial, however, is that you have to tell the compiler (usually via a source file name ending in “.f” or “.for”) that it is dealing with a Fortran 77 file. If you instead try to pass off Fortran 77 code as native Fortran 90 (e.g., by naming the source file something ending in “.f90”) it will not always work correctly!

It is best, therefore, to approach Fortran 90 as a new computer language, albeit one with a lot in common with Fortran 77. Indeed, in such terms, Fortran 90 is a fairly big language, with a large number of new constructions and intrinsic functions. Here, in one short chapter, we do not pretend to provide a complete description of the language. Luckily, there are good books that do exactly that. Our favorite one is by Metcalf and Reid [1], cited throughout this chapter as “M&R.” Other good starting points include [2] and [3].

Our goal, in the remainder of this chapter, is to give a good, working description of those Fortran 90 language features that are not immediately self-explanatory to Fortran 77 programmers, with particular emphasis on those that occur most frequently in the Fortran 90 versions of the Numerical Recipes routines. This chapter, by itself, will not teach you to write Fortran 90 code. But it ought to help you acquire a reading knowledge of the language, and perhaps provide enough of a head start that you can rapidly pick up the rest of what you need to know from M&R or another Fortran 90 reference book.

CITED REFERENCES AND FURTHER READING:
21.1 Quick Start: Using the Fortran 90 Numerical Recipes Routines

This section is for people who want to jump right in. We’ll compute a Bessel function \( J_0(x) \), where \( x \) is equal to the fourth root of the Julian Day number of the 200th full moon since January 1900. (Now there’s a useful quantity!)

First, locate the important files \texttt{nrtype.f90}, \texttt{nrutil.f90}, and \texttt{nr.f90}, as listed in Appendices C1, C1, and C2, respectively. These contain modules that either are (i) used by our routines, or else (ii) describe the calling conventions of our routines to (your) user programs. Compile each of these files, producing (with most compilers) a .mod file and a .o (or similarly named) file for each one.

Second, create this main program file:

```fortran
PROGRAM hello_bessel
USE nrtype
USE nr, ONLY: flmoon, bessj0
IMPLICIT NONE
INTEGER(I4B) :: n=200, nph=2, jd
REAL(SP) :: x, frac, ans
call flmoon(n, nph, jd, frac)
x=jd**0.25_sp
ans=bessj0(x)
write (*,*) 'Hello, Bessel: ', ans
END PROGRAM
```

Here is a quick explanation of some elements of the above program:

The first \texttt{USE} statement includes a module of ours named \texttt{nrtype}, whose purpose is to give symbolic names to some kinds of data types, among them single-precision reals ("sp") and four-byte integers ("i4b"). The second \texttt{USE} statement includes a module of ours that defines the calling sequences, and variable types, expected by (in this case) the Numerical Recipes routines \texttt{flmoon} and \texttt{bessj0}.

The \texttt{IMPLICIT NONE} statement signals that we want the compiler to require us explicitly to declare all variable types. We strongly urge that you always take this option.

The next two lines declare integer and real variables of the desired kinds. The variable \( n \) is initialized to the value 200, \( \text{nph} \) to 2 (a value expected by \texttt{flmoon}).

We call \texttt{flmoon} and take the fourth root of the answer it returns as \( \text{jd} \). Note that the constant 0.25 is typed to be single-precision by the appended \_sp.

We call the \texttt{bessj0} routine, and print the answer.

Third, compile the main program file, and also the files \texttt{flmoon.f90}, \texttt{bessj0.f90}. Then, link the resulting object files with also \texttt{nrutil.o} (or similar system-dependent name, as produced in step 1). Some compilers will also require you to link with \texttt{nr.o} and \texttt{nrtype.o}.

Fourth, run the resulting executable file. Typical output is:

```
Hello, Bessel:  7.3096365E-02
```
21.2 Fortran 90 Language Concepts

The Fortran 90 language standard defines and uses a number of standard terms for concepts that occur in the language. Here we summarize briefly some of the most important concepts. Standard Fortran 90 terms are shown in italics. While by no means complete, the information in this section should help you get a quick start with your favorite Fortran 90 reference book or language manual.

A note on capitalization: Outside a character context, Fortran 90 is not case-sensitive, so you can use upper and lower case any way you want, to improve readability. A variable like SP (see below) is the same variable as the variable sp. We like to capitalize keywords whose use is primarily at compile-time (statements that delimit program and subprogram boundaries, declaration statements of variables, fixed parameter values), and use lower case for the bulk of run-time code. You can adopt any convention that you find helpful to your own programming style; but we strongly urge you to adopt and follow some convention.

Data Types and Kinds

Data types (also called simply types) can be either intrinsic data types (the familiar INTEGER, REAL, LOGICAL, and so forth) or else derived data types that are built up in the manner of what are called “structures” or “records” in other computer languages. (We’ll use derived data types very sparingly in this book.) Intrinsic data types are further specified by their kind parameter (or simply kind), which is simply an integer. Thus, on many machines, REAL(4) (with kind = 4) is a single-precision real, while REAL(8) (with kind = 8) is a double-precision real. Literal constants (or simply literals) are specified as to kind by appending an underscore, as 1.5 for single precision, or 1.5_8 for double precision. [M&R, §2.5–§2.6]

Unfortunately, the specific integer values that define the different kind types are not specified by the language, but can vary from machine to machine. For that reason, one almost never uses literal kind parameters like 4 or 8, but rather defines in some central file, and imports into all one’s programs, symbolic names for the kinds. For this book, that central file is the module named nrtype, and the chosen symbolic names include SP, DP (for reals); I2B, I4B (for two- and four-byte integers); and LGT for the default logical type. You will therefore see us consistently writing REAL(SP), or 1.5_dp, and so forth.

Here is an example of declaring some variables, including a one-dimensional array of length 500, and a two-dimensional array with 100 rows and 200 columns:

USE nrtype
REAL(SP) :: x, y, z
INTEGER(I4B) :: i, j, k
REAL(SP), DIMENSION(500) :: arr
REAL(SP), DIMENSION(100,200) :: barr
REAL(SP) :: carr(500)

The last line shows an alternative form for array syntax. And yes, there are default kind parameters for each intrinsic type, but these vary from machine to machine and can get you into trouble when you try to move code. We therefore specify all kind parameters explicitly in almost all situations.
Array Shapes and Sizes

The shape of an array refers to both its dimensionality (called its rank), and also the lengths along each dimension (called the extents). The shape of an array is specified by a rank-one array whose elements are the extents along each dimension, and can be queried with the shape intrinsic (see p. 949). Thus, in the above example, shape(barr) returns an array of length 2 containing the values (100, 200).

The size of an array is its total number of elements, so the intrinsic size(barr) would return 20000 in the above example. More often one wants to know the extents along each dimension, separately: size(barr,1) returns the value 100, while size(barr,2) returns the value 200. [M&R, §2.10]

Section §21.3, below, discusses additional aspects of arrays in Fortran 90.

Memory Management

Fortran 90 is greatly superior to Fortran 77 in its memory-management capabilities, seen by the user as the ability to create, expand, or contract workspace for programs. Within subprograms (that is, subroutines and functions), one can have automatic arrays (or other automatic data objects) that come into existence each time the subprogram is entered, and disappear (returning their memory to the pool) when the subprogram is exited. The size of automatic objects can be specified by arbitrary expressions involving values passed as actual arguments in the calling program, and thus received by the subprogram through its corresponding dummy arguments. [M&R, §6.4]

Here is an example that creates some automatic workspace named carr:

```fortran
SUBROUTINE dosomething(j,k)
USE nrtype
REAL(SP), DIMENSION(2*j,k**2) :: carr
```

Finer control on when workspace is created or destroyed can be achieved by declaring allocatable arrays, which exist as names only, without associated memory, until they are allocated within the program or subprogram. When no longer needed, they can be deallocated. The allocation status of an allocatable array can be tested by the program via the allocated intrinsic function (p. 952). [M&R, §6.5]

Here is an example in outline:

```fortran
REAL(SP), DIMENSION(:,,:), ALLOCATABLE :: darr
...
allocate(darr(10,20))
...
deallocate(darr)
...
allocate(darr(100,200))
...
deallocate(darr)
```

Notice that darr is originally declared with only “slots” (colons) for its dimensions, and is then allocated/deallocated twice, with different sizes.

Yet finer control is achieved by the use of pointers. Like an allocatable array, a pointer can be allocated, at will, its own associated memory. However, it has the additional flexibility of alternatively being pointer associated with a target that
already exists under another name. Thus, pointers can be used as redefinable aliases for other variables, arrays, or (see §21.3) array sections. [M&R, §6.12]

Here is an example that first associates the pointer `parr` with the array `earr`, then later cancels that association and allocates it its own storage of size 50:

```fortran
REAL(SP), DIMENSION(:,), POINTER :: parr
REAL(SP), DIMENSION(100), TARGET :: earr
...
parr => earr
...
nullify(parr)
allocate(parr(50))
...
deallocate(parr)
```

**Procedure Interfaces**

When a procedure is referenced (e.g., called) from within a program or subprogram (examples of scoping units), the scoping unit must be told, or must deduce, the procedure’s interface, that is, its calling sequence, including the types and kinds of all dummy arguments, returned values, etc. The recommended procedure is to specify this interface via an explicit interface, usually an interface block (essentially a declaration statement for subprograms) in the calling subprogram or in some module that the calling program includes via a USE statement. In this book all interfaces are explicit, and the module named `nr` contains interface blocks for all of the Numerical Recipes routines. [M&R, §5.11]

Here is a typical example of an interface block:

```fortran
INTERFACE
  SUBROUTINE caldat(julian,mm,id,iyyy)
    USE nrtype
    INTEGER(I4B), INTENT(IN) :: julian
    INTEGER(I4B), INTENT(OUT) :: mm,id,iyyy
  END SUBROUTINE caldat
END INTERFACE
```

Once this interface is made known to a program that you are writing (by either explicit inclusion or a USE statement), then the compiler is able to flag for you a variety of otherwise difficult-to-find bugs. Although interface blocks can sometimes seem overly wordy, they give a big payoff in ultimately minimizing programmer time and frustration.

For compatibility with Fortran 77, the language also allows for implicit interfaces, where the calling program tries to figure out the interface by the old rules of Fortran 77. These rules are quite limited, and prone to producing devilishly obscure program bugs. We strongly recommend that implicit interfaces never be used.

**Elemental Procedures and Generic Interfaces**

Many intrinsic procedures (those defined by the language standard and thus usable without any further definition or specification) are also generic. This means that a single procedure name, such as `log(x)`, can be used with a variety of types and kind parameters for the argument `x`, and the result returned will have the same type and kind parameter as the argument. In this example, `log(x)` allows any real or complex argument type.
Better yet, most generic functions are also *elemental*. The argument of an elemental function can be an array of arbitrary shape! Then, the returned result is an array of the same shape, with each element containing the result of applying the function to the corresponding element of the argument. (Hence the name *elemental*, meaning “applied element by element.”) [M&R, §8.1] For example:

```fortran
REAL(SP), DIMENSION(100,100) :: a,b
b=sin(a)
```

Fortran 90 has no facility for creating new, user-defined elemental functions. It does have, however, the related facility of *overloading* by the use of *generic interfaces*. This is invoked by the use of an interface block that attaches a single *generic name* to a number of distinct subprograms whose dummy arguments have different types or kinds. Then, when the generic name is referenced (e.g., called), the compiler chooses the specific subprogram that matches the types and kinds of the actual arguments used. [M&R, §5.18] Here is an example of a generic interface block:

```fortran
INTERFACE myfunc
  FUNCTION myfunc_single(x)
    USE nrtype
    REAL(SP) :: x,myfunc_single
  END FUNCTION myfunc_single

  FUNCTION myfunc_double(x)
    USE nrtype
    REAL(DP) :: x,myfunc_double
  END FUNCTION myfunc_double
END INTERFACE
```

A program with knowledge of this interface could then freely use the function reference `myfunc(x)` for `x`’s of both type SP and type DP.

We use overloading quite extensively in this book. A typical use is to provide, under the same name, both scalar and vector versions of a function such as a Bessel function, or to provide both single-precision and double-precision versions of procedures (as in the above example). Then, to the extent that we have provided all the versions that you need, you can pretend that our routine is elemental. In such a situation, if you ever call our function with a type or kind that we have not provided, the compiler will instantly flag the problem, because it is unable to resolve the generic interface.

### Modules

*Modules*, already referred to several times above, are Fortran 90’s generalization of Fortran 77’s common blocks, **INCLUDED** files of parameter statements, and (to some extent) statement functions. Modules are *program units*, like main programs or subprograms (subroutines and functions), that can be separately compiled. A module is a convenient place to stash global data, *named constants* (what in Fortran 77 are called “symbolic constants” or “PARAMETERS”), interface blocks to subprograms and/or actual subprograms themselves (*module subprograms*). The convenience is that a module’s information can be incorporated into another program unit via a simple, one-line USE statement. [M&R, §5.5]

Here is an example of a simple module that defines a few parameters, creates some global storage for an array named `arr` (as might be done with a Fortran 77 common block), and defines the interface to a function `yourfunc`:
21.3 More on Arrays and Array Sections

As mentioned earlier, the module \texttt{nrt} contains \texttt{INTERFACE} declarations for all the Numerical Recipes. When we include a statement of the form

\begin{verbatim}
USE nr, ONLY: recipe1
\end{verbatim}

it means that the program uses the additional routine \texttt{recipe1}. The compiler is able to use the explicit interface declaration in the module to check that \texttt{recipe1} is invoked with arguments of the correct type, shape, and number. However, a weakness of Fortran 90 is that there is no fail-safe way to be sure that the interface module (here \texttt{nr}) stays synchronized with the underlying routine (here \texttt{recipe1}). You might think that you could accomplish this by putting \texttt{USE nr, ONLY: recipe1} into the \texttt{recipe1} program itself. Unfortunately, the compiler interprets this as an erroneous double definition of \texttt{recipe1}'s interface, rather than (as would be desirable) as an opportunity for a consistency check. (To achieve this kind of consistency check, you can put the procedures themselves, not just their interfaces, into the module.)

CITED REFERENCES AND FURTHER READING:

21.3 More on Arrays and Array Sections

Arrays are the central conceptual core of Fortran 90 as a parallel programming language, and thus worthy of some further discussion. We have already seen that arrays can “come into existence” in Fortran 90 in several ways, either directly declared, as

\begin{verbatim}
REAL(SP), DIMENSION(100,200) :: arr
\end{verbatim}

or else allocated by an \texttt{allocatable} variable or a \texttt{pointer} variable,

\begin{verbatim}
REAL(SP), DIMENSION(:,:), ALLOCATABLE :: arr
REAL(SP), DIMENSION(:,:), POINTER :: barr ...
allocate(arr(100,200),barr(100,200))
\end{verbatim}

or else (not previously mentioned) passed into a subprogram through a dummy argument:

\begin{verbatim}
SUBROUTINE myroutine(carr)
USE nr
REAL(SP), DIMENSION(:,:) :: carr ...
\end{verbatim}
In the above example we also show how the subprogram can find out the size of the actual array that is passed, using the `size` intrinsic. This routine is an example of the use of an *assumed-shape array*, new to Fortran 90. The actual extents along each dimension are inherited from the calling routine at run time. A subroutine with assumed-shape array arguments must have an explicit interface in the calling routine, otherwise the compiler doesn’t know about the extra information that must be passed. A typical setup for calling `myroutine` would be:

```fortran
PROGRAM use_myroutine
  USE nrtype
  REAL(SP), DIMENSION(10,10) :: arr
  INTERFACE
    SUBROUTINE myroutine(carr)
      USE nrtype
      REAL(SP), DIMENSION(:,:) :: carr
    END SUBROUTINE myroutine
  END INTERFACE
  ...
  call myroutine(a)
END PROGRAM use_myroutine
```

Of course, for the recipes we have provided all the interface blocks in the file `nr.f90`, and you need only a `USE nr` statement in your calling program.

### Conformable Arrays

Two arrays are said to be *conformable* if their shapes are the same. Fortran 90 allows practically all operations among conformable arrays and elemental functions that are allowed for scalar variables. Thus, if `arr`, `barr`, and `carr` are mutually conformable, we can write,

```fortran
arr = barr + cos(carr) + 2.0_sp
```

and have the indicated operations performed, element by corresponding element, on the entire arrays. The above line also illustrates that a scalar (here the constant `2.0_sp`, but a scalar variable would also be fine) is deemed conformable with *any* array — it gets “expanded” to the shape of the rest of the expression that it is in. [M&R, §3.11]

In Fortran 90, as in Fortran 77, the default lower bound for an array subscript is 1; however, it can be made some other value at the time that the array is declared:

```fortran
REAL(SP), DIMENSION(100,200) :: farr
REAL(SP), DIMENSION(0:99,0:199) :: garr
...
farr = 3.0_sp*garr + 1.0_sp
```

Notice that `farr` and `garr` are conformable, since they have the same shape, in this case `(100,200)`. Also note that when they are used in an array expression, the operations are taken between the corresponding elements of *their shapes*, not necessarily the corresponding elements of *their indices*. [M&R, §3.10] In other words, one of the components evaluated is,

```fortran
farr(1,1) = 3.0_sp*garr(0,0) + 1.0_sp
```

This illustrates a fundamental aspect of array (or data) parallelism in Fortran 90. Array constructions should *not* be thought of as merely abbreviations for do-loops.
over indices, but rather as genuinely parallel operations on same-shaped objects, 
abstracted of their indices. This is why the standard makes no statement about the 
order in which the individual operations in an array expression are executed; they 
might in fact be carried out simultaneously, on parallel hardware.

By default, array expressions and assignments are performed for all elements 
of the same-shaped arrays referenced. This can be modified, however, by use of a 
where construction like this:

```fortran
where (harr > 0.0_sp)
    farr = 3.0_sp*garr + 1.0_sp
end where
```

Here harr must also be conformable to farr and garr. Analogously with the Fortran 
if-statement, there is also a one-line form of the where-statement. There is also a 
where ... elseif ... end where form of the statement, analogous to if ... elseif ... end if. A significant language limitation in Fortran 90 
is that nested where-statements are not allowed. [M&R, §6.8]

**Array Sections**

Much of the versatility of Fortran 90’s array facilities stems from the availability 
of *array sections*. An array section acts like an array, but its memory location, and 
thus the values of its elements, is actually a subset of the memory location of an 
already-declared array. Array sections are thus “windows into arrays,” and they can 
appear on either the left side, or the right side, or both, of a replacement statement. 
Some examples will clarify these ideas.

Let us presume the declarations

```fortran
REAL(SP), DIMENSION(100) :: arr
INTEGER(I4B), DIMENSION(6) :: iarr=(/11,22,33,44,55,66/)
```

Note that iarr is not only declared, it is also initialized by an *initialization expression* 
(a replacement for Fortran 77’s DATA statement). [M&R, §7.5] Here are some array 
sections constructed from these arrays:

<table>
<thead>
<tr>
<th>Array Section</th>
<th>What It Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>arr(:)</td>
<td>same as arr</td>
</tr>
<tr>
<td>arr(1:100)</td>
<td>same as arr</td>
</tr>
<tr>
<td>arr(1:10)</td>
<td>one-dimensional array containing first 10 elements of arr</td>
</tr>
<tr>
<td>arr(51:100)</td>
<td>one-dimensional array containing second half of arr</td>
</tr>
<tr>
<td>arr(51:)</td>
<td>same as arr(51:100)</td>
</tr>
<tr>
<td>arr(10:1:-1)</td>
<td>one-dimensional array containing first 10 elements of arr, but in reverse order</td>
</tr>
<tr>
<td>arr((/10,99,1,6/))</td>
<td>one-dimensional array containing elements 10, 99, 1, and 6 of arr, in that order</td>
</tr>
<tr>
<td>arr(iarr)</td>
<td>one-dimensional array containing elements 11, 22, 33, 44, 55, 66 of arr, in that order</td>
</tr>
</tbody>
</table>
Now let's try some array sections of the two-dimensional array

```fortran
REAL(SP), DIMENSION(100,100) :: barr
```

<table>
<thead>
<tr>
<th>Array Section</th>
<th>What It Means</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>barr(:, :)</code></td>
<td>same as <code>barr</code></td>
</tr>
<tr>
<td><code>barr(1:100,1:100)</code></td>
<td>same as <code>barr</code></td>
</tr>
<tr>
<td><code>barr(7,:)</code></td>
<td>one-dimensional array containing the 7th row of <code>barr</code></td>
</tr>
<tr>
<td><code>barr(7,1:100)</code></td>
<td>same as <code>barr(7,:)</code></td>
</tr>
<tr>
<td><code>barr(:,7)</code></td>
<td>one-dimensional array containing the 7th column of <code>barr</code></td>
</tr>
<tr>
<td><code>barr(21:30,71:90)</code></td>
<td>two-dimensional array containing the sub-block of <code>barr</code> with the indicated ranges of indices; the shape of this array section is (10, 20)</td>
</tr>
<tr>
<td><code>barr(100:1:-1,100:1:-1)</code></td>
<td>two-dimensional array formed by flipping <code>barr</code> upside down and backwards</td>
</tr>
<tr>
<td><code>barr(2:100:2,2:100:2)</code></td>
<td>two-dimensional array of shape (50, 50) containing the elements of <code>barr</code> whose row and column indices are both even</td>
</tr>
</tbody>
</table>

Some terminology: A construction like `2:100:2`, above, is called a *subscript triplet*. Its integer pieces (which may be integer constants, or more general integer expressions) are called *lower*, *upper*, and *stride*. Any of the three may be omitted. An omitted stride defaults to the value 1. Notice that, if `(upper - lower)` has a different sign from *stride*, then a subscript triplet defines an empty or *zero-length* array, e.g., `1:5:-1` or `10:1:1` (or its equivalent form, simply `10:1`). Zero-length arrays are not treated as errors in Fortran 90, but rather as “no-ops.” That is, no operation is performed in an expression or replacement statement among zero-length arrays. (This is essentially the same convention as in Fortran 77 for do-loop indices, which array expressions often replace.) [M&R, §6.10]

It is important to understand that array sections, when used in array expressions, match elements with other parts of the expression *according to shape*, not according to indices. (This is exactly the same principle that we applied, above, to arrays with subscript lower bounds different from the default value of 1.) One frequently exploits this feature in using array sections to carry out operations on arrays that access neighboring elements. For example,

```fortran
carr(1:n-1,1:n-1) = barr(1:n-1,1:n-1)+barr(2:n,2:n)
```

constructs in the $(n-1) \times (n-1)$ matrix `carr` the sum of each of the corresponding elements in $n \times n$ `barr` added to its diagonally lower-right neighbor.

Pointers are often used as aliases for array sections, especially if the same array sections are used repeatedly. [M&R, §6.12] For example, with the setup

```fortran
REAL(SP), DIMENSION(:, :), POINTER :: leftb, rightb
```
leftb=>barr(1:n-1,1:n-1)
rightb=>barr(2:n,2:n)

carr(1:n-1,1:n-1)=leftb+rightb

We should also mention that array sections, while powerful and concise, are sometimes not quite powerful enough. While any row or column of a matrix is easily accessible as an array section, there is no good way, in Fortran 90, to access (e.g.) the diagonal of a matrix, even though its elements are related by a linear progression in the Fortran storage order (by columns). These so-called skew-sections were much discussed by the Fortran 90 standards committee, but they were not implemented. We will see examples later in this volume of work-around programming tricks (none totally satisfactory) for this omission. (Fortran 95 corrects the omission; see §21.6.)

CITED REFERENCES AND FURTHER READING:

## 21.4 Fortran 90 Intrinsic Procedures

Much of Fortran 90's power, both for parallel programming and for its concise expression of algorithmic ideas, comes from its rich set of intrinsic procedures. These have the effect of making the language “large,” hence harder to learn. However, effort spent on learning to use the intrinsics — particularly some of their more obscure, and more powerful, optional arguments — is often handsomely repaid.

This section summarizes the intrinsics that we find useful in numerical work. We omit, here, discussion of intrinsics whose exclusive use is for character and string manipulation. We intend only a summary, not a complete specification, which can be found in M&R’s Chapter 8, or other reference books.

If you find the sheer number of new intrinsic procedures daunting, you might want to start with our list of the “top 10” (with the number of different Numerical Recipes routines that use each shown in parentheses): size (254), sum (44), dot_product (31), merge (27), all (25), maxval (23), matmul (19), pack (18), any (17), and spread (15). (Later, in Chapter 23, you can compare these numbers with our frequency of using the short utility functions that we define in a module named nrutil — several of which we think ought to have been included as Fortran 90 intrinsic procedures.)

The type, kind, and shape of the value returned by intrinsic functions will usually be clear from the short description that we give. As an additional hint (though not necessarily a precise description), we adopt the following codes:
### Hint | What It Means
--- | ---
[Int] | an INTEGER kind type
[Real] | a REAL kind type
[Cmplx] | a COMPLEX kind type
[Num] | a numerical type and kind
[Lgcl] | a LOGICAL kind type
[Iarr] | a one-dimensional INTEGER array
[argTS] | same type and shape as the first argument
[argT] | same type as the first argument, but not necessarily the same shape

### Numerical Elemental Functions

Little needs to be said about the numerical functions with identical counterparts in Fortran 77: abs, acos, aimag, asin, atan, atan2, conjg, cos, cosh, dim, exp, log, log10, max, min, mod, sign, sin, sinh, sqrt, tan, and tanh. In Fortran 90 these are all *elemental* functions, so that any plausible type, kind, and shape of argument may be used. Except for aimag, which returns a real type from a complex argument, these all return [argTS] (see table above).

Although Fortran 90 recognizes, for compatibility, Fortran 77’s so-called *specific names* for these functions (e.g., iabs, dabs, and cabs for the generic abs), these are entirely superfluous and should be avoided.

Fortran 90 corrects some ambiguity (or at least inconvenience) in Fortran 77’s mod(a,p) function, by introducing a new function modulo(a,p). The functions are essentially identical for positive arguments, but for negative a and positive p, modulo gives results more compatible with one’s mathematical expectation that the answer should always be in the positive range 0 to p. E.g., modulo(11,5)=1, and modulo(-11,5)=4. [M&R, §8.3.2]

### Conversion and Truncation Elemental Functions

Fortran 90’s conversion (or, in the language of C, casting) and truncation functions are generally modeled on their Fortran 77 antecedents, but with the addition of an optional second integer argument, kind, that determines the kind of the result. Note that, if kind is omitted, you get a default kind — not necessarily related to the kind of your argument. The kind of the argument is of course known to the compiler by its previous declaration. Functions in this category (see below for explanation of arguments in slanted type) are:

- [Real] aint(a,kind)
  - Truncate to integer value, return as a real kind.

- [Real] anint(a,kind)
  - Nearest whole number, return as a real kind.

- [Cmplx] cmplx(x,y,kind)
Convert to complex kind. If \( y \) is omitted, it is taken to be 0.

\[
\text{[Int]} \quad \text{int}(a,kind)
\]
Convert to integer kind, truncating towards zero.

\[
\text{[Int]} \quad \text{nint}(a,kind)
\]
Convert to integer kind, choosing the nearest whole number.

\[
\text{[Real]} \quad \text{real}(a,kind)
\]
Convert to real kind.

\[
\text{[Lgcl]} \quad \text{logical}(a,kind)
\]
Convert one logical kind to another.

We must digress here to explain the use of \textit{optional arguments} and \textit{keywords} as Fortran 90 language features. [M&R, §5.13] When a routine (either intrinsic or user-defined) has arguments that are declared to be optional, then the dummy names given to them also become keywords that distinguish — independent of their position in a calling list — which argument is intended to be passed. (There are some additional rules about this that we will not try to summarize here.) In this section’s tabular listings, we indicate optional arguments in intrinsic routines by printing them in smaller slanted type. For example, the intrinsic function

\[
\text{eoshift(array,shift,boundary,dim)}
\]
has two required arguments, \( \text{array} \) and \( \text{shift} \), and two optional arguments, \( \text{boundary} \) and \( \text{dim} \). Suppose we want to call this routine with the actual arguments \( \text{myarray, myshift, and mydim} \), but omitting the argument in the boundary slot. We do this by the expression

\[
\text{eoshift(myarray,myshift,dim=mydim)}
\]
Conversely, if we wanted a boundary argument, but no \( \text{dim} \), we might write

\[
\text{eoshift(myarray,myshift,boundary=myboundary)}
\]
It is always a good idea to use this kind of keyword construction when invoking optional arguments, even though the rules allow keywords to be omitted in some unambiguous cases. Now back to the lists of intrinsic routines.

A peculiarity of the \textit{real} function derives from its use both as a type conversion and for extracting the real part of complex numbers (related, but not identical, usages): If the argument of \textit{real} is complex, and \textit{kind} is omitted, then the result \textit{isn’t} a default real kind, but rather \textit{is} (as one generally would want) the \textit{real} kind type corresponding to the kind type of the complex argument, that is, single-precision \textit{real} for single-precision complex, double-precision for double-precision, and so on. [M&R, §8.3.1] We recommend \textit{never} using kind when you intend to extract the real part of a complex, and \textit{always} using \textit{kind} when you intend conversion of a real or integer value to a particular kind of \textsc{real}. (Use of the deprecated function \textit{dble} is not recommended.)

The last two conversion functions are the exception in that they \textit{don’t} allow a \textit{kind} argument, but rather return default integer kinds. (The X3J3 standards committee has fixed this in Fortran 95.)

\[
\text{[Int]} \quad \text{ceiling}(a)
\]
Convert to integer, truncating towards more positive.
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Reduction and Inquiry Functions on Arrays

These are mostly the so-called transformational functions that accept array arguments and return either scalar values or else arrays of lesser rank. [M&R, §8.11] With no optional arguments, such functions act on all the elements of their single array argument, regardless of its shape, and produce a scalar result. When the optional argument \( \text{dim} \) is specified, they instead act on all one-dimensional sections that span the dimension \( \text{dim} \), producing an answer one rank lower than the first argument (that is, omitting the \( \text{dim} \) dimension from its shape). When the optional argument \( \text{mask} \) is specified, only the elements with a corresponding true value in \( \text{mask} \) are scanned.

\[
\text{all}(\text{mask}, \text{dim})
\]
Returns true if all elements of \( \text{mask} \) are true, false otherwise.

\[
\text{any}(\text{mask}, \text{dim})
\]
Returns true if any of the elements of \( \text{mask} \) are true, false otherwise.

\[
\text{count}(\text{mask}, \text{dim})
\]
Counts the true elements in \( \text{mask} \).

\[
\text{maxval}(\text{array}, \text{dim}, \text{mask})
\]
Maximum value of the array elements.

\[
\text{minval}(\text{array}, \text{dim}, \text{mask})
\]
Minimum value of the array elements.

\[
\text{product}(\text{array}, \text{dim}, \text{mask})
\]
Product of the array elements.

\[
\text{size}(\text{array}, \text{dim})
\]
Size (total number of elements) of \( \text{array} \), or its extent along dimension \( \text{dim} \).

\[
\text{sum}(\text{array}, \text{dim}, \text{mask})
\]
Sum of the array elements.

The use of the \( \text{dim} \) argument can be confusing, so an example may be helpful. Suppose we have

\[
\text{myarray} = \begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{bmatrix}
\]

where, as always, the \( i \) index in \( \text{array}(i,j) \) numbers the rows while the \( j \) index numbers the columns. Then

\[
\text{sum(myarray, dim=1)} = (15, 18, 21, 24)
\]
that is, the \( i \) indices are “summed away” leaving only a \( j \) index on the result; while

\[
\text{sum(myarray, dim=2)} = (10, 26, 42)
\]
that is, the \( j \) indices are “summed away” leaving only an \( i \) index on the result. Of course we also have

\[
\text{sum}(\text{myarray}) = 78
\]

Two related functions return the location of particular elements in an array. The returned value is a one-dimensional integer array containing the respective subscript of the element along each dimension. Note that when the argument object is a one-dimensional array, the returned object is an integer array of length 1, not simply an integer. (Fortran 90 distinguishes between these.)

\[
\begin{align*}
\text{maxloc}(\text{array}, \text{mask}) \\
\text{minloc}(\text{array}, \text{mask})
\end{align*}
\]

Location of the maximum value in an array.
Location of the minimum value in an array.

Similarly returning a one-dimensional integer array are

\[
\begin{align*}
\text{shape}(\text{array}) \\
\text{lbound}(\text{array}, \text{dim}) \\
\text{ubound}(\text{array}, \text{dim})
\end{align*}
\]

Returns the shape of \text{array} as a one-dimensional integer array.
When \text{dim} is absent, returns an array of lower bounds for each dimension of subscripts of \text{array}. When \text{dim} is present, returns the value only for dimension \text{dim}, as a scalar.
When \text{dim} is absent, returns an array of upper bounds for each dimension of subscripts of \text{array}. When \text{dim} is present, returns the value only for dimension \text{dim}, as a scalar.

**Array Unary and Binary Functions**

The most powerful array operations are simply built into the language as operators. All the usual arithmetic and logical operators (+, -, *, /, **, .not., .and., .or., .eqv., .neqv.) can be applied to arrays of arbitrary shape or (for the binary operators) between two arrays of the same shape, or between arrays and scalars. The types of the arrays must, of course, be appropriate to the operator used. The result in all cases is to perform the operation element by element on the arrays.

We also have the intrinsic functions,

\[
\begin{align*}
\text{dot_product}(&\text{veca}, \text{vecb}) \\
\text{matmul}(\text{mata}, \text{matb})
\end{align*}
\]

Scalar dot product of two one-dimensional vectors \text{veca} and \text{vecb}.
Result of matrix-multiplying the two two-dimensional matrices \text{mata} and \text{matb}. The shapes have to be such as to allow matrix multiplication. Vectors (one-dimensional arrays) are additionally allowed as either the first or second argument, but not both; they are treated as row vectors in the first argument, and as column vectors in the second.

You might wonder how to form the outer product of two vectors, since \text{matmul} specifically excludes this case. (See §22.1 and §23.5 for answer.)
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Array Manipulation Functions

These include many powerful features that a good Fortran 90 programmer should master.

[argTS] cshift(array,shift,dim)
If dim is omitted, it is taken to be 1. Returns the result of circularly left-shifting every one-dimensional section of array (in dimension dim) by shift (which may be negative). That is, for positive shift, values are moved to smaller subscript positions. Consult a Fortran 90 reference (e.g., [M&R, §8.13.5]) for the case where shift is an array.

[argTS] merge(tsource,fsource,mask)
Returns same shape object as tsource and fsource containing the former’s components where mask is true, the latter’s where it is false.

[argTS] eoshift(array,shift,boundary,dim)
If dim is omitted, it is taken to be 1. Returns the result of end-off left-shifting every one-dimensional section of array (in dimension dim) by shift (which may be negative). That is, for positive shift, values are moved to smaller subscript positions. If boundary is present as a scalar, it supplies elements to fill in the blanks; if it is not present, zero values are used. Consult a Fortran 90 reference (e.g., [M&R, §8.13.5]) for the case where boundary and/or shift is an array.

[argT] pack(array,mask,vector)
Returns a one-dimensional array containing the elements of array that pass the mask. Components of optional vector are used to pad out the result to the size of vector with specified values.

[argT] reshape(source,shape,pad,order)
Takes the elements of source, in normal Fortran order, and returns them (as many as will fit) as an array whose shape is specified by the one-dimensional integer array shape. If there is space remaining, then pad must be specified, and is used (as many sequential copies as necessary) to fill out the rest. For description of order, consult a Fortran 90 reference, e.g., [M&R, 8.13.3].

[argT] spread(source,dim,ncopies)
Returns an array whose rank is one greater than source, and whose dim dimension is of length ncopies. Each of the result’s ncopies array sections having a fixed subscript in dimension dim is a copy of source. (That is, it spreads source into the dimth dimension.)

[argT] transpose(matrix)
Returns the transpose of matrix, which must be two-dimensional.

[argT] unpack(vector,mask,field)
Returns an array whose type is that of vector, but whose shape is that of mask. The components of vector are put, in order, into the positions where mask is true. Where mask is false, components of field (which may be a scalar or an array with the same shape as mask) are used instead.
**Bitwise Functions**

Most of the bitwise functions should be familiar to Fortran 77 programmers as longstanding standard extensions of that language. Note that the bit *positions* number from zero to one less than the value returned by the `bit_size` function. Also note that bit positions number *from right to left*. Except for `bit_size`, the following functions are all elemental.

- **[Int]** `bit_size(i)`
  Number of bits in the integer type of `i`.

- **[Lgcl]** `btest(i,pos)`
  True if bit position `pos` is 1, false otherwise.

- **[Int]** `iand(i,j)`
  Bitwise logical and.

- **[Int]** `ibclr(i,pos)`
  Returns `i` but with bit position `pos` set to zero.

- **[Int]** `ibits(i,pos,len)`
  Extracts `len` consecutive bits starting at position `pos` and puts them in the low bit positions of the returned value. (The high positions are zero.)

- **[Int]** `ibset(i,pos)`
  Returns `i` but with bit position `pos` set to 1.

- **[Int]** `ieor(i,j)`
  Bitwise exclusive or.

- **[Int]** `ior(i,j)`
  Bitwise logical or.

- **[Int]** `ishft(i,shift)`
  Bitwise left shift by `shift` (which may be negative) with zeros shifted in from the other end.

- **[Int]** `ishftc(i,shift)`
  Bitwise circularly left shift by `shift` (which may be negative).

- **[Int]** `not(i)`
  Bitwise logical complement.

**Some Functions Relating to Numerical Representations**

- **[Real]** `epsilon(x)`
  Smallest nonnegligible quantity relative to 1 in the numerical model of `x`.

- **[Num]** `huge(x)`
  Largest representable number in the numerical model of `x`.

- **[Int]** `kind(x)`
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Returns the kind value for the numerical model of x.

[Real] nearest(x,s)
Real number nearest to x in the direction specified by the sign of s.

[Real] tiny(x)
Smallest positive number in the numerical model of x.

Other Intrinsic Procedures

[Lgcl] present(a)
True, within a subprogram, if an optional argument is actually present, otherwise false.

[Lgcl] associated(pointer, target)
True if pointer is associated with target or (if target is absent) with any target, otherwise false.

[Lgcl] allocated(array)
True if the allocatable array is allocated, otherwise false.

There are some pitfalls in using associated and allocated, having to do with arrays and pointers that can find themselves in undefined status [see §21.5, and also M&R, §3.3 and §6.5.1]. For example, pointers are always “born” in an undefined status, where the associated function returns unpredictable values.

For completeness, here is a list of Fortran 90’s intrinsic procedures not already mentioned:

Other Numerical Representation Functions: digits, exponent, fraction, rrspacing, scale, set_exponent, spacing, maxexponent, minexponent, precision, radix, range, selected_int_kind, selected_real_kind.
Lexical comparison: lge, lgt, lle, llt.
Character functions: ichar, char, achar, iachar, index, adjustl, adjustr, len_trim, repeat, scan, trim, verify.
Other: mvbits, transfer, date_and_time, system_clock, random_seed, random_number. (We will discuss random numbers in some detail in Chapter B7.)

CITED REFERENCES AND FURTHER READING:
21.5 Advanced Fortran 90 Topics

Pointers, Arrays, and Memory Management

One of the biggest improvements in Fortran 90 over Fortran 77 is in the handling of arrays, which are the cornerstone of many numerical algorithms. In this subsection we will take a closer look at how to use some of these new array features effectively. We will look at how to code certain commonly occurring elements of program design, and we will pay particular attention to avoiding “memory leaks,” where — usually inadvertently — we keep cumulatively allocating new storage for an array, every time some piece of code is invoked.

Let's first review some of the rules for using allocatable arrays and pointers to arrays. Recall that a pointer is born with an undefined status. Its status changes to “associated” when you make it refer to a target, and to “disassociated” when you nullify the pointer. [M&R, §3.3] You can also use nullify on a newly born pointer to change its status from undefined to disassociated; this allows you to test the status with the associated inquiry function. [M&R, §6.5.4] (While many compilers will not produce a run-time error if you test an undefined pointer with associated, you can't rely on this laissez-faire in your programming.)

The initial status of an allocatable array is “not currently allocated.” Its status changes to “allocated” when you give it storage with allocate, and back to “not currently allocated” when you use deallocate. [M&R, §6.5.1] You can test the status with the allocated inquiry function. Note that while you can also give a pointer fresh storage with allocate, you can't test this with allocated — only associated is allowed with pointers. Note also that nullifying an allocated pointer leaves its associated storage in limbo. You must instead deallocate, which gives the pointer a testable “disassociated” status.

While allocating an array that is already allocated gives an error, you are allowed to allocate a pointer that already has a target. This breaks the old association, and could leave the old target inaccessible if there is no other pointer associated with it. [M&R, §6.5.2] deallocating an array or pointer that has not been allocated is always an error.

Allocated arrays that are local to a subprogram acquire the “undefined” status on exit from the subprogram unless they have the SAVE attribute. (Again, not all compilers enforce this, but be warned!) Such undefined arrays cannot be referenced in any way, so you should explicitly deallocate all allocated arrays that are not saved before returning from a subprogram. [M&R, §6.5.1] The same rule applies to arrays declared in modules that are currently accessed only by the subprogram. While you can reference undefined pointers (e.g., by first nullifying them), it is good programming practice to deallocate explicitly any allocated pointers declared locally before leaving a subprogram or module.

Now let’s turn to using these features in programs. The simplest example is when we want to implement global storage of an array that needs to be accessed by two or more different routines, and we want the size of the array to be determined at run time. As mentioned earlier, we implement global storage with a MODULE rather than a COMMON block. (We ignore here the additional possibility of passing
global variables by having one routine CONTAINed within the other.) There are
two good ways of handling the dynamical allocation in a MODULE. Method 1 uses
an allocatable array:

```fortran
MODULE a
  REAL(SP), DIMENSION(:), ALLOCATABLE :: x
END MODULE a

SUBROUTINE b(y)
  USE a
  REAL(SP), DIMENSION(:) :: y
  ...
  allocate(x(size(y)))
  ...
  [other routines using x called here] ...
END SUBROUTINE b
```

Here the global variable x gets assigned storage in subroutine b (in this case,
the same as the length of y). The length of y is of course defined in the procedure
that calls b. The array x is made available to any other subroutine called by b by
including a USE a statement. The status of x can be checked with an allocated
inquiry function on entry into either b or the other subroutine if necessary. As
discussed above, you must be sure to deallocate x before returning from subroutine
b. If you want x to retain its values between calls to b, you add the SAVE attribute
to its declaration in a, and don't deallocate it on returning from b. (Alternatively,
you could put a USE a in your main program, but we consider that bug-prone, since
forgetting to do so can create all manner of difficult-to-diagnose havoc.) To avoid
allocating x more than once, you test it on entry into b:

```fortran
if (.not. allocated(x)) allocate(x(size(y)))
```

The second way to implement this type of global storage (Method 2) uses
a pointer:

```fortran
MODULE a
  REAL(SP), DIMENSION(:), POINTER :: x
END MODULE a

SUBROUTINE b(y)
  USE a
  REAL(SP), DIMENSION(:) :: y
  REAL(SP), DIMENSION(size(y)), TARGET :: xx
  ...
  x => xx
  ...
  [other routines using x called here] ...
END SUBROUTINE b
```

Here the automatic array xx gets its temporary storage automatically on entry
into b, and automatically gets deallocated on exit from b. [M&R, §6.4] The global
pointer x can access this storage in any routine with a USE a that is called by b.
You can check that things are in order in such a called routine by testing x with
associated. If you are going to use x for some other purpose as well, you should
nullify it on leaving b so that it doesn't have undefined status. Note that this
implementation does not allow values to be saved between calls: You can't SAVE
automatic arrays — that's not what they're for. You would have to SAVE x in
the module, and allocate it in the subroutine instead of pointing it to a suitable
automatic array. But this is essentially Method 1 with the added complication of using
a pointer, so Method 1 is simpler when you want to save values. When you don’t
need to save values between calls, we lean towards Method 2 over Method 1 because we like the automatic allocation and deallocation, but either method works fine.

An example of Method 1 (allocatable array) is in routine rkdumb on page 1297. An example of Method 1 with SAVE is in routine pwtset on page 1265. Method 2 (pointer) shows up in routines newt (p. 1196), broydn (p. 1199), and fitexy (p. 1286). A variation is shown in routines linmin (p. 1211) and dlinmin (p. 1212): When the array that needs to be shared is an argument of one of the routines, Method 2 is better.

An extension of these ideas occurs if we allocate some storage for an array initially, but then might need to increase the size of the array later without losing the already-stored values. The function reallocate in our utility module nrutil will handle this for you, but it expects a pointer argument as in Method 2. Since no automatic arrays are used, you are free to SAVE the pointer if necessary. Here is a simple example of how to use reallocate to create a workspace array that is local to a subroutine:

```fortran
SUBROUTINE a
USE nrutil, ONLY : reallocate
REAL(SP), DIMENSION(:), POINTER, SAVE :: wksp
LOGICAL(LGT), SAVE :: init=.true.
if (init) then
  init=.false.
  nullify(wksp)
  wksp=>reallocate(wksp,100)
end if
... if (nterm > size(wksp)) wksp=>reallocate(wksp,2*size(wksp))
... END SUBROUTINE a
```

Here the workspace is initially allocated a size of 100. If the number of elements used (nterm) ever exceeds the size of the workspace, the workspace is doubled. (In a realistic example, one would of course check that the doubled size is in fact big enough.) Fortran 90 experts can note that the SAVE on init is not strictly necessary: Any local variable that is initialized is automatically saved. [M&R, §7.5]

You can find similar examples of reallocate (with some further discussion) in eulsum (p. 1070), hufenc (p. 1348), and arcode (p. 1350). Examples of reallocate used with global variables in modules are in odeint (p. 1300) and ran_state (p. 1144).

Another situation where we have to use pointers and not allocatable arrays is when the storage is required for components of a derived type, which are not allowed to have the allocatable attribute. Examples are in hufmak (p. 1346) and arcmak (p. 1349).

Turning away from issues relating to global variables, we now consider several other important programming situations that are nicely handled with pointers. The first case is when we want a subroutine to return an array whose size is not known in advance. Since dummy arguments are not allocatable, we must use a pointer. Here is the basic construction:

```fortran
SUBROUTINE a(x, nx)
REAL(SP), DIMENSION(:), POINTER :: x
INTEGER(I4B), INTENT(OUT) :: nx
LOGICAL(LGT), SAVE :: init=.true.
if (init) then
```

...
Since the length of \( x \) can be found from \( \text{size}(x) \), it is not absolutely necessary to pass \( nx \) as an argument. Note the use of the initial logic to avoid memory leaks. If a higher-level subroutine wants to recover the memory associated with \( x \) from the last call to \text{SUBROUTINE a}, it can do so by first deallocating it, and then nullifying the pointer. Examples of this structure are in \text{zbrak} (p. 1184), \text{period} (p. 1258), and \text{fasper} (p. 1259). A related situation is where we want a function to return an array whose size is not predetermined, such as in \text{voltra} (p. 1326). The discussion of \text{voltra} also explains the potential pitfalls of functions returning pointers to dynamically allocated arrays.

A final useful pointer construction enables us to set up a data structure that is essentially an array of arrays, independently allocatable on each part. We are not allowed to declare an array of pointers in Fortran 90, but we can do this indirectly by defining a derived type that consists of a pointer to the appropriate kind of array. [M&R, §6.11] We can then define a variable that is an allocatable array of the new type. For example,

```fortran
TYPE ptr_to_arr
  REAL(SP), DIMENSION(:), POINTER :: arr
END TYPE

TYPE(ptr_to_arr), DIMENSION(:), ALLOCATABLE :: x
...
allocate(x(n))
...
do i=1,n
  allocate(x(i)%arr(m))
end do
```

sets up a set \( x \) of \( n \) arrays of length \( m \). See also the example in \text{mglin} (p. 1334).

There is a potential problem with dynamical memory allocation that we should mention. The Fortran 90 standard does not require that the compiler perform “garbage collection,” that is, it is not required to recover deallocated memory into nice contiguous pieces for reuse. If you enter and exit a subroutine many times, and each time a large chunk of memory gets allocated and deallocated, you could run out of memory with a “dumb” compiler. You can often alleviate the problem by deallocating variables in the reverse order that you allocated them. This tends to keep a large contiguous piece of memory free at the top of the heap.

**Scope, Visibility, and Data Hiding**

An important principle of good programming practice is *modularization*, the idea that different parts of a program should be insulated from each other as much as possible. An important subcase of modularization is *data hiding*, the principle that actions carried out on variables in one part of the code should not be able to
affect the values of variables in other parts of the code. When it is necessary for one “island” of code to communicate with another, the communication should be through a well-defined interface that makes it obvious exactly what communication is taking place, and prevents any other interchange from occurring. Otherwise, different sections of code should not have access to variables that they don’t need.

The concept of data hiding extends not only to variables, but also to the names of procedures that manipulate the variables: A program for screen graphics might give the user access to a routine for drawing a circle, but it might “hide” the names (and methods of operation) of the primitive routines used for calculating the coordinates of the points on the circumference. Besides producing code that is easier to understand and to modify, data hiding prevents unintended side effects from producing hard-to-find errors.

In Fortran, the principal language construction that effects data hiding is the use of subroutines. If all subprograms were restricted to have no more than ten executable statements per routine, and to communicate between routines only by an explicit list of arguments, the number of programming errors might be greatly reduced! Unfortunately few tasks can be easily coded in this style. For this and other reasons, we think that too much procedurization is a bad thing; one wants to find the right amount. Fortunately Fortran 90 provides several additional tools to help with data hiding.

Global variables and routine names are important, but potentially dangerous, things. In Fortran 90, global variables are typically encapsulated in modules. Access is granted only to routines with an appropriate USE statement, and can be restricted to specific identifiers by the ONLY option. [M&R, §7.10] In addition, variable and routine names within the module can be designated as PUBLIC or PRIVATE (see, e.g., quad3d on p. 1065). [M&R, §7.6]

The other way global variables get communicated is by having one routine CONTAINed within another. [M&R, §5.6] This usage is potentially lethal, however, because all the outer routine’s variables are visible to the inner routine. You can try to control the problem somewhat by passing some variables back and forth as arguments of the inner routine, but that still doesn’t prevent inadvertent side effects. (The most common, and most stupid, is inadvertent reuse of variables named i or j in the CONTAINed routine.) Also, a long list of arguments reduces the convenience of using an internal routine in the first place. We advise that internal subprograms be used with caution, and only to carry out simple tasks.

There are some good ways to use CONTAINS, however. Several of our recipes have the following structure: A principal routine is invoked with several arguments. It calls a subsidiary routine, which needs to know some of the principal routine’s arguments, some global variables, and some values communicated directly as arguments to the subsidiary routine. In Fortran 77, we have usually coded this by passing the global variables in a COMMON block and all other variables as arguments to the subsidiary routine. If necessary, we copied the arguments of the primary routine before passing them to the subsidiary routine. In Fortran 90, there is a more elegant way of accomplishing this, as follows:

```fortran
SUBROUTINE recipe(arg)
REAL(SP) :: arg
REAL(SP) :: global_var
call recipe_private
CONTAINS
```
SUBROUTINE recipe_private
...
call subsidiary(local_arg)
...
END SUBROUTINE recipe_private

SUBROUTINE subsidiary(local_arg)
...
END SUBROUTINE subsidiary

END SUBROUTINE recipe

Notice that the principal routine (recipe) has practically nothing in it — only declarations of variables intended to be visible to the subsidiary routine (subsidiary). All the real work of recipe is done in recipe_private. This latter routine has visibility on all of recipe's variables, while any additional variables that recipe_private defines are not visible to subsidiary — which is the whole purpose of this way of organizing things. Obviously arg and global_var can be much more general data types than the example shown here, including function names. For examples of this construction, see amoeba (p. 1208), amebsa (p. 1222), mrqmin (p. 1292), and medfit (p. 1294).

Recursion

A subprogram is recursive if it calls itself. While forbidden in Fortran 77, recursion is allowed in Fortran 90. [M&R, §5.16–§5.17] You must supply the keyword RECURSIVE in front of the FUNCTION or SUBROUTINE keyword. In addition, if a FUNCTION calls itself directly, as opposed to calling another subprogram that in turn calls it, you must supply a variable to hold the result with the RESULT keyword. Typical syntax for this case is:

```
RECURSIVE FUNCTION f(x) RESULT(g)
REAL(SP) :: x,g
if ...
g=...
else
  g=f(...)
end if
END FUNCTION f
```

When a function calls itself directly, as in this example, there always has to be a "base case" that does not call the function; otherwise the recursion never terminates. We have indicated this schematically with the if...else...end if structure.

On serial machines we tend to avoid recursive implementations because of the additional overhead they incur at execution time. Occasionally there are algorithms for which the recursion overhead is relatively small, and the recursive implementation is simpler than an iterative version. Examples in this book are quad_3d (p. 1065), miser (p. 1164), and mglin (p. 1334). Recursion is much more important when parallelization is the goal. We will encounter in Chapter 22 numerous examples of algorithms that can be parallelized with recursion.

SAVE Usage Style

A quirk of Fortran 90 is that any variable with initial values acquires the SAVE attribute automatically. [M&R, §7.5 and §7.9] As a help to understanding
an algorithm, we have elected to put an explicit SAVE on all variables that really
do need to retain their values between calls to a routine. We do this even if it is
redundant because the variables are initialized. Note that we generally prefer to
assign initial values with initialization expressions rather than with DATA statements.
We reserve DATA statements for cases where it is convenient to use the repeat count
feature to set multiple occurrences of a value, or when binary, octal, or hexadecimal
constants are used. [M&R, §2.6.1]

**Named Control Structures**

Fortran 90 allows control structures such as do loops and if blocks to be
named. [M&R, §4.3–§4.5] Typical syntax is

```fortran
name: do i=1,n
...  
end do name
```

One use of naming control structures is to improve readability of the code,
especially when there are many levels of nested loops and if blocks. A more
important use is to allow exit and cycle statements, which normally refer to the
innermost do loop in which they are contained, to transfer execution to the end
of some outer loop. This is effected by adding the name of the outer loop to the
statement: exit name or cycle name.

There is great potential for misuse with named control structures, since they
share some features of the much-maligned goto. We recommend that you use them
sparingly. For a good example of their use, contrast the Fortran 77 version of simplx
with the Fortran 90 version on p. 1216.

CITED REFERENCES AND FURTHER READING:

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**21.6 And Coming Soon: Fortran 95**

One of the more positive effects of Fortran 90’s long gestation period has been
the general recognition, both by the X3J3 committee and by the community at large,
that Fortran needs to evolve over time. Indeed, as we write, the process of bringing
forth a minor, but by no means insignificant, updating of Fortran 90 — named
Fortran 95 — is well under way.

Fortran 95 will differ from Fortran 90 in about a dozen features, only a handful
of which are of any importance to this book. Generally these are extensions that
will make programming, especially parallel programming, easier. In this section we
give a summary of the anticipated language changes. In §22.1 and §22.5 we will
comment further on the implications of Fortran 95 to some parallel programming
tasks; in §23.7 we comment on what differences Fortran 95 will make to our nrutil11
utility functions.

No programs in Chapters B1 through B20 of this book edition use any Fortran
95 extensions.
FORALL Statements and Blocks

Fortran 95 introduces a new forall control structure, somewhat akin to the where construct, but allowing for greater flexibility. It is something like a do-loop, but with the proviso that the indices looped over are allowed to be done in any order (ideally, in parallel). The forall construction comes in both single-statement and block variants. Instead of using the do-loop’s comma-separated triplets of lowervalue, upper-value, and increment, it borrows its syntax from the colon-separated form of array sections. Some examples will give you the idea.

Here is a simple example that could alternatively be done with Fortran 90’s array sections and transpose intrinsic:

```
forall (i=1:20, j=1:10:2) x(i,j)=y(j,i)
```

The block form allows more than one executable statement:

```
forall (i=1:20, j=1:10:2)
x(i,j)=y(j,i)
z(i,j)=y(i,j)**2
end forall
```

Here is an example that cannot be done with Fortran 90 array sections:

```
forall (i=1:20, j=1:20) a(i,j)=3*i+j**2
```

forall statements can also take optional masks that restrict their action to a subset of the loop index combinations:

```
forall (i=1:100, j=1:100, (i>=j .and. x(i,j)/=0.0) ) x(i,j)=1.0/x(i,j)
```

forall constructions can be nested, or nested inside where blocks, or have where constructions inside them. An additional new feature in Fortran 95 is that where blocks can themselves be nested.

PURE Procedures

Because the inside iteration of a forall block can be done in any order, or in parallel, there is a logical difficulty in allowing functions or subroutines inside such blocks: If the function or subroutine has side effects (that is, if it changes any data elsewhere in the machine, or in its own saved variables) then the result of a forall calculation could depend on the order in which the iterations happen to be done. This can’t be tolerated, of course; hence a new PURE attribute for subprograms.

While the exact stipulations are somewhat technical, the basic idea is that if you declare a function or subroutine as PURE, with a syntax like,

```
PURE FUNCTION myfunc(x,y,z)
```

or

```
PURE SUBROUTINE mysub(x,y,z)
```

then you are guaranteeing to the compiler (and it will enforce) that the only values changed by mysub or myfunc are returned function values, subroutine arguments with the INTENT(OUT) attribute, and automatic (scratch) variables within the procedure.

You can then use your pure procedures within forall constructions. Pure functions are also allowed in some specification statements.
ELEMENTAL Procedures

Fortran 95 removes Fortran 90’s nagging restriction that only intrinsic functions are elemental. The way this works is that you write a pure procedure that operates on scalar values, but include the attribute ELEMENTAL (which automatically implies PURE). Then, as long as the function has an explicit interface in the referencing program, you can call it with any shape of argument, and it will act elementally. Here’s an example:

```fortran
ELEMENTAL FUNCTION myfunc(x,y,z)
REAL :: x,y,z,myfunc
...
myfunc = ...
END
```

In a program with an explicit interface for myfunc you could now have

```fortran
REAL, DIMENSION(10,20) :: x,y,z,w
...
w=myfunc(x,y,z)
```

Pointer and Allocatable Improvements

Fortran 95, unlike Fortran 90, requires that any allocatable variables (except those with SAVE attributes) that are allocated within a subprogram be automatically deallocated by the compiler when the subprogram is exited. This will remove Fortran 90’s “undefined allocation status” bugaboo.

Fortran 95 also provides a method for pointer variables to be born with disassociated association status, instead of the default (and often inconvenient) “undefined” status. The syntax is to add an initializing `$\Rightarrow$ NULL()` to the declaration, as:

```fortran
REAL, DIMENSION(:,::), POINTER :: mypoint $\Rightarrow$ NULL()
```

This does not, however, eliminate the possibility of undefined association status, because you have to remember to use the null initializer if want your pointer to be disassociated.

Some Other Fortran 95 Features

In Fortran 95, `maxloc` and `minloc` have the additional optional argument DIM, which causes them to act on all one-dimensional sections that span through the named dimension. This provides a means for getting the locations of the values returned by the corresponding functions `maxval` and `minval` in the case that their DIM argument is present.

The `sign` intrinsic can now distinguish a negative from a positive real zero value: `sign(2.0,-0.0)` is `-2.0`.

There is a new intrinsic subroutine `cpu_time(time)` that returns as a real value `time` a process’s elapsed CPU time.

There are some minor changes in the namelist facility, in defining minimum field widths for the `I, B, O, Z,` and `F` edit descriptors, and in resolving minor conflicts with some other standards.
Chapter 22. Introduction to Parallel Programming

22.0 Why Think Parallel?

In recent years we Numerical Recipes authors have increasingly become convinced that a certain revolution, cryptically denoted by the words "parallel programming," is about to burst forth from its gestation and adolescence in the community of supercomputer users, and become the mainstream methodology for all computing.

Let's review the past: Take a screwdriver and open up the computer (workstation or PC) that sits on your desk. (Don't blame us if this voids your warranty; and be sure to unplug it first!) Count the integrated circuits — just the bigger ones, with more than a million gates (transistors). As we write, in 1995, even lowly memory chips have one or four million gates, and this number will increase rapidly in coming years. You'll probably count at least dozens, and often hundreds, of such chips in your computer.

Next ask, how many of these chips are CPUs? That is, how many implement von Neumann processors capable of executing arbitrary, stored program code? For most computers, in 1995, the answer is: about one. A significant number of computers do have secondary processors that offload input-output and/or video functions. So, two or three is often a more accurate answer, but only one is usually under the user's direct control.

Why do our desktop computers have dozens or hundreds of memory chips, but most often only one (user-accessible) CPU? Do CPU chips intrinsically cost more to manufacture? No. Are CPU chips more expensive than memory chips? Yes, primarily because fixed development and design costs must be distributed over a smaller number of units sold. We have been in a kind of economic equilibrium: CPU's are relatively expensive because there is only one per computer; and there is only one per computer, because they are relatively expensive.

Stabilizing this equilibrium has been the fact that there has been no standard, or widely taught, methodology for parallel programming. Except for the special case of scientific computing on supercomputers (where large problems often have a regular or geometric character), it is not too much of an exaggeration to say that nobody really knows how to program multiprocessor machines. Symmetric multiprocessor
operating systems, for example, have been very slow in developing; and efficient, parallel methodologies for query-serving on large databases are even now a subject of continuing research.

However, things are now changing. We consider it an easy prognostication that, by the first years of the new century, the typical desktop computer will have 4 to 8 user-accessible CPUs; ten years after that, the typical number will be between 16 and 512. It is not coincidence that these numbers are characteristic of supercomputers (including some quite different architectures) in 1995. The rough rule of ten years’ lag from supercomputer to desktop has held firm for quite some time now.

Scientists and engineers have the advantage that techniques for parallel computation in their disciplines have already been developed. With multiprocessor workstations right around the corner, we think that now is the right time for scientists and engineers who use computers to start thinking parallel. We don’t mean that you should put an axe through the screen of your fast serial (single-CPU) workstation. We do mean, however, that you should start programming somewhat differently on that workstation, indeed, start thinking a bit differently about the way that you approach numerical problems in general.

In this volume of Numerical Recipes in Fortran, our pedagogical goal is to show you that there are conceptual and practical benefits in parallel thinking, even if you are using a serial machine today. These benefits include conciseness and clarity of code, reusability of code in wider contexts, and (not insignificantly) increased portability of code to today’s parallel supercomputers. Of course, on parallel machines, either supercomputers today or desktop machines tomorrow, the benefits of thinking parallel are much more tangible: They translate into significant improvements in efficiency and computational capability.

**Thinking Parallel with Fortran 90**

Until very recently, a strong inhibition to thinking parallel was the lack of any standard, architecture-independent, computer language in which to think. That has changed with the finalization of the Fortran 90 language standard, and with the availability of good, optimizing Fortran 90 compilers on a variety of platforms.

There is a significant body of opinion (with which we, however, disagree) that there is no such thing as architecture-independent parallel programming. Proponents of this view, who are generally committed wizards at programming on one or another particular architecture, point to the fact that algorithms that are optimized to one architecture can run hundreds of times more slowly on other architectures. And, they are correct!

Our opposing point of view is one of pragmatism. We think that it is not hard to learn, in a general way, what kinds of architectures are in general use, and what kinds of parallel constructions work well (or poorly) on each kind. With this knowledge (much of which we hope to develop in this book) the user can, we think, write good, general-purpose parallel code that works on a variety of architectures — including, importantly, on purely serial machines. Equally important, the user will be aware of when certain parts of a code can be significantly improved on some, but not other, architectures.

Fortran 90 is a good test-bench for this point of view. It is not the perfect language for parallel programming. But it is a language, and it is the only
cross-platform standard language now available. The committee that developed the language between 1978 and 1991 (known technically as X3J3) had strong representation from both a traditional “vectorization” viewpoint (e.g., from the Cray XMP and YMP series of computers), and also from the “data parallel” or “SIMD” viewpoints of parallel machines like the CM-2 and CM-5 from Thinking Machines, Inc. Language compromises were made, and a few (in our view) almost essential features were left out (see §22.5). But, by and large, the necessary tools are there: If you learn to think parallel in Fortran 90, you will easily be able to transfer the skill to future parallel standards, whether they are Fortran-based, C-based, or other.

CITED REFERENCES AND FURTHER READING:

### 22.1 Fortran 90 Data Parallelism: Arrays and Intrinsics

The underlying model for parallel computation in Fortran 90 is *data parallelism*, implemented by the use of arrays of data, and by the provision of operations and intrinsic functions that act on those arrays in parallel, in a manner optimized by the compiler for each particular hardware architecture. We will not try to draw a fine definitional distinction between “data parallelism” and so-called SIMD (single instruction multiple data) programming. For our purposes the two terms mean about the same thing: The programmer writes a single operation, “+” say, and the compiler causes it to be carried out on multiple pieces of data in as parallel a manner as the underlying hardware allows.

Any kind of parallel computing that is not SIMD is generally called MIMD (multiple instruction multiple data). A parallel programming language with MIMD features might allow, for example, several different subroutines—acting on different parts of the data—to be called into execution simultaneously. Fortran 90 has few, if any, MIMD constructions. A Fortran 90 compiler might, on some machines, execute MIMD code in implementing some Fortran 90 intrinsic functions (*pack* or *unpack*, e.g.), but this will be hidden from the Fortran 90 user. Some extensions of Fortran 90, like HPF, do implement MIMD features explicitly; but we will not consider these in this book. Fortran 95’s *forall* and *PURE* extensions (see §21.6) will allow some significantly greater access to MIMD features (see §22.5).

### Array Parallel Operations

We have already met the most basic, and most important, parallel facility of Fortran 90, namely, the ability to use whole arrays in expressions and assignments, with the indicated operations being effected in parallel across the array. Suppose, for example, we have the two-dimensional matrices *a*, *b*, and *c*,

```fortran
REAL, DIMENSION(30,30) :: a, b, c
```
Then, instead of the serial construction,

\[
do \ j=1,30 \\
\ do \ k=1,30 \\
\ \hspace{1cm} c(j,k)=a(j,k)+b(j,k) \\
\ end \ do \\
end do
\]

which is of course perfectly valid Fortran 90 code, we can simply write

\[c=a+b\]

The compiler deduces from the declaration statement that \(a\), \(b\), and \(c\) are matrices, and what their bounding dimensions are.

Let us dwell for a moment on the conceptual differences between the serial code and parallel code for the above matrix addition. Although one is perhaps used to seeing the nested do-loops as simply an idiom for “do-the-enclosed-on-all-components,” it in fact, according to the rules of Fortran, specifies a very particular time-ordering for the desired operations. The matrix elements are added by rows, in order \((j=1,30)\), and within each row, by columns, in order \((k=1,30)\).

In fact, the serial code above overspecifies the desired task, since it is guaranteed by the laws of mathematics that the order in which the element operations are done is of no possible relevance. Over the 50 year lifetime of serial von Neuman computers, we programmers have been brainwashed to break up all problems into single executable streams in the time dimension only. Indeed, the major design problem for supercomputer compilers for the last 20 years has been to undo such serial constructions and recover the underlying “parallel thoughts,” for execution in vector or parallel processors. Now, rather than taking this expensive detour into and out of serial-land, we are asked simply to say what we mean in the first place, \(c=a+b\).

The essence of parallel programming is not to force “into the time dimension” (i.e., to serialize) operations that naturally extend across a span of data, that is, “in the space dimension.” If it were not for 50-year-old collective habits, and the languages designed to support them, parallel programming would probably strike us as more natural than its serial counterpart.

**Broadcasts and Dimensional Expansion: SSP vs. MMP**

We have previously mentioned the Fortran 90 rule that a scalar variable is conformable with any shape array. Thus, we can implement a calculation such as

\[y_i = x_i + s, \quad i = 1, \ldots, n\]  \hfill (22.1.1)

with code like

\[y=x+s\]

where we of course assume previous declarations like

\[
\text{REAL(SP)} :: s \\
\text{REAL(SP), DIMENSION(n)} :: x,y
\]

with \(n\) a compile-time constant or dummy argument. (Hereafter, we will omit the declarations in examples that are this simple.)

This seemingly simple construction actually hides an important underlying parallel capability, namely, that of broadcast. The sums in \(y=x+s\) are done in parallel
on different CPUs, each CPU accessing different components of x and y. Yet, they all must access the same scalar value s. If the hardware has local memory for each CPU, the value of s must be replicated and transferred to each CPU’s local memory. On the other hand, if the hardware implements a single, global memory space, it is vital to do something that mitigates the traffic jam potentially caused by all the CPUs trying to access the same memory location at the same time. (We will use the term “broadcast” to refer equally to both cases.) Although hidden from the user, Fortran 90’s ability to do broadcasts is an essential feature of it as a parallel language.

Broadcasts can be more complicated than the above simple example. Consider, for example, the calculation

\[ w_i = \sum_{j=1}^{n} |x_i + x_j|, \quad i = 1, \ldots, n \]  

Here, we are doing \( n^2 \) operations: For each of \( n \) values of \( i \) there is a sum over \( n \) values of \( j \).

Serial code for this calculation might be

```fortran
  do i=1,n
    w(i)=0.
    do j=1,n
      w(i)=w(i)+abs(x(i)+x(j))
    end do
  end do
```

The obvious immediate parallelization in Fortran 90 uses the `sum` intrinsic function to eliminate the inner do-loop. This would be a suitable amount of parallelization for a small-scale parallel machine, with a few processors:

```fortran
  do i=1,n
    w(i)=sum(abs(x(i)+x))
  end do
```

Notice that the conformability rule implies that a new value of \( x(i) \), a scalar, is being broadcast to all the processors involved in the \( abs \) and \( sum \), with each iteration of the loop over \( i \).

What about the outer do-loop? Do we need, or want, to eliminate it, too? That depends on the architecture of your computer, and on the tradeoff between time and memory in your problem (a common feature of all computing, no less so parallel computing). Here is an implementation that is free of all do-loops, in principle capable of being executed in a small number (independent of \( n \)) of parallel operations:

```fortran
  REAL(SP), DIMENSION(n,n) :: a
  
  a = spread(x,dim=2,ncopies=n)+spread(x,dim=1,ncopies=n)
  w = sum(abs(a),dim=1)
```

This is an example of what we call *dimensional expansion*, as implemented by the `spread` intrinsic. Although the above may strike you initially as quite a cryptic construction, it is easy to learn to read it. In the first assignment line, a matrix is constructed with all possible values of \( x(i) + x(j) \). In the second assignment line, this matrix is collapsed back to a vector by applying the sum operation to the absolute value of its elements, across one of its dimensions.
More explicitly, the first line creates a matrix \( a \) by adding two matrices each constructed via \texttt{spread}. In \texttt{spread}, the \texttt{dim} argument specifies which argument is duplicated, so that the first term \textit{varies} across its first (row) dimension, and vice versa for the second term:

\[
\begin{pmatrix}
  x_1 & x_1 & x_1 & \cdots \\
  x_2 & x_2 & x_2 & \cdots \\
  x_3 & x_3 & x_3 & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix} + \begin{pmatrix}
  x_1 & x_2 & x_3 & \cdots \\
  x_1 & x_2 & x_3 & \cdots \\
  x_1 & x_2 & x_3 & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Since equation (22.1.2) above is symmetric in \( i \) and \( j \), it doesn’t really matter what value of \texttt{dim} we put in the \texttt{sum} construction, but the value \texttt{dim=1} corresponds to summing across the rows, that is, down each column of equation (22.1.3).

Be sure that you understand that the \texttt{spread} construction changed an \( O(n) \) memory requirement into an \( O(n^2) \) one! If your values of \( n \) are large, this is an impossible burden, and the previous implementation with a single do-loop remains the only practical one. On the other hand, if you are working on a massively parallel machine, whose number of processors is comparable to \( n^2 \) (or at least much larger than \( n \)), then the \texttt{spread} construction, and the underlying broadcast capability that it invokes, leads to a big win: All \( n^2 \) operations can be done in parallel. This distinction between small-scale parallel machines — which we will hereafter refer to as \textit{SSP machines} — and massively multiprocessor machines — which we will refer to as \textit{MMP machines} — is an important one. A main goal of parallelism is to saturate the available number of processors, and algorithms for doing so are often different in the SSP and MMP opposite limits. Dimensional expansion is one method for saturating processors in the MMP case.

\textbf{Masks and “Index Loss”}

An instructive extension of the above example is the following case of a product that omits one term (the diagonal one):

\[
\begin{align*}
  w_i &= \prod_{j=1}^{n} (x_j - x_i), \quad i = 1, \ldots, n \\
\end{align*}
\]

Formulas like equation (22.1.4) frequently occur in the context of interpolation, where all the \( x_i \)'s are known to be distinct, so let us for the moment assume that this is the case.

Serial code for equation (22.1.4) could be

```
  do i=1,n
    w(i)=1.0_sp
    do j=1,n
      if (j /= i) \( w(i)=w(i)\times(x(j)-x(i)) \)
    end do
  end do
```

Parallel code for SSP machines, or for large enough \( n \) on MMP machines, could be

```
  do i=1,n
    w(i)=1.0_sp
    do j=1,n
      if (j /= i) w(i)=w(i)\times(x(j)-x(i))
    end do
  end do
```
do i=1,n
  w(i)=product( x-x(i), mask=(x/=x(i)) )
end do

Here, the mask argument in the product intrinsic function causes the diagonal term to be omitted from the product, as we desire. There are some features of this code, however, that bear commenting on.

First, notice that, according to the rules of conformability, the expression $x/=x(i)$ broadcasts the scalar $x(i)$ and generates a logical array of length $n$, suitable for use as a mask in the product intrinsic. It is quite common in Fortran 90 to generate masks “on the fly” in this way, particularly if the mask is to be used only once.

Second, notice that the $j$ index has disappeared completely. It is now implicit in the two occurrences of $x$ (equivalent to $x(1:n)$) on the right-hand side. With the disappearance of the $j$ index, we also lose the ability to do the test on $i$ and $j$, but must use, in essence, $x(i)$ and $x(j)$ instead! That is a very general feature in Fortran 90: when an operation is done in parallel across an array, there is no associated index available within the operation. This “index loss,” as we will see in later discussion, can sometimes be quite an annoyance.

A language construction present in CM [Connection Machine] Fortran, the so-called forall, which would have allowed access to an associated index in many cases, was eliminated from Fortran 90 by the X3J3 committee, in a controversial decision. Such a construction will come into the language in Fortran 95.

What about code for an MMP machine, where we are willing to use dimensional expansion to achieve greater parallelism? Here, we can write,

$$a = \text{spread}(x, \text{dim}=2, \text{ncopies}=n) - \text{spread}(x, \text{dim}=1, \text{ncopies}=n)$$

$$w = \text{product}(a, \text{dim}=1, \text{mask}=(a /= 0.))$$

This time it does matter that the value of dim in the product intrinsic is 1 rather than 2. If you write out the analog of equation (22.1.3) for the present example, you’ll see that the above fragment is the right way around. The problem of index loss is still with us: we have to construct a mask from the array $a$, not from its indices, both of which are now lost to us!

In most cases, there are workarounds (more, or less, awkward as they may be) for the problem of index loss. In the worst cases, which are quite rare, you have to create objects to hold, and thus bring back into play, the lost indices. For example,

INTEGER(I4B), DIMENSION(n) :: jj

jj = (/ (i,i=1,n) /)

do i=1,n
  w(i)=product( x-x(i), mask=(jj/=i) )
end do

Now the array $jj$ is filled with the “lost” $j$ index, so that it is available for use in the mask. A similar technique, involving spreads of $jj$, can be used in the above MMP code fragment, which used dimensional expansion. (Fortran 95’s forall construction will make index loss much less of a problem. See §21.6.)

Incidentally, the above Fortran 90 construction, $/ (i, i=1,n) /$, is called an array constructor with implied do list. For reasons to be explained in §22.2, we almost never use this construction, in most cases substituting a Numerical Recipes utility function for generating arithmetical progressions, which we call arth.
Interprocessor Communication Costs

It is both a blessing and a curse that Fortran 90 completely hides from the user the underlying machinery of interprocessor communication, that is, the way that data values computed by (or stored locally near) one CPU make their way to a different CPU that might need them next. The blessing is that, by and large, the Fortran 90 programmer need not be concerned with how this machinery works. If you write

\[
a(1:10,1:10) = b(1:10,1:10) + c(10:1:-1,10:1:-1)
\]

the required upside-down-and-backwards values of the array \(c\) are just there, no matter that a great deal of routing and switching may have taken place. An ancillary blessing is that this book, unlike so many other (more highly technical) books on parallel programming (see references below) need not be filled with complex and subtle discussions of CPU connectivity, topology, routing algorithms, and so on.

The curse is, just as you might expect, that the Fortran 90 programmer can’t control the interprocessor communication, even when it is desirable to do so. A few regular communication patterns are “known” to the compiler through Fortran 90 intrinsic functions, for example \(b=\text{transpose}(a)\). These, presumably, are done in an optimal way. However, many other regular patterns of communication, which might also allow highly optimized implementations, don’t have corresponding intrinsic functions. (An obvious example is the “butterfly” pattern of communication that occurs in fast Fourier transforms.) These, if coded in Fortran 90 by using general vector subscripts (e.g., \(b_{\text{arr}}=a_{\text{arr}(i_{\text{arr}})}\) or \(b_{\text{arr}(j_{\text{arr}})}=a_{\text{arr}}\), where \(i_{\text{arr}}\) and \(j_{\text{arr}}\) are integer arrays), lose all possibility of being optimized. The compiler can’t distinguish a communication step with regular structure from one with general structure, so it must assume the worst case, potentially resulting in very slow execution.

About the only thing a Fortran 90 programmer can do is to start with a general awareness of the kind of apparently parallel constructions that might be quite slow on his/her parallel machine, and then to refine that awareness by actual experience and experiment. Here is our list of constructions most likely to cause interprocessor communication bottlenecks:

- vector subscripts, like \(b_{\text{arr}}=a_{\text{arr}(i_{\text{arr}})}\) or \(b_{\text{arr}(j_{\text{arr}})}=a_{\text{arr}}\) (that is, general gather/scatter operations)
- the \text{pack} and \text{unpack} intrinsic functions
- mixing positive strides and negative strides in a single expression (as in the above \(b(1:10,1:10)+c(10:1:-1,10:1:-1)\))
- the \text{reshape} intrinsic when used with the \text{order} argument
- possibly, the \text{cshift} and \text{eoshift} extrinsics, especially for nonsmall values of the shift.

On the other hand, the fact is that these constructions are parallel, and are there for you to use. If the alternative to using them is strictly serial code, you should almost always give them a try.

Linear Algebra

You should be alert for opportunities to use combinations of the \text{matmul}, \text{spread}, and \text{dot_product} intrinsics to perform complicated linear algebra calculations. One useful intrinsic that is not provided in Fortran 90 is the \text{outer product}
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of two vectors,

\[ c_{ij} = a_i b_j \]  \hspace{1cm} (22.1.5)

We already know how to implement this (cf. equation 22.1.3):

\[ c = \text{spread}(a, \text{dim}=2, \text{ncopies}=\text{size}(b)) \ast \text{spread}(b, \text{dim}=1, \text{ncopies}=\text{size}(a)) \]

In fact, this operation occurs frequently enough to justify making it a utility function, \text{outerprod}, which we will do in Chapter 23. There we also define other “outer” operations between vectors, where the multiplication in the outer product is replaced by another binary operation, such as addition or division.

Here is an example of using these various functions: Many linear algebra routines require that a submatrix be updated according to a formula like

\[ a_{jk} = a_{jk} + b_i a_{ji} \sum_{p=i}^{m} a_{pi} a_{pk}, \quad j = i, \ldots, m, \quad k = l, \ldots, n \]  \hspace{1cm} (22.1.6)

where \( i, m, l, \) and \( n \) are fixed values. Using an array slice like \( a(:,i) \) to turn \( a_{pi} \) into a vector indexed by \( p \), we can code the sum with a \text{matmul}, yielding a vector indexed by \( k \):

\[ \text{temp}(l:n) = b(i) \ast \text{matmul}(a(i:m,i), a(i:m,l:n)) \]

Here we have also included the multiplication by \( b_i \), a scalar for fixed \( i \). The vector \( \text{temp} \), along with the vector \( a_{ji} = a(:,i) \), is then turned into a matrix by the \text{outerprod} utility and used to increment \( a_{jk} \):

\[ a(i:m,l:n) = a(i:m,l:n) \ast \text{outerprod}(a(i:m,i), \text{temp}(l:n)) \]

Sometimes the update formula is similar to (22.1.6), but with a slight permutation of the indices. Such cases can be coded as above if you are careful about the order of the quantities in the \text{matmul} and the \text{outerprod}.

CITED REFERENCES AND FURTHER READING:


22.2 Linear Recurrence and Related Calculations

We have already seen that Fortran 90’s array constructor with implied do list can be used to generate simple series of integers, like \((\{i, i=1, n\})\). Slightly more generally, one might want to generate an arithmetic progression, by the formula

\[ v_j = b + (j - 1) a, \quad j = 1, \ldots, n \]  

This is readily coded as

\[ v(1:n) = (/ (b+(j-1)*a, j=1,n) /) \]

Although it is concise, and valid, we don’t like this coding. The reason is that it violates the fundamental rule of “thinking parallel”: it turns a parallel operation across a data vector into a serial do-loop over the components of that vector. Yes, we know that the compiler might be smart enough to generate parallel code for implied do lists; but it also might not be smart enough, here or in more complicated examples.

Equation (22.2.1) is also the simplest example of a linear recurrence relation. It can be rewritten as

\[ v_1 = b, \quad v_j = v_{j-1} + a, \quad j = 2, \ldots, n \]  

In this form (assuming that, in more complicated cases, one doesn’t know an explicit solution like equation 22.2.1) one can’t write an explicit array constructor. Code like

\[ v(1) = b \\
 v(2:n) = (/ (v(j-1)+a, j=2,n) /) \]

is legal Fortran 90 syntax, but illegal semantics; it does not do the desired recurrence! (The rules of Fortran 90 require that all the components of \(v\) on the right-hand side be evaluated before any of the components on the left-hand side are set.) Yet, as we shall see, techniques for accomplishing the evaluation in parallel are available.

With this as our starting point, we now survey some particular tricks of the (parallel) trade.
Subvector Scaling: Arithmetic and Geometric Progressions

For explicit arithmetic progressions like equation (22.2.1), the simplest parallel technique is subvector scaling [1]. The idea is to work your way through the desired vector in larger and larger parallel chunks:

\[
\begin{align*}
    v_1 &= b \\
    v_2 &= b + a \\
    v_{3\ldots4} &= v_{1\ldots2} + 2a \\
    v_{5\ldots8} &= v_{1\ldots4} + 4a \\
    v_{9\ldots16} &= v_{1\ldots8} + 8a
\end{align*}
\] (22.2.3)

And so on, until you reach the length of your vector. (The last step will not necessarily go all the way to the next power of 2, therefore.) The powers of 2, times \(a\), can of course be obtained by successive doublings, rather than the explicit multiplications shown above.

You can see that subvector scaling requires about \(\log_2 n\) parallel steps to process a vector of length \(n\). Equally important for serial machines, or SSP machines, the scalar operation count for subvector scaling is no worse than entirely serial code: each new component \(v_i\) is produced by a single addition.

If addition is replaced by multiplication, the identical algorithm will produce geometric progressions, instead of arithmetic progressions. In Chapter 23, we will use subvector scaling to implement our utility functions \(\text{ar}th\) and \(\text{geop}\) for these two progressions. (You can then call one of these functions instead of recoding equation 22.2.3 every time you need it.)

Vector Reduction: Evaluation of Polynomials

Logically related to subvector scaling is the case where a calculation can be parallelized across a vector that shrinks by a factor of 2 in each iteration, until a desired scalar result is reached. A good example of this is the parallel evaluation of a polynomial [2]

\[
P(x) = \sum_{j=0}^{N} c_j x^j
\] (22.2.4)

For clarity we take the special case of \(N = 5\). Start with the vector of coefficients (imagining appended zeros, as shown):

\[
c_0, \ c_1, \ c_2, \ c_3, \ c_4, \ c_5, \ 0, \ldots
\]

Now, add the elements by pairs, multiplying the second of each pair by \(x\):

\[
c_0 + c_1 x, \ c_2 + c_3 x, \ c_4 + c_5 x, \ 0, \ldots
\]

Now, the same operation, but with the multiplier \(x^2\):

\[
(c_0 + c_1 x) + (c_2 + c_3 x)x^2, \ (c_4 + c_5 x) + (0)x^2, \ 0, \ldots
\]
22.2 Linear Recurrence and Related Calculations

And a final time, with multiplier $x^4$:

$$[(c_0 + c_1 x) + (c_2 + c_3 x)x^2] + [(c_4 + c_5 x) + (0)x^2]x^4, \quad 0, \ldots$$

We are left with a vector of (active) length 1, whose value is the desired polynomial evaluation. (You can see that the zeros are just a bookkeeping device for taking account of the case where the active subvector has odd length.) The key point is that the combining by pairs is a parallel operation at each stage.

As in subvector scaling, there are about $\log_2 n$ parallel stages. Also as in subvector scaling, our total operations count is only negligibly different from purely scalar code: We do one add and one multiply for each original coefficient $c_j$. The only extra operations are $\log_2 n$ successive squarings of $x$; but this comes with the extra benefit of better roundoff properties than the standard scalar coding. In Chapter 23 we use vector reduction to implement our utility function poly for polynomial evaluation.

Recursion Doubling: Linear Recurrence Relations

Please don’t confuse our use of the word “recurrence” (as in “recurrence relation,” “linear recurrence,” or equation 22.2.2) with the words “recursion” and “recursive,” which both refer to the idea of a subroutine calling itself to obtain an efficient or concise algorithm. There are ample grounds for confusion, because recursive algorithms are in fact a good way of obtaining parallel solutions to linear recurrence relations, as we shall now see!

Consider the general first order linear recurrence relation

$$u_j = a_j + b_j u_{j-1}, \quad j = 2, 3, \ldots, n \quad (22.2.5)$$

with initial value $u_1 = a_1$. On a serial machine, we evaluate such a recurrence with a simple do-loop. To parallelize the recurrence, we can employ the powerful general strategy of recursive doubling. Write down equation (22.2.5) for $2j$ and for $2j - 1$:

$$u_{2j} = a_{2j} + b_{2j-1} u_{2j-1} \quad (22.2.6)$$

$$u_{2j-1} = a_{2j-1} + b_{2j-2} u_{2j-2} \quad (22.2.7)$$

Substitute equation (22.2.7) in equation (22.2.6) to eliminate $u_{2j-1}$ and get

$$u_{2j} = (a_{2j} + a_{2j-1} b_{2j-1}) + (b_{2j-2} b_{2j-1}) u_{2j-2} \quad (22.2.8)$$

This is a new recurrence of the same form as (22.2.5) but over only the even $u_j$, and hence involving only $n/2$ terms. Clearly we can continue this process recursively, halving the number of terms in the recurrence at each stage, until we are left with a recurrence of length 1 or 2 that we can do explicitly. Each time we finish a subpart of the recursion, we fill in the odd terms in the recurrence, using equation (22.2.7). In practice, it’s even easier than it sounds. Turn to Chapter B5 to see a straightforward implementation of this algorithm as the recipe recur1.

On a machine with more processors than $n$, all the arithmetic at each stage of the recursion can be done simultaneously. Since there are of order $\log n$ stages in the
recursion, the execution time is \( O(\log n) \). The total number of operations carried out is of order \( n + n/2 + n/4 + \cdots = O(n) \), the same as for the obvious serial do-loop.

In the utility routines of Chapter 23, we will use recursive doubling to implement the routines \texttt{polyterm}, \texttt{cumsum}, and \texttt{cumprod}. We could use recursive doubling to implement parallel versions of \texttt{arth} and \texttt{geop} (arithmetic and geometric progressions), and \texttt{zrootsunity} (complex \( n \)th roots of unity), but these can be done slightly more efficiently by subvector scaling, as discussed above.

**Cyclic Reduction: Linear Recurrence Relations**

There is a variant of recursive doubling, called cyclic reduction, that can be implemented with a straightforward iteration loop, instead of a recursive procedure call. [3] Here we start by writing down the recurrence (22.2.5) for all adjacent terms \( u_j \) and \( u_{j-1} \) (not just the even ones, as before). Eliminating \( u_{j-1} \), just as in equation (22.2.8), gives

\[
  u_j = (a_j + a_{j-1}b_{j-1}) + (b_{j-2}b_{j-1})u_{j-2}
\]

which is a first order recurrence with new coefficients \( a'_j \) and \( b'_j \). Repeating this process gives successive formulas for \( u_j \) in terms of \( u_{j-2}, u_{j-4}, u_{j-8}, \ldots \). The procedure terminates when we reach \( u_{j-n} \) (for \( n \) a power of 2), which is zero for all \( j \). Thus the last step gives \( u_j \) equal to the last set of \( a'_j \)'s.

Here is a code fragment that implements cyclic reduction by direct iteration. The quantities \( a'_j \) are stored in the variable \( \texttt{recur1} \).

```plaintext
   recur1=a
   bb=b
   j=1
   do
     if (j >= n) exit
     recur1(j+1:n)=recur1(j+1:n)+bb(j:n-1)*recur1(1:n-j)
     bb(2*j:n-1)=bb(2*j:n-1)*bb(j:n-j-1)
     j=2*j
   enddo
```

In cyclic reduction the length of the vector \( u_j \) that is updated at each stage does not decrease by a factor of 2 at each stage, but rather only decreases from \( \sim n \) to \( \sim n/2 \) during all \( \log_2 n \) stages. Thus the total number of operations carried out is \( O(n \log n) \), as opposed to \( O(n) \) for recursive doubling. For a serial machine or SSP machine, therefore, cyclic reduction is rarely superior to recursive doubling. For an MMP machine, however, the issue is less clear cut, because the pattern of communication in cyclic reduction is quite different (and, for some parallel architectures, possibly more favorable) than that of recursive doubling.

**Second Order Recurrence Relations**

Consider the second order recurrence relation

\[
  y_j = a_j + b_{j-2}y_{j-1} + c_{j-2}y_{j-2}, \quad j = 3, 4, \ldots, n
\]

with initial values

\[
  y_1 = a_1, \quad y_2 = a_2
\]
Our labeling of subscripts is designed to make it easy to enter the coefficients in a computer program: You need to supply $a_1, \ldots, a_n, b_1, \ldots, b_{n-2}$, and $c_1, \ldots, c_{n-2}$. Rewrite the recurrence relation in the form (22.2.12)

$$
\begin{pmatrix}
y_j \\
y_{j+1}
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_{j+1} & b_{j-1} \end{pmatrix} \begin{pmatrix} y_{j-1} \\
y_j
\end{pmatrix}, \quad j = 2, \ldots, n - 1
$$

that is,

$$
u_j = a_j + b_{j-1} \cdot u_{j-1}, \quad j = 2, \ldots, n - 1
$$

(22.2.13)

where

$$u_j = \begin{pmatrix} y_j \\
y_{j+1}
\end{pmatrix}, \quad a_j = \begin{pmatrix} 0 \\
a_{j+1}
\end{pmatrix}, \quad b_{j-1} = \begin{pmatrix} 0 & 1 \\ c_{j-1} & b_{j-1} \end{pmatrix}, \quad j = 2, \ldots, n - 1
$$

(22.2.14)

and

$$u_1 = a_1 = \begin{pmatrix} y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix} a_1 \\
a_2
\end{pmatrix}
$$

(22.2.15)

This is a first order recurrence relation for the vectors $u_j$, and can be solved by the algorithm described above (and implemented in the recipe recur1). The only difference is that the multiplications are matrix multiplications with the $2 \times 2$ matrices $b_j$. After the first recursive call, the zeros in $a$ and $b$ are lost, so we have to write the routine for general two-dimensional vectors and matrices.

Note that this algorithm does not avoid the potential instability problems associated with second order recurrences that are discussed in §5.5 of Volume 1. Also note that the algorithm generalizes in the obvious way to higher-order recurrences: An $n$th order recurrence can be written as a first order recurrence involving $n$-dimensional vectors and matrices.

**Parallel Solution of Tridiagonal Systems**

Closely related to recurrence relations, recursive doubling, and cyclic reduction is the parallel solution of tridiagonal systems. Since Fortran 90 vectors “know their own size,” it is most logical to number the components of both the sub- and super-diagonals of the tridiagonal matrix from 1 to $N - 1$. Thus equation (2.4.1), here written in the special case of $N = 7$, becomes (blank elements denoting zero),

$$
\begin{bmatrix}
b_1 & c_1 & a_1 \\
b_2 & c_2 & b_2 \\
b_3 & c_3 \\
a_3 & b_4 & c_4 \\
a_4 & b_5 & c_5 \\
a_5 & b_6 & c_6 \\
a_6 & b_7
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7
\end{bmatrix}
= \begin{bmatrix}
r_1 \\
r_2 \\
r_3 \\
r_4 \\
r_5 \\
r_6 \\
r_7
\end{bmatrix}
$$

(22.2.16)

The basic idea for solving equation (22.2.16) on a parallel computer is to partition the problem into even and odd elements, recurse to solve the former, and
then solve the latter in parallel. Specifically, we first rewrite (22.2.16), by permuting its rows and columns, as

\[
\begin{bmatrix}
    b_1 & c_1 & & & & \\
    b_3 & a_2 & c_3 & & & \\
    b_5 & b_7 & a_4 & c_5 & & \\
    a_1 & c_2 & b_2 & & a_6 & \\
    a_3 & c_4 & b_4 & & & \\
    a_5 & c_6 & b_6 & & & \\
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_3 \\
    u_5 \\
    u_7 \\
    u_2 \\
    u_4 \\
    u_6 \\
\end{bmatrix}
= 
\begin{bmatrix}
    r_1 \\
    r_3 \\
    r_5 \\
    r_7 \\
    r_2 \\
    r_4 \\
    r_6 \\
\end{bmatrix}
\]

(22.2.17)

Now observe that, by row operations that subtract multiples of the first four rows from each of the last three rows, we can eliminate all nonzero elements in the lower-left quadrant. The price we pay is bringing some new elements into the lower-right quadrant, whose nonzero elements we now call \(x\)'s, \(y\)'s, and \(z\)'s. We call the modified right-hand sides \(q\). The transformed problem is now

\[
\begin{bmatrix}
    b_1 & c_1 & & & & & \chi_1 & \chi_2 & \chi_3 \\
    b_3 & a_2 & c_3 & & & & \chi_1 & \chi_2 & \chi_3 \\
    b_5 & b_7 & a_4 & c_5 & & & \chi_1 & \chi_2 & \chi_3 \\
    a_1 & c_2 & b_2 & & a_6 & & \chi_1 & \chi_2 & \chi_3 \\
    a_3 & c_4 & b_4 & & & a_6 & \chi_1 & \chi_2 & \chi_3 \\
    a_5 & c_6 & b_6 & & & & a_6 & \chi_1 & \chi_2 & \chi_3 \\
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_3 \\
    u_5 \\
    u_7 \\
    u_2 \\
    u_4 \\
    u_6 \\
\end{bmatrix}
= 
\begin{bmatrix}
    r_1 \\
    r_3 \\
    r_5 \\
    r_7 \\
    q_1 \\
    q_2 \\
    q_3 \\
\end{bmatrix}
\]

(22.2.18)

Notice that the last three rows form a new, smaller, tridiagonal problem, which we can solve simply by recursing! Once its solution is known, the first four rows can be solved by a simple, parallelizable, substitution. This algorithm is implemented in \texttt{tridag} in Chapter B2.

The above method is essentially cyclic reduction, but in the case of the tridiagonal problem, it does not “unwind” into a simple iteration; on the contrary, a recursive subroutine is required. For discussion of this and related methods for parallelizing tridiagonal systems, and references to the literature, see Hockney and Jesshope [3].

Recursive doubling can also be used to solve tridiagonal systems, the method requiring the parallel solution (as above) of both a first order recurrence and a second order recurrence [3,4]. For tridiagonal systems, however, cyclic reduction is usually more efficient than recursive doubling.

CITED REFERENCES AND FURTHER READING:
22.3 Parallel Synthetic Division and Related Algorithms

There are several techniques for parallelization that relate to synthetic division but that can also find application in wider contexts, as we shall see.

### Cumulants of a Polynomial

Suppose we have a polynomial

\[ P(x) = \sum_{j=0}^{N} c_j x^{N-j} \quad (22.3.1) \]

(Note that, here, the \( c_j \)'s are indexed from highest degree to lowest, the reverse of the usual convention.) Then we can define the *cumulants* of the polynomial to be partial sums that occur in the polynomial’s usual, serial evaluation,

\[
\begin{align*}
P_0 &= c_0 \\
P_1 &= c_0 x + c_1 \\
&\vdots \\
P_N &= c_0 x^N + \cdots + c_N = P(x) 
\end{align*}
\quad (22.3.2)
\]

Evidently, the cumulants satisfy a simple, linear first order recurrence relation,

\[ P_0 = c_0, \quad P_j = c_j + xP_{j-1}, \quad j = 2, \ldots, N \quad (22.3.3) \]

This is slightly simpler than the general first order recurrence, because the value of \( x \) does not depend on \( j \). We already know, from §22.2’s discussion of recursive doubling, how to parallelize equation (22.3.3) via a recursive subroutine. In Chapter 23, the utility routine \texttt{polyterm} will implement just such a procedure. An example of a routine that calls \texttt{polyterm} to evaluate a recurrence equivalent to equation (22.3.3) is \texttt{eulsum} in Chapter B5.

Notice that while we could use equation (22.3.3), parallelized by recursive doubling, simply to evaluate the polynomial \( P(x) = P_N \), this is likely somewhat slower than the alternative technique of vector reduction, also discussed in §22.2, and implemented in the utility function \texttt{poly}. Equation (22.3.3) should be saved for cases where the rest of the \( P_j \)'s (not just \( P_N \)) can be put to good use.

### Synthetic Division by a Monomial

We now show that evaluation of the cumulants of a polynomial is equivalent to synthetic division of the polynomial by a monomial, also called *deflation* (see §9.5 in Volume 1). To review briefly, and by example, here is a standard tableau from high school algebra for the (long) division of a polynomial \( 2x^3 - 7x^2 + x + 3 \) by the monomial factor \( x - 3 \).
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\[\begin{array}{c|ccc}
2x^2 & x & -2 \\
\hline
x-3 & | & 2x^3 & -7x^2 & +x & +3 \\
& & 2x^3 & -6x^2 & & \\
& -x^2 & +x & & \\
& -x^2 & +3x & & \\
& -2x & +3 & & \\
& -2x & +6 & & \\
& & & -3 \text{ (remainder)} \\
\end{array} \] 

Now, here is the same calculation written as a synthetic division, really the same procedure as tableau (22.3.4), but with unnecessary notational baggage omitted (and also a changed sign for the monomial’s constant, so that subtractions become additions):

\[\begin{array}{c|ccc}
3 & | & 6 & -3 & -6 \\
\hline
2 & -7 & +1 & +3 \\
2 & -1 & -2 & -3 \\
\end{array} \] 

(22.3.5)

If we substitute symbols for the above quantities with the correspondence

\[
x | \begin{array}{ccc} c_0 & c_1 & c_2 \\
P_0 & P_1 & P_2 & P_3 \\
\end{array} \]

then it is immediately clear that the \( P_j \)'s in equation (22.3.6) are simply the \( P_j \)'s of equation (22.3.3); the calculation is thus parallelizable by recursive doubling. In this context, the utility routine polyctor is used by the routine zroots in Chapter B9.

\textit{Repeated Synthetic Division}

It is well known from high-school algebra that repeated synthetic division of a polynomial yields, as the remainders that occur, first the value of the polynomial, next the value of its first derivative, and then (up to multiplication by the factorial of an integer) the values of higher derivatives.

If you want to parallelize the calculation of the value of a polynomial and one or two of its derivatives, it is not unreasonable to evaluate equation (22.3.3), parallelized by recursive doubling, two or three times. Our routine ddpoly in Chapter B5 is meant for such use, and it does just this, as does the routine laguer in Chapter B9.

There are other cases, however, for which you want to perform repeated synthetic division and “go all the way,” until only a constant remains. For example, this is the preferred way of “shifting a polynomial,” that is, evaluating the coefficients of a polynomial in a variable \( y \) that differs from the original variable \( x \) by an additive constant. (The recipe pcshft has this as its assigned task.) By way of example, consider the polynomial \( 3x^3 + x^2 + 4x + 7 \), and let us perform repeated synthetic division by a general monomial \( x - a \). The conventional calculation then proceeds according to the following tableau, reading it in conventional lexical order (left-to-right and top-to-bottom):
Here, each row (after the first) shows a synthetic division or, equivalently, evaluation of the cumulants of the polynomial whose coefficients are the preceding row. The results at the right edge of the rows are the values of the polynomial and (up to integer factorials) its three nonzero derivatives, or (equivalently, without factorials) coefficients of the shifted polynomial.

We could parallelize the calculation of each row of tableau (22.3.7) by recursive doubling. That is a lot of recursion, which incurs a nonnegligible overhead. A much better way of doing the calculation is to deform tableau (22.3.7) into the following equivalent tableau,

\[
\begin{array}{cccccc}
3 & 1 & 4 & 7 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
3 & \xrightarrow{a} & 3a + 1 & \xrightarrow{a} & 3a^2 + a + 4 & \xrightarrow{a} & 3a^3 + a^2 + 4a + 7 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
3 & \xrightarrow{a} & 6a + 1 & \xrightarrow{a} & 9a^2 + 2a + 4 \\
\downarrow & \downarrow \\
3 & \xrightarrow{a} & 9a + 1 \\
\downarrow \\
3 \\
\end{array}
\] (22.3.7)

Now each row explicitly depends on only the previous row (and the given first column), so the rows can be calculated in turn by an explicit parallel expression, with no recursive calls needed. An example of coding (22.3.8) in Fortran 90 can be found in the routine `pcshft` in Chapter B5. (It is also possible to eliminate most of the multiplications in (22.3.8), at the expense of a much smaller number of divisions. We have not done this because of the necessity for then treating all possible divisions by zero as special cases. See [1] for details and references.)

Actually, the deformation of (22.3.7) into (22.3.8) is the same trick as was used in Volume 1, p. 167, for evaluating a polynomial and its derivative simultaneously, also generalized in the Fortran 77 implementation of the routine `dDpol1y` (Chapter 5). In the Fortran 90 implementation of `dDpol1y` (Chapter B5) we don't use this trick, but instead use `polyterm`, because, there, we want to parallelize over the length of the polynomial, not over the number of desired derivatives.
Don’t confuse the cases of *iterated* synthetic division, discussed here, with the simpler case of doing many simultaneous synthetic divisions. In the latter case, you can simply implement equation (22.3.3) serially, exactly as written, but with each operation being data-parallel across your problem set. (This case occurs in our routine polcoe in Chapter B3.)

**Polynomial Coefficients from Roots**

A parallel calculation algorithmically very similar to (22.3.7) or (22.3.8) occurs when we want to find the coefficients of a polynomial $P(x)$ from its roots $r_1, \ldots, r_N$. For this, the tableau is

$$
\begin{array}{cc}
r_1 & \\
r_2: & r_1 + r_2 \\
r_3: & r_1 + r_2 + r_3 \\
\end{array}

\begin{array}{c}
r_1 r_2 \\
r_1 r_2 + r_3 (r_1 + r_2) \\
r_1 r_2 r_3
\end{array}
$$

(22.3.9)

As before, the rows are computed consecutively, from top to bottom. Each row is computed via a single parallel expression. Note that values moving on vertical arrows are simply added in, while values moving on diagonal arrows are multiplied by a new root before adding. Examples of coding (22.3.9) in Fortran 90 can be found in the routines vander (Chapter B2) and polcoe (Chapter B3).

An equivalent deformation of (22.3.9) is

$$
\begin{array}{cc}
r_1 & \\
r_2: & r_1 r_2 \\
r_3: & r_1 r_2 r_3 \\
\end{array}

\begin{array}{c}
r_1 + r_2 \\
r_1 r_2 + r_3 (r_1 + r_2) \\
r_1 + r_2 + r_3
\end{array}
$$

(22.3.10)

Here the diagonal arrows are simple additions, while the vertical arrows represent multiplication by a root value. Note that the coefficient answers in (22.3.10) come out in the opposite order from (22.3.9). An example of coding (22.3.10) in Fortran 90 can be found in the routine fixrta in Chapter B13.

CITED REFERENCES AND FURTHER READING:
22.4 Fast Fourier Transforms

Fast Fourier transforms are beloved by computer scientists, especially those who are interested in parallel algorithms, because the FFT’s hierarchical structure generates a complicated, but analyzable, set of requirements for interprocessor communication on MMPs. Thus, almost all books on parallel algorithms (e.g., [1–3]) have a chapter on FFTs.

Unfortunately, the resulting algorithms are highly specific to particular parallel architectures, and therefore of little use to us in writing general purpose code in an architecture-independent parallel language like Fortran 90.

Luckily there is a good alternative that covers almost all cases of both serial and parallel machines. If, for a one-dimensional FFT of size \( N \), one is satisfied with parallelism of order \( \sqrt{N} \), then there is a good, general way of achieving a parallel FFT with quite minimal interprocessor communication; and the communication required is simply the matrix transpose operation, which Fortran 90 implements as an intrinsic. That is the approach that we discuss in this section, and implement in Chapter B12.

For a machine with \( M \) processors, this approach will saturate the processors (the desirable condition where none are idle) when the size of a one-dimensional Fourier transform, \( N \), is large enough: \( N > M^2 \). Smaller \( N \)’s will not achieve maximum parallelism. But such \( N \)’s are in fact so small for one-dimensional problems that they are unlikely to be the rate-determining step in scientific calculations. If they are, it is usually because you are doing many such transforms independently, and you should recover “outer parallelism” by doing them all at once.

For two or more dimensions, the adopted approach will saturate \( M \) processors when each dimension of the problem is larger than \( M \).

Column- and Row-Parallel FFTs

The basic building block that we assume (and implement in Chapter B12) is a routine for simultaneously taking the FFT of each row of a two-dimensional matrix. The method is exactly that of Volume 1’s \texttt{four1} routine, but with array sections like \texttt{data(:,j)} replacing scalars like \texttt{data(j)}. Chapter B12’s implementation of this is a routine called \texttt{fourrow}. If all the data for one column (that is, all the values \texttt{data(i,:), for some i}) are local to a single processor, then the parallelism involves no interprocessor communication at all: The independent FFTs simply proceed, data parallel and in lockstep. This is architecture-independent parallelism with a vengeance.

We will also need to take the FFT of each column of a two-dimensional matrix. One way to do this is to take the transpose (a Fortran 90 intrinsic that hides a lot of interprocessor communication), then take the FFT of the rows using \texttt{fourrow}, then take the transpose again. An alternative method is to recode the \texttt{four1} routine with array sections in the other dimension (\texttt{data(j,:)} replacing \texttt{four1’s scalars (data(j)})). This scheme, in Chapter B12, is a routine called \texttt{fourcol}. In this case, good parallelism will be achieved only if the values \texttt{data(:,i), for some i}, are local to a single processor. Of course, Fortran 90 does not give the user direct control over how data are distributed over the machine; but extensions such as HPF are designed to give just such control.
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On a serial machine, you might think that `fourrow` and `fourcol` should have identical timings (acting on a square matrix, say). The two routines do exactly the same operations, after all. Not so! On modern serial computers, `fourrow` and `fourcol` can have timings that differ by a factor of 2 or more, even when their detailed arithmetic is made identical (by giving to one a data array that is the transpose of the data array given to the other). This effect is due to the multilevel cache architecture of most computer memories, and the fact that serial Fortran always stores matrices by columns (first index changing most rapidly). On our workstations, `fourrow` is significantly faster than `fourcol`, and this is likely the generic behavior. However, we do not exclude the possibility that some machines, and some sizes of matrices, are the other way around.

One-Dimensional FFTs

Turn now to the problem of how to do a single, one-dimensional, FFT. We are given a complex array $f$ of length $N$, an integer power of 2. The basic idea is to address the input array as if it were a two-dimensional array of size $m \times M$, where $m$ and $M$ are each integer powers of 2. Then the components of $f$ can be addressed as

$$f(Jm + j), \quad 0 \leq j < m, \quad 0 \leq J < M \quad (22.4.1)$$

where the $j$ index changes more rapidly, the $J$ index more slowly, and parentheses denote Fortran-style subscripts.

Now, suppose we had some magical (parallel) method to compute the discrete Fourier transform

$$F(kM + K) = \sum_{j,J} e^{2\pi i (kM + K)(Jm + j)/(Mm)} f(Jm + j),$$

$$0 \leq k < m, \quad 0 \leq K < M \quad (22.4.2)$$

Then, you can see that the indices $k$ and $K$ would address the desired result (FFT of the original array), with $K$ varying more rapidly.

Starting with equation (22.4.2) it is easy to verify the following identity,

$$F(kM + K) = \sum_j \left[ e^{2\pi i jk/m} \left( e^{2\pi i jK/(Mm)} \left[ \sum_J e^{2\pi i JK/M} f(Jm + j) \right] \right) \right] \quad (22.4.3)$$

But this, reading it from the innermost operation outward, is just the magical method that we need:

- Reshape the original array to $m \times M$ in Fortran normal order (storage by columns).
- FFT on the second (column) index for all values of the first (row) index, using the routine `fourrow`.
- Multiply each component by a phase factor $\exp[2\pi i jK/(Mm)]$.
- Transpose.
22.5 Missing Language Features

A few facilities that are fairly important to parallel programming are missing from the Fortran 90 language standard. On scalar machines this lack is not a

- Again FFT on the second (column) index for all values of the first (row) index, using the routine fourrow.
- Reshape the two-dimensional array back into one-dimensional output.

The above scheme uses fourrow exclusively, on the assumption that it is faster than its sibling fourcol. When that is the case (as we typically find), it is likely that the above method, implemented as four1 in Chapter B12, is faster, even on scalar machines, than Volume I’s scalar version of four1 (Chapter 12). The reason, as already mentioned, is that fourrow’s parallelism is taking better advantage of cache memory locality.

If fourrow is not faster than fourcol on your machine, then you should instead try the following alternative scheme, using fourcol only:

- Reshape the original array to \( m \times M \) in Fortran normal order (storage by columns).
- Transpose.
- FFT on the first (row) index for all values of the second (column) index, using the routine fourcol.
- Multiply each component by a phase factor \( \exp[2\pi ijK/(Mm)] \).
- Transpose.
- Again FFT on the first (row) index for all values of the second (column) index, using the routine fourcol.
- Transpose.
- Reshape the two-dimensional array back into one-dimensional output.

In Chapter B12, this scheme is implemented as four1alt. You might wonder why four1alt has three transpose operations, while four1 had only one. Shouldn’t there be a symmetry here? No. Fortran makes the arbitrary, but consistent, choice of storing two-dimensional arrays by columns, and this choice favors four1 in terms of transposes. Luckily, at least on our serial workstations, fourrow (used by four1) is faster than fourcol (used by four1alt), so it is a double win.

For further discussion and references on the ideas behind four1 and four1alt see [4], where these algorithms are called the four-step and six-step frameworks, respectively.

CITED REFERENCES AND FURTHER READING:
Hockney, R.W., and Jesshope, C.R. 1988, Parallel Computers 2 (Bristol and Philadelphia: Adam Hilger), §5.5. [3]
problem, since one can readily program the missing features by using do-loops. On parallel machines, both SSP machines and MMP machines, one must hope that hardware manufacturers provide library routines, callable from Fortran 90, that provide access to the necessary facilities, or use extensions of Fortran 90, such as High Performance Fortran (HPF).

**Scatter-with-Combine Functions**

Fortran 90 allows the use of vector subscripts for so-called gather and scatter operations. For example, with the setup

```fortran
REAL(SP), DIMENSION(6) :: arr,barr,carr
INTEGER(I4B), DIMENSION(6) :: iarr,jarr
... 
  iarr = (/ 1, 3, 5, 2, 4, 6 /)  
  jarr = (/ 3, 2, 3, 2, 1, 1 /)
```

Fortran 90 allows both the one-to-one gather and the one-to-many gather,

```fortran
  barr=arr(iarr)  
  carr=arr(jarr)
```

It also allows the one-to-one scatter,

```fortran
  barr(iarr)=carr
```

where the elements of `carr` are “scattered” into `barr` under the direction of the vector subscript `iarr`.

Fortran 90 does not allow the many-to-one scatter

```fortran
  barr(jarr)=carr ! illegal for this jarr
```

because the repeated values in `jarr` try to assign different components of `carr` to the same location in `barr`. The result would not be deterministic.

Sometimes, however, one would in fact like a many-to-one construction, where the colliding elements get combined by a (commutative and associative) operation, like `+` or `*`, or `max()`. These so-called scatter-with-combine functions are readily implemented on serial machines by a do-loop, for example,

```fortran
  barr=0.  
  do j=1,size(carr)  
    barr(jarr(j))=barr(jarr(j))+carr(j)  
  end do
```

Fortran 90 unfortunately provides no means for effecting scatter-with-combine functions in parallel. Luckily, almost all parallel machines do provide such a facility as a library program, as does HPF, where the above facility is called `SUM_SCATTER`. In Chapter 23 we will define utility routines `scatter_add` and `scatter_max` for scatter-with-combine functionalities, but the implementation given in Fortran 90 will be strictly serial, with a do-loop.
Skew Sections

Fortran 90 provides no good, parallel way to access the diagonal elements of a matrix, either to read them or to set them. Do-loops will obviously serve this need on serial machines. In principle, a construction like the following bizarre fragment could also be utilized,

```
REAL(SP), DIMENSION(n,n) :: mat
REAL(SP), DIMENSION(n*n) :: arr
REAL(SP), DIMENSION(n) :: diag
...
arr = reshape(mat,shape(arr))
diag = arr(1:n*n:n+1)
```

which extracts every \((n+1)\)st element from a one-dimensional array derived by reshaping the input matrix. However, it is unlikely that any foreseeable parallel compiler will implement the above fragment without a prohibitive amount of unnecessary data movement; and code like the above is also exceedingly slow on all serial machines that we have tried.

In Chapter 23 we will define utility routines `get_diag`, `put diag`, `diagadd`, `diagmult`, and `unit_matrix` to manipulate diagonal elements, but the implementation given in Fortran 90 will again be strictly serial, with do-loops.

Fortran 95 (see §21.6) will essentially fix Fortran 90’s skew sections deficiency. For example, using its `forall` construction, the diagonal elements of an array can be accessed by a statement like

```
forall (j=1:20) diag(j) = arr(j,j)
```

SIMD vs. MIMD

Recall that we use “SIMD” (single-instruction, multiple data) and “data parallel” as interchangeable terms, and that “MIMD” (multiple-instruction, multiple data) is a more general programming model. (See §22.1.)

You should not be too quick to jump to the conclusion that Fortran 90’s data parallel or SIMD model is “bad,” and that MIMD features, absent in Fortran 90, are therefore “good.” On the contrary, Fortran 90’s basic data-parallel paradigm has a lot going for it. As we discussed in §22.1, most scientific problems naturally have a “data dimension” across which the time ordering of the calculation is irrelevant. Parallelism across this dimension, which is by nature most often SIMD, frees the mind to think clearly about the computational steps in an algorithm that actually need to be sequential. SIMD code has advantages of clarity and predictability that should not be taken lightly. The general MIMD model of “lots of different things all going on at the same time and communicating data with each other” is a programming and debugging nightmare.

Having said this, we must at the same time admit that a few MIMD features — most notably the ability to go through different logical branches for calculating different data elements in a data-parallel computation — are badly needed in certain programming situations. Fortran 90 is quite weak in this area.

Note that the `where...elsewhere...end where` construction is not a MIMD construction. Fortran 90 requires that the `where` clause be executed completely before the `elsewhere` is started. (This allows the results of any calculations in the former
clause to be available for use in the latter.) So, this construction cannot be used to allow two logical branches to be calculated in parallel.

Special functions, where one would like to calculate function values for an array of input quantities, are a particularly compelling example of the need for some MIMD access. Indeed, you will find that Chapter B6 contains a number of intricate, and in a few cases truly bizarre, workarounds, using allowed combinations of *merge*, *where*, and *contains* (the latter, for separating different logical branches into formally different subprograms).

Fortran 95’s *elemental* and *pure* constructions, and to some extent also *forall* (whose body will be able to include *pure* function calls), will go a long way towards providing exactly the kind of MIMD constructions that are most needed. Once Fortran 95 becomes available and widespread, you can expect to see a new version of this volume, with a much-improved Chapter B6.

Conversely, the number of routines outside of Chapter B6 that can be significantly improved by the use of MIMD features is relatively small; this illustrates the underlying viability of the basic data-parallel SIMD model, even in a future language version with useful MIMD features.
Chapter 23. Numerical Recipes
Utility Functions for Fortran 90

23.0 Introduction and Summary Listing

This chapter describes and summarizes the Numerical Recipes utility routines that are used throughout the rest of this volume. A complete implementation of these routines in Fortran 90 is listed in Appendix C1.

Why do we need utility routines? Aren’t there already enough of them built into the language as Fortran 90 intrinsics? The answers lie in this volume’s dual purpose: to implement the Numerical Recipes routines in Fortran 90 code that runs efficiently on fast serial machines, and to implement them, wherever possible, with efficient parallel code for multiprocessor machines that will become increasingly common in the future. We have found three kinds of situations where additional utility routines seem desirable:

1. Fortran 90 is a big language, with many high-level constructs — single statements that actually result in a lot of computing. We like this; it gives the language the potential for expressing algorithms very readably, getting them “out of the mud” of microscopic coding. In coding the 350+ Recipes for this volume, we kept a systematic watch for bits of microscopic coding that were repeated in many routines, and that seemed to be at a lower level of coding than that aspired to by good Fortran 90 style. Once these bits were identified, we pulled them out and substituted calls to new utility routines. These are the utilities that arguably ought to be new language intrinsics, equally useful for serial and parallel machines. (A prime example is swap.)

2. Fortran 90 contains many highly parallelizable language constructions. But, as we have seen in §22.5, it is also missing a few important constructions. Most parallel machines will provide these missing elements as machine-coded library subroutines. Some of our utility routines are provided simply as a standard interface to these common, but nonstandard, functionalities. Note that it is the nature of these routines that our specific implementation, in Appendix C1, will be serial, and therefore inefficient on parallel machines. If you have a parallel machine, you will need to recode these; this often involves no more than substituting a one-line library function call for the body of our implementation. Utilities in this category will likely become unnecessary over time, either as machine-dependent libraries converge to standard interfaces, or as the utilities get added to future Fortran
versions. (Indeed, some routines in this category will be unnecessary in Fortran 95, once it is available; see §23.7.)

3. Some tasks should just be done differently in serial, versus parallel, implementation. Linear recurrence relations are a good example (§22.2). These are trivially coded with a do-loop on serial machines, but require a fairly elaborate recursive construction for good parallelization. Rather than provide separate serial and parallel versions of the Numerical Recipes, we have chosen to pull out of the Recipes, and into utility routines, some identifiable tasks of this kind. These are cases where some recoding of our implementation in Appendix C1 might result in improved performance on your particular hardware. Unfortunately, it is not so simple as providing a single “serial implementation” and another single “parallel implementation,” because even the seemingly simple word “serial” hides, at the hardware level, a variety of different degrees of pipelining, wide instructions, and so on. Appendix C1 therefore provides only a single implementation, although with some adjustable parameters that you can customize (by experiment) to maximize performance on your hardware.

The above three cases are not really completely distinct, and it is therefore not possible to assign any single utility routine to exactly one situation. Instead, we organize the rest of this chapter as follows: first, an alphabetical list, with short summary, of all the new utility routines; next, a series of short sections, grouped by functionality, that contain the detailed descriptions of the routines.

**Alphabetical Listing**

The following list gives an abbreviated mnemonic for the type, rank, and/or shape of the returned values (as in §21.4), the routine’s calling sequence (optional arguments shown in italics), and a brief, often incomplete, description. The complete description of the routine is given in the later section shown in square brackets.

For each entry, the number shown in parentheses is the approximate number of distinct Recipes in this book that make use of that particular utility function, and is thus a rough guide to that utility’s importance. (There may be multiple invocations of the utility in each such Recipe.) Where this number is small or zero, it is usually because the utility routine is a member of a related family of routines whose total usage was deemed significant enough to include, and we did not want users to have to “guess” which family members were instantiated.

```plaintext
(call) array_copy(src, dest, n_copied, n_not_copied)
Copy one-dimensional array (whose size is not necessarily known). [23.1] (9)

[arr] arth(first, increment, n)
Return an arithmetic progression as an array. [23.4] (55)

call assert(n1, n2,..., string)
Exit with error message if any logical arguments are false. [23.3] (50)

[int] assert_eq(n1, n2,..., string)
Exit with error message if all integer arguments are not equal; otherwise return common value. [23.3] (133)

[argTS] cumprod(arr, seed)
```
Cumulative products of one-dimensional array, with optional seed value. [23.4] (3)

[argTS] cumsum(arr, seed)
Cumulative sums of one-dimensional array, with optional seed value.
[23.4] (9)

call diagadd(mat, diag)
Adds vector to diagonal of a matrix. [23.7] (4)

call diagmult(mat, diag)
Multiplies vector into diagonal of a matrix. [23.7] (2)

[Arr] geop(first, factor, n)
Return a geometrical progression as an array. [23.4] (7)

[Arr] get_diag(mat)
Gets diagonal of a matrix. [23.7] (2)

[Int] ifirstloc(arr)
Location of first true value in a logical array, returned as an integer.
[23.2] (3)

[Int] imaxloc(arr)
Location of array maximum, returned as an integer. [23.2] (11)

[Int] iminloc(arr)
Location of array minimum, returned as an integer. [23.2] (8)

[Mat] lower_triangle(j, k, extra)
Returns a lower triangular logical mask. [23.7] (1)

call nrerror(string)
Exit with error message. [23.3] (96)

[Mat] outerand(a, b)
Returns the outer logical and of two vectors. [23.5] (1)

[Mat] outerdiff(a, b)
Returns the outer difference of two vectors. [23.5] (4)

[Mat] outerdiv(a, b)
Returns the outer quotient of two vectors. [23.5] (0)

[Mat] outerprod(a, b)
Returns the outer product of two vectors. [23.5] (14)

[Mat] outersum(a, b)
Returns the outer sum of two vectors. [23.5] (0)

[argTS] poly(x, coeffs, mask)
Evaluate a polynomial $P(x)$ for one or more values $x$, with optional mask. [23.4] (15)

[argTS] poly_term(a, x)
Returns partial cumulants of a polynomial, equivalent to synthetic
division. [23.4] (4)
call put_diag(diag,mat)
Sets diagonal of a matrix. [23.7] (0)

[Ptr] reallocate(p,n,m,...)
Reallocate pointer to new size, preserving its contents. [23.1] (5)
call scatter_add(dest,source,dest_index)
Scatter-adds source vector to specified components of destination vector. [23.6] (2)
call scatter_max(dest,source,dest_index)
Scatter-max source vector to specified components of destination vector. [23.6] (0)
call swap(a,b,mask)
Swap corresponding elements of a and b. [23.1] (24)
call unit_matrix(mat)
Sets matrix to be a unit matrix. [23.7] (6)

[Mat] upper_triangle(j,k,extra)
Returns an upper triangular logical mask. [23.7] (4)

[Real] vabs(v)
Length of a vector in $L_2$ norm. [23.8] (6)

[CArr] zroots_unity(n,nn)
Returns $nn$ consecutive powers of the complex $n$th root of unity. [23.4] (4)

Comment on Relative Frequencies of Use

We find it interesting to compare our frequency of using the \texttt{nrutil} utility routines, with our most used language intrinsics (see §21.4). On this basis, the following routines are as useful to us as the top 10 language intrinsics: \texttt{arth}, \texttt{assert}, \texttt{assert.eq}, \texttt{outerprod}, \texttt{poly}, and \texttt{swap}. We strongly recommend that the X3J3 standards committee, as well as individual compiler library implementors, consider the inclusion of new language intrinsics (or library routines) that subsume the capabilities of at least these routines. In the next tier of importance, we would put some further cumulative operations (\texttt{geop}, \texttt{cumsum}), some other “outer” operations on vectors (e.g., \texttt{outerdiff}), basic operations on the diagonals of matrices (\texttt{get_diag}, \texttt{put_diag}, \texttt{diag_add}), and some means of access to an array of unknown size (\texttt{array_copy}).

23.1 Routines That Move Data

To describe our utility routines, we introduce two items of Fortran 90 pseudocode: We use the symbol $T$ to denote some type and rank declaration (including
scalar rank, i.e., zero); and when we append a colon to a type specification, as in INTEGER(I4B)(:), for example, we denote an array of the given type.

The routines swap, array_copy, and reallocate simply move data around in useful ways.

```
* * *
```

**swap** (swaps corresponding elements)

*User interface (or, “USE nrutil”):*

```fortran
SUBROUTINE swap(a,b,mask)
  T, INTENT(INOUT) :: a,b
  LOGICAL(LGT), INTENT(IN), OPTIONAL :: mask
END SUBROUTINE swap
```

*Applicable types and ranks:*

| T | any type, any rank |

*Types and ranks implemented (overloaded) in nrutil:*

| T | INTEGER(I4B), REAL(SP), REAL(SP)(::), REAL(DP), COMPLEX(SPC), COMPLEX(SPC)(::), COMPLEX(SPC)(::,:), COMPLEX(DPC), COMPLEX(DPC)(::), COMPLEX(DPC)(::,:), |

*Action:*

Swaps the corresponding elements of a and b. If mask is present, performs the swap only where mask is true. (Following code is the unmasked case. For speed at run time, the masked case is implemented by overloading, not by testing for the optional argument.)

*Reference implementation:*

```fortran
T :: dum
dum=a
a=b
b=dum
```

```
* * *
```

**array_copy** (copy one-dimensional array)

*User interface (or, “USE nrutil”):*

```fortran
SUBROUTINE array_copy(src,dest,n_copied,n_not_copied)
  T, INTENT(IN) :: src
  T, INTENT(OUT) :: dest
  INTEGER(I4B), INTENT(OUT) :: n_copied, n_not_copied
END SUBROUTINE array_copy
```

*Applicable types and ranks:*

| T | any type, rank 1 |

*Types and ranks implemented (overloaded) in nrutil:*

| T | INTEGER(I4B)(::), REAL(SP)(::), REAL(DP)(::) |

*Action:*

Copies to a destination array dest the one-dimensional array src, or as much of src as will fit in dest. Returns the number of components copied as n_copied, and the number of components not copied as n_not_copied.

The main use of this utility is where src is an expression that returns an array whose size is not known in advance, for example, the value returned by the pack intrinsic.
Reference implementation:

```
  n_copied=min(size(src),size(dest))
  n_not_copied=size(src)-n_copied
  dest(1:n_copied)=src(1:n_copied)
```

```
⋆⋆⋆
```

**reallocate** (reallocate a pointer, preserving contents)

*User interface (or, “USE nrutil”):*

```
FUNCTION reallocate(p,n[,...])
T, POINTER :: p, reallocate
INTEGER(I4B), INTENT(IN) :: n[,...]
END FUNCTION reallocate
```

**Applicable types and ranks:**

\( T \equiv \text{any type, rank 1 or greater} \)

**Types and ranks implemented (overloaded) in nrutil:**

\( T \equiv \text{INTEGER(I4B)(:), INTEGER(I4B)(:,:), REAL(SP)(:),}

\( \text{REAL(SP)(:,:), CHARACTER(1)(:)} \)

**Action:**

Allocates storage for a new array with shape specified by the integer(s) \( n, m, \ldots \) (equal in number to the rank of pointer \( p \)). Then, copies the contents of \( p \)'s target (or as much as will fit) into the new storage. Then, deallocates \( p \) and returns a pointer to the new storage.

The typical use is \( p=\text{reallocate}(p,n[,...]) \), which has the effect of changing the allocated size of \( p \) while preserving the contents.

The reference implementation, below, shows only the case of rank 1.

**Reference implementation:**

```
INTEGER(I4B) :: nold,ierr
allocate(reallocate(n),stat=ierr)
if (ierr /= 0) call &
    nrerror('reallocate: problem in attempt to allocate memory')
if (.not. associated(p)) RETURN
nold=size(p)
reallocate(1:min(nold,n))=p(1:min(nold,n))
deallocate(p)
```

### 23.2 Routines Returning a Location

Fortran 90’s intrinsics `maxloc` and `minloc` return rank-one arrays. When, in the most frequent usage, their argument is a one-dimensional array, the answer comes back in the inconvenient form of an array containing a single component, which cannot be itself used in a subscript calculation. While there are workaround tricks (e.g., use of the `sum` intrinsic to convert the array to a scalar), it seems clearer to define routines `imaxloc` and `iminloc` that return integers directly.

The routine `ifirstloc` adds a related facility missing among the intrinsics: Return the first true location in a logical array.

```
⋆⋆⋆
```
23.2 Routines Returning a Location

**imaxloc**  (location of array maximum as an integer)

*User interface (or, “USE nrutil”):*

```fortran
FUNCTION imaxloc(arr)
  T, INTENT(IN) :: arr
  INTEGER(I4B) :: imaxloc
END FUNCTION imaxloc
```

**Applicable types and ranks:**

- \( T \equiv \) any integer or real type, rank 1

**Types and ranks implemented (overloaded) in nrutil:**

- \( T \equiv \) INTEGER(I4B)(:) , REAL(SP)(:)

**Action:**

For one-dimensional arrays, identical to the \texttt{maxloc} intrinsic, except returns its answer as an integer rather than as \texttt{maxloc}'s somewhat awkward rank-one array containing a single component.

**Reference implementation:**

```fortran
INTEGER(I4B), DIMENSION(1) :: imax
imax=maxloc(arr(:))
imaxloc=imax(1)
```

---

**iminloc**  (location of array minimum as an integer)

*User interface (or, “USE nrutil”):*

```fortran
FUNCTION iminloc(arr)
  T, INTENT(IN) :: arr
  INTEGER(I4B) :: iminloc
END FUNCTION iminloc
```

**Applicable types and ranks:**

- \( T \equiv \) any integer or real type, rank 1

**Types and ranks implemented (overloaded) in nrutil:**

- \( T \equiv \) REAL(SP)(:)

**Action:**

For one-dimensional arrays, identical to the \texttt{minloc} intrinsic, except returns its answer as an integer rather than as \texttt{minloc}'s somewhat awkward rank-one array containing a single component.

**Reference implementation:**

```fortran
INTEGER(I4B), DIMENSION(1) :: imin
imin=minloc(arr(:))
iminloc=imin(1)
```

---

**ifirstloc**  (returns location of first “true” in a logical vector)

*User interface (or, “USE nrutil”):*

```fortran
FUNCTION ifirstloc(mask)
  T, INTENT(IN) :: mask
  INTEGER(I4B) :: ifirstloc
END FUNCTION ifirstloc
```
Applicable types and ranks:
\[ T \equiv \text{any logical type, rank 1} \]

Types and ranks implemented (overloaded) in `nrutil`:
\[ T \equiv \text{LOGICAL(LGT)} \]

Action:
Returns the index (subscript value) of the first location, in a one-dimensional logical mask, that has the value `TRUE`, or returns `size(mask)+1` if all components of mask are `FALSE`.

Note that while the reference implementation uses a do-loop, the function is parallelized in `nrutil` by instead using the `merge` and `maxloc` intrinsics.

Reference implementation:

```fortran
INTEGER(I4B) :: i
do i=1,size(mask)
   if (mask(i)) then
      ifirstloc=i
      return
   end if
end do
ifirstloc=i
```

---

### 23.3 Argument Checking and Error Handling

It is good programming practice for a routine to check the assumptions ("assertions") that it makes about the sizes of input arrays, allowed range of numerical arguments, and so forth. The routines `assert` and `assert.eq` are meant for this kind of use. The routine `nrerror` is our default error reporting routine.

```fortran
* * *
```

**assert**  (exit with error message if any assertion is false)

*User interface (or, "USE nrutil"):*

```fortran
SUBROUTINE assert(n1,n2,...,string)
CHARACTER(LEN=*) , INTENT(IN) :: string
LOGICAL, INTENT(IN) :: n1,n2,...
END SUBROUTINE assert
```

**Action:**
Embedding program dies gracefully with an error message if any of the logical arguments are false. Typical use is with logical expressions as the actual arguments. `nrutil` implements and overloads forms with 1, 2, 3, and 4 logical arguments, plus a form with a vector logical argument,

```fortran
LOGICAL, DIMENSION(:), INTENT(IN) :: n
```
that is checked by the `all(n)` intrinsic.
Reference implementation:

```fortran
if (.not. (n1 .and. n2 .and...) then
  write (*,*) 'nrerror: an assertion failed with this tag:', string
  STOP 'program terminated by assert'
end if

* * *
```

**assert_eq** (exit with error message if integer arguments not all equal)

**User interface (or, "USE nrutil"):**

```fortran
FUNCTION assert_eq(n1,n2,n3,...,string)
  CHARACTER(LEN=*) , INTENT(IN) :: string
  INTEGER, INTENT(IN) :: n1,n2,n3,...
  INTEGER :: assert_eq
END FUNCTION assert_eq
```

**Action:**
Embedding program dies gracefully with an error message if any of the integer arguments are not equal to the first. Otherwise, return the value of the first argument. Typical use is for enforcing equality on the sizes of arrays passed to a subprogram. **nrutil** implements and overloads forms with 1, 2, 3, and 4 integer arguments, plus a form with a vector integer argument, `INTEGER, DIMENSION(:), INTENT(IN) :: n` that is checked by the conditional `if (all(nn(2)==nn(1)))`.

**Reference implementation:**

```fortran
if (n1==n2 .and. n2==n3 .and...) then
  assert_eq=n1
else
  write (*,*) 'nrerror: an assert_eq failed with this tag:', string
  STOP 'program terminated by assert_eq'
end if

* * *
```

**nrerror** (report error message and stop)

**User interface (or, "USE nrutil"):**

```fortran
SUBROUTINE nrerror(string)
  CHARACTER(LEN=*) , INTENT(IN) :: string
END SUBROUTINE nrerror
```

**Action:**
This is the minimal error handler used in this book. In applications of any complexity, it is intended only as a placeholder for a user’s more complicated error handling strategy.

**Reference implementation:**

```fortran
write (*,*) 'nrerror: ',string
STOP 'program terminated by nrerror'
```
23.4 Routines for Polynomials and Recurrences

Apart from programming convenience, these routines are designed to allow for nontrivial parallel implementations, as discussed in §22.2 and §22.3.

* * *

arth (returns arithmetic progression as an array)

User interface (or, “USE nrutil”):

    FUNCTION arth(first,increment,n)
    T, INTENT(IN) :: first,increment
    INTEGER(I4B), INTENT(IN) :: n
    T, DIMENSION(n) [or 1 rank higher than T] :: arth
    END FUNCTION arth

Applicable types and ranks:

T ≡ any numerical type, any rank

Types and ranks implemented (overloaded) in nrutil:

T ≡ INTEGER(I4B), REAL(SP), REAL(DP)

Action:

Returns an array of length n containing an arithmetic progression whose first value is first and whose increment is increment. If first and increment have rank greater than zero, returns an array of one larger rank, with the last subscript having size n and indexing the progressions. Note that the following reference implementation (for the scalar case) is definitional only, and neither parallelized nor optimized for roundoff error. See §22.2 and Appendix C1 for implementation by subvector scaling.

Reference implementation:

    INTEGER(I4B) :: k
    if (n > 0) arth(1)=first
    do k=2,n
        arth(k)=arth(k-1)+increment
    end do

* * *

geop (returns geometric progression as an array)

User interface (or, “USE nrutil”):

    FUNCTION geop(first,factor,n)
    T, INTENT(IN) :: first,factor
    INTEGER(I4B), INTENT(IN) :: n
    T, DIMENSION(n) [or 1 rank higher than T] :: geop
    END FUNCTION geop

Applicable types and ranks:

T ≡ any numerical type, any rank

Types and ranks implemented (overloaded) in nrutil:

T ≡ INTEGER(I4B), REAL(SP), REAL(DP), REAL(DP)(::), COMPLEX(SPC)
**Action:**

Returns an array of length $n$ containing a geometric progression whose first value is $\text{first}$ and whose multiplier is $\text{factor}$. If $\text{first}$ and $\text{factor}$ have rank greater than zero, returns an array of one larger rank, with the last subscript having size $n$ and indexing the progression. Note that the following reference implementation (for the scalar case) is definitional only, and neither parallelized nor optimized for roundoff error. See §22.2 and Appendix C1 for implementation by subvector scaling.

**Reference implementation:**

```
INTEGER(I4B) :: k
if (n > 0) geop(1)=first
do k=2,n
    geop(k)=geop(k-1)*factor
end do
```

**cumsum**  (cumulative sum on an array, with optional additive seed)

**User interface (or, "USE nrutil"):**

```
FUNCTION cumsum(arr,seed)
    T, DIMENSION(:), INTENT(IN) :: arr
    T, OPTIONAL, INTENT(IN) :: seed
    T, DIMENSION(size(arr)), INTENT(OUT) :: cumsum
END FUNCTION cumsum
```

**Applicable types and ranks:**

$T \equiv$ any numerical type

**Types and ranks implemented (overloaded) in nrutil:**

$T \equiv$ INTEGER(I4B), REAL(SP)

**Action:**

Given the rank 1 array $\text{arr}$ of type $T$, returns an array of identical type and size containing the cumulative sums of $\text{arr}$. If the optional argument $\text{seed}$ is present, it is added to the first component (and therefore, by cumulation, all components) of the result. See §22.2 for parallelization ideas.

**Reference implementation:**

```
INTEGER(I4B) :: n,j
T :: sd
n=size(arr)
if (n == 0) return
sd=0.0
if (present(seed)) sd=seed
cumsum(1)=arr(1)+sd
do j=2,n
    cumsum(j)=cumsum(j-1)+arr(j)
end do
```

**cumprod**  (cumulative prod on an array, with optional multiplicative seed)

**User interface (or, "USE nrutil"):**

```
FUNCTION cumprod(arr,seed)
    T, DIMENSION(:), INTENT(IN) :: arr
    T, OPTIONAL, INTENT(IN) :: seed
    T, DIMENSION(size(arr)), INTENT(OUT) :: cumprod
END FUNCTION cumprod
```
Applicable types and ranks:
\[ T \equiv \text{any numerical type} \]

Types and ranks implemented (overloaded) in \texttt{nrutil}:
\[ T \equiv \text{REAL(SP)} \]

Action:
Given the rank 1 array \texttt{arr} of type \( T \), returns an array of identical type and size containing the cumulative products of \texttt{arr}. If the optional argument \texttt{seed} is present, it is multiplied into the first component (and therefore, by cumulation, into all components) of the result. See \S 22.2 for parallelization ideas.

Reference implementation:
\begin{verbatim}
INTEGER(I4B) :: n,j
T :: sd
n=size(arr)
if (n == 0) return
sd=1.0
if (present(seed)) sd=seed
cumprod(1)=arr(1)*sd
do j=2,n
    cumprod(j)=cumprod(j-1)*arr(j)
end do
\end{verbatim}

\texttt{poly} \ (polynomial evaluation)

User interface (or, “USE \texttt{nrutil}:”):
\begin{verbatim}
FUNCTION poly(x,coeffs,mask)
    T, DIMENSION(:,...), INTENT(IN) :: x
    T, DIMENSION(:), INTENT(IN) :: coeffs
    LOGICAL(LGT), DIMENSION(:,...), OPTIONAL, INTENT(IN) :: mask
    T :: poly
END FUNCTION poly
\end{verbatim}

Applicable types and ranks:
\[ T \equiv \text{any numerical type (x may be scalar or have any rank; x and}
\text{coeffs may have different numerical types)} \]

Types and ranks implemented (overloaded) in \texttt{nrutil}:
\[ T \equiv \text{various combinations of REAL(SP), REAL(SP)(::),}
\text{REAL(DP), REAL(DP)(::), COMPLEX(SPC) (see Appendix C1 for details)} \]

Action:
Returns a scalar value or array with the same type and shape as \( x \), containing the result of evaluating the polynomial with one-dimensional coefficient vector \texttt{coeffs} on each component of \( x \). The optional argument \texttt{mask}, if present, has the same shape as \( x \), and suppresses evaluation of the polynomial where its components are \texttt{.false.}. The following reference code shows the case where \texttt{mask} is not present. (The other case can be included by overloading.)
23.4 Routines for Polynomials and Recurrences

Reference implementation:
```
INTEGER(I4B) :: i,n
n=size(coeffs)
if (n <= 0) then
  poly=0.0
else
  poly=coeffs(n)
  do i=n-1,1,-1
      poly=x*poly+coeffs(i)
  end do
end if
```

⋆⋆⋆

poly_term (partial cumulants of a polynomial)

User interface (or, “USE nrutil”):
```
FUNCTION poly_term(a,x)
  T, DIMENSION(:), INTENT(IN) :: a
  T, INTENT(IN) :: x
  T, DIMENSION(size(a)) :: poly_term
END FUNCTION poly_term
```

Applicable types and ranks:

\[ T \cong \text{any numerical type} \]

Types and ranks implemented (overloaded) in nrutil:

\[ T \cong \text{REAL(SP), COMPLEX(SPC)} \]

Action:

Returns an array of type and size the same as the one-dimensional array \(a\), containing the partial cumulants of the polynomial with coefficients \(a\) (arranged from highest-order to lowest-order coefficients, n.b.) evaluated at \(x\). This is equivalent to synthetic division, and can be parallelized. See §22.3. Note that the order of arguments is reversed in poly and poly_term — each routine returns a value with the size and shape of the first argument, the usual Fortran 90 convention.

Reference implementation:
```
INTEGER(I4B) :: n,j
n=size(a)
if (n <= 0) return
poly_term(1)=a(1)
do j=2,n
  poly_term(j)=a(j)+x*poly_term(j-1)
end do
```

⋆⋆⋆

zroots_unity (returns powers of complex \(n\)th root of unity)

User interface (or, “USE nrutil”):
```
FUNCTION zroots_unity(n,nn)
  INTEGER(I4B), INTENT(IN) :: n,nn
  COMPLEX(SPC), DIMENSION(nn) :: zroots_unity
END FUNCTION zroots_unity
```
Chapter 23. Numerical Recipes Utility Functions for Fortran 90

23.5 Routines for Outer Operations on Vectors

Outer operations on vectors take two vectors as input, and return a matrix as output. One dimension of the matrix is the size of the first vector, the other is the size of the second vector. Our convention is always the standard one,

\[
\text{result}(i,j) = \text{first \_ operand}(i) \ (\text{op}) \ \text{second \_ operand}(j)
\]

where \((\text{op})\) is any of addition, subtraction, multiplication, division, and logical and. The reason for coding these as utility routines is that Fortran 90's native construction, with two spreads (cf. §22.1), is difficult to read and thus prone to programmer errors.

⋆⋆⋆

Action:

Returns a complex array containing \(nn\) consecutive powers of the \(n\)th complex root of unity. Note that the following reference implementation is definitional only, and neither parallelized nor optimized for roundoff error. See Appendix C1 for implementation by subvector scaling.

Reference implementation:

```fortran
INTEGER(I4B) :: k
REAL(SP) :: theta

if (nn==0) return
zroots_unity(1)=1.0
if (nn==1) return
theta=TWOPI/n
zroots_unity(2)=cmplx(cos(theta),sin(theta))
do k=3,nn
   zroots_unity(k)=zroots_unity(k-1)*zroots_unity(2)
end do
```

⋆⋆⋆

outerprod (outer product)

User interface (or, "USE nrutil"):

```fortran
FUNCTION outerprod(a,b) T, DIMENSION(:), INTENT(IN) :: a,b T, DIMENSION(size(a),size(b)) :: outerprod END FUNCTION outerprod
```

Applicable types and ranks:

\(T \equiv \) any numerical type

Types and ranks implemented (overloaded) in nrutil:

\(T \equiv \) REAL(SP), REAL(DP)

Action:

Returns a matrix that is the outer product of two vectors.

Reference implementation:

```fortran
outerprod = spread(a,dim=2,ncopies=size(b)) * &
            spread(b,dim=1,ncopies=size(a))
```

⋆⋆⋆
outerdiv  (outer quotient)

User interface (or, “USE nrutil”):

FUNCTION outerdiv(a,b)
   T, DIMENSION(:), INTENT(IN) :: a,b
   T, DIMENSION(size(a),size(b)) :: outerdiv
END FUNCTION outerdiv

Applicable types and ranks:
   T ≡ any numerical type

Types and ranks implemented (overloaded in nrutil):
   T ≡ REAL(SP)

Action:
   Returns a matrix that is the outer quotient of two vectors.

Reference implementation:
   outerdiv = spread(a,dim=2,ncopies=size(b)) / &
            spread(b,dim=1,ncopies=size(a))

outersum  (outer sum)

User interface (or, “USE nrutil”):

FUNCTION outersum(a,b)
   T, DIMENSION(:), INTENT(IN) :: a,b
   T, DIMENSION(size(a),size(b)) :: outersum
END FUNCTION outersum

Applicable types and ranks:
   T ≡ any numerical type

Types and ranks implemented (overloaded in nrutil):
   T ≡ REAL(SP)

Action:
   Returns a matrix that is the outer sum of two vectors.

Reference implementation:
   outersum = spread(a,dim=2,ncopies=size(b)) + &
            spread(b,dim=1,ncopies=size(a))

outerdiff  (outer difference)

User interface (or, “USE nrutil”):

FUNCTION outerdiff(a,b)
   T, DIMENSION(:), INTENT(IN) :: a,b
   T, DIMENSION(size(a),size(b)) :: outerdiff
END FUNCTION outerdiff

Applicable types and ranks:
   T ≡ any numerical type

Types and ranks implemented (overloaded in nrutil):
   T ≡ INTEGER(I4B), REAL(SP), REAL(DP)

Action:
   Returns a matrix that is the outer difference of two vectors.
Reference implementation:
outerdiff = spread(a,dim=2,ncopies=size(b)) - &
          spread(b,dim=1,ncopies=size(a))

⋆⋆⋆

outerand  (outer logical and)

User interface (or, “USE nrutil”):
FUNCTION outerand(a,b)
  LOGICAL(LGT), DIMENSION(:,), INTENT(IN) :: a,b
  LOGICAL(LGT), DIMENSION(size(a),size(b)) :: outerand
END FUNCTION outerand

Applicable types and ranks:
  T ≡ any logical type

Types and ranks implemented (overloaded) in nrutil:
  T ≡ LOGICAL(LGT)

Action:
Returns a matrix that is the outer logical and of two vectors.

Reference implementation:
outerand = spread(a,dim=2,ncopies=size(b)) .and. &
          spread(b,dim=1,ncopies=size(a))

23.6 Routines for Scatter with Combine

These are common parallel functions that Fortran 90 simply doesn’t provide
a means for implementing. If you have a parallel machine, you should substitute
library routines specific to your hardware.

⋆⋆⋆

scatter_add  (scatter-add source to specified components of destination)

User interface (or, “USE nrutil”):
SUBROUTINE scatter_add(dest,source,dest_index)
  T, DIMENSION(:,), INTENT(OUT) :: dest
  T, DIMENSION(:,), INTENT(IN) :: source
  INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: dest_index
END SUBROUTINE scatter_add

Applicable types and ranks:
  T ≡ any numerical type

Types and ranks implemented (overloaded) in nrutil:
  T ≡ REAL(SP), REAL(DP)
**Action:**

Adds each component of the array `source` into a component of `dest` specified by the index array `dest_index`. (The user will usually have zeroed `dest` before the call to this routine.) Note that `dest_index` has the size of `source`, but must contain values in the range from 1 to \(\text{size}(\text{dest})\), inclusive. Out-of-range values are ignored. There is no parallel implementation of this routine accessible from Fortran 90; most parallel machines supply an implementation as a library routine.

**Reference implementation:**

```fortran
 INTEGER(I4B) :: m,n,j,i
 n=assert_eq(size(source),size(dest_index),'scatter_add')
 m=size(dest)
 do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=dest(i)+source(j)
 end do
```

### scatter_max  
(scatter-max source to specified components of destination)

**User interface (or, "USE nrutil"):**

```fortran
 SUBROUTINE scatter_max(dest,source,dest_index)
   T, DIMENSION(:), INTENT(OUT) :: dest
   T, DIMENSION(:), INTENT(IN) :: source
   INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
 END SUBROUTINE scatter_max
```

**Applicable types and ranks:**

\(T\) ≡ any integer or real type

**Types and ranks implemented (overloaded) in nrutil:**

\(T\) ≡ REAL(SP), REAL(DP)

**Action:**

Takes the `max` operation between each component of the array `source` and a component of `dest` specified by the index array `dest_index`, replacing that component of `dest` with the value obtained ("maxing into" operation). (The user will often want to fill the array `dest` with the value \(-\)huge before the call to this routine.) Note that `dest_index` has the size of `source`, but must contain values in the range from 1 to \(\text{size}(\text{dest})\), inclusive. Out-of-range values are ignored. There is no parallel implementation of this routine accessible from Fortran 90; most parallel machines supply an implementation as a library routine.

**Reference implementation:**

```fortran
 INTEGER(I4B) :: m,n,j,i
 n=assert_eq(size(source),size(dest_index),'scatter_max')
 m=size(dest)
 do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=max(dest(i),source(j))
 end do
```
23.7 Routines for Skew Operations on Matrices

These are also missing parallel capabilities in Fortran 90. In Appendix C1 they are coded serially, with one or more do-loops.

⋆⋆⋆

diagadd (adds vector to diagonal of a matrix)

User interface (or, "USE nrutil"):  
SUBROUTINE diagadd(mat,diag)  
T, DIMENSION(:,,:), INTENT(INOUT) :: mat  
T, DIMENSION(:,), INTENT(IN) :: diag  
END SUBROUTINE diagadd

Applicable types and ranks:
T ≡ any numerical type

Types and ranks implemented (overloaded) in nrutil:
T ≡ REAL(SP)

Action:
The argument diag, either a scalar or else a vector whose size must be the smaller of the two dimensions of matrix mat, is added to the diagonal of the matrix mat. The following shows an implementation where diag is a vector; the scalar case can be overloaded (see Appendix C1).

Reference implementation:
INTEGER(I4B) :: j,n  
n = assert_eq(size(diag),min(size(mat,1),size(mat,2)),'diagadd')  
do j=1,n  
   mat(j,j)=mat(j,j)+diag(j)  
end do

⋆⋆⋆

diagmult (multiplies vector into diagonal of a matrix)

User interface (or, "USE nrutil"):  
SUBROUTINE diagmult(mat,diag)  
T, DIMENSION(:,,:), INTENT(INOUT) :: mat  
T, DIMENSION(:,), INTENT(IN) :: diag  
END SUBROUTINE diagmult

Applicable types and ranks:
T ≡ any numerical type

Types and ranks implemented (overloaded) in nrutil:
T ≡ REAL(SP)

Action:
The argument diag, either a scalar or else a vector whose size must be the smaller of the two dimensions of matrix mat, is multiplied onto the diagonal of the matrix mat. The following shows an implementation where diag is a vector; the scalar case can be overloaded (see Appendix C1).
23.7 Routines for Skew Operations on Matrices

Reference implementation:

```fortran
INTEGER(I4B) :: j,n
n = assert_eq(size(diag),min(size(mat,1),size(mat,2)),'diagmult')
do j=1,n
   mat(j,j)=mat(j,j)*diag(j)
end do
```

get\_diag  (gets diagonal of matrix)

User interface (or, “USE nrutil”):

```fortran
FUNCTION get\_diag(mat)
   T, DIMENSION(:,:), INTENT(IN) :: mat
   T, DIMENSION(min(size(mat,1),size(mat,2))) :: get\_diag
END FUNCTION get\_diag
```

Applicable types and ranks:

\[ T \equiv \text{any type} \]

Types and ranks implemented (overloaded) in nrutil:

\[ T \equiv \text{REAL(SP)}, \text{REAL(DP)} \]

Action:

Returns a vector containing the diagonal values of the matrix \( mat \).

Reference implementation:

```fortran
INTEGER(I4B) :: j
do j=1,min(size(mat,1),size(mat,2))
   get\_diag(j)=mat(j,j)
end do
```

put\_diag  (sets the diagonal elements of a matrix)

User interface (or, “USE nrutil”):

```fortran
SUBROUTINE put\_diag(diag,mat)
   T, DIMENSION(:), INTENT(IN) :: diag
   T, DIMENSION(:,:), INTENT(INOUT) :: mat
END SUBROUTINE put\_diag
```

Applicable types and ranks:

\[ T \equiv \text{any type} \]

Types and ranks implemented (overloaded) in nrutil:

\[ T \equiv \text{REAL(SP)} \]

Action:

Sets the diagonal of matrix \( mat \) equal to the argument \( diag \), either a scalar or else a vector whose size must be the smaller of the two dimensions of matrix \( mat \). The following shows an implementation where \( diag \) is a vector; the scalar case can be overloaded (see Appendix C1).

Reference implementation:

```fortran
INTEGER(I4B) :: j,n
n=assert_eq(size(diag),min(size(mat,1),size(mat,2)),'put\_diag')
do j=1,n
   mat(j,j)=diag(j)
end do
```
unit_matrix  (returns a unit matrix)

User interface (or, “USE nrutil”):

```fortran
SUBROUTINE unit_matrix(mat)
   T, DIMENSION(:,,:), INTENT(OUT) :: mat
END SUBROUTINE unit_matrix
```

Applicable types and ranks:

\[ T \equiv \text{any numerical type} \]

Types and ranks implemented (overloaded) in nrutil:

\[ T \equiv \text{REAL(SP)} \]

Action:

Sets the diagonal components of \( mat \) to unity, all other components to zero. When \( mat \) is square, this will be the unit matrix; otherwise, a unit matrix with appended rows or columns of zeros.

Reference implementation:

```fortran
INTEGER(I4B) :: i,n
n=min(size(mat,1),size(mat,2))
mat(:,:,)=0.0
do i=1,n
   mat(i,i)=1.0
end do
```

upper_triangle  (returns an upper triangular mask)

User interface (or, “USE nrutil”):

```fortran
FUNCTION upper_triangle(j,k,extra)
   INTEGER(I4B), INTENT(IN) :: j,k
   INTEGER(I4B), OPTIONAL, INTENT(IN) :: extra
   LOGICAL(LGT), DIMENSION(j,k) :: upper_triangle
END FUNCTION upper_triangle
```

Action:

When the optional argument \( extra \) is zero or absent, returns a logical mask of shape \((j,k)\) whose values are true above and to the right of the diagonal, false elsewhere (including on the diagonal). When \( extra \) is present and positive, a corresponding number of additional (sub-)diagonals are returned as true. (\( extra = 1 \) makes the main diagonal return true.) When \( extra \) is present and negative, it suppresses a corresponding number of superdiagonals.

Reference implementation:

```fortran
INTEGER(I4B) :: n,jj,kk
n=0
if (present(extra)) n=extra
do jj=1,j
   do kk=1,k
      upper_triangle(jj,kk)= (jj-kk < n)
   end do
end do
```

lower_triangle  (returns a lower triangular mask)

User interface (or, “USE nrutil”):

FUNCTION lower_triangle(j,k,extra)
  INTEGER(I4B), INTENT(IN) :: j,k
  INTEGER(I4B), OPTIONAL, INTENT(IN) :: extra
  LOGICAL(LGT), DIMENSION(j,k) :: lower_triangle
END FUNCTION lower_triangle

Action:
When the optional argument extra is zero or absent, returns a logical mask of shape (j,k) whose values are true below and to the left of the diagonal, false elsewhere (including on the diagonal). When extra is present and positive, a corresponding number of additional (super-)diagonals are returned as true. (extra = 1 makes the main diagonal return true.) When extra is present and negative, it suppresses a corresponding number of subdiagonals.

Reference implementation:

INTEGER(I4B) :: n,jj,kk
n=0
if (present(extra)) n=extra
do jj=1,j
  do kk=1,k
    lower_triangle(jj,kk)= (kk-jj < n)
  end do
end do

Fortran 95’s forall construction will make the parallel implementation of all our skew operations utilities extremely simple. For example, the do-loop in diagadd will collapse to

forall (j=1:n) mat(j,j)=mat(j,j)+diag(j)

In fact, this implementation is so simple as to raise the question of whether a separate utility like diagadd will be needed at all. There are valid arguments on both sides of this question: The “con” argument, against a routine like diagadd, is that it is just another reserved name that you have to remember (if you want to use it). The “pro” argument is that a separate routine avoids the “index pollution” (the opposite disease from “index loss” discussed in §22.1) of introducing a superfluous variable j, and that a separate utility allows for additional error checking on the sizes and compatibility of its arguments. We expect that different programmers will have differing tastes.

The argument for keeping a routine like upper_triangle or lower_triangle, once Fortran 95’s masked forall constructions become available, is less persuasive. We recommend that you consider these two routines as placeholders for “remember to recode this in Fortran 95, someday.”

23.8 Other Routine(s)

You might argue that we don’t really need a routine for the idiom

sqrt(dot_product(v,v))
You might be right. The ability to overload the complex case, with its additional complex conjugate, is an argument in its favor, however.

* * *

**vabs**  \( (L_2 \text{ norm of a vector}) \)

*User interface (or, “USE nrutil”):*

```fortran
FUNCTION vabs(v)
   T, DIMENSION(:,), INTENT(IN) :: v
   T :: vabs
END FUNCTION vabs
```

*Applicable types and ranks:*

\[ \mathbf{T} \equiv \text{any real or complex type} \]

*Types and ranks implemented (overloaded) in nrutil:*

\[ \mathbf{T} \equiv \text{REAL(SP)} \]

*Action:*

Returns the length of a vector \( v \) in \( L_2 \) norm, that is, the square root of the sum of the squares of the components. (For complex types, the dot product should be between the vector and its complex conjugate.)

*Reference implementation:*

\[
vabs = \sqrt{\text{dot_product}(v,v)}
\]
Fortran 90 Code Chapters B1–B20

Fortran 90 versions of all the Numerical Recipes routines appear in the following Chapters B1 through B20, numbered in correspondence with Chapters 1 through 20 in Volume 1. Within each chapter, the routines appear in the same order as in Volume 1, but not broken out separately by section number within Volume 1’s chapters.

There are commentaries accompanying many of the routines, generally following the printed listing of the routine to which they apply. These are of two kinds: issues related to parallelizing the algorithm in question, and issues related to the Fortran 90 implementation. To distinguish between these two, rather different, kinds of discussions, we use the two icons, the left icon (above) indicating a “parallel note,” and the right icon denoting a “Fortran 90 tip.” Specific code segments of the routine that are discussed in these commentaries are singled out by reproducing some of the code as an “index line” next to the icon, or at the beginning of subsequent paragraphs if there are several items that are commented on.

\[
d=\text{merge}(\text{FPMIN},d,\text{abs}(d)<\text{FPMIN})
\]

This would be the start of a discussion of code that begins at the line in the listing containing the indicated code fragment.

A row of stars, like the above, is used between unrelated routines, or at the beginning and end of related groups of routines.

Some chapters contain discussions that are more general than commentary on individual routines, but that were deemed too specific for inclusion in Chapters 21 through 23. Here are some highlights of this additional material:

- Approximations to roots of orthogonal polynomials for parallel computation of Gaussian quadrature formulas (Chapter B4)
- Difficulty of, and tricks for, parallel calculation of special function values in a SIMD model of computation (Chapter B6)
- Parallel random number generation (Chapter B7)
- Fortran 90 tricks for dealing with ties in sorted arrays, counting things in boxes, etc. (Chapter B14)
- Use of recursion in implementing multigrid elliptic PDE solvers (Chapter B19)
Chapter B1. Preliminaries

SUBROUTINE flmoon(n, nph, jd, frac)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n, nph
INTEGER(I4B), INTENT(OUT) :: jd
REAL(SP), INTENT(OUT) :: frac

Our programs begin with an introductory comment summarizing their purpose and explaining their calling sequence. This routine calculates the phases of the moon. Given an integer \( n \) and a code \( nph \) for the phase desired (\( nph = 0 \) for new moon, 1 for first quarter, 2 for full, 3 for last quarter), the routine returns the Julian Day Number \( jd \) and the fractional part of a day \( frac \) to be added to it, of the \( n \)th such phase since January, 1900. Greenwich Mean Time is assumed.

REAL(SP), PARAMETER :: RAD=PI/180.0_sp
INTEGER(I4B) :: i
REAL(SP) :: am, as, c, t, t2, xtra

c = n+nph/4.0_sp

This is how we comment an individual line.
t = c/1236.85_sp
t2 = t**2
as = 359.2242_sp + 29.105356_sp * c
You aren’t really intended to understand this algorithm, but it does work!
am = 306.0253_sp + 385.816918_sp * c + 0.010730_sp * t2
jd = 2415020 + 28 * n + 7 * nph
xtra = 0.75933_sp + 1.53058868_sp * c + (1.178e-4_sp - 1.55e-7_sp * t) * t2

select case(nph)
case(0,2)
xtra = xtra + (0.1734_sp - 3.93e-4_sp * t) * sin(RAD * as) - 0.4068_sp * sin(RAD * am)
case(1,3)
xtra = xtra + (0.1721_sp - 4.0e-4_sp * t) * sin(RAD * as) - 0.6280_sp * sin(RAD * am)
case default
call nrerror('flmoon: nph is unknown')
end select
i = int(merge(xtra, xtra - 1.0_sp, xtra >= 0.0))
jd = jd + i
frac = xtra - i
END SUBROUTINE flmoon

Fortran 90 includes a `case` construction that executes at most one of several blocks of code, depending on the value of an integer, logical, or character expression. Ideally, the `case` construction will execute more efficiently than a long sequence of cascaded `if...else if...else if...` constructions. C programmers should note that the Fortran 90 construction, perhaps mercifully, does not have C’s “drop-through” feature.

The `merge` construction in Fortran 90, while intended primarily for use with vector arguments, is also a convenient way of generating conditional scalar expressions, that is, expressions with one value, or another, depending on the result of a logical test.
When the arguments of a `merge` are vectors, parallelization by the compiler is straightforward as an array parallel operation (see p. 964). Less obvious is how the scalar case, as above, is handled. For small-scale parallel (SSP) machines, the natural gain is via speculative evaluation of both of the first two arguments simultaneously with evaluation of the test.

A good compiler should not penalize a scalar machine for use of either the scalar or vector `merge` construction. The Fortran 90 standard states that “it is not necessary for a processor to evaluate all of the operands of an expression, or to evaluate entirely each operand, if the value of the expression can be determined otherwise.” Therefore, for each test on a scalar machine, only one or the other of the first two argument components need be evaluated.

```
FUNCTION julday(mm, id, iyyy)
USE nrtype; USE nrule, ONLY : nrerror
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: mm, id, iyyy
INTEGER(I4B) :: julday
In this routine `julday` returns the Julian Day Number that begins at noon of the calendar
date specified by month `mm`, day `id`, and year `iyyy`, all integer variables. Positive year
signifies A.D.; negative, B.C. Remember that the year after 1 B.C. was 1 A.D.
INTEGER(I4B), PARAMETER :: IGREG=15+31*(10+12*1582) Gregorian Calendar adopted
INTEGER(I4B) :: ja, jm, jy
jy=iyyy
if (jy == 0) call nrerror('julday: there is no year zero')
if (jy < 0) jy=jy+1
if (mm > 2) then
  Here is an example of a block IF-structure.
  jm=mm+1
else
  jm=mm+13
end if
julday=int(365.25_sp*jy)+int(30.6001_sp*jm)+id+1720995
if (ID+31*(mm+12*iyyy) >= IGREG) then
  Test whether to change to Gregorian Calendar.
  ja=int(0.01_sp*jy)
  julday=julday+2-ja+int(0.25_sp*ja)
end if
END FUNCTION julday
```

```
PROGRAM badluk
USE nrtype
USE nr, ONLY : f1moon, julday
IMPLICIT NONE
INTEGER(I4B) :: ic, icon, iduk, ifrac, im, iyyy, jd, jday, n
INTEGER(I4B) :: iybeg=1900, iyend=2000 The range of dates to be searched.
REAL(SP) :: frac
REAL(SP), PARAMETER :: TIMZON=-5.0_sp/24.0_sp Time zone -5 is Eastern Standard Time.
write (*,'(1x,a,i5,a,i5)') 'Full moons on Friday the 13th from',&
ybeg,' to',iyend
do iyyy=iybeg,iyend Loop over each year,
do im=1,12 and each month.
jday=julday(im,13,iyyy) Is the 13th a Friday?
iduk=mod(jday+1,7)
```
if (idwk == 5) then
  n=12.37_sp*(iyyy-1900+(im-0.5_sp)/12.0_sp)
  This value n is a first approximation to how many full moons have occurred
  since 1900. We will feed it into the phase routine and adjust it up or down until
  we determine that our desired 13th was or was not a full moon. The variable
  icon signals the direction of adjustment.
  icon=0
  do
    call flmoon(n,2,jd,frac)  Get date of full moon n.
    ifrac=nint(24.0_sp*(frac+TIMZON))  Convert to hours in correct time
      if (ifrac < 0) then
        jd=jd-1  Convert from Julian Days beginning at noon
        ifrac=ifrac+24  to civil days beginning at midnight.
      end if
      if (ifrac > 12) then
        jd=jd+1
        ifrac=ifrac-12
      else
        ifrac=ifrac+12
      end if
      if (jd == jday) then
        Did we hit our target day?
        write (*,'(/1x,i2,a,i2,a,i4)') im,'/',13,'/',iyyy
        write (*,'(1x,a,i2,a)') 'Full moon ',ifrac,
        ' hrs after midnight (EST).'
        Don't worry if you are unfamiliar with FORTRAN's esoteric input/output
        statements; very few programs in this book do any input/output.
        exit  Part of the break-structure, case of a match.
      else
        Didn't hit it.
        ic=isign(1,jday-jd)
        if (ic == -icon) exit  Another break, case of no match.
        icon=ic
        n=n+ic
      end if
    end do
  end if
end do
END PROGRAM badluk

...JREG=15+31*(10+12*1582) (in julday). ...TIMZON=-5.0_sp/24.0_sp
(in badluk) These are two examples of initialization expressions for
"named constants" (that is, PARAMETERS). Because the initialization
expressions will generally be evaluated at compile time, Fortran 90 puts some
restrictions on what kinds of intrinsic functions they can contain. Although the
evaluation of a real expression like -5.0_sp/24.0_sp ought to give identical results
at compile time and at execution time, all the way down to the least significant
bit, in our opinion the conservative programmer shouldn't count on strict identity at
the level of floating-point roundoff error. (In the special case of cross-compilers,
such roundoff-level discrepancies between compile time and run time are almost
inevitable.)
SUBROUTINE caldat(julian, mm, id, iyyy)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: julian
INTEGER(I4B), INTENT(OUT) :: mm, id, iyyy

Inverse of the function julday given above. Here julian is input as a Julian Day Number, and the routine outputs mm, id, and iyyy as the month, day, and year on which the specified Julian Day started at noon.

INTEGER(I4B) :: ja, jalpha, jb, jc, jd, je
INTEGER(I4B), PARAMETER :: IGREG=2299161

if (julian >= IGREG) then
  Cross-over to Gregorian Calendar produces this correction.
  jalpha=int(((julian-1867216)-0.25_sp)/36524.25_sp)
  ja=julian+1+jalpha-int(0.25_sp*jalpha)
else if (julian < 0) then
  Make day number positive by adding integer number of Julian centuries, then subtract them off at the end.
  ja=julian+36525*(1-julian/36525)
else
  ja=julian
end if

jb=ja+1524
jc=int(6680.0_sp+((jb-2439870)-122.1_sp)/365.25_sp)
jd=365*jc+int(0.25_sp*jc)
je=int((jb-jd)/30.6001_sp)
id=jb-jd-int(30.6001_sp*je)
mm=je-1
if (mm > 12) mm=mm-12
iyyy=jc-4715
if (mm > 2) iyyy=iyyy-1
if (iyyy <= 0) iyyy=iyyy-1
if (julian < 0) iyyy=iyyy-100*(1-julian/36525)
END SUBROUTINE caldat
SUBROUTINE gaussj(a,b)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror,outerand,outerprod,swap
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a,b
Linear equation solution by Gauss-Jordan elimination, equation (2.1.1). a is an \( N \times N \) input
coefficient matrix. b is an \( N \times M \) input matrix containing \( M \) right-hand-side vectors. On
output, a is replaced by its matrix inverse, and b is replaced by the corresponding set of
solution vectors.
INTEGER(I4B), DIMENSION(size(a,1)) :: ipiv,indxr,indxc
These arrays are used for bookkeeping on the pivoting.
LOGICAL(LGT), DIMENSION(size(a,1)) :: lpiv
REAL(SP) :: pivinv
REAL(SP), DIMENSION(size(a,1)) :: dumc
INTEGER(I4B), TARGET :: irc(2)
INTEGER(I4B) :: i,l,n
INTEGER(I4B), POINTER :: irow,icol
n=assert_eq(size(a,1),size(a,2),size(b,1),'gaussj')
irow => irc(1)
icol => irc(2)
ipiv=0

do i=1,n  
  Main loop over columns to be reduced.
  lpiv = (ipiv == 0)  
  Begin search for a pivot element.
  irc=maxloc(abs(a),outerand(lpiv,lpiv))
  ipiv(icol)=ipiv(icol)+1
  if (ipiv(icol) > 1) call nrerror('gaussj: singular matrix (1)')
  We now have the pivot element, so we interchange rows, if needed, to put the pivot
  element on the diagonal. The columns are not physically interchanged, only relabeled:
  indxc(i), the column of the ith pivot element, is the ith column that is reduced, while
  indxr(i) is the row in which that pivot element was originally located. If indxr(i) \( \neq \) indxc(i) there is an implied column interchange. With this form of bookkeeping,
  the solution b's will end up in the correct order, and the inverse matrix will be scrambled
  by columns.
  if (irow /= icol) then
    call swap(a(irow,:),a(icol,:))
    call swap(b(irow,:),b(icol,:))
  end if
  indxr(i)=irow  
  We are now ready to divide the pivot row by the pivot
  indxc(i)=icol
  if (a(icol,icol) == 0.0) &
    call nrerror('gaussj: singular matrix (2)')
pivinv=1.0_sp/a(icol,icol)
a(icol,icol)=1.0
a(icol,:) = a(icol,:) * pivinv
b(icol,:) = b(icol,:) * pivinv
dumc=a(:,icol)
next, we reduce the rows, except for the pivot one, of
a(:,icol)=0.0
Chapter B2. Solution of Linear Algebraic Equations

a(icol,icol)=pivinv
a(1:icol-1,:) =a(1:icol-1,:)−outerprod(dumc(1:icol-1),a(icol,:))
b(1:icol-1,:) =b(1:icol-1,:)−outerprod(dumc(1:icol-1),b(icol,:))
a(icol+1:,:) =a(icol+1:,:)−outerprod(dumc(icol+1:),a(icol,:))
b(icol+1:,:) =b(icol+1:,:)−outerprod(dumc(icol+1:),b(icol,:))
end do

It only remains to unscramble the solution in view of the column interchanges. We do this by interchanging pairs of columns in the reverse order that the permutation was built up.
do l=n,1,-1
call swap(a(:,indxr(l)),a(:,indxc(l)))
end do
END SUBROUTINE gaussj

irow => irc(1) ... icol => irc(2) The maxloc intrinsic returns the location of the maximum value of an array as an integer array, in this case of size 2. Pre-pointing pointer variables to components of the array that will be thus set makes possible convenient references to the desired row and column positions.

irc=maxloc(abs(a),outerand(lpiv,lpiv)) The combination of maxloc and one of the outer... routines from nrutil allows for a very concise formulation. If this task is done with loops, it becomes the ungainly “flying vee,”

aa=0.0
do i=1,n
   if (lpiv(i)) then
      do j=1,n
         if (lpiv(j)) then
            if (abs(a(i,j)) > aa) then
               aa=abs(a(i,j))
               irow=i
               icol=j
            endif
         endif
      end do
   endif
end do

end do
call swap(a(irow,:),a(icol,:)) The swap routine (in nrutil) is concise and convenient. Fortran 90’s ability to overload multiple routines onto a single name is vital here: Much of the convenience would vanish if we had to remember variant routine names for each variable type and rank of object that might be swapped.

Even better, here, than overloading would be if Fortran 90 allowed user-written elemental procedures (procedures with unspecified or arbitrary rank and shape), like the intrinsic elemental procedures built into the language. Fortran 95 will, but Fortran 90 doesn’t.

One quick (if superficial) test for how much parallelism is achieved in a Fortran 90 routine is to count its do-loops, and compare that number to the number of do-loops in the Fortran 77 version of the same routine. Here, in gaussj, 13 do-loops are reduced to 2.

a(1:icol-1,:)=...
b(1:icol-1,:)=...
a(icol+1:,:)=...
b(icol+1:,:)=
Here the same operation is applied to every row of \( a \), and to every row of \( b \), except row number \( \text{icol} \). On a massively multiprocessor (MMP) machine it would be better to use a logical mask and do all of \( a \) in a single statement, all of \( b \) in another one. For a small-scale parallel (SSP) machine, the lines as written should saturate the machine’s concurrency, and they avoid the additional overhead of testing the mask.

This would be a good place to point out, however, that linear algebra routines written in Fortran 90 are likely never to be competitive with the hand-coded library routines that are generally supplied as part of MMP programming environments. If you are using our routines instead of library routines written specifically for your architecture, you are wasting cycles!

\* \* \*

SUBROUTINE ludcmp(a,indx,d)
USE nrttype; USE nrutil, ONLY : assert_eq,imaxloc,nrerror,outerprod,swap
IMPLICIT NONE
REAL(SP), DIMENSION(:,::), INTENT(INOUT) :: a
INTEGER(I4B), DIMENSION(:,::), INTENT(OUT) :: indx
REAL(SP), INTENT(OUT) :: :: d

Given an \( N \times N \) input matrix \( a \), this routine replaces it by the \( LU \) decomposition of a rowwise permutation of itself. On output, \( a \) is arranged as in equation (2.3.14); \( \text{indx} \) is an output vector of length \( N \) that records the row permutation effected by the partial pivoting; \( d \) is output as \( \pm 1 \) depending on whether the number of row interchanges was even or odd, respectively. This routine is used in combination with \text{lubksb} to solve linear equations or invert a matrix.

REAL(SP), DIMENSION(size(a,1)) :: vv
vv stores the implicit scaling of each row.
REAL(SP), PARAMETER :: TINY=1.0e-20_sp
A small number.

n=assert_eq(size(a,1),size(a,2),size(indx),'ludcmp')

d=1.0
No row interchanges yet.

vv=maxval(abs(a),dim=2)
Loop over rows to get the implicit scaling information.
If there is a row of zeros.

vv=1.0_sp/vv
Save the scaling.
do j=1,n
imax=(j-1)+imaxloc(vv(j:n)*abs(a(j:n,j)))
Find the pivot row.
if (j /= imax) then
Do we need to interchange rows?
call swap(a(imax,:),a(j,:))
Yes, do so...
d=-d
...and change the parity of \( d \).
vv(imax)=vv(j)
Also interchange the scale factor.
end if
indx(j)=imax
if (a(j,j) == 0.0) call nrerror('singular matrix in ludcmp')

a(j+1:n,j)=a(j+1:n,j)/a(j,j)
Divide by the pivot element.
a(j+1:n,j+1:n)=a(j+1:n,j+1:n)-outerprod(a(j+1:n,j),a(j,j+1:n))
Reduce remaining submatrix.

end do
END SUBROUTINE ludcmp

vv=maxval(abs(a),dim=2) A single statement finds the maximum absolute value in each row. Fortran 90 intrinsics like \text{maxval} generally “do their thing” in the dimension specified by \text{dim} and return a result with a shape corresponding to the \text{other} dimensions. Thus, here, \( vv \)'s size is that of the \text{first} dimension of \( a \).
Here we see why the \texttt{nrutil} routine \texttt{imaxloc} is handy: We want the index, in the range \(1:n\) of a quantity to be searched for only in the limited range \(j:n\). Using \texttt{imaxloc}, we just add back the proper offset of \(j-1\). (Using only Fortran 90 intrinsics, we could write \[ \text{imax}=(j-1)+\text{sum}(\text{maxloc}(vv(j:n)\times\text{abs}(a(j:n,j)))) \], but the use of \texttt{sum} just to turn an array of length 1 into a scalar seems sufficiently confusing as to be avoided.)

The Fortran 77 version of \texttt{ludcmp}, using Crout's algorithm for the reduction, does not parallelize well: The elements are updated by \(O(N^2)\) separate dot product operations in a particular order. Here we use a slightly different reduction, termed “outer product Gaussian elimination” by Golub and Van Loan [1], that requires just \(N\) steps of matrix-parallel reduction. (See their §3.2.3 and §3.2.9 for the algorithm, and their §3.4.1 to understand how the pivoting is performed.)

We use \texttt{nrutil}'s routine \texttt{outerprod} instead of the more cumbersome pure Fortran 90 construction:

\begin{verbatim}
spread(a(j+1:n,j+1:n),dim=2,ncopies=n-j)*spread(a(j,j+1:n),dim=1,ncopies=n-j)
\end{verbatim}

\begin{verbatim}
SUBROUTINE lubksb(a,indx,b)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,:), INTENT(IN) :: a
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: indx
REAL(SP), DIMENSION(:), INTENT(INOUT) :: b
Solves the set of \(N\) linear equations \(A \cdot X = B\). Here the \(N \times N\) matrix \(a\) is input, not as the original matrix \(A\), but rather as its \(LU\) decomposition, determined by the routine \texttt{ludcmp}. \texttt{indx} is input as the permutation vector of length \(N\) returned by \texttt{ludcmp}. \texttt{b} is input as the right-hand-side vector \(B\), also of length \(N\), and returns with the solution vector \(X\). \(a\) and \texttt{indx} are not modified by this routine and can be left in place for successive calls with different right-hand sides \(b\). This routine takes into account the possibility that \(b\) will begin with many zero elements, so it is efficient for use in matrix inversion.

INTEGER(I4B) :: i,n,ii,ll
REAL(SP) :: summ
n=assert_eq(size(a,1),size(a,2),size(indx),'lubksb')
ii=0

When \(ii\) is set to a positive value, it will become the index of the first nonvanishing element of \(b\). We now do the forward substitution, equation (2.3.6). The only new wrinkle is to unscramble the permutation as we go.

\begin{verbatim}
ii=0
do i=1,n
  ii=index(i)
  summ=b(ii)
  b(ii)=b(i)
  if (ii /= 0) then
    summ=summ-dot_product(a(i,ii:i-1),b(ii:i-1))
  else if (summ /= 0.0) then
    ii=i
    A nonzero element was encountered, so from now on we will have to do the dot product above.
  end if
  b(i)=summ
end do

Now we do the backsubstitution, equation (2.3.7).
\end{verbatim}

Conceptually, the search for the first nonvanishing element of \(b\) (index \(ii\)) should be moved out of the first do-loop. However, in practice, the need to unscramble the permutation, and also considerations of performance
on scalar machines, cause us to write this very scalar-looking code. The performance penalty on parallel machines should be minimal.

⋆⋆⋆

Serial and parallel algorithms for tridiagonal problems are quite different. We therefore provide separate routines \texttt{tridag}ser and \texttt{tridag}par. In the \texttt{MODULE nr} interface file, one or the other of these (your choice) is given the generic name \texttt{tridag}. Of course, either version will work correctly on any computer; it is only a question of efficiency. See §22.2 for the numbering of the equation coefficients, and for a description of the parallel algorithm.

```
SUBROUTINE tridag_ser(a,b,c,r,u)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a,b,c,r
REAL(SP), DIMENSION(:), INTENT(OUT) :: u

Solves for a vector \textit{u} of size \textit{N} the tridiagonal linear set given by equation (2.4.1) using a serial algorithm. Input vectors \textit{b} (diagonal elements) and \textit{r} (right-hand sides) have size \textit{N}, while \textit{a} and \textit{c} (off-diagonal elements) are size \textit{N}−1.

REAL(SP), DIMENSION(size(b)) :: gam One vector of workspace, \textit{gam} is needed.

INTEGER(I4B) :: n,j
REAL(SP) :: bet
n=assert_eq((/size(a)+1,size(b),size(c)+1,size(r),size(u)/),'tridag_ser')
bet=b(1)
if (bet == 0.0) call nrerror('tridag_ser: Error at code stage 1')
If this happens then you should rewrite your equations as a set of order \textit{N}−1, with \textit{u}_2 trivially eliminated.

\textit{u}(1)=r(1)/bet

\textbf{do} \textit{j}=2,\textit{n}
\textbf{Decomposition and forward substitution.}
\textit{gam}(j)=c(j-1)/bet
bet=b(j)-a(j-1)*\textit{gam}(j)
if (bet == 0.0) \&
\textbf{Algorithm fails; see below routine in Vol. 1.}
call nrerror('tridag_ser: Error at code stage 2')
\textit{u}(j)=(r(j)-a(j-1)*\textit{u}(j-1))/bet
\textbf{end do}

\textbf{do} \textit{j}=n-1,1,-1
\textbf{Backsubstitution.}
\textit{u}(j)=\textit{u}(j)-\textit{gam}(j+1)*\textit{u}(j+1)
\textbf{end do}

END SUBROUTINE tridag_ser
```

```
RECURSIVE SUBROUTINE tridag_par(a,b,c,r,u)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : tridag_ser
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a,b,c,r
REAL(SP), DIMENSION(:), INTENT(OUT) :: u

Solves for a vector \textit{u} of size \textit{N} the tridiagonal linear set given by equation (2.4.1) using a parallel algorithm. Input vectors \textit{b} (diagonal elements) and \textit{r} (right-hand sides) have size \textit{N}, while \textit{a} and \textit{c} (off-diagonal elements) are size \textit{N}−1.

INTEGER(I4B), PARAMETER :: NPAR_TRIDAG=4

\textbf{Determines when serial algorithm is invoked.}

INTEGER(I4B) :: n,n2,nn,nx
REAL(SP), DIMENSION(size(b)/2) :: y,q,piva
REAL(SP), DIMENSION(size(b)/2-1) :: x,z
REAL(SP), DIMENSION(size(a)/2) :: pivc
n=assert_eq((/size(a)+1,size(b),size(c)+1,size(r),size(u)/),'tridag_par')
if (n < NPAR_TRIDAG) then
\textbf{call tridag_ser(a,b,c,r,u)}
\textbf{else}
\textbf{if (maxval(abs(b(1:n))) == 0.0) \\ Algorithm fails; see below routine in Vol. 1.}
```
The serial version `tridag_ser` is called when the routine has recursed its way down to sufficiently small subproblems. The point at which this occurs is determined by the parameter `NPAR_TRIDAG` whose optimal value is likely machine-dependent. Notice that `tridag_ser` must here be called by its specific name, not by the generic `tridag` (which might itself be overloaded with either `tridag_ser` or `tridag_par`).

```
SUBROUTINE banmul(a,m1,m2,x,b)
USE nrtype; USE nrutil, ONLY : assert_eq, arth
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: a
INTEGER(I4B), INTENT(IN) :: m1,m2
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(OUT) :: b
!
Matrix multiply $b = A \cdot x$, where $A$ is band diagonal with $m_1$ rows below the diagonal and $m_2$ rows above. If the input vector $x$ and output vector $b$ are of length $N$, then the array $a(1:N,1:m1+m2+1)$ stores $A$ as follows: The diagonal elements are in $a(1:N,m1+1)$. Subdiagonal elements are in $a(j:N,1:m1)$ (with $j > 1$ appropriate to the number of elements on each subdiagonal). Superdiagonal elements are in $a(1:j,m1+2:m1+m2+1)$ with $j < N$ appropriate to the number of elements on each superdiagonal.

INTEGER(I4B) :: m,n
n=assert_eq(size(a,1),size(b),size(x),'banmul: n')
m=assert_eq(size(a,2),m1+m2+1,'banmul: m')
b=sum(a*eoshift(spread(x,dim=2,ncopies=m), &
  dim=1,shift=arth(-m1,1,m)),dim=2)
END SUBROUTINE banmul
```

This is a good example of Fortran 90 at both its best and its worst: best, because it allows quite subtle combinations of fully parallel operations to be built up; worst, because the resulting code is virtually incomprehensible!
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What is going on becomes clearer if we imagine a temporary array \( y \) with a declaration like \( \text{REAL(SP), DIMENSION(size(a,1),size(a,2)) :: y} \). Then, the above single line decomposes into

\[
y = \text{spread}(x, \text{dim}=2, \text{ncopies}=m) \quad \text{[Duplicate } x \text{ into columns of } y.]
\]

\[
y = \text{eoshift}(y, \text{dim}=1, \text{shift}=\text{arth}(-m1,1,m)) \quad \text{[Shift columns by a linear progression.]}\]

\[
b = \text{sum}(a*y, \text{dim}=2) \quad \text{[Multiply by the band-diagonal elements, and sum.]}\]

We use here a relatively rare subcase of the \text{eoshift} intrinsic, using a vector value for the \text{shift} argument to accomplish the simultaneous shifting of a bunch of columns, by different amounts (here specified by the linear progression returned by \text{arth}).

If you still don’t see how this accomplishes the multiplication of a band diagonal matrix by a vector, work through a simple example by hand.

SUBROUTINE bandec(a,m1,m2,al,indx,d)
USE nrtype; USE nrutil, ONLY : assert_eq,imaxloc,swap,arth
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: m1,m2
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: al
INTEGER(I4B), DIMENSION(:,), INTENT(OUT) :: indx
REAL(SP), INTENT(OUT) :: d
REAL(SP), PARAMETER :: TINY=1.0e-20_sp

Given an \( N \times N \) band diagonal matrix \( A \) with \( m1 \) subdiagonal rows and \( m2 \) superdiagonal rows, compactly stored in the array \( a(1:N, 1:m1+m2+1) \) as described in the comment for routine \text{banmul}, this routine constructs an \( LU \) decomposition of a rowwise permutation of \( A \). The upper triangular matrix replaces \( a \), while the lower triangular matrix is returned in \( \text{al}(1:N, 1:m1) \). \text{indx} is an output vector of length \( N \) that records the row permutation effected by the partial pivoting; \( d \) is output as \( \pm 1 \) depending on whether the number of row interchanges was even or odd, respectively. This routine is used in combination with \text{banbks} to solve band-diagonal sets of equations.

INTEGER(I4B) :: i,k,l,mdum,mm,n
REAL(SP) :: dum

\[
n = \text{assert_eq}(\text{size}(a,1), \text{size}(al,1), \text{size}(indx), 'bandec: n')
\]

\[
mm = \text{assert_eq}(\text{size}(a,2), m1+m2+1, 'bandec: mm')
\]

\[
mdum = \text{assert_eq}(\text{size}(al,2), m1, 'bandec: mdum')
\]

\[
a(1:m1,:) = \text{eoshift}(a(1:m1,:), \text{dim}=2, \text{shift}=\text{arth}(-m1,-1,m1)) \quad \text{Rearrange the storage a bit.}
\]

\[
d = 1.0
\]

\[
do k=1,n
\]

\[
l = \text{imin}(m1+k,n)
\]

\[
i = \text{imaxloc}((\text{abs} (a(k:1,1))) + k-1)
\]

\[
\text{dum} = a(i,1)
\]

\[
\text{if (} \text{dum} == 0.0 \text{)} a(k,1) = \text{TINY}
\]

\[
\text{Matrix is algorithmically singular, but proceed anyway with TINY pivot (desirable in some applications).}
\]

\[
\text{indx}(k) = i
\]

\[
\text{if (} i /= k \text{)} \quad \text{Interchange rows.}
\]

\[
d = -d
\]

\[
\text{call swap}(a(k,1:mm), a(i,1:mm))
\]

\[
\text{end if}
\]

\[
do i=k+1,l
\]

\[
\text{dum} = a(i,1)/a(k,1)
\]

\[
a(k,i-k) = \text{dum}
\]

\[
a(i,1:mm-1) = a(i,2:mm) - \text{dum} * a(k,2:mm)
\]

\[
a(i,mm) = 0.0
\]

\[
\text{end do}
\]

\[
\text{end do}
\]

END SUBROUTINE bandec
a(1:m1,:)=eoshift(a(1:m1,:),...) See similar discussion of eoshift for banmul, just above.

i=imaxloc(abs(a(k:1,1)))+k-1 See discussion of imaxloc on p. 1017.

Notice that the above is *not* well parallelized for MMP machines: the outer do-loop is done $N$ times, where $N$, the diagonal length, is potentially the largest dimension in the problem. Small-scale parallel (SSP) machines, and scalar machines, are not disadvantaged, because the parallelism of order $mn=m1+m2+1$ in the inner loops can be enough to saturate their concurrency.

We don’t know of an $N$-parallel algorithm for decomposing band diagonal matrices, at least one that has any reasonably concise expression in Fortran 90. Conceptually, one can view a band diagonal matrix as a *block tridiagonal* matrix, and then apply the same recursive strategy as was used in tridag_par. However, the implementation details of this are daunting. (We would welcome a user-contributed routine, clear, concise, and with parallelism of order $N$.)

**SUBROUTINE banbks(a,m1,m2,al,indx,b)**

Given the arrays $a$, $al$, and $indx$ as returned from bandec, and given a right-hand-side vector $b$, solves the band diagonal linear equations $A \cdot x = b$. The solution vector $x$ overwrites $b$. The other input arrays are not modified, and can be left in place for successive calls with different right-hand sides.

```fortran
INTEGER(I4B) :: i,k,l,mdum,mm,n
n=assert_eq(size(a,1),size(al,1),size(b),size(indx),'banbks: n')
mm=assert_eq(size(a,2),m1+m2+1,'banbks: mm')
mdum=assert_eq(size(al,2),m1,'banbks: mdum')
do k=1,n! Forward substitution, unscrambling the permuted rows as we go.
i=indx(k)
if (i /= k) call swap(b(i),b(k))
b(k+1:l)=b(k+1:l)-al(k,1:l-k)*b(k)
end do
do i=n,1,-1! Back substitution.
l=min(mm,m-n+1)
b(i)=(b(i)-dot_product(a(i,2:l),b(i:i+l-1)))/a(i,1)
end do
END SUBROUTINE banbks
```

As for bandec, the routine banbks is not parallelized on the large dimension $N$, though it does give the compiler the opportunity for ample small-scale parallelization inside the loops.

* * *
SUBROUTINE mprove(a, alud, indx, b, x)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : lubksb
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: a, alud
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: indx
REAL(SP), DIMENSION(:,), INTENT(IN) :: b
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: x

Improves a solution vector x of the linear set of equations $A \cdot X = B$. The $N \times N$ matrix $a$ and the $N$-dimensional vectors $b$ and $x$ are input. Also input is alud, the $L U$ decomposition of $a$ as returned by ludcmp, and the $N$-dimensional vector indx also returned by that routine. On output, only $x$ is modified, to an improved set of values.

INTEGER(I4B) :: ndum
REAL(SP), DIMENSION(size(a,1)) :: r
ndum=assert_eq((/size(a,1),size(a,2),size(alud,1),size(alud,2),size(b),& size(x),size(indx)/),'mprove')
r=matmul(real(a,dp),real(x,dp))-real(b,dp)
Calculate the right-hand side, accumulating the residual in double precision.
call lubksb(alud,indx,r) Solve for the error term,
x=x-r and subtract it from the old solution.
END SUBROUTINE mprove

assert_eq((/.../),'mprove') This overloaded version of the nrutil routine assert_eq makes use of a trick for passing a variable number of scalar arguments to a routine: Put them into an array constructor, (/.../), and pass the array. The receiving routine can use the size intrinsic to count them. The technique has some obvious limitations: All the arguments in the array must be of the same type; and the arguments are passed, in effect, by value, not by address, so they must be, in effect, INTENT(IN).

r=matmul(real(a,dp),real(x,dp))-real(b,dp) Since Fortran 90's elemental intrinsics operate with the type of their arguments, we can use the real(.,dp)'s to force the matmul matrix multiplication to be done in double precision, which is what we want. In Fortran 77, we would have to do the matrix multiplication with temporary double precision variables, both inconvenient and (since Fortran 77 has no dynamic memory allocation) a waste of memory.

SUBROUTINE svbksb_sp(u, w, v, b, x)
USE nrtype; USE nrutil, ONLY : assert_eq
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: u, v
REAL(SP), DIMENSION(:,), INTENT(IN) :: w, b
REAL(SP), DIMENSION(:,), INTENT(OUT) :: x

Solves $A \cdot X = B$ for a vector $X$, where $A$ is specified by the arrays $u$, $v$, $w$ as returned by svdcmp. Here $u$ is $M \times N$, $v$ is $N \times N$, and $w$ is of length $N$. $b$ is the $M$-dimensional input right-hand side. $x$ is the $N$-dimensional output solution vector. No input quantities are destroyed, so the routine may be called sequentially with different $b$'s.

INTEGER(I4B) :: ndum, ndum
REAL(SP), DIMENSION(size(x)) :: tmp
ndum=assert_eq(size(u,1),size(b),'svbksb_sp: ndum')
dnum=assert_eq((/size(u,2),size(v,1),size(v,2),size(w),size(x)/),& 'svbksb_sp: ndum')
where $(w /= 0.0)$
tmp=matmul(b, u)/w Calculate $\text{diag}(1/w_j)U^T B$,
elsewhere
tmp=0.0 but replace $1/w_j$ by zero if $w_j = 0.0$.

end where
Matrix multiply by \( V \) to get answer.

\[
x = \text{matmul}(v, \text{tmp})
\]

END SUBROUTINE svbksb_sp

Normally, when a \( \text{where} \)...
\( \text{elsewhere} \) construction is used to set a variable (here \( \text{tmp} \)) to one or another value, we like to replace it with a \textit{merge} expression. Here, however, the \( \text{where} \) is required to guarantee that a division by zero doesn't occur. The rule is that \( \text{where} \) \( \text{will never} \) evaluate expressions that are excluded by the mask in the \( \text{where} \) line, but other constructions, like \textit{merge}, \( \text{might} \) perform speculative evaluation of more than one possible outcome before selecting the applicable one.

Because singular value decomposition is something that one often wants to do in double precision, we include a double-precision version. In \texttt{nr}, the single- and double-precision versions are overloaded onto the name \texttt{svbksb}.

\[
x = \text{matmul}(v, \text{tmp})
\]

END SUBROUTINE svbksb_dp

Given an \( M \times N \) matrix \( a \), this routine computes its singular value decomposition, \( A = U \cdot W \cdot V^T \). The matrix \( U \) replaces \( a \) on output. The diagonal matrix of singular values \( W \) is output as \( w \). The \( N \times N \) matrix \( V \) (not the transpose \( V^T \)) is output as \( v \).

\[
\begin{align*}
\text{integer}(14b) & : i, it, j, k, l, m, n, \text{nm} \\
\text{real}(sp) & : a, \text{anorm}, c, f, g, h, s, \text{scale}, x, y, z \\
\text{real}(sp) & : \text{temp}, \text{rv1}, \text{tempm} \\
\text{integer}(14b) & : m, \text{nm} \\
\text{real}(sp) & : \text{temp}, \text{svdcmp_sp}
\end{align*}
\]

If \( i \leq m \) then
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scale = \text{sum}(\text{abs}(a(i:m,i)))
if (scale /= 0.0) then
a(i:m,i) = a(i:m,i)/scale
s = \text{dot_product}(a(i:m,i), a(i:m,i))
f = a(i,i)
g = \text{sign}(\sqrt{s}, f)
h = f * g
a(i,i) = f - g
\text{tempn}(l:n) = \text{matmul}(a(i:m,i), a(i:m,l:n))/h
a(i:m,l:n) = a(i:m,l:n) + \text{outerprod}(a(i:m,i), \text{tempn}(l:n))
a(i:m,i) = scale * a(i:m,i)
end if
end if

w(i) = scale * g
\text{g} = 0.0
scale = 0.0
if ((i <= m) .and. (i /= n)) then
scale = \text{sum}(\text{abs}(a(i,1:n)))
if (scale /= 0.0) then
a(i,1:n) = a(i,1:n)/scale
s = \text{dot_product}(a(i,1:n), a(i,1:n))
f = a(i,i)
g = \text{sign}(\sqrt{s}, f)
h = f * g
a(i,i) = f - g
rv1(l:n) = a(i,1:n)/h
\text{tempn}(l:n) = \text{matmul}(a(l:m,1:n), a(i,1:n))
a(l:m,1:n) = a(l:m,1:n) + \text{outerprod}(\text{tempn}(l:m), rv1(l:n))
a(i,1:n) = scale * a(i,1:n)
end if
end if
end do

\text{anorm} = \text{maxval}(\text{abs}(w) + \text{abs}(rv1))
do i = n, 1, -1
\text{Accumulation of right-hand transformations.}
if (i < n) then
if (g /= 0.0) then
v(l:n,i) = (a(i,l:n)/a(i,l))/g
\text{Double division to avoid possible underflow.}
\text{tempn}(l:n) = \text{matmul}(a(i,l:n), v(l:n,l:n))
v(l:n,1:n) = v(l:n,1:n) + \text{outerprod}(v(l:n,i), \text{tempn}(l:n))
end if
v(i,1:n) = 0.0
v(l:n,i) = 0.0
end if
v(i,i) = 1.0
\text{g} = rv1(i)
l = i
end do

\text{do i = min(m,n), 1, -1}
\text{Accumulation of left-hand transformations.}
l = i + 1
\text{g} = w(i)
a(1,1:n) = 0.0
if (g /= 0.0) then
\text{g} = 1.0_sp/g
\text{tempn}(l:n) = (\text{matmul}(a(l:m,i), a(l:m,1:n))/a(i,i))*g
a(i:m,1:n) = a(i:m,1:n) + \text{outerprod}(a(i:m,i), \text{tempn}(l:n))
a(i,1:n) = a(i,1:n)*g
else
a(i,m,i) = 0.0
end if
\text{a(i,i)} = a(i,i) + 1.0_sp
end do

\text{do k = n, 1, -1}
\text{Diagonalization of the bidiagonal form: Loop over singular values, and over allowed iterations.}
do its = 1, 30
\text{do l = k, 1, -1}
\text{Test for splitting.}
nm=l-1
if ((abs(rv1(l))+anorm) == anorm) exit
Note that rv1(l) is always zero, so can never fall through bottom of loop.
if ((abs(w(nm))+anorm) == anorm) then
  c=0.0  Cancellation of rv1(l), if l > 1.
s=1.0
  do i=l,k
    f=s*rv1(i)
    rv1(i)=c*rv1(i)
    g=w(i)
h=pythag(f,g)
w(i)=h
h=1.0_sp/h
c= (g*h)
s=-(f*h)
tempm(1:m)=a(1:m,nm)
a(1:m,nm)=a(1:m,nm)*c+a(1:m,i)*s
a(1:m,i)=-tempm(1:m)*s+a(1:m,i)*c
  end do
  exit
end if
end do
z=w(k)
if (l == k) then
Convergence.
if (z < 0.0) then
  Singular value is made nonnegative.
w(k)=-z
v(1:n,k)=-v(1:n,k)
end if
exit
end if
if (its == 30) call nrerror(‘svdcmp_sp: no convergence in svdcmp’) x=w(l)
Shift from bottom 2-by-2 minor.

nm=k-1
y=w(nm)
g=rv1(nm)
h=rv1(k)
f=((y-z)*(y+z)+(g-h)*(g+h))/(2.0_sp*h*y)
g=pythag(f,1.0_sp)
f=((x-z)*(x+z)+h*((y/(f+sign(g,f)))-h))/x
c=1.0
Next QR transformation:
s=1.0
  do j=l,nm
    i=j+1
g=rv1(i)
y=w(i)
h=sg
z=pythag(f,h)
rv1(j)=z
c=f/z
s=h/z
f= (x*c)+(g*s)
g=-(x*c)+(g*c)
h=y*s
y=y*c
tempn(1:n)=v(1:n,j)
v(1:n,j)=v(1:n,j)*c+v(1:n,i)*s
v(1:n,i)=-tempn(1:n)*s+v(1:n,i)*c
z=pythag(f,h)
w(j)=z
  end do
if (z /= 0.0) then
  Rotation can be arbitrary if z = 0.
z=1.0_sp/z
c=f*z
Chapter B2. Solution of Linear Algebraic Equations

The SVD algorithm implemented above does not parallelize very well. There are two parts to the algorithm. The first, reduction to bidiagonal form, can be parallelized. The second, the iterative diagonalization of the bidiagonal form, uses QR transformations that are intrinsically serial. There have been proposals for parallel SVD algorithms [2], but we do not have sufficient experience with them yet to recommend them over the well-established serial algorithm.

As with svbksb, single- and double-precision versions of the routines are overloaded onto the name svdcmp in nr.

```fortran
SUBROUTINE svdcmp_sp(a,w,v)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror,outerprod
USE nr, ONLY : pythag

IMPLICIT NONE
REAL(DP), DIMENSION(:,,:), INTENT(INOUT) :: a
REAL(DP), DIMENSION(:), INTENT(OUT) :: w
REAL(DP), DIMENSION(:,,:), INTENT(OUT) :: v
INTEGER(I4B) :: i,its,j,k,l,m,n,nm
REAL(DP) :: anorm,c,f,g,h,s,scale,x,y,z
REAL(DP), DIMENSION(size(a,1)) :: tempm
REAL(DP), DIMENSION(size(a,2)) :: rv1,tempn
m=size(a,1)

n=assert_eq(size(a,2),size(v,1),size(v,2),size(w),'svdcmp_dp')
g=0.0
scale=0.0
do i=1,n
  l=i+1
  rv1(i)=scale*g
  g=0.0
  scale=0.0
  if (i <= m) then
    scale=sum(abs(a(i:m,i)))
    if (scale /= 0.0) then
      a(i:m,i)=a(i:m,i)/scale
      s=dot_product(a(i:m,i),a(i:m,i))
      f=a(i,i)
      g=sign(sqrt(s),f)
  end if
end do
rv1(l)=0.0
rv1(k)=f
w(k)=x
end do
END SUBROUTINE svdcmp_sp
```

```fortran
tempm(l:m)=matmul...a(i:m,i)=...outerprod... Here is an example of an update as in equation (22.1.6). In this case $b_i$ is independent of $i$: It is simply $1/h$.
```

```fortran
SUBROUTINE svdcmp_dp(a,w,v)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror,outerprod
USE nr, ONLY : pythag

IMPLICIT NONE
REAL(DP), DIMENSION(:,,:), INTENT(INOUT) :: a
REAL(DP), DIMENSION(:), INTENT(OUT) :: w
REAL(DP), DIMENSION(:,,:), INTENT(OUT) :: v
INTEGER(I4B) :: i,its,j,k,l,m,n,nm
REAL(DP) :: anorm,c,f,g,h,s,scale,x,y,z
REAL(DP), DIMENSION(size(a,1)) :: tempm
REAL(DP), DIMENSION(size(a,2)) :: rv1,tempn
m=size(a,1)

n=assert_eq(size(a,2),size(v,1),size(v,2),size(w),'svdcmp_dp')
g=0.0
scale=0.0
do i=1,n
  l=i+1
  rv1(i)=scale*g
  g=0.0
  scale=0.0
  if (i <= m) then
    scale=sum(abs(a(i:m,i)))
    if (scale /= 0.0) then
      a(i:m,i)=a(i:m,i)/scale
      s=dot_product(a(i:m,i),a(i:m,i))
      f=a(i,i)
      g=sign(sqrt(s),f)
  end if
end do
rv1(l)=0.0
rv1(k)=f
w(k)=x
end do
END SUBROUTINE svdcmp_dp
```
\[
\begin{align*}
    h &= fg - s \\
    a(i,i) &= f - g \\
    \text{tempn}(l:n) &= \text{matmul}(a(i:m,i), a(i:m,l:n))/h \\
    a(i:m,l:n) &= a(i:m,l:n) + \text{outerprod}(a(i:m,i), \text{tempn}(l:n)) \\
    a(i:m,i) &= \text{scale} \times a(i:m,i)
\end{align*}
\]

end if

end if

\[
\begin{align*}
    w(i) &= \text{scale} \times g \\
    g &= 0.0 \\
    \text{scale} &= 0.0
\end{align*}
\]

if \(((i < m) \land (i \neq n))\) then

\[
\begin{align*}
    \text{scale} &= \text{sum}(|a(i,1:n)|) \\
    \text{if} (\text{scale} \neq 0.0) \text{ then} \\
    a(i,1:n) &= a(i,1:n)/\text{scale} \\
    s &= \text{dot_product}(a(i,1:n), a(i,1:n)) \\
    f &= a(i,1) \\
    g &= \text{sign}(\sqrt{s}, f) \\
    h &= fg - s \\
    a(i,1:n) &= f - g \\
    \text{rv1}(1:n) &= a(i,1:n)/h \\
    \text{tempm}(l:m) &= \text{matmul}(a(l:m,1:n), a(i,1:n)) \\
    a(l:m,1:n) &= a(l:m,1:n) + \text{outerprod}(\text{tempm}(l:m), \text{rv1}(1:n)) \\
    a(i,1:n) &= \text{scale} \times a(i,1:n)
\end{align*}
\]

end if

end if

end do

\[
\begin{align*}
    \text{anorm} &= \max(|w| + |\text{rv1}|) \\
    \text{do } i = n, 1, -1
    \text{if } (i < n) \text{ then} \\
    \text{if } (g \neq 0.0) \text{ then} \\
    v(l:n,i) &= (a(i,1:n)/a(i,1))/g \\
    \text{tempn}(l:n) &= \text{matmul}(a(i,1:n), v(l:n,1:n)) \\
    v(l:n,1:n) &= v(l:n,1:n) + \text{outerprod}(v(l:n,1), \text{tempn}(l:n))
\end{align*}
\]

end if

end if

v(i,l:n) = 0.0
v(l:n,i) = 0.0

end if

v(i,i) = 1.0

\[
\begin{align*}
    g &= \text{rv1}(1) \\
    l &= i
\end{align*}
\]

end do

\[
\begin{align*}
    \text{do } i = \min(m, n), 1, -1 \\
    l &= i+1 \\
    a(i,1:n) &= 0.0 \\
    \text{if } (g \neq 0.0) \text{ then} \\
    g &= 1.0/|g| \\
    \text{tempn}(l:n) &= \text{matmul}(a(l:m,1:n), a(l:m,1:n))/a(i,i)*g \\
    a(i,m,1:n) &= a(i,m,1:n) + \text{outerprod}(a(i,m,i), \text{tempn}(l:n)) \\
    a(i,m,i) &= a(i,m,i)*g \\
    \text{else} \\
    a(i,m,i) &= 0.0
\end{align*}
\]

end if

\[
\begin{align*}
    a(i,i) &= a(i,i) + 1.0_{dp}
\end{align*}
\]

end do

\[
\begin{align*}
    \text{do } k = n, 1, -1 \\
    \text{do } \text{its} = 1, 30 \\
    \text{do } l = k, 1, -1 \\
    \text{do } m = l-1 \\
    \text{if } ((|\text{rv1}(1)| + \text{anorm}) = \text{anorm}) \text{ exit} \\
    \text{if } ((|w(nm)| + \text{anorm}) = \text{anorm}) \text{ then} \\
    c &= 0.0 \\
    s &= 1.0 \\
    \text{do } i = 1, k
\end{align*}
\]
Chapter B2. Solution of Linear Algebraic Equations

\[ f = s \times rv1(i) \]
\[ rv1(i) = c \times rv1(i) \]
if \((\text{abs}(f) + \text{anorm}) == \text{anorm})\) exit
\[ g = rv1(i) \]
\[ h = \text{pythag}(f, g) \]
\[ w(i) = h \]
\[ h = 1.0 \times \text{dp}/h \]
\[ c = (g \times h) \]
\[ s = -(f \times h) \]
\[ \text{tempm}(1:m) = a(1:m,nm) \]
\[ a(1:m,nm) = a(1:m,nm) \times c + a(1:m,i) \times s \]
\[ a(1:m,i) = -\text{tempm}(1:m) \times s + a(1:m,i) \times c \]
end do
exit
end if
end do

\[ z = w(k) \]
if \((l == k)\) then
if \((z < 0.0)\) then
\[ w(k) = -z \]
\[ v(1:n,k) = -v(1:n,k) \]
end if
exit
end if
end if
if \((\text{its} == 30)\) call nrerror('svdcmp_dp: no convergence in svdcmp')
\[ x = w(1) \]
\[ nm = k - 1 \]
\[ y = w(nm) \]
\[ g = rv1(nm) \]
\[ h = rv1(k) \]
\[ f = ((y - z) \times (y + z) + (g - h) \times (g + h)) / (2.0 \times \text{dp} \times h \times y) \]
\[ g = \text{pythag}(f, 1.0 \times \text{dp}) \]
\[ f = ((x - z) \times (x + z) + h \times ((y / (f \times \text{sign}(g, f)) - h)) / x \]
\[ c = 1.0 \]
\[ s = 1.0 \]
do \[ j = l, nm \]
\[ i = j + 1 \]
\[ y = w(i) \]
\[ h = s \times g \]
\[ g = c \times g \]
\[ z = \text{pythag}(f, h) \]
\[ rv1(j) = z \]
\[ c = f / z \]
\[ s = h / z \]
\[ f = (x \times c) + (g \times s) \]
\[ g = -(x \times s) + (g \times c) \]
\[ h = y \times s \]
\[ y = y \times c \]
\[ \text{tempn}(1:n) = v(1:n,j) \]
\[ v(1:n,j) = v(1:n,j) \times c + v(1:n,i) \times s \]
\[ v(1:n,i) = -\text{tempn}(1:n) \times s + v(1:n,i) \times c \]
\[ z = \text{pythag}(f, h) \]
\[ v(j) = z \]
if \((z /= 0.0)\) then
\[ z = 1.0 \times \text{dp}/z \]
\[ c = f \times z \]
\[ s = h \times z \]
end if
\[ f = (c \times g) \times (a \times y) \]
\[ x = -(s \times g) \times (c \times y) \]
\[ \text{tempm}(1:m) = a(1:m,j) \]
\[ a(1:m,j) = a(1:m,j) \times c + a(1:m,i) \times s \]
\[ a(1:m,i) = -\text{tempm}(1:m) \times s + a(1:m,i) \times c \]
FUNCTION pythag_sp(a,b)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a, b
REAL(SP) :: pythag_sp

Computes $(a^2+b^2)^{1/2}$ without destructive underflow or overflow.
REAL(SP) :: absa, absb
absa=abs(a)
absb=abs(b)
if (absa > absb) then
    pythag_sp=absa*sqrt(1.0_sp+(absb/absa)**2)
else
    if (absb == 0.0) then
        pythag_sp=0.0
    else
        pythag_sp=absb*sqrt(1.0_sp+(absa/absb)**2)
    end if
end if
END FUNCTION pythag_sp

FUNCTION pythag_dp(a,b)
USE nrtype
IMPLICIT NONE
REAL(DP), INTENT(IN) :: a, b
REAL(DP) :: pythag_dp
REAL(DP) :: absa, absb

absa=abs(a)
absb=abs(b)
if (absa > absb) then
    pythag_dp=absa*sqrt(1.0_dp+(absb/absa)**2)
else
    if (absb == 0.0) then
        pythag_dp=0.0
    else
        pythag_dp=absb*sqrt(1.0_dp+(absa/absb)**2)
    end if
end if
END FUNCTION pythag_dp

*  *  *
SUBROUTINE cyclic(a,b,c,alpha,beta,r,x)
USE nrtype; USE nrutil, ONLY : assert,assert_eq
USE nr, ONLY : tridag
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN):: a,b,c,r
REAL(SP), INTENT(IN) :: alpha,beta
REAL(SP), DIMENSION(:), INTENT(OUT):: x
Solves the "cyclic" set of linear equations given by equation (2.7.9). a, b, c, and r are input vectors, while x is the output solution vector, all of the same size. alpha and beta are the corner entries in the matrix. The input is not modified.
INTEGER(I4B) :: n
REAL(SP) :: fact,gamma
REAL(SP), DIMENSION(size(x)) :: bb,u,z
n=assert_eq((/size(a),size(b),size(c),size(r),size(x)/),'cyclic')
call assert(n > 2, 'cyclic arg')
gamma=-b(1)  Avoid subtraction error in forming bb(1).
bb(1)=b(1)-gamma  Set up the diagonal of the modified tridiagonal system.
bb(n)=b(n)-alpha*beta/gamma
bb(2:n-1)=b(2:n-1)
call tridag(a(2:n),bb,c(1:n-1),r,x)  Solve A · x = r.
u(1)=gamma  Set up the vector u.
u(n)=alpha
u(2:n-1)=0.0
call tridag(a(2:n),bb,c(1:n-1),u,z)  Solve A · z = u.
fact=(x(1)+beta*x(n)/gamma)/(1.0_sp+z(1)+beta*z(n)/gamma)  Form v · x/(1+v · z).
x=x-fact*z  Now get the solution vector x.
END SUBROUTINE cyclic

The parallelism in cyclic is in tridag. Users with multiprocessor machines will want to be sure that, in nrutil, they have set the name tridag to be overloaded with tridag_par instead of tridag_ser.

⋆⋆⋆

The routines sprsin, sprsax, sprstx, sprstp, and sprsdia give roughly equivalent functionality to the corresponding Fortran 77 routines, but they are not plug compatible. Instead, they take advantage of (and illustrate) several Fortran 90 features that are not present in Fortran 77.

In the module nrtype we define a TYPE sprs2_sp for two-dimensional sparse, square, matrices, in single precision, as follows

```
TYPE sprs2_sp
  INTEGER(I4B) :: n,len
  REAL(SP), DIMENSION(:,), POINTER :: val
  INTEGER(I4B), DIMENSION(:,), POINTER :: irow
  INTEGER(I4B), DIMENSION(:,), POINTER :: jcol
END TYPE sprs2_sp
```

This has much less structure to it than the "row-indexed sparse storage mode" used in Volume 1. Here, a sparse matrix is just a list of values, and corresponding lists giving the row and column number that each value is in. Two integers n and len give, respectively, the underlying size (number of rows or columns) in the full matrix, and the number of stored nonzero values. While the previously used row-indexed scheme can be somewhat more efficient for serial machines, it does not parallelize conveniently, while this one does (though with some caveats; see below).
SUBROUTINE sprsin_sp(a,thresh,sa)
USE nrtype; USE nrutil, ONLY : arth,assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,:), INTENT(IN) :: a
REAL(SP), INTENT(IN) :: thresh
TYPE(sprs2_sp), INTENT(OUT) :: sa

Converts a square matrix \( a \) to sparse storage format as \( sa \). Only elements of \( a \) with magnitude \( \geq \) \( \text{thresh} \) are retained.

INTEGER(I4B) :: n,len
LOGICAL(LGT), DIMENSION(size(a,1),size(a,2)) :: mask
n=assert_eq(size(a,1),size(a,2),'sprsin_sp')
mask=abs(a)>thresh
len=count(mask) How many elements to store?
allocate(sa%val(len),sa%irow(len),sa%jcol(len))

sa%n=n
sa%len=len
sa%val=pack(a,mask) Grab the values, row, and column numbers.

sa%irow=pack(spread(arth(1,1,n),2,n),mask)

sa%jcol=pack(spread(arth(1,1,n),1,n),mask)
END SUBROUTINE sprsin_sp

SUBROUTINE sprsin_dp(a,thresh,sa)
USE nrtype; USE nrutil, ONLY : arth,assert_eq
IMPLICIT NONE
REAL(DP), DIMENSION(:,:), INTENT(IN) :: a
REAL(DP), INTENT(IN) :: thresh
TYPE(sprs2_dp), INTENT(OUT) :: sa
INTEGER(I4B) :: n,len
LOGICAL(LGT), DIMENSION(size(a,1),size(a,2)) :: mask
n=assert_eq(size(a,1),size(a,2),'sprsin_dp')
mask=abs(a)>thresh
len=count(mask)
allocate(sa%val(len),sa%irow(len),sa%jcol(len))

sa%n=n
sa%len=len
sa%val=pack(a,mask)

sa%irow=pack(spread(arth(1,1,n),2,n),mask)

sa%jcol=pack(spread(arth(1,1,n),1,n),mask)
END SUBROUTINE sprsin_dp

Note that the routines \( \text{sprsin}_{\text{sp}} \) and \( \text{sprsin}_{\text{dp}} \) — single and double precision versions of the same algorithm — are overloaded onto the name \( \text{sprsin} \) in module \( \text{nr} \). We supply both forms because the routine \( \text{linbcg} \), below, works in double precision.

The trick here is to use the same mask, \( \text{abs}(a)>\text{thresh} \), in three consecutive \text{pack} expressions, thus guaranteeing that the corresponding elements of the array argument get selected for packing. The first time, we get the desired matrix element values. The second time (above code fragment), we construct a matrix with each element having the value of its row number. The third time, we construct a matrix with each element having the value of its column number.
SUBROUTINE sprsax_sp(sa,x,b)
USE nrtype; USE nrutil, ONLY : assert_eq,scatter_add
IMPLICIT NONE
TYPE(sprs2_sp), INTENT(IN) :: sa
REAL(SP), DIMENSION (:), INTENT(IN) :: x
REAL(SP), DIMENSION (:), INTENT(OUT) :: b
   INTEGER(I4B) :: ndum
   ndum=assert_eq(sa%n,size(x),size(b),'sprsax_sp')
   b=0.0_sp
   call scatter_add(b,sa%val*x(sa%jcol),sa%irow)
END SUBROUTINE sprsax_sp

SUBROUTINE sprsax_dp(sa,x,b)
USE nrtype; USE nrutil, ONLY : assert_eq,scatter_add
IMPLICIT NONE
TYPE(sprs2_dp), INTENT(IN) :: sa
REAL(DP), DIMENSION (:), INTENT(IN) :: x
REAL(DP), DIMENSION (:), INTENT(OUT) :: b
   INTEGER(I4B) :: ndum
   ndum=assert_eq(sa%n,size(x),size(b),'sprsax_dp')
   b=0.0_dp
   call scatter_add(b,sa%val*x(sa%jcol),sa%irow)
END SUBROUTINE sprsax_dp

call scatter_add(b,sa%val*x(sa%jcol),sa%irow)
Since more than one component of the middle vector argument will, in general, need to be added into the same component of b, we must resort to a call to the nrutil routine scatter_add to achieve parallelism. However, this parallelism is achieved only if a parallel version of scatter_add is available! As we have discussed previously (p. 984), Fortran 90 does not provide any scatter-with-combine (here, scatter-with-add) facility, insisting instead that indexed operations yield non-colliding addresses. Luckily, almost all parallel machines do provide such a facility as a library program. In HPF, for example, the equivalent of scatter_add is SUM_SCATTER.

The call to scatter_add above is equivalent to the do-loop

   b=0.0
   do k=1,sa%len
      b(sa%irow(k))=b(sa%irow(k))+sa%val(k)*x(sa%jcol(k))
   end do

SUBROUTINE sprstx_sp(sa,x,b)
USE nrtype; USE nrutil, ONLY : assert_eq,scatter_add
IMPLICIT NONE
TYPE(sprs2_sp), INTENT(IN) :: sa
REAL(SP), DIMENSION (:), INTENT(IN) :: x
REAL(SP), DIMENSION (:), INTENT(OUT) :: b
   INTEGER(I4B) :: ndum
   ndum=assert_eq(sa%n,size(x),size(b),'sprstx_sp')
   b=0.0_sp
   call scatter_add(b,sa%val*x(sa%irow),sa%jcol)
END SUBROUTINE sprstx_sp
SUBROUTINE sprstx_dp(sa,x,b)
USE nrtype; USE nrutil, ONLY : assert_eq,scatter_add
IMPLICIT NONE
TYPE(sprs2_dp), INTENT(IN) :: sa
REAL(DP), DIMENSION (:), INTENT(IN) :: x
REAL(DP), DIMENSION (:), INTENT(OUT) :: b
INTEGER(I4B) :: ndum
ndum=assert_eq(sa%n,size(x),size(b),'sprstx_dp')
b=0.0_dp
call scatter_add(b,sa%val*x(sa%irow),sa%jcol)
END SUBROUTINE sprstx_dp

Precisely the same comments as for sprsax apply to sprstx. The call to scatter_add is here equivalent to

\[
b=0.0 \\
do k=1,sa%len \\
   b(sa%jcol(k))=b(sa%jcol(k))+sa%val(k)*x(sa%irow(k)) \\
end do
\]

SUBROUTINE sprstp(sa)
USE nrtype
IMPLICIT NONE
TYPE(sprs2_sp), INTENT(INOUT) :: sa
Replaces sa, in sparse matrix format, by its transpose.
INTEGER(I4B), DIMENSION(:), POINTER :: temp
temp=>sa%irow
We need only swap the row and column pointers.
sa%irow=>sa%jcol
sa%jcol=>temp
END SUBROUTINE sprstp

SUBROUTINE sprsdiag_sp(sa,b)
USE nrtype; USE nrutil, ONLY : array_copy,assert_eq
IMPLICIT NONE
TYPE(sprs2_sp), INTENT(IN) :: sa
REAL(SP), DIMENSION(:), INTENT(OUT) :: b
Extracts the diagonal of a matrix sa in sparse matrix format into a vector b.
REAL(SP), DIMENSION(size(b)) :: val
INTEGER(I4B) :: k,1,ndum,nerr
INTEGER(I4B), DIMENSION(size(b)) :: i
LOGICAL(LGT), DIMENSION(:), ALLOCATABLE :: mask
ndum=assert_eq(sa%n,size(b),'sprsdiag_sp')
l=sa%len
allocate(mask(l))
mask = (sa%irow(1:l) == sa%jcol(1:l))  Find diagonal elements.
call array_copy(pack(sa%val(1:l),mask),val,k,nerr)  Grab the values...
i(1:k)=pack(sa%irow(1:l),mask)  ...and their locations.
deallocate(mask)
b=0.0  Zero b because zero values not stored in sa.
b(i(1:k))=val(1:k)  Scatter values into correct slots.
END SUBROUTINE sprsdiag_sp
SUBROUTINE sprsdiag_dp(sa,b)
USE nrtype; USE nrutil, ONLY : array_copy,assert_eq
IMPLICIT NONE
TYPE(sprs2_dp), INTENT(IN) :: sa
REAL(DP), DIMENSION(:), INTENT(OUT) :: b
REAL(DP), DIMENSION(size(b)) :: val
INTEGER(I4B) :: k,l,ndum,nerr
INTEGER(I4B), DIMENSION(size(b)) :: i
LOGICAL(LGT), DIMENSION(:), ALLOCATABLE :: mask
ndum=assert_eq(sa%n,size(b),'sprsdiag_dp')
l=sa%len
allocate(mask(l))
mask = (sa%irow(1:l) == sa%jcol(1:l))
call array_copy(pack(sa%val(1:l),mask),val,k,nerr)
i(1:k)=pack(sa%irow(1:l),mask)
deallocate(mask)
b=0.0
b(i(1:k))=val(1:k)
END SUBROUTINE sprsdiag_dp

We use the nrutil routine array_copy because we don't know in advance how many nonzero diagonal elements will be selected by mask. Of course we could count them with a count(mask), but this is an extra step, and inefficient on scalar machines.

Using the same mask, we pick out the corresponding locations of the diagonal elements. No need to use array_copy now, since we know the value of k.

Finally, we can put each element in the right place. Notice that if the sparse matrix is ill-formed, with more than one value stored for the same diagonal element (which should not happen!) then the vector subscript i(1:k) is a “many-one section” and its use on the left-hand side is illegal.

* * *

SUBROUTINE linbcg(b,x,itol,tol,itmax,iter,err)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : atimes,asolve,snrm
IMPLICIT NONE
REAL(DP), DIMENSION(:), INTENT(IN) :: b
REAL(DP), DIMENSION(:), INTENT(INOUT) :: x
REAL(DP), PARAMETER :: EPS=1.0e-14_dp
DOUBLE PRECISION is a good idea in this routine.
REAL(DP), DIMENSION(:), INTENT(IN) :: itol,itmax
REAL(DP), DIMENSION(:), INTENT(INOUT) :: tol
REAL(DP), DIMENSION(:), INTENT(INOUT) :: iter
REAL(DP), PARAMETER :: EPS=1.0e-14_dp

Solves \( A \cdot x = b \) for \( x \), given \( b \) of the same length, by the iterative biconjugate gradient method. On input \( x \) should be set to an initial guess of the solution (or all zeros); \( itol \) is 1,2,3, or 4, specifying which convergence test is applied (see text); \( itmax \) is the maximum number of allowed iterations; and \( tol \) is the desired convergence tolerance. On output, \( x \) is reset to the improved solution, \( iter \) is the number of iterations actually taken, and \( err \) is the estimated error. The matrix \( A \) is referenced only through the user-supplied routines atimes, which computes the product of either \( A \) or its transpose on a vector; and asolve,
which solves $A \cdot x = b$ or $A^T \cdot x = b$ for some preconditioner matrix $\tilde{A}$ (possibly the trivial diagonal part of $A$).

INTEGER(4B) :: n
REAL(DP) :: ak, akden, bk, bkden, bknum, bnrm, dxnrm, znrm, zm1nrm
REAL(DP), DIMENSION(size(b)) :: p, pp, r, rr, z, zz
n=assert_eq(size(b),size(x),'linbcg')
iter=0

call atimes(x,r,0)  
Calculate initial residual. Input to atimes is $x(1:n)$, output is $r(1:n)$; the final 0 indicates that the matrix (not its transpose) is to be used.

r=b-r
rr=r
! call atimes(r,rr,0)  
Uncomment this line to get the “minimum residual” variant of the algorithm.

select case(itol)
  case(1)
    bnrm=snrm(b,itol)
    call asolve(r,z,0)  
    Input to asolve is $r(1:n)$, output is $z(1:n)$; the final 0 indicates that the matrix $\tilde{A}$ (not its transpose) is to be used.
  case(2)
    call asolve(b,z,0)
    bnrm=snrm(z,itol)
    call asolve(r,z,0)
  case(3:4)
    call asolve(b,z,0)
    bnrm=snrm(z,itol)
    call asolve(r,z,0)
    znrm=snrm(z,itol)
  case default
    call nrerror('illegal itol in linbcg')
end select

do
  Main loop.
  if (iter > itmax) exit
  iter=iter+1
  call asolve(rr,zz,1)  
  Final 1 indicates use of transpose matrix $\tilde{A}^T$.
  bknum=dot_product(z,rr)
  if (iter == 1) then
    p=z
    pp=zz
  else
    bk=bknum/bkden
    p=bk*p+z
    pp=bk*pp+zz
  end if
  bkden=bknum
  call atimes(p,z,0)
  akden=dot_product(z,pp)
  ak=bknum/akden
  call atimes(pp,zz,1)
  x=x+ak*p
  r=r-ak*z
  rr=rr-ak*zz
  call asolve(r,z,0)  
  Solve $A \cdot z = r$ and check stopping criterion.
  select case(itol)
  case(1)
    err=snrm(r,itol)/bnrm
  case(2)
    err=snrm(z,itol)/bnrm
  case(3:4)
    zm1nrm=znrm
    znrm=snrm(z,itol)
    if (abs(zm1nrm-znrm) > EPS*znrm) then
      dxnrm=abs(ak)*snrm(p,itol)
      err=znrm/abs(zm1nrm-znrm)*dxnrm
    else
      err=znrm/bnrm
    end if
  end select
  cycle
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end if
xnrm=snrm(x,itol)
if (err <= 0.5_dp*xnrm) then
  err=err/xnrm
else
  err=znrm/bnrm
  cycle
end if
end select
write (*,*) ' iter=',iter,' err=',err
if (err <= tol) exit
end do
END SUBROUTINE linbcg

case default...call nrerror('illegal itol in linbcg')  It's always a good idea to trap errors when the value of a case construction is supplied externally to the routine, as here.

FUNCTION snrm(sx,itol)
USE nrtype
IMPLICIT NONE
REAL(DP), DIMENSION(:,), INTENT(IN) :: sx
INTEGER(I4B), INTENT(IN) :: itol
REAL(DP) :: snrm
Compute one of two norms for a vector sx, as signaled by itol. Used by linbcg.
if (itol <= 3) then
  snrm=sqrt(dot_product(sx,sx))  Vector magnitude norm.
else
  snrm=maxval(abs(sx))  Largest component norm.
end if
END FUNCTION snrm

SUBROUTINE atimes(x,r,itrnsp)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : sprsax,sprstx  DOUBLE PRECISION versions of sprsax and sprstx.
USE xlinbcg_data  The matrix is accessed through this module.
REAL(DP), DIMENSION(:,), INTENT(IN) :: x
REAL(DP), DIMENSION(:,), INTENT(IN) :: r
INTEGER(I4B), INTENT(IN) :: itrnsp
INTEGER(I4B) :: n
n=assert_eq(size(x),size(r),'atimes')
if (itrnsp == 0) then
  call sprsax(sa,x,r)
else
  call sprstx(sa,x,r)
end if
END SUBROUTINE atimes
**Chapter B2. Solution of Linear Algebraic Equations**

```fortran
SUBROUTINE asolve(b,x,itrnsp)
  USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
  USE nr, ONLY : sprsdiag DOUBLE PRECISION version of sprsdiag.
  USE xlinbcg_data  The matrix is accessed through this module.
  REAL(DP), DIMENSION(:), INTENT(IN) :: b
  REAL(DP), DIMENSION(:), INTENT(OUT) :: x
  INTEGER(I4B), INTENT(IN) :: itrnsp
  INTEGER(I4B) :: ndum
  ndum=assert_eq(size(b),size(x),'asolve')
  call sprsdiag(sa,x)
  if (any(x == 0.0)) call nrerror('asolve: singular diagonal matrix')
  x=b/x
END SUBROUTINE asolve
```

The routines `atimes` and `asolve` are examples of user-supplied routines that interface `linbcg` to a user-supplied method for multiplying the user’s sparse matrix by a vector, and for solving the preconditioner matrix equation. Here, we have used these routines to connect `linbcg` to the sparse matrix machinery developed above. If we were instead using the different sparse matrix machinery of Volume I, we would modify `atimes` and `asolve` accordingly.

### FUNCTION vander(x,q)

```fortran
FUNCTION vander(x,q)
  USE nrtype; USE nrutil, ONLY : assert_eq,outerdiff
  IMPLICIT NONE
  REAL(DP), DIMENSION(:), INTENT(IN) :: x,q
  REAL(DP), DIMENSION(size(x)) :: vander
  Solves the Vandermonde linear system \( \sum_{i=1}^{N} x^{k-1} w_i = q_k \) \( (k = 1, \ldots, N) \). Input consists of the vectors \( x \) and \( q \) of length \( N \). The solution \( w \) (also of length \( N \)) is returned in \( vander \).
  REAL(DP), DIMENSION(size(x)) :: c
  REAL(DP), DIMENSION(size(x),size(x)) :: a
  INTEGER(I4B) :: i,n
  n=assert_eq(size(x),size(q),'vander')
  if (n == 1) then
    vander(1)=q(1)
  else
    c(:)=0.0  Initialize array.
    if (any(c(:) == 0.0)) call nrerror('vander: singular diagonal matrix')
    do i=2,n
      c(n+1-i:n-1)=c(n+1-i:n-1)-x(i)*c(n+2-i:n)
      c(n)=c(n)-x(i)
    end do
    a(:,1)=-c(1)/x(:)
    do i=2,n
      a(:,i)=-(c(i)-a(:,i-1))/x(:)
    end do
    vander(:)=matmul(a,q)/vander(:)
  end if
END FUNCTION vander
```
Here is an example of the coding of equation (22.1.4). Since in this case the product is over the second index \((n \text{ in } x_j - x_n)\), we have \(\dim=2\) in the product.

**FUNCTION** `toeplz(r,y)`

**USE** nrtype; USE nrutil, ONLY : assert_eq,nrerror

**IMPLICIT NONE**

**REAL(SP), DIMENSION(:), INTENT(IN) :: r, y**

**REAL(SP), DIMENSION(size(y)) :: toeplz**

Solves the Toeplitz system \(\sum_{j=1}^{N} R(N+i-j)x_j = y_i \) \((i = 1, \ldots, N)\). The Toeplitz matrix need not be symmetric. \(y\) \((\text{of length } N)\) and \(R\) \((\text{of length } 2N - 1)\) are input arrays; the solution \(x\) \((\text{of length } N)\) is returned in `toeplz`.

**INTEGER(I4B) :: m,m1,n,ndum**

**REAL(SP) :: sd,sgd,sgn,shn,sxn**

**REAL(SP), DIMENSION(size(y)) :: g,h,t**

\(n = \text{size}(y)\)

\(\text{ndum} = \text{assert_eq}(2\cdot\text{size}(r), \text{‘toeplz: ndum’})\)

**if (r(n) == 0.0) call nrerror(’toeplz: initial singular minor’)**

\(\text{toeplz}(1) = y(1)/r(n)\)

**Initialize for the recursion.**

**if (n == 1) RETURN**

\(g(1) = r(n-1)/r(n)\)

\(h(1) = r(n+1)/r(n)\)

**Main loop over the recursion.**

\(m1 = m + 1\)

\(\text{sxn} = -y(m1) + \text{dot_product}(r(n+1:n+m), \text{toeplz}(m:1:-1))\)

\(\text{Compute numerator and denominator for } x_.\)

\(\text{sd} = -r(n) + \text{dot_product}(r(n+1:n+m), g(1:m))\)

**if (sd == 0.0) exit**

\(\text{toeplz}(m1) = \text{sxn}/\text{sd}\)

whence \(x\).

\(\text{toeplz}(1:m) = \text{toeplz}(1:m) - \text{g}(m:1:-1)\)

\(\text{Compute numerator and denominator for G}.\)

\(\text{sgn} = -r(n-m1) + \text{dot_product}(r(n-m:n-1), g(1:m))\)

**if (m1 == m) RETURN**

\(\text{shn} = -r(n+m1) + \text{dot_product}(r(n+m:n+1:-1), h(1:m))\)

\(\text{sgd} = -r(n) + \text{dot_product}(r(n-m:n-1), h(1:m))\)

**if (sd == 0.0 .or. sgd == 0.0) exit**

\(\text{g}(m1) = \text{sgn}/\text{sgd}\)

\(\text{h}(m1) = \text{shn}/\text{sd}\)

\(\text{t}(1:m) = \text{g}(1:m)\)

\(\text{g}(1:m) = \text{g}(1:m) - \text{g}(m1) \cdot \text{h}(m:1:-1)\)

\(\text{h}(1:m) = \text{h}(1:m) - \text{h}(m1) \cdot \text{t}(m:1:-1)\)

**Back for another recurrence.**

**if (m > n) call nrerror(’toeplz: sanity check failed in routine’)**

**call nrerror(’toeplz: singular principal minor’)**

**END FUNCTION** `toeplz`

**SUBROUTINE** `choldc(a,p)`

**USE** nrtype; USE nrutil, ONLY : assert_eq,nrerror

**IMPLICIT NONE**

**REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a**

**REAL(SP), DIMENSION(:), INTENT(OUT) :: p**

Given an \(N \times N\) positive-definite symmetric matrix \(a\), this routine constructs its Cholesky decomposition, \(A = L \cdot L^T\). On input, only the upper triangle of \(a\) need be given; it is not modified. The Cholesky factor \(L\) is returned in the lower triangle of \(a\), except for its diagonal elements, which are returned in \(p\), a vector of length \(N\).

**INTEGER(I4B) :: i,n**

**REAL(SP) :: summ**

\(n = \text{assert_eq}(\text{size}(a,1), \text{size}(a,2), \text{size}(p), \text{‘choldc’})\)

**do i=1,n**
### Chapter B2. Solution of Linear Algebraic Equations

#### Section 1039

```fortran
! Solution of linear algebraic equations

! Define parameters

! Example: Solve the system of equations
! a(1,1)x_1 + a(1,2)x_2 = b(1)
! a(2,1)x_1 + a(2,2)x_2 = b(2)

! Using Cholesky decomposition

! Cholesky decomposition:
! A = L^T * L

! Cholesky solver:
! x = L^{-1} * y
! y = L^{-T} * b

! Example:
! a(1,1) = 2, a(1,2) = 1, a(2,1) = 1, a(2,2) = 3
! b(1) = 5, b(2) = 8

! Cholesky decomposition:
! L =
! | 1  0  |
! | 1/2  1 |

! Cholesky solver:
! x =
! | x_1 |
! | 1/2 x_2 |

! Solve for x_1 and x_2
```

### Example

Suppose we have the following system of equations:

\[ a(1,1)x_1 + a(1,2)x_2 = b(1) \]
\[ a(2,1)x_1 + a(2,2)x_2 = b(2) \]

Using Cholesky decomposition, we can solve this system efficiently. First, we perform Cholesky decomposition on the matrix \( A \):

\[ A = L^T \cdot L \]

where \( L \) is a lower triangular matrix. Then, we solve for \( L \cdot y = b \) and \( L^T \cdot x = y \) to find \( x \) and \( y \) respectively.

### Implementation

```fortran
! Cholesky decomposition

! Cholesky solver
```
Chapter B2. Solution of Linear Algebraic Equations

\[ d(k) = -\text{scale}\cdot\sigma \]
\[ a(k:n,k+1:n) = a(k:n,k+1:n) - \text{outerprod}(a(k:n,k), & \]
\[ \text{matmul}(a(k:n,k), a(k:n,k+1:n)) / c(k) \]

end if
end do

\[ d(n) = a(n,n) \]

if \( d(n) = 0.0 \) sing = true.

END SUBROUTINE qrdcmp

\[ a(k:n,k+1:n) = a(k:n,k+1:n) - \text{outerprod}\ldots\text{matmul}\ldots \]

See discussion of equation (22.1.6).

SUBROUTINE qrsolv(a,c,d,b)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : rsolv
IMPLICIT NONE
REAL(SP), DIMENSION(:,:), INTENT(IN) :: a
REAL(SP), DIMENSION(:), INTENT(IN) :: c,d
REAL(SP), DIMENSION(:), INTENT(INOUT) :: b

Solves the set of \( n \) linear equations \( A \cdot x = b \). The \( n \times n \) matrix \( a \) and the \( n \)-dimensional vectors \( c \) and \( d \) are input as the output of the routine \( \text{qrdcmp} \) and are not modified. \( b \) is input as the right-hand-side vector of length \( n \), and is overwritten with the solution vector on output.

INTEGER(I4B) :: j,n
REAL(SP) :: tau
n = assert_eq( (/size(a,1), size(a,2), size(b), size(c), size(d)/), 'qrsolv')
do j = 1, n-1
   Form \( Q^T \cdot b \).
   \[ \tau = \text{dot_product}(a(j:n,j), b(j:n))/c(j) \]
   \[ b(j:n) = b(j:n) - \tau \cdot a(j:n,j) \]
end do
call rsolv(a,d,b) Solve \( R \cdot x = Q^T \cdot b \).
END SUBROUTINE qrsolv

SUBROUTINE rsolv(a,d,b)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,:), INTENT(IN) :: a
REAL(SP), DIMENSION(:), INTENT(IN) :: d
REAL(SP), DIMENSION(:), INTENT(INOUT) :: b

Solves the set of \( n \) linear equations \( R \cdot x = b \), where \( R \) is an upper triangular matrix stored in \( a \) and \( d \). The \( n \times n \) matrix \( a \) and the vector \( d \) of length \( n \) are input as the output of the routine \( \text{qrdcmp} \) and are not modified. \( b \) is input as the right-hand-side vector of length \( n \), and is overwritten with the solution vector on output.

INTEGER(I4B) :: i,n
n = assert_eq( (/size(a,1), size(a,2), size(b), size(d)/), 'rsolv')
b(n) = b(n)/d(n)
do i = n-1, 1, -1
   \[ b(i) = (b(i) - \text{dot_product}(a(i,i+1:n), b(i+1:n)))/d(i) \]
end do
END SUBROUTINE rsolv

\[
\star \quad \star \quad \star
\]
SUBROUTINE qrupdt(r, qt, u, v)
USE nrtype; USE nrutil, ONLY : assert_eq, ifirstloc
USE nr, ONLY : rotate, pythag
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: r, qt
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: u
REAL(SP), DIMENSION(:,), INTENT(IN) :: v

Given the QR decomposition of some $n \times n$ matrix, calculates the QR decomposition of the matrix $Q \cdot (R + u \otimes v)$. Here $r$ and $qt$ are $n \times n$ matrices, $u$ and $v$ are $n$-dimensional vectors. Note that $Q^T$ is input and returned in $qt$.

INTEGER(I4B) :: i, k, n
n=assert_eq((/size(r,1),size(r,2),size(qt,1),size(qt,2),size(u),
size(v)/),'qrupdt')
k=n+1-ifirstloc(u(n:1:-1) /= 0.0)          Find largest $k$ such that $u(k) \neq 0$.
if (k < 1) k=1
   do i=k-1,1,-1
      Transform $R + u \otimes v$ to upper Hessenberg.
      call rotate(r,qt,i,u(i),-u(i+1))
      u(i)=pythag(u(i),u(i+1))
   end do
   r(1,:)=r(1,:)+u(1)*v
   do i=1,k-1
      Transform upper Hessenberg matrix to upper triangular.
      call rotate(r,qt,i,r(i,i),-r(i+1,i))
   end do
END SUBROUTINE qrupdt

The function ifirstloc in nrutil returns the first occurrence of .true. in a logical vector. See the discussion of the analogous routine imaxloc on p. 1017.

SUBROUTINE rotate(r, qt, i, a, b)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), TARGET, INTENT(INOUT) :: r, qt
INTEGER(I4B), INTENT(IN) :: i
REAL(SP), INTENT(IN) :: a, b

Given $n \times n$ matrices $r$ and $qt$, carry out a Jacobi rotation on rows $i$ and $i+1$ of each matrix. $a$ and $b$ are the parameters of the rotation: $\cos \theta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin \theta = \frac{b}{\sqrt{a^2+b^2}}$.

REAL(SP), DIMENSION(size(r,1)) :: temp
INTEGER(I4B) :: n
REAL(SP) :: c, fact, s
n=assert_eq(size(r,1),size(r,2),size(qt,1),size(qt,2), 'rotate')
if (a == 0.0) then              Avoid unnecessary overflow or underflow.
   c=0.0
   s=sign(1.0_sp,b)
else if (abs(a) > abs(b)) then
   fact=b/a
   c=sign(1.0_sp/sqrt(1.0_sp+fact**2),a)
   s=fact*c
else
   fact=a/b
   s=sign(1.0_sp/sqrt(1.0_sp+fact**2),b)
   c=fact*s
end if

temp(i:n)=r(i,i:n)              Premultiply $r$ by Jacobi rotation.
r(i,i:n)=c*temp(i:n)+s*r(i+1,i:n)
r(i+1,i:n)=s*temp(i:n)+c*r(i+1,i:n)

temp=qt(i,:)
qt(i,:)=c*temp+s*qt(i+1,:)
qt(i+1,:)=s*temp+c*qt(i+1,:)
END SUBROUTINE rotate
CITED REFERENCES AND FURTHER READING:
Chapter B3. Interpolation and Extrapolation

SUBROUTINE polint(xa,ya,x,y,dy)
USE nrtype; USE nrutil, ONLY : assert_eq, iminloc, nrerror
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: xa,ya
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: y,dy
  Given arrays xa and ya of length N, and given a value x, this routine returns a value y, and an error estimate dy. If \( P(x) \) is the polynomial of degree \( N-1 \) such that \( P(xa_i) = ya_i, \ i = 1, \ldots, N \), then the returned value \( y = P(x) \).

INTEGER(14B) :: m,n,ns
REAL(SP), DIMENSION(size(xa)) :: c,d,den,ho
n=assert_eq(size(xa),size(ya),'polint')
c=ya
Initialize the tableau of c's and d's.
d=ya

ho=x-xa
Find index ns of closest table entry.
y=y(ns)
This is the initial approximation to y.
ns=ns-1
do m=1,n-1
  For each column of the tableau,
den(1:n-m)=ho(1:n-m)-ho(1+m:n)
  we loop over the current c's and d's and update them.
  if (any(den(1:n-m) == 0.0)) &
  call nrerror('polint: calculation failure')
  This error can occur only if two input xa's are (to within roundoff) identical.
den(1:n-m)=(c(2:n-m+1)-d(1:n-m))/den(1:n-m)
d(1:n-m)=ho(1+m:n)*den(1:n-m)
Here the c's and d's are updated.
c(1:n-m)=ho(1:n-m)*den(1:n-m)
if (2*ns < n-m) then
  After each column in the tableau is completed, we decide
  dy=c(ns+1)
  which correction, c or d, we want to add to our accumulating value of y, i.e., which path to take through the tableau—forking up or down. We do this in such a way as to take the most "straight line" route through the tableau to its apex, updating ns accordingly to keep track of where we are. This route keeps the partial approximations centered (as far as possible) on the target x. The last dy added is thus the error indication.
  else
    dy=d(ns)
  end if
  ns=ns-1
end do
y=y+dy
end do
END SUBROUTINE polint

SUBROUTINE ratint(xa,ya,x,y,dy)
USE nrtype; USE nrutil, ONLY : assert_eq, iminloc, nrerror
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: xa,ya
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: y,dy
  Given arrays xa and ya of length N, and given a value of x, this routine returns a value of y and an accuracy estimate dy. The value returned is that of the diagonal rational function, evaluated at x, that passes through the N points (xa_i, ya_i), i = 1, \ldots, N.

INTEGER(14B) :: m,n,ns
REAL(SP), DIMENSION(size(xa)) :: c,d,dd,h,t
REAL(SP), PARAMETER :: TINY=1.0e-25_sp
A small number.
n=assert_eq(size(xa),size(ya),'ratint')
h=xa-x
ns=iminloc(abs(h))
y=ya(ns)
if (x == xa(ns)) then
dy=0.0
RETURN
end if
c=ya
d=ya+TINY
The TINY part is needed to prevent
ns=ns-1
a rare zero-over-zero condition.
do m=1,n-1
h will never be zero, since this was
t(1:n-m)=(xa(1:n-m)-x)*d(1:n-m)/h(1+m:n)
tested in the initializing loop.
dd(1:n-m)=t(1:n-m)-c(2:n+m1)
This error condition indicates that
dd(1:n-m)=c(2:n+m1)-d(1:n-m)/dd(1:n-m)
c(1:n-m)=t(1:n-m)*dd(1:n-m)
if (2*ns < n-m) then
   dy=c(ns+1)
dy=d(ns)
else
   ns=ns-1
   dy=dy
end if
end do
END SUBROUTINE ratint

***

SUBROUTINE spline(x,y,yp1,ypn,y2)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : tridag
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y
REAL(SP), INTENT(IN) :: yp1,ypn
REAL(SP), DIMENSION(:,), INTENT(OUT) :: y2
Given arrays x and y of length N containing a tabulated function, i.e.,
y_i = f(x_i), with x_1 < x_2 < ... < x_N, and given values yp1 and yp for the first
derivative of the interpolating function at points 1 and N, respectively, this routine returns an array y2 of length N
that contains the second derivatives of the interpolating function at the tabulated points x_i. If yp1 and/or yp for
the corresponding boundary condition for a natural spline, with zero second derivative on that
boundary.
INTEGER(I4B) :: n
REAL(SP), DIMENSION(size(x)) :: a,b,c,r
n=assert_eq(size(x),size(y),size(y2),'spline')
c(1:n-1)=x(2:n)-x(1:n-1)
Set up the tridiagonal equations.
r(1:n-1)=6.0_sp*(y(2:n)-y(1:n-1))/c(1:n-1))
r(2:n-1)=r(2:n-1)-r(1:n-2)
a(2:n-1)=c(2:n-1)
b(2:n-1)=2.0_sp*c(2:n-1)+a(2:n-1)
b(1)=1.0
b(n)=1.0
if (yp1 > 0.99e30_sp) then
   The lower boundary condition is set either to be “nat-
   rial”
   r(1)=0.0
   c(1)=0.0
else
   or else to have a specified first derivative.
   r(1)=3.0_sp/(x(2)-x(1))*((y(2)-y(1))/(x(2)-x(1))-yp1)
The upper boundary condition is set either to be "natural"

\[
\begin{align*}
\text{if } (\text{ypn} > 0.99e30_{-sp}) \text{ then} & \\
    r(n) &= 0.0 \\
    a(n) &= 0.0
\end{align*}
\]

or else to have a specified first derivative.

\[
\begin{align*}
    r(n) &= \left(-3.0_{-sp}/(x(n) - x(n-1))\right) \times \left((y(n) - y(n-1))/(x(n) - x(n-1)) - ypn\right) \\
    a(n) &= 0.5
\end{align*}
\]

END SUBROUTINE spline

FUNCTION splint(xa, ya, y2a, x)
USE nrtype; USE nrutil, ONLY : assert_eq, nrerror
USE nr, ONLY: locate
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: xa, ya, y2a
REAL(SP), INTENT(IN) :: x
REAL(SP) :: splint

Given the arrays \(xa\) and \(ya\), which tabulate a function (with the \(xa\)’s in increasing or decreasing order), and given the array \(y2a\), which is the output from \(spline\) above, and given a value of \(x\), this routine returns a cubic-spline interpolated value. The arrays \(xa\), \(ya\) and \(y2a\) are all the same size.

INTEGER(I4B) :: khi, klo, n
REAL(SP) :: a, b, h

\[n = \text{assert_eq}(	ext{size}(xa), \text{size}(ya), \text{size}(y2a), \text{’splint’})\]

\[klo = \text{max} \left(\min(\text{locate}(xa, x), n-1), 1\right)\]

We will find the right place in the table by means of \(\text{locate}\)’s bisection algorithm. This is optimal if sequential calls to this routine are at random values of \(x\). If sequential calls are in order, and closely spaced, one would do better to store previous values of \(klo\) and \(khi\) and test if they remain appropriate on the next call.

\[khi = klo + 1\]

\(klo\) and \(khi\) now bracket the input value of \(x\).

\[h = xa(khi) - xa(klo)\]

\[\text{if } (h == 0.0) \text{ call nrerror(’bad xa input in splint’)}\]

The \(xa\)’s must be distinct.

\[a = (xa(khi) - x)/h\]

Cubic spline polynomial is now evaluated.

\[b = (x - xa(klo))/h\]

\[\text{splint} = a*ya(klo) + b*ya(khi) + ((a**3 - a)*y2a(klo) + (b**3 - b)*y2a(khi))*(h**2)/6.0_{-sp}\]

END FUNCTION splint

FUNCTION locate(xx, x)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: xx
REAL(SP), INTENT(IN) :: x
INTEGER(I4B) :: locate

Given an array \(xx(1:N)\), and given a value \(x\), returns a value \(j\) such that \(x\) is between \(xx(j)\) and \(xx(j + 1)\). \(xx\) must be monotonic, either increasing or decreasing. \(j = 0\) or \(j = N\) is returned to indicate that \(x\) is out of range.

INTEGER(I4B) :: n, j, j1, jm, ju
LOGICAL :: ascnd
Chapter B3. Interpolation and Extrapolation

The use of bisection is perhaps a sin on a genuinely parallel machine, but (since the process takes only logarithmically many sequential steps) it is at most a small sin. One can imagine a “fully parallel” implementation like,

k=iminloc(abs(x-xx))
if ((x < xx(k)) .eqv. (xx(1) < xx(n))) then
locate=k-1
else
locate=k
end if

Problem is, unless the number of physical (not logical) processors participating in the minloc is larger than \(N\), the length of the array, this “parallel” code turns a \(\log N\) algorithm into one scaling as \(N\), quite an unacceptable inefficiency. So we prefer to be small sinners and bisect.

SUBROUTINE hunt(xx,x,jlo)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: jlo
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: xx

Given an array xx(1:N), and given a value x, returns a value jlo such that x is between xx(jlo) and xx(jlo+1). xx must be monotonic, either increasing or decreasing. jlo = 0 or jlo = N is returned to indicate that x is out of range. jlo on input is taken as the initial guess for jlo on output.

INTEGER(I4B) :: n,inc,jhi,jm
LOGICAL :: ascnd

n=size(xx)
ascnd = (xx(n) >= xx(1)) True if ascending order of table, false otherwise.
if (jlo <= 0 .or. jlo > n) then Input guess not useful. Go immediately to bisection.
jlo=0
jhi=n+1
else
inc=1
if (x >= xx(jlo) .equiv. ascnd) then Hunt up:
do
jhi=jlo+inc
if (jhi > n) then Done hunting, since off end of table.
      jlo=jhi-inc
    end do
locate=jlo
else
locate=jhi-1
end if
Chapter B3. Interpolation and Extrapolation

\[ jhi = n + 1 \]

\[ \text{exit} \]

else
\[ \text{if (} x < xx(jhi) \text{ .eqv. ascnd) exit} \]
\[ jlo = jhi \]
\[ \text{Not done hunting,} \]
\[ \text{inc = inc + inc} \]
\[ \text{so double the increment} \]
end if
end do
and try again.
else
Hunt down:
\[ jhi = jlo \]
do
\[ jlo = jhi - inc \]
if (jlo < 1) then
\[ \text{Done hunting, since off end of table.} \]
\[ jlo = 0 \]
\[ \text{exit} \]
else
\[ \text{if (} x \geq xx(jlo) \text{ .eqv. ascnd) exit} \]
\[ jhi = jlo \]
\[ \text{Not done hunting,} \]
\[ \text{inc = inc + inc} \]
\[ \text{so double the increment} \]
end if
end do
and try again.
end if
end if
Done hunting, value bracketed.
do
Hunt is done, so begin the final bisection phase:
\[ \text{if (} jhi - jlo <= 1 \text{) then} \]
\[ \text{if (} x = xx(n) \text{) jlo = n - 1} \]
\[ \text{if (} x = xx(1) \text{) jlo = 1} \]
\[ \text{exit} \]
else
\[ jm = (jhi + jlo) / 2 \]
\[ \text{if (} x \geq xx(jm) \text{ .eqv. ascnd) then} \]
\[ jlo = jm \]
else
\[ jhi = jm \]
end if
end if
end do
END SUBROUTINE hunt

FUNCTION polcoe(x,y)
USE nrtype; USE nrutil, ONLY : assert_eq, outerdiff
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y
REAL(SP), DIMENSION(size(x)) :: polcoe

Given same-size arrays x and y containing a tabulated function \( y_i = f(x_i) \), this routine returns a same-size array of coefficients \( c_j \), such that \( y_i = \sum_j c_j x_i^{j-1} \).

INTEGER(I4B) :: i,k,n
REAL(SP), DIMENSION(size(x)) :: s
REAL(SP), DIMENSION(size(x),size(x)) :: a
n=assert_eq(size(x),size(y),'polcoe')
s=0.0
Coefficients \( s_i \) of the master polynomial \( P(x) \) are found by recurrence.
do i=2,n
s(n)=x(1)
s(n+1:i-1:n-1)=s(n+1:i-1:n-1)-x(i)*s(n+2:i:n)
s(n)=s(n)-x(i)
end do
a=outerdiff(x,x) \[ \text{Make vector} \ w_j = \prod_{k \neq j} (x_j - x_k), \text{using polcoe for temporary storage.} \]
polcoe=product(a,dim=2,mask=a /= 0.0)
Now do synthetic division by $x - x_j$. The division for all $x_j$ can be done in parallel (on a parallel machine), since the $j$ in the loop below is over $j$.

```fortran
a(:,1) = s(1)/x(:)
do k=2,n
   a(:,k) = -(s(k) - a(:,k-1))/x(:)
end do
s = y/polcoe
polcoe = matmul(s, a)  ! Solve linear system.
END FUNCTION polcoe
```

For a description of the coding here, see §22.3, especially equation (22.3.9). You might also want to compare the coding here with the Fortran 77 version, and also look at the description of the method on p. 84 in Volume 1. The Fortran 90 implementation here is in fact much closer to that description than is the Fortran 77 method, which goes through some acrobatics to roll the synthetic division and matrix multiplication into a single set of two nested loops. The price we pay, here, is storage for the matrix $a$. Since the degree of any useful polynomial is not a very large number, this is essentially no penalty.

Also worth noting is the way that parallelism is brought to the required synthetic division. For a single such synthetic division (e.g., as accomplished by the `nrutil` routine `poly_term`), parallelism can be obtained only by recursion. Here things are much simpler, because we need a whole bunch of simultaneous and independent synthetic divisions; so we can just do them in the obvious, data-parallel, way.

```fortran
FUNCTION polcof(xa, ya)
USE nrtype, USE nrutil, ONLY : assert_eq,iminloc
USE nr, ONLY : polint
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: xa, ya
REAL(SP), DIMENSION(size(xa)) :: polcof
Given same-size arrays $xa$ and $ya$ containing a tabulated function $ya_i = f(xa_i)$, this routine returns a same-size array of coefficients $c_j$ such that $ya_i = \sum_{j} c_j xa_i^{-1}$.

INTEGER(I4B) :: j, k, m, n
REAL(SP) :: dy
REAL(SP), DIMENSION(size(xa)) :: x, y
n = assert_eq(size(xa), size(ya), 'polcof')
x = xa
y = ya
do j=1,n
   m = n - j
   call polint(x(1:m), y(1:m), 0.0_sp, polcof(j), dy)
   Use the polynomial interpolation routine of §3.1 to extrapolate to $x = 0$.
   k = iminloc(abs(x(1:m)))  ! Find the remaining $x_k$ of smallest absolute value, where $x(1:m) /= 0.0$
   y(1:m) = (y(1:m) - polcof(j))/x(1:m)  ! reduce all the terms,
   y(k+1:m) = y(k+1:m)  ! and eliminate $x_k$.
   x(k+1:m) = x(k+1:m)
end do
END FUNCTION polcof
```

* * *

FUNCTION polcof(xa, ya)
USE nrtype; USE nrutil, ONLY : assert_eq, iminloc
USE nr, ONLY : polint
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: xa, ya
REAL(SP), DIMENSION(size(xa)) :: polcof
Given same-size arrays $xa$ and $ya$ containing a tabulated function $ya_i = f(xa_i)$, this routine returns a same-size array of coefficients $c_j$ such that $ya_i = \sum_{j} c_j xa_i^{-1}$.

INTEGER(I4B) :: j, k, m, n
REAL(SP) :: dy
REAL(SP), DIMENSION(size(xa)) :: x, y
n = assert_eq(size(xa), size(ya), 'polcof')
x = xa
y = ya
do j=1,n
   m = n - j
   call polint(x(1:m), y(1:m), 0.0_sp, polcof(j), dy)
   Use the polynomial interpolation routine of §3.1 to extrapolate to $x = 0$.
   k = iminloc(abs(x(1:m)))  ! Find the remaining $x_k$ of smallest absolute value, where $x(1:m) /= 0.0$
   y(1:m) = (y(1:m) - polcof(j))/x(1:m)  ! reduce all the terms,
   y(k+1:m) = y(k+1:m)  ! and eliminate $x_k$.
   x(k+1:m) = x(k+1:m)
end do
END FUNCTION polcof
SUBROUTINE polin2(x1a,x2a,ya,x1,x2,y,dy)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : polint
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x1a,x2a
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: ya
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP), INTENT(OUT) :: y,dy

Given arrays x1a of length M and x2a of length N of independent variables, and an $M \times N$ array of function values ya, tabulated at the grid points defined by x1a and x2a, and given values x1 and x2 of the independent variables, this routine returns an interpolated function value y, and an accuracy indication dy (based only on the interpolation in the x1 direction, however).

INTEGER(I4B) :: j,m,ndum
REAL(SP), DIMENSION(size(x1a)) :: ymtmp
REAL(SP), DIMENSION(size(x2a)) :: yntmp
m=assert_eq(size(x1a),size(ya,1),'polin2: m')
ndum=assert_eq(size(x2a),size(ya,2),'polin2: ndum')
do j=1,m
   yntmp=ya(j,:)
copy row into temporary storage.
call polint(x2a,yntmp,x2,ymtmp(j),dy)
   Interpolate answer into temporary storage.
end do

call polint(x1a,ymtmp,x1,y,dy)
Do the final interpolation.
END SUBROUTINE polin2

SUBROUTINE bcucof(y,y1,y2,y12,d1,d2,c)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: d1,d2
REAL(SP), DIMENSION(4), INTENT(IN) :: y,y1,y2,y12
REAL(SP), DIMENSION(4,4), INTENT(OUT) :: c

Given arrays y, y1, y2, and y12, each of length 4, containing the function, gradients, and cross derivative at the four grid points of a rectangular grid cell (numbered counterclockwise from the lower left), and given d1 and d2, the length of the grid cell in the 1- and 2-directions, this routine returns the $4 \times 4$ table c that is used by routine bcuint for bicubic interpolation.

REAL(SP), DIMENSION(16) :: x
REAL(SP), DIMENSION(16,16) :: wt
DATA wt /1,0,-3,2,4*0,-3,0,-6,2,0,-6,4,&
  8*0,3,0,-9,6,-2,0,6,-4,10*0,9,-6,2*0,-6,4,2*0,3,-2,6*0,-9,6,&
  2*0,6,-4,4*0,1,0,-3,2,-2,0,6,-4,10*0,-3,2,8*0,-1,0,3,-2,1,0,-3,&
  2,10*0,-3,2,2*0,3,-2,6*0,3,-2,2*0,-6,4,2*0,3,-2,0,1,-2,1,5*0,&
  -3,6,-3,0,2,-4,2,9,0,3,-6,3,0,-2,4,-2,10*0,-3,3,2*0,2,-2,2*0,&
  -1,1,6*0,3,-3,2*0,-2,2,5,0,1,-2,1,0,-2,4,-2,0,1,-2,1,9*0,-1,2,&
  -1,0,1,-2,1,10*0,1,-1,2*0,-1,1,6*0,-1,1,2*0,2,-2,2*0,-1,1/,

x(1:4)=y Pack a temporary vector x.
x(5:8)=y1*d1
x(9:12)=y2*d2
x(13:16)=y12*d1*d2
x=matmul(wt,x) Matrix multiply by the stored table.
c=reshape(x,(/4,4/),order=(/2,1/)) Unpack the result into the output table.
END SUBROUTINE bcucof
It is a powerful technique to combine the matmul intrinsic with reshape's of the input or output. The idea is to use matmul whenever the calculation can be cast into the form of a linear mapping between input and output objects. Here the order=(/2,1/) parameter specifies that we want the packing to be by rows, not by Fortran's default of columns. (In this two-dimensional case, it's the equivalent of applying transpose.)

SUBROUTINE bcuint(y,y1,y2,y12,x1l,x1u,x2l,x2u,x1,x2,ansy,ansy1,ansy2)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : bcucof
IMPLICIT NONE
REAL(SP), DIMENSION(4), INTENT(IN) :: y,y1,y2,y12
REAL(SP), INTENT(IN) :: x1l,x1u,x2l,x2u,x1,x2
REAL(SP), INTENT(OUT) :: ansy,ansy1,ansy2

Bicubic interpolation within a grid square. Input quantities are y,y1,y2,y12 (as described in bcucof); x1l and x1u, the lower and upper coordinates of the grid square in the 1-direction; x2l and x2u likewise for the 2-direction; and x1, x2, the coordinates of the desired point for the interpolation. The interpolated function value is returned as ansy, and the interpolated gradient values as ansy1 and ansy2. This routine calls bcucof.

INTEGER(I4B) :: i
REAL(SP) :: t,u
REAL(SP), DIMENSION(4,4) :: c

CALL bcucof(y,y1,y2,y12,x1u-x1l,x2u-x2l,c) \ Get the c's.

IF (x1u == x1l .OR. x2u == x2l) CALL & nrerror('bcuint: problem with input values - boundary pair equal?')

T=(x1-x1l)/(x1u-x1l) \ Equation (3.6.4).
U=(x2-x2l)/(x2u-x2l)

Ancy=0.0
Ancy2=0.0
Ancy1=0.0

DO I=4,1,-1 \ Equation (3.6.6).

Ancy=T*Ansy+((C(I,4)*U+C(I,3))*U+C(I,2))*U+C(I,1)
Ancy2=T*Ancy2+(3.0Sp*C(I,4)*U+2.0Sp*C(I,3))*U+C(I,2)
Ancy1=U*Ancy1+(3.0Sp*C(4,I)*T+2.0Sp*C(3,I))*T+C(2,I)

END DO

Ansy1=Ancy1/(x1u-x1l)
Ancy2=Ancy2/(x2u-x2l)

END SUBROUTINE bcuint

SUBROUTINE splie2(x1a,x2a,ya,y2a)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : spline
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x1a,x2a
REAL(SP), DIMENSION(:,:), INTENT(IN) :: ya
REAL(SP), DIMENSION(:,:), INTENT(OUT) :: y2a

Given an M × N tabulated function y, and N tabulated independent variables x2a, this routine constructs one-dimensional natural cubic splines of the rows of ya and returns the second derivatives in the M × N array y2a. (The array x1a is included in the argument list merely for consistency with routine splin2.)

INTEGER(I4B) :: j,m,ndum
M=assert_eq(size(x1a),size(ya,1),size(y2a,1),'splie2: m')
Ndum=assert_eq(size(x2a),size(ya,2),size(y2a,2),'splie2: ndum')

DO J=1,M

CALL spline(x2a,ya(j,:),1.0e30_sp,1.0e30_sp,y2a(j,:))

END SUBROUTINE splie2

*   *   *

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Values $1 \times 10^{30}$ signal a natural spline.

```
FUNCTION splin2(x1a,x2a,ya,y2a,x1,x2)
USE nrttype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : spline, splint
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x1a,x2a
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: ya,y2a
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP) :: splin2

Given $x_{1a}, x_{2a}, y_a$ as described in splie2 and $y_{2a}$ as produced by that routine; and given
a desired interpolating point $x_1, x_2$; this routine returns an interpolated function value by
bicubic spline interpolation.

INTEGER(I4B) :: j,m,ndum
REAL(SP), DIMENSION(size(x1a)) :: yytmp,y2tmp2
m=assert_eq(size(x1a),size(ya,1),size(y2a,1),'splin2: m')
ndum=assert_eq(size(x2a),size(ya,2),size(y2a,2),'splin2: ndum')
do j=1,m
   yytmp(j)=splint(x2a,ya(j,:),y2a(j,:),x2)
   Perform $m$ evaluations of the row splines constructed by splie2, using the one-dimensional
   spline evaluator splint.
endo

call spline(x1a,yytmp,1.0e30_sp,1.0e30_sp,y2tmp2)
Construct the one-dimensional column spline and evaluate it.
splin2=splint(x1a,yytmp,y2tmp2,x1)
END FUNCTION splin2
```

end do
END SUBROUTINE splie2
Chapter B4. Integration of Functions

SUBROUTINE trapzd(func,a,b,s,n)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE

This routine computes the nth stage of refinement of an extended trapezoidal rule. func is input as the name of the function to be integrated between limits a and b, also input. When called with n=1, the routine returns as s the crudest estimate of \( \int_a^b f(x) \, dx \). Subsequent calls with n=2,3,... (in that sequential order) will improve the accuracy of s by adding \( 2^{n-2} \) additional interior points. s should not be modified between sequential calls.

REAL(SP) :: del,fsum
INTEGER(I4B) :: it
if (n == 1) then
   s=0.5_sp*(b-a)*sum(func( (/ a,b /) ))
else
   it=2***(n-2)
   del=(b-a)/it
   fsum=sum(func(arth(a+0.5_sp*del,del,it)))
   s=0.5_sp*(s+del*fsum)
end if
END SUBROUTINE trapzd

While most of the quadrature routines in this chapter are coded as functions, trapzd is a subroutine because the argument s that returns the function value must also be supplied as an input parameter. We could change the subroutine into a function by declaring s to be a local variable with the SAVE attribute. However, this would prevent us from being able to use the routine recursively to do multidimensional quadrature (see quad3d on p. 1065). When s is left as an argument, a fresh copy is created on each recursive call. As a SAVE’d variable, by contrast, its value would get overwritten on each call, and the code would not be properly “re-entrant.”

\[ s=0.5\, s_{\text{old}} + \text{func}(\text{arith}(a, b)) \]

Note how we use the (/.../) construct to supply a set of scalar arguments to a vector function.

* * *

1052
FUNCTION qtrap(func,a,b)
  USE nrtype; USE nrutil, ONLY : nrerror
  USE nr, ONLY : trapzd
  IMPLICIT NONE
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP) :: qtrap
  INTERFACE
    FUNCTION func(x)
      USE nrtype
      REAL(SP), DIMENSION(:), INTENT(IN) :: x
      REAL(SP), DIMENSION(size(x)) :: func
    END FUNCTION func
  END INTERFACE
  INTEGER(I4B), PARAMETER :: JMAX=20
  REAL(SP), PARAMETER :: EPS=1.0e-6_sp, UNLIKELY=-1.0e30_sp
  Returns the integral of the function func from a to b. The parameter EPS should be set to the desired fractional accuracy and JMAX so that 2 to the power JMAX-1 is the maximum allowed number of steps. Integration is performed by the trapezoidal rule.

  REAL(SP) :: olds
  INTEGER(I4B) :: j
  olds=UNLIKELY
  do j=1,JMAX
    call trapzd(func,a,b,qtrap,j)
    if (j < 5) then
      Avoid spurious early convergence.
      if (abs(qtrap-olds) < EPS*abs(olds) .or. &
          (qtrap == 0.0 .and. olds == 0.0)) RETURN
    end if
    olds=qtrap
  end do

  call nrerror('qtrap: too many steps')
END FUNCTION qtrap

FUNCTION qsimp(func,a,b)
  USE nrtype; USE nrutil, ONLY : nrerror
  USE nr, ONLY : trapzd
  IMPLICIT NONE
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP) :: qsimp
  INTERFACE
    FUNCTION func(x)
      USE nrtype
      REAL(SP), DIMENSION(:), INTENT(IN) :: x
      REAL(SP), DIMENSION(size(x)) :: func
    END FUNCTION func
  END INTERFACE
  INTEGER(I4B), PARAMETER :: JMAX=20
  REAL(SP), PARAMETER :: EPS=1.0e-6_sp, UNLIKELY=-1.0e30_sp
  Returns the integral of the function func from a to b. The parameter EPS should be set to the desired fractional accuracy and JMAX so that 2 to the power JMAX-1 is the maximum allowed number of steps. Integration is performed by Simpson's rule.

  INTEGER(I4B) :: j
  REAL(SP) :: os,ost,st
  ost=UNLIKELY
  os= UNLIKELY
  do j=1,JMAX
    call trapzd(func,a,b,qsimp,j)
    qsimp=(4.0_sp*st-ost)/3.0_sp
    if (j < 5) then
      Avoid spurious early convergence.
      if (abs(qsimp-os) < EPS*abs(os) .or. &
          (qsimp == 0.0 .and. os == 0.0)) RETURN
    end if
    os=qsimp
  end do
END FUNCTION qsimp
FUNCTION qromb(func,a,b)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : polint,trapzd
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP) :: qromb
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: JMAX=20,JMAXP=JMAX+1,K=5,KM=K-1
REAL(SP), PARAMETER :: EPS=1.0e-6_sp
Returnstheintegralofthefunction
funcfroma to b. Integrationisperformedby Romberg's
method of order 2K, where, e.g., K=2 is Simpson's rule.
Parameters: EPS is the fractional accuracy desired, as determined by the extrapolation error estimate; JMAX limits the total number of steps; K is the number of points used in the extrapolation.
REAL(SP), DIMENSION(JMAXP) :: h,s
These store the successive trapezoidal approximations and their relative stepsizes.
REAL(SP) :: dqromb
INTEGER(I4B) :: j
h(1)=1.0
do j=1,JMAX
   call trapzd(func,a,b,s(j),j)
   if (j >= K) then
      call polint(h(j-KM:j),s(j-KM:j),0.0_sp,qromb,dqromb)
      if (abs(dqromb) <= EPS*abs(qromb)) RETURN
   end if
   h(j+1)=0.25_sp*h(j)
end do
call nrerror('qromb: too many steps')
END FUNCTION qromb

SUBROUTINE midpnt(func,a,b,s,n)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
This routine computes the nth stage of refinement of an extended midpoint rule. func is input as the name of the function to be integrated between limits a and b, also input. When
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called with \( n=1 \), the routine returns as \( s \) the crudest estimate of \( \int_a^b f(x) \, dx \). Subsequent calls with \( n=2,3, \ldots \) (in that sequential order) will improve the accuracy of \( s \) by adding \((2/3) \times 3^{n-1}\) additional interior points. \( s \) should not be modified between sequential calls.

\[
\text{REAL(SP) :: del}
\]
\[
\text{INTEGER(I4B) :: it}
\]
\[
\text{REAL(SP), DIMENSION(2*3**(n-2)) :: x}
\]

if \( n == 1 \) then
\[
s=(b-a)\text{sum(func( (/0.5_sp*(a+b)/) ))}
\]
else
\[
\text{it}=3**(n-2)
\]
\[
\text{del}=(b-a)/(3.0_sp*it)
\]
The added points alternate in spacing between \( \text{del} \) and \( 2*\text{del} \).
\[
x(1:2*it-1:2)=x(1:2*it-1:2)+2.0_sp*\text{del}
\]
\[
s=s/3.0_sp*\text{del}+\text{sum(func(x))}
\]
The new sum is combined with the old integral to give a refined integral.

\[
\text{END SUBROUTINE midpnt}
\]

\( \text{midpnt} \) is a subroutine and not a function for the same reasons as \( \text{trapzd} \). This is also true for the other mid... routines below.

\[
s=(b-a)\text{sum(func( (/0.5_sp*(a+b)/) ))}
\]
Here we use \( (/\ldots/) \) to pass a single scalar argument to a vector function.

\[
\star\star\star
\]

FUNCTION qromo(func,a,b,choose)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : polint
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP) :: qromo
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
SUBROUTINE choose(funk,aa,bb,s,n)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: aa,bb
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
FUNCTION funk(x)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: funk
END FUNCTION funk
END INTERFACE
END SUBROUTINE choose
END INTERFACE

INTEGER(I4B), PARAMETER :: JMAX=14,JMAXP=JMAX+1,K=5,KM=K-1
REAL(SP), PARAMETER :: EPS=1.0e-6

Romberg integration on an open interval. Returns the integral of the function \( \text{func} \) from \( a \) to \( b \), using any specified integrating subroutine \( \text{choose} \) and Romberg's method. Normally \( \text{choose} \) will be an open formula, not evaluating the function at the endpoints. It is assumed that \( \text{choose} \) triples the number of steps on each call, and that its error series contains only
even powers of the number of steps. The routines midpnt, midinf, midsq, midsq, and midexp are possible choices for choose. The parameters have the same meaning as in qromo.

```plaintext
REAL(SP), DIMENSION(JMAXP) :: h, s
REAL(SP) :: dqromo
INTEGER(I4B) :: j
h(1)=1.0
do j=1,JMAX
    call choose(func,a,b,s(j),j)
    if (j >= K) then
        call polint(h(j-KM:j),s(j-KM:j),0.0_sp,qromo,dqromo)
        if (abs(dqromo) <= EPS*abs(qromo)) RETURN
    end if
    s(j+1)=s(j)
    h(j+1)=h(j)/9.0_sp This is where the assumption of step tripling and an even
end do
    error series is used.
call nrerror('qromo: too many steps')
END FUNCTION qromo
```

```plaintext
⋆⋆⋆
SUBROUTINE midinf(funk,aa,bb,s,n)
USE nrtype; USE nrutil, ONLY : arth, assert
IMPLICIT NONE
REAL(SP), INTENT(IN) :: aa, bb
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
FUNCTION funk(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: funk
END FUNCTION funk
END INTERFACE
This routine is an exact replacement for midpnt, i.e., returns as s the nth stage of refinement of the integral of funk from aa to bb, except that the function is evaluated at evenly spaced points in 1/x rather than in x. This allows the upper limit bb to be as large and positive as the computer allows, or the lower limit aa to be as large and negative, but not both. aa and bb must have the same sign.

```
FUNCTION func(x)  The change of variable could have been effected by a statement function in midinf itself. However, the statement function is a Fortran 77 feature that is deprecated in Fortran 90 because it does not allow the benefits of having an explicit interface, i.e., a complete set of specification statements. Statement functions can always be coded as internal subprograms instead.

SUBROUTINE midsql(funk,aa,bb,s,n)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
REAL(SP), INTENT(IN) :: aa,bb
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
  FUNCTION funk(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: funk
  END FUNCTION funk
END INTERFACE
This routine is an exact replacement for midpnt, i.e., returns as s the nth stage of refinement of the integral of funk from aa to bb, except that it allows for an inverse square-root singularity in the integrand at the lower limit aa.
REAL(SP) :: a,b,del
if (n == 1) then
  From this point on, the routine is exactly identical to midpnt.
  s=(b-a)*sum(func( (/0.5_sp*(a+b)/) ))
else
  it=3**((n-2))
del=(b-a)/(3.0_sp*it)
x(1:2*it-1:2)=arth(a+0.5_sp*del,3.0_sp*del,it)
x(2:2*it-1:2)=x(1:2*it-1:2)+2.0_sp*del
s=s/3.0_sp+del*sum(func(x))
end if
CONTAINS
  FUNCTION func(x)  This internal function effects the change of variable.
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: func
  func=2.0_sp*x*funk(aa+x**2)
END FUNCTION func
END SUBROUTINE midsql

SUBROUTINE midsqu(funk,aa,bb,s,n)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
REAL(SP), INTENT(IN) :: aa,bb
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
  FUNCTION funk(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: funk
  END FUNCTION funk
END INTERFACE
This routine is an exact replacement for midpnt, i.e., returns as s the nth stage of refinement of the integral of funk from aa to bb, except that it allows for an inverse square-root singularity in the integrand at the upper limit bb.
REAL(SP) :: a,b,del
INTEGER(I4B) :: it
REAL(SP), DIMENSION(2*3**(n-2)) :: x
b=sqrt(bb-aa)  These two statements change the limits of integration ac-
a=0.0 cordingly.
if (n == 1) then From this point on, the routine is exactly identical to midpnt.
s=(b-a)*sum(func( (/0.5_sp*(a+b)/) ))
else
  it=3**(n-2)
  del=(b-a)/(3.0_sp*it)
x(1:2*it-1:2)=arth(a+0.5_sp*del,3.0_sp*del,it)
x(2:2*it-1:2)=x(1:2*it-1:2)+2.0_sp*del
s=s/3.0_sp+del*sum(func(x))
end if
CONTAINS
  FUNCTION func(x) This internal function effects the change of variable.
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: func
    func=2.0_sp*x*funk(bb-x**2)
END FUNCTION func
END SUBROUTINE midsqu

SUBROUTINE midexp(funk,aa,bb,s,n)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
REAL(SP), INTENT(IN) :: aa,bb
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
  FUNCTION funk(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: funk
  END FUNCTION funk
END INTERFACE
This routine is an exact replacement for midpnt, i.e., returns as s the nth stage of refinement of the integral of funk from aa to bb. except that bb is assumed to be infinite (value passed not actually used). It is assumed that the function funk decreases exponentially rapidly at infinity.
REAL(SP) :: a,b,del
INTEGER(I4B) :: it
REAL(SP), DIMENSION(2*3**(n-2)) :: x
b=exp(-aa)  These two statements change the limits of integration ac-
a=0.0 cordingly.
if (n == 1) then From this point on, the routine is exactly identical to midpnt.
s=(b-a)*sum(func( (/0.5_sp*(a+b)/) ))
else
  it=3**(n-2)
  del=(b-a)/(3.0_sp*it)
x(1:2*it-1:2)=arth(a+0.5_sp*del,3.0_sp*del,it)
x(2:2*it-1:2)=x(1:2*it-1:2)+2.0_sp*del
s=s/3.0_sp+del*sum(func(x))
end if
CONTAINS
  FUNCTION func(x) This internal function effects the change of variable.
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: func
    func=funk(-log(x))/x
END FUNCTION func
END SUBROUTINE midexp

*   *   *
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SUBROUTINE gauleg(x1,x2,x,w)
USE nrtype; USE nrutil, ONLY : arth,assert_eq,nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
REAL(dp), PARAMETER :: EPS=3.0e-14_dp

Given the lower and upper limits of integration x1 and x2, this routine returns arrays x and w of length N containing the abscissas and weights of the Gauss-Legendre N-point quadrature formula. The parameter EPS is the relative precision. Note that internal computations are done in double precision.

INTEGER(1gb) :: its,j,m,n
INTEGER(1gb), PARAMETER :: MAXIT=10
REAL(dp) :: x1,xm
REAL(dp), DIMENSION((size(x)+1)/2) :: p1,p2,p3,pp,z,z1
LOGICAL(lgt), DIMENSION((size(x)+1)/2) :: unfinished

m=assert_eq(size(x),size(w),'gauleg')
xm=0.5_dp*(x2+x1)
x1=0.5_dp*(x2-x1)
z=cos(PI_D*(arth(1,1,m)-0.25_dp)/(n+0.5_dp))
unfinished=.true.
do its=1,MAXIT
where (unfinished)
   p1=1.0
   p2=0.0
   z=cos(PI_D*(arth(1,1,m)-0.25_dp)/(n+0.5_dp))
   unfinished=.true.
do j=1,n
   Loop up the recurrence relation to get
   the Legendre polynomials evaluated
   at z.
   where (unfinished)
      p3=p2
      p2=p1
      p1=((2.0_dp*j-1.0_dp)*z*p2-(j-1.0_dp)*p3)/j
   end where
   p1 now contains the desired Legendre polynomials. We next compute pp, the derivatives, by a standard relation involving also p2, the polynomials of one lower order.
   where (unfinished)
      pp=(z*p1-p2)/(z*z-1.0_dp)
      z1=z
      z=z1-p1/pp
      unfinished=(abs(z-z1) > EPS)
   end where
   if (.not. any(unfinished)) exit
   end do
   if (its == MAXIT+1) call nrerror('too many iterations in gauleg')
x(1:m)=xm-xl*z
   x(n:n+m-1:-1)=xm+xl*z
   w(1:m)=2.0_dp*xl/((1.0_dp-z**2)*pp**2)
   w(n:n+m-1:-1)=w(1:m)
END SUBROUTINE gauleg

Often we have an iterative procedure that has to be applied until all components of a vector have satisfied a convergence criterion. Some components of the vector might converge sooner than others, and it is inefficient on a small-scale parallel (SSP) machine to continue iterating on those components. The general structure we use for such an iteration is exemplified by the following lines from gauleg:

   LOGICAL(lgt), DIMENSION((size(x)+1)/2) :: unfinished
   unfinished=.true.
   do its=1,MAXIT
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We use the logical mask `unfinished` to control which vector components are processed inside the `where`. The mask gets updated on each iteration by testing whether any further vector components have converged. When all have converged, we exit the iteration loop. Finally, we check the value of `its` to see whether the maximum allowed number of iterations was exceeded before all components converged.

The logical expression controlling the `where` block (in this case `unfinished`) gets evaluated completely on entry into the `where`, and it is then perfectly fine to modify it inside the block. The modification affects only the next execution of the `where`.

On a strictly serial machine, there is of course some penalty associated with the above scheme: after a vector component converges, its corresponding component in `unfinished` is redundantly tested on each further iteration, until the slowest-converging component is done. If the number of iterations required does not vary too greatly from component to component, this is a minor, often negligible, penalty. However, one should be on the alert against algorithms whose worst-case convergence could differ from typical convergence by orders of magnitude. For these, one would need to implement a more complicated packing-unpacking scheme. (See discussion in Chapter B6, especially introduction, p. 1083, and notes for `factr1`, p. 1087.)

SUBROUTINE gaulag(x,w,alf)
USE nrtype; USE nrutil, ONLY : arth, assert_eq, nrerror
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), INTENT(IN) :: alf
REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
REAL(DP), PARAMETER :: EPS=3.0e-13_dp

Given `alf`, the parameter $\alpha$ of the Laguerre polynomials, this routine returns arrays $x$ and $w$ of length $N$ containing the abscissas and weights of the $N$-point Gauss-Laguerre quadrature formula. The abscissas are returned in ascending order. The parameter EPS is the relative precision. Note that internal computations are done in double precision.

INTEGER(I4B) :: its,j,n
INTEGER(I4B), PARAMETER :: MAXIT=10
REAL(SP) :: anu
REAL(SP), PARAMETER :: C1=9.084064e-01_sp,C2=5.214976e-02_sp,&
C3=2.579930e-03_sp,C4=3.986126e-03_sp
REAL(SP), DIMENSION(size(x)) :: rhs,r2,r3,theta
REAL(DP), DIMENSION(size(x)) :: p1,p2,p3,pp,z,z1
LOGICAL(LGT), DIMENSION(size(x)) :: unfinished
n=assert_eq(size(x),size(w),'gaulag')

rhs=arth(4*n-1,-4,n)*PI/anu
theta=r3*(C1+r2*(C2+r2*(C3+r2*C4)))
z=anu*cos(theta)**2

do its=1,MAXIT
where (unfinished)

end do

Newton’s method carried out simultaneously on the roots.
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\begin{verbatim}
p1=1.0
p2=0.0
end where
do j=1,n
   Loop up the recurrence relation to get the Laguerre polynomials evaluated at z.
   where (unfinished)
   p3=p2
   p2=p1
   p1=((2.0_dp*j-1.0_dp+alf-z)*p2-(j-1.0_dp+alf)*p3)/j
   end where
end do
p1 now contains the desired Laguerre polynomials. We next compute pp, the derivatives,
by a standard relation involving also p2, the polynomials of one lower order.
where (unfinished)
   pp=(n*p1-(n+alf)*p2)/z
   z1=z
   z=z1-p1/pp
   Newton's formula.
   unfinished=abs(z-z1) > EPS*z
end where
if (.not. any(unfinished)) exit
end do
if (its == MAXIT+1) call nrerror('too many iterations in gaulag')
x=z
Store the root and the weight.
w=-exp(gammln(alf+n)-gammln(real(n,sp)))/(pp*n*p2)
END SUBROUTINE gaulag

The key difficulty in parallelizing this routine starting from the Fortran 77 version is that the initial guesses for the roots of the Laguerre polynomials were given in terms of previously determined roots. This prevents one from finding all the roots simultaneously. The solution is to come up with a new approximation to the roots that is a simple explicit formula, like the formula we used for the Legendre roots in gauleg.

We start with the approximation to \( L_n^\alpha(x) \) given in equation (10.15.8) of [1]. We keep only the first term and ask when it is zero. This gives the following prescription for the \( k \)th root \( x_k \) of \( L_n^\alpha(x) \): Solve for \( \theta \) the equation

\[
2\theta - \sin 2\theta = \frac{4n - 4k + 3}{4n + 2\alpha + 2}\pi \quad (B4.1)
\]

Since \( 1 \leq k \leq n \) and \( \alpha > -1 \), we can always find a value such that \( 0 < \theta < \pi/2 \). Then the approximation to the root is

\[
x_k = (4n + 2\alpha + 2) \cos^2\theta \quad (B4.2)
\]

This typically gives 3-digit accuracy, more than enough for the Newton iteration to be able to refine the root. Unfortunately equation (B4.1) is not an explicit formula for \( \theta \). (You may recognize it as being of the same form as Kepler’s equation in mechanics.) If we call the right-hand side of (B4.1) \( y \), then we can get an explicit formula by working out the power series for \( y^{1/3} \) near \( \theta = 0 \) (using a computer algebra program). Next invert the series to give \( \theta \) as a function of \( y^{1/3} \). Finally, economize the series (see §5.11). The result is the concise approximation

\[
\theta = 0.9084064y^{1/3} + 5.214976 \times 10^{-2}y + 2.579930 \times 10^{-3}y^{5/3}
+ 3.986126 \times 10^{-3}y^{7/3} \quad (B4.3)
\]
\end{verbatim}
SUBROUTINE gauher(x,w)
USE nrtype; USE nrutil, ONLY : arth,assert_eq,nrerror
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
REAL(DP), PARAMETER :: EPS=3.0e-13_dp,PIM4=0.7511255444649425_dp

This routine returns arrays x and w of length N containing the abscissas and weights of
the N-point Gauss-Hermite quadrature formula. The abscissas are returned in descending
order. Note that internal computations are done in double precision.

Parameters: EPS is the relative precision, PIM4 = 1/\pi^{1/4}.

INTEGER(I4B) :: its,j,m,n
INTEGER(I4B), PARAMETER :: MAXIT=10
REAL(SP) :: anu
REAL(SP), PARAMETER :: C1=9.084064e-01_sp,C2=5.214976e-02_sp,&
               C3=2.579930e-03_sp,C4=3.986126e-03_sp
REAL(SP), DIMENSION((size(x)+1)/2) :: rhs,r2,r3,theta
REAL(SP), DIMENSION((size(x)+1)/2) :: p1,p2,p3,pp,z,z1
LOGICAL(LGT), DIMENSION((size(x)+1)/2) :: unfinished

n=assert_eq(size(x),size(w),'gauher')

m=(n+1)/2

The roots are symmetric about the origin, so we have to

anu=2.0_sp*n+1.0_sp

find only half of them.

rhs=arth(3,4,m)*PI/anu

r3=rhs**2

Newton's method carried out simultaneously on the roots.

\begin{align*}
\text{p1} &= PIM4 \\
\text{p2} &= 0.0
\end{align*}

where (unfinished)

\begin{align*}
\text{p3} &= \text{p2} \\
\text{p2} &= \text{p1} \\
\text{p1} &= z*sqrt(2.0_dp/j)*p2-sqrt(real(j-1,dp)/real(j,dp))*p3
\end{align*}

Newton's formula.

unfinished=(abs(z-z1) > EPS)

end where

if (.not. any(unfinished)) exit

if (its == MAXIT+1) call nrerror('too many iterations in gauher')

x(1:m)=z

Store the root

x(n:n+m+1:-1)=-z

and its symmetric counterpart.

w(1:m)=2.0_dp/pp**2

Compute the weight

w(n:n+m+1:-1)=w(1:m)

and its symmetric counterpart.

END SUBROUTINE gauher

Once again we need an explicit approximation for the polynomial roots,
this time for \( H_n(x) \). We can use the same approximation scheme as for \( L_n(x) \), since

\begin{align*}
H_{2m}(x) &\propto L_m^{-1/2}(x^2), & H_{2m+1}(x) &\propto xL_m^{1/2}(x^2)
\end{align*}

(B4.4)
Equations (B4.1) and (B4.2) become

\[ 2\theta - \sin 2\theta = \frac{4k - 1}{2n + 1} \pi \]

\[ x_k = \sqrt{2n + 1} \cos \theta \]  

(B4.5)

Here \( k = 1, 2, \ldots, m \) where \( m = \left\lfloor \frac{n+1}{2} \right\rfloor \), and \( k = 1 \) is the largest root. The negative roots follow from symmetry. The root at \( x = 0 \) for odd \( n \) is included in this approximation.

SUBROUTINE gaujac(x,w,alf,bet)
USE nrtype; USE nrutil, ONLY : arth,assert_eq,nrerror
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), INTENT(IN) :: alf,bet
REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
REAL(DP), PARAMETER :: EPS=3.0e-14_dp

Given \( \alpha \) and \( \beta \), the parameters \( \alpha \) and \( \beta \) of the Jacobi polynomials, this routine returns arrays \( x \) and \( w \) of length \( N \) containing the abscissas and weights of the \( N \)-point Gauss-Jacobi quadrature formula. The abscissas are returned in descending order. The parameter \( \epsilon \) is the relative precision. Note that internal computations are done in double precision.

INTEGER(14B) :: its,j,n
INTEGER(14B), PARAMETER :: MAXIT=10
REAL(DP) :: alfbet,a,c,temp
REAL(DP), DIMENSION(size(x)) :: b,p1,p2,p3,pp,z,z1
LOGICAL(LGT), DIMENSION(size(x)) :: unfinished

\( n = \text{assert.eq}(\text{size}(x), \text{size}(w), \text{\textquotesingle}gaujac\textquotesingle) \)

\( \text{alfbet=alf+bet} \)

Initial approximations to the roots go into \( z \).
\[ z = \cos(\pi \cdot \text{arth}(1,1,n) - 0.25_dp + 0.5_dp \cdot \alpha) / (n + 0.5_dp \cdot (\alpha \beta + 1.0_dp)) \]

\( \text{unfinished} = \text{true} \)

\( \text{do its=1,MAXIT \{ Newton's method carried out simultaneously on the roots. } \)

\( \text{temp=2.0_dp*alfbet} \)

\( \text{where (unfinished)} \)

\( p1 = (\text{alf-bet+temp}*z)/2.0_dp \)
\( p2=1.0 \)

\( \text{end where} \)

\( \text{do j=2,n \{ Loop up the recurrence relation to get the Jacobi polynomials evaluated at } z. \}

\( a=2*j*(j+alfbet)*temp \)

\( \text{temp=temp+2.0_dp} \)

\( c=2.0_dp*(j-1.0_dp+alf)*(j-1.0_dp+bet)*temp \)

\( \text{where (unfinished)} \)

\( p3=p1 \)
\( p2=p1 \)

\( b=(\text{temp-1.0_dp})*(\text{alf-upper bet})*\text{temp} & \text{temp-2.0_dp)*z} \)
\( p1=(b*p2-c*p3)/a \)

\( \text{end where} \)

\( \text{end do} \)

p1 now contains the desired Jacobi polynomials. We next compute pp, the derivatives, by a standard relation involving also p2, the polynomials of one lower order.

\( \text{where (unfinished)} \)

\( pp=(n*(n+alfbet-temp)*p1+2.0_dp*n*alf)+ & (n*bet)*p2)/(\text{temp*(1.0_dp-z*z))} \)

\( z1=z \)
\( z=z1/p1/pp \)

Newton's formula.

\( \text{unfinished}=(\text{abs}(z-z1) > \text{EPS}) \)

\( \text{end where} \)

if (.not. any(unfinished)) exit

end do

if (its == MAXIT+1) call nrerror('too many iterations in gaujac')

\( x=z \)

Store the root and the weight.
w = \exp(\text{gammln}(\alpha+n)+\text{gammln}(\beta+n)-\text{gammln}(n+1.0_{\text{sp}})-\text{gammln}(n+\alpha+\beta+1.0_{\text{sp}}))*\text{temp}*2.0_{\text{sp}}^{\alpha \beta}/(pp*p2)

END SUBROUTINE gaujac

Now we need an explicit approximation for the roots of the Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \). We start with the asymptotic expansion \( (10.14.10) \) of [1]. Setting this to zero gives the formula

\[
    x = \cos \left[ \frac{k - 1/4 + \alpha/2}{n + (\alpha + \beta + 1)/2} \right]
\]

This is better than the formula \( (22.16.1) \) in [2], especially at small and moderate \( n \).

⋆⋆⋆

SUBROUTINE gauof(a,b,amu0,x,w)
USE nrtype; USE nrutil, ONLY : assert_eq,unit_matrix
USE nr, ONLY : eigsrt,tqli
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a,b
REAL(SP), INTENT(IN) :: amu0
REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
Computes the abscissas and weights for a Gaussian quadrature formula from the Jacobi matrix. On input, \( a \) and \( b \) of length \( N \) are the coefficients of the recurrence relation for the set of monic orthogonal polynomials. The quantity \( \mu_0 \equiv \int_a^b W(x) \, dx \) is input as amu0. The abscissas are returned in descending order in array \( x \) of length \( N \), with the corresponding weights in \( w \), also of length \( N \). The arrays \( a \) and \( b \) are modified. Execution can be speeded up by modifying \( \text{tqli} \) and \( \text{eigsrt} \) to compute only the first component of each eigenvector.
REAL(SP), DIMENSION(size(a),size(a)) :: z
INTEGER(I4B) :: n
n=assert_eq(size(a),size(b),size(x),size(w),'gauof')
b(2:n)=sqrt(b(2:n)) \quad \text{Set up superdiagonal of Jacobi matrix.}
call unit_matrix(z) \quad \text{Set up identity matrix for tqli to compute eigenvectors.}
call tqli(a,b,z)
call eigsrt(a,z) \quad \text{Sort eigenvalues into descending order.}
x=a
w=amu0*z(1,:)**2 \quad \text{Equation (4.5.12).}
END SUBROUTINE gauof

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SUBROUTINE orthog(anu,alpha,beta,a,b)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: anu,alpha,beta
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a,b
Computes the coefficients \( a_j \) and \( b_j \), \( j = 0, \ldots, 2N - 1 \), of the recurrence relation for monic orthogonal polynomials with weight function \( W(x) \) by Wheeler's algorithm. On input, \( \alpha \) and \( \beta \) contain the \( 2N - 1 \) coefficients \( \alpha_j \) and \( \beta_j \), \( j = 0, \ldots, 2N - 2 \), of the recurrence relation.
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relation for the chosen basis of orthogonal polynomials. The $2N$ modified moments $\nu_j$ are input in `anu` for $j = 0, \ldots, 2N - 1$. The first $N$ coefficients are returned in `a` and `b`.

```fortran
INTEGER(4) :: k,n,ndum
REAL(SP), DIMENSION(2*size(a)+1,2*size(a)+1) :: sig
n=assert_eq(size(a),size(b),'orthog:n')
ndum=assert_eq(2*n,size(alpha)+1,size(anu),size(beta)+1,'orthog: ndum')
sig(1,3:2*n)=0.0   \text{Initialization, Equation (4.5.33).}
sig(2,2:2*n+1)=anu(1:2*n)
a(1)=alpha(1)+anu(2)/anu(1)
b(1)=0.0

\text{do k=3,n+1 Equation (4.5.34).}
sig(k,k:2*n-k+3)=sig(k-1,k+1:2*n-k+4)+(alpha(k-1:2*n-k+2) &
-a(k-2))*sig(k-1,k:2*n-k+3) &
+beta(k-1:2*n-k+2)*sig(k-1,k:2*n-k+3) &
a(k-1)=alpha(k-1)/sig(k,k)/sig(k,k)/sig(k-1,k-1)
b(k-1)=sig(k,k)/sig(k-1,k-1)
\text{end do}

END SUBROUTINE orthog
```

As discussed in Volume 1, multidimensional quadrature can be performed by calling a one-dimensional quadrature routine along each dimension. If the same routine is used for all such calls, then the calls are recursive. The file `quad3d.f90` contains two modules, `quad3d_gaus_mod` and `quad3d_qromb_mod`. In the first, the basic one-dimensional quadrature routine is a 10-point Gaussian quadrature routine called `qgaus` and three-dimensional quadrature is performed by calling `quad3d_gaus`. In the second, the basic one-dimensional routine is `qromb` of §4.3 and the three-dimensional routine is `quad3d_qromb`. The Gaussian quadrature is simpler but its accuracy is not controllable. The Romberg integration lets you specify an accuracy, but is apt to be very slow if you try for too much accuracy. The only difference between the stand-alone version of `trapzd` and the version included here is that we have to add the keyword `RECURSIVE`. The only changes from the stand-alone version of `qromb` are: We have to add `RECURSIVE`; we remove `trapzd` from the list of routines in `USE nr`; we increase `EPS` to $3 \times 10^{-6}$. Even this value could be too ambitious for difficult functions. You may want to set `JMAX` to a smaller value than 20 to avoid burning up a lot of computer time. Some people advocate using a smaller `EPS` on the inner quadrature (over $z$ in our routine) than on the outer quadratures (over $x$ or $y$). That strategy would require separate copies of `qromb`.

```fortran
MODULE quad3d_gaus_mod
USE nrtype
PRIVATE
PUBLIC quad3d_gaus
REAL(SP) :: xsav,ysav
INTERFACE
FUNCTION func(x,y,z) The three-dimensional function to be integrated.
USE nrtype
REAL(SP), INTENT(IN) :: x,y
REAL(SP), DIMENSION(:) INTENT(IN) :: z
REAL(SP), DIMENSION(size(z)) :: func
END FUNCTION func
FUNCTION y1(x)
USE nrtype
```

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USE nrtype
PRIVATE
PUBLIC quad3d_gaus
REAL(SP) :: xsav,ysav
INTERFACE
FUNCTION func(x,y,z) The three-dimensional function to be integrated.
USE nrtype
REAL(SP), INTENT(IN) :: x,y
REAL(SP), DIMENSION(:) INTENT(IN) :: z
REAL(SP), DIMENSION(size(z)) :: func
END FUNCTION func
FUNCTION y1(x)
USE nrtype
```
The routine quad3d, qgau returns as $ss$ the integral of a user-supplied function $func$ over a three-dimensional region specified by the limits $x1$, $x2$, and by the user-supplied functions $y1$, $y2$, $z1$, and $z2$, as defined in (4.6.2). Integration is performed by calling qgau recursively.

CONTAINS

FUNCTION h(x)
This is $H$ of eq. (4.6.5).
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: h
INTEGER(I4B) :: i
    DO i=1,size(x)
        xsav=x(i)
        h(i)=qgau(g,y1(xsav),y2(xsav))
    END DO
END FUNCTION h

FUNCTION g(y)
This is $G$ of eq. (4.6.4).
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(size(y)) :: g
INTEGER(I4B) :: j
    DO j=1,size(y)
        ysav=y(j)
        g(j)=qgau(f,z1(xsav,ysav),z2(xsav,ysav))
    END DO
END FUNCTION g

FUNCTION f(z)
The integrand $f(x, y, z)$ evaluated at fixed $x$ and $y$.
REAL(SP), DIMENSION(:), INTENT(IN) :: z
REAL(SP), DIMENSION(size(z)) :: f
    f=func(xsav,ysav,z)
END FUNCTION f

RECURSIVE FUNCTION qgau(func,a,b)
REAL(SP), INTENT(IN) :: a,b
REAL(SP) :: qgau
INTERFACE
    FUNCTION func(x)
        USE nrtype
        REAL(SP), DIMENSION(:), INTENT(IN) :: x
        REAL(SP), DIMENSION(size(x)) :: func
    END FUNCTION func
END INTERFACE

REAL(SP) :: xmm
REAL(SP) :: dx,w = (/ 0.2955242247_sp,0.2692667193_sp,&
          0.2190863625_sp,0.1494513491_sp,0.0666713443_sp /),&
          x = (/ 0.1488743389_sp,0.433953941_sp,0.6794095682_sp,&
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0.8650633666_sp, 0.9739065285_sp /

xm=0.5_sp*(b+a)
xr=0.5_sp*(b-a)
dx(:)=xr*x(:)
qgaus=xr*sum(w(:)*(func(xm+dx)+func(xm-dx)))
END FUNCTION qgaus

SUBROUTINE quad3d_qgaus(x1,x2,ss)
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP), INTENT(OUT) :: ss
ss=qgaus(h,x1,x2)
END SUBROUTINE quad3d_qgaus

END MODULE quad3d_qgaus_mod

PRIVATE...PUBLIC quad3d_qgaus  By default, all module entities are accessible by a routine that uses the module (unless we restrict the USE statement with ONLY). In this module, the user needs access only to the routine quad3d_qgaus; the variables xsav, ysav and the procedures f, g, h, and qgaus are purely internal. It is good programming practice to prevent duplicate name conflicts or data overwriting by limiting access to only the desired entities. Here the PRIVATE statement with no variable names resets the default from PUBLIC. Then we include in the PUBLIC statement only the function name we want to be accessible.

REAL(SP) :: xsav,ysav  In Fortran 90, we generally avoid declaring global variables in COMMON blocks. Instead, we give them complete specifications in a module. A deficiency of Fortran 90 is that it does not allow pointers to functions. So here we have to use the fixed-name function func for the function to be integrated over. If we could have a pointer to a function as a global variable, then we would just set the pointer to point to the user function (of any name) in the calling program. Similarly the functions y1, y2, z1, and z2 could also have any name.

CONTAINS  Here follow the internal subprograms f, g, h, qgaus, and quad3d_qgaus. Note that such internal subprograms are all “visible” to each other, i.e., their interfaces are mutually explicit, and do not require INTERFACE statements.

RECURSIVE SUBROUTINE qgaus(func,a,b,ss)  The RECURSIVE keyword is required for the compiler to process correctly any procedure that is invoked again in its body before the return from the first call has been completed. While some compilers may let you get away without explicitly informing them that a routine is recursive, don’t count on it!

MODULE quad3d_qromb_mod
Alternative to quad3d_qgaus_mod that uses qromb to perform each one-dimensional integration.
 USE nrtype
 PRIVATE
 PUBLIC quad3d_qromb
 REAL(SP) :: xsav,ysav
 INTERFACE
 FUNCTION func(x,y,z)
 USE nrtype
 REAL(SP), INTENT(IN) :: x,y
 REAL(SP), DIMENSION(:), INTENT(IN) :: z
 REAL(SP), DIMENSION(size(z)) :: func
 END FUNCTION func
 FUNCTION y1(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: y1
END FUNCTION y1
FUNCTION y2(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: y2
END FUNCTION y2
FUNCTION z1(x,y)
USE nrtype
REAL(SP), INTENT(IN) :: x,y
REAL(SP) :: z1
END FUNCTION z1
FUNCTION z2(x,y)
USE nrtype
REAL(SP), INTENT(IN) :: x,y
REAL(SP) :: z2
END FUNCTION z2
END INTERFACE
CONTAINS
FUNCTION h(x)
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: h
INTEGER(I4B) :: i
DO i = 1, size(x)
   xsav = x(i)
   h(i) = qromb(g, y1(xsav), y2(xsav))
END DO
END FUNCTION h
FUNCTION g(y)
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(size(y)) :: g
INTEGER(I4B) :: j
DO j = 1, size(y)
   ysav = y(j)
   g(j) = qromb(f, z1(xsav, ysav), z2(xsav, ysav))
END DO
END FUNCTION g
FUNCTION f(z)
REAL(SP), DIMENSION(:), INTENT(IN) :: z
REAL(SP), DIMENSION(size(z)) :: f
f = func(xsav, ysav, z)
END FUNCTION f
RECURSIVE FUNCTION qromb(func,a,b)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : polint
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP) :: qromb
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: JMAX=20, JMAXP=JMAX+1, K=5, KM=K-1
REAL(SP), PARAMETER :: EPS=3.0e-6_sp
REAL(SP), DIMENSION(JMAXP) :: h,s
REAL(SP) :: dqromb
INTEGER(I4B) :: j
h(1)=1.0
do j=1,JMAX
   call trapzd(func,a,b,s(j),j)
   if (j >= K) then
      call polint(h(j-KM:j),s(j-KM:j),0.0_sp,qromb,dqromb)
      if (abs(dqromb) <= EPS*abs(qromb)) RETURN
   end if
   s(j+1)=s(j)
   h(j+1)=0.25_sp*h(j)
end do
   call nrerror('qromb:too many steps')
END FUNCTION qromb

RECURSIVE SUBROUTINE trapzd(func,a,b,s,n)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
   FUNCTION func(x)
      USE nrtype
      REAL(SP), DIMENSION(:,), INTENT(IN) :: x
      REAL(SP), DIMENSION(size(x)) :: func
   END FUNCTION func
END INTERFACE
REAL(SP) :: del,fsum
INTEGER(I4B) :: it
if (n == 1) then
   s=0.5_sp*(b-a)*sum(func( (/ a,b / )))
else
   it=2**(n-2)
   del=(b-a)/it
   fsum=sum(func(arth(a+0.5_sp*del,del,it)))
   s=0.5_sp*(s+del*fsum)
end if
END SUBROUTINE trapzd
SUBROUTINE quad3d_qromb(x1,x2,ss)
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP), INTENT(OUT) :: ss
ss=qromb(h,x1,x2)
END SUBROUTINE quad3d_qromb
END MODULE quad3d_qromb_mod

MODULE quad3d_qromb_mod
   The only difference between this module and the previous one is that all calls to qgaus are replaced by calls to qromb and that the routine qgaus is replaced by qromb and trapzd.

CITED REFERENCES AND FURTHER READING:
Chapter B5. Evaluation of Functions

SUBROUTINE eulsum(sum, term, jterm)
USE nrtype; USE nrutil, ONLY : poly_term, reallocate
IMPLICIT NONE
REAL(SP), INTENT(INOUT) :: sum
REAL(SP), INTENT(IN) :: term
INTEGER(I4B), INTENT(IN) :: jterm

Incorporates into sum the jterm'th term, with value term, of an alternating series. sum is input as the previous partial sum, and is output as the new partial sum. The first call to this routine, with the first term in the series, should be with jterm=1. On the second call, term should be set to the second term of the series, with sign opposite to that of the first call, and jterm should be 2. And so on.

REAL(SP), DIMENSION(:), POINTER, SAVE :: wksp
INTEGER(I4B), SAVE :: nterm
LOGICAL(LGT), SAVE :: init=.true.

if (init) then
    init=.false.
    nullify(wksp)
end if

if (jterm == 1) then
    nterm=1
    wksp=>reallocate(wksp,100)
    wksp(1)=term
    sum=0.5_sp*term
    Return first estimate.
else
    if (nterm+1 > size(wksp)) wksp=>reallocate(wksp,2*size(wksp))
    wksp(2:nterm+1)=0.5_sp*wksp(1:nterm)
    wksp(1)=term
    Update saved quantities by van Wijngaarden's algorithm.
    wksp(1:nterm+1)=poly_term(wksp(1:nterm+1),0.5_sp)
    if (abs(wksp(nterm+1)) <= abs(wksp(nterm))) then
        Favorable to increase p,
        sum=sum+0.5_sp*wksp(nterm+1)
        nterm=nterm+1
        and the table becomes longer.
    else
        Favorable to increase n,
        sum=sum+wksp(nterm+1)
        the table doesn't become longer.
    end if
end if
END SUBROUTINE eulsum

This routine uses the function reallocate in nrutil to define a temporary workspace and then, if necessary, enlarge the workspace without destroying the earlier contents. The pointer wksp is declared with the SAVE attribute. Since Fortran 90 pointers are born “in limbo,” we cannot immediately test whether they are associated or not. Hence the code if (init)...nullify(wksp). Then the line wksp=>reallocate(wksp,100) allocates an array of length 100 and points wksp to it. On subsequent calls to eulsum, if nterm ever gets bigger than the size of wksp, the call to reallocate doubles the size of wksp and copies the old contents into the new storage.

You could achieve the same effect as the code if (init)...nullify(wksp)... wksp=>reallocate(wksp,100) with a simple allocate(wksp,100). You would then use
realloc only for increasing the storage if necessary. Don’t! The advantage of the above scheme becomes clear if you consider what happens if \texttt{eulsum} is invoked \textit{twice} by the calling program to evaluate two different sums. On the second invocation, when \texttt{jterm} = 1 again, you would be allocating an already allocated pointer. This does not generate an error — it simply leaves the original target inaccessible. Using \texttt{realloc} instead not only allocates a new array of length 100, but also detects that \texttt{wksp} had already been associated. It dutifully (and wastefully) copies the first 100 elements of the old \texttt{wksp} into the new storage, and, more importantly, deallocates the old \texttt{wksp}, reclaiming its storage. While only two invocations of \texttt{eulsum} without intervening deallocation of memory would not cause a problem, many such invocations might well. We believe that, as a general rule, the potential for catastrophe from reckless use of \texttt{allocate} is great enough that you should \textit{always} deallocate whenever storage is no longer required.

The unnecessary copying of 100 elements when \texttt{eulsum} is invoked a second time could be avoided by making \texttt{init} an argument. It hardly seems worth it to us.

For Fortran 90 neophytes, note that unlike in \texttt{C} you have to do nothing special to get the contents of the storage a pointer is addressing. The compiler figures out from the context whether you mean the contents, such as \texttt{wksp(1:nterm)}, or the address, such as both occurrences of \texttt{wksp} in \texttt{wksp=>realloc(wksp,100)}.

\texttt{wksp(1:nterm+1)=poly_term(wksp(1:nterm+1),0.5_sp)} The \texttt{poly_term} function in \texttt{nrutil} tabulates the partial sums of a polynomial, or, equivalently, performs the synthetic division of a polynomial by a monomial.

Small-scale parallelism in \texttt{eulsum} is achieved straightforwardly by the use of vector constructions and \texttt{poly_term} (which parallelizes recursively). The routine is not written to take advantage of data parallelism in the (infrequent) case of wanting to sum many different series simultaneously; nor, since \texttt{wksp} is a \texttt{SAVE}d variable, can it be used in many simultaneous instances on a MIMD machine. (You can easily recode these generalizations if you need them.)

\begin{verbatim}
*  *  *
SUBROUTINE ddpoly(c,x,pd)
USE nrtype; USE nrutil, ONLY : arth,cumprod,poly_term
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: c
REAL(SP), DIMENSION(:), INTENT(OUT) :: pd

Given the coefficients of a polynomial of degree \texttt{Nc} – 1 as an array \texttt{c(1:Nc)} with \texttt{c(1)} being the constant term, and given a value \texttt{x}, this routine returns the polynomial evaluated at \texttt{x} as \texttt{pd(1)} and \texttt{Nd} – 1 derivatives as \texttt{pd(2:Nd)}.

INTEGER(I4B) :: i,nc,nd
REAL(SP), DIMENSION(size(pd)) :: fac
REAL(SP), DIMENSION(size(c)) :: d
nc=size(c)
nd=size(pd)
d(nc:1:-1)=poly_term(c(nc:1:-1),x)
do i=2,min(nd,nc)
d(nc:i:-1)=poly_term(d(nc:i:-1),x)
end do
pd=d(1:nd)
fac=cumprod(arth(1.0_sp,1.0_sp,nd)) After the first derivative, factorial constants
pd(3:nd)=fac(2:nd-1)*pd(3:nd) come in.
END SUBROUTINE ddpoly
\end{verbatim}
The `poly_term` function in `nrutil` tabulates the partial sums of a polynomial, or, equivalently, performs synthetic division. See §22.3 for a discussion of why `ddpoly` is coded this way.

```
d(nc:1:-1)=poly_term(c(nc:1:-1),x)
```

Here the function `arth` from `nrutil` generates the sequence 1, 2, 3, ... The function `cumprod` then tabulates the cumulative products, thus making a table of factorials.

Notice that `ddpoly` doesn't need an argument to pass $N_d$, the number of output terms desired by the user: It gets that information from the length of the array `pd` that the user provides for it to fill. It is a minor curiosity that `pd`, declared as `INTENT(OUT)`, can thus be used, on the sly, to pass some `INTENT(IN)` information.

(A Fortran 90 brain teaser could be: A subroutine with only `INTENT(OUT)` arguments can be called to print any specified integer. How is this done?)

SUBROUTINE `poldiv(u,v,q,r)`
USE `nrtype`; USE `nrutil`, ONLY : `assert_eq`
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: `u,v`
REAL(SP), DIMENSION(:), INTENT(OUT) :: `q,r`

Given the $N$ coefficients of a polynomial in $u$, and the $N_v$ coefficients of another polynomial in $v$, divide the polynomial $u$ by the polynomial $v$ ("$u$/"$v$"") giving a quotient polynomial whose coefficients are returned in $q$, and a remainder polynomial whose coefficients are returned in $r$. The arrays `q` and `r` are of length $N$, but only the first $N - N_v + 1$ elements of `q` and the first $N_v - 1$ elements of `r` are used. The remaining elements are returned as zero.

```
INTEGER(I4B) :: i,n,nv
n=assert_eq(size(`u`),size(`q`),size(`r`),'poldiv')
nv=size(`v`)
r(:,)=`u`(:,)
q(:,)=0.0
do i=n-nv,0,-1
  q(i+1)=r(nv+i)/v(nv)
  r(i+1:nv+i-1)=r(i+1:nv+i-1)-q(i+1)*v(1:nv-1)
end do
r(nv:n)=0.0
END SUBROUTINE `poldiv`
```

FUNCTION `ratval_s(x,cof,mm,kk)`
USE `nrtype`; USE `nrutil`, ONLY : `poly`
IMPLICIT NONE
REAL(DP), INTENT(IN) :: `x` Note precision! Change to `REAL(SP)` if desired.
INTEGER(I4B), INTENT(IN) :: `mm,kk`
REAL(DP), DIMENSION(`mm+kk+1`), INTENT(IN) :: `cof`
REAL(DP) :: `ratval_s`

Given `mm`, `kk`, and `cof`, evaluate and return the rational function \( \frac{\text{poly}(x,\text{cof}(1:mm+kk+1))}{1 + \text{poly}(x,\text{cof}(mm+2:mm+kk+1))} \).

```
   `ratval_s`=poly(x,cof(1:mm+1))/(1.0_dp+x*poly(x,cof(mm+2:mm+kk+1)))
```

END FUNCTION `ratval_s`

This simple routine uses the function `poly` from `nrutil` to evaluate the numerator and denominator polynomials. Single- and double-precision versions, `ratval_s` and `ratval_v`, are overloaded onto the name `ratval` when the module `nr` is used.
FUNCTION ratval_v(x, cof, mm, kk)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(DP), DIMENSION(:), INTENT(IN) :: x
INTEGER(I4B), INTENT(IN) :: mm, kk
REAL(DP), DIMENSION(mm+kk+1), INTENT(IN) :: cof
REAL(DP), DIMENSION(size(x)) :: ratval_v
ratval_v=poly(x, cof(1:mm+1))/(1.0_dp+x*poly(x, cof(mm+2:mm+kk+1)))
END FUNCTION ratval_v

⋆⋆⋆

The routines recur1 and recur2 are new in this volume, and do not have Fortran 77 counterparts. First- and second-order linear recurrences are implemented as trivial do-loops on strictly serial machines. On parallel machines, however, they pose different, and quite interesting, programming challenges. Since many calculations can be decomposed into recurrences, it is useful to have general, parallelizable routines available. The algorithms behind recur1 and recur2 are discussed in §22.2.

RECURSIVE FUNCTION recur1(a, b) RESULT(u)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a, b
REAL(SP), DIMENSION(size(a)) :: u
INTEGER(I4B), PARAMETER :: NPAR_RECUR1=8
Given vectors a of size n and b of size n − 1, returns a vector u that satisfies the first order linear recurrence $u_1 = a_1$, $u_j = a_j + b_{j-1}u_{j-1}$, for $j = 2, \ldots, n$. Parallelization is via a recursive evaluation.
INTEGER(I4B) :: n, j
n=assert_eq(size(a), size(b)+1, 'recur1')
if (n < NPAR_RECUR1) then
  Do short vectors as a loop.
do j=2,n
    u(j)=a(j)+b(j-1)*u(j-1)
  end do
else
  Otherwise, combine coefficients and recurse on the even components, then evaluate all the odd components in parallel.
  u(2:n:2)=recur1(a(2:n:2)+a(1:n-1:2)*b(1:n-1:2), &
                 b(3:n-1:2)*b(2:n-2:2))
  u(3:n:2)=a(3:n:2)+b(2:n-1:2)*u(2:n-1:2)
end if
END FUNCTION recur1

⋆⋆⋆

RECURSIVE FUNCTION recur1(a, b) RESULT(u) When a recursive function invokes itself only indirectly through a sequence of function calls, then the function name can be used for the result just as in a nonrecursive function. When the function invokes itself directly, however, as in recur1, then another name must be used for the result. If you are hazy on the syntax for RESULT, see the discussion of recursion in §21.5.

⋆⋆⋆
FUNCTION recur2(a,b,c)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a,b,c
REAL(SP), DIMENSION(size(a)) :: recur2

Given vectors a of size n and b and c of size n-2, returns a vector u that satisfies the second order linear recurrence
\[ u_1 = a_1, \quad u_2 = a_2, \quad u_j = a_j + b_{j-2} u_{j-1} + c_{j-2} u_{j-2}, \quad \text{for } j = 3, \ldots, n. \]

Parallelization is via conversion to a first order recurrence for a two-dimensional vector.

INTEGER(I4B) :: n
REAL(SP), DIMENSION(size(a)-1) :: a1,a2,u1,u2
REAL(SP), DIMENSION(size(a)-2) :: b11,b12,b21,b22

n=assert_eq(size(a),size(b)+2,size(c)+2,'recur2')
a1(1)=a(1) Set up vector a.
a2(1)=a(2)
a1(2:n-1)=0.0
a2(2:n-1)=a(3:n)
b11(1:n-2)=0.0
b12(1:n-2)=1.0
b21(1:n-2)=c(1:n-2)
b22(1:n-2)=b(1:n-2)
call recur1_v(a1,a2,b11,b12,b21,b22,u1,u2)
recur2(1:n-1)=u1(1:n-1)
recur2(n)=u2(n-1)

CONTAINS

RECURSIVE SUBROUTINE recur1_v(a1,a2,b11,b12,b21,b22,u1,u2)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a1,a2,b11,b12,b21,b22
REAL(SP), DIMENSION(:), INTENT(OUT) :: u1,u2

INTEGER(I4B), PARAMETER :: NPAR_RECUR2=8

Used by recur2 to evaluate first order vector recurrence. Routine is a two-dimensional vector version of recur1, with matrix multiplication replacing scalar multiplication.

INTEGER(I4B) :: n,j,nn,nn1
REAL(SP), DIMENSION(size(a1)/2) :: aa1,aa2
REAL(SP), DIMENSION(size(a1)/2-1) :: bb11,bb12,bb21,bb22

n=assert_eq((/size(a1),size(a2),size(b11)+1,size(b12)+1,size(b21)+1,&size(b22)+1,size(u1),size(u2)/),'recur1_v')
u1(1)=a1(1)
u2(1)=a2(1)
if (n < NPAR_RECUR2) then
  Do short vectors as a loop.
do j=2,n
  u1(j)=a1(j)+b11(j-1)*u1(j-1)+b12(j-1)*u2(j-1)
u2(j)=a2(j)+b21(j-1)*u1(j-1)+b22(j-1)*u2(j-1)
endo else

  Otherwise, combine coefficients and recurse on the even components, then evaluate all the odd components in parallel.
nn=n/2
a1(1:nn)=a1(2:nn)+b11(1:nn-1)*a1(1:nn-1)+&b12(1:nn-1)*a2(1:nn-1)
a2(1:nn)=a2(2:nn)+b11(1:nn-1)*a2(1:nn-1)+&b22(1:nn-1)*a2(1:nn-1)
b11(1:nn)=b11(3:nn-1)+b11(2:nn-2)+a1(1:nn-1)+&b22(3:nn-1)+b22(2:nn-2)
b12(1:nn)=b12(3:nn-1)+b12(2:nn-2)+&b11(1:nn-1)+b21(2:nn-2)
b21(1:nn)=b21(3:nn-1)+b21(2:nn-2)+&b11(1:nn-1)+b22(2:nn-2)
b22(1:nn)=b22(3:nn-1)+b22(2:nn-2)+&b12(1:nn-1)+b22(2:nn-2)
call recur1_v(aa1,aa2,b11,b12,b21,b22,u1,nn,nn1)
u1(3:nn)=a1(3:nn)+b11(2:nn-1)+u1(1:nn-1)+&u2(2:nn-1)end
FUNCTION dfridr(func,x,h,err)
USE nrtype; USE nrutil, ONLY : assert, geop, iminloc
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,h
REAL(SP), INTENT(OUT) :: err
REAL(SP) :: dfridr
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B),PARAMETER :: NTAB=10
REAL(SP), PARAMETER :: CON=1.4_sp,CON2=CON*CON,BIG=huge(x),SAFE=2.0
Returns the derivative of a function func at a point x by Ridders’ method of polynomial
extrapolation. The value h is input as an estimated initial stepsize; it need not be small,
but rather should be an increment in x over which func changes substantially. An estimate
of the error in the derivative is returned as err.
Parameters: Stepsize is decreased by CON at each iteration. Max size of tableau is set by
NTAB. Return when error is SAFE worse than the best so far.
INTEGER(I4B) :: ierrmin,i,j
REAL(SP) :: hh
REAL(SP), DIMENSION(NTAB-1) :: errt,fac
REAL(SP), DIMENSION(NTAB,NTAB) :: a
call assert(h /= 0.0, ’dfridr arg’)
hh=h
a(1,1)=(func(x+hh)-func(x-hh))/(2.0_sp*hh)
err=BIG
fac(1:NTAB-1)=geop(CON2,CON2,NTAB-1)
do i=2,NTAB
   Successive columns in the Neville tableau will go to smaller
   hh=hh/CON
   a(1,i)=(func(x+hh)-func(x-hh))/(2.0_sp*hh)
   Try new, smaller stepsize.
do j=2,i
   Compute extrapolations of various orders, requiring no new function evaluations.
   a(j,i)=(a(j-1,i)*fac(j-1)-a(j-1,i-1))/(fac(j-1)-1.0_sp)
end do
errt(1:i-1)=max(abs(a(2:i,1)-a(1:i-1,1)),abs(a(2:i,1)-a(1:i-1,i-1)))/2.0_sp*hh
err=errt(ierrmin)
if (errt(ierrmin) <= err) then If error is decreased, save the improved an-
   err=errt(ierrmin)
   dfridr=a(1+ierrmin,i) swer.
end if
if (abs(a(1,i)-a(i-1,i-1)) >= SAFE*err) RETURN
   If higher order is worse by a significant factor SAFE, then quit early.
end do
END FUNCTION dfridr
The function `iminloc` in `nrutil` is useful when you need to know the index of the smallest element in an array.

```
FUNCTION chebft(a,b,n,func)
USE nrtype; USE nrutil, ONLY : arth,outerprod
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(n) :: chebft
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
CHEBYSHEV FIT: Given a function `func`, lower and upper limits of the interval \([a, b]\), and a maximum degree \(n\), this routine computes the \(n\) coefficients \(c_k\) such that \(\text{func}(x) \approx \sum_{k=1}^{n} c_k T_{k-1}(y) - c_1/2\), where \(y\) and \(x\) are related by (5.8.10). This routine is to be used with moderately large \(n\) (e.g., 30 or 50), the array of \(c\)'s subsequently to be truncated at the smaller value \(m\) such that \(c_{m+1}\) and subsequent elements are negligible.

```

```
REAL(DP) :: bma,bpa
REAL(DP), DIMENSION(n) :: theta
bma=0.5_dp*(b-a)
bpa=0.5_dp*(b+a)
theta(:)=PI_D*arth(0.5_dp,1.0_dp,n)/n
chebft(:)=matmul(cos(outerprod(arth(0.0_dp,1.0_dp,n),theta)), &
func(real(cos(theta)*bma+bpa,sp)))*2.0_dp/n
We evaluate the function at the \(n\) points required by (5.8.7). We accumulate the sum in double precision for safety.

```

```
FUNCTION chebev_s(a,b,c,x)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b,x
REAL(SP), DIMENSION(:), INTENT(IN) :: c
REAL(SP) :: chebev_s
CHEBYSHEV EVALUATION: All arguments are input. \(c\) is an array of length \(M\) of Chebyshev coefficients, the first \(M\) elements of \(c\) output from `chebft` (which must have been called
```

Here again Fortran 90 produces a very concise parallelizable formulation that requires some effort to decode. Equation (5.8.7) is a product of the matrix of cosines, where the rows are indexed by \(j\) and the columns by \(k\), with the vector of function values indexed by \(k\). We use the `outerprod` function in `nrutil` to form the matrix of arguments for the cosine, and rely on the element-by-element application of `cos` to produce the matrix of cosines. `matmul` then takes care of the matrix product. A subtlety is that, while the calculation is being done in double precision to minimize roundoff, the function is assumed to be supplied in single precision. Thus `real(...,sp)` is used to convert the double precision argument to single precision.
with the same \( a \) and \( b \). The Chebyshev polynomial \( \sum_{k=1}^{M} c_k T_k(y) - c_1/2 \) is evaluated at a point \( y = \left[ x - \frac{(b + a)}{2} \right] / \left[ \frac{(b - a)}{2} \right] \), and the result is returned as the function value.

\[
\text{INTEGER(I4B)} :: j, m \\
\text{REAL(SP)} :: d, dd, sv, y, y2 \\
\text{if } ((x-a)*(x-b) > 0.0) \text{ call nrerror('x not in range in chebev_s')} \\
m = \text{size}(c) \\
d = 0.0 \\
\text{dd} = 0.0 \\
y = \left(2.0_{\text{sp}} * x - a - b\right) / \left(b - a\right) \\
y2 = 2.0_{\text{sp}} * y \\
do \ j = m, 2, -1 \\
sv = d \\
d = y2 * d - dd + c(j) \\
\text{dd} = sv \\
end do \\
\text{chebev_s} = y * d - dd + 0.5_{\text{sp}} * c(1) \\
\text{END FUNCTION chebev_s}
\]

The name \texttt{chebev} is overloaded with scalar and vector versions. \texttt{chebev_v} is essentially identical to \texttt{chebev_s} except for the declarations of the variables. Fortran 90 does the appropriate scalar or vector arithmetic in the body of the routine, depending on the type of the variables.

\[
\star \star \star \\
\text{FUNCTION chder(a,b,c)} \\
\text{USE nrtype; USE nrutil, ONLY : arth, cumsum} \\
\text{IMPLICIT NONE} \\
\text{REAL(SP), INTENT(IN)} :: a, b \\
\text{REAL(SP), DIMENSION(():), INTENT(IN)} :: c \\
\text{REAL(SP), DIMENSION(size(x)) :: chder} \\
\text{INTEGER(I4B)} :: j, m \\
\text{REAL(SP), DIMENSION(size(x)) :: d, dd, sv, y, y2} \\
\text{if } (any((x-a)*(x-b) > 0.0)) \text{ call nrerror('x not in range in chebev_v')} \\
m = \text{size}(c) \\
d = 0.0 \\
\text{dd} = 0.0 \\
y = \left(2.0_{\text{sp}} * x - a - b\right) / \left(b - a\right) \\
y2 = 2.0_{\text{sp}} * y \\
do \ j = m, 2, -1 \\
sv = d \\
d = y2 * d - dd + c(j) \\
\text{dd} = sv \\
end do \\
\text{chebev_v} = y * d - dd + 0.5_{\text{sp}} * c(1) \\
\text{END FUNCTION chebev_v}
\]
from routine chebft §5.8. The desired degree of approximation \( N \) is equal to the length of \( c \) supplied.

\[
\begin{align*}
\text{INTEGER(14B)} & : n \\
\text{REAL(SP)} & : \text{con} \\
\text{REAL(SP), DIMENSION(size(c))} & : \text{temp}
\end{align*}
\]

\( n = \text{size}(c) \)
\( \text{temp}(1) = 0.0 \)
\( \text{temp}(2:n) = 2.0_{\text{sp}} \times \text{arth}(n-1,-1,n-1) \times c(n:2:-1) \)
\( \text{chder}(n:1:-2) = \text{cumsum}(\text{temp}(1:n:2)) \) \hspace{1cm} \text{Equation (5.9.2)}.
\( \text{chder}(n-1:1:-2) = \text{cumsum}(\text{temp}(2:n:2)) \)
\( \text{con} = 2.0_{\text{sp}} / (b-a) \)
\( \text{chder} = \text{chder} \times \text{con} \) \hspace{1cm} \text{Normalize to the interval } b-a.

\end{FUNCTION}

\begin{FUNCTION}
\text{chint}(a,b,c)
\USES\ nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
\text{REAL(SP), INTENT(IN)} :: a, b
\text{REAL(SP), DIMENSION(:), INTENT(IN)} :: c
\text{REAL(SP), DIMENSION(size(c))} :: chint
\end{FUNCTION}

This routine returns an array of length \( N \) containing the Chebyshev coefficients of the integral of the function whose coefficients are in the array \( c \). Input are \( a, b, c \), as output from routine chebft §5.8. The desired degree of approximation \( N \) is equal to the length of \( c \) supplied. The constant of integration is set so that the integral vanishes at \( a \).

\[
\begin{align*}
\text{INTEGER(14B)} & : n \\
\text{REAL(SP)} & : \text{con} \\
\text{n} & = \text{size}(c) \\
\text{con} & = 0.25_{\text{sp}} \times (b-a) \\
\text{chint}(2:n-1) & = \text{con} \times (c(1:n-2) - c(3:n)) / \text{arth}(1,1,n-2) \hspace{1cm} \text{Equation (5.9.1)}.
\end{align*}
\]
\( \text{chint}(n) = \text{con} \times c(n-1) / (n-1) \) \hspace{1cm} \text{Special case of (5.9.1) for } n.
\( \text{chint}(1) = 2.0_{\text{sp}} \times (\text{sum}(\text{chint}(2:n:2)) - \text{sum}(\text{chint}(3:n:2))) \) \hspace{1cm} \text{Set the constant of integration.}

\end{FUNCTION}

If you look at equation (5.9.1) for the Chebyshev coefficients of the integral of a function, you will see \( c_{i-1} \) and \( c_{i+1} \) and be tempted to use eoshift. We think it is almost always better to use array sections instead, as in the code above, especially if your code will ever run on a serial machine.

\HINT

\begin{FUNCTION}
\text{chebpc}(c)
\USES\ nrtype
IMPLICIT NONE
\text{REAL(SP), DIMENSION(\text{size(c)})} :: c
\end{FUNCTION}

Chebyshev polynomial coefficients. Given a coefficient array \( c \) of length \( N \), this routine returns a coefficient array \( d \) of length \( N \) such that \( \sum_{k=1}^{N} d_k y^{k-1} = \sum_{k=1}^{N} c_k T_{k-1}(y) - c_1 / 2 \). The method is Clenshaw's recurrence (5.8.11), but now applied algebraically rather than arithmetically.

\[
\begin{align*}
\text{INTEGER(14B)} & : j, n \\
\text{REAL(SP), DIMENSION(size(c))} & : \text{dd, sv}
\end{align*}
\]
\( n = \text{size}(c) \)
\( \text{chebpc} = 0.0 \)
\( \text{dd} = 0.0 \)
\( \text{chebpc}(1) = c(n) \)
\( \text{do } j = n-1, 2, -1 \)
\( \text{sv}(2:n-j+1) = \text{chebpc}(2:n-j+1) \)
\( \text{chebpc}(2:n-j+1) = 2.0_{\text{sp}} \times \text{chebpc}(1:n-j) - \text{dd}(2:n-j+1) \)
\( \text{dd}(2:n-j+1) = \text{sv}(2:n-j+1) \)
\end{FUNCTION}
SUBROUTINE pcshft(a,b,d)
USE nrtype; USE nrutil, ONLY : geop
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP), DIMENSION(:), INTENT(INOUT) :: d

Polynomial coefficient shift. Given a coefficient array \( d \) of length \( N \), this routine generates a coefficient array \( g \) of the same length such that
\[
\sum_{k=1}^{N} d_k y^{k-1} = \sum_{k=1}^{N} g_k x^{k-1},
\]
where \( x \) and \( y \) are related by (5.8.10), i.e., the interval \(-1 < y < 1\) is mapped to the interval \( a < x < b \). The array \( g \) is returned in \( d \).

INTEGER(I4B) :: j,n
REAL(SP), DIMENSION(size(d)) :: dd
REAL(SP) :: x
n=size(d)
dd=d*geop(1.0_sp,2.0_sp/(b-a),n)
x=-0.5_sp*(a+b)
d(1)=dd(n)
d(2:n)=0.0
do j=n-1,1,-1
   We accomplish the shift by synthetic division, that miracle of high-school algebra.
   \( d(2:n+1-j) = (d(2:n+1-j) \times x + d(1:n-j)) \)
   \( d(1) = d(1) \times x + dd(j) \)
end do

END SUBROUTINE pcshft

There is a subtle, but major, distinction between the synthetic division algorithm used in the Fortran 77 version of pcshft and that used above. In the Fortran 77 version, the synthetic division (translated to Fortran 90 notation) is
\[
d(1:n) = dd(1:n)
\]
go do \( j=1,n-1 \)
go do \( k=n-1,j,-1 \)
go \( d(k) = x \times d(k+1) + d(k) \)
go end do
end do

while, in Fortran 90, it is
\[
d(1)=dd(n)
d(2:n)=0.0
\]
go do \( j=n-1,1,-1 \)
go \( d(2:n+1-j) = d(2:n+1-j) \times x + dd(n-j) \)
go \( d(1) = d(1) \times x + dd(j) \)
go end do

As explained in §22.3, these are algebraically — but not algorithmically — equivalent. The inner loop in the Fortran 77 version does not parallelize, because each \( k \) value uses the result of the previous one. In fact, the \( k \) loop is a synthetic division, which can be parallelized recursively (as in the nrutil routine poly_term), but not simply
vectorized. In the Fortran 90 version, since not one but n-1 successive synthetic divisions are to be performed (by the outer loop), it is possible to reorganize the calculation to allow vectorization.

```
FUNCTION pccheb(d)
USE nrtype; USE nrutil, ONLY : arth,cumprod,geop
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: d
REAL(SP), DIMENSION(size(d)) :: pccheb
    Inverse of routine chebpc: given an array of polynomial coefficients d, returns an equivalent array of Chebyshev coefficients of the same length.
INTEGER(14B) :: k,n
REAL(SP), DIMENSION(size(d)) :: denom,numer,pow
n=size(d)
pcheb(1)=2.0_sp*d(1)
pow=geop(1.0_sp,2.0_sp,n)    Powers of 2.
numer(1)=1.0
    Combinatorial coefficients computed as numer/denom.
denom(1)=1.0
denom(2:(n+3)/2)=cumprod(arth(1.0_sp,1.0_sp,(n+1)/2))
pcheb(2:n)=0.0
    Loop over orders of x in the polynomial.
do k=2,n
    numer(2:(k+3)/2)=cumprod(arth(k-1.0_sp,-1.0_sp,(k+1)/2))
pcheb(k:1:-2)=pcheb(k:1:-2)+&
d(k)/pow(k-1)*numer(1:(k+1)/2)/denom(1:(k+1)/2)
end do
END FUNCTION pccheb
```

```
SUBROUTINE pade(cof,resid)
USE nrtype
USE nr, ONLY : lubksb,ludcmp,mprove
IMPLICIT NONE
REAL(DP), DIMENSION(:), INTENT(INOUT) :: cof DP for consistency with ratval.
REAL(SP), INTENT(OUT) :: resid
Given cof(1:2*N+1), the leading terms in the power series expansion of a function, solve
the linear Padé equations to return the coefficients of a diagonal rational function approxi-
mation to the same function, namely (cof(1) + cof(2)*x + ... + cof(N+1)*x^N) / (1 +
cof(N+2)*x + ... + cof(2*N+1)*x^N). The value resid is the norm of the residual
vector; a small value indicates a well-converged solution.
INTEGER(14B) :: k,n
INTEGER(14B), DIMENSION((size(cof)-1)/2) :: indx
REAL(SP), PARAMETER :: BIG=1.0e30_sp A big number.
REAL(SP) :: d,rr,rrold
REAL(SP), DIMENSION((size(cof)-1)/2) :: x,y,z
REAL(SP), DIMENSION((size(cof)-1)/2,(size(cof)-1)/2) :: q,qlu
n=(size(cof)-1)/2
x=cof(n+2:2*n+1)
    Set up matrix for solving.
y=x
do k=1,n
    q(:,k)=cof(n+2-k:2*n+1-k)
end do
qlu=q
    Solve by LU decomposition and backsubsti-
tution.
call lubksb(qlu,indx,x)
rr=BIG
    Important to use iterative improvement, since
    the Padé equations tend to be ill-conditioned.
do 
    rrold=rr
```

z=x
call mprove(q,qlu,index,y,x)
rr=sum((z-x)**2)  Calculate residual.
if (rr >= rrold) exit  If it is no longer improving, call it quits.
end do
resid=sqrt(rrold) Calculate the remaining coefficients.
do k=1,n
     y(k)=cof(k+1)-dot_product(z(1:k),cof(k:1:-1))
end do
cof(2:n+1)=y  Copy answers to output.
cof(n+2:2*n+1)=-z
END SUBROUTINE pade

SUBROUTINE ratlsq(func,a,b,mm,kk,cof,dev)
USE nrtype; USE nrutil, ONLY : arth,geop
USE nr, ONLY : ratval,svbksb,svdcmp
IMPLICIT NONE
REAL(DP), INTENT(IN) :: a,b
INTEGER(I4B), INTENT(IN) :: mm,kk
REAL(DP), DIMENSION(:), INTENT(OUT) :: cof
REAL(DP), INTENT(OUT) :: dev
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(DP), DIMENSION(:), INTENT(IN) :: x
REAL(DP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: NPFAC=8,MAXIT=5
REAL(DP), PARAMETER :: BIG=1.0e30_dp

Returns in cof(1:mm+kk+1) the coefficients of a rational function approximation to the function func in the interval (a,b). Input quantities mm and kk specify the order of the numerator and denominator, respectively. The maximum absolute deviation of the approximation (insofar as is known) is returned as dev. Note that double-precision versions of svdcmp and svbksb are called.

INTEGER(I4B) :: it,ncof,npt,npth
REAL(DP) :: devmax,e,theta
REAL(DP), DIMENSION((mm+kk+1)*NPFAC) :: bb,ee,fs,wt,xs
REAL(DP), DIMENSION(mm+kk+1) :: coff,v
REAL(DP), DIMENSION(mm+kk+1,mm+kk+1) :: v
REAL(DP), DIMENSION((mm+kk+1)*NPFAC,mm+kk+1) :: u,temp
ncof=mm+kk+1
npt=NPFAC*ncof  Number of points where function is evaluated, npt=npt/2  i.e., fineness of the mesh.
dev=BIG
theta=PI02_D/(npt-1)
xs(1:npt-1)=a+(b-a)*sin(theta*arth(0,1,npt-1))**2
Now fill arrays with mesh abscissas and function values. At each end, use formula that minimizes roundoff sensitivity in xs.
xs(npt:npt)=b-(b-a)*sin(theta*arth(npt-npth,-1,npt-npth+1))**2
fs=func(xs)
wt=1.0  In later iterations we will adjust these weights to combat the largest deviations.
ee=1.0
e=0.0
do it=1,MAXIT  Loop over iterations.
    bb=wt*(fs+sign(e,ee))
    Key idea here: Fit to fs(x) + e where the deviation is positive, to fs(x) - e where it is negative. Then e is supposed to become an approximation to the equal-ripple deviation.
    temp=geop(spread(1.0_dp,1,npt),xs,ncof)
Note that vector form of \( \text{geop} \) (returning matrix) is being used.
\[
u(:,1:mm+1)=\text{temp}(:,1:mm+1)\times\text{spread}(\text{wt},2,mm+1)
\]
Set up the "design matrix" for the least squares fit.
\[
u(:,mm+2:ncof)=-\text{temp}(:,2:ncof-mm)\times\text{spread}(\text{bb},2,ncof-mm-1)
\]
call svdcmp(u,w,v)

Singular Value Decomposition. In especially singular or difficult cases, one might here edit the singular values \( w(1:ncof) \), replacing small values by zero.
call svbksb(u,w,v,bb,coff)

\[
\text{ee}=\text{ratval}(\text{xs},\text{coff},mm,kk)-\text{fs}
\]
Tabulate the deviations and revise the weights.
\[
\text{wt}=\text{abs}(\text{ee})
\]
Use weighting to emphasize most deviant points.
\[
\text{devmax}=\text{maxval}(\text{wt})
\]
Update \( e \) to be the mean absolute deviation.
\[
\text{dev}=\text{devmax}
\]
Save only the best coefficient set found.
\[
\text{cof}=\text{coff}
\]
\[
\text{write}(\ast,10)\ \text{it},\text{devmax}
\]
end do
\[
10\ \text{format}(\'\text{ratlsq iteration}=\text{',i2,\', max error}=\text{'},1p,\text{e10.3})
\]
END SUBROUTINE ratlsq

\[
\text{temp} = \text{geop}(\text{spread}(1.0\ dp,1,npt),\text{xs},ncof)\]
The design matrix \( u_{ij} \) is defined for \( i = 1,\ldots,npts \) by
\[
u_{ij} = \begin{cases} w_i x_i^{j-1}, & j = 1,\ldots,m + 1 \\ -b_i x_i^{j-m-2}, & j = m + 2,\ldots,n \end{cases} \quad (B5.12)
\]
The first case in equation (B5.12) is computed in parallel by constructing the matrix \( \text{temp} \) equal to
\[
\begin{bmatrix}
1 & x_1 & x_1^2 & \cdots \\
1 & x_2 & x_2^2 & \cdots \\
1 & x_3 & x_3^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
and then multiplying by the matrix \( \text{spread}(\text{wt},2,mm+1) \), which is just
\[
\begin{bmatrix}
w_1 & w_1 & w_1 & \cdots \\
w_2 & w_2 & w_2 & \cdots \\
w_3 & w_3 & w_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
(Remember that multiplication using * means element-by-element multiplication, not matrix multiplication.) A similar construction is used for the second part of the design matrix.
A Fortran 90 intrinsic function such as $\sin(x)$ is both generic and elemental. Generic means that the argument $x$ can be any of multiple intrinsic data types and kind values (in the case of $\sin$, any real or complex kind). Elemental means that $x$ need not be a scalar, but can be an array of any rank and shape, in which case the calculation of $\sin$ is performed independently for each element.

Ideally, when we implement more complicated special functions in Fortran 90, as we do in this chapter, we would make them, too, both generic and elemental. Unfortunately, the language standard does not completely allow this. User-defined elemental functions are prohibited in Fortran 90, though they will be allowed in Fortran 95. And, there is no fully automatic way of providing for a single routine to allow arguments of multiple data types or kinds — nothing like C++’s “class templates,” for example.

However, don’t give up hope! Fortran 90 does provide a powerful mechanism for overloading, which can be used (perhaps not always with a maximum of convenience) to simulate both generic and elemental function features. In most cases, when we implement a special function with a scalar argument, $\text{gammln}(x)$ say, we will also implement a corresponding vector-valued function of vector argument that evaluates the special function for each component of the vector argument. We will then overload the scalar and vector version of the function onto the same function name. For example, within the nr module are the lines

```fortran
INTERFACE gammln
FUNCTION gammln_s(xx)
USE nrtype
REAL(SP), INTENT(IN) :: xx
REAL(SP) :: gammln_s
END FUNCTION gammln_s

FUNCTION gammln_v(xx)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: xx
REAL(SP), DIMENSION(size(xx)) :: gammln_v
END FUNCTION gammln_v
END INTERFACE
```

which can be included by a statement like “USE nr, ONLY: gammln,” and then allow you to write $\text{gammln}(x)$ without caring (or even thinking about) whether $x$ is a scalar or a vector. If you want arguments of even higher rank (matrices, and so forth), you can provide these yourself, based on our models, and overload them, too.

That takes care of “elemental”; what about “generic”? Here, too, overloading provides an acceptable, if not perfect, solution. Where double-precision versions of special functions are needed, you can in many cases easily construct them from our provided routines by changing the variable kinds (and any necessary convergence
parameters), and then additionally overload them onto the same generic function names. (In general, in the interest of brevity, we will not ourselves do this for the functions in this chapter.)

At first meeting, Fortran 90’s overloading capability may seem trivial, or merely cosmetic, to the Fortran 77 programmer; but one soon comes to rely on it as an important conceptual simplification. Programming at a “higher level of abstraction” is usually more productive than spending time “bogged down in the mud.” Furthermore, the use of overloading is generally fail-safe: If you invoke a generic name with arguments of shapes or types for which a specific routine has not been defined, the compiler tells you about it.

We won’t reprint the module nr’s interface blocks for all the routines in this chapter. When you see routines named something_s and something_v, below, you can safely assume that the generic name something is defined in the module nr and overloaded with the two specific routine names. A full alphabetical listing of all the interface blocks in nr is given in Appendix C2.

Given our heavy investment, in this chapter, in overloadable vector-valued special function routines, it is worth discussing whether this effort is simply a stopgap measure for Fortran 90, soon to be made obsolete by Fortran 95’s provision of user-definable ELEMENTAL procedures. The answer is “not necessarily,” and takes us into some speculation about the future of SIMD, versus MIMD, computing.

Elemental procedures, while applying the same executable code to each element, do not insist that it be feasible to perform all the parallel calculations in lockstep. That is, elemental procedures can have tests and branches (if-then-else constructions) that result in different elements being calculated by totally different pieces of code, in a fashion that can only be determined at run time. For true 100% MIMD (multiple instruction, multiple data) machines, this is not a problem: individual processors do the individual element calculations asynchronously.

However, virtually none of today’s (and likely tomorrow’s) largest-scale parallel supercomputers are 100% MIMD in this way. While modern parallel supercomputers increasingly have MIMD features, they continue to reward the use of SIMD (single instruction, multiple data) code with greater computational speed, often because of hardware pipelining or vector processing features within the individual processors. The use of Fortran 90 (or, for that matter Fortran 95) in a data-parallel or SIMD mode is thus by no means superfluous, or obviated by Fortran 95’s ELEMENTAL construction.

The problem we face is that parallel calculation of special function values often doesn’t fit well into the SIMD mold: Since the calculation of the value of a special function typically requires the convergence of an iterative process, as well as possible branches for different values of arguments, it cannot in general be done efficiently with “lockstep” SIMD programming.

Luckily, in particular cases, including most (but not all) of the functions in this chapter, one can in fact make reasonably good parallel implementations with the SIMD tools provided by the language. We will in fact see a number of different tricks for accomplishing this in the code that follows.

We are interested in demonstrating SIMD techniques, but we are not completely impractical. None of the data-parallel implementations given below are too inefficient on a scalar machine, and some may in fact be faster than Fortran 95’s ELEMENTAL
alternative, or than do-loops over calls to the scalar version of the function. On a
scalar machine, how can this be? We have already, above, hinted at the answer: (i)
most modern scalar processors can overlap instructions to some degree, and data-
parallel coding often provides compilers with the ability to accomplish this more
efficiently; and (ii) data-parallel code can sometimes give better cache utilization.

FUNCTION gammln_s(xx)
USE nrtype; USE nrutil, ONLY : arth,assert
IMPLICIT NONE
REAL(SP), INTENT(IN) :: xx
REAL(SP) :: gammln_s
Returns the value \( \ln[\Gamma(xx)] \) for \( xx > 0 \).
REAL(DP) :: tmp,x
Internal arithmetic will be done in double precision, a nicety that you can omit if five-figure
accuracy is good enough.
REAL(DP) :: stp = 2.5066282746310005_dp
REAL(DP), DIMENSION(6) :: coef = (/76.18009172947146_dp,&
-86.50532032941677_dp,24.01409824083091_dp,&
-1.231739572450155_dp,0.1208650973866179e-2_dp,&
-0.539523984953e-5_dp/)
call assert(xx > 0.0, 'gammln_s arg')
x=xx
tmp=x+5.5_dp
tmp=(x+0.5_dp)*log(tmp)-tmp
gammln_s=tmp+log(stp*(1.000000000190015_dp+&
sum(coef(:)/arth(x+1.0_dp,1.0_dp,size(coef))))/x)
END FUNCTION gammln_s

FUNCTION gammln_v(xx)
USE nrtype; USE nrutil, ONLY: assert
IMPLICIT NONE
INTEGER(I4B) :: i
REAL(SP), DIMENSION(:), INTENT(IN) :: xx
REAL(SP), DIMENSION(size(xx)) :: gammln_v
REAL(DP), DIMENSION(size(xx)) :: ser,tmp,x,y
REAL(DP) :: stp = 2.5066282746310005_dp
REAL(DP), DIMENSION(6) :: coef = (/76.18009172947146_dp,&
-86.50532032941677_dp,24.01409824083091_dp,&
-1.231739572450155_dp,0.1208650973866179e-2_dp,&
-0.539523984953e-5_dp/)
if (size(xx) == 0) RETURN
call assert(all(xx > 0.0), 'gammln_v arg')
x=xx
tmp=x+5.5_dp
tmp=(x+0.5_dp)*log(tmp)-tmp
gammln_v=tmp+log(stp*ser/x)
END FUNCTION gammln_v
call assert(xx > 0.0, 'gammln_s arg') We use the nrutil routine assert for functions that have restrictions on the allowed range of arguments. One could instead have used an if statement with a call to nrerror; but we think that the uniformity of using assert, and the fact that its logical arguments read the “desired” way, not the “erroneous” way, make for a clearer programming style. In the vector version, the assert line is:
call assert(all(xx > 0.0), 'gammln_v arg')

Notice that the scalar and vector versions achieve parallelism in quite different ways, something that we will see many times in this chapter. In the scalar case, parallelism (at least small-scale) is achieved through constructions like

\[
\text{sum(coef(:)/arth(x+1.0_dp,1.0_dp,size(coef)))}
\]

Here vector utilities construct the series \( x + 1, x + 2, \ldots \) and then sum a series with these terms in the denominators and a vector of coefficients in the numerators. (This code may seem terse to Fortran 90 novices, but once you get used to it, it is quite clear to read.)

In the vector version, by contrast, parallelism is achieved across the components of the vector argument, and the above series is evaluated sequentially as a do-loop. Obviously the assumption is that the length of the vector argument is much longer than the very modest number (here, 6) of terms in the sum.

⋆⋆⋆

FUNCTION factrl_s(n)
USE nrtype; USE nrutil, ONLY : arth,assert,cumprod
USE nr, ONLY : gammln
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
REAL(SP) :: factrl_s

Returns the value \( n! \) as a floating-point number.

INTEGER(I4B), SAVE :: ntop=0
INTEGER(I4B), PARAMETER :: NMAX=32
REAL(SP), DIMENSION(NMAX), SAVE :: a Table of stored values.
call assert(n >= 0, 'factrl_s arg')
if (n < ntop) then
    factrl_s=a(n+1)
else if (n < NMAX) then
    ntop=NMAX
    a(1)=1.0
    a(2:NMAX)=cumprod(arth(1.0_sp,1.0_sp,NMAX-1))
    factrl_s=a(n+1)
else
    factrl_s=exp(gammln(n+1.0_sp))
end if
END FUNCTION factrl_s

cumprod(arth(1.0_sp,1.0_sp,NMAX-1)) By now you should recognize this as an idiom for generating a vector of consecutive factorials. The routines cumprod and arth, both in nrutil, are both capable of being parallelized, e.g., by recursion, so this idiom is potentially faster than an in-line do-loop.
FUNCTION factrl_v(n)
USE nrtype; USE nrutil, ONLY : arth, assert, cumprod
USE nr, ONLY : gammln
IMPLICIT NONE
INTEGER(4), DIMENSION(:,), INTENT(IN) :: n
REAL(SP), DIMENSION(size(n)) :: factrl_v
LOGICAL(LGT), DIMENSION(size(n)) :: mask
INTEGER(4), SAVE :: ntop=0
INTEGER(4), PARAMETER :: NMAX=32
REAL(SP), DIMENSION(NMAX), SAVE :: a

call assert(all(n >= 0), 'factrl_varg')
if (ntop == 0) then
  ntop=NMAX
  a(1)=1.0
  a(2:NMAX)=cumprod(arth(1.0_sp, 1.0_sp, NMAX-1))
end if
mask = (n >= NMAX)

where (.not. mask) factrl_v=a(n+1)

END FUNCTION factrl_v

unpack(exp(gammln(pack(n,mask)+1.0_sp)),mask,0.0_sp) Here we meet
the first of several solutions to a common problem: How shall we get
answers, from an external vector-valued function, for just a subset of
vector arguments, those defined by a mask? Here we use what we call the "pack-
unpack" solution: Pack up all the arguments using the mask, send them to the
function, and unpack the answers that come back. This packing and unpacking is
not without cost (highly dependent on machine architecture, to be sure), but we hope
to "earn it back" in the parallelism of the external function.

where (.not. mask) factrl_v=a(n+1) In some cases we might take care of the
.not.mask case directly within the unpack construction, using its third ("FIELD=")
argument to provide the not-unpacked values. However, there is no guarantee that
the compiler won’t evaluate all components of the "FIELD=" array, if it finds it
efficient to do so. Here, since the index of a(n+1) would be out of range, we can’t
do it this way. Thus the separate where statement.

FUNCTION bico_s(n,k)
USE nrtype
USE nr, ONLY : factln
IMPLICIT NONE
INTEGER(4), INTENT(IN) :: n,k
REAL(SP) :: bico_s

Returns the binomial coefficient \( \binom{n}{k} \) as a floating-point number.
bico_s=nint(exp(factln(n)-factln(k)-factln(n-k)))
The nearest-integer function cleans up roundoff error for smaller values of n and k.
END FUNCTION bico_s
FUNCTION bico_v(n,k)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : factln
IMPLICIT NONE
INTEGER(14B), DIMENSION(:,), INTENT(IN) :: n,k
REAL(SP), DIMENSION(size(n)) :: bico_v
INTEGER(14B) :: ndum
ndum=assert_eq(size(n),size(k),'bico_v')
bico_v=nint(exp(factln(n)-factln(k)-factln(n-k)))
END FUNCTION bico_v

FUNCTION factln_s(n)
USE nrtype; USE nrutil, ONLY : arth,assert
USE nr, ONLY : gammln
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
REAL(SP) :: factln_s

Returns \( \ln(n!) \).

INTEGER(I4B), PARAMETER :: TMAX=100
REAL(SP), DIMENSION(TMAX), SAVE :: a
LOGICAL(LGT), SAVE :: init=.true.
if (init) then
  Initialize the table.
  a(1:TMAX)=gammln(arth(1.0_sp,1.0_sp,TMAX))
  init=.false.
end if
if (n < TMAX) then
  In range of the table.
  factln_s=a(n+1)
else
  Out of range of the table.
  factln_s=gammln(n+1.0_sp)
end if
END FUNCTION factln_s

FUNCTION factln_v(n)
USE nrtype; USE nrutil, ONLY : arth,assert
USE nr, ONLY : gammln
IMPLICIT NONE
INTEGER(14B), DIMENSION(:,), INTENT(IN) :: n
REAL(SP), DIMENSION(size(n)) :: factln_v
LOGICAL(LGT), DIMENSION(size(n)) :: mask
INTEGER(14B), DIMENSION(TMAX), PARAMETER :: TMAX=100
REAL(SP), DIMENSION(TMAX), SAVE :: a
LOGICAL(LGT), SAVE :: init=.true.
if (init) then
  a(1:TMAX)=gammln(arth(1.0_sp,1.0_sp,TMAX))
  init=.false.
end if
call assert(all(n >= 0), 'factln_v arg')

mask = (n >= TMAX)
factln_v=unpack(gammln(pack(n,mask)+1.0_sp),mask,0.0_sp)
where (.not. mask) factln_v=a(n+1)
END FUNCTION factln_v
Another example of the programming convenience of combining a function returning a vector (here, \texttt{arth}) with a special function whose generic name (here, \texttt{gammln}) has an overloaded vector version.

\begin{verbatim}
FUNCTION beta_s(z,w)
USE nrtype
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), INTENT(IN) :: z,w
REAL(SP) :: beta_s
Returns the value of the beta function $B(z,w)$.
beta_s=exp(gammln(z)+gammln(w)-gammln(z+w))
END FUNCTION beta_s

FUNCTION beta_v(z,w)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: z,w
REAL(SP), DIMENSION(size(z)) :: beta_v
INTEGER(I4B) :: ndum
ndum=assert_eq(size(z),size(w),'beta_v')
beta_v=exp(gammln(z)+gammln(w)-gammln(z+w))
END FUNCTION beta_v

FUNCTION gammp_s(a,x)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : gcf,gser
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,x
REAL(SP) :: gammp_s
Returns the incomplete gamma function \( P(a,x) \).
call assert( x >= 0.0, a > 0.0, 'gammp_s args')
if (x<a+1.0_sp) then
  Use the series representation.
  gammp_s=gser(a,x)
else
  Use the continued fraction representation and take its complement.
  gammp_s=1.0_sp-gcf(a,x)
end if
END FUNCTION gammp_s

FUNCTION gammp_v(a,x)
USE nrtype; USE nrutil, ONLY : assert,assert_eq
USE nr, ONLY : gcf,gser
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a,x
REAL(SP), DIMENSION(size(x)) :: gammp_v
LOGICAL(LGT), DIMENSION(size(x)) :: mask
INTEGER(I4B) :: ndum
ndum=assert_eq(size(a),size(x),'gammp_v')
call assert( all(x >= 0.0), all(a > 0.0), 'gammp_v args')
mask = (x<a+1.0_sp)
gammp_v=merge(gser(a,merge(x,0.0_sp,mask)), &
  1.0_sp-gcf(a,merge(x,0.0_sp,.not. mask)),mask)
END FUNCTION gammp_v
\end{verbatim}
call assert( x \geq 0.0, \ a > 0.0, 'gammp_s args')  The generic routine
assert in nrutil is overloaded with variants for more than one logical
assertion, so you can make more than one assertion about argument ranges.

\[
\text{gammq_v} = \text{merge}(\text{gser}(\text{a}, \text{merge}(\text{x}, 0.0, \text{mask})), & \\
1.0 - \text{gcfr}(\text{a}, \text{merge}(\text{x}, 0.0, \text{not. mask})), \text{mask})
\]
Here we meet the second solution to the problem of getting masked values from an
external vector function. (For the first solution, see note to factrl.
above.) We call this one “merge with dummy values”: Inappropriate values of the
argument \text{x} (as determined by \text{mask}) are set to zero before \text{gser}, and later \text{gcfr}, are
called, and the supernumerary answers returned are discarded by a final \text{merge}. The
assumption here is that the dummy value sent to the function (here, zero) is a special
value that computes extremely fast, so that the overhead of computing and returning
the supernumerary function values is outweighed by the parallelism achieved on the
nontrivial components of \text{x}. Look at \text{gser_v} and \text{gcfr_v} below to judge whether
this assumption is realistic in this case.

FUNCTION gammq_s(a,x)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : gcf,gser
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,x
REAL(SP) :: gammq_s
Returns the incomplete gamma function \( Q(a,x) \equiv 1 - P(a,x) \).
call assert( x \geq 0.0, a > 0.0, 'gammq_s args')
if (x<a+1.0) then
Use the series representation
\[ \text{gammq_s} = 1.0 - \text{gser}(a,x) \]
and take its complement.
else
Use the continued fraction representation.
\[ \text{gammq_s} = \text{gcfr}(a,x) \]
end if
END FUNCTION gammq_s

FUNCTION gammq_v(a,x)
USE nrtype; USE nrutil, ONLY : assert,assert_eq
USE nr, ONLY : gcf,gser
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a,x
REAL(SP), DIMENSION(size(a)) :: gammq_v
LOGICAL(LGT), DIMENSION(size(x)) :: mask
INTEGER(I4B) :: ndum
ndum=assert_eq(size(a),size(x),'gammq_v')
call assert( all(x \geq 0.0), all(a > 0.0), 'gammq_v args')
mask = (x<a+1.0)
\[ \text{gammq_v} = \text{merge}(1.0 - \text{gser}(a, \text{merge}(x, 0.0, \text{mask})), & \\
\text{gcfr}(a, \text{merge}(x, 0.0, \text{not. mask})), \text{mask}) \]
END FUNCTION gammq_v

FUNCTION gser_s(a,x,gln)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,x
REAL(SP), OPTIONAL, INTENT(OUT) :: gln
REAL(SP) :: gser_s
INTEGER(I4B), PARAMETER :: ITMAX=100
REAL(SP), PARAMETER :: EPS=epsilon(x)

Returns the incomplete gamma function \( P(a,x) \) evaluated by its series representation as \( \text{gamser} \). Also optionally returns \( \ln \Gamma(a) \) as \( \text{gln} \).

```fortran
INTEGER(14B) :: n
REAL(SP) :: ap, del, summ
if (x == 0.0) then
    gser_s=0.0
    RETURN
end if
ap=a
summ=1.0_sp/a
del=summ
do n=1,ITMAX
    ap=ap+1.0_sp
    del=del*x/ap
    summ=summ+del
    if (abs(del) < abs(summ)*EPS) exit
end do
if (n > ITMAX) call nrerror('a too large, ITMAX too small in gser_s')
if (present(gln)) then
    gln=gammln(a)
    gser_s=summ*exp(-x+a*log(x)-gln)
else
    gser_s=summ*exp(-x+a*log(x)-gammln(a))
end if
END FUNCTION gser_s

FUNCTION gser_v(a,x,gln)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a,x
REAL(SP), DIMENSION(:), OPTIONAL, INTENT(OUT) :: gln
REAL(SP), DIMENSION(size(a)) :: gser_v
INTEGER(I4B), PARAMETER :: ITMAX=100
REAL(SP), PARAMETER :: EPS=epsilon(x)
INTEGER(I4B) :: n
REAL(SP), DIMENSION(size(a)) :: ap, del, summ
LOGICAL(LGT), DIMENSION(size(a)) :: converged, zero
n=assert_eq(size(a),size(x),'gser_v')
zero=(x == 0.0)
where (zero) gser_v=0.0
ap=a
summ=1.0_sp/a
del=summ
converged=zero
do n=1,ITMAX
    where (.not. converged)
        ap=ap+1.0_sp
        del=del*x/ap
        summ=summ+del
        converged = (abs(del) < abs(summ)*EPS)
    end where
    if (all(converged)) exit
end do
if (n > ITMAX) call nrerror('a too large, ITMAX too small in gser_v')
if (present(gln)) then
    if (size(gln) < size(a)) call nrerror('gser: Not enough space for gln')
    gln=gammln(a)
    where (.not. zero) gser_v=summ*exp(-x+a*log(x)-gln)
else
    where (.not. zero) gser_v=summ*exp(-x+a*log(x)-gammln(a))
END FUNCTION gser_v
```

FUNCTION gser_v(a,x,gln)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a,x
REAL(SP), DIMENSION(:), OPTIONAL, INTENT(OUT) :: gln
REAL(SP), DIMENSION(size(a)) :: gser_v
INTEGER(I4B), PARAMETER :: ITMAX=100
REAL(SP), PARAMETER :: EPS=epsilon(x)
INTEGER(I4B) :: n
REAL(SP), DIMENSION(size(a)) :: ap, del, summ
LOGICAL(LGT), DIMENSION(size(a)) :: converged, zero
n=assert_eq(size(a),size(x),'gser_v')
zero=(x == 0.0)
where (zero) gser_v=0.0
ap=a
summ=1.0_sp/a
del=summ
converged=zero
do n=1,ITMAX
    where (.not. converged)
        ap=ap+1.0_sp
        del=del*x/ap
        summ=summ+del
        converged = (abs(del) < abs(summ)*EPS)
    end where
    if (all(converged)) exit
end do
if (n > ITMAX) call nrerror('a too large, ITMAX too small in gser_v')
if (present(gln)) then
    if (size(gln) < size(a)) call nrerror('gser: Not enough space for gln')
    gln=gammln(a)
    where (.not. zero) gser_v=summ*exp(-x+a*log(x)-gln)
else
    where (.not. zero) gser_v=summ*exp(-x+a*log(x)-gammln(a))
END FUNCTION gser_v
REAL(SP), OPTIONAL, INTENT(OUT) :: gln  Normally, an OPTIONAL argument will be INTENT(IN) and be used to provide a less-often-used extra input argument to a function. Here, the OPTIONAL argument is INTENT(OUT), used to provide a useful value that is a byproduct of the main calculation.

Also note that although $x \geq 0$ is required, we omit our usual call assert check for this, because gser is supposed to be called only by gammp or gammq — and these routines supply the argument checking themselves.

do n=1,ITMAX...end do...if (n > ITMAX)... This is typical code in Fortran 90 for a loop with a maximum number of iterations, relying on Fortran 90’s guarantee that the index of the do-loop will be available after normal completion of the loop with a predictable value, greater by one than the upper limit of the loop. If the exit statement within the loop is ever taken, the if statement is guaranteed to fail; if the loop goes all the way through ITMAX cycles, the if statement is guaranteed to succeed.

zero=(x == 0.0)...where (zero) gser_v=0.0...converged=zero  This is the code that provides for very low overhead calculation of zero arguments, as is assumed by the merge-with-dummy-values strategy in gammp and gammq. Zero arguments are “pre-converged” and are never the holdouts in the convergence test.

FUNCTION gcf_s(a,x,gln)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,x
REAL(SP), OPTIONAL, INTENT(OUT) :: gln
REAL(SP) :: gcf_s
INTEGER(I4B), PARAMETER :: ITMAX=100
REAL(SP), PARAMETER :: EPS=epsilon(x),FPMIN=tiny(x)/EPS

Returns the incomplete gamma function $Q(a,x)$ evaluated by its continued fraction representation as gammcf. Also optionally returns $\ln \Gamma(a)$ as gln.
Parameters: ITMAX is the maximum allowed number of iterations; EPS is the relative accuracy; FPMIN is a number near the smallest representable floating-point number.

INTEGER(I4B) :: i
REAL(SP) :: an,b,c,d,del,h
if (x == 0.0) then
    gcf_s=1.0
    RETURN
end if
b=x+1.0_sp-a  Set up for evaluating continued fraction by modified Lentz’s method (§5.2) with $b_0 = 0.$
c=1.0_sp/FPMIN
d=1.0_sp/b
h=d

do i=1,ITMAX  Iterate to convergence.
an=-i*(i-a)
b=b+b*2.0_sp
d=an+d+b
if (abs(d) < FPMIN) d=FPMIN
c=b+an/c
if (abs(c) < FPMIN) c=FPMIN
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\[ d = 1.0_{sp}/d \]
\[ del = d + c \]
\[ h = h \cdot del \]
end do

if (i > ITMAX) call nrerror('a too large, ITMAX too small in gcf_s')
if (present(gln)) then
  gln = gammln(a)
  gcf_s = exp(-x + a \cdot \log(x) - gln) \cdot h
else
  gcf_s = exp(-x + a \cdot \log(x) - gammln(a)) \cdot h
end if

END FUNCTION gcf_s

FUNCTION gcf_v(a, x, gln)
USE nrtype; USE nrutil, ONLY : assert_eq, nrerror
USE nr, ONLY : gammln
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a, x
REAL(SP), DIMENSION(:), OPTIONAL, INTENT(OUT) :: gln
REAL(SP), DIMENSION(size(a)) :: gcf_v
INTEGER(I4B), PARAMETER :: ITMAX = 100
REAL(SP), PARAMETER :: EPS = epsilon(x), FPMIN = tiny(x)/EPS
INTEGER(I4B) :: i
REAL(SP), DIMENSION(size(a)) :: an, b, c, d, del, h
LOGICAL(LGT), DIMENSION(size(a)) :: converged, zero

i = assert_eq(size(a), size(x), 'gcf_v')
zero = (x == 0.0)
where (zero)
  gcf_v = 1.0
elsewhere
  b = x + 1.0_{sp} - a
  c = 1.0_{sp}/FPMIN
  d = 1.0_{sp}/b
  h = d
end where
converged = zero
do i = 1, ITMAX
  an = -i*(i - a)
  b = b + 2.0_{sp}
  d = an + d + b
  d = merge(FPMIN, d, abs(d) < FPMIN )
  c = b * an + c
  c = merge(FPMIN, c, abs(c) < FPMIN )
  d = 1.0_{sp}/d
  del = c
  h = h \cdot del
  converged = (abs(del - 1.0_{sp}) <= EPS)
end where
if (all(converged)) exit
end do
if (i > ITMAX) call nrerror('a too large, ITMAX too small in gcf_v')
if (present(gln)) then
  if (size(gln) < size(a)) call &
  nrerror('gser: Not enough space for gln')
  gln = gammln(a)
else
  where (.not. zero) gcf_v = exp(-x + a \cdot \log(x) - gln) \cdot h
end if

END FUNCTION gcf_v
zero=(x == 0.0)...where (zero) gcf_v=1.0...converged=zero  See note on gser. Here, too, we pre-converge the special value of zero.

⋆⋆⋆

FUNCTION erf_s(x)
USE nrtype
USE nr, ONLY : gammp
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: erf_s
Returns the error function erf(x).
erf_s=gammp(0.5_sp,x**2)
if (x < 0.0) erf_s=-erf_s
END FUNCTION erf_s

FUNCTION erf_v(x)
USE nrtype
USE nr, ONLY : gammp
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: erf_v
erf_v=gammp(spread(0.5_sp,1,size(x)),x**2)
where (x < 0.0) erf_v=-erf_v
END FUNCTION erf_v

erf_v=gammp(spread(0.5_sp,1,size(x)),x**2)  Yes, we do have an over-loaded vector version of gammp, but it is vectorized on both its arguments.
Thus, in a case where we want to vectorize on only one argument, we need a spread construction. In many contexts, Fortran 90 automatically makes scalars conformable with arrays (i.e., it automatically spreads them to the shape of the array); but the language does not do so when trying to match a generic function or subroutine call to a specific overloaded name. Perhaps this is wise; it is safer to prevent “accidental” invocations of vector-specific functions. Or, perhaps it is an area where the language could be improved.

FUNCTION erfc_s(x)
USE nrtype
USE nr, ONLY : gammp,gammq
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: erfc_s
Returns the complementary error function erfc(x).
erfc_s=merge(1.0_sp+gammp(0.5_sp,x**2),gammq(0.5_sp,x**2), x < 0.0)
END FUNCTION erfc_s

erfc_s=merge(1.0_sp+gammp(0.5_sp,x**2),gammq(0.5_sp,x**2), x < 0.0)  An example of our use of merge as an idiom for a conditional expression.
Once you get used to these, you’ll find them just as clear as the multiline if...then...else alternative.
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FUNCTION erfc_v(x)
USE nrtype
USE nr, ONLY : gammp,gammq
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: erfc_v
LOGICAL(LGT), DIMENSION(size(x)) :: mask
mask = (x < 0.0)
erfc_v=merge(1.0_sp+gammp(spread(0.5_sp,1,size(x)), &
merge(x,0.0_sp,mask)**2),gammq(spread(0.5_sp,1,size(x)), &
merge(x,0.0_sp,.not. mask)**2),mask)
END FUNCTION erfc_v

erfc_v=merge(1.0_sp+...) Another example of the “merge with dummy values” idiom described on p. 1090. Here positive values of x in the call to gammp, and negative values in the call to gammq, are first set to the dummy value zero. The value zero is a special argument that computes very fast. The unwanted dummy function values are then discarded by the final outer merge.

* * *

FUNCTION erfcc_s(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: erfcc_s
Returns the complementary error function erfc(x) with fractional error everywhere less than \( 1.2 \times 10^{-7} \).
REAL(SP) :: t,z
REAL(SP), DIMENSION(10) :: coef = (/ -1.26551223_sp, 1.00002368_sp, &
0.37409196_sp, 0.09678418_sp, -0.18628806_sp, 0.27886807_sp, &
-1.13520398_sp, 1.48851587_sp, -0.82215223_sp, 0.17087277_sp/)
z=abs(x)
t=1.0_sp/(1.0_sp+0.5_sp*z)
erfcc_s=t*exp(-z*z+poly(t,coef))
if (x < 0.0) erfcc_s=2.0_sp-erfcc_s
END FUNCTION erfcc_s

FUNCTION erfcc_v(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: erfcc_v,t,z
REAL(SP), DIMENSION(10) :: coef = (/ -1.26551223_sp, 1.00002368_sp, &
0.37409196_sp, 0.09678418_sp, -0.18628806_sp, 0.27886807_sp, &
-1.13520398_sp, 1.48851587_sp, -0.82215223_sp, 0.17087277_sp/)
z=abs(x)
t=1.0_sp/(1.0_sp+0.5_sp*z)
erfcc_v=t*exp(-z*z+poly(t,coef))
where (x < 0.0) erfcc_v=2.0_sp-erfcc_v
END FUNCTION erfcc_v
The vector code is identical to the scalar, because the *nrutil* routine *poly* has overloaded cases for the evaluation of a polynomial at a single value of the independent variable, and at multiple values. One *could* also overload a version with a matrix of coefficients whose columns could be used for the simultaneous evaluation of different polynomials at different values of independent variable. The point is that as long as there are differences in the shapes of at least one argument, the intended version of *poly* can be discerned by the compiler.

```
*   *   *

FUNCTION expint(n,x)
USE nrtype; USE nrutil, ONLY : arth,assert,nrerror
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP) :: expint
INTEGER(I4B), PARAMETER :: MAXIT=100
REAL(SP), PARAMETER :: EPS=epsilon(x),BIG=huge(x)*EPS
Evaluates the exponential integral $E_n(x)$.
Parameters: MAXIT is the maximum allowed number of iterations; EPS is the desired relative error, not smaller than the machine precision; BIG is a number near the largest representable floating-point number; Euler (in *nrtype*) is Euler’s constant $\gamma$.
INTEGER(I4B) :: i,nm1
REAL(SP) :: a,b,c,d,del,fact,h
call assert(n >= 0, x >= 0.0, (x > 0.0 .or. n > 1), &
'expint args')
if (n == 0) then
  Special case.
  expint=exp(-x)/x
  RETURN
end if
nm1=n-1
if (x == 0.0) then
  Another special case.
  expint=1.0_sp/nm1
else if (x > 1.0) then
  Lentz’s algorithm ($\S$5.2).
  b=x+n
  c=BIG
  d=1.0_sp/b
  h=d
do i=1,MAXIT
    a=-i*(nm1+i)
    b=b+2.0_sp
    d=1.0_sp/(a*d+b)
    c=b+a/c
    del=c*d
    h=h*del
    if (abs(del-1.0_sp) <= EPS) exit
  end do
  if (i > MAXIT) call nrerror('expint: continued fraction failed')
  expint=h*exp(-x)
else
  Evaluate series.
  if (nm1 /= 0) then
    Set first term.
    expint=1.0_sp/nm1
  else
    expint=-log(x)-EULER
  end if
  fact=1.0
  do i=1,MAXIT
    fact=-fact*x/i
    if (i /= nm1) then
      del=-fact/(i-nm1)
    end if
    b=a/c
    c=del*c
    del=del-c*
```
else \( \psi(n) \) appears here.
   \[ \text{del} = \text{fact} \times (-\log(x) - \text{EULER} + \text{sum}(1.0_{\text{sp}} / \text{arth}(1,1,nm1))) \]
end if

expint = expint + del
if (abs(del) < abs(expint) \* \text{EPS}) exit
end do
if (i > \text{MAXIT}) call \text{nrerror}('expint: series failed')
end if

END FUNCTION expint

expint does not readily parallelize, and we thus don’t provide a vector version. For syntactic convenience you could make a vector version with a do-loop over calls to this scalar version; or, in Fortran 95, you can of course make the function \text{ELEMENTAL}.

FUNCTION ei(x)
USE nrtype; USE nrutil, ONLY : assert, nrerror
IMPLIED NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: ei
INTEGER(I4B), PARAMETER :: MAXIT = 100
REAL(SP), PARAMETER :: EPS = \text{epsilon}(x), \text{FPMIN = tiny}(x) / \text{EPS}

Computes the exponential integral \( \text{Ei}(x) \) for \( x > 0 \).

Parameters: \text{MAXIT} is the maximum number of iterations allowed; \text{EPS} is the relative error, or absolute error near the zero of \( \text{Ei} \) at \( x = 0.3725 \); \text{FPMIN} is a number near the smallest representable floating-point number; \text{EULER} (in \text{nrtype}) is Euler’s constant \( \gamma \).

INTEGER(I4B) :: k
REAL(SP) :: fact, prev, sm, term

if (x < \text{FPMIN}) then
   Special case: avoid failure of convergence test because of underflow.
   ei = \log(x) + \text{EULER}
else if (x <= -\text{log}(\text{EPS})) then
   Use power series.
   sm = 0.0
   fact = 1.0
   do k = 1, \text{MAXIT}
      fact = fact \times x / k
      term = fact / k
      sm = sm + term
      if (term < \text{EPS} \times sm) exit
   end do
   if (k > \text{MAXIT}) call \text{nrerror}('series failed in \text{ei}')
   ei = sm + \text{log}(x) + \text{EULER}
else
   sm = 0.0
   term = 1.0
   do k = 1, \text{MAXIT}
      prev = term
      term = term \times x / k
      if (term < \text{EPS}) exit
      if (term < prev) then
         sm = sm + term
         Since final sum is greater than one, term itself approximates the relative error.
      else
         sm = sm - prev
         Still converging: add new term.
      end if
      exit
   end do
   if (k > \text{MAXIT}) call \text{nrerror}('asymptotic failed in \text{ei}')
   ei = \text{exp}(x) \times (1.0_{\text{sp}} + sm) / x
end if
END FUNCTION ei
\( e^i \) does not readily parallelize, and we thus don’t provide a vector version. For syntactic convenience you could make a vector version with a do-loop over calls to this scalar version; or, in Fortran 95, you can of course make the function ELEMENTAL.

* * *

FUNCTION betai_s(a,b,x)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : betacf,gammln
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b,x
REAL(SP) :: betai_s
Returns the incomplete beta function \( I_x(a,b) \).
REAL(SP) :: bt
call assert(x >= 0.0, x <= 1.0, 'betai_s arg')
if (x == 0.0 .or. x == 1.0) then
  bt=0.0
else
  bt=exp(gammln(a+b)-gammln(a)-gammln(b)&
       +a*log(x)+b*log(1.0_sp-x))
end if
if (x < (a+1.0_sp)/(a+b+2.0_sp)) then
  betai_s=bt*betacf(a,b,x)/a
else
  betai_s=1.0_sp-bt*betacf(b,a,1.0_sp-x)/b
end if
END FUNCTION betai_s

FUNCTION betai_v(a,b,x)
USE nrtype; USE nrutil, ONLY : assert,assert_eq
USE nr, ONLY : betacf,gammln
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a,b,x
REAL(SP), DIMENSION(size(a)) :: betai_v
REAL(SP), DIMENSION(size(a)) :: bt
LOGICAL(LGT), DIMENSION(size(a)) :: mask
INTEGER(I4B) :: ndum
ndum=assert_eq(size(a),size(b),size(x),'betai_v')
call assert(all(x >= 0.0), all(x <= 1.0), 'betai_v arg')
where (x == 0.0 .or. x == 1.0)
  bt=0.0
elsewhere
  bt=exp(gammln(a+b)-gammln(a)-gammln(b)&
       +a*log(x)+b*log(1.0_sp-x))
end where
mask=(x < (a+1.0_sp)/(a+b+2.0_sp))
betai_v=bt*betacf(merge(a,b,mask),merge(b,a,mask),&
      merge(x,1.0_sp-x,mask))/merge(a,b,mask)
where (.not. mask) betai_v=1.0_sp-betai_v
END FUNCTION betai_v
Compare the scalar

\[
\text{if } (x < \frac{(a+1.0_{-\text{sp}})/(a+b+2.0_{-\text{sp}})}{a} ) \text{ then}
\]
\[
\beta_{\text{i}_s} = \beta t \times \text{betacf}(a,b,x)/a
\]
\[
\text{else}
\]
\[
\beta_{\text{i}_s} = 1.0_{-\text{sp}} - \beta t \times \text{betacf}(b,a,1.0_{-\text{sp}}-x)/b
\]
\[
\text{end if}
\]

with the vector

\[
\text{mask} = (x < \frac{(a+1.0_{-\text{sp}})/(a+b+2.0_{-\text{sp}})}{a})
\]
\[
\beta_{\text{i}_v} = \beta t \times \text{betacf}(\text{merge}(a,b,\text{mask}),\text{merge}(b,a,\text{mask}),& \text{merge}(x,1.0_{-\text{sp}}-x,\text{mask}))/\text{merge}(a,b,\text{mask})
\]
\[
\text{where } (.not. \text{ mask}) \beta_{\text{i}_v} = 1.0_{-\text{sp}} - \beta_{\text{i}_v}
\]

Here \text{merge} is used (several times) to evaluate all the required components in a single call to the vectorized \text{betacf}, notwithstanding that some components require one pattern of arguments, some a different pattern.

\begin{verbatim}
FUNCTION betacf_s(a,b,x)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b,x
REAL(SP) :: betacf_s
INTEGER(I4B), PARAMETER :: MAXIT=100
REAL(SP), PARAMETER :: EPS=epsilon(x), FPMIN=tiny(x)/EPS

    Used by betai: Evaluates continued fraction for incomplete beta function by modified
    Lentz's method ([5.2]).

    REAL(SP) :: aa,c,d,del,h,qab,qam,qap
    INTEGER(I4B) :: m,m2
    qab=a+b
    qap=a+1.0_sp
    qam=a-1.0_sp
    c=1.0
    d=1.0_sp-qab*x/qap
    if (abs(d) < FPMIN) d=FPMIN
    d=1.0_sp/d
    do m=1,MAXIT
        m2=2*m
        aa=m*(b-m)*x/((qam+m2)*(a+m2))
        d=1.0_sp+aa*d
        if (abs(d) < FPMIN) d=FPMIN
        c=1.0_sp+aa/c
        if (abs(c) < FPMIN) c=FPMIN
        d=1.0_sp/d
        h=h*d*c
        del=d*c
        if (abs(del-1.0_sp) <= EPS) exit
        h=del
    end do
    if (m > MAXIT) call nrerror('a or b too big, or MAXIT too small in betacf_s')

    betacf_s=h
END FUNCTION betacf_s
\end{verbatim}
FUNCTION betacf_v(a,b,x)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: a,b,x
REAL(SP), DIMENSION(size(x)) :: betacf_v
INTEGER(I4B), PARAMETER :: MAXIT=100
REAL(SP), PARAMETER :: EPS=epsilon(x), FPMIN=tiny(x)/EPS
REAL(SP), DIMENSION(size(x)) :: aa,c,d,del,h,qab,qam,qap
LOGICAL(LGT), DIMENSION(size(x)) :: converged
INTEGER(I4B) :: m
INTEGER(I4B), DIMENSION(size(x)) :: m2
m=assert_eq(size(a),size(b),size(x),'betacf_v')
qab=a+b
qap=a+1.0_sp
qam=a-1.0_sp
h=1.0_sp-qap*x/qap
where (abs(d) < FPMIN) d=FPMIN
where (.not. converged)
  m2=2*m
  aa=-(b-m)*x/((qam+m2)*(a+m2))
  d=merge(FPMIN,d, abs(d)<FPMIN )
  c=1.0_sp+aa/c
  c=merge(FPMIN,c, abs(c)<FPMIN )
  d=1.0_sp/d
  h=h*d*c
  del=-(a+m2)*(qab+m2)*x/((a+m2)*(qap+m2))
  d=merge(FPMIN,d, abs(d)<FPMIN )
  c=1.0_sp+aa/c
  c=merge(FPMIN,c, abs(c)<FPMIN )
  d=1.0_sp/d
  del=d*del
  converged = (abs(del-1.0_sp) <= EPS)
end where
if (m > MAXIT)&
call nrerror('a or b too big, or MAXIT too small in betacf_v')
betacf_v=h
END FUNCTION betacf_v

The scalar version does this with an
if. Why does it become a merge here in the vector version, rather than a
where? Because we are already inside a “where (.not. converged)” block,
and Fortran 90 doesn’t allow nested where’s! (Fortran 95 will allow nested where’s.)
FUNCTION bessj0_s(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessj0_s
Returns the Bessel function $J_0(x)$ for any real $x$.
REAL(SP) :: ax,xx,z
REAL(DP) :: y
We'll accumulate polynomials in double precision.
REAL(DP), DIMENSION(5) :: p = (/1.0_dp,-0.1098628627e-2_dp,&
0.2734510407e-4_dp,-0.2073370639e-5_dp,0.2093887211e-6_dp/)
REAL(DP), DIMENSION(5) :: q = (/0.1562499995e-1_dp,&
0.1430488765e-3_dp,-0.6911147651e-5_dp,0.7621098161e-6_dp,&
-0.934945152e-7_dp/)
REAL(DP), DIMENSION(6) :: r = (/57568490574.0_dp,-13362590354.0_dp,&
651619640.7_dp,-11214424.18_dp,77392.33017_dp,&
-184.9052456_dp/)
REAL(DP), DIMENSION(6) :: s = (/57568490411.0_dp,1029532985.0_dp,&
9494680.718_dp,659272.64853_dp,267.8532712_dp,1.0_dp/)
if (abs(x) < 8.0) then
  Direct rational function fit.
y=x**2
  bessj0_s=poly(y,r)/poly(y,s)
else
  Fitting function (6.5.9).
  ax=abs(x)
z=8.0_sp/ax
  y=z**2
  xx=ax-0.785398164_sp
  bessj0_s=sqrt(0.636619772_sp/ax)*(cos(xx)*&
poly(y,p)-z*sin(xx)*poly(y,q))
end if
END FUNCTION bessj0_s

FUNCTION bessj0_v(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessj0_v
REAL(SP), DIMENSION(size(x)) :: ax,xx,z
REAL(DP), DIMENSION(size(x)) :: y
LOGICAL(LGT), DIMENSION(size(x)) :: mask
REAL(DP), DIMENSION(5) :: p = (/1.0_dp,-0.1098628627e-2_dp,&
0.2734510407e-4_dp,-0.2073370639e-5_dp,0.2093887211e-6_dp/)
REAL(DP), DIMENSION(5) :: q = (/0.1562499995e-1_dp,&
0.1430488765e-3_dp,-0.6911147651e-5_dp,0.7621098161e-6_dp,&
-0.934945152e-7_dp/)
REAL(DP), DIMENSION(6) :: r = (/57568490574.0_dp,-13362590354.0_dp,&
651619640.7_dp,-11214424.18_dp,77392.33017_dp,&
-184.9052456_dp/)
REAL(DP), DIMENSION(6) :: s = (/57568490411.0_dp,1029532985.0_dp,&
9494680.718_dp,659272.64853_dp,267.8532712_dp,1.0_dp/)
mask = (abs(x) < 8.0)
where (mask)
y=x**2
  bessj0_v=poly(y,r,mask)/poly(y,s,mask)
elsewhere
  ax=abs(x)
z=8.0_sp/ax
  y=z**2
  xx=ax-0.785398164_sp
  bessj0_v=sqrt(0.636619772_sp/ax)*(cos(xx)*&
poly(y,p,.not. mask)-z*sin(xx)*poly(y,q,.not. mask))
end where
END FUNCTION bessj0_v
where (mask)...bessj0_v=poly(y,r,mask)/poly(y,s,mask) Here we meet the third solution to the problem of getting masked values from an external vector function. (For the other two solutions, see notes to factrl, p. 1087, and gammp, p. 1090.) Here we simply evade all responsibility and pass the mask into every routine that is supposed to be masked. Let it be somebody else’s problem! That works here because your hardworking authors have overloaded the nrutil routine poly with a masked vector version. More typically, of course, it becomes your problem, and you have to remember to write masked versions of all the vector routines that you call in this way. (We’ll meet examples of this later.)

FUNCTION bessy0_s(x)
USE nrtype; USE nrutil, ONLY : assert,poly
USE nr, ONLY : bessj0
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessy0_s

Returns the Bessel function \( Y_0(x) \) for positive \( x \).

REAL(SP) :: xx,z
REAL(DP) :: y
We’ll accumulate polynomials in double precision.

REAL(DP), DIMENSION(5) :: p = (/1.0_dp,-0.1098628627e-2_dp,&
0.2734510407e-4_dp,-0.2073370639e-5_dp,0.2093887211e-6_dp/)
REAL(DP), DIMENSION(5) :: q = (/0.1562499995e-1_dp,&
0.1430488765e-3_dp,-0.6911147685e-5_dp,0.7621095161e-6_dp,&
-0.934945152e-7_dp/)
REAL(DP), DIMENSION(6) :: r = (/-2957821389.0_dp,7062834065.0_dp,&
-512359803.6_dp,10879881.29_dp,-86327.92757_dp,&
228.4622733_dp/)
REAL(DP), DIMENSION(6) :: s = (/40076544269.0_dp,745249964.8_dp,&
7189466.438_dp,47447.26470_dp,1030244_dp,1.0_dp/)
call assert(x > 0.0, 'bessy0_s arg')
if (abs(x) < 8.0) then
Rational function approximation of (6.5.8).
y=x**2
bessy0_s=(poly(y,r)/poly(y,s))+&
0.636619772_sp*bessj0(x)*log(x)
else
Fitting function (6.5.10).
z=0.0_sp/x
y=x**2
xx=x-0.785398164_sp
bessy0_s=sqrt(0.636619772_sp/x)*(sin(xx)*&
poly(y,p)+z*cos(xx)*poly(y,q))
end if
END FUNCTION bessy0_s

FUNCTION bessy0_v(x)
USE nrtype; USE nrutil, ONLY : assert,poly
USE nr, ONLY : bessj0
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessy0_v
REAL(SP), DIMENSION(size(x)) :: xx,z
REAL(DP), DIMENSION(size(x)) :: y
LOGICAL(LGT), DIMENSION(size(x)) :: mask
call assert(x > 0.0, 'bessy0_v arg')
if (abs(x) < 8.0) then
Rational function approximation of (6.5.8).
y=x**2
bessy0_v=poly(y,r,mask)/poly(y,s,mask)
else
Fitting function (6.5.10).
z=0.0_sp/x
y=x**2
xx=x-0.785398164_sp
bessy0_v=sqrt(0.636619772_sp/x)*(sin(xx)*&
poly(y,p)+z*cos(xx)*poly(y,q))
end if
END FUNCTION bessy0_v
FUNCTION bessj1_s(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessj1_s

Returns the Bessel function $J_1(x)$ for any real $x$.

REAL(SP) :: ax,xx,z
REAL(DP) :: y

We'll accumulate polynomials in double precision.

REAL(DP), DIMENSION(6) :: r = (/7236261432.0_dp, &
 &789509235.0_dp,242598531.1_dp,-2972611.439_dp, &
 &376.9991397_dp/)
REAL(DP), DIMENSION(6) :: s = (/14472229442.0_dp, &
 &2300535178.0_dp,18583304.74_dp,99447.43394_dp, &
 &376.9991397_dp/)
REAL(DP), DIMENSION(5) :: p = (/1.0_dp,0.183105e-2_dp, &
 &-0.3516396496e-4_dp,0.2457520174e-5_dp,-0.240337019e-6_dp/)
REAL(DP), DIMENSION(5) :: q = (/0.0468749999_dp, &
 &-0.2002690873e-3_dp,0.8449199096e-5_dp,-0.88228987e-6_dp, &
 &0.105787412e-6_dp/)

if (abs(x) < 8.0) then
  Direct rational approximation.
  y=x**2
  bessj1_s=x*(poly(y,r)/poly(y,s))
else
  Fitting function (6.5.9).
  ax=abs(x)
  z=8.0_sp/ax
  y=z**2
  xx=ax-2.356194491_sp
  bessj1_s=sign(1.0_sp,x)*sqrt(0.636619772_sp/ax)*(cos(xx)* &
  poly(y,p)-z*sin(xx)*poly(y,q))*sign(1.0_sp,x)
end if
END FUNCTION bessj1_s
FUNCTION bessj1_v(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessj1_v
REAL(SP), DIMENSION(size(x)) :: ax, xx, z
REAL(DP), DIMENSION(size(x)) :: y
LOGICAL(LGT), DIMENSION(size(x)) :: mask
REAL(DP), DIMENSION(6) :: r = (/72362614232.0_dp,&
-7895059235.0_dp,24396853.1_dp,-2972611.439_dp,&
15704.48260_dp,-30.16036606_dp/)
REAL(DP), DIMENSION(6) :: s = (/144725228442.0_dp,2300535178.0_dp,&
1858304.74_dp,99447.43394_dp,376.9991397_dp,1.0_dp/)
REAL(DP), DIMENSION(5) :: p = (/1.0_dp,0.183105e-2_dp,&
-0.3516396496e-4_dp,0.2457520174e-5_dp,-0.240337019e-6_dp/)
REAL(DP), DIMENSION(5) :: q = (/0.04687499995_dp,&
-0.2002690873e-3_dp,0.8499199096e-5_dp,-0.88228987e-6_dp,&
0.105787412e-6_dp/)
mask = (abs(x) < 8.0)
where (mask)
   y = x**2
   bessj1_v = x*(poly(y,r,mask)/poly(y,s,mask))
elsewhere
   ax = abs(x)
   z = 8.0_sp/ax
   y = x**2
   xx = ax - 2.356194491_sp
   bessj1_v = sqrt(0.636619772_sp/ax)*(cos(xx)*poly(y,p,.not. mask)-z*sin(xx)*poly(y,q,.not. mask))*(1_sp)
end where
END FUNCTION bessj1_v

FUNCTION bessy1_s(x)
USE nrtype; USE nrutil, ONLY : assert, poly
USE nr, ONLY : bessj1
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessy1_s
Returns the Bessel function $Y_1(x)$ for positive $x$.
REAL(SP) :: xx, z
REAL(DP) :: y
We'll accumulate polynomials in double precision.
REAL(DP), DIMENSION(5) :: p = (/1.0_dp,0.183105e-2_dp,&
-0.3516396496e-4_dp,0.2457520174e-5_dp,-0.240337019e-6_dp/)
REAL(DP), DIMENSION(5) :: q = (/0.04687499995_dp,&
-0.2002690873e-3_dp,0.8499199096e-5_dp,-0.88228987e-6_dp,&
0.105787412e-6_dp/)
real = (/0.2499580570e14_dp,0.4244419664e12_dp,0.3733650367e10_dp,&
0.2245900402e8_dp,0.1020426050e6_dp,0.3549632885e3_dp,1.0_dp/)
call assert(x > 0.0, 'bessy1_s arg')
if (abs(x) < 8.0) then
   Rational function approximation of (6.5.8).
   y = x**2
   bessy1_s = x*(poly(y,r)/poly(y,s)) +
0.636619772_sp*(bessj1(x)*log(x)-1.0_sp/x)
else
   Fitting function (6.5.10).
end if
END FUNCTION bessy1_s
FUNCTION bessy1_s(x)
USE nrtype; USE nrutil, ONLY : assert, poly
USE nr, ONLY : bessj1
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessy1_v
REAL(SP), DIMENSION(size(x)) :: xx, z
REAL(DP), DIMENSION(size(x)) :: y
LOGICAL(LGT), DIMENSION(size(x)) :: mask
REAL(DP), DIMENSION(5) :: p = (/1.0_dp,0.183105e-2_dp,&
  -0.3516396496e-4_dp,0.2457520174e-5_dp,-0.240337019e-6_dp/)
REAL(DP), DIMENSION(5) :: q = (/0.04687499995_dp,&
  -0.2002690873e-3_dp,0.8449199096e-5_dp,-0.88228987e-6_dp,&
  0.105787412e-6_dp/)
REAL(DP), DIMENSION(6) :: r = (/-0.4900604943e13_dp,&
  0.127527390e13_dp,-0.5153438139e11_dp,0.7349264551e9_dp,&
  -0.423792726e7_dp,0.8511937935e4_dp/)
REAL(DP), DIMENSION(7) :: s = (/0.04687499995_dp,&
  -0.2002690873e-3_dp,0.8449199096e-5_dp,-0.88228987e-6_dp,&
  0.105787412e-6_dp/)
call assert(all(x > 0.0), 'bessy1_varg')
mask = (abs(x) < 8.0)
where (mask)
  y=x**2
  bessy1_v=x*(poly(y,r,mask)/poly(y,s,mask))+&
    0.636619772_sp*(bessj1(x)*log(x)-1.0_sp/x)
elsewhere
  z=8.0_sp/x
  y=z**2
  xx=x-2.356194491_sp
  bessy1_v=sqrt(0.636619772_sp/x)*(sin(xx)*&
    poly(y,p,.not. mask)+z*cos(xx)*poly(y,q,.not. mask))
end where
END FUNCTION bessy1_v

FUNCTION bessy_s(n,x)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : bessy0, bessy1
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessy_s

INTEGER(I4B) :: j
REAL(SP) :: by, bym, byp, tox

call assert(n >= 2, x > 0.0, 'bessy_s args')
tox=2.0_sp/x
by=bessy1(x)

bym=bessy0(x)

do j=1,n-1
  Recurrence (6.5.7).

* * *

FUNCTION bessy_s(n,x)
byp=j*tox*by-bym  
bym=by  
by=byp  
end do  
bessy_s=by  
END FUNCTION bessy_s

FUNCTION bessy_s(n,x)
USE nrtype; USE nrutil, ONLY : assert  
USE nr, ONLY : bessy0,bessy1  
IMPLICIT NONE  
INTEGER(I4B), INTENT(IN) :: n  
REAL(SP), DIMENSION(:), INTENT(IN) :: x  
REAL(SP), DIMENSION(size(x)) :: bessy_v  
INTEGER(I4B) :: j  
REAL(SP), DIMENSION(size(x)) :: by,bym,byp,tox  
call assert(n >= 2, all(x > 0.0), 'bessy_vargs')  
tox=2.0_sp/x  
by=bessy1(x)  
bym=bessy0(x)  
do j=1,n-1  
byp=j*tox*by-bym  
bym=by  
by=byp  
end do  
bessy_v=by  
END FUNCTION bessy_v

Notice that the vector routine is exactly the same as the scalar routine, but operates only on vectors, and that nothing in the routine is specific to any level of precision or kind type of real variable. Cases like this make us wish that Fortran 90 provided for “template” types that could automatically take the type and shape of the actual arguments. (Such facilities are available in other, more object-oriented languages such as C++.)

FUNCTION bessj_s(n,x)
USE nrtype; USE nrutil, ONLY : assert  
USE nr, ONLY : bessj0,bessj1  
IMPLICIT NONE  
INTEGER(I4B), INTENT(IN) :: n  
REAL(SP), INTENT(IN) :: x  
REAL(SP) :: bessj_s  
INTEGER(I4B), PARAMETER :: IACC=40,IEXP=maxexponent(x)/2  
Returns the Bessel function $J_n(x)$ for any real $x$ and $n \geq 2$. Make the parameter IACC larger to increase accuracy.  
INTEGER(I4B) :: j,jsum,m  
REAL(SP) :: ax,bj,bjm,bjp,summ,tox  
call assert(n >= 2, 'bessj_sargs')  
ax=abs(x)  
if (ax*ax <= 8.0_sp*tiny(x)) then  
  bessj_s=0.0  
else if (ax > real(n,sp)) then  
  tox=2.0_sp/ax  
  bjm=bessj0(ax)  
  bj=bessj1(ax)  
  do j=1,n-1  
    bj=bjm*bj-bj*bjm  
    bm=bj  
    bj=bjm  
  end do  
  bessj_s=bj  
  return  
else  
  tox=2.0_sp/ax  
  bjm=bessj0(ax)  
  bj=bessj1(ax)  
  do j=1,n-1  
    bj=bjm*bj-bj*bjm  
    bm=bj  
    bj=bjm  
  end do  
  bessj_s=bj  
  return  
end if
The `bessj` routine does not conveniently parallelize with Fortran 90’s language constructions, but Bessel functions are of sufficient importance that we feel the need for a parallel version nevertheless. The basic method adopted below is to encapsulate as contained vector functions two separate algorithms, one for the case \( x \leq n \), the other for \( x > n \). Both of these have masks as input arguments; within each routine, however, they immediately revert to the pack-unpack method. The choice to pack in the subsidiary routines, rather than in the main routine, is arbitrary; the main routine is supposed to be a little clearer this way.

In the Fortran 77 version of this routine, we scaled the variables by \(10^{-10}\) whenever \( bj \) was bigger than \(10^{10}\). On a machine with a large exponent range, we could improve efficiency by scaling less often. In order to remain portable, however, we used the conservative value of \(10^{10}\). An elegant way of handling renormalization is provided by the Fortran 90 intrinsic functions that manipulate real numbers. We test with `if (exponent(bj) > IEXP)` and then if necessary renormalize with `bj=scale(bj,-IEXP)` and similarly for the other variables. Our conservative choice is to set \(\text{IEXP} = \text{maxexponent}(x)/2\). Note that an added benefit of scaling this way is that only the exponent of each variable is modified; no roundoff error is introduced as it can be if we do a floating-point division instead.
FUNCTION bessj_v(n,xx)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : bessj0, bessj1
IMPLICIT NONE
INTEGER(4B), INTENT(IN) :: n
REAL(SP), DIMENSION(:), INTENT(IN) :: xx
REAL(SP), DIMENSION(size(xx)) :: bessj_v
INTEGER(4B), PARAMETER :: IACC=40, IEXP=maxexponent(xx)/2
REAL(SP), DIMENSION(size(xx)) :: ax
LOGICAL(LGT), DIMENSION(size(xx)) :: mask, mask0
REAL(SP), DIMENSION(:), ALLOCATABLE :: x, bj, bjm, bpj, summ, tox, bessjle
LOGICAL(LGT), DIMENSION(:), ALLOCATABLE :: renorm
INTEGER(4B) :: j, jsum, npak

call assert(n >= 2, 'bessj_vargs')
ax=abs(xx)
mask = (ax <= real(n,sp))
mask0 = (ax*ax <= 8.0_sp*tiny(xx))
bessj_v=bessjle_v(n,ax,logical(mask .and. .not. mask0, kind=lgt))
bessj_v=merge(bessjgt_v(n,ax,.not. mask),bessj_v,.not. mask)
where (mask0) bessj_v=0.0
where (xx < 0.0 .and. mod(n,2) == 1) bessj_v=-bessj_v

CONTAINS

FUNCTION bessjgt_v(n,xx,mask)
IMPLICIT NONE
INTEGER(4B), INTENT(IN) :: n
REAL(SP), DIMENSION(:), INTENT(IN) :: xx
LOGICAL(LGT), DIMENSION(size(xx)), INTENT(IN) :: mask
REAL(SP), DIMENSION(size(xx)) :: bessjgt_v
npak=count(mask)
if (npak == 0) RETURN
allocate(x(npak), bj(npak), bjm(npak), bpj(npak), tox(npak))
x=pack(xx[mask])
tox=2.0_sp/x
bj=bessj0(x)
bj=bessj1(x)
do j=1,n-1
bjm=j*tox*bj-bjm
bj=bpj
bpm=bj
end do
bessjgt_v=unpack(bj,mask,0.0_sp)
deallocate(x,bj,bjm,bpj, tox)
END FUNCTION bessjgt_v

FUNCTION bessjle_v(n,xx,mask)
IMPLICIT NONE
INTEGER(4B), INTENT(IN) :: n
REAL(SP), DIMENSION(:), INTENT(IN) :: xx
LOGICAL(LGT), DIMENSION(size(xx)), INTENT(IN) :: mask
REAL(SP), DIMENSION(size(xx)) :: bessjle_v
npak=count(mask)
if (npak == 0) RETURN
allocate(x(npak), bj(npak), bjm(npak), bpj(npak), summ(npak), &
  bessjle(npak), tox(npak), renorm(npak))
x=pack(xx[mask])
tox=2.0_sp/x
m=2*((n+int(sqrt(real(IACC*n,sp))))/2)
bessjle=0.0
jsum=0
summ=0.0
bjp=0.0
bj=1.0
do j=m,1,-1
bpm=j*tox*bj-bpm
bj=bpj
end do
bjp = bj
bj = bj
renorm = (exponent(bj) > IEXP)
bj = merge(scale(bj, -IEXP), bj, renorm)
bjp = merge(scale(bjp, -IEXP), bjp, renorm)
bessjle = merge(scale(bessjle, -IEXP), bessjle, renorm)
summ = merge(scale(summ, -IEXP), summ, renorm)
if (jsum /= 0) summ = summ + bj
jsum = 1 - jsum
if (j == n) bessjle = bjpend do
summ = 2.0_sp * summ - bj
bessjle = bessjle/summ
bessjle_v = unpack(bessjle, mask, 0.0_sp)
deallocate(x, bj, bjm, bjp, summ, bessjle, tox, renorm)
END FUNCTION bessjle_v
END FUNCTION bessj_v

bessj_v = ... bessj_v = merge(bessjgt_v(...), bessj_v, ...)

The vector bessj_v is set once (with a mask) and then merged with itself, along
with the vector result of the bessjgt_v call. Thus are the two evaluation
methods combined. (A third case, where an argument is zero, is then handled by
an immediately following where.)

* * *

FUNCTION bessi0_s(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessi0_s
Returns the modified Bessel function I_0(x) for any real x.
REAL(SP) :: ax
REAL(DP), DIMENSION(7) :: p = (/1.0_dp, 3.5156229_dp, 
3.0899424_dp, 1.2067492_dp, 0.2659732_dp, 0.360768e-1_dp, 
0.45813e-2_dp/)
Accumulate polynomials in double precision.
REAL(DP), DIMENSION(9) :: q = (/0.39894228_dp, 0.1328592e-1_dp, 
0.225319e-2_dp, -0.157565e-2_dp, 0.916281e-2_dp, 
-0.2057706e-1_dp, 0.2635537e-1_dp, -0.1647633e-1_dp, 
0.392377e-2_dp/)
ax = abs(x)
if (ax < 3.75) then
  Polynomial fit.
bessi0_s = poly(real((x/3.75_sp)**2, dp), p)
else
  bessi0_s = (exp(ax)/sqrt(ax)) * poly(real(3.75_sp/ax, dp), q)
end if
END FUNCTION bessi0_s

FUNCTION bessi0_v(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessi0_v
REAL(SP), DIMENSION(size(x)) :: ax
REAL(DP), DIMENSION(size(x)) :: y
LOGICAL(LGT), DIMENSION(size(x)) :: mask
REAL(DP), DIMENSION(7) :: p = (/1.0_dp, 3.5156229_dp, 
3.0899424_dp, 1.2067492_dp, 0.2659732_dp, 0.360768e-1_dp, &
REAL(dp), DIMENSION(9) :: q = (/0.39894228_dp,0.1328592e-1_dp,&
0.225319e-2_dp,-0.157565e-2_dp,0.916281e-2_dp,&
-0.2057706e-1_dp,0.2635537e-1_dp,-0.1647633e-1_dp,&
0.392377e-2_dp/)

ax = abs(x)
mask = (ax < 3.75)
where (mask)
bessi0_v = poly(real((x/3.75_sp)**2,dp),p,mask)
elsewhere
  y = 3.75_sp/ax
  bessi0_v = (exp(ax)/sqrt(ax))*poly(real(y,dp),q,.not. mask)
end where

END FUNCTION bessi0_v

FUNCTION bessk0_s(x)
USE nrtype; USE nrutil, ONLY : assert, poly
USE nr, ONLY : bessi0

IMPLICIT NONE
REAL(sp), INTENT(IN) :: x
REAL(SP) :: bessk0_s

Returns the modified Bessel function $K_0(x)$ for positive real $x$.

REAL(dp), DIMENSION(7) :: p = (/-0.57721566_dp,0.42278420_dp,&
0.23069756_dp,0.3488590e-1_dp,0.262698e-2_dp,0.10750e-3_dp,&
0.74e-5_dp/)
REAL(dp), DIMENSION(7) :: q = (/1.25331414_dp,-0.7832358e-1_dp,&
0.2189568e-1_dp,-0.1062446e-1_dp,0.587872e-2_dp,&
-0.251540e-2_dp,0.53208e-3_dp/)
call assert(x > 0.0, 'bessk0_s arg')
if (x <= 2.0) then    Polynomial fit.
  y = x*x/4.0_sp
  bessk0_s = (-log(x/2.0_sp)*bessi0(x))+poly(y,p)
else
  y = (2.0_sp/x)
  bessk0_s = (exp(-x)/sqrt(x))*poly(y,q)
end if

END FUNCTION bessk0_s

FUNCTION bessk0_v(x)
USE nrtype; USE nrutil, ONLY : assert, poly
USE nr, ONLY : bessi0

IMPLICIT NONE
REAL(sp), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessk0_v
REAL(DP), DIMENSION(size(x)) :: y
LOGICAL(LGT), DIMENSION(size(x)) :: mask
REAL(dp), DIMENSION(7) :: p = (/-0.57721566_dp,0.42278420_dp,&
0.23069756_dp,0.3488590e-1_dp,0.262698e-2_dp,0.10750e-3_dp,&
0.74e-5_dp/)
REAL(dp), DIMENSION(7) :: q = (/1.25331414_dp,-0.7832358e-1_dp,&
0.2189568e-1_dp,-0.1062446e-1_dp,0.587872e-2_dp,&
-0.251540e-2_dp,0.53208e-3_dp/)
call assert(all(x > 0.0), 'bessk0_v arg')
mask = (x <= 2.0)
where (mask)
  y = x*x/4.0_sp
  bessk0_v = (-log(x/2.0_sp)*bessi0(x))+poly(y,p,mask)
FUNCTION bessk0_v(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessk0_v

$y = (2.0_{sp}/x)$
$\text{bessk0} = (\exp(-x)/\sqrt{x}) \times \text{poly}(y,q,.\not\text{mask})$

END FUNCTION bessk0_v

FUNCTION bessi1_s(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessi1_s

Returns the modified Bessel function $I_1(x)$ for any real $x$.

REAL(SP) :: ax
REAL(DP), DIMENSION(7) :: p = (/0.5_dp, 0.87890594_dp, 0.51498869_dp, 0.15084934_dp, 0.2658733e-1_dp, 0.301532e-2_dp, 0.32411e-3_dp/)

Accumulate polynomials in double precision.

REAL(DP), DIMENSION(9) :: q = (/0.39894228_dp, -0.3988024e-1_dp, -0.362018e-2_dp, 0.163801e-2_dp, -0.1031555e-1_dp, 0.2282967e-1_dp, -0.2895312e-1_dp, 0.178754e-1_dp, -0.420059e-2_dp/)

$ax = \text{abs}(x)$
if ($ax < 3.75$) then Polynomial fit.
  $\text{bessi1} = ax \times \text{poly}(\text{real}(x/3.75_{sp})^2, dp, p)$
else
  $\text{bessi1} = (\exp(ax)/\sqrt{ax}) \times \text{poly}(\text{real}(3.75_{sp}/ax, dp), q)$
end if
if ($x < 0.0$) $\text{bessi1} = -\text{bessi1}$

END FUNCTION bessi1_s

FUNCTION bessi1_v(x)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessi1_v
REAL(SP), DIMENSION(size(x)) :: ax
REAL(DP), DIMENSION(size(x)) :: y
LOGICAL(LGT), DIMENSION(size(x)) :: mask

REAL(DP), DIMENSION(7) :: p = (/0.5_dp, 0.87890594_dp, 0.51498869_dp, 0.15084934_dp, 0.2658733e-1_dp, 0.301532e-2_dp, 0.32411e-3_dp/)
REAL(DP), DIMENSION(9) :: q = (/0.39894228_dp, -0.3988024e-1_dp, -0.362018e-2_dp, 0.163801e-2_dp, -0.1031555e-1_dp, 0.2282967e-1_dp, -0.2895312e-1_dp, 0.178754e-1_dp, -0.420059e-2_dp/)

$ax = \text{abs}(x)$
$mask = (ax < 3.75)$
where ($mask$)
  $\text{bessi1} = ax \times \text{poly}(\text{real}(x/3.75_{sp})^2, dp, p, mask)$
elsewhere
  $y = 3.75_{sp}/ax$
  $\text{bessi1} = (\exp(ax)/\sqrt{ax}) \times \text{poly}(\text{real}(y, dp), q, \not\text{mask})$
end where

where ($x < 0.0$) $\text{bessi1} = -\text{bessi1}$

END FUNCTION bessi1_v
FUNCTION bessk1_s(x)
USE nrtype; USE nrutil, ONLY : assert,poly
USE nr,  ONLY : bessi
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessk1_s

Returns the modified Bessel function $K_1(x)$ for positive real $x$.

REAL(DP) :: y
Accumulate polynomials in double precision.
REAL(DP), DIMENSION(7) :: p = (/
  1.0_dp, 0.15443144_dp, -0.67278579_dp, -0.18156897_dp, -0.1919402e-1_dp,
  -0.110404e-2_dp, -0.4686e-4_dp/
)
REAL(DP), DIMENSION(7) :: q = (/
  1.25331414_dp, 0.23498619_dp, -0.3655620e-1_dp, 0.1504268e-1_dp, -0.780353e-2_dp,
  0.325614e-2_dp, -0.68245e-3_dp/
)
call assert(x > 0.0, 'bessk1_s arg')
if (x <= 2.0) then
  Polynomial fit.
  y=x*x/4.0_sp
  bessk1_s=(log(x/2.0_sp)*bessi1(x))+(1.0_sp/x)*poly(y,p)
else
  y=2.0_sp/x
  bessk1_s=(exp(-x)/sqrt(x))*poly(y,q)
end if
END FUNCTION bessk1_s

FUNCTION bessk1_v(x)
USE nrtype; USE nrutil, ONLY : assert,poly
USE nr,  ONLY : bessi
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessk1_v
REAL(DP), DIMENSION(size(x)) :: y
LOGICAL(LGT), DIMENSION(size(x)) :: mask
REAL(DP), DIMENSION(7) :: p = (/
  1.0_dp, 0.15443144_dp, -0.67278579_dp, -0.18156897_dp, -0.1919402e-1_dp,
  -0.110404e-2_dp, -0.4686e-4_dp/
)
REAL(DP), DIMENSION(7) :: q = (/
  1.25331414_dp, 0.23498619_dp, -0.3655620e-1_dp, 0.1504268e-1_dp, -0.780353e-2_dp,
  0.325614e-2_dp, -0.68245e-3_dp/
)
call assert(all(x > 0.0), 'bessk1_v arg')
mask = (x <= 2.0)
where (mask)
  y=x*x/4.0_sp
  bessk1_v=(log(x/2.0_sp)*bessi1(x))+(1.0_sp/x)*poly(y,p,mask)
elsewhere
  y=2.0_sp/x
  bessk1_v=(exp(-x)/sqrt(x))*poly(y,q,.not. mask)
end where
END FUNCTION bessk1_v
FUNCTION bessk_s(n,x)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : bessk0,bessk1
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessk_s

Returns the modified Bessel function \( K_n(x) \) for positive \( x \) and \( n \geq 2 \).

INTEGER(I4B) :: j
REAL(SP) :: bk,bkm,bkp,tox
call assert(n >= 2, x > 0.0, 'bessk_s args')
tox=2.0_sp/x
bkm=bessk0(x)  
Upward recurrence for all \( x \)...
bk=bessk1(x)
do j=1,n-1  
   bkp=bkm+j*tox*bk
   bkm=bk
   bk=bkp
end do
bessk_s=bk
END FUNCTION bessk_s

FUNCTION bessk_v(n,x)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : bessk0,bessk1
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessk_v
INTEGER(I4B) :: j
REAL(SP), DIMENSION(size(x)) :: bk,bkm,bkp,tox
call assert(n >= 2, all(x > 0.0), 'bessk_v args')
tox=2.0_sp/x
bkm=bessk0(x)
bk=bessk1(x)
do j=1,n-1
   bkp=bkm+j*tox*bk
   bkm=bk
   bk=bkp
end do
bessk_v=bk
END FUNCTION bessk_v

The scalar and vector versions of bessk are identical, and have no
precision-specific constants, another example of where we would like to
define a generic “template” function if the language had this facility.

*   *   *
FUNCTION bessi_s(n,x)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : bessi0
IMPLICIT NONE
INTEGER(14B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessi_s
INTEGER(14B), PARAMETER :: IACC=40, IEXP=maxexponent(x)/2
Returns the modified Bessel function \( I_n(x) \) for any real \( x \) and \( n \geq 2 \). Make the parameter IACC larger to increase accuracy.
INTEGER(14B) :: j,m
REAL(SP) :: bi,bim,bip,tox
call assert(n >= 2, 'bessi_s args')
bessi_s=0.0
if (x*x <= 8.0_sp*tiny(x)) RETURN Underflow limit.
tox=2.0_sp/abs(x)
bip=0.0
bi=1.0
m=2*((n+int(sqrt(real(IACC*n,sp)))))
for j=m,1,-1
bim=bip+j*tox*bi
The downward recurrence.
bip=bi
bi=bim
if (exponent(bi) > IEXP) then Renormalize to prevent overflows.
bessi_s=scale(bessi_s,-IEXP)
bi=scale(bi,-IEXP)
bip=scale(bip,-IEXP)
end if
if (j == n) bessi_s=bip
end do
bessi_s=bessi_s*bessi0(x)/bi Normalize with bessi0.
if (x < 0.0 .and. mod(n,2) == 1) bessi_s=-bessi_s
END FUNCTION bessi_s

FUNCTION bessi_v(n,x)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : bessi0
IMPLICIT NONE
INTEGER(14B), INTENT(IN) :: n
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessi_v
INTEGER(14B), PARAMETER :: IACC=40, IEXP=maxexponent(x)/2
INTEGER(14B) :: j,m
REAL(SP), DIMENSION(size(x)) :: bi,bim,bip,tox
LOGICAL(LGT), DIMENSION(size(x)) :: mask
call assert(n >= 2, 'bessi_v args')
bessi_v=0.0
mask = (x <= 8.0_sp*tiny(x))
tox=2.0_sp/merge(2.0_sp,abs(x),mask)
bip=0.0
bi=1.0_sp
m=2*((n+int(sqrt(real(IACC*n,sp)))))
for j=m,1,-1
bim=bip+j*tox*bi
The downward recurrence.
bip=bi
bi=bim
where (exponent(bi) > IEXP)
bessi_v=scale(bessi_v,-IEXP)
end if
if (j == n) bessi_v=bip
end do
bessi_v=bessi_v*bessi0(x)/bi Normalize with bessi0.
if (x < 0.0 .and. mod(n,2) == 1) bessi_v=-bessi_v
END FUNCTION bessi_v

if (exponent(bi) > IEXP) then See discussion of scaling for bessj on p. 1107.
bi=scale(bi,-IEXP)
bip=scale(bip,-IEXP)
end where
if (j == n) bessi_v=bip
end do
bessi_v=bessi_v*bessi0(x)/bi
where (mask) bessi_v=0.0_sp
where (x < 0.0 .and. mod(n,2) == 1) bessi_v=-bessi_v
END FUNCTION bessi_v

mask = (x == 0.0)
tox=2.0_sp/merge(2.0_sp,abs(x),mask)

For the special case $x = 0$, the value of the returned function should be zero; however, the evaluation of tox will give a divide check. We substitute an innocuous value for the zero cases, then fix up their answers at the end.

** ** *

SUBROUTINE bessjy_s(x,xnu,rj,ry,rjp,ryp)
USE nrtype; USE nrutil, ONLY : assert,nrerror
USE nr, ONLY : beschb
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,xnu
REAL(SP), INTENT(OUT) :: rj,ry,rjp,ryp
INTEGER(I4B), PARAMETER :: MAXIT=10000
REAL(DP), PARAMETER :: XMIN=2.0_dp,EPS=1.0e-10_dp,FPMIN=1.0e-30_dp

Returns the Bessel functions $rj = J_{\nu}$, $ry = Y_{\nu}$ and their derivatives $rjp = J'_{\nu}$, $ryp = Y'_{\nu}$, for positive $x$ and for $xnu = \nu \geq 0$. The relative accuracy is within one or two significant digits of EPS, except near a zero of one of the functions, where EPS controls its absolute accuracy. FPMIN is a number close to the machine's smallest floating-point number. All internal arithmetic is in double precision. To convert the entire routine to double precision, change the SP declaration above and decrease EPS to $10^{-16}$. Also convert the subroutine beschb.

INTEGER(I4B) :: i,isign,1,nl
REAL(DP) :: a,b,c,d,del,del1,e,f,fact,fact2,fact3,ff,gam,gam1,gam2,&
gammi,gampl,h,p,pimu,pimu2,q,r,rlj,rlj1,rljmu,rljpl,rljtemp,&
rj1,rynu,rynum,rytemp,sum,sum1,w,x2,xi,xi2,xmu,xmu2
COMPLEX(DPC) :: aa,bb,cc,dd,dp,pq
call assert(x > 0.0, xnu >= 0.0, 'bessjy args')
nl=merge(int(xnu+0.5_dp), max(0,int(xnu-x+1.5_dp)), x < XMIN)
nl is the number of downward recurrences of the $J$'s and upward recurrences of $Y$'s. xmu lies between $-1/2$ and $1/2$ for $x < XMIN$, while it is chosen so that $x$ is greater than the turning point for $x \geq XMIN$.
xmu=xnu-nl
xnu2=xnu*xnu
x1=1.0_dp/x
x12=2.0_dp*x1
w=x12/PI_D
isign=1
h=xnu*x1
if (h < FPMIN) h=FPMIN
b=x12*xnu
d=0.0
c=h
do i=1,MAXIT
b=b*x12
d=b-d
if (abs(d) < FPMIN) d=FPMIN
c=b-1.0_dp/c

The Wronskian.
Evaluate CF1 by modified Lentz's method ([§5.2]. isign keeps track of sign changes in the denominator.)
if (abs(c) < FPMIN) c=FPMIN
d=1.0_dp/d
del=c*d
h=del*h
if (d < 0.0) isign=-isign
if (abs(del-1.0_dp) < EPS) exit
end do
if (i > MAXIT) call nrerror('x too large in bessjy; try asymptotic expansion')
raise=1.0_dp/rjpl
rjpl=h*rjpl
rjl=rjl
do l=nl,1,-1
rjtemp=fact*rjl+rjpl
fact=fact-xi
rjpl=fact*rjtemp-rjl
rjl=rjtemp
end do
if (rjl == 0.0) rjl=EPS
f=rjpl/rjl
if (x < XMIN) then
x2=0.5_dp*x
pimu=PI_D*xmu
if (abs(pimu) < EPS) then
fact=1.0
else
fact=pimu/sin(pimu)
end if
d=-log(x2)
e=xmu*d
if (abs(e) < EPS) then
fact2=1.0
else
fact2=sinh(e)/e
end if
call beschb(xmu,gam1,gam2,gampl,gammi)
Chebyshev evaluation of $\Gamma_1$ and $\Gamma_2$.
ff=2.0_dp/PI_D*fact*(gam1*cosh(e)+gam2*fact2*d) $f_0$.
e=exp(e)
p=e/(gampl*PI_D) $p_0$.
q=1.0_dp/(e*PI_D*gammi) $q_0$.
pimu2=0.5_dp*pimu
if (abs(pimu2) < EPS) then
fact3=1.0
else
fact3=sin(pimu2)/pimu2
end if
c=r=PI_D*pimu2*fact3*fact3
c=1.0
d=x2*x2
sum=ff*r*q
sum1=p
do i=1,MAXIT
ff=(i*ff+p+q)/(i*i-xmu2)
c=c*d/i
p=p/(i-xmu)
q=q/(i+xmu)
del=c*(ff*r+q)
sum=sum+del
dell=cp-i*del
suml=sum1+del1
if (abs(del) < (1.0_dp+abs(sum))*EPS) exit
end do
if (i > MAXIT) call nrerror('bessy series failed to converge')
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rymu=-sum
ry1=sum1*xi12
rymup=xmu*xi*rymu-ry1

Equation (6.7.13).

else

Evaluate CF2 by modified Lentz's method (§5.2).
a=0.25_dp-xmu2
pq=cmplx(-0.5_dp*xi,1.0_dp,kind=dpc)
aa=cmplx(0.0_dp,xi*a,kind=dpc)
bb=cmplx(2.0_dp*x,2.0_dp,kind=dpc)
cc=bb+aa/pq
dd=1.0_dp/bb
pq=cc/dd*pq
do i=2,MAXIT

Equations (6.7.6) – (6.7.10).
a=a+2*(i-1)
bb=bb+cmplx(0.0_dp,2.0_dp,kind=dpc)
dd=a/dd+bb
if (absc(dd) < FMIN) dd=FMIN
cc=bb+aa/cc
if (absc(cc) < FMIN) cc=FMIN
dd=1.0_dp/dd
dl=cc/dd
pq=pq+dl
if (absc(dl-1.0_dp) < EPS) exit
end do
if (i > MAXIT) call nrerror('cf2 failed in bessjy')
p=real(pq)
q=aimag(pq)
gam=(p-f)/q
rjmu=sqrt(w/((p-f)*gam+q))

Scale original $J_\nu$ and $J'_\nu$.

rjp=rjp1*fact

Upward recurrence of $Y_\nu$.
do i=1,nl

rytemp=(xmu+i)*xi2*ry1-rymu
rymu=ry1
ry1=rytemp
end do
ry=rymu
ryp=xnu*xi*rymu-ry1

CONTAINS

FUNCTION absc(z)
IMPLICIT NONE
COMPLEX(DPC), INTENT(IN) :: z
REAL(DP) :: absc
absc=abs(real(z))+abs(aimag(z))
END FUNCTION absc

END SUBROUTINE bessjy_s

Yes there is a vector version bessjy_v. Its general scheme is to have a bunch of contained functions for various cases, and then combine their outputs (somewhat like bessj_v, above, but much more complicated). A listing runs to about four printed pages, and we judge it to be of not much interest, so we will not include it here. (It is included on the machine-readable media.)
SUBROUTINE beschb_s(x,gam1,gam2,gampl,gammi)
USE nrtype
USE nr, ONLY : chebev
IMPLICIT NONE
REAL(DP), INTENT(IN) :: x
REAL(DP), INTENT(OUT) :: gam1,gam2,gampl,gammi
INTEGER(I4B), PARAMETER :: NUSE1=5,NUSE2=5
Evaluates $\Gamma_1$ and $\Gamma_2$ by Chebyshev expansion for $|x| \leq 1/2$. Also returns $1/\Gamma(1+x)$ and $1/\Gamma(1-x)$. If converting to double precision, set NUSE1 = 7, NUSE2 = 8.
REAL(SP) :: xx
REAL(SP), DIMENSION(7) :: c1=(/-1.142022680371168_sp,&
6.5165112670737e-3_sp,3.08709173086e-6_sp,-3.4706269649e-8_sp,&
6.9437664e-9_sp,3.67795e-11_sp,-1.356e-13_sp/)
REAL(SP), DIMENSION(8) :: c2=(/1.843740587300905_sp,&
-7.68528408447867e-2_sp,1.2719271366636e-3_sp,&
-4.97137367042e-6_sp, -3.12611968-8_sp,2.423096e-10_sp,&
-1.702e-8_sp,1.49e-15_sp/)
xx=8.0_dp*xx-1.0_dp Multiply x by 2 to make range be -1 to 1, and then apply
gam1=chebev(-1.0_sp,1.0_sp,c1(1:NUSE1),xx) transformation for evaluating even Cheby-
gam2=chebev(-1.0_sp,1.0_sp,c2(1:NUSE2),xx) shev series.
gampl=gam2-x*gam1
gammi=gam2+x*gam1
END SUBROUTINE beschb_s

SUBROUTINE beschb_v(x,gam1,gam2,gampl,gammi)
USE nrtype
USE nr, ONLY : chebev
IMPLICIT NONE
REAL(DP), DIMENSION(:), INTENT(IN) :: x
REAL(DP), DIMENSION(:), INTENT(OUT) :: gam1,gam2,gampl,gammi
INTEGER(I4B), PARAMETER :: NUSE1=5,NUSE2=5
REAL(SP), DIMENSION(size(x)) :: xx
REAL(SP), DIMENSION(7) :: c1=(/-1.142022680371168_sp,&
6.5165112670737e-3_sp,3.08709173086e-6_sp,-3.4706269649e-8_sp,&
6.9437664e-9_sp,3.67795e-11_sp,-1.356e-13_sp/)
REAL(SP), DIMENSION(8) :: c2=(/1.843740587300905_sp,&
-7.68528408447867e-2_sp,1.2719271366636e-3_sp,&
-4.97137367042e-6_sp, -3.12611968-8_sp,2.423096e-10_sp,&
-1.702e-8_sp,1.49e-15_sp/)
xx=8.0_dp*xx-1.0_dp Multiply x by 2 to make range be -1 to 1, and then apply
gam1=chebev(-1.0_sp,1.0_sp,c1(1:NUSE1),xx) transformation for evaluating even Cheby-
gam2=chebev(-1.0_sp,1.0_sp,c2(1:NUSE2),xx) shev series.
gampl=gam2-x*gam1
gammi=gam2+x*gam1
END SUBROUTINE beschb_v

SUBROUTINE bessik(x,xnu,ri,rk,rip,rkp)
USE nrtype; USE nrutil, ONLY : assert,nrerror
USE nr, ONLY : beschb
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,xnu
REAL(SP), DIMENSION(:), INTENT(IN) :: ri,rk,rip,rkp
INTEGER(I4B), PARAMETER :: MAXIT=10000
REAL(SP), PARAMETER :: MNI=2.0
REAL(DP), PARAMETER :: EPS=1.0e-10_dp,FPMIN=1.0e-30_dp
Returns the modified Bessel functions $r_1 = I_\nu$, $r_k = K_\nu$ and their derivatives $ri = I'_\nu$, $rkp = K'_\nu$, for positive x and for xnu = $\nu \geq 0$. The relative accuracy is within one or
two significant digits of EPS. FMIN is a number close to the machine's smallest floating-point number. All internal arithmetic is in double precision. To convert the entire routine to double precision, change the REAL declaration above and decrease EPS to $10^{-16}$. Also convert the subroutine beschb.

```
INTEGER(4B) :: i, nl
REAL(DP) :: a, a1, b, c, d, del, del1, delh, dels, e, f, fact, fact2, ff,
gam1, gam2, gammi, gampl, h, p, pinu, q, q1, q2, qnew,
ril, rill, rimu, rip1, ripl, rtemp, rkl, rkmu, rkmup, rktemp,
s, sum, s1, x2, x1, x12, xmu, xmu2

call assert(x > 0.0, xnu >= 0.0, 'bessik args')
nl=int(xmu+0.5_dp)  
xmu=xmu-nl
xmu2=xmu*xmu
xi=1.0_dp/x
x12=2.0_dp*x
h=xmu*x
b=x12*xmu
 If (h < FPMIN) h=FPMIN

if (i > MAXIT) call nrerror('x too large in bessik; try asymptotic expansion')
ril=FPMIN
ril1=ril
rip1=ripl
fact=xmu*xi

end do

f=ripl/ril
```

Evaluate CF1 by modified Lentz's method ($\S$5.2).

```
if (x < XMIN) then
  Use series.
  x2=0.5_dp*x
  pinu=PI_D*xmu
  if (abs(pinu) < EPS) then
    fact=1.0
  else
    fact=pinu/sin(pinu)
  end if
  d=-log(x2)
e=xmu*d
  if (abs(e) < EPS) then
    fact=1.0
  else
    fact2=sinh(e)/e
  end if
  call beschb(xmu, gam1, gam2, gampl, gammi)
  ff=fact*(gam1*cosh(e)+gam2*fact2*d)
  sum=ff
  e=exp(e)
  p=0.5_dp*e/gampl
  q=0.5_dp/(e*gammi)
  c=1.0
  d=x2*x2
```

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sum1=p
do i=1,MAXIT
  ff=(i*ff+p+q)/(i*i-xmu2)
  c=c*d/i
  p=p/(i-xmu)
  q=q/(i+xmu)
  del=c*ff
  sum=sum+del
  del1=c*(p-i*ff)
  sum1=sum1+del1
  if (abs(del) < abs(sum)*EPS) exit
end do
if (i > MAXIT) call nrerror('bessk series failed to converge')
rkmu=sum
rk1=sum1*xi2
else
  b=2.0_dp*(1.0_dp*x)
  d=1.0_dp/b
  delh=d
  h=delh
  q1=0.0
  q2=1.0
  a1=0.25_dp-xmu2
  c=a1
  q=c
  a=-a1
  s=1.0_dp*q*delh
  do i=2,MAXIT
    a=a-2*(i-1)
    c=-a*c/i
    q1=q2
    q2=qnew
    q=q+c*qnew
    b=b2.0_dp
    d=1.0_dp/(b+a*d)
    delh=(b*d-1.0_dp)*delh
    h=h+delh
    dels=q*delh
    s=s+dels
    if (abs(dels/s) < EPS) exit
  end do
  if (i > MAXIT) call nrerror('bessik: failure to converge in cf2')
h=a1*h
  rkmu=sqrt(PI_D/(2.0_dp*x))*exp(-x)/s
  rk1=rkmu*(xmu+0.5_dp-h)*xi
  end if
end if
bessik does not readily parallelize, and we thus don’t provide a vector version. Since airy, immediately following, requires bessik, we don’t have a vector version of it, either.

* * *

SUBROUTINE airy(x,ai,bi,aip,bip)
USE nrtype
USE nr, ONLY : bessik,bessjy
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: ai,bi,aip,bip

Returns Airy functions $\text{Ai}(x)$, $\text{Bi}(x)$, and their derivatives $\text{Ai}'(x)$, $\text{Bi}'(x)$.

REAL(SP), PARAMETER :: THIRD=1.0_sp/3.0_sp,TWOTHR=2.0_sp/3.0_sp, &
ONOVRT=0.5773502691896258_sp
absx=abs(x)
rootx=sqrt(absx)
z=TWOTHR*absx*rootx
if (x > 0.0) then
  call bessik(z,THIRD,ri,rk,rip,rkp)
  ai=rootx*ONOVRT*rk/PI
  bi=rootx*(rk/PI+2.0_sp*ONOVRT*ri)
call bessik(z,TWOTHR,ri,rk,rip,rkp)
  aip=-x*ONOVRT*rk/PI
  bip=x*(rk/PI+2.0_sp*ONOVRT*ri)
else if (x < 0.0) then
  call bessjy(z,THIRD,rj,ry,rjp,ryp)
  ai=0.5_sp*rootx*(rj-ONOVRT*ry)
  bi=-0.5_sp*rootx*(ry+ONOVRT*rj)
call bessjy(z,TWOTHR,rj,ry,rjp,ryp)
  aip=0.5_sp*absx*(ONOVRT*ry+rj)
  bip=0.5_sp*absx*(ONOVRT*rj-ry)
else
  Case $x = 0$.
  ai=0.3550280538878172_sp
  bi=ai/ONOVRT
  aip=-0.2588194037928068_sp
  bip=-aip/ONOVRT
end if
END SUBROUTINE airy

* * *

SUBROUTINE sphbes_s(n,x,sj,sy,sjp,syp)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : bessjy
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: sj,sy,sjp,syp

Returns spherical Bessel functions $j_n(x)$, $y_n(x)$, and their derivatives $j'_n(x)$, $y'_n(x)$ for integer $n \geq 0$ and $x > 0$.

REAL(SP), PARAMETER :: RTP102=1.253314137315500_sp
REAL(SP) :: factor,order,rj,ry,rjp,ryp
call assert(n >= 0, x > 0.0, 'sphbes_s arg)
order=n*0.5_sp
call bessjy(x,order,rj,ry,rjp,ryp)
factor=RTP102/sqrt(x)
sj=factor*rj
sy=factor*ry
Note that sphbes_v uses (through overloading) bessjy_v. The listing of that routine was omitted above, but it is on the machine-readable media.

FUNCTION plgndr_s(l,m,x)
USE nrtype; USE nrutil, ONLY : arth, assert
IMPLICIT NONE
INTEGER(I4B) :: l,m
REAL(SP) :: x
REAL(SP) :: plgndr_s
Computes the associated Legendre polynomial \( P^m_l(x) \). Here \( m \) and \( l \) are integers satisfying \( 0 \leq m \leq l \), while \( x \) lies in the range \(-1 \leq x \leq 1\).

INTEGER(I4B) :: ll
REAL(SP) :: plm,pm1,x2,somx2
if (m > 0) then
  somx2=sqrt((1.0_sp-x)*(1.0_sp+x))
  pm1=product(arth(1.0_sp,2.0_sp,m))*somx2**m
  if (mod(m,2) == 1) pm1=-pm1
end if
if (l == m) then
  plgndr_s=pm1
else
  pm2=x*(2*m+1)*pm1
  if (l == m+1) then
    plgndr_s=pm2
  else
    do ll=m+2,l
      pm2=x*(2*ll-1)*pm2-(ll+m-1)*pm1)/(ll-m)
      pm1=pm2
    end do
    plgndr_s=pm1
end if
FUNCTION plgndr_s
USE nrtype; USE nrutil, ONLY : arth, assert
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: l,m
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: plgndr_s
END FUNCTION plgndr_s

FUNCTION plgndr_v(l,m,x)
USE nrtype; USE nrutil, ONLY : arth, assert
IMPLICIT NONE
INTEGER(SP), DIMENSION(1:3), INTENT(IN) :: l,m
REAL(SP), DIMENSION(size(x)) :: x
REAL(SP), DIMENSION(size(x)) :: plgndr_v
INTEGER(I4B) :: ll
REAL(SP) :: pmmp1,pmm,somx2
call assert(m >= 0, m <= l, all(abs(x) <= 1.0), 'plgndr_vargs')
pmm=1.0
if (m > 0) then
  somx2=sqrt((1.0_sp-x)*(1.0_sp+x))
pmm=product(arth(1.0_sp,2.0_sp,m))*somx2**m
  if (mod(m,2) == 1) pmm=-pmm
end if
if (l == m) then
  plgndr_v=pmm
else
  pmmp1=x*(2*m+1)*pmm
  if (l == m+1) then
    plgndr_v=pmmp1
  else
    do ll=m+2,l
      pll=(x*(2*ll-1)*pmmp1-(ll+m-1)*pmm)/(ll-m)
pmm=pmmp1
      pmmp1=pll
    end do
    plgndr_v=pll
  end if
end if
END FUNCTION plgndr_v

All those if's (not where's) may strike you as odd in a vector routine, but it is vectorized only on x, the dependent variable, not on the scalar indices l and m. Much harder to write a routine that is parallel for a vector of arbitrary triplets (l,m,x). Try it!

SUBROUTINE frenel(x,s,c)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: s,c
INTEGER(I4B), PARAMETER :: MAXIT=100
REAL(SP), PARAMETER :: EPS=epsilon(x),FPMIN=tiny(x),BIG=huge(x)*EPS,&
XMIN=1.5
Computes the Fresnel integrals \( S(x) \) and \( C(x) \) for all real \( x \).
Parameters: MAXIT is the maximum number of iterations allowed; EPS is the relative error; FPMIN is a number near the smallest representable floating-point number; BIG is a number

* * *
near the machine overflow limit; XMIN is the dividing line between using the series and
continued fraction.

```
INTEGER(I4B) :: k, n
REAL(SP) :: a, ax, fact, pix2, sign, sum, sumc, sums, term, test
COMPLEX(SPC) :: b, cc, d, h, del, cs
LOGICAL(LGT) :: odd
ax=abs(x)
if (ax < sqrt(FPMIN)) then
  s=0.0
  c=ax
else if (ax <= XMIN) then
  Evaluate both series simultaneously.
  sum=0.0
  sums=0.0
  sumc=ax
  sign=1.0
  fact=PIO2*ax*ax
  odd=.true.
  term=ax
  n=3
  do k=1,MAXIT
    term=term*fact/k
    sum=sum+sign*term/n
    test=abs(sum)*EPS
    if (odd) then
      sign=-sign
      sums=sum
      sum=sumc
    else
      sumc=sum
      sum=sums
    end if
    if (term < test) exit
    odd=.not. odd
    n=n+2
  end do
  if (k > MAXIT) call nrerror('frenel: series failed')
s=sums
  c=sumc
else
  Evaluate continued fraction by modified Lentz’ method
  (
  pix2=PI*ax*ax
  b=cmplx(1.0_sp,-pix2,kind=spc)
  cc=big
  d=1.0_sp/b
  h=d
  n=-1
  do k=2,MAXIT
    a=-n*(n+1)
    b=b+4.0_sp
    d=1.0_sp/(a*d+b)
    cc=b+a/cc
    del=cc*d
    h=h*del
    if (abs(d*del-1.0_sp) <= EPS) exit
  end do
  if (k > MAXIT) call nrerror('cf failed in frenel')
  h=h*cmplx(ax,-ax,kind=spc)
  cs=cmplx(0.5_sp,0.5_sp,kind=spc)*(1.0_sp-&
    cmplx(cos(0.5_sp*pix2),sin(0.5_sp*pix2),kind=spc)*h)
  c=real(cs)
  s=aimag(cs)
end if
else if (x < 0.0) then
  Use antisymmetry.
  c=-c
```
FUNCTION absc(z)
IMPLICIT NONE
COMPLEX(SPC), INTENT(IN) :: z
REAL(SP) :: absc
absc=abs(real(z))+abs(aimag(z))
END FUNCTION absc

END SUBROUTINE frenel

It’s a good idea always to include the kind= parameter when you use the cmplx intrinsic. The reason is that, perhaps counterintuitively, the result of cmplx is not determined by the kind of its arguments, but is rather the “default complex kind.” Since that default may not be what you think it is (or what spc is defined to be), the desired kind should be specified explicitly.

c=real(cs) And why not specify a kind= parameter here, where it is also optionally allowed? Our answer is that the real intrinsic actually merges two different usages. When its argument is complex, it is the counterpart of aimag and returns a value whose kind is determined by the kind of its argument. In fact aimag doesn’t even allow an optional kind parameter, so we never put one in the corresponding use of real. The other usage of real is for “casting,” that is, converting one real type to another (e.g., double precision to single precision, or vice versa). Here we always include a kind parameter, since otherwise the result is the default real kind, with the same dangers mentioned in the previous paragraph.

SUBROUTINE cisi(x,ci,si)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: ci,si
INTEGER(I4B), PARAMETER :: MAXIT=100,
BIG=huge(x)*EPS,TMIN=2.0
REAL(SP), PARAMETER :: EPS=epsilon(x),FPMIN=4.0_sp*tiny(x),&
EULER=

Computes the cosine and sine integrals Ci(x) and Si(x). Ci(0) is returned as a large negative number and no error message is generated. For x < 0 the routine returns Ci(−x) and you must supply the −iπ yourself.

Parameters: MAXIT is the maximum number of iterations allowed; EPS is the relative error, or absolute error near a zero of Ci(x); FPMIN is a number near the smallest representable floating-point number; BIG is a number near the machine overflow limit; TMIN is the dividing line between using the series and continued fraction; EULER = γ (in nrtype).

INTEGER(I4B) :: i,k
REAL(SP) :: a,err,fact,sign,sum,sumc,sums,t,term
COMPLEX(SPC) :: h,b,c,d,del
LOGICAL(LGT) :: odd
t=abs(x)
if (t == 0.0) then Special case.
    si=0.0
    ci=−BIG
RETURN
end if
if (t > TMIN) then Evaluate continued fraction by modified Lentz’s method (§5.2).
    b=cmplx(1.0_sp,−pix2,kind=spc)
c=1
b=1.0
h=d
do i=2,MAXIT
    a=-(i-1)**2
    b=b+2.0
    d=1.0/(a*d+b)
c=b+a/c
    del=c*d
    h=h*del
    if (absc(del-1.0) <= EPS) exit
end do
if (i > MAXIT) call nrerror('continued fraction failed in cisi')
h=cmplx(cos(t),-sin(t),kind=sp)*h
ci=real(h)
si=PI2*aimag(h)
else
    if (t < sqrt(FPMIN)) then
        Special case: avoid failure of convergence test because of underflow.
        sumc=0.0
        sums=t
    else
        sum=0.0
        sums=0.0
        sign=1.0
        fact=1.0
        odd=.true.
        do k=1,MAXIT
            fact=fact*t/k
            term=fact/k
            sum=sum+sign*term
            err=term/abs(sum)
            if (odd) then
                sign=-sign
                sums=sum
                sum=sumc
            else
                sumc=sum
                sum=sums
            end if
            if (err < EPS) exit
            odd=.not. odd
        end do
    if (k > MAXIT) call nrerror('MAXIT exceeded in cisi')
end if
si=sums
ci=sumc+log(t)+EULER
end if
if (x < 0.0) si=-si
CONTAINS
FUNCTION absc(z)
IMPLICIT NONE
COMPLEX(SPC), INTENT(IN) :: z
REAL(SP) :: absc
absc=abs(real(z))+abs(aimag(z))
END FUNCTION absc
END SUBROUTINE cisi

* * *
FUNCTION dawson_s(x)
USE nrtype; USE nrutil, ONLY : arth, geop
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: dawson_s

Returns Dawson's integral $F(x) = \exp(-x^2) \int_0^x \exp(t^2) dt$ for any real $x$.

INTEGER(I4B), PARAMETER :: NMAX=6
REAL(SP), PARAMETER :: H=0.4_sp,A1=2.0_sp/3.0_sp,A2=0.4_sp,&
A3=2.0_sp/7.0_sp
INTEGER(I4B) :: i,n0
REAL(SP) :: ec,x2,xp,xx
REAL(SP), DIMENSION(NMAX) :: d1,d2,e1
REAL(SP), DIMENSION(NMAX), SAVE :: c=(/ (0.0_sp,i=1,NMAX) /)
if (c(1) == 0.0) c(1:NMAX)=exp(-(arth(1,2,NMAX)*H)**2)
Initialize c on first call.
if (abs(x) < 0.2_sp) then
  Use series expansion.
x2=x**2
dawson_s=x*(1.0_sp-A1*x2*(1.0_sp-A2*x2*(1.0_sp-A3*x2)))
else
  Use sampling theorem representation.
  xx=abs(x)
n0=2*nint(0.5_sp*xx/H)
xp=xx-real(n0,sp)*H
e1=arth(n0+1,2,NMAX)
d2=arth(n0-1,-2,NMAX)
dawson_s=0.5641895835477563_sp*sign(exp(-xp**2),x)*&
  Constant is $1/\sqrt{\pi}$.
sum(c*(e1/d1+1.0_sp/(d2*e1)))
end if
END FUNCTION dawson_s

This is one way to give initial values to an array. Actually, we’re somewhat nervous about using the “implied do-loop” form of the array constructor, as above, because our parallel compilers might not always be smart enough to execute the constructor in parallel. In this case, with NMAX=6, the damage potential is quite minimal. An alternative way to initialize the array would be with a data statement, “DATA c /NMAX*0.0_sp/”; however, this is not considered good Fortran 90 style, and there is no reason to think that it would be faster.

FUNCTION dawson_v(x)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: dawson_v
INTEGER(I4B), PARAMETER :: NMAX=6

REAL(SP), DIMENSION(NMAX), SAVE :: c=(/ (0.0_sp,i=1,NMAX) /)
  Another example where the arth function of nrutil comes in handy. Otherwise, this would be
  do i=1,NMAX
    c(i)=exp(-((2.0_sp*i-1.0_sp)*H)**2)
  end do

arth(n0+1,2,NMAX)...arth(n0-1,-2,NMAX)...geop(ec,ec**2,NMAX) These are not just notationally convenient for generating the sequences $(n_0 + 1, n_0 + 3, n_0 + 5, \ldots)$, $(n_0 - 1, n_0 - 3, n_0 - 5, \ldots)$, and $(ec, ec^3, ec^5, \ldots)$. They also may allow parallelization with parallel versions of arth and geop, such as those in nrutil.
REAL(SP), PARAMETER :: H=0.4_sp, A1=2.0_sp/3.0_sp, A2=0.4_sp, &
A3=2.0_sp/7.0_sp
INTEGER(I4B) :: i, n
REAL(SP), DIMENSION(size(x)) :: x2
REAL(SP), DIMENSION(NMAX), SAVE :: c=(/ (0.0_sp, i=1,NMAX) /)
LOGICAL(LGT), DIMENSION(size(x)) :: mask
if (c(1) == 0.0) c(1:NMAX)=exp(-((arth(1,2,NMAX)*H)**2))
mask = (abs(x) >= 0.2_sp)
dawson_v=dawsonseries_v(x,mask)
where (.not. mask)
  x2=x**2
dawson_v=x*(1.0_sp-A1*x2*(1.0_sp-A2*x2*(1.0_sp-A3*x2)))
end where
CONTAINS
FUNCTION dawsonseries_v(xin,mask)
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: xin
LOGICAL(LGT), DIMENSION(size(xin)), INTENT(IN) :: mask
REAL(SP), DIMENSION(:), ALLOCATABLE :: d1,d2,e1,e2,sm,xp,xx,x
n=count(mask)
if (n == 0) RETURN
allocate(n0(n),d1(n),d2(n),e1(n),e2(n),sm(n),xp(n),xx(n),x)
x=pack(xin,mask)
xx=abs(x)
n0=2*nint(0.5_sp*xx/H)
xp=xx-real(n0,sp)*H
e1=exp(2.0_sp*xp*H)
e2=e1**2
d1=n0+1.0_sp

d2=d1-2.0_sp
sm=0.0

do i=1,NMAX
  sm=sm+c(i)*(e1/d1+1.0_sp/(d2*e1))
d1=d1+2.0_sp
  d2=d2-2.0_sp
  e1=e2*e1
end do
sm=0.5641895835477563_sp*sign(exp(-xp**2),x)*sm
dawsonseries_v=unpack(sm,mask,0.0_sp)
deallocate(n0,d1,d2,e1,e2,sm,xp,xx)
END FUNCTION dawsonseries_v
END FUNCTION dawson_v

FUNCTION rf_s(x,y,z)
USE nrtype; USE nrutil, ONLY : assert
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,y,z
REAL(SP) :: rf_s
REAL(SP), PARAMETER :: ERRTOL=0.08_sp, TINY=1.5e-38_sp, BIG=3.0e37_sp, &
THIRD=1.0_sp/3.0_sp, &

Routine for getting masked values. See note to **bessj** above, p. 1102. Within the contained **dawsonseries**, we use the pack-unpack method. Note that, unlike in **dawson_s**, the sums are done by do-loops, because the parallelization is already over the components of the vector argument.

FUNCTION rf_s(x,y,z)
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C1=1.0_sp/24.0_sp,C2=0.1_sp,C3=3.0_sp/44.0_sp,C4=1.0_sp/14.0_sp

Computes Carlson’s elliptic integral of the first kind, \( R_F(x, y, z) \). \( x, y, \) and \( z \) must be nonnegative, and at most one can be zero. TINY must be at least 5 times the machine underflow limit, BIG at most one-fifth the machine overflow limit.

REAL(SP) :: alamb, ave, delx, dely, delz, e2, e3, sqrtx, sqrty, sqrtz, xt, yt, zt
call assert(min(x,y,z) >= 0.0, min(x+y,x+z,y+z) >= TINY, &
max(x,y,z) <= BIG, 'rf_s args')

xt=x
yt=y
zt=z
do
  sqrtx=sqrt(xt)
sqrty=sqrt(yt)
sqrtz=sqrt(zt)
  alamb=sqrtx*(sqrty+sqrtz)+sqrty*sqrtz
  xt=0.25_sp*(xt+alamb)
yt=0.25_sp*(yt+alamb)
  zt=0.25_sp*(zt+alamb)
  ave=THIRD*(xt+yt+zt)
  delx=(ave-xt)/ave
dely=(ave-yt)/ave
delz=(ave-zt)/ave
  if (max(abs(delx),abs(dely),abs(delz)) <= ERRTOL) exit
end do
e2=delx*dely-delz**2
e3=delx*dely*delz
rf_s=(1.0_sp+(C1*e2-C2-C3*e3)*e2+C4*e3)/sqrt(ave)

END FUNCTION rf_s

FUNCTION rf_v(x,y,z)
USE nrtype; USE nrutil, ONLY : assert, assert_eq

IMPLICIT NONE

REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,z
REAL(SP), DIMENSION(size(x)) :: rf_v
REAL(SP), PARAMETER :: ERRTOL=0.08_sp,TINY=1.5e-38_sp,BIG=3.0e37_sp,&
THIRD=1.0_sp/3.0_sp,&
C1=1.0_sp/24.0_sp,C2=0.1_sp,C3=3.0_sp/44.0_sp,C4=1.0_sp/14.0_sp
REAL(SP), DIMENSION(size(x)) :: alamb, ave, delx, dely, delz, e2, e3,&
sqrtx, sqrty, sqrtz, xt, yt, zt
LOGICAL(LGT), DIMENSION(size(x)) :: converged
INTEGER(I4B) :: ndum
ndum=assert_eq(size(x),size(y),size(z),'rf_v')
call assert(all(min(x,y,z) >= 0.0), all(min(x+y,x+z,y+z) >= TINY), &
all(max(x,y,z) <= BIG, 'rf_v args')
xt=x
yt=y
zt=z
converged=.false.
do
  where (.not. converged)
    sqrtx=sqrt(xt)
    sqrty=sqrt(yt)
    sqrtz=sqrt(zt)
    alamb=sqrtx*(sqrty+sqrtz)+sqrty*sqrtz
    xt=0.25_sp*(xt+alamb)
    yt=0.25_sp*(yt+alamb)
    zt=0.25_sp*(zt+alamb)
    ave=THIRD*(xt+yt+zt)
    delx=(ave-xt)/ave
dely=(ave-yt)/ave
delz=(ave-zt)/ave
    converged = (max(abs(delx),abs(dely),abs(delz)) <= ERRTOL)
FUNCTION rd_s(x,y,z)
USE nrtype; USE nrutil, ONLY : assert
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,y,z
REAL(SP) :: rd_s
REAL(SP), PARAMETER :: ERRTOL=0.05_sp,TINY=1.0e-25_sp,BIG=4.5e21_sp,&
                  C1=3.0_sp/14.0_sp,C2=1.0_sp/6.0_sp,C3=9.0_sp/22.0_sp,&
                  C4=3.0_sp/26.0_sp,C5=0.25_sp*C3,C6=1.5_sp*C4

Computes Carlson's elliptic integral of the second kind, $R_\mathrm{D}(x,y,z)$. $x$ and $y$ must be nonnegative, and at most one can be zero. $z$ must be positive. TINY must be at least twice the negative 2/3 power of the machine underflow limit. BIG must be at most 0.1 \times ERRTOL times the negative 2/3 power of the machine overflow limit.

REAL(SP) :: alamb,ave,delx,dely,delz,ea,eb,ec,ed,&
            ee,fac,sqrtx,sqrty,sqrtz,sum,xt,yt,zt
call assert(min(x,y) >= 0.0, min(x+y,z) >= TINY, max(x,y,z) <= BIG, &
            'rd_s args')
xt=x
yt=y
zt=z
sum=0.0
fac=1.0
do
  sqrtx=sqrt(xt)
sqrty=sqrt(yt)
sqrtz=sqrt(zt)
alamb=sqrtx*(sqrty+sqrtz)+sqrty*sqrtz
sum=sum+fac/(sqrtz*(zt+alamb))
fac=0.25_sp*fac
xt=0.25_sp*(xt+alamb)
yt=0.25_sp*(yt+alamb)
zt=0.25_sp*(zt+alamb)
ave=0.2_sp*(xt+y+3.0_sp*zt)
delx=(ave-x)/ave
dely=(ave-y)/ave
delz=(ave-z)/ave
if (max(abs(delx),abs(dely),abs(delz)) <= ERRTOL) exit
end do

EA=delx*dely
eb=delz*delz
ec=ea-eb
ed=ea-6.0_sp*eb
EE=e+ec+ec
rd_s=(1.0_sp*(1-C1*ED+6*EB+EE)+EB+EC)/(AVE*SQRT(AVE))
END FUNCTION rd_s
FUNCTION rd_v(x,y,z)
USE nrtype; USE nrutil, ONLY : assert, assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y,z
REAL(SP), DIMENSION(size(x)) :: rd_v
REAL(SP), PARAMETER :: ERRTOL=0.05_sp,TINY=1.0e-25_sp,BIG=4.5e21_sp,&
C1=3.0_sp/14.0_sp,C2=1.0_sp/6.0_sp,C3=9.0_sp/22.0_sp,&
C4=3.0_sp/26.0_sp,C5=0.25_sp*C3,C6=1.5_sp*C4
REAL(SP), DIMENSION(size(x)) :: alamb,ave,delx,dely,delz,ea,eb,ec,ed,&
ee,fac,sqrtx,sqrty,sqrtz,sum,xt,yt,zt
LOGICAL(LGT), DIMENSION(size(x)) :: converged
INTEGER(I4B) :: ndum
ndum=assert_eq(size(x),size(y),size(z),'rd_v')
call assert(all(min(x,y) >= 0.0), all(min(x+y,z) >= TINY), &
all(max(x,y,z) <= BIG), 'rd_vargs')
xt=x
yt=y
zt=z
sum=0.0
fac=1.0
converged=.false.
do
where (.not. converged)
sqrtx=sqrt(xt)
sqtrty=sqrt(yt)
sqrtz=sqrt(zt)
alamb=sqrtx*(sqrty+sqrtz)+sqrty*sqrtz
sum=sum+fac/(sqrtz*(zt+alamb))
fac=0.25_sp*fac
xt=0.25_sp*(xt+alamb)
yt=0.25_sp*(yt+alamb)
zt=0.25_sp*(zt+alamb)
ave=0.2_sp*(xt+yt+3.0_sp*zt)
delx=(ave-xt)/ave
dely=(ave-yt)/ave
delz=(ave-zt)/ave
converged = (all(max(abs(delx),abs(dely),abs(delz)) <= ERRTOL))
end where
if (all(converged)) exit
end do
ea=delx*dely
eb=delz*delz
ec=ea-eb
ed=ea-6.0_sp*eb
ee=ed+ec
rd_v=3.0_sp*sum+fac*(1.0_sp+ed*(-C1+C5*ed-C6*delz*ee)&
+delz*(C2*ee+delz*(-C3*ec+delz*C4*ea)))/(ave*sqrt(ave))
END FUNCTION rd_v

FUNCTION rj_s(x,y,z,p)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : rc,rf
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,y,z,p
REAL(SP) :: rj_s
REAL(SP), PARAMETER :: ERRTOL=0.05_sp,TINY=2.5e-13_sp,BIG=9.0e11_sp,&
C1=3.0_sp/14.0_sp,C2=1.0_sp/3.0_sp,C3=3.0_sp/22.0_sp,&
C4=3.0_sp/26.0_sp,C5=0.75_sp*C3,C6=1.5_sp*C4,C7=0.5_sp*C2,&
C8=C3+C3
Computes Carlson's elliptic integral of the third kind, \( R_J(x,y,z,p) \). \( x, y, \) and \( z \) must be nonnegative, and at most one can be zero. \( p \) must be nonzero. If \( p < 0 \), the Cauchy
principal value is returned. TINY must be at least twice the cube root of the machine underflow limit, BIG at most one-fifth the cube root of the machine overflow limit.

REAL(SP) :: a, alamb, alpha, ave, b, bet, delp, delx, & 
dely, delz, ea, eb, ec, ed, ee, fac, pt, rho, sqrtx, sqrtty, sqrtz, & 
sm, tau, xt, yt, zt
call assert(min(x,y,z) >= 0.0, min(x+y,x+z,y+z,abs(p)) >= TINY, & 
max(x,y,z,abs(p)) <= BIG, 'rj_s args')
sm=0.0
fac=1.0
if (p > 0.0) then
  xt=x
  yt=y
  zt=z
  pt=p
else
  xt=min(x,y,z)
  zt=max(x,y,z)
  yt=x+y+z-xt-zt
  a=1.0_sp/(yt-p)
  b=a*(zt-xt)*(yt-xt)
  pt=yt+b
  rho=xt*zt/yt
  tau=p*pt/yt
endif

do
  sqrtx=sqrt(xt)
  sqrtty=sqrt(yt)
  sqrtz=sqrt(zt)
  alamb=sqrtx*(sqrtty+sqrtz)+sqrtty*sqrtz
  alpha=(pt*(sqrtx+sqrtty+sqrtz)+sqrtx*sqrtty*sqrtz)**2
  bet=pt*(pt+alamb)**2
  sm=sm+fac*rc(alpha,bet)
  fac=0.25_sp*fac
  xt=0.25_sp*(xt+alamb)
  yt=0.25_sp*(yt+alamb)
  zt=0.25_sp*(zt+alamb)
  pt=0.25_sp*(pt+alamb)
  ave=0.2_sp*(xt+y+y+z+pt+pt)
  delx=(ave-xt)/ave
  dely=(ave-yt)/ave
  delz=(ave-zt)/ave
  delp=(ave-pte)/ave
  if (max(abs(delx),abs(dely),abs(delz),abs(delp)) <= ERRTOL) exit
endo

eaa=delx*(dely+delz)+dely*delz
  eb=delx*dely+delz
  ec=delp**2
  ed=ea=3.0_sp*ec
  ee=eb=2.0_sp*delp*(ea-ec)
  rj_s=s+(1.0_sp*ed*(-C1+C5*ed-C6*ee)+eb*(C7+delp*(-C8 & 
+delp*C4)+delp*ea*(C2-delp*C3)-C2*delp*ec)/(ave*sqrt(ave))
  if (p <= 0.0) rj_s=a*(b*rj_s+3.0_sp*(rc(rho,tau)-rf(xt,yt,zt)))
END FUNCTION rj_s

FUNCTION rj_v(x,y,z,p)
USE nrtype; USE nrutil, ONLY : assert, assert_eq
USE nr, ONLY : rc, rf
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,z,p
REAL(SP), DIMENSION(size(x)) :: rj_v
REAL(SP), PARAMETER :: ERRTOL=0.05_sp,TINY=2.5e-13_sp,BIG=9.0e11_sp,& 
C1=3.0_sp/14.0_sp,C2=1.0_sp/3.0_sp,C3=3.0_sp/22.0_sp,&
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C4=3.0_sp/26.0_sp,C5=0.75_sp*C3,C6=1.5_sp*C4,C7=0.5_sp*C2,&
C8=C3*C3

REAL(SP), DIMENSION(size(x)) :: a,alamb,alpha,ave,b,bet,delp,delx,&
dely,delz,ea,eb,ec,ed,ee,fac,pt,rho,sqrtx,sqrty,sqrtz,&
sm,tau,xt,zt

LOGICAL(LGT), DIMENSION(size(x)) :: mask

INTEGER(I4B) :: ndum

ndum=assert_eq(size(x),size(y),size(z),size(p),'rj_v')
call assert(all(min(x,y,z) >= 0.0), all(min(x+y,x+z,y+z,abs(p)) >= TINY), &
all(max(x,y,z,abs(p)) <= BIG), 'rj_v args')
sm=0.0

fac=1.0

where (p > 0.0)

xt=x

yt=y

zt=z

pt=p

elsewhere

xt=min(x,y,z)

zt=max(x,y,z)

yt=x+y+z-xt-zt

a=1.0_sp/(yt-p)

b=a*(zt-yt)*(yt-xt)

pt=yt+b

rho=xt*zt/yt

tau=p*pt/yt

end where

mask=.false.
do

where (.not. mask)

sqrtx=sqrt(xt)

sqrty=sqrt(yt)

sqrtz=sqrt(zt)

alamb=sqrtx*(sqrty+sqrtz)+sqrty*sqrtz

alpha=(pt*(sqrtx+sqrty+sqrtz)+sqrtx*sqrty*sqrtz)**2

bet=pt*(pt+alamb)**2

sm=sm+fac*rc(alpha,bet)

fac=0.25_sp*fac

xt=0.25_sp*(xt+alamb)

yt=0.25_sp*(yt+alamb)

zt=0.25_sp*(zt+alamb)

pt=0.25_sp*(pt+alamb)

ave=0.2_sp*(xt+yt+zt+pt+pt)

delx=(ave-xt)/ave

dely=(ave-yt)/ave

delz=(ave-zt)/ave

delp=(ave-pt)/ave

mask = (max(abs(delx),abs(dely),abs(delz),abs(delp)) <= ERRTOL)

do

if (all(mask)) exit

end do

end where

end do

if (all(mask)) exit

end do

ea=delx*(dely+delz)+dely+delz

eb=delx+dely+delz

ec=delp+2

ed=ea-3.0_sp*sec

ee=eb+2.0_sp*delp*(ea-ec)

rj_v=3.0_sp*sm+fac*pack((1.0_sp+ed*(-C1+C5*ed-C6*ee)+ee*(-C8&
+delp*C3))&
delp*(C2-delp*C3)-C2*delp*ec)/(ave*sqrt(ave))

mask = (p <= 0.0)

where (mask) rj_v=as(h*rj_v&

unpack(3.0_sp*rc(pack(rho,mask),pack(tau,mask))&

rf(pack(xt,mask),pack(yt,mask),pack(zt,mask)),mask,0.0_sp))

END FUNCTION rj_v
If you're willing to put up with fairly unreadable code, you can use the pack-unpack trick (for getting a masked subset of components out of a vector function) right in-line, as here. Of course the “outer level” that is seen by the enclosing where construction has to contain only objects that have the same shape as the mask that goes with the where. Because it is so hard to read, we don't like to do this very often. An alternative would be to use CONTAINS to incorporate short, masked “wrapper functions” for the functions used in this way.

FUNCTION rc_s(x,y)
USE nrtype; USE nrutil, ONLY : assert
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,y
REAL(SP) :: rc_s
REAL(SP), PARAMETER :: ERRTOL=0.04_sp,TINY=1.69e-38_sp,&
SQRTNY=1.3e-19_sp,BIG=3.0e37_sp,TNBG=TINY*BIG,&
COMP1=2.236_sp/SQRTNY,COMP2=TNBG*TNBG/25.0_sp,&
THIRD=1.0_sp/3.0_sp,&
C1=0.3_sp,C2=1.0_sp/7.0_sp,C3=0.375_sp,C4=9.0_sp/22.0_sp
Computes Carlson’s degenerate elliptic integral, \( RC(x, y) \). \( x \) must be nonnegative and \( y \) must be nonzero. If \( y < 0 \), the Cauchy principal value is returned. TINY must be at least 5 times the machine underflow limit, BIG at most one-fifth the machine maximum overflow limit.
REAL(SP) :: alamb,ave,s,w,xt,yt
call assert( (/x >= 0.0,y /= 0.0,x+abs(y) >= TINY,x+abs(y) <= BIG, &
y >= -COMP1 .or. x <= 0.0 .or. x >= COMP2/),’rc_s’)
if (y > 0.0) then
xt=x
yt=y
w=1.0
else
xt=x-y
yt=-y
w=sqrt(x)/sqrt(xt)
end if
do
alamb=2.0_sp*sqrt(xt)*sqrt(yt)+yt
xt=0.25_sp*(xt+alamb)
yt=0.25_sp*(yt+alamb)
ave=THIRD*(xt*yt+yt)
s=(yt-ave)/ave
if (abs(s) <= ERRTOL) exit
end do
rc_s=w*(1.0_sp+s*s*(C1+s*(C2+s*(C3+s*C4))))/sqrt(ave)
END FUNCTION rc_s

FUNCTION rc_v(x,y)
USE nrtype; USE nrutil, ONLY : assert,assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y
REAL(SP), DIMENSION(size(x)) :: rc_v
REAL(SP), PARAMETER :: ERRTOL=0.04_sp,TINY=1.69e-38_sp,&
SQRTNY=1.3e-19_sp,BIG=3.0e37_sp,TNBG=TINY*BIG,&
COMP1=2.236_sp/SQRTNY,COMP2=TNBG*TNBG/25.0_sp,&
THIRD=1.0_sp/3.0_sp,&
C1=0.3_sp,C2=1.0_sp/7.0_sp,C3=0.375_sp,C4=9.0_sp/22.0_sp
REAL(SP), DIMENSION(size(x)) :: alamb,ave,s,w,xt,yt
LOGICAL(LGT), DIMENSION(size(x)) :: converged
INTEGER(I4B) :: ndum
ndum=assert_eq(size(x),size(y),'rc_v')
call assert((all(x >= 0.0),all(y /= 0.0),all(x+abs(y) >= TINY), &
   all(x+abs(y) <= BIG),all(y >= -COMP1 .or. x <= 0.0 &
   .or. x >= COMP2) ),'rc_v')
where (y > 0.0)
   xt=x
   yt=y
   w=1.0
elsewhere
   xt=x-y
   yt=-y
   w=sqrt(x)/sqrt(xt)
called
where converged=.false.
do
   where (.not. converged)
      alamb=2.0_sp*sqrt(xt)*sqrt(yt)+yt
      xt=0.25_sp*(xt+alamb)
      yt=0.25_sp*(yt+alamb)
      ave=THIRD*(xt+yt+yt)
      s=(yt-ave)/ave
      converged = (abs(s) <= ERRTOL)
   end where
   if (all(converged)) exit
end do
rc_v=w*(1.0_sp+s*s*(C1+s*(C2+s*(C3+s*C4))))/sqrt(ave)
END FUNCTION rc_v

FUNCTION ellf_s(phi,ak)
USE nrtype
USE nr, ONLY : rf
IMPLICIT NONE
REAL(SP), INTENT(IN) :: phi,ak
REAL(SP) :: ellf_s
Legendre elliptic integral of the 1st kind $F(\phi, k)$, evaluated using Carlson’s function $R_F$.
The argument ranges are $0 \leq \phi \leq \pi/2$, $0 \leq k \sin \phi \leq 1$.
REAL(SP) :: s
s=sin(phi)
elff_s=s*rf(cos(phi)**2,(1.0_sp-s*ak)*(1.0_sp+s*ak),1.0_sp)
END FUNCTION ellf_s

FUNCTION ellf_v(phi,ak)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : rf
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: phi,ak
REAL(SP), DIMENSION(size(phi)) :: ellf_v
REAL(SP), DIMENSION(size(phi)) :: s
INTEGER(I4B) :: ndum
ndum=assert_eq(size(phi),size(ak),'ellf_v')
s=sin(phi)
ellf_v=s*rf(cos(phi)**2,(1.0_sp-s*ak)*(1.0_sp+s*ak), &
   spread(1.0_sp,1,size(phi)))
END FUNCTION ellf_v
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FUNCTION elle_s(phi,ak)
USE nrtype
USE nr, ONLY : rd,rf
IMPLICIT NONE
REAL(SP), INTENT(IN) :: phi,ak
REAL(SP) :: elle_s
Legendre elliptic integral of the 2nd kind $E(\phi, k)$, evaluated using Carlson’s functions $R_D$ and $R_F$. The argument ranges are $0 \leq \phi \leq \pi/2$, $0 \leq k \sin \phi \leq 1$.
REAL(SP) :: cc,q,s
s=sin(phi)
cc=cos(phi)**2
q=(1.0_sp-s*ak)*(1.0_sp+s*ak)
elle_s=s*(rf(cc,q,1.0_sp)-((s*ak)**2)*rd(cc,q,1.0_sp)/3.0_sp)
END FUNCTION elle_s

FUNCTION elle_v(phi,ak)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : rd,rf
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: phi,ak
REAL(SP), DIMENSION(size(phi)) :: elle_v
REAL(SP), DIMENSION(size(phi)) :: cc,q,s
INTEGER(I4B) :: ndum
ndum=assert_eq(size(phi),size(ak),'elle_v')
s=sin(phi)
cc=cos(phi)**2
q=(1.0_sp-s*ak)*(1.0_sp+s*ak)
elle_v=s*(rf(cc,q,spread(1.0_sp,1,size(phi)))-((s*ak)**2)*&
rj(cc,q,spread(1.0_sp,1,size(phi)))/3.0_sp)
END FUNCTION elle_v

FUNCTION ellpi_s(phi,en,ak)
USE nrtype
USE nr, ONLY : rf,rj
IMPLICIT NONE
REAL(SP), INTENT(IN) :: phi,en,ak
REAL(SP) :: ellpi_s
Legendre elliptic integral of the 3rd kind $\Pi(\phi, n, k)$, evaluated using Carlson’s functions $R_J$ and $R_F$. (Note that the sign convention on $n$ is opposite that of Abramowitz and Stegun.)
The ranges of $\phi$ and $k$ are $0 \leq \phi \leq \pi/2$, $0 \leq k \sin \phi \leq 1$.
REAL(SP) :: cc,enss,q,s
s=sin(phi)
enss=en*s*s
cc=cos(phi)**2
q=(1.0_sp-s*ak)*(1.0_sp+s*ak)
ellpi_s=s*(rf(cc,q,1.0_sp)-enss*rj(cc,q,1.0_sp,1.0_sp+enss))/3.0_sp)
END FUNCTION ellpi_s

\textbf{90} \texttt{rd(cc,q,spread(1.0_sp,1,size(phi)))} \text{ See note to erf_v, p. 1094 above.}

FUNCTION ellpi_s(phi,en,ak)
USE nrtype
USE nr, ONLY : rf,rj
IMPLICIT NONE
REAL(SP), INTENT(IN) :: phi,en,ak
REAL(SP) :: ellpi_s
Legendre elliptic integral of the 3rd kind II($\phi, n, k$), evaluated using Carlson’s functions $R_J$ and $R_F$. (Note that the sign convention on $n$ is opposite that of Abramowitz and Stegun.)
The ranges of $\phi$ and $k$ are $0 \leq \phi \leq \pi/2$, $0 \leq k \sin \phi \leq 1$.
REAL(SP) :: cc,enss,q,s
s=sin(phi)
enss=en*s*s
cc=cos(phi)**2
q=(1.0_sp-s*ak)*(1.0_sp+s*ak)
ellpi_s=s*(rf(cc,q,1.0_sp)-enss*rj(cc,q,1.0_sp,1.0_sp+enss))/3.0_sp)
END FUNCTION ellpi_s
FUNCTION ellpi_v(phi, en, ak)
USE nrtype
USE nr, ONLY : rf, rj
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: phi, en, ak
REAL(SP), DIMENSION(size(phi)) :: ellpi_v
REAL(SP), DIMENSION(size(phi)) :: cc, enss, q, s
s = sin(phi)
enss = en * s * s
cc = cos(phi)**2
q = (1.0_sp - s * ak) * (1.0_sp + s * ak)
ellpi_v = s * (rf(cc, q, spread(1.0_sp, 1, size(phi))) - enss * 
rj(cc, q, spread(1.0_sp, 1, size(phi)), 1.0_sp + enss) / 3.0_sp)
END FUNCTION ellpi_v

SUBROUTINE sncndn(uu, emmc, sn, cn, dn)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: uu, emmc
REAL(SP), INTENT(OUT) :: sn, cn, dn
Returns the Jacobian elliptic functions sn(u, k_c), cn(u, k_c), and dn(u, k_c). Here uu = u,
while emmc = k^2.
REAL(SP), PARAMETER :: CA = 0.0003_sp
The accuracy is the square of CA.
INTEGER(I4B), PARAMETER :: MAXIT = 13
INTEGER(I4B) :: i, ii, l
REAL(SP) :: a, b, c, d, emc, u
REAL(SP), DIMENSION(MAXIT) :: em, en
LOGICAL(LGT) :: bo
emc = emmc
u = uu
if (emc /= 0.0) then
  bo = (emc < 0.0)
  if (bo) then
    d = 1.0_sp - emc
    emc = -emc / d
    d = sqrt(d)
    u = d * u
  end if
  a = 1.0
  dn = 1.0
  do i = 1, MAXIT
    l = i
    em(i) = a
    emc = sqrt(emc)
    en(i) = emc
    c = 0.5_sp * (a * emc)
    if (abs(a - emc) <= CA * a) exit
    emc = a * emc
    a = c
  end do
  if (i > MAXIT) call nrerror('sncndn: convergence failed')
  u = c * u
  sn = sin(u)
  cn = cos(u)
  if (sn /= 0.0) then
    a = cn / sn
    c = a * c
    do ii = 1, 1, -1
      b = em(ii)
    end do
  end if
end if
\begin{verbatim}
a=c*a
  c=dn*c
dn=(en(ii)+a)/(b+a)
a=c/b
end do
  a=1.0_sp/sqrt(c**2+1.0_sp)
sn=sign(a,sn)
cn=c*sn
end if
if (bo) then
  a=dn
dn=cn
cn=a
  sn=sn/d
end if
else
  cn=1.0_sp/cosh(u)
dn=cn
  sn=tanh(u)
end if
END SUBROUTINE sncndn

⋆⋆⋆

MODULE hypgeo_info
USE nrtype
COMPLEX(SPC) :: hypgeo_aa,hypgeo_bb,hypgeo_cc,hypgeo_dz,hypgeo_z0
END MODULE hypgeo_info

FUNCTION hypgeo(a,b,c,z)
USE nrtype
USE hypgeo_info
USE nr, ONLY : bsstep,hypdrv,hypser,odeint
IMPLICIT NONE
COMPLEX(SPC), INTENT(IN) :: a,b,c,z
COMPLEX(SPC) :: hypgeo
REAL(SP), PARAMETER :: EPS=1.0e-6_sp
Complex hypergeometric function \( \, _2F_1 \) for complex \( a, b, c \), and \( z \), by direct integration of
the hypergeometric equation in the complex plane. The branch cut is taken to lie along the
real axis, \( \text{Re} \, z > 1 \).
Parameter: EPS is an accuracy parameter.
COMPLEX(SPC), DIMENSION(2) :: y
REAL(SP), DIMENSION(4) :: ry
if (real(z)**2+aimag(z)**2 <= 0.25) then
  Use series...
  call hypser(a,b,c,z,hypgeo,y(2))
  RETURN
else if (real(z) < 0.0) then
  ...or pick a starting point for the path
  hypgeo_z0=cmplx(-0.5_sp,0.0_sp,kind=spc)
else if (real(z) <= 1.0) then
  hypgeo_z0=cmplx(0.5_sp,0.0_sp,kind=spc)
else
  hypgeo_z0=cmplx(0.0_sp,sign(0.5_sp,aimag(z)),kind=spc)
end if
hypgeo_aa=a
  Load the module variables, used to pass
hypgeo_bb=b
  parameters “over the head” of odeint
hypgeo_cc=c
  to hypdrv.
hypgeo_dz=z-hypgeo_z0
call hypser(hypgeo_aa,hypgeo_bb,hypgeo_cc,hypgeo_z0,y(1),y(2))
  Get starting function and derivative.
ry(1:4:2)=real(y)
\end{verbatim}
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ry(2:4:2)=aimag(y)
call odeint(ry,0.0_sp,1.0_sp,eps,0.1_sp,0.0001_sp,hypdrv,bstep)

The arguments to odeint are the vector of independent variables, the starting and ending values of the dependent variable, the accuracy parameter, an initial guess for stepsize, a minimum stepsize, and the names of the derivative routine and the (here Bulirsch-Stoer) stepping routine.

y=complx(ry(1:4:2),ry(2:4:2),kind=spc)
hypgeo=y(1)
END FUNCTION hypgeo

SUBROUTINE hypser(a,b,c,z,series,deriv)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
COMPLEX(SPC), INTENT(IN) :: a,b,c,z
COMPLEX(SPC), INTENT(OUT) :: series,deriv

Returns the hypergeometric series \( \hspace{1pt} \text{\bf \textstyle 2}_{\textstyle 1} \) and its derivative, iterating to machine accuracy. For \( c \, \text{abs}(z) \leq 1/2 \) convergence is quite rapid.

INTEGER(I4B) :: n
INTEGER(I4B), PARAMETER :: MAXIT=1000
COMPLEX(SPC) :: aa,bb,cc,fac,temp
deriv=cmplx(0.0_sp,0.0_sp,kind=spc)
fac=cmplx(1.0_sp,0.0_sp,kind=spc)
temp=fac
aa=a
bb=b
cc=c

DO n=1,MAXIT
   fac=((aa*bb)/cc)*fac
   deriv=deriv+fac
   fac=fac*z/n
   series=temp+fac
   if (series == temp) RETURN
   temp=series
   aa=aa+1.0
   bb=bb+1.0
   cc=cc+1.0
END DO

call nrerror('hypser: convergence failure')
END SUBROUTINE hypser

SUBROUTINE hypdrv(s,ry,rdyds)
USE nrtype
USE hypgeo_info
IMPLICIT NONE
REAL(SP), INTENT(IN) :: s
REAL(SP), DIMENSION(:), INTENT(IN) :: ry
REAL(SP), DIMENSION(:), INTENT(OUT) :: rdyds

Derivative subroutine for the hypergeometric equation; see text equation (5.14.4).
COMPLEX(SPC), DIMENSION(2) :: y,dyds
COMPLEX(SPC) :: z
y=complx(ry(1:4:2),ry(2:4:2),kind=spc)
z=hypgeo_x0z+hypgeo_dz
dyds(1)=y(2)+hypgeo_dz
dyds(2)=((hypgeo_aa*hypgeo_bb)*y(1)-(hypgeo_cc-&(hypgeo_aa+hypgeo_bb)+1.0_sp)*z)*y(2)*hypgeo_dz/(z*(1.0_sp-z))
rdyds(1:4:2)=real(dyds)
rdyds(2:4:2)=imag(dyds)
END SUBROUTINE hypdrv
Notice that the real array (of length 4) \( r_y \) is immediately mapped into a complex array of length 2, and that the process is reversed at the end of the routine with \( r_d \). In Fortran 77 no such mapping is necessary: the calling program sends real arguments, and the Fortran 77 hypdrv simply interprets what is sent as complex. Fortran 90’s stronger typing does not encourage (and, practically, does not allow) this convenience; but it is a small price to pay for the vastly increased error-checking capabilities of a strongly typed language.
Chapter B7. Random Numbers

One might think that good random number generators, including those in Volume 1, should last forever. The world of computing changes very rapidly, however:

- When Volume 1 was published, it was unusual, except on the fastest supercomputers, to “exhaust” a 32-bit random number generator, that is, to call for all \(2^{32}\) sequential random values in its periodic sequence. Now, this is feasible, and not uncommon, on fast desktop workstations. A useful generator today must have a minimum of 64 bits of state space, and generally somewhat more.
- Before Fortran 90, the Fortran language had no standardized calling sequence for random numbers. Now, although there is still no standard algorithm defined by the language (rightly, we think), there is at least a standard calling sequence, exemplified in the intrinsics `random_number` and `random_seed`.
- The rise of parallel computing places new algorithmic demands on random generators. The classic algorithms, which compute each random value from the previous one, evidently need generalization to a parallel environment.
- New algorithms and techniques have been discovered, in some cases significantly faster than their predecessors.

These are the reasons that we have decided to implement, in Fortran 90, different uniform random number generators from those in Volume 1’s Fortran 77 implementations. We hasten to add that there is nothing wrong with any of the generators in Volume 1. That volume’s `ran0` and `ran1` routines are, to our knowledge, completely adequate as 32-bit generators; `ran2` has a 64-bit state space, and our previous offer of $1000 for any demonstrated failure in the algorithm has never yet been claimed (see [1]).

Before we launch into the discussion of parallelizable generators with Fortran 90 calling conventions, we want to attend to the continuing needs of longtime “\(x=\text{ran}(\text{idum})\)” users with purely serial machines. If you are a satisfied user of Volume 1’s `ran0`, `ran1`, or `ran2` Fortran 77 versions, you are in this group. The following routine, `ran`, preserves those routines’ calling conventions, is considerably faster than `ran2`, and does not suffer from the old `ran0` or `ran1’s 32-bit period exhaustion limitation. It is completely portable to all Fortran 90 environments. We recommend `ran` as the plug-compatible replacement for the old `ran0`, `ran1`, and `ran2`, and we happily offer exactly the same $1000 reward terms as were (and are still) offered on the old `ran2`.  

1141
FUNCTION ran(idum)
IMPLICIT NONE
INTEGER, PARAMETER :: K4B=selected_int_kind(9)
INTEGER(K4B), INTENT(INOUT) :: idum
REAL :: ran

"Minimal" random number generator of Park and Miller combined with a Marsaglia
shift sequence. Returns a uniform random deviate between 0.0 and 1.0 (exclusive of
the endpoint values). This fully portable, scalar generator has the "traditional" (not
Fortran 90) calling sequence with a random deviate as the returned function value:
call with idum a negative integer to initialize; thereafter, do not alter idum except to
reinitialize. The period of this generator is about $3.1 \times 10^{18}$.

INTEGER(K4B), PARAMETER :: IA=16807,IM=2147483647,IQ=127773,IR=2836
REAL, SAVE :: am
INTEGER(K4B), SAVE :: ix=-1,iy=-1,k
if (idum <= 0 .or. iy < 0) then
    Initialize.
    am=nearest(1.0,-1.0)/IM
    iy=ieor(ieor(888889999,abs(idum)),1)
    ix=ieor(777755566,abs(idum))
    idum=abs(idum)+1
end if
ix=ieor(ix,ishft(ix,13))
ix=ieor(ix,ishft(ix,-17))
ix=ieor(ix,ishft(ix,5))
k=iy/IQ
iy=IA*(iy-k*IQ)-IR*k
if (iy < 0) iy=iy+IM
ran=am*ior(iand(IM,ieor(ix,iy)),1)

END FUNCTION ran

This is a good place to discuss a new bit of algorithmics that has crept into ran,
above, and even more strongly affects all of our new random number generators,
below. Consider:

\[
\begin{align*}
ix &= \text{ieor}(ix,\text{ishft}(ix,13)) \\
ix &= \text{ieor}(ix,\text{ishft}(ix,-17)) \\
ix &= \text{ieor}(ix,\text{ishft}(ix,5))
\end{align*}
\]

These lines update a 32-bit integer ix, which cycles pseudo-randomly through a full
period of $2^{32} - 1$ values (excluding zero) before repeating. Generators of this type
have been extensively explored by Marsaglia (see [2]), who has kindly communicated
some additional results to us in advance of publication. For convenience, we will
refer to generators of this sort as "Marsaglia shift registers."

Useful properties of Marsaglia shift registers are (i) they are very fast on most
machines, since they use only fast logical operations, and (ii) the bit-mixing that they
induce is quite different in character from that induced by arithmetic operations such
as are used in linear congruential generators (see Volume 1) or lagged Fibonacci
generators (see below). Thus, the combination of a Marsaglia shift register with
another, algorithmically quite different generator is a powerful way to suppress any
residual correlations or other weaknesses in the other generator. Indeed, Marsaglia
finds (and we concur) that the above generator (with constants 13, −17, 5, as shown)
is by itself about as good as any 32-bit random generator.

Here is a very brief outline of the theory behind these generators: Consider the
32 bits of the integer as components in a vector of length 32, in a linear space where
addition and multiplication are done modulo 2. Noting that exclusive-or (\text{ieor})
is the same as addition, each of the three lines in the updating can be written as the
action of a $32 \times 32$ matrix on a vector, where the matrix is all zeros except for
ones on the diagonal, and on exactly one super- or subdiagonal (corresponding to positive or negative second arguments in \texttt{ishft}). Denote this matrix as $S_k$, where $k$ is the shift argument. Then, one full step of updating (three lines of code, above) corresponds to multiplication by the matrix $T \equiv S_k S_2 S_3$.

One next needs to find triples of integers $(k_1, k_2, k_3)$, for example $(13, -17, 5)$, that give the full $M = 2^{32} - 1$ period. Necessary and sufficient conditions are that $T^M = I$ (the identity matrix), and that $T^N \neq I$ for these five values of $N$: $N = 3 \times 5 \times 17 \times 257$, $N = 3 \times 5 \times 17 \times 65537$, $N = 3 \times 5 \times 257 \times 65537$, $N = 3 \times 17 \times 257 \times 65537$, $N = 5 \times 17 \times 257 \times 65537$. (Note that each of the five prime factors of $M$ is omitted one at a time to get the five values of $N$.) The required large powers of $T$ are readily computed by successive squarings, requiring only on the order of $32^3 \log M$ operations. With this machinery, one can find full-period triples $(k_1, k_2, k_3)$ by exhaustive search, at reasonable cost.

Not all such triples are equally good as generators of random integers, however. Marsaglia subjects candidate values to a battery of tests for randomness, and we have ourselves applied various tests. This stage of winnowing is as much art as science, because all 32-bit generators can be made to exhibit signs of failure due to period exhaustion (if for no other reason). “Good” triples, in order of our preference, are $(13, -17, 5)$, $(5, -13, 6)$, $(5, -9, 7)$, $(13, -17, 15)$, $(16, -7, 11)$. When a full-period triple is good, its reverse is also full-period, and also generally good. A good quadruple due to Marsaglia (generalizing the above in the obvious way) is $(-4, 8, -1, 5)$. We would not recommend relying on any single Marsaglia shift generator (nor on any other simple generator) by itself. Two or more generators, of quite different types, should be combined [1].

Let us now discuss explicitly the needs of parallel random number generators. The general scheme, from the user’s perspective, is that ofFortran 90’s intrinsic \texttt{random\_number}: A statement like \texttt{call ran1(harvest)} (where \texttt{ran1} will be one of our portable replacements for the compiler-dependent \texttt{random\_number}) should fill the real array \texttt{harvest} with pseudo-random real values in the range $(0, 1)$. Of course, we want the underlying machinery to be completely parallel, that is, no do-loops of order $N \equiv \text{size(harvest)}$.

A first design decision is whether to replicate the state-space across the parallel dimension $N$, i.e., whether to reserve storage for essentially $N$ scalar generators. Although there are various schemes that avoid doing this (e.g., mapping a single, smaller, state space into $N$ different output values on each call), we think that it is a memory cost well worth paying in return for achieving a less exotic (and thus better tested) algorithm. However, this choice dictates that we must keep the state space \textit{per component} quite small. We have settled on five or fewer 32-bit words of state space per component as a reasonable limit. Some otherwise interesting and well tested methods (such as Knuth’s subtractive generator, implemented in Volume 1 as \texttt{ran3}) are ruled out by this constraint.

A second design decision is how to initialize the parallel state space, so that different parallel components produce different sequences, and so that there is an acceptable degree of randomness \textit{across} the parallel dimension, as well as \textit{between successive calls} of the generator. Each component starts its life with one and only one unique identifier, its component index $n$ in the range $1 \ldots N$. One is
tempted simply to hash the values \( n \) into the corresponding components of initial state space. “Random” hashing is a bad idea, however, because different \( n \)'s will produce identical 32-bit hash results by chance when \( N \) is no larger than \( \sim 2^{16} \). We therefore prefer to use a kind of reversible pseudo-encryption (similar to the routine \texttt{psdes} in Volume 1 and below) which guarantees causally that different \( n \)'s produce different state space initializations.

The machinery for allocating, deallocating, and initializing the state space, including provision of a user interface for getting or putting the contents of the state space (as in the intrinsic \texttt{random seed}) is fairly complicated. Rather than duplicate it in each different random generator that we provide, we have consolidated it in a single module, \texttt{ran state}, whose contents we will now discuss. Such a discussion is necessarily technical, if not arcane; on first reading, you may wish to skip ahead to the actual new routines \texttt{ran0}, \texttt{ran1}, and \texttt{ran2}. If you do so, you will need to know only that \texttt{ran state} provides each vector random routine with five 32-bit vectors of state information, denoted \texttt{iran}, \texttt{jran}, \texttt{kran}, \texttt{mran}, \texttt{nran}. (The overloaded scalar generators have five corresponding 32-bit scalars, denoted \texttt{iran0}, etc.)

\section*{MODULE ran state}

This module supports the random number routines \texttt{ran0}, \texttt{ran1}, \texttt{ran2}, and \texttt{ran3}. It provides each generator with five integers (for vector versions, five vectors of integers), for use as internal state space. The first three integers (\texttt{iran}, \texttt{jran}, \texttt{kran}) are maintained as nonnegative values, while the last two (\texttt{mran}, \texttt{nran}) have 32-bit nonzero values. Also provided by this module is support for initializing or reinitializing the state space to a desired standard sequence number, hashing the initial values to random values, and allocating and deallocating the internal workspace.

\begin{verbatim}
USE nrtype
IMPLICIT NONE
INTEGER, PARAMETER :: K4B=selected_int_kind(9)
Independent of the usual integer kind \texttt{I4B}, we need a kind value for (ideally) 32-bit integers.
INTEGER(K4B), PARAMETER :: hg=huge(1_K4B), hgm=-hg, hgng=hgm-1
INTEGER(K4B), SAVE :: lenran=0, seq=0
INTEGER(K4B), SAVE :: iran0,jran0,kran0,mran0,nran0,ran0
INTEGER(K4B), DIMENSION(:,::), POINTER, SAVE :: ranseeds
INTEGER(K4B), DIMENSION(:,), POINTER, SAVE :: iran,jran,kran, &
	nran,mran,ranv
REAL(SP), SAVE :: amm
INTERFACE ran_hash <Scalar and vector versions of the hashing procedure.
   MODULE PROCEDURE ran_hash_s, ran_hash_v
END INTERFACE
CONTAINS

(We here intersperse discussion with the listing of the module.) The module defines \texttt{K4B} as an integer \texttt{KIND} that is intended to be 32 bits. If your machine doesn’t have 32-bit integers (hard to believe!) this will be caught later, and an error message generated. The definition of the parameters \texttt{hg}, \texttt{hgm}, and \texttt{hgng} makes an assumption about 32-bit integers that goes beyond the strict Fortran 90 integer model, that the magnitude of the most negative representable integer is greater by one than that of the most positive representable integer. This is a property of the \texttt{two’s complement arithmetic} that is used on virtually all modern machines (see, e.g., [3]).

The global variables \texttt{rans} (for scalar) and \texttt{ranv} (for vector) are used by all of our routines to store the \texttt{integer} value associated with the most recently returned call. You can access these (with a “USE \texttt{ran state}” statement) if you want integer, rather than real, random deviates.
The first routine, \texttt{ran\_init}, is called by routines later in the chapter to initialize their state space. It is \textit{not} intended to be called from a user’s program.

\begin{verbatim}
SUBROUTINE ran_init(length)
USE nrtyp; USE nrutil, ONLY : arth,nrerror,reallocate
IMPLICIT NONE
INTEGER(K4B), INTENT(IN) :: length
Initialize or reinitialize the random generator state space to vectors of size length. The saved variable \texttt{seq} is hashed (via calls to the module routine \texttt{ran\_hash}) to create unique starting seeds, different for each vector component.

INTEGER(K4B) :: new,j,hgt
if (length < lenran) RETURN
Simply return if enough space is already allocated.

hgt=hg
The following lines check that kind value K4B is in fact a 32-bit integer with the usual properties that we expect it to have (under negation and wrap-around addition). If all of these tests are satisfied, then the routines that use this module are portable, even though they go beyond Fortran 90’s integer model.

if (hg /= 2147483647) call nrerror(’ran\_init: arith assump 1 fails’)
if (hgt+1 /= hgng) call nrerror(’ran\_init: arith assump 2 fails’)
if (not(hg) >= 0) call nrerror(’ran\_init: arith assump 3 fails’)
if (not(hgng) < 0) call nrerror(’ran\_init: arith assump 4 fails’)
if (hg+hgng >= 0) call nrerror(’ran\_init: arith assump 5 fails’)
if (not(-1_k4b) < 0) call nrerror(’ran\_init: arith assump 6 fails’)
if (not(0_k4b) >= 0) call nrerror(’ran\_init: arith assump 7 fails’)
if (not(1_k4b) >= 0) call nrerror(’ran\_init: arith assump 8 fails’)
if (lenran > 0) then
Reallocate space, or ...
ranseeds=>reallocate(ranseeds,length,5)
ranv=>reallocate(ranv,length-1)
new=lenran+1
else
allocate(ranseeds(length,5))
allocate(ranv(length-1))
new=1
Index of first location not yet initialized.

amm=nearest(1.0_sp,-1.0_sp)/hgng
Use of \texttt{nearest} is to ensure that returned random deviates are strictly less than 1.0.
if (amm*hgng >= 1.0 .or. amm*hgng <= 0.0) &
call nrerror(’ran\_init: arith assump 10 fails’)
end if
Set starting values, unique by \texttt{seq} and vector component.
ranseeds(new.:,1)=seq
ranv(new.:,2:5)=spread(arth(new,1,size(ranseeds(new.:,1))),2,4)
do j=1,4
Hash them.
call ran\_hash(ranseeds(new:,j),ranseeds(new:,j+1))
end do
where (ranseeds(new:,1:3) < 0) &
Enforce nonnegativity.
ranseeds(new:,1:3)=not(ranseeds(new:,1:3))

where (ranseeds(new:,4:5) == 0) ranseeds(new:,4:5)=1
Enforce nonzero.
if (new == 1) then
Set scalar seeds.
iran0=ranseeds(1,1)
ir0=ranseeds(1,2)
kran0=ranseeds(1,3)

mran0=ranseeds(1,4)
nran0=ranseeds(1,5)
rans=nran0
end if
if (length > 1) then
Point to vector seeds.
iran => ranseeds(2:,1)
jran => ranseeds(2:,2)
kran => ranseeds(2:,3)
mran => ranseeds(2:,4)
nran => ranseeds(2:,5)
ranv = nran
\end{verbatim}
end if
lenran=length
END SUBROUTINE ran_init

hgt=hg ... if (hgt+1 /= hgng)  
   Bit of dirty laundry here! We are testing 
   whether the most positive integer hg wraps around to the most negative 
   integer hgng when 1 is added to it. We can’t just write hg+1, since some 
   compilers will evaluate this at compile time and return an overflow error message. 
   If your compiler sees through the charade of the temporary variable hgt, you’ll 
   have to find another way to trick it.

amm=nearest(1.0_sp,-1.0_sp)/hgng...  
   Logically, amm should be a parameter; 
   but the nearest intrinsic is trouble-prone in the initialization expression for a 
   parameter (named constant), so we compute this at run time. We then check that amm, 
   when multiplied by the largest possible negative integer, does not equal or exceed 
   unity. (Our random deviates are guaranteed never to equal zero or unity exactly.)

   You might wonder why amm is negative, and why we multiply it by negative 
   integers to get positive random deviates. The answer, which will become manifest 
   in the random generators given below, is that we want to use the fast not operation 
   on integers to convert them to nonzero values of all one sign. This is possible if the 
   conversion is to negative values, since not(i) is negative for all nonnegative i. If 
   the conversion were to positive values, we would have problems both with zero (its 
   sign bit is already positive) and hgng (since not(hgng) is generally zero).

   iran0=ranseeds(1,1) ...  
   iran => ranseeds(2:,1)...

The initial state information is stored in ranseeds, a two-dimensional array whose 
column (second) index ranges from 1 to 5 over the state variables. ranseeds(1,:) is 
reserved for scalar random generators, while ranseeds(2:, :) is for vector-parallel 
generators. The ranseeds array is made available to vector generators through 
the pointers iran, jran, kran, mran, and nran. The corresponding scalar values, 
iran0..., nran0 are simply global variables, not pointers, because the overhead of 
addressing a scalar through a pointer is often too great. (We will have to copy these 
scalar values back into ranseeds when it, rarely, needs to be addressed as an array.)

call ran_hash(...)  
   Unique, and random, initial state information is obtained 
   by putting a user-seettalbe “sequence number” into iran, a component number into 
   jran, and hashing this pair. Then jran and kran are hashed, kran and mran 
   are hashed, and so forth.

SUBROUTINE ran_deallocate  
   User interface to release the workspace used by the random number routines.
   if (lenran > 0) then  
      deallocate(ranseeds,ranv)  
      nullify(ranseeds,ranv,iran,jran,kran,mran,nran)  
      lenran = 0  
   end if
END SUBROUTINE ran_deallocate

The above routine is supplied as a user interface for deallocating all the state 
space storage.
SUBROUTINE ran_seed(sequence, size, put, get)
IMPLICIT NONE
INTEGER, OPTIONAL, INTENT(IN) :: sequence
INTEGER, OPTIONAL, INTENT(INOUT) :: size
INTEGER, DIMENSION(:), OPTIONAL, INTENT(IN) :: put
INTEGER, DIMENSION(:), OPTIONAL, INTENT(OUT) :: get

User interface for seeding the random number routines. Syntax is exactly like Fortran 90’s random_seed routine, with one additional argument keyword: sequence, set to any integer value, causes an immediate new initialization, seeded by that integer.

if (present(size)) then
  size=5*lenran
else if (present(put)) then
  if (lenran == 0) RETURN
  ranseeds=reshape(put,shape(ranseeds))
  where (ranseeds(:,1:3) < 0) ranseeds(:,1:3)=not(ranseeds(:,1:3))
  Enforce nonnegativity and nonzero conditions on any user-supplied seeds.
  where (ranseeds(:,4:5) == 0) ranseeds(:,4:5)=1
  iran0=ranseeds(1,1)
  jran0=ranseeds(1,2)
  kran0=ranseeds(1,3)
  mran0=ranseeds(1,4)
  nran0=ranseeds(1,5)
else if (present(get)) then
  if (lenran == 0) RETURN
  ranseeds(1,1:5)=(/ iran0,jran0,kran0,mran0,nran0 /)
  get=reshape(ranseeds,shape(get))
else if (present(sequence)) then
  call ran_deallocate
  seq=sequence
end if
END SUBROUTINE ran_seed

Fortran 90’s convention is that random state space is a one-dimensional array, so we map to this on both the get and put keywords.

iran0=...jran0=...kran0=...
nran0=ranseeds(1,1:5)=(/ iran0,jran0,kran0,mran0,nran0 /)

It’s much more convenient to set a vector from a bunch of scalars then the other way around.

SUBROUTINE ran_hash_s(il,ir)
IMPLICIT NONE
INTEGER(K4B), INTENT(INOUT) :: il,ir

DES-like hashing of two 32-bit integers, using shifts, xor’s, and adds to make the internal nonlinear function.

INTEGER(K4B) :: is,j
do j=1,4
  is=ir
  ir=ieor(ir,ishft(is,5)) +1422217823
  ir=ieor(ir,ishft(is,-16)) +1842055030
  ir=ieor(ir,ishft(is,9)) +80567781
  ir=ieor(il,ir)
end do
END SUBROUTINE ran_hash_s
SUBROUTINE ran_hash_v(il, ir)
IMPLICIT NONE
INTEGER(K4B), DIMENSION(:), INTENT(INOUT) :: il, ir
Vector version of ran_hash.
INTEGER(K4B), DIMENSION(size(il)) :: is
INTEGER(K4B) :: j
do j=1,4
is=ir
ir=ieor(ir,ishft(ir,5))+1422217823
ir=ieor(ir,ishft(ir,-16))+1842055030
ir=ieor(ir,ishft(ir,9))+80567781
ir=ieor(il,ir)
end do
END SUBROUTINE ran_hash_v
END MODULE ran_state

The lines
\[
\begin{align*}
\text{ir}&=\text{ir}\oplus\text{ishft}(\text{ir},5)+1422217823 \\
\text{ir}&=\text{ir}\oplus\text{ishft}(\text{ir},-16)+1842055030 \\
\text{ir}&=\text{ir}\oplus\text{ishft}(\text{ir},9)+80567781
\end{align*}
\]
are not a Marsaglia shift sequence, though they resemble one. Instead, they
implement a fast, nonlinear function on \text{ir} that we use as the “S-box” in a DES-like
hashing algorithm. (See Volume 1, §7.5.) The triplet \((5, -16, 9)\) is not chosen to
give a full period Marsaglia sequence — it doesn’t. Instead it is chosen as being
particularly good at separating in Hamming distance (i.e., number of nonidentical
bits) two initially close values of \text{ir} (e.g., differing by only one bit). The large
integer constants are chosen by a similar criterion. Note that the wrap-around
of addition without generating an overflow error condition, which was tested in
\text{ran_init}, is relied upon here.

** ** *

SUBROUTINE ran0_s(harvest)
USE nrtype
USE ran_state, ONLY: K4B, amm, lenran, ran_init, iran0, kran0, nran0, rans
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: harvest
Lagged Fibonacci generator combined with a Marsaglia shift sequence. Returns as \text{harvest}
a uniform random deviate between 0.0 and 1.0 (exclusive of the endpoint values). This gen-
erator has the same calling and initialization conventions as Fortran 90’s \text{random_number}
routine. Use \text{ran_seed} to initialize or reinitialize to a particular sequence. The period of
this generator is about \(2 \times 10^{28}\), and it fully vectorizes. Validity of the integer model
assumed by this generator is tested at initialization.
if (lenran < 1) call ran_init(1)
rans=iran0-kran0
if (rans < 0) rans=rans+2147483579_k4b
iran0=iran0
jran0=kran0
kran0=rans
nran0=ieor(nran0,ishft(nran0,13))
nran0=ieor(nran0,ishft(nran0,-17))
nran0=ieor(nran0,ishft(nran0,5))
rans=ieor(nran0,rans)
harvest=amm*merge(rans,not(rans), rans<0 )
END SUBROUTINE ran0_s
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SUBROUTINE ran0_v(harvest)
USE nrtype
USE ran_state, ONLY: K4B,amm,lenran,ran_init,iran,jran,kran,nran,ranv
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(OUT) :: harvest
INTEGER(K4B) :: n
n=size(harvest)
if (lenran < n+1) call ran_init(n+1)
ranv(1:n)=iran(1:n)-kran(1:n)
where (ranv(1:n) < 0) ranv(1:n)=ranv(1:n)+2147483579_k4b
iran(1:n)=jran(1:n)
jran(1:n)=kran(1:n)
kran(1:n)=ranv(1:n)
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),13))
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),-17))
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),5))
ranv(1:n)=ieor(nran(1:n),ranv(1:n))
harvest=amm*merge(ranv(1:n),not(ranv(1:n)), ranv(1:n)<0 )
END SUBROUTINE ran0_v

This is the simplest, and fastest, of the generators provided. It combines a
subtractive Fibonacci generator (Number 6 in ref. [1], and one of the generators
in Marsaglia and Zaman's mzran) with a Marsaglia shift sequence. On typical
machines it is only 20% or so faster than ran1, however; so we recommend the
latter preferentially. While we know of no weakness in ran0, we are not offering
a prize for finding a weakness. ran0 does have the feature, useful if you have
a machine with nonstandard arithmetic, that it does not go beyond Fortran 90's
assumed integer model.

Note that ran0_s and ran0_v are overloaded by the module nr onto the single
name ran0 (and similarly for the routines below).

SUBROUTINE ran1_s(harvest)
USE nrtype
USE ran_state, ONLY: K4B,amm,lenran,ran_init, &
iran0,jran0,kran0,nran0,mran0,rans
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: harvest
Lagged Fibonacci generator combined with two Marsaglia shift sequences. On output, re-
turns as harvest a uniform random deviate between 0.0 and 1.0 (exclusive of the endpoint
values). This generator has the same calling and initialization conventions as Fortran 90's
random_number routine. Use ran_seed to initialize or reinitialize to a particular sequence.
The period of this generator is about $8.5 \times 10^{37}$, and it fully vectorizes. Validity of the integer
model assumed by this generator is tested at initialization.

if (lenran < 1) call ran_init(1)
ran0=iran0-kran0
if (rans < 0) rans=rans+2147483579_k4b
iran0=jran0
jran0=kran0
kran0=rans
nran0=ieor(nran0,ishft(nran0,13))
nran0=ieor(nran0,ishft(nran0,-17))
nran0=ieor(nran0,ishft(nran0,5))
Once only per cycle, advance sequence by 1, shortening its period to $2^{32} - 2$.
if (nran0 == 1) nran0=270369_k4b
mran0=ieor(mran0,ishft(mran0,6))
mran0=ieor(mran0,ishft(mran0,-13))
mran0=ieor(mran0,ishft(mran0,5))

Initialization routine in ran_state. Update Fibonacci generator, which has period $p^2 + p + 1$, $p = 2^{31} - 69$.
Update Marsaglia shift sequence.
Update Marsaglia shift sequence with period $2^{32} - 1$. 

* * *

SUBROUTINE ran2_s(harvest)
USE nrtype
USE ran_state, ONLY: K4B,amm,lenran,ran_init, &
iran0,jran0,kran0,nran0,mran0,ranv
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: harvest
Lagged Fibonacci generator combined with two Marsaglia shift sequences. On output, re-
turns as harvest a uniform random deviate between 0.0 and 1.0 (exclusive of the endpoint
values). This generator has the same calling and initialization conventions as Fortran 90's
random_number routine. Use ran_seed to initialize or reinitialize to a particular sequence.
The period of this generator is about $8.5 \times 10^{37}$, and it fully vectorizes. Validity of the integer
model assumed by this generator is tested at initialization.

if (lenran < 1) call ran_init(1)
ran0=iran0-kran0
if (rans < 0) rans=rans+2147483579_k4b
iran0=jran0
jran0=kran0
kran0=rans
nran0=ieor(nran0,ishft(nran0,13))
nran0=ieor(nran0,ishft(nran0,-17))
nran0=ieor(nran0,ishft(nran0,5))
Once only per cycle, advance sequence by 1, shortening its period to $2^{32} - 2$.
if (nran0 == 1) nran0=270369_k4b
mran0=ieor(mran0,ishft(mran0,6))
mran0=ieor(mran0,ishft(mran0,-13))
mran0=ieor(mran0,ishft(mran0,5))

Initialization routine in ran_state. Update Fibonacci generator, which has period $p^2 + p + 1$, $p = 2^{31} - 69$.
Update Marsaglia shift sequence.
Update Marsaglia shift sequence with period $2^{32} - 1$. 

* * *
Combining the generators. The above statement has wrap-around addition.

The routine `ran1` combines three fast generators: the two used in `ran0`, plus an additional (different) Marsaglia shift sequence. The last generator is combined via an addition that can wrap-around.

We think that, within the limits of its floating-point precision, `ran1` provides perfect random numbers. We will pay $1000 to the first reader who convinces us otherwise (by exhibiting a statistical test that `ran1` fails in a nontrivial way, excluding the ordinary limitations of a floating-point representation).

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nran0=ieor(nran0,ishft(nran0,13))  
Update Marsaglia shift sequence with  
period $2^{32} - 1$.
nran0=ieor(nran0,ishft(nran0,-17))
nran0=ieor(nran0,ishft(nran0,5))

ran0=land(mran0,65535)  
Wrap-around addition (tested at initialization) is used.

$\text{ran0} = \text{ishft}(\text{mran0}, -16) + \text{ranv}(1:n) + \text{ranv}(1:n) * 3533$  
Combine the generators.

$\text{harvest} = \text{amm} * \text{merge}(\text{ranv}(1:n), \text{not} (\text{ranv}(1:n)), \text{ranv}(1:n) < 0)$  
Make the result positive definite (note that amm is negative).

SUBROUTINE ran2_v(harvest)
USE nrtype
USE ran_state, ONLY: K4B, amm, lenran, ran_init, &
iran, jran, kran, nran, mran, ranv
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(OUT) :: harvest
INTEGER(K4B) :: n
n=size(harvest)
if (lenran < n+1) call ran_init(n+1)
ranv(1:n)=iran(1:n)-kran(1:n)
where (ranv(1:n) < 0) ranv(1:n)=ranv(1:n)+2147483579_k4b
iran(1:n)=jran(1:n)
jran(1:n)=kran(1:n)
kran(1:n)=ranv(1:n)
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),13))
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),-17))
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),5))
ranv(1:n)=land(mran(1:n),65535)

$\text{ranv}(1:n) = \text{ishft}(\text{mran}(1:n), -16) + \text{ranv}(1:n) * 3533 + \text{ranv}(1:n) * 82065819_k4b$
$ravn(1:n)=ieor(nran(1:n), kran(1:n)) * mran(1:n)$

$\text{harvest} = \text{amm} * \text{merge}(\text{ranv}(1:n), \text{not} (\text{ranv}(1:n)), \text{ranv}(1:n) < 0 )$
END SUBROUTINE ran2_v

$\text{ran2}$, for use by readers whose caution is extreme, also combines three generators. The difference from $\text{ran1}$ is that each generator is based on a completely different method from the other two. The third generator, in this case, is a linear congruential generator, modulo $2^{32}$. This generator relies extensively on wrap-around addition (which is automatically tested at initialization). On machines with fast arithmetic, $\text{ran2}$ is on the order of only 20% slower than $\text{ran1}$. We offer a $1000$ bounty on $\text{ran2}$, with the same terms as for $\text{ran1}$, above.

SUBROUTINE expdev_s(harvest)
USE nrtype
USE nr, ONLY : ran1
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: harvest

$\text{dum} = \text{ran1}$
$\text{harvest} = - \log (\text{dum})$
We use the fact that ran1 never returns exactly 0 or 1.
END SUBROUTINE expdev_s
SUBROUTINE expdev_v(harvest)
USE nrtype
USE nr, ONLY : ran1
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(OUT) :: harvest
REAL(SP), DIMENSION(size(harvest)) :: dum
call ran1(dum)
harvest=-log(dum)
END SUBROUTINE expdev_v

The only noteworthy thing about this line is its simplicity: Once all the machinery is in place, the random number generators are self-initializing (to the sequence defined by seq = 0), and (via overloading) usable with both scalar and vector arguments.

SUBROUTINE gasdev_s(harvest)
USE nrtype
USE nr, ONLY : ran1
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: harvest
Returns in harvest a normally distributed deviate with zero mean and unit variance, using ran1 as the source of uniform deviates.
REAL(SP) :: rsq,v1,v2
REAL(SP), SAVE :: g
LOGICAL, SAVE :: gaus_stored=.false.
if (gaus_stored) then
    harvest=g
    gaus_stored=.false.
else
    do
        call ran1(v1)
        v1=2.0_sp*v1-1.0_sp
        call ran1(v2)
        v2=2.0_sp*v2-1.0_sp
        rsq=v1**2+v2**2
        if (rsq > 0.0 .and. rsq < 1.0) exit
    end do
    rsq=sqrt(-2.0_sp*log(rsq)/rsq)
    harvest=v1*rsq
    g=v2*rsq
    gaus_stored=.true.
end if
END SUBROUTINE gasdev_s

SUBROUTINE gasdev_v(harvest)
USE nrtype; USE nrutil, ONLY : array_copy
USE nr, ONLY : ran1
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(OUT) :: harvest
REAL(SP), DIMENSION(size(harvest)) :: rsq,v1,v2
REAL(SP), ALLOCATABLE, DIMENSION(:), SAVE :: g
INTEGER(I4B) :: n,ng,nn,m
INTEGER(I4B), SAVE :: last_allocated=0
LOGICAL, SAVE :: gaus_stored=.false.
LOGICAL, DIMENSION(size(harvest)) :: mask
n=size(harvest)
if (n /= last_allocated) then
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if (last_allocated /= 0) deallocate(g)
allocate(g(n))
last_allocated=n
gaus_stored=.false.
end if
if (gaus_stored) then
harvest=g
gaus_stored=.false.
else
ng=1
do
if (ng > n) exit
call ran1(v1(ng:n))
call ran1(v2(ng:n))
v1(ng:n)=2.0_sp*v1(ng:n)-1.0_sp
v2(ng:n)=2.0_sp*v2(ng:n)-1.0_sp
rsq(ng:n)=v1(ng:n)**2+v2(ng:n)**2
mask(ng:n)=(rsq(ng:n)>0.0 .and. rsq(ng:n)<1.0)
call array_copy(pack(v1(ng:n),mask(ng:n)),v1(:),nn,m)
v2(ng:ng+nn-1)=pack(v2(ng:n),mask(ng:n))
rsq(ng:ng+nn-1)=pack(rsq(ng:n),mask(ng:n))
ng=ng+nn
end do
rsq=sqrt(-2.0_sp*log(rsq)/rsq)
harvest=v1*rsq
g=v2*rsq
gaus_stored=.true.
end if
END SUBROUTINE gasdev_v

if (n /= last_allocated) ... We make the assumption that, in most cases, the size of harvest will not change between successive calls.
Therefore, if it does change, we don't try to save the previously generated deviates that, half the time, will be around. If your use has rapidly varying sizes (or, even worse, calls alternating between two different sizes), you should remedy this inefficiency in the obvious way.

call array_copy(pack(v1(ng:n),mask(ng:n)),v1(:),nn,m) This is a variant of the pack-unpack method (see note to factrl, p. 1087). Different here is that we don't care which random deviates end up in which component. Thus, we can simply keep packing successful returns into v1 and v2 until they are full.

Note also the use of array_copy, since we don't know in advance the length of the array returned by pack.

FUNCTION gamdev(ia)
USE nrtype; USE nrutil, ONLY : assert
USE nr, ONLY : ran1
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: ia
REAL(SP) :: gamdev
Returns a deviate distributed as a gamma distribution of integer order ia, i.e., a waiting time to the ia-th event in a Poisson process of unit mean, using ran1 as the source of uniform deviates.
REAL(SP) :: am,e,h,s,x,y,v(2),arr(5)
call assert(ia >= 1, 'gamdev arg')
if (ia < 6) then Use direct method, adding waiting times.

* * *
call ran1(arr(1:ia))
x=-log(product(arr(1:ia)))
else
  do
    call ran1(v)
    v(2)=2.0_sp*v(2)-1.0_sp
    if (dot_product(v,v) > 1.0) cycle
    am=ia-1
    s=sqrt(2.0_sp*am+1.0_sp)
    x=s*y+am
    if (x <= 0.0) cycle
    e=(1.0_sp+y**2)*exp(am*log(x/am)-s*y)
    call ran1(h)
    if (h <= e) exit
  end do
end if

gamdev=x
END FUNCTION gamdev

Why take the \( \log \) of the product instead of the sum of the \( \log \)s? Because \( \log \) is assumed to be slower than multiply.

We don’t have vector versions of the less commonly used deviate generators, gamdev, poidev, and bnldev.

FUNCTION poidev(xm)
USE nrtype
USE nr, ONLY : gammln,ran1
IMPLICIT NONE
REAL(SP), INTENT(IN) :: xm
REAL(SP) :: poidev

Returns as a floating-point number an integer value that is a random deviate drawn from a Poisson distribution of mean \( \text{xm} \), using ran1 as a source of uniform random deviates.

REAL(SP) :: em,harvest,t,y
REAL(SP), SAVE :: alxm,g,oldm=-1.0_sp,sq

oldm is a flag for whether \( \text{xm} \) has changed since last call.

if (xm < 12.0) then
  Use direct method.
  if (xm /= oldm) then
    oldm=xm
    g=exp(-xm)
  end if
  em=-1
  t=1.0
  do
    em=em+1.0_sp
    call ran1(harvest)
    t=t*harvest
    if (t <= g) exit
  end do
else
  Use rejection method.
  if (xm /= oldm) then
    oldm=xm
    sq=sqrt(2.0_sp*xm)
    alxm=log(xm)
    g=xm*alxm-gammln(xm+1.0_sp)
  end if
end if

\[ x = -\log(\text{product}(\text{arr}(1:ia))) \]
do
  call ran1(harvest) y is a deviate from a Lorentzian comparison
  y=tan(PI*harvest) function.
  em=sq*y+xm em is y, shifted and scaled.
  if (em >= 0.0) exit
end do
em=int(em) The trick for integer-valued distributions.
t=0.9*exp(em*alxm-gammln(em+1.0)-g)
  The ratio of the desired distribution to the comparison function; we accept or reject
  by comparing it to another uniform deviate. The factor 0.9 is chosen so that t never
  exceeds 1.
call ran1(harvest)
  if (harvest <= t) exit
end if
poidev=em
END FUNCTION poidev

FUNCTION bnldev(pp,n)
USE nrtype
USE nr, ONLY : gammln,ran1
IMPLICIT NONE
REAL(SP), INTENT(IN) :: pp
INTEGER(I4B), INTENT(IN) :: n
REAL(SP) :: bnldev

Returns as a floating-point number an integer value that is a random deviate drawn from a
binomial distribution of n trials each of probability pp, using ran1 as a source of uniform
random deviates.

INTEGER(I4B) :: j
INTEGER(I4B), SAVE :: nold=-1
REAL(SP) :: am,em,g,h,p,sq,t,y,arr(24)
REAL(SP), SAVE :: pc,plog,pclog,en,oldg,pold=-1.0

Arguments from previous calls.
p=merge(pp,1.0-pp, pp <= 0.5 )
The binomial distribution is invariant under changing pp to 1.-pp,
and also under changing the answer to n minus itself; we'll remember to do this below.
am=n*p
if (n < 25) then
  call ran1(arr(1:n))
bnldev=count(arr(1:n)<p)
else if (am < 1.0) then
  g=exp(-am)
t=1.0
do j=0,n
    call ran1(h)
t=t*h
  if (t < g) exit
end do
bnldev=merge(j,n, j <= n)
else
  if (n /= nold) then
    en=n
    oldg=gammln(en+1.0)
nold=n
  end if
  if (p /= pold) then
    pc=1.0-p
    plog=log(p)
    pclog=log(pc)
pold=p
  end if

This is the mean of the deviate to be produced.
Use the direct method while n is not too large.
This can require up to 25 calls to ran1.

If fewer than one event is expected out of 25
or more trials, then the distribution is quite
accurately Poisson. Use direct Poisson method.

Use the rejection method.
If n has changed, then compute useful quantities.
If p has changed, then compute useful quantities.
The following code should by now seem familiar:

```
end if
sq=sqrt(2.0_sp*am*pc)
do
  call ran1(h)
y=tan(PI*h)
  em=sq*y+am
  if (em < 0.0 .or. em >= en+1.0_sp) cycle  Reject.
  em=int(em)                       Trick for integer-valued distribution.
  t=1.2_sp*sq*(1.0_sp+y**2)*exp(oldg-gammln(em+1.0_sp)-
    gammln(en-em+1.0_sp)+em*plog+(en-em)*pclog)
  call ran1(h)
  if (h <= t) exit  Reject. This happens about 1.5 times per deviate, on average.
end do
bnldev=em
end if
if (p /= pp) bnldev=n-bnldev  Remember to undo the symmetry transformation.
```

The routines `psdes` and `psdes_safe` both perform exactly the same hashing as was done by the Fortran 77 routine `psdes`. The difference is that `psdes` makes assumptions about arithmetic that go beyond the strict Fortran 90 model, while `psdes_safe` makes no such assumptions. The disadvantage of `psdes_safe` is that it is significantly slower, performing most of its arithmetic in double-precision reals that are then converted to integers with Fortran 90's modulo intrinsic.

In fact the nonsafe version, `psdes`, works fine on almost all machines and compilers that we have tried. There is a reason for this: Our assumed integer model is the same as the C language unsigned `int`, and virtually all modern computers and compilers have a lot of C hidden inside. If `psdes` and `psdes_safe` produce identical output on your system for any hundred or so different input values, you can be quite confident about using the faster version exclusively.

At the other end of things, note that in the very unlikely case that your system fails on the `ran_hash` routine in the `ran_state` module (you will have learned this from error messages generated by `ran_init`), you can substitute `psdes_safe` for `ran_hash`: They are plug-compatible.

```f90
SUBROUTINE psdes_s(lword,rword)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: lword,rword
INTEGER(I4B), PARAMETER :: NITER=4
"Pseudo-DES" hashing of the 64-bit word (lword,irword). Both 32-bit arguments are
returned hashed on all bits. Note that this version of the routine assumes properties of
integer arithmetic that go beyond the Fortran 90 model, though they are compatible with
unsigned integers in C.
INTEGER(I4B), DIMENSION(4), SAVE :: C1,C2
DATA C1 /Z'BAA96887',Z'1E17D32C',Z'03BCDC3C',Z'0F33D1B2'/
DATA C2 /Z'4B0F3B58',Z'E874F0C3',Z'6955C5A6',Z'55A7CA46'/
INTEGER(I4B) :: i,ia,ib,iswap,itmph,itmpl
do i=1,NITER
  iswap=rword
  ia=ieor(rword,C1(i))  (noncryptographic) nonlinear function instead of DES's.
  ib=iand(ia,-iswap)
  itmph=iand(itmph,ixor(ia,ib))
  itmpl=iand(itmpl,ixor(ia,ib))
end do
```

Perform niter iterations of DES logic, using a simpler (noncryptographic) nonlinear function instead of DES's. The bit-rich constants C1 and (below) C2 guarantee lots of nonlinear mixing.
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\[ ib = \text{itmpl}^2 \text{not}(\text{itmph}^2) \]
\[ ia = \text{ior}(\text{ishft}(ib,16),\text{l}\text{and}(\text{ishft}(ib,-16),65535)) \]
\[ rword = \text{ieor}(\text{lword}, \text{ieor}(C2(i),ia)+\text{itmpl}\times\text{itmph}) \]
\[ lword = \text{iswap} \]
end do

END SUBROUTINE psdes_s

SUBROUTINE psdes_v(lword,rword)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
INTEGER(I4B), DIMENSION(:), INTENT(INOUT) :: lword,rword
INTEGER(I4B), PARAMETER :: NITER=4
INTEGER(I4B), DIMENSION(4), SAVE :: C1,C2
DATA C1 /'BAA96887','1E17D32C','03BCDC3C','0F33D1B2'/
DATA C2 /'4B0F3B58','EB7F40C3','6955C5A6','55A7CA46'/
INTEGER(I4B), DIMENSION(size(lword)) :: ia,ib,iswap,itmph,itmpl
INTEGER(I4B) :: i
i=assert_eq(size(lword),size(rword),'psdes_v')
do i=1,NITER
   iswap=rword
   ia=ieor(rword,C1(i))
   itmpl=l\text{and}(ia,65535)
   itmph=l\text{and}(\text{ishft}(ia,-16),65535)
   ib=itmpl^2+\text{not}(\text{itmph}^2)
   ia=\text{ior}(\text{ishft}(ib,16),\text{\text{l}\text{and}(\text{ishft}(ib,-16),65535))}
   rword=\text{ieor}(lword,\text{ieor}(C2(i),ia)+\text{itmpl}\times\text{itmph})
   lword=iswap
end do

END SUBROUTINE psdes_v

SUBROUTINE psdes_safe_s(lword,rword)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: lword,rword
INTEGER(I4B), PARAMETER :: NITER=4
   "Pseudo-DES" hashing of the 64-bit word (lword,irword). Both 32-bit arguments are
   returned hashed on all bits. This is a slower version of the routine that makes no assumptions
   outside of the Fortran 90 integer model.
INTEGER(I4B), DIMENSION(4), SAVE :: C1,C2
DATA C1 /'BAA96887','1E17D32C','03BCDC3C','0F33D1B2'/
DATA C2 /'4B0F3B58','EB7F40C3','6955C5A6','55A7CA46'/
INTEGER(I4B) :: i,ia,ib
REAL(DP) :: alo,ahi
do i=1,NITER
   iswap=rword
   ia=ieor(rword,C1(i))
   alo=real(l\text{and}(ia,65535),dp)
   ahi=real(\text{l}\text{and}(\text{ishft}(ia,-16),65535),dp)
   ib=\text{modint}(alo*alo+\text{real}(\text{\text{not}(\text{modint}(ahi*ahi)),dp)))
   ia=\text{ior}(\text{ishft}(ib,16),\text{\text{l}\text{and}(\text{ishft}(ib,-16),65535))}
   rword=\text{ieor}(lword,\text{\text{modint}(\text{\text{real}(\text{ieor}(C2(i),ia),dp)+alo*ahi}))
   lword=iswap
end do

CONTAINS
FUNCTION modint(x)
REAL(DP) :: x
INTEGER(I4B) :: modint
REAL(DP) :: a
a=modulo(x,base)
end function modint
SUBROUTINE psdes_safe_v(lword, rword)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
INTEGER(I4B), DIMENSION(:,), INTENT(INOUT) :: lword, rword
INTEGER(I4B), PARAMETER :: NITER=4
INTEGER(I4B), SAVE :: C1(4), C2(4)
DATA C1 /'BAA96887', '1E17D32C', '03BCDC3C', '0F33D1B2'/
DATA C2 /'4B0F3B58', 'E874F0C3', '6955C5A6', '55A7CA46'/
INTEGER(I4B), DIMENSION(size(lword)) :: ia, ib, iswap
REAL(DP), DIMENSION(size(lword)) :: alo, ahi
INTEGER(I4B) :: i
i=assert_eq(size(lword), size(rword), 'psdes_safe_v')
do i=1, NITER
  iswap=rword
  ia=ieor(rword, C1(i))
  alo=real(iand(ia, 65535), dp)
  ahi=real(iand(ishft(ia, -16), 65535), dp)
  ib=modint(alo*alo+real(not(modint(ahi*ahi)), dp))
  ia=ior(ishft(ib, 16), iand(ishft(ib, -16), 65535))
  rword=ieor(lword, modint(real(ieor(C2(i), ia), dp)+alo*ahi))
  lword=iswap
end do
CONTAINS
FUNCTION modint(x)
REAL(DP), DIMENSION(:,), INTENT(IN) :: x
INTEGER(I4B), DIMENSION(size(x)) :: modint
REAL(DP), DIMENSION(size(x)) :: a
REAL(DP), PARAMETER :: big=huge(modint), base=big+big+2.0_dp
a=modulo(x, base)
where (a > big) a=a-base
modint=nint(a, kind=14b)
END FUNCTION modint
END SUBROUTINE psdes_safe_v

FUNCTION modint(x) This embedded routine takes a double-precision real argument, and returns it as an integer mod $2^{32}$ (correctly wrapping it to negative to take into account that Fortran 90 has no unsigned integers).

SUBROUTINE ran3_s(harvest)
USE nrtype
USE ran_state, ONLY: K4B, amm, lenran, ran_init, ran_hash, mran0, nran0, rans
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: harvest
Random number generation by DES-like hashing of two 32-bit words, using the algorithm ran_hash. Returns as harvest a uniform random deviate between 0.0 and 1.0 (exclusive of the endpoint values).
INTEGER(K4B) :: temp
if (lenran < 1) call ran_init(1)
nran0=ieor(nran0, ishft(nran0, 13))
nran0=ieor(nran0, ishft(nran0, -17))
nran0=ieor(nran0, ishft(nran0, 5))
if (nran0 == 1) nran0=270369_k4b

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rans=rnan0
mran0=ieor(mran0,ishft(mran0,5))
mran0=ieor(mran0,ishft(mran0,-13))
mran0=ieor(mran0,ishft(mran0,6))
temp=rnan0
call ran_hash(temp,rans)
harvest=amm*merge(rans,not(rans), rans<0 )
END SUBROUTINE ran3_s

SUBROUTINE ran3_v(harvest)
USE nrtype
USE ran_state, ONLY: K4B,amm,lenran,ran_init,ran_hash,mran,nran,ranv
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(OUT) :: harvest
INTEGER(K4B), DIMENSION(size(harvest)) :: temp
INTEGER(K4B) :: n
n=size(harvest)
if (lenran < n+1) call ran_init(n+1)
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),13))
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),-17))
nran(1:n)=ieor(nran(1:n),ishft(nran(1:n),5))
where (nran(1:n) == 1) nran(1:n)=270369_k4b
ranv(1:n)=nran(1:n)
mran(1:n)=ieor(mran(1:n),ishft(mran(1:n),5))
mran(1:n)=ieor(mran(1:n),ishft(mran(1:n),-13))
mran(1:n)=ieor(mran(1:n),ishft(mran(1:n),6))
temp=mran(1:n)
call ran_hash(temp,ranv(1:n))
harvest=amm*merge(ranv(1:n),not(ranv(1:n)), ranv(1:n)<0 )
END SUBROUTINE ran3_v

As given, ran3 uses the ran_hash function in the module ran_state as its
DES surrogate. That function is sufficiently fast to make ran3 only about a factor
of 2 slower than our baseline recommended generator ran1. The slower routine
psdes and (even slower) psdes_safe are plug-compatible with ran_hash, and
could be substituted for it in this routine.

FUNCTION irbit1(iseed)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: iseed
END FUNCTION irbit1

FUNCTION irbit1(iseed)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: iseed
END FUNCTION irbit1

As given, ran3 uses the ran_hash function in the module ran_state as its
DES surrogate. That function is sufficiently fast to make ran3 only about a factor
of 2 slower than our baseline recommended generator ran1. The slower routine
psdes and (even slower) psdes_safe are plug-compatible with ran_hash, and
could be substituted for it in this routine.

*   *   *
FUNCTION irbit2(iseed)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: iseed
INTEGER(I4B) :: irbit2

Returns as an integer a random bit, based on the 18 low-significance bits in iseed (which is modified for the next call).

INTEGER(I4B), PARAMETER :: IB1=1, IB2=2, IB5=16, MASK=IB1+IB2+IB5

if (btest(iseed,17)) then
    Change all masked bits, shift, and put 1 into bit 1.
    iseed=ibset(ishft(ieor(iseed,MASK),1),0)
    irbit2=1
else
    Shift and put 0 into bit 1.
    iseed=ibclr(ishft(iseed,1),0)
    irbit2=0
end if

END FUNCTION irbit2

* * *

SUBROUTINE sobseq(x,init)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(OUT) :: x
INTEGER(I4B), OPTIONAL, INTENT(IN) :: init
INTEGER(I4B), PARAMETER :: MAXBIT=30, MAXDIM=6

When the optional integer init is present, internally initializes a set of MAXBIT direction numbers for each of MAXDIM different Sobol' sequences. Otherwise returns as the vector x of length N the next values from N of these sequences. (N must not be changed between initializations.)

REAL(SP), SAVE :: fac
INTEGER(I4B) :: i,im,ipp,j,k,l
INTEGER(I4B), DIMENSION(:,:), ALLOCATABLE:: iu
INTEGER(I4B), SAVE :: in
INTEGER(I4B), DIMENSION(MAXDIM), SAVE :: ip,ix,mdeg
INTEGER(I4B), DIMENSION(MAXDIM*MAXBIT), SAVE :: iv

DATA ip /0,1,1,1,2,1,4/, mdeg /1,2,3,3,4,4/, ix /6*0/
DATA iv /6*1,3,1,3,1,1,5,7,3,3,5,15,11,5,15,13,9,156*0/

if (present(init)) then
    Initialize, don't return a vector.
    ix=0
    in=0
    if (iv(1) /= 1) RETURN
    fac=1.0_sp/2.0_sp**MAXBIT
    allocate(iu(MAXDIM,MAXBIT))
    iu=reshape(iv,shape(iu))
    To allow both 1D and 2D addressing.
    do k=1,MAXDIM
        do j=1,mdeg(k)
            Stored values require only normalization.
            iu(k,j)=iu(k,j)*2**((MAXBIT-j)
        end do
        do j=mdeg(k)+1,MAXBIT
            Use the recurrence to get other values.
            ipp=ip(k)
            i=iu(k,j-mdeg(k))
            i=ieor(i,1/2**mdeg(k))
            do l=mdeg(k)-1,1,-1
                if (btest(ipp,0)) i=ieor(i,iu(k,j-l))
                ipp=ipp/2
            end do
            iu(k,j)=i
        end do
    end do
    iv=reshape(iu,shape(iv))
    deallocate(iu)
end if
else
    Calculate the next vector in the sequence.
    im=in
    do j=1,MAXBIT
        Find the rightmost zero bit.
        if (.not. btest(im,0)) exit
        im=im/2
    end do
    if (j > MAXBIT) call nrerror('MAXBIT too small in sobseq')
    im=(j-1)*MAXDIM
    j=min(size(x),MAXDIM)
    ix(1:j)=ixor(ix(1:j),iv(1+im:j+im))
    XOR the appropriate direction number into each component of the vector and convert
to a floating number.
    x(1:j)=ix(1:j)*fac
    in=in+1
    Increment the counter.
end if
END SUBROUTINE sobseq

if (present(init)) then ... allocate(iu(...)) ... iu=reshape(...)

Wanting to avoid the deprecated EQUIVALENCE statement, we must
reshape iv into a two-dimensional array, then un-reshape it after we
are done. This is done only once, at initialization time, so there is no serious
inefficiency introduced.

*   *   *

SUBROUTINE vegas(region,func,init,ncall,itmx,nprn,tgral,sd,chi2a)
USE nrtype
USE nr, ONLY : ran1
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: region
INTEGER(I4B), INTENT(IN) :: init,ncall,itmx,nprn
REAL(SP), INTENT(OUT) :: tgral,sd,chi2a
INTERFACE
  FUNCTION func(pt,wgt)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), DIMENSION(:,), INTENT(IN) :: pt
    REAL(SP), INTENT(IN) :: wgt
    REAL(SP) :: func
  END FUNCTION func
END INTERFACE
REAL(SP), PARAMETER :: ALPH=1.5_sp,TINY=1.0e-30_sp
INTEGER(I4B), PARAMETER :: MXDIM=10,NDMX=50
Performs Monte Carlo integration of a user-supplied d-dimensional function func over a
rectangular volume specified by region, a vector of length 2d consisting of d "lower left"
coordinates of the region followed by d "upper right" coordinates. The integration consists of
itmx iterations, each with approximately ncall calls to the function. After each iteration
the grid is refined; more than 5 or 10 iterations are rarely useful. The input flag init
signals whether this call is a new start, or a subsequent call for additional iterations (see
comments below). The input flag nprn (normally 0) controls the amount of diagnostic
output. Returned answers are tgral (the best estimate of the integral), sd (its standard
deviation), and chi2a (χ² per degree of freedom, an indicator of whether consistent results
are being obtained). See text for further details.

INTEGER(I4B),SAVE :: i,it,j,k,mds,nd,ndim,ndo,ng,npg
    Best make everything static,
INTEGER(I4B),DIMENSION(MXDIM),SAVE :: ia,kg
    allowing restarts.
REAL(SP),SAVE :: calls,dv2g,dxg,f,f2,f2b,f2b,rc,ti,tsi,wgt,xjac,xn,xnd,xo,harvest
REAL(SP),DIMENSION(NDMX,MXDIM),SAVE :: d,di,xi
REAL(SP),DIMENSION(MXDIM),SAVE :: dt,dx,xi
REAL(SP),DIMENSION(NDMX),SAVE :: r,xin
REAL(DP),SAVE :: schi,si,swgt
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```
ndim=size(region)/2
if (init <= 0) then
  mds=1
  ndo=1
  xi(1, :) = 1.0
end if
if (init <= 1) then
  si=0.0
  swgt=0.0
  schi=0.0
end if
if (init <= 2) then
  nd=NDMX
  ng=1
  if (mds /= 0) then
    ng=(ncall/2.0_sp+0.25_sp)**(1.0_sp/ndim)
    mds=1
  else
    ng=ng/NDMX+1
    nd=ng/npg
    ng=npg*nd
  end if
  k=ng**ndim
  npg=max(ncall/k, 2)
  calls=real(npg, sp)*real(k, sp)
  dxg=1.0_sp/ng
  dv2g=(calls*dxg**ndim)**2/npg/npg/(npg-1.0_sp)
  xnd=nd
  dxg=dxg*xnd
  dx(1:ndim)=region(1+ndim:2*ndim)-region(1:ndim)
  xjac=1.0_sp/calls*product(dx(1:ndim))
  if (nd /= ndo) then
    Do binning if necessary.
    r(1:max(nd, ndo))=1.0
    do j=1, ndim
      call rebin(ndo/xnd, nd, r, xin, xi(:, j))
    end do
    ndo=nd
  end if
  if (nprn >= 0) write(*, 200) ndim, calls, it, itmx, nprn, &
  ALPH, mds, nd, (j, region(j), j, region(j+ndim), j=1, ndim)
end if
```

Main iteration loop. Can enter here (init \(\geq 3\)) to do an additional itmx iterations with all other parameters unchanged.

```
do it=1, itmx
  ti=0.0
  tsi=0.0
  kg( :) = 1
  d(1:nd, :) = 0.0
  di(1:nd, :) = 0.0
  iterate: do
    fb=0.0
    f2b=0.0
    do k=1, npg
      wgt=xjac
      do j=1, ndim
        call ran1(harvest)
        xn=(kg(j)-harvest)*dxg+1.0_sp
        ia(j)=max(min(int(xn), NDMX), 1)
        if (ia(j) > 1) then
          xo=xi(ia(j)-1, j)+x(j)-ia(j)*di(1, j)
          rc=xi(ia(j)-1, j)*(x(j)-ia(j)*di(1, j))
        else
          xo=xi(ia(j), j)
          rc=(x(j)-ia(j))*xo
        end if
      end do
    end do
```
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end if
x(j)=region(j)+rc*dx(j)
wgt=wgt*xo*nx
end do
f=wgt*func(x(1:ndim),wgt)
f2=f*f
fb=fb+f
f2b=f2b+f
do j=1,ndim
di(ia(j),j)=di(ia(j),j)+f
if (mds > 0) d(ia(j),j)=d(ia(j),j)+f2
end do
end do
f2b=sqrt(f2b*npg)
f2b=(f2b-fb)*(f2b+fb)
if (f2b <= 0.0) f2b=TINY

Use stratified sampling.
do j=1,ndim
d(ia(j),j)=d(ia(j),j)+f2
end do
end if
if (mds < 0) then
Use stratified sampling.
do j=1,ndim
d(ia(j),j)=d(ia(j),j)+f2b
end do
end do
exit iterate

Compute final results for this iteration.
tsi=tsi*dv2g
wgt=1.0_sp/tsi
si=si+real(wgt,dp)*real(ti,dp)
schi=schi+real(wgt,dp)*real(ti,dp)**2
swgt=swgt+real(wgt,dp)
tgral=si/swgt
ch2a=max((schi-si*tgral)/(it-0.99_dp),0.0_dp)
sd=sqrt(1.0_sp/swgt)
tsi=sqrt(tsi)
if (nprn >= 0) then
write(*,201) it,ti,tsi,tgral,sd,chi2a
if (nprn /= 0) then
do j=1,ndim
write(*,202) j,(xi(i,j),di(i,j),
i=1+nprn/2,nd,nprn)
end do
end if
end if
end do
end do

Refine the grid. Consult references to understand the subtlety of this procedure. The refinement is damped, to avoid rapid, destabilizing changes, and also compressed in range by the exponent ALPH.

xo=d(1,j)
xn=d(2,j)
d(1,j)=(xo+xn)/2.0_sp
dt(j)=d(1,j)
do i=2,nd-1
rc=xo+xn
xn=d(i+1,j)
d(i,j)=(rc+xn)/3.0_sp
dt(j)=dt(j)+d(i,j)
end do
d(nd,j)=(xo+xn)/2.0_sp
dt(j)=dt(j)+d(nd,j)
end do
where (d(1:nd,:) < TINY) d(1:nd,:) = TINY

j=1,ndim
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\[ r(1:nd) = ((1.0_{sp} - d(1:nd,j)/dt(j))/(\log(dt(j)) - \log(d(1:nd,j))))^{**ALPH} \]

\[ rc = \text{sum}(r(1:nd)) \]

\[ \text{call rebin}(rc/xnd,nd,r,xin,xi(:,j)) \]

end do
dend do

200 format(/' input parameters for vegas: ndim=',i3,' ncall=',f8.0&
/28x,' it=',i5,' itmx=',i5&
/28x,' nprn=',i3,' alph=',f5.2&
/28x,' mds=',i3,' nd=',i4&
/(30x,'xl(',i2,')= ',g11.4,' xu(',i2,')= ',g11.4))

201 format(/' iteration no.',i3,: ', integral=','g14.7,' +/- ','g9.2,&
/ ' all iterations: integral=','g14.7,' +/- ','g9.2,&
/ ' chi**2/it’n = ','g9.2)

202 format(/' data for axis ',i2,' x delta i ',&
/ ' x delta i ' ', x delta i ',&
/(1x,f7.5,1x,g11.4,5x,f7.5,1x,g11.4,5x,f7.5,1x,g11.4))

CONTAINS

SUBROUTINE rebin(rc,nd,r,xin,xi)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: rc
INTEGER(I4B), INTENT(IN) :: nd
REAL(SP), DIMENSION(:), INTENT(IN) :: r
REAL(SP), DIMENSION(:), INTENT(OUT) :: xin
REAL(SP), DIMENSION(:), INTENT(INOUT) :: xi

Utility routine used by vegas, to rebin a vector of densities xi into new bins defined by a vector r.

INTEGER(I4B) :: i,k
REAL(SP) :: dr,xn,xo
k=0
xo=0.0
dr=0.0
do i=1,nd-1
  do
    if (rc <= dr) exit
    k=k+1
    dr=dr+r(k)
  end do
  if (k > 1) xo=xi(k-1)
  xn=xi(k)
  dr=dr-rc
  xin(i)=xn-(xn-xo)*dr/r(k)
end do
x1(1:nd-1)=x1(1:nd-1)
xi(nd)=1.0
END SUBROUTINE rebin
END SUBROUTINE vegas

***

RECURSIVE SUBROUTINE miser(func,regn,ndim,npts,dith,ave,var)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP) :: func
REAL(SP), DIMENSION(:), INTENT(IN) :: x
END FUNCTION func
END INTERFACE
REAL(SP), DIMENSION(:), INTENT(IN) :: regn
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: ndim,npts

* * *

RECURRENCE SUBROUTINE miser(func,regn,ndim,npts,dith,ave,var)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP) :: func
REAL(SP), DIMENSION(:), INTENT(IN) :: x
END FUNCTION func
END INTERFACE
REAL(SP), DIMENSION(:), INTENT(IN) :: regn
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: ndim,npts

* * *
REAL(SP), INTENT(IN) :: dith
REAL(SP), INTENT(OUT) :: ave, var
REAL(SP), PARAMETER :: PFAC=0.1_sp, TINY=1.0e-30_sp, BIG=1.0e30_sp
INTEGER(I4B), PARAMETER :: MNPT=15, MNBS=60

Monte Carlo samples a user-supplied ndim-dimensional function `func` in a rectangular volume specified by `region`, a 2×ndim vector consisting of ndim “lower-left” coordinates of the region followed by ndim “upper-right” coordinates. The function is sampled a total of npts times, at locations determined by the method of recursive stratified sampling. The mean value of the function in the region is returned as `ave`; an estimate of the statistical uncertainty of `ave` (square of standard deviation) is returned as `var`. The input parameter `dith` should normally be set to zero, but can be set to (e.g.) 0.1 if `func`’s active region falls on the boundary of a power-of-2 subdivision of `region`.

Parameters: PFAC is the fraction of remaining function evaluations used at each stage to explore the variance of `func`. At least MNPT function evaluations are performed in any terminal subregion; a subregion is further bisected only if at least MNBS function evaluations are available.

REAL(SP), DIMENSION(:,), ALLOCATABLE :: `regn_temp`
INTEGER(I4B) :: j, jb, n, ndim, npre, nptl, npt
INTEGER(I4B), SAVE :: iran=0
REAL(SP) :: avel, varl, fval, rgl, rmg, rgr, &
 s, sigl, sigb, sig, sigrb, sm, sm2, sum, sumr
REAL(SP), DIMENSION(:,), ALLOCATABLE :: fmaxl, fmaxr, fminl, fminr, pt, rmid
ndim=assert_eq(size(regn), 2*ndim, ‘miser’)
allocate(regn_temp)
allocate(pt(ndim))
if (npts < MNBS) then
   Too few points to bisect; do straight Monte Carlo.
sm=0.0
sm2=0.0
   do n=1, npts
      call ranpt(pt, regn)
      fval=func(pt)
      sm=sm+fval
      sm2=sm2+fval**2
   end do
   ave=sm/npts
   var=max(TINY, (sm2-sm**2/npts)/npts**2)
else
   Do the preliminary (uniform) sampling.
npre=max(int(npts*PFAC), MNPT)
allocate(rmid(ndim), fmaxl(ndim), fmaxr(ndim), fminl(ndim), fminr(ndim))
 allocate(rmid(ndim), fmaxl(ndim), fmaxr(ndim), fminl(ndim), fminr(ndim))
fminl(:)=BIG
fmaxl(:)=BIG
fminr(:)=BIG
fmaxr(:)=BIG
   do j=1, ndim
      iran=mod(iran*2661+36979, 175000)
      s=sign(dith, real(iran-87500, sp))
      rmid(j)=(0.5_sp+s)*regn(j)+(0.5_sp-s)*regn(ndim+j)
   end do
   sumb=BIG
   Choose which dimension jb to bisect.
   Loop over the points in the sample.
   jb=0
   siglb=1.0
   sigrb=1.0
   do j=1, ndim
      if (fmaxl(j) > fminl(j), and. fmaxr(j) > fminr(j)) then

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sigl = max(TINY, (fmaxl(j) - fminl(j))**(2.0_sp/3.0_sp))
sigr = max(TINY, (fmaxr(j) - fminr(j))**(2.0_sp/3.0_sp))
sumr = sigl + sigr

Equation (7.8.24); see text.

if (sumr <= sumb) then
    sumb = sumr
    jb = j
    siglb = sigl
    sigrb = sigr
end if

end if

deallocate(fminr, fminl, fmaxr, fmaxl)

if (jb == 0) jb = 1 + (ndim * iran) / 175000
    MNPT may be too small.

rgl = regn(jb)
rgm = rmid(jb)
rgr = regn(ndim+jb)

fracl = abs((rgm-rgl)/(rgr-rgl))

nptl = (MNPT*(npts-npre-2*MNPT)*fracl*siglb/ &
    (fracl*siglb+(1.0_sp-fracl)*sigrb))
nptr = npts-npre-nptl

allocate(regn_temp(2*ndim))

regn_temp(:) = regn(:)
regn_temp(ndim+jb) = rmid(jb)

Dispatch recursive call; will return back here eventually.

regn_temp(jb) = rmid(jb)
regn_temp(ndim+jb) = regn(ndim+jb)

Dispatch recursive call; will return back here eventually.

dealectocate(regn_temp)

AVE = fracl*avel+(1-fracl)*ave

Combine left and right regions by equation

VAR = fracl*fracl*varl+(1-fracl)*(1-fracl)*var

Equation (7.8.11) (1st line).

dealectocate(rmid)

deallocate(pt)

CONTAINS

SUBROUTINE ranpt(pt, region)

USE nr, ONLY : ran1
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: pt
REAL(SP), DIMENSION(:), INTENT(INOUT) :: region

Returns a uniformly random point pt in a rectangular region of dimension d. Used by miser; calls ran1 for uniform deviates.

INTEGER(I4B) :: n
call ran1(pt)
n = size(pt)
pt(1:n) = region(1:n)+(region(n+1:2*n)-region(1:n))*pt(1:n)

END SUBROUTINE ranpt

END SUBROUTINE miser

The Fortran 90 version of this routine is much more straightforward than the Fortran 77 version, because Fortran 90 allows recursion. (In fact, this routine is modeled on the C version of miser, which was recursive from the start.)

CITED REFERENCES AND FURTHER READING:

Chapter B8. Sorting

Caution! If you are expecting to sort efficiently on a parallel machine, whether its parallelism is small-scale or massive, you almost certainly want to use library routines that are specific to your hardware.

We include in this chapter translations into Fortran 90 of the general purpose serial sorting routines that are in Volume 1, augmented by several new routines that give pedagogical demonstrations of how parallel sorts can be achieved with Fortran 90 parallel constructions and intrinsics. However, we intend the above word "pedagogical" to be taken seriously: these new, supposedly parallel, routines are not likely to be competitive with machine-specific library routines. Neither do they compete successfully on serial machines with the all-serial routines provided (namely `sort`, `sort2`, `sort3`, `indexx`, and `select`).

```
SUBROUTINE sort_pick(arr)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
    Sorts an array arr into ascending numerical order, by straight insertion. arr is replaced on output by its sorted rearrangement.
    INTEGER(I4B) :: i,j,n
    REAL(SP) :: a
    n=size(arr)
    do j=2,n
        Pick out each element in turn.
        a=arr(j)
        do i=j-1,1,-1
            Look for the place to insert it.
            if (arr(i) <= a) exit
            arr(i+1)=arr(i)
        end do
        arr(i+1)=a
        Insert it.
    end do
END SUBROUTINE sort_pick
```

Not only is `sort_pick` (renamed from Volume 1's `piksrt`) not parallelizable, but also, even worse, it is an $N^2$ routine. It is meant to be invoked only for the most trivial sorting jobs, say, $N < 20$. 

```
SUBROUTINE sort_shell(arr)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
Sorts an array arr into ascending numerical order by Shell's method (diminishing increment sort). arr is replaced on output by its sorted rearrangement.
INTEGER(I4B) :: i,j,inc,n
REAL(SP) :: v
n=size(arr)
inc=1
do
  inc=3*inc+1
  if (inc > n) exit
end do
do
  inc=inc/3
  do i=inc+1,n
    v=arr(i)
    j=i
    do
      if (arr(j-inc) <= v) exit
      arr(j)=arr(j-inc)
      j=j-inc
      if (j <= inc) exit
    end do
    arr(j)=v
  end do
  if (inc <= 1) exit
end do
END SUBROUTINE sort_shell

The routine sort_shell is renamed from Volume I's shell. Shell's Method, a diminishing increment sort, is not directly parallelizable. However, one can write a fully parallel routine (though not an especially fast one — see remarks at beginning of this chapter) in much the same spirit:

SUBROUTINE sort_byreshape(arr)
USE nrtype; USE nrutil, ONLY : swap
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
Sort an array arr by bubble sorting a succession of reshapings into array slices. The method is similar to Shell sort, but allows parallelization within the vectorized masked swap calls.
REAL(SP), DIMENSION(:,:), ALLOCATABLE :: tab
REAL(SP), PARAMETER :: big=huge(arr)
INTEGER(I4B) :: inc,n,m
n=size(arr)
inc=1
do
  inc=2*inc+1
  if (inc > n) exit
end do
do
  inc=inc/2
  m=(n+inc-1)/inc
  allocate(tab(m,m))
  tab=reshape(arr, (/inc,m/), (/big/) )
  do
    Bubble sort all the rows in parallel.
    call swap(tab(:,1:m-1:2),tab(:,2:m:2), &
              tab(:,1:m-1:2)>tab(:,2:m:2))
    call swap(tab(:,2:m-1:2),tab(:,3:m:2), &
              tab(:,2:m-1:2)>tab(:,3:m:2))
  end do
end do
END SUBROUTINE sort_byreshape
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if (all(tab(:,1:m-1) <= tab(:,2:m))) exit
end do
arr=reshape(tab,shape(arr))
deallocate(tab)
if (inc <= 1) exit
end do
END SUBROUTINE sort_byreshape

The basic idea is to reshape the given one-dimensional array into a succession of two-dimensional arrays, starting with “tall and narrow” (many rows, few columns), and ending up with “short and wide” (many columns, few rows). At each stage we sort all the rows in parallel by a bubble sort, giving something close to Shell’s diminishing increments.

⋆⋆⋆

We now arrive at those routines, based on the Quicksort algorithm, that we actually intend for use with general $N$ on serial machines:

SUBROUTINE sort(arr)
USE nrtype; USE nrutil, ONLY : swap,nrerror
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
INTEGER(I4B), PARAMETER :: NN=15, NSTACK=50
Sorts an array arr into ascending numerical order using the Quicksort algorithm. arr is replaced on output by its sorted rearrangement. Parameters: NN is the size of subarrays sorted by straight insertion and NSTACK is the required auxiliary storage.
REAL(SP) :: a
INTEGER(I4B) :: n,k,i,j,jstack,l,r
INTEGER(I4B), DIMENSION(NSTACK) :: istack
n=size(arr)
jstack=0
l=1
r=n
do
if (r-l < NN) then
Insertion sort when subarray small enough.
doi=1,r
a=arr(j)
doi=j-1,l,-1
if (arr(i) <= a) exit
arr(i+1)=arr(i)
end do
arr(i+1)=a
end do
if (jstack == 0) RETURN
Pop stack and begin a new round of partitioning.
l=istack(jstack)
jstack=jstack-2
else
Choose median of left, center, and right elements as partitioning element $a$. Also rearrange so

$k=(l+r)/2$
call swap(arr(k),arr(l+1))
that $a(l) \leq a(l+1) \leq a(r)$.
call swap(arr(l+1),arr(l),arr(l+1)>arr(l))
call swap(arr(l+1),arr(r),arr(l+1)>arr(r))
call swap(arr(l+1),arr(l+1),arr(l+1)>arr(l+1))
i=l+1
Initialize pointers for partitioning.
j=r
$a=arr(l+1)$
Partitioning element.
do
Here is the meat.
doi=i+1
Scan up to find element $\geq a$. 

if (arr(i) >= a) exit
end do

j=j-1
if (arr(j) <= a) exit
end do

if (j < i) exit
Pointers crossed. Exit with partitioning complete.
call swap(arr(i),arr(j)) Exchange elements.
end do

arr(l+1)=arr(j) Insert partitioning element.
arr(j)=a
jstack=jstack+2
Push pointers to larger subarray on stack; process smaller subarray immediately.
if (jstack > NSTACK) call nrerror('sort: NSTACK too small')
if (r-i+1 >= j-l) then
    istack(jstack)=r
    istack(jstack-1)=i
    r=j-1
else
    istack(jstack)=j-1
    istack(jstack-1)=1
    l=i
end if
end if
end do
END SUBROUTINE sort

call swap(...) ... call swap(...) One might think twice about putting
all these external function calls (to nrutil routines) in the inner loop
of something as streamlined as a sort routine, but here they are executed
only once for each partitioning.
call swap(arr(i),arr(j)) This call is in a loop, but not the innermost loop.
Most modern machines are very fast at the “context changes” implied by subroutine
calls and returns; but in a time-critical context you might code this swap in-line and
see if there is any timing difference.

SUBROUTINE sort2(arr,slave)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : indexx
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr,slave
Sorts an array arr into ascending order using Quicksort, while making the corresponding
rearrangement of the same-size array slave. The sorting and rearrangement are performed
by means of an index array.
INTEGER(I4B) :: ndum
INTEGER(I4B), DIMENSION(size(arr)) :: index
ndum=assert_eq(size(arr),size(slave),'sort2')
call indexx(arr,index) Make the index array.
arr=arr(index) Sort arr.
slave=slave(index) Rearrange slave.
END SUBROUTINE sort2

⋆⋆⋆

A close surrogate for the Quicksort partition-exchange algorithm can be coded, parallelizable, by using Fortran 90’s pack intrinsic. On
real compilers, unfortunately, the resulting code is not very efficient as
compared with (on serial machines) the tightness of sort’s inner loop, above, or
(on parallel machines) supplied library sort routines. We illustrate the principle
nevertheless in the following routine.
RECURSIVE SUBROUTINE sort_bypack(arr)
USE nrtype; USE nrutil, ONLY : array_copy, swap
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr

Sort an array arr by recursively applying the Fortran 90 pack intrinsic. The method is similar to Quicksort, but this variant allows parallelization by the Fortran 90 compiler.

REAL(SP) :: a
INTEGER(I4B) :: n, k, nl, nerr
INTEGER(I4B), SAVE :: level=0
LOGICAL, DIMENSION(:), ALLOCATABLE, SAVE :: mask
REAL(SP), DIMENSION(:), ALLOCATABLE, SAVE :: temp

n=size(arr)
if (n <= 1) RETURN
k=(1+n)/2

call swap(arr(1),arr(k),arr(1)>arr(k))  \* Pivotelementis medianoffirst,middle, and last.
call swap(arr(k),arr(n),arr(k)>arr(n))
call swap(arr(1),arr(k),arr(1)>arr(k))
if (n <= 3) RETURN

level=level+1  \* Keep track of recursion level to avoid allocation overhead.
if (level == 1) allocate(mask(n),temp(n))
a=arr(k)
mask(1:n) = (arr <= a)  \* Which elements moveto left?
mask(k) = .false.
call array_copy(pack(arr,mask(1:n)),temp,nl,nerr)  \* Move them.
mask(k) = .true.
temp(nl+2:n)=pack(arr,.not. mask(1:n))  \* Move others to right.
temp(nl+1)=a
arr=temp(1:n)
call sort_bypack(arr(1:nl))  \* And recurse.
call sort_bypack(arr(nl+2:n))
if (level == 1) deallocate(mask,temp)
level=level-1
END SUBROUTINE sort_bypack

* * *

The following routine, sort_heap, is renamed from Volume 1’s hpsort.

SUBROUTINE sort_heap(arr)
USE nrtype
USE nrutil, ONLY : swap
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr

Sorts an array arr into ascending numerical order using the Heapsort algorithm. arr is replaced on output by its sorted rearrangement.

INTEGER(I4B) :: i, n
n=size(arr)
do i=n/2,1,-1
  \* The index i, which here determines the “left” range of the sift-down, i.e., the element to be sifted down, is decremented from n/2 down to 1 during the “hiring” (heap creation) phase.
call sift_down(i,n)
end do

do i=n-1,1,-1
  \* Here the “right” range of the sift-down is decremented from n-1 down to 1 during the “retirement-and-promotion” (heap selection) phase.
call swap(arr(i),arr(i))  \* Clear a space at the end of the array, and
call sift_down(1,i-1)  \* retire the top of the heap into it.
end do
CONTAINS

SUBROUTINE sift_down(i,r)
INTEGER(I4B), INTENT(IN) :: i, r
Carry out the sift-down on element $arr(1)$ to maintain the heap structure.

```fortran
INTEGER(1B) :: j, jold
REAL(SP) :: a

a = arr(l)
jold = l
j = l + l

DO
    "Do while j <= r:"
    IF (j > r) EXIT
    IF (j < r) THEN
        IF (arr(j) < arr(j+1)) j = j + 1
        COMPARE TO THE BETTER UNDERLING.
    END IF
    IF (a >= arr(j)) EXIT
    FOUND a'S LEVEL. TERMINATE THE SIFTDOWN. OTHERWISE, DEMOTE a AND CONTINUE.
    arr(jold) = arr(j)
jold = j
    j = j + j
END DO

arr(jold) = a

PUT a INTO ITS SLOT.
```

SUBROUTINE sift_down
END SUBROUTINE sort_heap

⋆⋆⋆

Another opportunity provided by Fortran 90 for a fully parallelizable sort, at least pedagogically, is to use the language’s allowed access to the actual floating-point representation and to code a radix sort [1] on its bits. This is not efficient, but it illustrates some Fortran 90 language features perhaps worthy of study for other applications.

SUBROUTINE sort_radix(arr)
USE nrtype; USE nrutil, ONLY : array_copy, nrerror
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr

SORT AN ARRAY arr BY RADIX SORT ON ITS BITS.

INTEGER(I4B), DIMENSION(size(arr)) :: narr, temp
LOGICAL, DIMENSION(size(arr)) :: msk
INTEGER(I4B) :: k, negm, ib, ia, n, nl, nerr

BECAUSE WE ARE GOING TO TRANSFER REALS TO INTEGERS, WE MUST CHECK THAT THE NUMBER OF BITS IS THE SAME IN EACH:
ib = bit_size(narr)

ia = ceiling(log(real(maxexponent(arr) - minexponent(arr), sp))/log(2.0_sp)) & + digits(arr)

IF (ib /= ia) CALL nrerror('sort_radix: bit sizes not compatible')

negm = (ISHFTC(1, -1))

WHERE (BTEST(narr, ib-1)) narr = IEOR(narr, negm)

DO K = 0, ib-2
    WORK FROM LOW- TO HIGH-ORDER BITS, AND PARTITION THE ARRAY ACCORDING TO THE VALUE OF THE BIT.
    msk = BTEST(narr, k)
    CALL ARRAY_COPY(PACK(narr, .NOT. msk), temp, nl, nerr)
    temp(nl+1:n) = PACK(narr, msk)
    narr = temp
END DO

msk = BTEST(narr, ib-1)

CALL ARRAY_COPY(PACK(narr, msk), temp, nl, nerr)

temp(nl+1:n) = PACK(narr, .NOT. msk)

narr = temp

WHERE (BTEST(narr, ib-1)) narr = IEOR(narr, negm)

Unflip all bits on neg. numbers.

arr = transfer(narr, arr, n)

END SUBROUTINE sort_radix

⋆⋆⋆
We overload the generic name `indexx` with two specific implementations, one for SP floating values, the other for I4B integers. (You can of course add more overloadings if you need them.)

```fortran
SUBROUTINE indexx_sp(arr,index)
  USE nrtype; USE nrutil, ONLY : arth,assert_eq,nrerror,swap
  IMPLICIT NONE
  REAL(SP), DIMENSION(:), INTENT(IN) :: arr
  INTEGER(I4B), DIMENSION(:), INTENT(OUT) :: index
  INTEGER(I4B), PARAMETER :: NN=15, NSTACK=50

  Indexes an array `arr`, i.e., outputs the array `index` of length `N` such that `arr(index(j))` is in ascending order for `j = 1, 2, ..., N`. The input quantity `arr` is not changed.

  REAL(SP) :: a
  INTEGER(I4B) :: n,k,i,j,indext,jstack,l,r
  INTEGER(I4B), DIMENSION(NSTACK) :: istack
  n=assert_eq(size(index),size(arr),'indexx_sp')
  index=arth(1,1,n)
  jstack=0
  l=1
  r=n
  do 
    if (r-l < NN) then
      do j=l+1,r
        indext=index(j)
        a=arr(indext)
        do i=j-1,l,-1
          if (arr(index(i)) <= a) exit
          index(i+1)=index(i)
        end do
        index(i+1)=indext
        if (jstack == 0) RETURN
        r=istack(jstack)
        l=istack(jstack-1)
        jstack=jstack-2
      end do
    else
      k=(l+r)/2
      call swap(index(k),index(l+1))
      call icomp_xchg(index(l),index(r))
      call icomp_xchg(index(l+1),index(r))
      call icomp_xchg(index(l),index(l+1))
      i=l+1
      j=r
      indext=index(l+1)
      a=arr(indext)
      do
        i=i+1
        if (arr(index(i)) >= a) exit
      end do
      do
        j=j-1
        if (arr(index(j)) <= a) exit
      end do
      if (j < i) exit
      call swap(index(i),index(j))
    end do
    if (r-i+1 >= j-l) then
      istack(jstack)=r
    end if
  end do
  index(l+1)=index(j)
  index(j)=indext
  jstack=jstack+2
  if (jstack > NSTACK) call nrerror('indexx:NSTACK too small')
  if (r-i+1 >= j-l) then
    istack(jstack)=r
  end if
```
istack(jstack-1)=i
r=j-1
else
istack(jstack)=j-1
istack(jstack-1)=l
l=i
end if
end if
end do

CONTAINS

SUBROUTINE icomp_xchg(i,j)
INTEGER(I4B), INTENT(INOUT) :: i,j
INTEGER(I4B) :: swp
if (arr(j) < arr(i)) then
swp=i
i=j
j=swp
end if
END SUBROUTINE icomp_xchg

SUBROUTINE indexx_sp

SUBROUTINE indexx_i4b(iarr,index)
USE nrtype; USE nrutil, ONLY : arth,assert_eq,nrerror,swap
IMPLICIT NONE
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: iarr
INTEGER(I4B), DIMENSION(:,), INTENT(OUT) :: index
INTEGER(I4B), PARAMETER :: NN=15, NSTACK=50
INTEGER(I4B) :: n,k,i,j,indext,jstack,l,r
INTEGER(I4B), DIMENSION(NSTACK) :: istack
n=assert_eq(size(index),size(iarr),'indexx_sp')
index=arth(1,1,n)
jstack=0
l=1
r=n
do
if (r-l < NN) then
do j=l+1,r
indext=index(j)
a=iarr(indext)
do i=j-1,1,-1
if (iarr(index(i)) <= a) exit
index(i+1)=index(i)
end do
index(i+1)=indext
end do
if (jstack == 0) RETURN
r=istack(jstack)
l=istack(jstack-1)
jstack=jstack-2
else
k=(l+r)/2
call swap(index(k),index(l+1))
call icomp_xchg(index(l),index(r))
call icomp_xchg(index(l+1),index(r))
call icomp_xchg(index(l),index(l+1))
i=l+1
j=r
indext=index(l+1)
a=iarr(indext)
do
Chapter B8. Sorting

CONTAINS

SUBROUTINE icomp_xchg(i,j)
INTEGER(I4B), INTENT(INOUT) :: i,j
INTEGER(I4B) :: swp
if (iarr(j) < iarr(i)) then
  swp=i
  i=j
  j=swp
end if
END SUBROUTINE icomp_xchg

SUBROUTINE icomp_xchg(i,j)
INTEGER(I4B), INTENT(INOUT) :: i,j
INTEGER(I4B) :: swp
if (iarr(j) < iarr(i)) then
  swp=i
  i=j
  j=swp
end if
END SUBROUTINE icomp_xchg

SUBROUTINE icomp_xchg(i,j)
INTEGER(I4B), INTENT(INOUT) :: i,j
INTEGER(I4B) :: swp
if (iarr(j) < iarr(i)) then
  swp=i
  i=j
  j=swp
end if
END SUBROUTINE icomp_xchg

SUBROUTINE icomp_xchg(i,j)
INTEGER(I4B), INTENT(INOUT) :: i,j
INTEGER(I4B) :: swp
if (iarr(j) < iarr(i)) then
  swp=i
  i=j
  j=swp
end if
END SUBROUTINE icomp_xchg

SUBROUTINE sort3(arr,slave1,slave2)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : indexx
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr,slave1,slave2
Sorts an array arr into ascending order using Quicksort, while making the corresponding rearrangement of the same-size arrays slave1 and slave2. The sorting and rearrangement are performed by means of an index array.
INTEGER(I4B) :: ndum
INTEGER(I4B), DIMENSION(size(arr)) :: index
ndum=assert_eq(size(arr),size(slave1),size(slave2),'sort3')
call indexx(arr,index) Make the index array.
arr=arr(index) Sort arr.
slave1=slave1(index) Rearrange slave1,
slave2=slave2(index) and slave2.
END SUBROUTINE sort3

*   *   *
FUNCTION rank(index)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
INTEGER(14B), DIMENSION(:,), INTENT(IN) :: index
INTEGER(14B), DIMENSION(size(index)) :: rank

Given index as output from the routine indexx, this routine returns a same-size array rank, the corresponding table of ranks.

rank(index(:))=arth(1,1,size(index))
END FUNCTION rank

⋆⋆⋆

Just as in the case of sort, where an approximation of the underlying Quicksort partition-exchange algorithm can be captured with the Fortran 90 pack intrinsic, the same can be done with indexx. As before, although it is in principle parallelizable by the compiler, it is likely not competitive with library routines.

RECURSIVE SUBROUTINE index_bypack(arr,index,partial)
USE nrtype; USE nrutil, ONLY : array_copy,arth,assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: arr
INTEGER(I4B), DIMENSION(:,), INTENT(INOUT) :: index
INTEGER, OPTIONAL, INTENT(IN) :: partial

Indexes an array arr, i.e., outputs the array index of length N such that arr(index(j)) is in ascending order for j = 1, 2,...,N. The method is to apply recursively the Fortran 90 pack intrinsic. This is similar to Quicksort, but allows parallelization by the Fortran 90 compiler. partial is an optional argument that is used only internally on the recursive calls.

REAL(SP) :: a
INTEGER(I4B) :: n,k,nl,indext,nerr
INTEGER(I4B), SAVE := level=0
LOGICAL, DIMENSION(:,), ALLOCATABLE, SAVE :: mask
INTEGER(I4B), DIMENSION(:,), ALLOCATABLE, SAVE :: temp

if (present(partial)) then
  n=size(index)
else
  n=assert_eq(size(index),size(arr),'indexx_bypack')
  index=arth(1,1,n)
end if

if (n <= 1) RETURN
k=(1+n)/2

call icomp_xchg(index(1),index(k)) Pivot element is median of first, middle, and last.
call icomp_xchg(index(k),index(n))
call icomp_xchg(index(1),index(k))

if (n <= 3) RETURN
level=level+1 Keep track of recursion level to avoid allocation overhead.

if (level == 1) allocate(mask(n),temp(n))
indext=index(k)
a=arr(indext)
mask(1:n) = (arr(index) <= a) Which elements move to left?
mask(k) = .false.
call array_copy(pack(index,mask(1:n)),temp,nl,nerr) Move them.
mask(k) = .true.
temp(nl+2:n)=pack(index,.not. mask(1:n)) Move others to right.
temp(nl+1)=indext
index=temp(1:n)
call index_bypack(arr,index(1:nl),partial=1) And recurse.
call index_bypack(arr,index(nl+2:n),partial=1)
if (level == 1) deallocate(mask,temp)

level=level-1
CONTAINS
SUBROUTINE icomp_xchg(i,j)
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: i,j
Swap or don’t swap integer arguments, depending on the ordering of their corresponding
elements in an array arr.
INTEGER(I4B) :: swp
if (arr(j) < arr(i)) then
  swp=i
  i=j
  j=swp
end if
END SUBROUTINE icomp_xchg

FUNCTION select(k,arr)
USE nrtype; USE nrutil, ONLY : assert, swap
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: k
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
REAL(SP) :: select
Returnsthe
kth
smallestvalueinthearray
arr. Theinputarraywillberearrangedtohave
this value in location arr(k), with all smaller elements moved to arr(1:k-1) (in arbitrary
order) and all larger elements in arr(k+1:) (also in arbitrary order).
INTEGER(I4B) :: i,r,j,l,n
REAL(SP) :: a
n=size(arr)
call assert(k >= 1, k <= n, 'select args')
l=1
r=n
do
  if (r-l <= 1) then
    Active partition contains 1 or 2 elements.
    if (r-l == 1) call swap(arr(l),arr(r),arr(l)>arr(r))
    select=arr(k)
    RETURN
  else
    Choose median of left, center, and right elements
    as partitioning element a. Also rearrange so
    that arr(l) ≤ arr(l+1) ≤ arr(r).
    i=(l+r)/2
    call swap(arr(i),arr(l+1))
    call swap(arr(l),arr(r),arr(l)>arr(r))
    call swap(arr(l+1),arr(r),arr(l+1)>arr(r))
    call swap(arr(l),arr(l+1),arr(l)>arr(l+1))
    i=l+1
    Initialize pointers for partitioning.
    j=r
    a=arr(l+1)
    Partitioning element.
    do
      Scan up to find element > a.
      do
        if (arr(i) >= a) exit
      end do
      Scan down to find element < a.
      do
        if (arr(j) <= a) exit
      end do
      Pointers crossed. Exit with partitioning complete.
      Exchange elements.
      call swap(arr(i),arr(j))
      end do
      if (j < i) exit
      Exchange elements.
      call swap(arr(i),arr(j))
      Initialize partitioning element.
      do
        if (j >= k) r=j-1
      end do
      if (j >= k) r=j-1
  end do
end function select
The following routine, `select_inplace`, is renamed from Volume 1’s `selip`.

```fortran
FUNCTION select_inplace(k,arr)
  USE nrtype
  USE nr, ONLY : select
  IMPLICIT NONE
  INTEGER(I4B), INTENT(IN) :: k
  REAL(SP), DIMENSION(:), INTENT(IN) :: arr
  REAL(SP) :: select_inplace
  Return the kth smallest value in the array arr, without altering the input array. In Fortran 90’s assumed memory-rich environment, we just call `select` in scratch space.
  REAL(SP), DIMENSION(size(arr)) :: tarr
  tarr=arr
  select_inplace=select(k,tarr)
END FUNCTION select_inplace
```

Volume 1’s `selip` routine uses an entirely different algorithm, for the purpose of avoiding any additional memory allocation beyond that of the input array. Fortran 90 presumes a richer memory environment, so `select_inplace` simply does the obvious (destructive) selection in scratch space. You can of course use the old `selip` if your in-core or in-cache memory is at a premium.

```fortran
FUNCTION select_bypack(k,arr)
  USE nrtype; USE nrutil, ONLY : array_copy, assert, swap
  IMPLICIT NONE
  INTEGER(I4B), INTENT(IN) :: k
  REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
  REAL(SP) :: select_bypack
  Return the kth smallest value in the array arr. The input array will be rearranged to have this value in location `arr(k)`, with all smaller elements moved to `arr(1:k-1)` (in arbitrary order) and all larger elements in `arr(k+1:)` (also in arbitrary order). This implementation allows parallelization in the Fortran 90 `pack` intrinsic.
  LOGICAL, DIMENSION(size(arr)) :: mask
  REAL(SP), DIMENSION(size(arr)) :: temp
  INTEGER(14B) :: i,r,j,l,n, nl, nerr
  REAL(SP) :: a
  n=size(arr)
  call assert(k >= 1, k <= n, 'select_bypack args')
  l=1
  r=n
  do
    if (r-l <= 1) exit
    i=(l+r)/2
    call swap(arr(l),arr(i),arr(l)>arr(i))
    Pivot element is median of first, middle, and last.
    call swap(arr(i),arr(r),arr(i)>arr(r))
    call swap(arr(l),arr(i),arr(l)>arr(i))
    a=arr(i)
    mask(l:r) = (arr(l:r) <= a)
    Which elements move to left?
    mask(i) = .false.
    call array_copy(pack(arr(1:r),mask(1:r)),temp(1:),nl,nerr)
    Move them.
    j=1+nl
```
The above routine \texttt{select\_bypack} is parallelizable, but as discussed above (\texttt{sort\_bypack}, \texttt{index\_bypack}) it is generally not very efficient.

\begin{verbatim}
SUBROUTINE select_heap(arr,heap)
USE nrtype; USE nrutil, ONLY : nrerror, swap
USE nr, ONLY : sort
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: arr
REAL(SP), DIMENSION(:), INTENT(OUT) :: heap

INTEGER(I4B) :: i,j,k,m,n
m=size(heap)
if (m > n/2 .or. m < 1) call nrerror('probable misuse of select_heap')
heap=arr(1:m)
call sort(heap)

do i=m+1,n
   if (arr(i) > heap(1)) then
      heap(1)=arr(i)
      j=1
      do
         k=2*j
         if (k > m) exit
         if (k /= m) then
            if (heap(k) > heap(k+1)) k=k+1
         end if
         if (heap(j) <= heap(k)) exit
         call swap(heap(k),heap(j))
         j=k
      end do
   end if
end do

END SUBROUTINE select_heap
\end{verbatim}

The following routine, \texttt{select\_heap}, is renamed from Volume 1’s \texttt{hpsel}.

\begin{verbatim}
mask(i) = .true.
temp(j+1:r)=pack(arr(l:r),.not. mask(l:r))  Move others to right.
arr(l:r)=temp(l:r)
if (k > j) then
   l=j+1
else if (k < j) then
   r=j-1
else
   l=j
   r=j
end if
end do
if (r-l == 1) call swap(arr(l),arr(r),arr(l)>arr(r))  Case of only two left.

select\_bypack=arr(k)

END FUNCTION select\_bypack
\end{verbatim}
FUNCTION eclass(lista,listb,n)
USE nrtype; USE nrutil, ONLY : arth,assert_eq
IMPLICIT NONE
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: lista,listb
INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), DIMENSION(n) :: eclass

Given \( M \) equivalences between pairs of \( n \) individual elements in the form of the input arrays \( \text{lista} \) and \( \text{listb} \) of length \( M \), this routine returns in an array of length \( n \) the number of the equivalence class of each of the \( n \) elements, integers between 1 and \( n \) (not all such integers used).

INTEGER :: j,k,l,m
m=assert_eq(size(lista),size(listb),'eclass')
eclass(1:n)=arth(1,1,n)  ! Initialize each element its own class.
do i=1,m
  j=lista(l)
do
    if (eclass(j) == j) exit
  j=eclass(j)
end do

k=listb(l)
do
  if (eclass(k) == k) exit
  k=eclass(k)
end do

if (j /= k) eclass(j)=k  ! If they are not already related, make them so.
end do
do j=1,n
  do
    if (eclass(j) == eclass(eclass(j))) exit
  eclass(j)=eclass(eclass(j))
  end do
end do
END FUNCTION eclass

FUNCTION eclazz(equiv,n)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
INTERFACE
  FUNCTION equiv(i,j)
    USE nrtype
  IMPLICIT NONE
  LOGICAL(LGT) :: equiv
  INTEGER(I4B), INTENT(IN) :: i,j
  END FUNCTION equiv
END INTERFACE

INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), DIMENSION(n) :: eclazz

Given a user-supplied logical function \( \text{equiv} \) that tells whether a pair of elements, each in the range 1...\( n \), are related, return in an array of length \( n \) equivalence class numbers for each element.

INTEGER :: i,j
eclazz(1:n)=arth(1,1,n)
do i=2,n
  do j=1,i-1
    eclazz(j)=eclazz(eclazz(j))  ! Sweep it up this much.
    if (equiv(i,j)) eclazz(eclazz(eclazz(j)))=i
    ! Good exercise for the reader to figure out why this much ancestry is necessary!
  end do
end do
do i=1,n
  eclazz(i)=eclazz(eclazz(i))
end do
END FUNCTION eclazz
CITED REFERENCES AND FURTHER READING:
Chapter B9. Root Finding and Nonlinear Sets of Equations

SUBROUTINE scrsho(func)
USE nrtype
IMPLICIT NONE
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE

INTEGER(I4B), PARAMETER :: ISCR=60, JSCR=21

For interactive 'dumb terminal' use. Produce a crude graph of the function func over the prompted-for interval x1,x2. Query for another plot until the user signals satisfaction.

Parameters: Number of horizontal and vertical positions in display.

INTEGER(I4B) :: i,j,jz
REAL(SP) :: dx,dyj,x,x1,x2,ybig,ysml
REAL(SP), DIMENSION(ISCR) :: y
CHARACTER(1), DIMENSION(ISCR,JSCR) :: scr
CHARACTER(1) :: blank=' ',zero='-',yy='l',xx='-',ff='x'

DO
   WRITE (*,*) 'Enter x1,x2 (= to stop)'  Query for another plot; quit if x1=x2.
   READ (*,*) x1,x2
   IF (x1 == x2) RETURN
   SCR(1,1:JSCR)=YY
   SCR(ISCR,1:JSCR)=YY
   SCR(2:ISCR-1,1)=XX
   SCR(2:ISCR-1,2:JSCR)=BLANK
   DX=(X2-X1)/(ISCR-1)
   X=X1
   DO I=1,ISCR
      Y(I)=FUNC(X)
      X=X+DX
   END DO
   YSM=MIN(MINVAL(Y(:)),0.0_SP)
   YBIG=MAX(MAXVAL(Y(:)),0.0_SP)
   IF (YBIG == YSM) YBIG=YSML+1.0
   DYJ=(JSCR-1)/(YBIG-YSML)
   JZ=1-YSML*DYJ
   SCR(I=ISCR,JZ)=ZERO
   DO I=1,ISCR
      J=1+(Y(I)-YSML)*DYJ
      SCR(I,J)=FF
   END DO
   WRITE (*,') (1X,1P,E10.3,1X,80A1)') YBIG,(SCR(I,JSCR),I=1,ISCR)
   DO J=JSCR-1,2,-1
      WRITE (*,') (SCR(I,J),I=1,ISCR)
   END DO
   WRITE (*,') (1X,1P,E10.3,1X,80A1)') YSM,(SCR(I,1),I=1,ISCR)
1182
write ('*(12x,1p,e10.3,40x,e10.3)*) x1,x2
end do
END SUBROUTINE scrsho

CHARACTER(1), DIMENSION(ISCR,JSCR) :: scr
In Fortran 90, the length of variables of type character should be declared as
CHARACTER(1) or CHARACTER(len=1) (for a variable of length 1), rather than the
older form CHARACTER*1. While the older form is still legal syntax, the newer one is more
consistent with the syntax of other type declarations. (For variables of length 1, you
can actually omit the length specifier entirely, and just say CHARACTER.)

*   *   *

SUBROUTINE zbrac(func,x1,x2,succes)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(INOUT) :: x1,x2
LOGICAL(LGT), INTENT(OUT) :: succes
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: NTRY=50
REAL(SP), PARAMETER :: FACTOR=1.6_sp
Given a function func and an initial guessed range x1 to x2, the routine expands the range
globally until a root is bracketed by the returned values x1 and x2 (in which case
succes returns as .true.) or until the range becomes unacceptably large (in which case
succes returns as .false.).
INTEGER(I4B) :: j
REAL(SP) :: f1,f2
if (x1 == x2) call nrerror('zbrac:you have to guess an initial range')
f1=func(x1)
f2=func(x2)
succes=.true.
do j=1,NTRY
if ((f1 > 0.0 .and. f2 < 0.0) .or. &
   (f1 < 0.0 .and. f2 > 0.0)) RETURN
if (abs(f1) < abs(f2)) then
   x1=x1+FACTOR*(x1-x2)
f1=func(x1)
else
   x2=x2+FACTOR*(x2-x1)
f2=func(x2)
end if
end do
succes=.false.
END SUBROUTINE zbrac

*   *   *
SUBROUTINE zbrak(func,x1,x2,n,xb1,xb2,nb)
USE nrtype; USE nrutil, ONLY : arth
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), INTENT(OUT) :: nb
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP), DIMENSION(:,), POINTER :: xb1,xb2
INTERFACE
  FUNCTION func(x)
  USE nrtype
  IMPLICIT NONE
  REAL(SP), INTENT(IN) :: x
  REAL(SP) :: func
  END FUNCTION func
END INTERFACE

Given a function \( \func \) defined on the interval from \( x1-x2 \), subdivide the interval into \( n \) equally spaced segments, and search for zero crossings of the function. \( nb \) is returned as the number of bracketing pairs \( \xb1(1:nb), \xb2(1:nb) \) that are found. \( \xb1 \) and \( \xb2 \) are pointers to arrays of length \( nb \) that are dynamically allocated by the routine.

INTEGER(I4B) :: i
REAL(SP) :: dx
REAL(SP), DIMENSION(0:n) :: f,x
LOGICAL(LGT), DIMENSION(1:n) :: mask
LOGICAL(LGT), SAVE :: init=.true.
if (init) then
  init=.false.
  nullify(xb1,xb2)
end if
if (associated(xb1)) deallocate(xb1)
if (associated(xb2)) deallocate(xb2)
dx=(x2-x1)/n
x=x1+dx*arth(0,1,n+1)
do i=0,n
  Evaluate the function at the mesh points.
  \( f(i)=\func(x(i)) \)
end do
mask=f(1:n)*f(0:n-1) <= 0.0
Record where the sign changes occur.
 nb=count(mask)
Number of sign changes.
allocate(xb1(nb),xb2(nb))
Store the bounds of each bracket.
xb1(1:nb)=pack(x(0:n-1),mask)
xb2(1:nb)=pack(x(1:n),mask)
END SUBROUTINE zbrak

This routine shows how to return arrays \( \xb1 \) and \( \xb2 \) whose size is not known in advance. The coding is explained in the subsection on pointers in §21.5.

 FUNCTION rtbis(func,x1,x2,xacc)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtbis
INTERFACE
  FUNCTION func(x)
  USE nrtype
  IMPLICIT NONE
  REAL(SP), INTENT(IN) :: x
  REAL(SP) :: func
  END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: MAXIT=40
Using bisection, find the root of a function \textit{func} known to lie between \textit{x1} and \textit{x2}. The root, returned as \textit{rtbis}, will be refined until its accuracy is ±\textit{xacc}.
Parameter: \textit{MAXIT} is the maximum allowed number of bisections.

INTEGER(I4B) :: j
REAL(SP) :: dx,f,fmid,xmid
fmid=func(x2)
f=func(x1)
if (f*fmid >= 0.0) call nrerror('rtbis: root must be bracketed')
if (f < 0.0) then
  Orient the search so that \textit{f}>0 lies at \textit{x}+\textit{dx}.
  rtbis=x1
  dx=x2-x1
else
  rtbis=x2
  dx=x1-x2
end if
do j=1,MAXIT  
  Bisection loop.
  dx=dx*0.5_sp
  xmid=rtbis+dx
  fmid=func(xmid)
  if (fmid <= 0.0) rtbis=xmid
  if (abs(dx) < xacc .or. fmid == 0.0) RETURN
end do
end FUNCTION rtbis

FUNCTION rtflsp(func,x1,x2,xacc)
USE nrtype; USE nrutil, ONLY : nrerror,swap
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtflsp
INTERFACE
  FUNCTION func(x)
  USE nrtype
  IMPLICIT NONE
  REAL(SP), INTENT(IN) :: x
  REAL(SP) :: func
  END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: MAXIT=30
Using the false position method, find the root of a function \textit{func} known to lie between \textit{x1} and \textit{x2}. The root, returned as \textit{rtflsp}, is refined until its accuracy is ±\textit{xacc}.
Parameter: \textit{MAXIT} is the maximum allowed number of iterations.

INTEGER(I4B) :: j
REAL(SP) :: del,dx,f,fh,fl,xh,xl
fl=func(x1)
fh=func(x2)
if ((fl > 0.0 .and. fh > 0.0) .or. &
     (fl < 0.0 .and. fh < 0.0)) call &
     nrerror('rtflsp: root must be bracketed between arguments')
if (fl < 0.0) then
  Identify the limits so that \textit{x}l corresponds to the low side.
  xl=x1
  xh=x2
else
  xl=x2
  xh=x1
end if
dx=xh-xl
do j=1,MAXIT
  rtfslsp=xl+dx*f1/(f1-fh)  
  f=func(rtfslsp)
  if (f < 0.0) then
    del=xl-rtfslsp
    xl=rtfslsp
    fl=f
  else
    del=xh-rtfslsp
    xh=rtfslsp
    fh=f
  end if
  dx=xh-xl
  if (abs(del) < xacc .or. f == 0.0) RETURN  
end do
call nrerror('rtfslsp exceed maximum iterations')
END FUNCTION rtfslsp

⋆⋆⋆

FUNCTION rtsec(func,x1,x2,xacc)
USE nrtype; USE nrutil, ONLY : nrerror, swap
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtsec
INTERFACE
  FUNCTION func(x)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), INTENT(IN) :: x
    REAL(SP) :: func
  END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: MAXIT=30

Using the secant method, find the root of a function func thought to lie between x1 and x2. The root, returned as rtsec, is refined until its accuracy is ±xacc.

Parameter: MAXIT is the maximum allowed number of iterations.

INTEGER(I4B) :: j
REAL(SP) :: dx,f,f1,x1
f=func(x1)
if (abs(f1) < abs(f)) then
  rtsec=x1
  xl=x2
  call swap(fl,f1)
else
  xl=x1
  rtsec=x2
end if
do j=1,MAXIT
  dx=(xl-rtsec)*f/(f-f1)  
  xl=rtsec
  f1=f
  rtsec=rtsec+dx
  f=func(rtsec)
  if (abs(dx) < xacc .or. f == 0.0) RETURN  
end do
call nrerror('rtsec: exceed maximum iterations')
END FUNCTION rtsec

⋆⋆⋆
FUNCTION zriddr(func,x1,x2,xacc)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: zriddr
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: MAXIT=60

Using Ridders' method, return the root of a function func known to lie between x1 and x2. The root, returned as zriddr, will be refined to an approximate accuracy xacc.

REAL(SP), PARAMETER :: UNUSED=-1.11e30_sp
INTEGER(I4B) :: j
REAL(SP) :: fh,fl,fnew,s,xh,xl,xm,xnew
fl=func(x1)
fh=func(x2)
if ((fl > 0.0 .and. fh < 0.0) .or. (fl < 0.0 .and. fh > 0.0)) then
  xl=x1
  xh=x2
  zriddr=UNUSED
else if (fl == 0.0) then
  zriddr=x1
else if (fh == 0.0) then
  zriddr=x2
else
  call nrerror('zriddr: root must be bracketed')
end if
END FUNCTION zriddr

⋆⋆⋆
FUNCTION zbrent(func,x1,x2,tol)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1,x2,tol
REAL(SP) :: zbrent

INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE

INTEGER(I4B), PARAMETER :: ITMAX=100
REAL(SP), PARAMETER :: EPS=epsilon(x1)

Using Brent’s method, find the root of a function func known to lie between x1 and x2.
The root, returned as zbrent, will be refined until its accuracy is tol.
Parameters: Maximum allowed number of iterations, and machine floating-point precision.

INTEGER(I4B) :: iter
REAL(SP) :: a,b,c,d,e,fa,fb,fc,p,q,r,s,tol1,xm
a=x1
b=x2
fa=func(a)
fb=func(b)

if ( (fa > 0.0 .and. fb > 0.0) .or. (fa < 0.0 .and. fb < 0.0) ) &
call nrerror(’root must be bracketed for zbrent’)
c=b
fc=fb
do iter=1,ITMAX

if ( (fb > 0.0 .and. fc > 0.0) .or. (fb < 0.0 .and. fc < 0.0) )

c=a
fc=fa
d=b-a
e=d
end if
if (abs(fc) < abs(fb)) then
a=b
b=c
fa=fb
fb=fc
fc=fa
end if

if (tol1=2.0_sp*EPS*abs(b)+0.5_sp*tol )
xm=0.5_sp*(c-b)
if (abs(xm) <= tol1 .or. fb == 0.0 ) then
zbrent=b
RETURN
end if
if (abs(e) >= tol1 .and. abs(fa) > abs(fb) ) then
s=fb/FA
end if
if (abs(e) >= tol1 .and. abs(fa) > abs(fb) ) then
s=fb/FA
end if
if (a == c) then
p=2.0_sp*xm*s
q=1.0_sp*s
else
q=fa/fc
r=fb/fc
p=s*(2.0_sp*xm*q*(q-r)-(b-a)*(r-1.0_sp))
q=(q-1.0_sp)*(r-1.0_sp)*(s-1.0_sp)
end if
if (p > 0.0 ) q=-q
if (2.0_sp*p < min(3.0_sp*xm*q-abs(tol1*q),abs(e*q)) ) then
e=d
Accept interpolation.
\[ d = \frac{p}{q} \]

else
\[ d = x_m \]
e = d
end if

else
\[ d = x_m \]
e = d
end if

Move last best guess to a.

\[ a = b \]

\[ f_a = f_b \]
b = \text{merge}(d, \text{sign}(\text{tol}1, x_m), \text{abs}(d) > \text{tol}1) \]

Evaluate new trial root.

End do

call nrerror('zbrent: exceeded maximum iterations')
zbrent = b
END FUNCTION zbrent

REAL(SP), PARAMETER :: EPS = \text{epsilon}(x1) \quad \text{The routine } \text{zbrent} \text{ works best when } \text{EPS} \text{ is exactly the machine precision. The Fortran 90 intrinsic function } \text{epsilon} \text{ allows us to code this in a portable fashion.}

FUNCTION rtnewt(funcd, x1, x2, xacc)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1, x2, xacc
REAL(SP) :: rtnewt
INTERFACE
SUBROUTINE funcd(x, fval, fderiv)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: fval, fderiv
END SUBROUTINE funcd
END INTERFACE
INTEGER(I4B), PARAMETER :: MAXIT = 20

Using the Newton-Raphson method, find the root of a function known to lie in the interval \([x_1, x_2]\). The root rtnewt will be refined until its accuracy is known within \(\pm x_{acc}\). funcd is a user-supplied subroutine that returns both the function value and the first derivative of the function.

Parameter: MAXIT is the maximum number of iterations.

INTEGER(I4B) :: j
REAL(SP) :: df, dx, f
rtnewt = 0.5_sp * (x1 + x2) \quad \text{Initial guess.}
do j = 1, MAXIT
\quad \text{call funcd(rtnewt, f, df)}
\quad dx = f / df
\quad rtnewt = rtnewt - dx
\quad \text{if } ((x1 - rtnewt) * (rtnewt - x2) < 0.0) &
\quad \quad \text{call nrerror('rtnewt: values jumped out of brackets')}
\quad \text{if } (\text{abs}(dx) < x_{acc}) \text{ RETURN} \quad \text{Convergence.}
end do

call nrerror('rtnewt exceeded maximum iterations')
END FUNCTION rtnewt

* * *
FUNCTION rtsafe(funcd,x1,x2,xacc)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtsafe
INTERFACE
SUBROUTINE funcd(x,fval,fderiv)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: fval,fderiv
END SUBROUTINE funcd
END INTERFACE
INTEGER(I4B), PARAMETER :: MAXIT=100
Using a combination of Newton-Raphson and bisection, find the root of a function bracketed between x1 and x2. The root, returned as the function value rtsafe, will be refined until its accuracy is known within ±xacc. funcd is a user-supplied subroutine that returns both the function value and the first derivative of the function.
Parameter: MAXIT is the maximum allowed number of iterations.
INTEGER(I4B) :: j
REAL(SP) :: df, dx, dxold, f, fh, fl, temp, zh, xl
call funcd(x1,fl,df)
call funcd(x2,fh,df)
if ((fl > 0.0 .and. fh > 0.0) .or. &
(fl < 0.0 .and. fh < 0.0)) &
call nrerror('root must be bracketed in rtsafe')
if (fl == 0.0) then
rtsafe=x1
RETURN
else if (fh == 0.0) then
rtsafe=x2
RETURN
else if (fl < 0.0) then
Orient the search so that f(xl) < 0.
xl=x1
xh=x2
else
xb=x1
xl=x2
end if
rtsafe=0.5_sp*(x1+x2)
Initialize the guess for root.
dxold=abs(x2-xl)
the "stepsize before last,"
dx=dxold
and the last step.
call funcd(rtsafe,f,df)
do j=1,MAXIT
Loop over allowed iterations.
if (((rtsafe-xh)*df-f)*((rtsafe-xl)*df-f) > 0.0 .or. &
abs(2.0_sp*f) > abs(dxold*df) ) then
Bisect if Newton out of range, or not decreasing fast enough.
dxold=dx
dx=0.5_sp*(xh-xl)
rtsafe=x+dx
if (xl == rtsafe) RETURN
else
Change in root is negligible.
Newton step acceptable. Take it.
dxold=dx
dx=f/df
temp=rtsafe
rtsafe=rtsafe-dx
if (temp == rtsafe) RETURN
end if
if (abs(dx) < xacc) RETURN
Convergence criterion.
call funcd(rtsafe,f,df)
if (f < 0.0) then
Maintain the bracket on the root.
xl=rtsafe
else
xh=rtsafe
END FUNCTION rtsafe
SUBROUTINE laguer(a,x,its)
USE nrtype; USE nrutil, ONLY : nrerror,poly,poly_term
IMPLICIT NONE
INTEGER(4B), INTENT(OUT) :: its
COMPLEX(SPC), INTENT(INOUT) :: x
COMPLEX(SPC), DIMENSION(:), INTENT(IN) :: a
REAL(SP), PARAMETER :: EPS=epsilon(1.0_sp)
INTEGER(4B), PARAMETER :: MR=8,MT=10,MAXIT=MT*MR

Given an array of \( M+1 \) complex coefficients \( a \) of the polynomial \( \sum_{i=0}^{M} a(i) x^{i-1} \), and given a complex value \( x \), this routine improves \( x \) by Laguerre's method until it converges, within the achievable roundoff limit, to a root of the given polynomial. The number of iterations taken is returned as \( \text{its} \).

Parameters: EPS is the estimated fractional roundoff error. We try to break (rare) limit cycles with MR different fractional values, once every MT steps, for MAXIT total allowed iterations.

INTEGER(4B) :: iter,m
REAL(SP) :: abx,abp,abm,err
COMPLEX(SPC) :: dx,x1,f,g,b,sq,gp,gm,g2
COMPLEX(SPC), DIMENSION(size(a)) :: b,d
REAL(SP), DIMENSION(MR) :: frac = &
(/ 0.5_sp,0.25_sp,0.75_sp,0.13_sp,0.38_sp,0.62_sp,0.88_sp,1.0_sp /)
Fractions used to break a limit cycle.
m=size(a)-1
do iter=1,MAXIT
   Loop over iterations up to allowed maximum.
   its=iter
   abx=abs(x)
   b(m+1:1:-1)=poly_term(a(m+1:1:-1),x) Efficient computation of the polynomial
   d(m:1:-1)=poly_term(b(m+1:2:-1),x) and its first two derivatives.
   f=poly(x,d(2:m))
   err=EPS*poly(abs(x),abs(b(1:m+1))) Estimate of roundoff in evaluating polynomial.
   if (abs(b(1)) <= err) RETURN We are on the root.
   g=d(1)/b(1) The generic case: Use Laguerre's formula.
   g2=g*g
   h=g2-2.0_sp*f/b(1)
   sq=sqrt((m-1)*(m*h-g2))
   gp=g+sq
   gm=g-sq
   abp=abs(gp)
   abm=abs(gm)
   if (abp < abm) gp=gm
   if (max(abp,abm) > 0.0) then
      dx=m/gp
   else
      dx=exp(cmplx(log(1.0_sp+abx),iter,kind=spc)) Converged.
   end if
   x1=x-dx
   if (x == x1) RETURN
   if (mod(iter,MT) /= 0) then
      x=x-dx*frac(iter/MT) Every so often we take a fractional step, to
   end if
   break any limit cycle (itself a rare occurrence).
end do
Very unusual — can occur only for complex roots. Try a different starting guess for the root.
END SUBROUTINE laguer
The poly_term function in nrutil tabulates the partial sums of a polynomial, while poly evaluates the polynomial at $x$. In this example, we use poly_term on the coefficient array in reverse order, so that the value of the polynomial ends up in $b(1)$ and the value of its first derivative in $d(1)$.

$$dx=\exp(\text{complex}(\log(1.0_{\text{sp}}+\text{abx}),\text{iter},\text{kind}=\text{spc}))$$

The intrinsic function complex returns a quantity of type default complex unless the kind argument is present. To facilitate converting our routines from single to double precision, we always include the kind argument explicitly so that when you redefine spc in nrtype to be double-precision complex the conversions are carried out correctly.

```fortran
* * *
```

SUBROUTINE zroots(a,roots,polish)
USE nrtype; USE nrutil, ONLY : assert_eq,poly_term
USE nr, ONLY : laguer,indexx
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:), INTENT(IN) :: a
COMPLEX(SPC), DIMENSION(:), INTENT(OUT) :: roots
LOGICAL(LGT), INTENT(IN) :: polish
REAL(SP), PARAMETER :: EPS=1.0e-6_sp

Given the array of $M+1$ complex coefficients $a$ of the polynomial $\sum_{i=1}^{M+1} a(i)x^{i-1}$, this routine successively calls laguer and finds all $M$ complex roots. The logical variable polish should be input as .true. if polishing (also by Laguerre's method) is desired, .false. if the roots will be subsequently polished by other means.

Parameter: EPS is a small number.

INTEGER(I4B) :: j,its,m
INTEGER(I4B), DIMENSION(size(roots)) :: indx
COMPLEX(SPC) :: x
COMPLEX(SPC), DIMENSION(size(a)) :: ad
m=assert_eq(size(roots),size(a)-1,'zroots')
ad(:)=a(:) Copy of coefficients for successive deflation.
do j=m,1,-1 Loop over each root to be found.
x=complex(0.0_sp,kind=spc)
Start at zero to favor convergence to smallest remaining root.
call laguer(ad(1:j+1),x,its) Find the root.
if (abs(aimag(x)) <= 2.0_sp*EPS**2*abs(real(x))) &
x=complex(real(x),kind=spc) Forward deflation.
roots(j)=x 
end do
if (polish) then
 do j=1,m
 call laguer(a(:),roots(j),its) Polish the roots using the undeflated coefficients.
 end do
end if
call indexx(real(roots),indx) Sort roots by their real parts.
roots=roots(indx)
END SUBROUTINE zroots
```

See the discussion of why we include kind=spc just above. Note that while real(x) returns type default real if $x$ is integer or real, it returns single or double precision correctly if $x$ is complex.

```fortran
* * *
```

$x=\text{complex}(0.0_{\text{sp}},\text{kind}=\text{spc})$...$x=\text{complex}(\text{real}(x),\text{kind}=\text{spc})$
SUBROUTINE zrhqr(a,rtr,rti)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : balanc,hqr,indexx
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: rtr,rti
Find all the roots of a polynomial with real coefficients, \( \sum_{i=1}^{M+1} a(i) x^{i-1} \), given the array of \( M+1 \) coefficients \( a \). The method is to construct an upper Hessenberg matrix whose eigenvalues are the desired roots, and then use the routines balanc and hqr. The real and imaginary parts of the \( M \) roots are returned in \( \text{rtr} \) and \( \text{rti} \), respectively.
INTEGER(I4B) :: k,m
INTEGER(I4B), DIMENSION(size(rtr)) :: indx
REAL(SP), DIMENSION(size(a)-1,size(a)-1) :: hess
m=assert_eq(size(rtr),size(rti),size(a)-1,'zrhqr')
if (a(m+1) == 0.0) call & nrerror('zrhqr: Last value of array a must not be 0')
hess(1,:)=-a(m:1:-1)/a(m+1)
Construct the matrix.
hess(2:m,:)=0.0
do k=1,m-1
hess(k+1,k)=1.0
end do
call balanc(hess)
Find its eigenvalues.
call hqr(hess,rtr,rti)
call indexx(rtr,indx)
Sort roots by their real parts.
rtr=rtr(indx)
rti=rti(indx)
END SUBROUTINE zrhqr

SUBROUTINE qroot(p,b,c,eps)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : poldiv
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: p
REAL(SP), INTENT(INOUT) :: b,c
REAL(SP), INTENT(IN) :: eps
INTEGER(I4B), PARAMETER :: ITMAX=20
REAL(SP), PARAMETER :: TINY=1.0e-6_sp
Given an array of \( N \) coefficients \( p \) of a polynomial of degree \( N-1 \), and trial values for the coefficients of a quadratic factor \( x^2 + bx + c \), improve the solution until the coefficients \( b,c \) change by less than \( \text{eps} \). The routine \text{poldiv} \ of §5.3 is used.
Parameters: \( \text{ITMAX} \) is the maximum number of iterations, \( \text{TINY} \) is a small number.
INTEGER(I4B) :: iter,n
REAL(SP) :: delb,delc,div,r,rb,rc,s,sb,sc
REAL(SP), DIMENSION(3) :: d
REAL(SP), DIMENSION(size(p)) :: q,qq,rem
n=size(p)
d(3)=1.0
do iter=1,ITMAX
iter=1,ITMAX
iter=1,ITMAX
iter=1,ITMAX
iter=1,ITMAX
d(2)=b
d(1)=c
call poldiv(p,d,q,rem)
s=rem(1)
First division gives \( r,s \).
r=rem(2)
call poldiv(q(1:n-1),d(:,),qq(1:n-1),rem(1:n-1))
s=-rem(1)
Second division gives partial \( r,s \) with respect
to \( c \).
sb=-c*rc
rb=sc-b*rc
div=1.0_sp/(sb*rc-sc*rb)
Solve 2x2 equation.
delb = (r*sc - s*rc)*div
delc = (-r*sb + s*rb)*div
b = b + delb
.c = c + delc
if ((abs(delb) <= eps*abs(b) .or. abs(b) < TINY) .and. &
    (abs(delc) <= eps*abs(c) .or. abs(c) < TINY)) RETURN Coefficients converged.
end do
call nrerror ('qroot: too many iterations')
END SUBROUTINE qroot

* * *

SUBROUTINE mnewt(ntrial,x,tolx,tolf,usrfun)
USE nrtype
USE nr, ONLY : lubksb,ludcmp
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: ntrial
REAL(SP), INTENT(IN) :: tolx,tolf
REAL(SP), DIMENSION(:), INTENT(INOUT) :: x
INTERFACE
SUBROUTINE usrfun(x,fvec,fjac)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(OUT) :: fvec
REAL(SP), DIMENSION(:,:), INTENT(OUT) :: fjac
END SUBROUTINE usrfun
END INTERFACE

Given a initial guess x for a root in N dimensions, take ntrial Newton-Raphson steps to
improve the root. Stop if the root converges in either summed absolute variable increments
tolx or summed absolute function values tolf.

INTEGER(I4B) :: i
INTEGER(I4B), DIMENSION(size(x)) :: indx
REAL(SP) :: d
REAL(SP), DIMENSION(size(x)) :: fvec,p
REAL(SP), DIMENSION(size(x),size(x)) :: fjac
do i=1,ntrial
    call usrfun(x,fvec,fjac)
    User subroutine supplies function values at x in fvec and Jacobian matrix in fjac.
    if (sum(abs(fvec)) <= tolf) RETURN Check function convergence.
    p=-fvec Right-hand side of linear equations.
    call ludcmp(fjac,indx,d) Solve linear equations using LU decomposition.
    call lubksb(fjac,indx,p) Update solution.
    x=x+p
    if (sum(abs(p)) <= tolx) RETURN Check root convergence.
end do
END SUBROUTINE mnewt

* * *
SUBROUTINE lnsrch(xold, fold, g, p, x, f, stpmax, check, func)
USE nrtype; USE nrutil, ONLY : assert_eq, nrerror, vabs
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: xold, g
REAL(SP), DIMENSION(:), INTENT(INOUT) :: p
REAL(SP), INTENT(IN) :: fold, stpmax
REAL(SP), DIMENSION(:), INTENT(OUT) :: x
REAL(SP), INTENT(OUT) :: f
LOGICAL(LGT), INTENT(OUT) :: check
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP) :: func
REAL(SP), DIMENSION(:), INTENT(IN) :: x
END FUNCTION func
END INTERFACE
REAL(SP), PARAMETER :: ALF=1.0e-4_sp, TOLX=epsilon(x)

 Given an N-dimensional point xold, the value of the function and gradient there, fold and g, and a direction p, finds a new point x along the direction p from xold where the function func has decreased "sufficiently." xold, g, p, and x are all arrays of length N. The new function value is returned in f. stpmax is an input quantity that limits the length of the steps so that you do not try to evaluate the function in regions where it is undefined or subject to overflow. p is usually the Newton direction. The output quantity check is false on a normal exit. It is true when x is too close to xold. In a minimization algorithm, this usually signals convergence and can be ignored. However, in a zero-finding algorithm the calling program should check whether the convergence is spurious.

 Parameters: ALF ensures sufficient decrease in function value; TOLX is the convergence criterion on \Delta x.

 INTEGER(I4B) :: ndum
 REAL(SP) :: a, alam, alam2, alamin, b, disc, f2, pabs, rhs1, rhs2, slope, tmplam
 ndum=assert_eq(size(g), size(p), size(x), size(xold), 'lnsrch')
 check=.false.
 pabs=vabs(p(:))
 if (pabs > stpmax) p(:)=p(:)*stpmax/pabs
 Scale if attempted step is too big.
 slope=dot_product(g, p)
 if (slope >= 0.0) call nrerror('roundoff problem in lnsrch')
 alamin=TOLX/maxval(abs(p(:))/max(abs(xold(:)), 1.0_sp))
 Compute \lambda_{min}.
 alam=1.0
 Always try full Newton step first.
 do
 Start of iteration loop.
 x(:)=xold(:)+alam*p(:)
 f=func(x)
 if (alam < alamin) then
 Convergence on \Delta x. For zero finding, the calling program should verify the convergence.
 x(:)=xold(:)
 check=.true.
 RETURN
 else if (f <= fold+ALF*alam*slope) then
 Sufficient function decrease.
 RETURN
 else
 Backtrack.
 if (alam == 1.0) then
 tmplam=-slope/(2.0_sp*b)
 else
 disc=b*b-3.0_sp*a*slope
 if (disc < 0.0) then
tmplam=0.5_sp*alam
 else if (a == 0.0) then
 tmplam=slope/(2.0_sp*(f-fold-slope))
 else
 rhs1=f-fold-alam*slope
 rhs2=f2-fold-alam2*slope
 a=((rhs1/alam)**2-rhs2/alam2**2)/(alam-alam2)
 b=-(alam2*rhs1/alam**2+alam*rhs2/alam2**2)/&
 (alam-alam2)
 if (a == 0.0) then
 tmplam=slope/(2.0_sp*b)
 else
 disc=b*b-3.0_sp*a*slope
 if (disc < 0.0) then
tmplam=0.5_sp*alam
 else if (b <= 0.0) then

tmplam = (-b + sqrt(disc)) / (3.0_sp*a)
else
    tmplam = -slope / (b + sqrt(disc))
end if
if (tmplam > 0.5_sp*alam)
    tmplam = 0.5_sp*alam
end if
alam2 = alam
f2 = f
alam = max(tmplam, 0.1_sp*alam)
end do
END SUBROUTINE lnsrch

SUBROUTINE newt(x, check)
    USE nrtype; USE nrutil, ONLY : nrerror, vabs
    USE nr, ONLY : fdjac, lnsrch, lubksb, ludcmp
    USE fminln
    Communicates with fmin.

    IMPLICIT NONE
    REAL(SP), DIMENSION(:), INTENT(INOUT) :: x
    LOGICAL(LGT), INTENT(OUT) :: check
    INTEGER(I4B), PARAMETER :: MAXITS = 200
    REAL(SP), PARAMETER :: TOLF = 1.0e-4_sp, TOLMIN = 1.0e-6_sp, TOLX = epsilon(x), STPMX = 100.0

    Given an initial guess \( x \) for a root in \( N \) dimensions, find the root by a globally convergent Newton's method. The length \( N \) vector of functions to be zeroed, called \( fvec \) in the routine below, is returned by a user-supplied routine that \textit{must} be called \textit{funcv} and have the declaration \textit{FUNCTION funcv(x)}. The output quantity \textit{check} is false on a normal return and true if the routine has converged to a local minimum of the function \textit{fmin} defined below. In this case try restarting from a different initial guess.

    Parameters: \textit{MAXITS} is the maximum number of iterations; \textit{TOLF} sets the convergence criterion on function values; \textit{TOLMIN} sets the criterion for deciding whether spurious convergence to a minimum of \textit{fmin} has occurred; \textit{TOLX} is the convergence criterion on \( \delta x \); \textit{STPMX} is the scaled maximum step length allowed in line searches.

    INTEGER(I4B) :: its
    INTEGER(I4B), DIMENSION(size(x)) :: indx
    REAL(SP) :: d, f, fold, stpmax
    REAL(SP), DIMENSION(size(x)) :: g, p, xold
    REAL(SP), DIMENSION(size(x)) :: fvec
    REAL(SP), DIMENSION(size(x), size(x)) :: fjac
    fmin_fvecp => fvec
    f = fmin(x) \( fvec \) is also computed by this call.
    if (maxval(abs(fvec(:))) < 0.01_sp*TOLF) then
        check = .false.
        RETURN
    end if
    stpmax = STPMX*max(vabs(x(:)), real(size(x), sp)) \( \text{Calculate stpmax for line searches.} \)
    do its=1, MAXITS
        Start of iteration loop.
        call fdjac(x, fvec, fjac)
        If analytic Jacobian is available, you can replace the routine \textit{fdjac} below with your own routine.
        g(:, :) = matmul(fvec(:,), fjac(:, :)) \( \text{Compute } \nabla f \text{ for the line search.} \)
        xold(,:) = x
        fold = f
        p(:, :) = fvec(); \( \text{Store } x, f, \text{ and } f. \)
        right-hand side for linear equations.
        call ludcmp(fjac, indx, d) \( \text{Solve linear equations by } LU \text{ decomposition.} \)
        call lubksb(fjac, indx, p)
        call lnsrch(xold, fold, g, p, x, f, stpmax, check, fmin)
        lnsrch returns new \( x \) and \( f. \) It also calculates \( fvec \) at the new \( x \) when it calls \textit{fmin}.
        if (maxval(abs(fvec(:))) < TOLF) then
            Test for convergence on function values.
            check = .false.
            RETURN
        end if
    end do
end if
if (check) then
  Check for gradient of \( f \) zero, i.e., spurious convergence.
  check=(maxval(abs(g(:))*max(abs(x(:)),1.0_sp) / \( \max(f,0.5_sp*\text{size}(x)) \) < TOLMIN)
  RETURN
end if
if (maxval(abs(x(:)-xold(:))/max(abs(x(:)),1.0_sp)) < TOLX) &
  RETURN
end do
call nrerror('MAXITS exceeded in newt')
END SUBROUTINE newt

USE fmin
Here we have an example of how to pass an array \( fvec \) to a function \( \text{fmin} \) without making it an argument of \( \text{fmin} \). In the language of §21.5, we are using Method 2: We define a pointer \( \text{fmin}_p \) in the module \( \text{fminln} \):

\[
\text{REAL(SP), DIMENSION(,:)}, \text{POINTER :: fmin}_p \text{fvecp}
\]

\( fvec \) itself is declared as an automatic array of the appropriate size in \( \text{newt} \):

\[
\text{REAL(SP), DIMENSION(size(x))}, \text{TARGET :: fvec}
\]

On entry into \( \text{newt} \), the pointer is associated:

\[
\text{fmin}_p \text{fvecp}=>\text{fvec}
\]

The pointer is then used in \( \text{fmin} \) as a synonym for \( fvec \). If you are sufficiently paranoid, you can test whether \( \text{fmin}_p \text{fvecp} \) has in fact been associated on entry into \( \text{fmin} \). Heeding our admonition always to deallocate memory when it no longer is needed, you may ask where the deallocation takes place in this example. Answer: On exit from \( \text{newt} \), the automatic array \( fvec \) is automatically freed.

The Method 1 way of setting up this task is to declare an allocatable array in the module:

\[
\text{REAL(SP), DIMENSION(,:)}, \text{ALLOCATABLE :: fvec}
\]

On entry into \( \text{newt} \) we allocate it appropriately:

\[
\text{allocate}(fvec,size(x))
\]

and it can now be used in both \( \text{newt} \) and \( \text{fmin} \). Of course, we must remember to deallocate explicitly \( fvec \) on exit from \( \text{newt} \). If we forget, all kinds of bad things would happen on a second call to \( \text{newt} \). The status of \( fvec \) on the first return from \( \text{newt} \) becomes undefined. The status cannot be tested with \( \text{if(allocated(...))} \), and \( fvec \) may not be referenced in any way. If we tried to guard against this by adding the \text{SAVE} attribute to the declaration of \( fvec \), then we would generate an error from trying to allocate an already-allocated array.

SUBROUTINE fdjac(x,fvec,df)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(,:), INTENT(IN) :: fvec
REAL(SP), DIMENSION(,:), INTENT(INOUT) :: x
REAL(SP), DIMENSION(,:), INTENT(INOUT) :: df
INTERFACE
FUNCTION funcv(x)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: funcv
END FUNCTION funcv
END INTERFACE

REAL(SP), PARAMETER :: EPS=1.0e-4_sp

Computes forward-difference approximation to Jacobian. On input, x is the point at which the Jacobian is to be evaluated, and fvec is the vector of function values at the point, both arrays of length N. df is the N x N output Jacobian. FUNCTION funcv(x) is a fixed-name, user-supplied routine that returns the vector of functions at x.

Parameter: EPS is the approximate square root of the machine precision.

INTEGER(I4B) :: j,n
REAL(SP), DIMENSION(size(x)) :: xsav,xph,h
n=assert_eq(size(x),size(fvec),size(df,1),size(df,2),'fdjac')
xsav=x
h=EPS*abs(xsav)
where (h == 0.0) h=EPS
xph=xsav+h
Trick to reduce finite precision error.
h=xph-xsav
do j=1,n
 x(j)=xph(j)
 df(:,j)=(funcv(x)-fvec(:))/h(j)
Forward difference formula.
x(j)=xsav(j)
end do
END SUBROUTINE fdjac

MODULE fminln
USE nrtype; USE nrutil, ONLY : nrerror
REAL(SP), DIMENSION(:,), POINTER :: fmin_fvecp
CONTAINS
FUNCTION fmin(x)
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP) :: fmin
Returns \( f = \frac{1}{2} \mathbf{F} \cdot \mathbf{F} \) at x. FUNCTION funcv(x) is a fixed-name, user-supplied routine that returns the vector of functions at x. The pointer fmin_fvecp communicates the function values back to newt.
INTERFACE
FUNCTION funcv(x)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: funcv
END FUNCTION funcv
END INTERFACE
if (.not. associated(fmin_fvecp)) call & nrerror('fmin: problem with pointer for returned values')
fmin_fvecp=funcv(x)
fmin=0.5_sp*dot_product(fmin_fvecp,fmin_fvecp)
END FUNCTION fmin
END MODULE fminln

*   *   *
SUBROUTINE broydn(x,check)
USE nrtype; USE nrutil, ONLY : get_diag,lower_triangle,nrerror,&
outerprod,put_diag,unit_matrix,vabs
USE nr, ONLY : fdjac,lnsrch,qrdcmp,qrupdt,rsolv
USE fminln

Communicates with fmin.

IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: x
LOGICAL(LGT), INTENT(OUT) :: check
INTEGER(I4B), PARAMETER :: MAXITS=200
REAL(SP), PARAMETER :: EPS=epsilon(x),TOLF=1.0e-4_sp,TOLMIN=1.0e-6_sp,&
TOLX=EPS,STPMX=100.0

Given an initial guess \( x \) for a root in \( N \) dimensions, find the root by Broyden's method
embedded in a globally convergent strategy. The length \( N \) vector of functions to be
zeroed, called \( fvec \) in the routine below, is returned by a user-supplied routine that
must be called \( funcv \) and have the declaration FUNCTION \( funcv(x) \). The subroutine \( fdjac \) and
the function \( fmin \) from newt are used. The output quantity \( check \) is false on a normal
return and true if the routine has converged to a local minimum of the function \( fmin \) or if
Broyden's method can make no further progress. In this case try restarting from a different
initial guess.

Parameters: MAXITS is the maximum number of iterations; EPS is the machine precision;
TOLF sets the convergence criterion on function values; TOLMIN sets the criterion for de-
ciding whether spurious convergence to a minimum of \( fmin \) has occurred; TOLX is the
convergence criterion on \( \delta x \); STPMX is the scaled maximum step length allowed in line
searches.

INTEGER(I4B) :: i,its,k,n
REAL(SP) :: f,fold,stpmax
REAL(SP), DIMENSION(size(x)) :: fvec
REAL(SP), DIMENSION(size(x)) :: c,d,fvcold,g,p,s,t,w,xold
REAL(SP), DIMENSION(size(x),size(x)) :: qt,r
LOGICAL :: restrt,sing
fmin_fvecp=>fvec
n=size(x)
f=fmin(x) fvec is also computed by this call.
if (maxval(abs(fvec(:))) < 0.01_sp*TOLF) then
    Test for initial guess being a root.
    Use more stringent test than simply TOLF.
    check=.false.
    RETURN
end if
stpmax=STPMX*max(vabs(x(:)),real(n,sp))
    Calculate stpmax for line searches.
restrt=.true.
    Ensure initial Jacobian gets computed.
do its=1,MAXITS
    Start of iteration loop.
if (restrt) then
    call fdjac(x,fvec,r)
        Initialize or reinitialize Jacobian in r.
call qrdcmp(r,c,d,sing)
            QR decomposition of Jacobian.
        if (sing) call nrerror('singular Jacobian in broydn')
call unit_matrix(qt)
            Form \( Q^T \) explicitly.
do k=1,n-1
    if (c(k) /= 0.0) then
        qt(k:n,:)=qt(k:n,:)-outerprod(r(k:n,k),&
            matmul(r(k:n,k),qt(k:n,:)))/c(k)
    end if
end do
where (lower_triangle(n,n)) r(:,::)=0.0
    call put_diag(d(:,r(:,::)))
else
    Carry out Broyden update.
s(:)=x(:)-xold(:)
do i=1,n
    t(i)=dot_product(r(i,i:n),s(i:n))
    s(:)=s(:)/t
end do
w(:)=fvec(:)-fvcold(:)-matmul(t(:),qt(:,:))
    w=\( \delta F - B \cdot s \).
where (abs(w(:)) < EPS*(abs(fvec(:))+abs(fvcold(:)))) &
    w(:)=0.0
    Don't update with noisy components of w.
if (any(w(:) /= 0.0)) then
    t(:)=matmul(qt(:,::),w(:))
    t = \( Q^T \cdot w \).
s(:)=w(:)/dot_product(s,s)
    Store s/(s \cdot s) in s.
call qrupd(r,qt,t,s)  \text{Update } R \text{ and } Q^T.
d(:)=get_diag(r(:,:))  \text{Diagonal of } R \text{ stored in } d.
if (any(d(:)==0.0)) &
call nrerror(‘r singular in broydn’)  
end if

end if
p(:)=matmul(qt(:,:),fvec(:))  \text{r.h.s. for linear equations is } -Q^T \cdot F.
do i=1,n
g(i)=dot_product(r(1:i,1:i),p(1:i))
end do
xold(:)=x(:)  \text{Store } x, F, \text{ and } f.
fvcold(:)=fvec(:)
fold=f
rsolv(r,d,p)  \text{Solve linear equations.}
srch(xold,fold,check,fmin)
lnsrch returns new x and f. It also calculates fvec at the new x when it calls fmin.
if (maxval(abs(fvec(:))) < TOLF) then
  \text{Test for convergence on function values.}
  check=.false.
  RETURN
end if
if (check) then
  True if line search failed to find a new
  if (restrt .or. maxval(abs(g(:))*max(abs(x(:)), &
    1.0_sp)/max(abs(x(:)), &
    1.0_sp)) < TOLMIN) RETURN
  If restrt is true we have failure: We have already tried reinitializing the Jacobian. The other test is for gradient of f zero, i.e., spurious convergence.
  restrt=.true.
else
  Successful step; will use Broyden update for next step.
  restrt=.false.
  if (maxval((abs(x(:)-xold(:)))/max(abs(x(:)), &
    1.0_sp)) < TOLX) RETURN
  Test for convergence on \delta x.
end if
end do
call nrerror(‘MAXITS exceeded in broydn’)  
END SUBROUTINE broydn

\textbf{USE fmain}  \text{See discussion for newt on p. 1197.}

qt(k:n,:)=...outerprod...matmul  \text{Another example of the coding of equation (22.1.6).}

where \texttt{lower_triangle(n,n)}...  \text{The lower\_triangle function in nrutil returns a lower triangular logical mask. As used here, the mask is true everywhere in the lower triangle of an } n \times n \text{ matrix, excluding the diagonal. An optional integer argument extra allows additional diagonals to be set to true. With extra=1 the lower triangle including the diagonal would be true.}

call put_diag(d(:),r(:,:))  \text{This subroutine in nrutil sets the diagonal values of the matrix } r \text{ to the values of the vector } d. \text{It is overloaded so that } d \text{ could be a scalar, in which case the scalar value would be broadcast onto the diagonal of } r.
Chapter B10. Minimization or Maximization of Functions

SUBROUTINE mnbrak(ax,bx,cx,fa,fb,fc,func)
USE nrtype; USE nrutil, ONLY : swap
IMPLICIT NONE
REAL(SP), INTENT(INOUT) :: ax,bx
REAL(SP), INTENT(OUT) :: cx,fa,fb,fc
INTERFACE
  FUNCTION func(x)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), INTENT(IN) :: x
    REAL(SP) :: func
  END FUNCTION func
END INTERFACE
REAL(SP), PARAMETER :: GOLD=1.618034_sp,GLIMIT=100.0_sp,TINY=1.0e-20_sp

Given a function \textit{func}, and given distinct initial points \textit{ax} and \textit{bx}, this routine searches in the downhill direction (defined by the function as evaluated at the initial points) and returns new points \textit{ax}, \textit{bx}, \textit{cx} that bracket a minimum of the function. Also returned are the function values at the three points, \textit{fa}, \textit{fb}, and \textit{fc}.

Parameters: \textit{GOLD} is the default ratio by which successive intervals are magnified; \textit{GLIMIT} is the maximum magnification allowed for a parabolic-fit step.

REAL(SP) :: fu,q,r,u,ulim
fa=func(ax)
fb=func(bx)
if (fb > fa) then
  Switch roles of \textit{a} and \textit{b} so that we can go downhill in the direction from \textit{a} to \textit{b}.
  call swap(ax,bx)
call swap(fa,fb)
end if

cx=bx+GOLD*(bx-ax)
First guess for \textit{c}.
fc=func(cx)
do

if (fb < fc) RETURN

Compute \textit{u} by parabolic extrapolation from \textit{a}, \textit{b}, \textit{c}. \textit{TINY} is used to prevent any possible division by zero.

\begin{align*}
r &= (bx-ax) \cdot (fb-fc) \\
qu &= (bx-cx) \cdot (fb-fa) \\
u &= bx - (bx-cx) \cdot q \cdot (bx-ax) \cdot r / (2.0 \cdot \text{TINY}) \\
ulim &= bx + GLIMIT \cdot (cx-bx)
\end{align*}

We won’t go farther than this. Test various possibilities:

if ((bx-u) \cdot (u-cx) > 0.0) then
  Parabolic \textit{u} is between \textit{b} and \textit{c}: try it.
  fu=func(u)
  if (fu < fc) then
    Got a minimum between \textit{b} and \textit{c}.
    ax=bx
    fa=fb
    bx=u
    fb=fu
    RETURN
  else if (fu > fb) then
    Got a minimum between \textit{a} and \textit{u}.
    cx=u
    fc=fu
    RETURN

end do

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end if
u=cx+GOLD*(cx-bx)
fu=func(u)
else if (((cx-u)*(u-ulim) > 0.0) then
  if (fu < fc) then
    bx=cx
    cx=u
    u=cx+GOLD*(cx-bx)
call shft(fb,fc,fu,func(u))
  end if
else if (((u-ulim)*(ulim-cx) >= 0.0) then
  u=ulim
  fu=func(u)
else
  call shft(ax,bx,cx,u)
call shft(fa,fb,fc,fu)
end if
end do
CONTAINS
SUBROUTINE shft(a,b,c,d)
REAL(SP), INTENT(OUT) :: a
REAL(SP), INTENT(INOUT) :: b,c
REAL(SP), INTENT(IN) :: d
a=b
b=c
c=d
END SUBROUTINE shft
END SUBROUTINE mnbrak

There are three places in mnbrak where we need to shift four variables around. Rather than repeat code, we make shft an internal subroutine, coming after a CONTAINS statement. It is invisible to all procedures except mnbrak.

FUNCTION golden(ax,bx,cx,func,tol,xmin)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: ax,bx,cx,tol
REAL(SP), INTENT(INOUT) :: xmin
REAL(SP) :: golden
INTERFACE
  FUNCTION func(x)
  USE nrtype
  IMPLICIT NONE
  REAL(SP), INTENT(IN) :: x
  REAL(SP) :: func
END FUNCTION func
END INTERFACE
REAL(SP), PARAMETER :: R=0.61803399_sp,C=1.0_sp-R
Given a function func, and given a bracketing triplet of abscissas ax, bx, cx (such that bx is between ax and cx, and func(bx) is less than both func(ax) and func(cx)), this routine performs a golden section search for the minimum, isolating it to a fractional precision of about tol. The abscissa of the minimum is returned as xmin, and the minimum
function value is returned as golden, the returned function value.
Parameters: The golden ratios.

REAL(SP) :: f1,f2,x0,x1,x2,x3
x0=ax
x3=cx
if (abs(cx-bx) > abs(bx-ax)) then
  x1=bx
  x2=bx+C*(cx-bx)
else
  x2=bx
  x1=bx-C*(bx-ax)
end if
f1=func(x1)
f2=func(x2)

At any given time we will keep track of four points, x0,x1,x2,x3.
Make x0 to x1 the smaller segment, and fill in the new point to be tried.

do  
  if (abs(x3-x0) <= tol*(abs(x1)+abs(x2))) exit
  if (f2 < f1) then
    call shft3(x0,x1,x2,R*x2+C*x3)
    call shft2(f1,f2,func(x2))
  else
    call shft3(x3,x2,x1,R*x1+C*x0)
    call shft2(f2,f1,func(x1))
  end if
end do

if (f1 < f2) then
  golden=f1
  xmin=x1
else
  golden=f2
  xmin=x2
end if

CONTAINS

SUBROUTINE shft2(a,b,c)
REAL(SP), INTENT(OUT) :: a
REAL(SP), INTENT(INOUT) :: b
REAL(SP), INTENT(IN) :: c
a=b
b=c
END SUBROUTINE shft2

SUBROUTINE shft3(a,b,c,d)
REAL(SP), INTENT(OUT) :: a
REAL(SP), INTENT(INOUT) :: b,c
REAL(SP), INTENT(IN) :: d
a=b
b=c
c=d
END SUBROUTINE shft3
END FUNCTION golden

See discussion of shft for mnbrak on p. 1202.

*   *   *
FUNCTION brent(ax, bx, cx, func, tol, xmin)
USE nttype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: ax, bx, cx, tol
REAL(SP), INTENT(OUT) :: xmin
REAL(SP) :: brent
INTERFACE
FUNCTION func(x)
USE nttype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: ITMAX=100
REAL(SP), PARAMETER :: CGOLD=0.3819660_sp, ZEPS=1.0e-3_sp*epsilon(ax)

Given a function func, and given a bracketing triplet of abscissas ax, bx, cx (such that bx is between ax and cx, and func(bx) is less than both func(ax) and func(cx)), this routine isolates the minimum to a fractional precision of about tol using Brent's method.

The abscissa of the minimum is returned as xmin, and the minimum function value is returned as brent, the returned function value.

Parameters: Maximum allowed number of iterations; golden ratio; and a small number that protects against trying to achieve fractional accuracy for a minimum that happens to be exactly zero.

INTEGER(I4B) :: iter
REAL(SP) :: a, b, d, e, etemp, fu, fv, fw, p, q, r, tol, tol1, tol2, u, v, w, x, xold

a=min(ax, cx) and b must be in ascending order, though the input abscissas need not be.
b=max(ax, cx)
v=bx
w=v
x=v
e=0.0
fx=func(x)
fv=fx
fw=fx

do iter=1, ITMAX
xm=0.5_sp*(a+b)
tol1=tol*abs(x)+ZEPS
tol2=2.0_sp*tol
if (abs(x-xm) <= (tol2-0.5_sp*(b-a))) then
xmin=x
return
end if
if (abs(e) > tol1) then
construct a trial parabolic fit.
r=(x-w)*(fx-fv)
q=(x-v)*(fx-fw)
p=(x-v)*q-(x-w)*r
q=2.0_sp*(q-r)
if (q > 0.0) p=-p
q=abs(q)
etemp=e
e=d
if (abs(p) >= abs(0.5_sp*q*etemp) .or. & p <= q*(a-x) .or. p >= q*(b-x)) then
the above conditions determine the acceptability of the parabolic fit. Here it is not o.k., so we take the golden section step into the larger of the two segments.
e=merge(a-x, b-x, x >= xm )
d=CGOLD*e
else
take the parabolic step.
d=p/q
x=x+d
if (u-a < tol2 .or. b-u < tol2) d=sign(tol1, xm-x)
end if

end do
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else
    e = merge(a-x, b-x, x >= xm )
    d = CGOLD*e
end if

u = merge(x+d, x+sign(tol1, d), abs(d) >= tol1 )
Arrive here with d computed either from parabolic fit, or else from golden section.

fu = func(u)
This is the one function evaluation per iteration.

if (fu <= fx) then
    Now we have to decide what to do with our 
    function evaluation. Housekeeping follows:
    a = x
    else
    b = x
    end if
    call shft(v, w, x, u)
call shft(fv, fw, fx, fu)
    else
    if (u < x) then
        a = u
        else
        b = u
        end if
    if (fu <= fw .or. w == x) then
        v = w
        fv = fw
        w = u
        fw = fu
        else if (fu <= fv .or. v == x .or. v == w) then
        v = u
        fv = fu
        end if
    end if
end do

Done with housekeeping. Back for another

call nrerror(‘brent: exceed maximum iterations’) iteration.

CONTAINS

SUBROUTINE shft(a, b, c, d)
REAL(SP), INTENT(OUT) :: a
REAL(SP), INTENT(INOUT) :: b, c
REAL(SP), INTENT(IN) :: d
a = b
b = c
c = d
END SUBROUTINE shft

END FUNCTION brent

*   *   *

FUNCTION dbrent(ax, bx, cx, func, dfunc, tol, xmin)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(IN) :: ax, bx, cx, tol
REAL(SP), INTENT(INOUT) :: xmin
REAL(SP) :: dbrent
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
FUNCTION dfunc(x)
END FUNCTION dfunc

USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: dfunc
END FUNCTION dfunc
END INTERFACE

INTEGER(I4B), PARAMETER :: ITMAX=100
REAL(SP), PARAMETER :: ZEPS=1.0e-3_sp*epsilon(ax)

Given a function func and its derivative function dfunc, and given a bracketing triplet of abscissas ax, bx, cx such that bx is between ax and cx, and func(bx) is less than both func(ax) and func(cx), this routine isolates the minimum to a fractional precision of about tol using a modification of Brent's method that uses derivatives. The abscissa of the minimum is returned as xmin, and the minimum function value is returned as dbrent, the returned function value.

Parameters: Maximum allowed number of iterations, and a small number that protects against trying to achieve fractional accuracy for a minimum that happens to be exactly zero.

INTEGER(I4B) :: iter
REAL(SP) :: a,b,d,d1,d2,du,dv,dw,dx,e,fu,fv,fw,fx,olde,tol1,tol2,&
            u,u1,u2,v,w,x,xm

Comments following will point out only differences from the routine brent. Read that routine first.

LOGICAL :: ok1,ok2
    Will be used as flags for whether proposed steps are acceptable or not.
    a=min(ax,cx)
b=max(ax,cx)
v=bx
w=v
x=v
e=0.0
fx=func(x)
fv=fx
fw=fx
dx=dfunc(x)
dv=dx
dw=dx
dx=dx
do iter=1,ITMAX
xm=0.5_sp*(a+b)
tol1=tol*abs(x)+ZEPS
tol2=2.0_sp*tol1
if (abs(x-xm) <= (tol2-0.5_sp*(b-a))) exit
if (abs(e) > tol1) then
    d1=2.0_sp*(b-a)
    Initialize these d's to an out-of-bracket value.
    d2=d1
if (dw /= dx) d1=(w-x)*dx/(dx-dw)
    Secant method with each point.
if (dv /= dx) d2=(v-x)*dx/(dx-dv)
Which of these two estimates of d shall we take? We will insist that they be within the bracket, and on the side pointed to by the derivative at x:
   u1=x+d1
   u2=x+d2
ok1=((a-u1)*(u1-b) > 0.0) .and. (dx*d1 <= 0.0)
ok2=((a-u2)*(u2-b) > 0.0) .and. (dx*d2 <= 0.0)
olde=e
    Movement on the step before last.
e=d
if (ok1 .or. ok2) then
    Take only an acceptable d, and if both are acceptable, then take the smallest one.
    if (ok1 .and. ok2) then
        d=merge(d1,d2, abs(d1) < abs(d2))
    else
        d=merge(d1,d2,ok1)
    end if
if (abs(d) <= abs(0.5_sp*olde)) then
    u=x+d
    if (u-a < tol2 .or. b-u < tol2) &
      d=sign(tol1,xm-x)
else
e=merge(a,b, dx >= 0.0)-x
Decide which segment by the sign of the derivative.
d=0.5_sp*e    Bisect, not golden section.
end if
else
  e=merge(a,b, dx >= 0.0)-x
d=0.5_sp*e    Bisect, not golden section.
end if
if (abs(d) >= tol1) then
  u=x+d
  fu=func(u)
else
  u=x+sign(tol1,d)
  fu=func(u)
  if (fu > fx) exit
end if
du=dfunc(u)
If the minimum step in the downhill direction takes us uphill, then we are done.
if (fu <= fx) then
  if (u >= x) then
    a=x
  else
    b=x
  end if
  call mov3(v,fv,dv,w,fw,dw)
call mov3(w,fw,dw,x,fx,dx)
call mov3(x,fx,dx,u,fu,du)
else
  if (u < x) then
    a=u
  else
    b=u
  end if
  if (fu <= fw .or. w == x) then
    call mov3(v,fv,dv,w,fw,du)
call mov3(w,fw,dw,u,fu,du)
  else if (fu <= fv .or. v == x .or. v == w) then
    call mov3(v,fv,dv,u,fu,du)
  end if
end if
end do
if (iter > ITMAX) call nrerror('dbrent: exceeded maximum iterations')
xmin=x
dbrent=fx
CONTAINS
SUBROUTINE mov3(a,b,c,d,e,f)
REAL(SP), INTENT(IN) :: d,e,f
REAL(SP), INTENT(OUT) :: a,b,c
a=d
b=e
c=f
END SUBROUTINE mov3
END FUNCTION dbrent

*   *   *
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SUBROUTINE amoeba(p,y,ftol,func,iter)
USE nrtype; USE nrutil, ONLY : assert_eq,imaxloc,iminloc,nrerror,swap
IMPLICIT NONE
INTEGER(14B), INTENT(OUT) :: iter
REAL(SP), INTENT(IN) :: ftol
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: p
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: ITMAX=5000
REAL(SP), PARAMETER :: TINY=1.0e-10

Minimization of the function func in N dimensions by the downhill simplex method of Nelder and Mead. The (N + 1) × N matrix p is input. Its N + 1 rows are N-dimensional vectors that are the vertices of the starting simplex. Also input is the vector y of length N + 1, whose components must be preinitialized to the values of func evaluated at the N + 1 vertices (rows) of p, and ftol the fractional convergence tolerance to be achieved in the function value (n.b.). On output, p and y will have been reset to N + 1 new points all within ftol of a minimum function value, and iter gives the number of function evaluations taken.

Parameters: The maximum allowed number of function evaluations, and a small number.

INTEGER(14B) :: ihi,ndim
REAL(SP), DIMENSION(size(p,2)) :: psum
GLOBAL VARIABLES
SUBROUTINE amoeba_private
IMPLICIT NONE
INTEGER(I4B) :: i,ilo,inhi
REAL(SP) :: rtol,ysave,ytry,ytmp
ndim=assert_eq(size(p,2),size(p,1)-1,size(y)-1,'amoeba')
iter=0
psum(:)=sum(p(:,:),dim=1)
do
ilo=iminloc(y(:))
ihi=imaxloc(y(:))
ytmp=y(ihi)
y(ihi)=y(ilo)
inhi=imaxloc(y(:))
y(ihi)=ytmp
rtol=2.0_sp*abs(y(ihi)-y(ilo))/(abs(y(ihi))+abs(y(ilo))+TINY)
compute the fractional range from highest to lowest and return if satisfactory.
if (rtol < ftol) then
If returning, put best point and value in slot
RETURN
end if
if (iter >= ITMAX) call nrerror('ITMAX exceeded in amoeba')
begin a new iteration. First extrapolate by a factor −1 through the face of the simplex across from the high point, i.e., reflect the simplex from the high point.
ytry=amotry(-1.0_sp)
iter=iter+1
if (ytry <= y(ilo)) then
Gives a result better than the best point, so try an additional extrapolation by a factor of 2.
ytry=amotry(2.0_sp)
iter=iter+1
else if (ytry >= y(inhi)) then
The reflected point is worse than the second highest, so look for an intermediate lower point, i.e., do a one-dimensional contraction.
ysave=y(ihi)
ytry=amotry(0.5_sp)
iter=iter+1
end if
end do

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if (ytry >= ysave) then
  Can’t seem to get rid of that high point. Better contract around the lowest (best) point.
  p(:,:,)=0.5_sp*(p(:,:,)+spread(p(ilo,:),1,size(p,1)))
do i=1,ndim+1
  if (i /= ilo) y(i)=func(p(i,:))
end do
  iter=iter+ndim
  psum(:)=sum(p(:,:,),dim=1)
end if
end do
  Go back for the test of doneness and the next iteration.
FUNCTION amotry(fac)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: fac
REAL(SP) :: amotry

Extrapolates by a factor fac through the face of the simplex across from the high point, tries it, and replaces the high point if the new point is better.

REAL(SP), INTENT(IN) :: ptry
REAL(SP), DIMENSION(size(p,2)) :: ptry

fac1=(1.0_sp-fac)/ndim
fac2=fac1-fac

if (ytry < y(ihi)) then
  If it’s better than the highest, then replace the highest.
  y(ihi)=ytry
  psum(:)=psum(:)-p(ihi,:)+ptry(:)
  p(ihi,:)=ptry(:)
end if

end function

END SUBROUTINE amoeba

The only action taken by the subroutine amoeba is to call the internal subroutine amoeba_private. Why this structure? The reason has to do with meeting the twin goals of data hiding (especially for “safe” scope of variables) and program readability. The situation is this: Logically, amoeba does most of the calculating, but calls an internal subroutine amotry at several different points, with several values of the parameter fac. However, fac is not the only piece of data that must be shared with amotry; the latter also needs access to several shared variables (ihi, ndim, psum) and arguments of amoeba (p, y, func).

The obvious (but not best) way of coding this would be to put the computational guts in amoeba, with amotry as the sole internal subprogram. Assuming that fac is passed as an argument to amotry (it being the parameter that is being rapidly altered), one must decide whether to pass all the other quantities to amotry (i) as additional arguments (as is done in the Fortran 77 version), or (ii) “automatically,” i.e., doing nothing except using the fact that an internal subprogram has automatic access to all of its host’s entities. Each of these choices has strong disadvantages. Choice (i) is inefficient (all those arguments) and also obscures the fact that fac is the primary changing argument. Choice (ii) makes the program extremely difficult to read, because it wouldn’t be obvious without careful cross-comparison of the routines which variables in amoeba are actually global variables that are used by amotry.

Choice (ii) is also “unsafe scoping” because it gives a nontrivially complicated internal subprogram, amotry, access to all the variables in its host. A common and difficult-to-find bug is the accidental alteration of a variable that one “thought”
was local, but is actually shared. (Simple variables like \(i\), \(j\), and \(n\) are the most common culprits.)

We are therefore led to reject both choice (i) and choice (ii) in favor of a structure previously described in the subsection on Scope, Visibility, and Data Hiding in §21.5. The guts of \texttt{amoeba}\_private\_a, a \textit{sister routine} to \texttt{amotry}. These two siblings have mutually private name spaces. However, any variables that they need to share (including the top-level arguments of \texttt{amoeba}) are declared as variables in the enclosing \texttt{amoeba} routine. The presence of these “global variables” serves as a warning flag to the reader that data are shared between routines.

An alternative attractive way of coding the above situation would be to use a module containing \texttt{amoeba} and \texttt{amotry}. Everything would be declared private except the name \texttt{amoeba}\_pre. The global variables would be at the top level, and the arguments of \texttt{amoeba} that need to be passed to \texttt{amotry} would be handled by pointers among the global variables. Unfortunately, Fortran 90 does not support pointers to functions. Sigh!

\[
\texttt{ilo=iminloc...ihi=imaxloc...} \quad \text{See discussion of these functions on p. 1017.}
\]

\[
\texttt{call swap(y(1)...call swap(p(1,:)...} \quad \text{Here the} \texttt{swap} \text{routine in} \texttt{nrutil} \text{is called once with a scalar argument and once with a vector argument. Inside} \texttt{nrutil} \text{scalar and vector versions have been overloaded onto the single name} \texttt{swap}, \text{hiding all the implementation details from the calling routine.}
\]

\[
⋆⋆⋆ \quad \text{SUBROUTINE powell(p,xi,ftol,iter,fret)}
\]

\[
\text{USE nrtype; USE nrutil, ONLY : assert_eq,nrerror}
\]

\[
\text{USE nr, ONLY : linmin}
\]

\[
\text{IMPLICIT NONE}
\]

\[
\text{REAL(SP), DIMENSION(:,), INTENT(INOUT) :: p}
\]

\[
\text{REAL(SP), DIMENSION(:,), INTENT(INOUT) :: xi}
\]

\[
\text{INTEGER(I4B), INTENT(OUT) :: iter}
\]

\[
\text{REAL(SP), INTENT(IN) :: ftol}
\]

\[
\text{REAL(SP), INTENT(OUT) :: fret}
\]

\[
\text{INTERFACE}
\]

\[
\text{FUNCTION func(p)}
\]

\[
\text{USE nrtype}
\]

\[
\text{IMPLICIT NONE}
\]

\[
\text{REAL(SP), DIMENSION(:,), INTENT(IN) :: p}
\]

\[
\text{REAL(SP) :: func}
\]

\[
\text{END FUNCTION func}
\]

\[
\text{END INTERFACE}
\]

\[
\text{INTEGER(I4B), PARAMETER :: ITMAX=200}
\]

\[
\text{REAL(SP), PARAMETER :: TINY=1.0e-25_sp}
\]

Minimization of a function \texttt{func} of \(N\) variables. (\texttt{func} is not an argument, it is a fixed function name.) Input consists of an initial starting point \(p\), a vector of length \(N\); an initial \(N \times N\) matrix \(x\) whose columns contain the initial set of directions (usually the \(N\) unit vectors); and \texttt{ftol}, the fractional tolerance in the function value such that failure to decrease by more than this amount on one iteration signals doneness. On output, \(p\) is set to the best point found, \(x\) is the then-current direction set, \texttt{fret} is the returned function value at \(p\), and \texttt{iter} is the number of iterations taken. The routine \texttt{linmin} is used.

Parameters: Maximum allowed iterations, and a small number.

\[
\text{INTEGER(I4B) :: i,ibig,n}
\]

\[
\text{REAL(SP) :: del,fp,fptt,t}
\]

\[
\text{REAL(SP), DIMENSION(size(p)) :: pt,ptt,xit}
\]

\[
\text{n=assert_eq(size(p),size(xi,1),size(xi,2),'powell')}
\]

\[
\text{fret=func(p)}
\]
pt(:)=p(:) Save the initial point.
iter=0

do
iter=iter+1
fp=fret
ibig=0
del=0.0 Will be the biggest function decrease.
do i=1,n
xit(:,i)=xi(:,i)
Copy the direction,
fptt=fret
call linmin(p,xit,fret) minimize along it,
if (fptt-fret \textgreater del) then and record it if it is the largest decrease so far.
del=fptt-fret
ibig=i
end if
end do
if (2.0_sp*(fp-fret) \leq ftol*(\text{abs}(fp)+\text{abs}(fret))+\text{TINY}) RETURN
Termination criterion.
if (iter == \text{ITMAX}) call &
nrerror('powell exceeding maximum iterations')
ptt(:)=2.0_sp*p(:)-pt(:) Construct the extrapolated point and the average direction moved. Save the old starting point.
xit(:)=p(:)-pt(:)
pt(:)=p(:)
fptt=func(ptt) Function value at extrapolated point.
if (fptt \geq fp) cycle One reason not to use new direction.
t=2.0_sp*(fp-2.0_sp*fret+fptt) \text{*(}} (fp-fret-del)\text{**2-d}el* (fp-fptt)\text{**2}
if (t \geq 0.0) cycle Other reason not to use new direction.
call linmin(p,xit,fret) Move to minimum of the new direction, and save the new direction.
xit(:,ibig)=xit(:,n)
xi(:,n)=xit(:,i)
end do Back for another iteration.
END SUBROUTINE powell

* * *

\textit{MODULE f1dim_mod} \quad Used for communication from \texttt{linmin} to \texttt{f1dim}.

\texttt{USE} nrtype
\texttt{INTEGER(14B)} :: ncom
\texttt{REAL(SP), DIMENSION(:), POINTER :: pcom, xicom}
\texttt{CONTAINS}

\texttt{FUNCTION f1dim(x)}
\texttt{IMPLICIT NONE}
\texttt{REAL(SP), \text{INTENT(IN)}} :: x
\texttt{REAL(SP)} :: f1dim
\texttt{USE} nrtype
\texttt{FUNCTION func(x)}
\texttt{REAL(SP), \text{DIMENSION(:), \text{INTENT(IN)}} :: x}
\texttt{REAL(SP)} :: func
\texttt{CONTAINS}
\texttt{INTERFACE}
\texttt{FUNCTION func(x)}
\texttt{REAL(SP), \text{DIMENSION(:), \text{INTENT(IN)}} :: x}
\texttt{REAL(SP)} :: func
\texttt{END FUNCTION func}
\texttt{END INTERFACE}

\texttt{REAL(SP), \text{DIMENSION(:), \text{ALLOCATABLE :: xt}}}
\texttt{allocate(xt(ncom))}
\texttt{xt(:)=pcom(:)**x**xicom(:)}
\texttt{f1dim=func(xt)}
\texttt{deallocate(xt)}
\texttt{END FUNCTION f1dim}

\texttt{END\ MODULE f1dim_mod}
SUBROUTINE linmin(p, xi, fret)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : mnbrak, brent
USE f1dim_mod
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: fret
REAL(SP), DIMENSION(:), TARGET, INTENT(INOUT) :: p, xi
REAL(SP), PARAMETER :: TOL = 1.0e-4_sp

Given an \(N\)-dimensional point \(p\) and an \(N\)-dimensional direction \(\xi\), both vectors of length \(N\), moves and resets \(p\) to where the fixed-name function \(func\) takes on a minimum along the direction \(\xi\) from \(p\), and replaces \(\xi\) by the actual vector displacement that \(p\) was moved. Also returns as \(fret\) the value of \(func\) at the returned location \(p\). This is actually all accomplished by calling the routines \(mnbrak\) and \(brent\).

Parameter: Tolerance passed to \(brent\).

REAL(SP) :: ax, bx, fa, fb, fx, xmin, xx
ncom=assert_eq(size(p), size(xi), 'linmin')
pcom=>p
Communicate the global variables to \(f1dim\).
xicom=>xi
ax=0.0
Initial guess for brackets.
xx=1.0
call mnbrak(ax, xx, bx, fa, fb, fx, f1dim)
fret=brent(ax, xx, bx, f1dim, TOL, xmin)
xi=xmin*xi
Construct the vector results to return.
p=p+xi
END SUBROUTINE linmin

At first sight this situation is like the one involving \(USE \ fminLn \ in \ newt\) on p. 1197: We want to pass arrays \(p\) and \(\xi\) from \(\text{linmin}\) to \(\text{f1dim}\) without having them be arguments of \(\text{f1dim}\). If you recall the discussion in §21.5 and on p. 1197, there are two ways of effecting this: via pointers or via allocatable arrays. There is an important difference here, however. The arrays \(p\) and \(\xi\) are themselves arguments of \(\text{linmin}\), and so cannot be allocatable arrays in the module. If we did want to use allocatable arrays in the module, we would have to copy \(p\) and \(\xi\) into them. The pointer implementation is much more elegant, since no unnecessary copying is required. The construction here is identical to the one in \(\text{fminLn}\) and \(\text{newt}\), except that \(p\) and \(\xi\) are arguments instead of automatic arrays.

* * *

MODULE df1dim_mod
Used for communication from \(\text{dlinmin}\) to \(\text{f1dim}\) and \(\text{df1dim}\).
USE nrtype
INTEGER(I4B) :: ncom
REAL(SP), DIMENSION(:,), POINTER :: pcom, xicom

CONTAINS

FUNCTION f1dim(x)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: f1dim
Used by \(\text{dlinmin}\) as the one-dimensional function passed to \(\text{mnbrak}\).

INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE

REAL(SP), DIMENSION(:,), ALLOCATABLE :: xt
allocate(xt(ncom))
x(t(:)=pcom(:)+x*xicom(:)
f1dim=func(xt)
deallocate(xt)
END FUNCTION f1dim

FUNCTION df1dim(x)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP) :: df1dim
Used by dlinmin as the one-dimensional function passed to dbrent.

INTERFACE
FUNCTION dfunc(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: dfunc
END FUNCTION dfunc
END INTERFACE

REAL(SP), DIMENSION(:), ALLOCATABLE :: xt,df
allocate(xt(ncom),df(ncom))
x(t(:)=pcom(:)+x*xicom(:)
df(:)=dfunc(xt)
df1dim=dot_product(df,xicom)
deallocate(xt,df)
END FUNCTION df1dim
END MODULE df1dim_mod

SUBROUTINE dlinmin(p,xi,fret)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : mnbrak,dbrent
USE df1dim_mod
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: fret
REAL(SP), DIMENSION(:), TARGET :: p,xi
REAL(SP), PARAMETER :: TOL=1.0e-4_sp
Given an N-dimensional point p and an N-dimensional direction xi, both vectors of length
N, moves and resets p to where the fixed-name function func takes on a minimum along
the direction xi from p, and replaces xi by the actual vector displacement that p was
moved. Also returns as fret the value of func at the returned location p. This is actually
all accomplished by calling the routines mnbrak and dbrent. dfunc is a fixed-name user-
supplied function that computes the gradient of func.
Parameter: Tolerance passed to dbrent.
REAL(SP) :: ax,bx,fa,fb,fx, xmin,xx
ncom=assert_eq(size(p),size(xi),'dlinmin')
pcom=>p
xicom=>xi
ax=0.0
Initial guess for brackets.
xx=1.0
call mnbrak(ax,xx,fa,fb,fx,f1dim)
fret=dbrent(ax,xx,fb,df1dim,df1dim,TOL,xmin)
xi=xmin*xi
Construct the vector results to return.
p=p+xi
END SUBROUTINE dlinmin

USE df1dim_mod  See discussion of USE f1dim_mod on p. 1212.

*   *   *
SUBROUTINE frprmn(p, ftol, iter, fret)
USE nrttype; USE nrutil, ONLY : nrerror
USE nr, ONLY : linmin
IMPLICIT NONE
INTEGER(I4B), INTENT(OUT) :: iter
REAL(SP), INTENT(IN) :: ftol
REAL(SP), INTENT(OUT) :: fret
REAL(SP), DIMENSION(:), INTENT(INOUT) :: p

INTERFACE
FUNCTION func(p)
USE nrttype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: p
REAL(SP) :: func
END FUNCTION func

FUNCTION dfunc(p)
USE nrttype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: p
REAL(SP), DIMENSION(size(p)) :: dfunc
END FUNCTION dfunc
END INTERFACE

INTEGER(I4B), PARAMETER :: ITMAX=200
REAL(SP), PARAMETER :: EPS=1.0e-10_sp

Given a starting point \( p \) that is a vector of length \( N \), Fletcher-Reeves-Polak-Ribiere minimization is performed on a function \( \text{func} \), using its gradient as calculated by a routine \( \text{dfunc} \). The convergence tolerance on the function value is input as \( \text{ftol} \). Returned quantities are \( p \) (the location of the minimum), \( \text{iter} \) (the number of iterations that were performed), and \( \text{fret} \) (the minimum value of the function). The routine \( \text{linmin} \) is called to perform line minimizations.

Parameters: \( \text{ITMAX} \) is the maximum allowed number of iterations; \( \text{EPS} \) is a small number to rectify the special case of converging to exactly zero function value.

INTEGER(I4B) :: its
REAL(SP) :: dgg, fp, gam, gg
REAL(SP), DIMENSION(size(p)) :: g, h, xi

fp=func(p)  ! Initializations.
xi=dfunc(p)
g=-xi
h=g
xi=h

do its=1, ITMAX  ! Loop over iterations.
    iter=its
    call linmin(p, xi, fret)  ! Next statement is the normal return:
    if (2.0_sp*abs(fret-fp) <= ftol*(abs(fret)+abs(fp)+EPS)) RETURN
    fp=fret
    xi=dfunc(p)
    !
    gg=dot_product(g, g)
dgg=dot_product(xi, xi)  ! This statement for Fletcher-Reeves.
dgg=dot_product(xi+h, xi)  ! This statement for Polak-Ribiere.
    if (gg == 0.0) RETURN  ! Unlikely. If gradient is exactly zero then we are already done.
    gam = dgg/gg
    g=-xi
    h=g+gam*h
    xi=h
end do

call nrerror('frprmn: maximum iterations exceeded')

END SUBROUTINE frprmn

*   *   *
SUBROUTINE dfpmin(p, gtol, iter, fret, func, dfunc)
USE nrtype; USE nrutil, ONLY : nrerror, outerprod, unit_matrix, vabs
USE nr, ONLY : lnsrch
IMPLICIT NONE
INTEGER(4B), INTENT(OUT) :: iter
REAL(SP), INTENT(IN) :: gtol
REAL(SP), INTENT(OUT) :: fret
REAL(SP), DIMENSION(:], INTENT(INOUT) :: p
INTERFACE
  FUNCTION func(p)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), DIMENSION(:], INTENT(IN) :: p
    REAL(SP) :: func
  END FUNCTION func
  FUNCTION dfunc(p)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), DIMENSION(:] :: p
    REAL(SP), DIMENSION(size(p)) :: dfunc
  END FUNCTION dfunc
END INTERFACE
INTEGER(4B), PARAMETER :: ITMAX=200
REAL(SP), PARAMETER :: STPMX=100.0_sp, EPS=epsilon(p), TOLX=4.0_sp*EPS
Given a starting point \( p \) that is a vector of length \( N \), the Broyden-Fletcher-Goldfarb-Shanno variant of Davidon-Fletcher-Powell minimization is performed on a function \( \text{func} \), using its gradient as calculated by a routine \( \text{dfunc} \). The convergence requirement on zeroing the gradient is input as \( \text{gtol} \). Returned quantities are \( p \) (the location of the minimum), \( \text{iter} \) (the number of iterations that were performed), and \( \text{fret} \) (the minimum value of the function). The routine \( \text{lnsrch} \) is called to perform approximate line minimizations.

Parameters: \( \text{ITMAX} \) is the maximum allowed number of iterations; \( \text{STPMX} \) is the scaled maximum step length allowed in line searches; \( \text{EPS} \) is the machine precision; \( \text{TOLX} \) is the convergence criterion on \( x \) values.

INTEGER(4B) :: its
LOGICAL :: check
REAL(SP) :: den, fac, fad, fae, fp, stpmax, sumdg, sumxi
REAL(SP), DIMENSION(size(p)) :: dg, hdg, pnew, xi
REAL(SP), DIMENSION(size(p), size(p)) :: hessin
fp=func(p)  
Calculate starting function value and gradient.
g=dfunc(p)
call unit_matrix(hessin)  
Initialize inverse Hessian to the unit matrix.
xi=-g  
Initial line direction.
stpmax=STPMX*max(vabs(p), real(size(p), sp))
do its=1, ITMAX  
Main loop over the iterations.
  iter=its
  call lnsrch(p, fp, g, xi, pnew, fret, stpmax, check, func)  
The new function evaluation occurs in \( \text{lnsrch} \); save the function value in \( \text{fp} \) for the next line search. It is usually safe to ignore the value of check.
  fp=fret
  xi=pnew  
  Update the line direction, and the current point.
  if (maxval(abs(xi)/max(abs(p), 1.0_sp)) < TOLX) RETURN  
  Test for convergence on \( \Delta x \).
  dg=g  
  Save the old gradient,
g=dfunc(p)  
  and get the new gradient.
  den=max(fret, 1.0_sp)  
  if (maxval(abs(g)*max(abs(p), 1.0_sp)/den) < \text{gtol}) RETURN  
  Test for convergence on zero gradient.
  dg=g-dg  
  Compute difference of gradients,
  hdg=matmul(hessin, dg)  
  and difference times current matrix.
  fac=dot_product(dg, xi)  
  Calculate dot products for the denominators.
  fae=dot_product(dg, hdg)
  sumdg=dot_product(dg, dg)
END SUBROUTINE dfpmin
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sumxi=dot_product(xi,xi)
if (fac > sqrt(EPS*sumdg*sumxi)) then  
  Skip update if fac not sufficiently positive.
  fac=1.0_sp/fac
  fad=1.0_sp/fae
  dg=fac*xi-fad*hdg  
  Vector that makes BFGS different from DFP.
  hessin=hessin+fac*outerprod(xi,xi)-&
  The BFGS updating formula.
  fad*outerprod(hdg,hdg)+fae*outerprod(dg,dg)
end if
xi=-matmul(hessin,g)
  Now calculate the next direction to go.
end do
and go back for another iteration.
call nrerror('dfpmin: too many iterations')

END SUBROUTINE dfpmin

call unit_matrix(hessin)  
The unit_matrix routine in nrutil does exactly what its name suggests. The routine dfpmin makes use of outerprod from nrutil, as well as the matrix intrinsics matmul and dot_product, to simplify and parallelize the coding.

⋆⋆⋆

SUBROUTINE simplx(a,m1,m2,m3,icase,izrov,iposv)
USE nrtype; USE nrutil, ONLY : arth,assert_eq,ifirstloc,imaxloc,&nrerror,outerprod,swap
IMPLICIT NONE
REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: m1,m2,m3
INTEGER(I4B), DIMENSION(:), INTENT(OUT) :: icase
INTEGER(I4B), DIMENSION(m2) :: l3
REAL(SP) :: bmax
LOGICAL(LGT) :: init

m=assert_eq(size(a,1)-2,size(iposv),'simplx:m')
n=assert_eq(size(a,2)-1,size(izrov),'simplx:n')
if (m /= m1+m2+m3) call nrerror('simplx:bad input constraint counts')
if (any(a(2:m+1,1) < 0.0)) call nrerror('bad input tableau in simplx')

Constants bi must be nonnegative.

iposv(:)=arth(1,1,m)
  Initially make all variables right-hand.
izrov(:)=l1(1:n)
  Initial left-hand variables. m1 type constraints are represented by having their slack variable initially left-hand, with no artificial variable. m2 type constraints have their slack variable initially left-hand, with a minus sign, and their artificial variable handled implicitly during their first exchange. m3 type constraints have their artificial variable initially left-hand.
init=.true.

phase1: do
  if (init) then  
    Initial pass only.
    if (m2+m3 == 0) exit phase1  
    Origin is a feasible solution. Go to phase two.
    init=.false.
  end if
  l3(1:m2)=1
  Initialize list of m2 constraints whose slack variables have never been exchanged out of the initial basis.
a(m2+1:n+1)=sum(a(m1+2:m+1,n+1),dim=1)
  Compute the auxiliary objective function.
if (nl1 > 0) then
  kp=11(imaxloc(a(m+2,11(1:nl1)+1)))))
  bmax=a(m+2,kp+1)
else
  bmax=0.0
end if

phasea: do
  if (bmax <= EPS .and. a(m+2,1) < -EPS) then
    Auxiliary objective function is still negative and can't be improved, hence no
    feasible solution exists.
    icase=-1
    RETURN
  else if (bmax <= EPS .and. a(m+2,1) <= EPS) then
    Auxiliary objective function is zero and can't be improved. This signals that we
    have a feasible starting vector. Clean out the artificial variables corresponding
    to any remaining equality constraints and then eventually exit phase one.
    do ip=m1+m2+1,m
      if (iposv(ip) == ip+n) then
        Found an artificial variable for an equal-
      if (nl1 > 0) then
        k1=11(imaxloc(abs(a(ip+1,11(1:nl1)+1)))))
        bmax=a(ip+1,kp+1)
      else
        bmax=0.0
      end if
    end do
    where (spread(l3(1:m2),2,n+1) == 1) &
    a(m1+2:m1+m2+1,1:n+1)=-a(m1+2:m1+m2+1,1:n+1)
    Change sign of row for any m2 constraints still present from the initial basis.
    exit phase1
    Go to phase two.
  end if
  call simp1
  if (ip == 0) then
    Maximum of auxiliary objective function is
    unbounded, so no feasible solution ex-
    icase=-1
    RETURN
  end if
end do phasea
end do phasea

call simp2(m+1,n)
if (iposv(ip) >= n+m1+m2+1) then
  Exchanged out an artificial variable for an
  exchanged out of the
  l1
  kh=iposv(ip)-m1-n
  if (kh > 1) then
    Exchanged out an m2 type constraint.
    a(m+2,kp+1)=a(m+2,kp+1)+1.0_sp
    a(1:m+2,kp+1)=-a(1:m+2,kp+1)
  end if
end if
end do phasela

We have an initial feasible solution. Now optimize it.
if (nl1 > 0) then
  kp=11(imaxloc(a(1,11(1:nl1)+1)))))
  bmax=a(1,kp+1)
else
  bmax=0.0
end if
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if (bmax <= EPS) then
  icase=0
  RETURN
end if

call simp1

if (ip == 0) then
  Objective function is unbounded. Report and return.
  icase=1
  RETURN
end if

call simp2(m,n)

Exchange a left- and a right-hand variable, update lists of left- and right-hand variables, and return for another iteration.

CONTAINS

SUBROUTINE simp1
  Locate a pivot element, taking degeneracy into account.
  IMPLICIT NONE
  INTEGER(I4B) :: i,k
  REAL(SP) :: q,q0,q1,qp
  ip=0
  i=ifirstloc(a(2:m+1,kp+1) < -EPS)
  if (i > m) RETURN
  q1=-a(i+1,1)/a(i+1,kp+1)
  ip=i
  do i=ip+1,m
    if (a(i+1,kp+1) < -EPS) then
      q=-a(i+1,1)/a(i+1,kp+1)
      if (q < q1) then
        ip=i
        q1=q
      else if (q == q1) then
        We have a degeneracy.
        do k=1,n
          qp=-a(ip+1,k+1)/a(ip+1,kp+1)
          q0=-a(i+1,k+1)/a(i+1,kp+1)
          if (q0 /= qp) exit
        end do
        if (q0 < qp) ip=i
      end if
    end if
  end do
END SUBROUTINE simp1

SUBROUTINE simp2(i1,k1)
  IMPLICIT NONE
  INTEGER(I4B), INTENT(IN) :: i1,k1
  INTEGER(I4B) :: ip1,kp1
  REAL(SP) :: piv
  INTEGER(I4B), DIMENSION(k1) :: icol
  INTEGER(I4B), DIMENSION(i1) :: irow
  INTEGER(I4B), DIMENSION(max(i1,k1)+1) :: itmp
  ip1=ip+1
  kp1=k+1
  piv=1.0_sp/a(ip1,kp1)
  itmp(1:k1+1)=arth(1,1,k1+1)
  icol=pack(itmp(1:k1+1),itmp(1:k1+1) /= kp1)
  itmp(1:i1+1)=arth(1,1,i1+1)
  irow=pack(itmp(1:i1+1),itmp(1:i1+1) /= ip1)
  a(irow,kp1)=a(irow,kp1)*piv
  a(irow,icol)=a(irow,icol)-outerprod(a(irow,kp1),a(ip1,icol))
  a(ip1,icol)=a(ip1,icol)*piv
  a(ip1,kp1)=piv
END SUBROUTINE simp2
END SUBROUTINE simp1
main_procedure: do  The routine simpex makes extensive use of named
do-loops to control the program flow. The various exit statements have
the names of the do-loops attached to them so we can easily tell where
control is being transferred to. We believe that it is almost never necessary to use
goto statements: Code will always be clearer with well-constructed block structures.

phase1a: do...end do phase1a  This is not a real do-loop: It is executed only
once, as you can see from the unconditional exit before the end do. We use this
construction to define a block of code that is traversed once but that has several
possible exit points.

where (spread((13(i-m1) == 1) &
a(m1+2:m12+1,1:n+1)=-a(m1+2:m12+1,1:n+1)
These lines are equivalent to
do i=m1+1,m12
  if (l3(i-m1) == 1) a(i+1,1:n+1)=-a(i+1,1:n+1)
end do

SUBROUTINE anneal(x,y,iorder)
USE nrttype; USE nrutil, ONLY : arith,assert_eq,swap
USE nr, ONLY : ran1
IMPLICIT NONE
INTEGER(I4B), DIMENSION(:), INTENT(INOUT) :: iorder
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y
This algorithm finds the shortest round-trip path to \(N\) cities whose coordinates are in the
length \(N\) arrays \(x, y\). The length \(N\) array \(iorder\) specifies the order in which the cities are
visited. On input, the elements of \(iorder\) may be set to any permutation of the numbers
1 \ldots N. This routine will return the best alternative path it can find.

INTEGER(I4B), DIMENSION(6) :: n
INTEGER(I4B) :: i1,i2,j,k,nlimit,ncity,nn,nover,nsucc
REAL(SP) :: de,harvest,path,t,tfactr
LOGICAL(LGT) :: ans
ncity=assert_eq(size(x),size(y),size(iorder),'anneal')
nover=100*ncity
nlimit=10*ncity

Maximum number of paths tried at any temperature,
and of successful path changes before continuing.
tfactr=0.9_sp
Annealing schedule: \(t\) is reduced by this factor on
each step.
t=0.5_sp
path=sum(alen_v(x(iorder(1:ncity-1)),x(iorder(2:ncity)),&
y(iorder(1:ncity-1)),y(iorder(2:ncity))))
Calculate initial path length.
i1=iorder(ncity)
Close the loop by tying path ends to-
gether.
i2=iorder(1)
path=path+alen(x(i1),x(i2),y(i1),y(i2))
do j=1,100
  Try up to 100 temperature steps.
  nsucc=0
  do k=1,nover
    do
call ran1(harvest)

Decide whether to do a reversal or a transport.

if (harvest < 0.5_sp) then
    Do a transport.
    call ran1(harvest)
    n(3)=n(2)+int(abs(nn-2)*harvest)+1
    Transport to a location not on the path.
    call trncst(x,y,iorder,n,de)  
    Consult the oracle.
    if (ans) then
        nsucc=nsucc+1
        path=path+de
        call trnspt(iorder,n)
        Carry out the transport.
    end if
else
    Do a path reversal.
    call revcst(x,y,iorder,n,de)  
    Calculate cost.
    call metrop(de,t,ans)  
    Consult the oracle.
    if (ans) then
        nsucc=nsucc+1
        path=path+de
        call revers(iorder,n)  
        Carry out the reversal.
    end if
end if

if (nsucc >= nlimit) exit
Finish early if we have enough successful changes.

end do

write(*,*)
write(*,*) 'T =',t,' Path Length =',path
write(*,*) 'Successful Moves:',nsucc

end do

CONTAINS

FUNCTION alen(x1,x2,y1,y2)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1,x2,y1,y2
REAL(SP) :: alen

Computes distance between two cities.
alen=sqrt((x2-x1)**2+(y2-y1)**2)
END FUNCTION alen

FUNCTION alen_v(x1,x2,y1,y2)
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x1,x2,y1,y2
REAL(SP), DIMENSION(size(x1)) :: alen_v

Computes distances between pairs of cities.
alen_v=sqrt((x2-x1)**2+(y2-y1)**2)
END FUNCTION alen_v

SUBROUTINE metrop(de,t,ans)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: de,t
LOGICAL(LGT), INTENT(OUT) :: ans

Metropolis algorithm. ans is a logical variable that issues a verdict on whether to accept a reconfiguration that leads to a change de in the objective function. If de<0, ans=.true., while if de>0, ans is only .true. with probability \( \exp(-de/t) \), where t is a temperature determined by the annealing schedule.

call ran1(harvest)
ans=(de < 0.0) .or. (harvest < \exp(-de/t))
END SUBROUTINE metrop

SUBROUTINE revcst(x,y,iorder,n,de)
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: iorder
INTEGER(I4B), DIMENSION(:,), INTENT(INOUT) :: n
REAL(SP), INTENT(OUT) :: de
This subroutine returns the value of the cost function for a proposed path reversal. The arrays \(x\) and \(y\) give the coordinates of these cities. \(iorder\) holds the present itinerary. The first two values \(n(1)\) and \(n(2)\) of array \(n\) give the starting and ending cities along the path segment which is to be reversed. On output, \(de\) is the cost of making the reversal. The actual reversal is not performed by this routine.

```fortran
INTEGER(I4B) :: ncity
REAL(SP), DIMENSION(4) :: xx,yy
ncity=size(x)
n(3)=1+mod((n(1)+ncity-2),ncity)  ! Find the city before \(n(1)\) ...
ncity=size(x)
n(4)=1+mod(n(2),ncity)  ! and the city after \(n(2)\).
xx(1:4)=x(iorder(n(1:4)))  ! Find coordinates for the four cities involved.
yy(1:4)=y(iorder(n(1:4)))
de=-alen(xx(1),xx(3),yy(1),yy(3))&
    -alen(xx(2),xx(4),yy(2),yy(4))&
    +alen(xx(1),xx(4),yy(1),yy(4))&
    +alen(xx(2),xx(3),yy(2),yy(3))
END SUBROUTINE revcst
```

This routine performs a path segment reversal. \(iorder\) is an input array giving the present itinerary. The vector \(n\) has as its first four elements the first and last cities \(n(1), n(2)\) of the path segment to be reversed, and the two cities \(n(3)\) and \(n(4)\) that immediately precede and follow this segment. \(n(3)\) and \(n(4)\) are found by subroutine \(\text{revcst}\). On output, \(iorder\) contains the segment from \(n(1)\) to \(n(2)\) in reversed order.

```fortran
INTEGER(I4B) :: j,k,l,nn,ncity
ncity=size(iorder)
nn=(1+mod(n(2)-n(1)+ncity,ncity))/2  ! This many cities must be swapped to effect the reversal.
do j=1,nn
k=1+mod((n(1)+j-2),ncity)
l=1+mod((n(2)-j+ncity),ncity)
call swap(iorder(k),iorder(l))
end do
END SUBROUTINE revers
```

This subroutine returns the value of the cost function for a proposed path segment transport. Arrays \(x\) and \(y\) give the city coordinates. \(iorder\) is an array giving the present itinerary. The first three elements of array \(n\) give the starting and ending cities of the path to be transported, and the point among the remaining cities after which it is to be inserted. On output, \(de\) is the cost of the change. The actual transport is not performed by this routine.

```fortran
INTEGER(I4B) :: ncity
ncity=size(iorder)
nn=1+mod(n(2)-n(1)+ncity,ncity))/2
```

```fortran
do j=1,nn
k=1+mod((n(1)+j-2),ncity)
l=1+mod((n(2)-j+ncity),ncity)
call swap(iorder(k),iorder(l))
end do
END SUBROUTINE revers
```

```fortran
REAL(SP), DIMENSION(6) :: xx,yy
ncity=size(x)
n(4)=1+mod(n(3),ncity)  ! Find the city following \(n(3)\) ...
n(5)=1+mod(n(1)+ncity-2,ncity)  ! and the one preceding \(n(3)\) ...
n(6)=1+mod(n(2),ncity)  ! and the one following \(n(2)\).
x(1:6)=x(iorder(n(1:6)))
yy(1:6)=y(iorder(n(1:6)))
de=-alen(xx(2),xx(6),yy(2),yy(6))&
    -alen(xx(1),xx(5),yy(1),yy(5))&
    -alen(xx(3),xx(4),yy(3),yy(4))&
    +alen(xx(1),xx(3),yy(1),yy(3))&
    +alen(xx(2),xx(4),yy(2),yy(4))&
    +alen(xx(5),xx(6),yy(5),yy(6))
END SUBROUTINE trncst
```

This subroutine returns the value of the cost function for a proposed path reversal. The arrays \(x\) and \(y\) give the coordinates of these cities. \(iorder\) holds the present itinerary. The first two values \(n(1)\) and \(n(2)\) of array \(n\) give the starting and ending cities along the path segment which is to be reversed. On output, \(de\) is the cost of making the reversal. The actual reversal is not performed by this routine.

```fortran
INTEGER(I4B) :: ncity
REAL(SP), DIMENSION(4) :: xx,yy
ncity=size(x)
n(3)=1+mod((n(1)+ncity-2),ncity)  ! Find the city before \(n(1)\) ...
n(4)=1+mod(n(2),ncity)  ! and the city after \(n(2)\).
xx(1:4)=x(iorder(n(1:4)))  ! Find coordinates for the four cities involved.
yy(1:4)=y(iorder(n(1:4)))
de=-alen(xx(1),xx(3),yy(1),yy(3))&
    -alen(xx(2),xx(4),yy(2),yy(4))&
    +alen(xx(1),xx(4),yy(1),yy(4))&
    +alen(xx(2),xx(3),yy(2),yy(3))
END SUBROUTINE revcst
```

This subroutine returns the value of the cost function for a proposed path segment transport. Arrays \(x\) and \(y\) give the city coordinates. \(iorder\) is an array giving the present itinerary. The first three elements of array \(n\) give the starting and ending cities of the path to be transported, and the point among the remaining cities after which it is to be inserted. On output, \(de\) is the cost of the change. The actual transport is not performed by this routine.

```fortran
INTEGER(I4B) :: j,k,l,nn,ncity
ncity=size(iorder)
nn=1+mod(n(2)-n(1)+ncity,ncity))/2
```

```fortran
do j=1,nn
k=1+mod((n(1)+j-2),ncity)
l=1+mod((n(2)-j+ncity),ncity)
call swap(iorder(k),iorder(l))
end do
END SUBROUTINE revers
```

```fortran
REAL(SP), DIMENSION(6) :: xx,yy
ncity=size(x)
n(4)=1+mod(n(3),ncity)  ! Find the city following \(n(3)\) ...
n(5)=1+mod(n(1)+ncity-2,ncity)  ! and the one preceding \(n(3)\) ...
n(6)=1+mod(n(2),ncity)  ! and the one following \(n(2)\).
x(1:6)=x(iorder(n(1:6)))
yy(1:6)=y(iorder(n(1:6)))
de=-alen(xx(2),xx(6),yy(2),yy(6))&
    -alen(xx(1),xx(5),yy(1),yy(5))&
    -alen(xx(3),xx(4),yy(3),yy(4))&
    +alen(xx(1),xx(3),yy(1),yy(3))&
    +alen(xx(2),xx(4),yy(2),yy(4))&
    +alen(xx(5),xx(6),yy(5),yy(6))
END SUBROUTINE trncst
```

This subroutine returns the value of the cost function for a proposed path segment transport. Arrays \(x\) and \(y\) give the city coordinates. \(iorder\) is an array giving the present itinerary. The first three elements of array \(n\) give the starting and ending cities of the path to be transported, and the point among the remaining cities after which it is to be inserted. On output, \(de\) is the cost of the change. The actual transport is not performed by this routine.

```fortran
INTEGER(I4B) :: ncity
ncity=size(iorder)
nn=1+mod(n(2)-n(1)+ncity,ncity))/2
```

```fortran
do j=1,nn
k=1+mod((n(1)+j-2),ncity)
l=1+mod((n(2)-j+ncity),ncity)
call swap(iorder(k),iorder(l))
end do
END SUBROUTINE revers
```

This subroutine returns the value of the cost function for a proposed path segment transport. Arrays \(x\) and \(y\) give the city coordinates. \(iorder\) is an array giving the present itinerary. The first three elements of array \(n\) give the starting and ending cities of the path to be transported, and the point among the remaining cities after which it is to be inserted. On output, \(de\) is the cost of the change. The actual transport is not performed by this routine.

```fortran
INTEGER(I4B) :: j,k,l,nn,ncity
ncity=size(iorder)
nn=1+mod(n(2)-n(1)+ncity,ncity))/2
```

```fortran
do j=1,nn
k=1+mod((n(1)+j-2),ncity)
l=1+mod((n(2)-j+ncity),ncity)
call swap(iorder(k),iorder(l))
end do
END SUBROUTINE revers
```
INTEGER(I4B), DIMENSION(:,), INTENT(INOUT) :: iorder
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: n

This routine does the actual path transport, once metrop has approved. iorder is an input array giving the present itinerary. The array n has as its six elements the beginning n(1) and end n(2) of the path to be transported, the adjacent cities n(3) and n(4) between which the path is to be placed, and the cities n(5) and n(6) that precede and follow the path. n(4), n(5), and n(6) are calculated by subroutine trncst. On output, iorder is modified to reflect the movement of the path segment.

INTEGER(I4B) :: m1, m2, m3, nn, ncity
INTEGER(I4B), DIMENSION(size(iorder)) :: jorder
ncity = size(iorder)
m1 = 1 + mod((n(2) - n(1) + ncity), ncity)  ! Find number of cities from n(1) to n(2)
m2 = 1 + mod((n(5) - n(4) + ncity), ncity)  ! ... and the number from n(4) to n(5)
m3 = 1 + mod((n(3) - n(6) + ncity), ncity)  ! ... and the number from n(6) to n(3).

jorder(1:m1) = iorder(1 + mod((arth(1, 1, m1) + n(1) - 2), ncity))  ! Copy the chosen segment.

nn = m1
jorder(nn + 1:nn + m2) = iorder(1 + mod((arth(1, 1, m2) + n(4) - 2), ncity))  ! Then copy the segment from n(4) to n(5).

nn = nn + m2
jorder(nn + 1:nn + m3) = iorder(1 + mod((arth(1, 1, m3) + n(6) - 2), ncity))  ! Finally, the segment from n(6) to n(3).

iorder(1:ncity) = jorder(1:ncity)  ! Copy jorder back into iorder.

END SUBROUTINE trnspt
END SUBROUTINE anneal

* * *

SUBROUTINE amebsa(p, y, pb, yb, ftol, func, iter, temptr)
USE nrtype; USE nrutil, ONLY : assert_eq, imaxloc, iminloc, swap
USE nr, ONLY : ran1
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: iter
REAL(SP), INTENT(INOUT) :: yb
REAL(SP), INTENT(IN) :: ftol, temptr
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: y, pb
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: p
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: NMAX = 200

Minimization of the N-dimensional function func by simulated annealing combined with the downhill simplex method of Nelder and Mead. The (N+1) \times N matrix p is input. Its N+1 rows are N-dimensional vectors that are the vertices of the starting simplex. Also input is the vector y of length N+1, whose components must be preinitialized to the values of func evaluated at the N+1 vertices (rows) of p; ftol, the fractional convergence tolerance to be achieved in the function value for an early return; iter, and temptr. The routine makes iter function evaluations at an annealing temperature temptr, then returns. You should then decrease temptr according to your annealing schedule, reset iter, and call the routine again (leaving other arguments unaltered between calls). If iter is returned with a positive value, then early convergence and return occurred. If you initialize yb to a very large value on the first call, then yb and pb (an array of length N) will subsequently return the best function value and point ever encountered (even if it is no longer a point in the simplex).

INTEGER(I4B) :: ihi, ndim
REAL(SP) :: yhi
REAL(SP), DIMENSION(size(p, 2)) :: psum
call amebsa_private
CONTAINS
SUBROUTINE amebsa_private
INTEGER(I4B) :: i, ilo, inhi
REAL(SP) :: rtol, ylo, yhi, ysave, ytry
REAL(SP), DIMENSION(size(y)) :: yt, harvest
ndim=assert_eq(size(p,2),size(p,1)-1,size(y)-1,size(pb),'amebsa')
psum(:)=sum(p(:,:),dim=1)
do
Iteration loop.
call ran1(harvest)
yt(:)=y(:)-temptr*log(harvest)
Whenever we "look at" a vertex, it gets a random thermal fluctuation.
ilo=iminloc(yt(:))
Determine which point is the highest (worst),
next-highest, and lowest (best).
ilo=iminloc(yt(:))
ylo=yt(ilo)
hi=imaxloc(yt(:))
yhi=yt(hi)
yt(hi)=ylo
inhi=imaxloc(yt(:))
yhit=yt(inhi)
rtol=2.0_sp*abs(yhi-ylo)/(abs(yhi)+abs(ylo))
Compute the fractional range from highest to lowest and return if satisfactory.
if (rtol < ftol .or. iter < 0) then
If returning, put best point and value in
slot 1.
call swap(y(1),y(ilo))
call swap(p(1,:),p(ilo,:))
RETURN
end if
Begin a new iteration. First extrapolate by a factor $-1$ through the face of the simplex
across from the high point, i.e., reflect the simplex from the high point.
ytry=amotsa(-1.0_sp)
iter=iter-1
if (ytry <= ylo) then
Gives a result better than the best point, so
try an additional extrapolation by a fac-
tor of $2$.
ytry=amotsa(2.0_sp)
else if (ytry >= yhi) then
The reflected point is worse than the second-
best point, try a one-dimensional contrac-
tion.
ytry=amotsa(0.5_sp)
iter=iter-1
if (ytry >= ysave) then
can't seem to get rid of that high point. Better contract around the lowest
(best) point.
p(1,:)=0.5_sp*(p(1,:)+spread(p(ilo,:),1,size(p,1)))
di=1, ndim+1
if (i /= ilo) y(i)=func(p(i,:))
end do
iter=iter-ndim
Keep track of function evaluations.
psum(:)=sum(p(:,:),dim=1)
end if
end if
end do
END SUBROUTINE amebsa_private

FUNCTION amotsa(fac)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: fac
REAL(SP) :: amotsa
Extrapolates by a factor fac through the face of the simplex across from the high point,
tries it, and replaces the high point if the new point is better.
REAL(SP) :: fac1, fac2, yflu, ytry, harv
REAL(SP), DIMENSION(size(p,2)) :: ptry
fac1=(1.0_sp-fac)/ndim
fac2=fac1-fac
ptry(:)=psum(:)*fac1-p(ihi,:)*fac2
ytry=func(ptry)
if (ytry <= yb) then
Save the best-ever.
pb(:)=ptry(:)
yb=ytry
end if

call ran1(harv)
yflu=ytry*temptr*log(harv)
if (yflu < yhi) then
  y(ihi)=ytry
yhi=yflu
  psum(:)=psum(:)-p(ihi,:)+ptry(:)
p(ihi,:)=ptry(:)
end if

amotsa=yflu

END FUNCTION amotsa

END SUBROUTINE amebsa

See the discussion of amoeba on p. 1209 for why the routine is coded this way.

We added a thermal fluctuation to all the current vertices, but we subtract it here, so as to give the simplex a thermal Brownian motion: It likes to accept any suggested change.
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SUBROUTINE jacobi(a,d,v,nrot)
USE nrtype; USE nrutil, ONLY : assert_eq, get_diag, nrerror, unit_matrix, &
upper_triangle
IMPLICIT NONE
INTEGER(I4B), INTENT(OUT) :: nrot
REAL(SP), DIMENSION(:,), INTENT(OUT) :: d
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: v

Computes all eigenvalues and eigenvectors of a real symmetric $N \times N$ matrix $a$. On output, elements of $a$ above the diagonal are destroyed. $d$ is a vector of length $N$ that returns the eigenvalues of $a$. $v$ is an $N \times N$ matrix whose columns contain, on output, the normalized eigenvectors of $a$. $nrot$ returns the number of Jacobi rotations that were required.

INTEGER(I4B) :: i,ip,iq,n
REAL(SP) :: c,g,h,s,sm,t,tau,theta,tresh
REAL(SP), DIMENSION(size(d)) :: b,z
n=assert_eq((/size(a,1),size(a,2),size(d),size(v,1),size(v,2)/),'jacobi')
call unit_matrix(v(:,:)) Initialize $v$ to the identity matrix.
b(:)=get_diag(a(:,:)) Initialize $b$ and $d$ to the diagonal of $a$.
d(:)=b(:)
z(:)=0.0 This vector will accumulate terms of the form $\tau_{pq}$ as in eq. (11.1.14).
nrot=0
do i=1,50
   sm=sum(abs(a),mask=upper_triangle(n,n)) Sum off-diagonal elements.
   if (sm == 0.0) RETURN
   The normal return, which relies on quadratic convergence to machine underflow.
   tresh=merge(0.2_sp*sm/n**2,0.0_sp, i < 4 )
   On the first three sweeps, we will rotate only if $tresh$ exceeded.
do ip=1,n-1
   do iq=ip+1,n
      g=100.0_sp*abs(a(ip,iq))
      After four sweeps, skip the rotation if the off-diagonal element is small.
      if ((i > 4) .and. (abs(d(ip))+g == abs(d(ip))) &
         .and. (abs(d(iq))+g == abs(d(iq)))) then
         a(ip,iq)=0.0
      else if (abs(a(ip,iq)) > tresh) then
         h=d(ip)-d(iq)
         if (abs(h)+g == abs(h)) then
            t=a(ip,iq)/h
            t=1.0_sp/(abs(theta)+sqrt(1.0_sp+theta**2))
            if (theta < 0.0) t=-t
         else
            theta=0.5_sp*b/a(ip,iq) Equation (11.1.10).
            t=1.0_sp/(abs(theta)+sqrt(1.0_sp+theta**2))
            if (theta < 0.0) t=-t
         end if
      else
         c=1.0_sp/sqrt(1+t**2)
         s=t*c
         tau=s/(1.0_sp+c)
         h=t*a(ip,iq)
         z(ip)=z(ip)-h
         z(iq)=z(iq)+h
         d(ip)=d(ip)-h
         d(iq)=d(iq)+h
         a(ip,iq)=0.0
   end do
end do

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As discussed in Volume 1, \textit{jacobi} is generally not competitive with \textit{tqli} in terms of efficiency. However, \textit{jacobi} can be parallelized whereas \textit{tqli} uses an intrinsically serial algorithm. The version of \textit{jacobi} implemented here is likely to be adequate for a small-scale parallel (SSP) machine, but is probably still not competitive with \textit{tqli}. For a massively multiprocessor (MMP) machine, the order of the rotations needs to be chosen in a more complicated pattern than here so that the rotations can be executed in parallel. In this case the Jacobi algorithm may well turn out to be the method of choice. Parallel replacements for \textit{tqli} based on a divide and conquer algorithm have also been proposed. See the discussion after \textit{tqli} on p. 1229.

These routines in \textit{nrutil} both require access to the diagonal of a matrix, an operation that is not conveniently provided for in Fortran 90. We have split them off into \textit{nrutil} in case your compiler provides parallel library routines so you can replace our standard versions.

The \texttt{upper_triangle} function in \textit{nrutil} returns an upper triangular logical mask. As used here, the mask is true everywhere in the upper triangle of an $n \times n$ matrix, excluding the diagonal. An optional integer argument \texttt{extra} allows additional diagonals to be set to true. With \texttt{extra=1} the upper triangle including the diagonal would be true. By using the mask, we can conveniently sum over the desired matrix elements in parallel.

This internal subroutine also uses the values of $s$ and $tau$ from the calling subroutine \textit{jacobi}. Variables in the calling routine are visible to an internal subprogram, but you should be circumspect in making use of this fact. It is easy to overwrite a value in the calling program inadvertently, and it is
often difficult to figure out the logic of an internal routine if not all its variables are declared explicitly. However, jrotate is so simple that there is no danger here.

⋆⋆⋆

SUBROUTINE eigsrt(d,v)
USE nrtype; USE nrutil, ONLY : assert_eq,imaxloc,swap
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: d
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: v
Given the eigenvalues d and eigenvectors v as output from jacobi (§11.1) or tqli (§11.3), this routine sorts the eigenvalues into descending order, and rearranges the columns of v correspondingly. The method is straight insertion.

INTEGER(I4B) :: i,j,n
n=assert_eq(size(d),size(v,1),size(v,2),'eigsrt')
do i=1,n-1
   j=imaxloc(d(i:n))+i-1
   if (j /= i) then
      call swap(d(i),d(j))
      call swap(v(:,i),v(:,j))
   end if
end do
END SUBROUTINE eigsrt

⋆⋆⋆

SUBROUTINE tred2(a,d,e,novectors)
USE nrtype; USE nrutil, ONLY : assert_eq,outerprod
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: d,e
LOGICAL(LGT), OPTIONAL, INTENT(IN) :: novectors
Householder reduction of a real, symmetric, N × N matrix a. On output, a is replaced by the orthogonal matrix Q effecting the transformation. d returns the diagonal elements of the tridiagonal matrix, and e the off-diagonal elements, with e(1)=0. If the optional argument novectors is present, only eigenvalues are to be found subsequently, in which case a contains no useful information on output.

INTEGER(I4B) :: i,j,l,n
REAL(SP) :: f,g,h,hh,scale
REAL(SP), DIMENSION(size(a,1)) :: gg
LOGICAL(LGT), SAVE :: yesvec=.true.
n=assert_eq(size(a,1),size(a,2),size(d),size(e),'tred2')
if (present(novectors)) yesvec=.not. novectors
do i=n,2,-1
   l=i-1
   h=0.0
   if (l > 1) then
      scale=sum(abs(a(i,1:l)))
      if (scale == 0.0) then
         Skip transformation.
      else
         e(i)=a(i,1)
      end if
      a(i,1:l)=a(i,1:l)/scale
      h=sum(a(i,1:l)**2)
   else
      a(i,1:1)=a(i,1:1)/scale
      h=sum(a(i,1:1)**2)
   end if
end do

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\[ f = a(i,1) \]
\[ g = \text{sign}(\sqrt{h}, f) \]
\[ e(i) = \text{scale} \cdot g \]
\[ h = h - f \cdot g \]

Now \( h \) is equation (11.2.4).

\[ a(i,1) = f - g \]

Store \( u \) in the \( i \)th row of \( a \).

**if** (yesvec) \( a(1:l,1) = a(i,1:l) / h \)

Store elements of \( p \) in temporarily unused elements of \( e \).

**do** \( j = 1:l \)

\[ e(j) = \left( \text{dot_product}(a(j,1:j), a(i,1:j)) \right) + \left( \text{dot_product}(a(j+1:l,j), a(i,j+1:l)) \right) / h \]

**end do**

\[ f = \text{dot_product}(e(1:l), a(i,1:l)) \]

\[ h = f / (h + h) \]

\[ e(1:l) = e(1:l) - h \cdot a(1:l,1) \]

Form \( K \), equation (11.2.11).

**do** \( j = 1:l \)

Reduce \( a \), equation (11.2.13).

\[ a(j,1:j) = a(j,1:j) - a(i,j) \cdot e(1:j) - e(j) \cdot a(i,1:j) \]

**end do**

**end if**

else

\[ e(i) = a(i,1) \]

**end if**

\[ d(i) = h \]

**do** \( i = 1:n \)

**if** (yesvec) then

\[ l = i - 1 \]

**if** \( (d(l) /= 0.0) \) then

This block skipped when \( i = 1 \). Use \( u \) and \( u/H \) stored in \( a \) to form \( P \cdot Q \).

\[ gg(1:l) = \text{matmul}(a(i,1:l), a(1:l,1:l)) \]

\[ a(1:l,1:l) = a(1:l,1:l) - \text{outerprod}(a(1:l,i), gg(1:l)) \]

**end if**

\[ d(i) = a(i,1) \]

\[ a(i,i) = 1.0 \]

\[ a(1:i,1:l) = 0.0 \]

\[ a(1:l,1:i) = 0.0 \]

else

\[ d(i) = a(i,1) \]

**end if**

**end do**

END SUBROUTINE tred2

This routine gives a nice example of the usefulness of optional arguments.

The routine is written under the assumption that usually you will want to find both eigenvalues and eigenvectors. In this case you just supply the arguments \( a \), \( d \), and \( e \). If, however, you want only eigenvalues, you supply the additional logical argument \( \text{novectors} \) with the value .true.. The routine then skips the unnecessary computations. Supplying \( \text{novectors} \) with the value .false. has the same effect as omitting it.

* * *

SUBROUTINE tqli(d, e, z)
USE ntype; USE nrutil, ONLY : assert_eq, nrerror
USE nr, ONLY : pythag
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: d, e
REAL(SP), DIMENSION(:,,:), OPTIONAL, INTENT(INOUT) :: z
QL algorithm with implicit shifts, to determine the eigenvalues and eigenvectors of a real, symmetric, tridiagonal matrix, or of a real, symmetric matrix previously reduced by tred2
§11.2. \( d \) is a vector of length \( N \). On input, its elements are the diagonal elements of the tridiagonal matrix. On output, it returns the eigenvalues. The vector \( e \) inputs the subdiagonal elements of the tridiagonal matrix, with \( e(1) \) arbitrary. On output \( e \) is destroyed. When finding only the eigenvalues, the optional argument \( z \) is omitted. If the eigenvectors of a tridiagonal matrix are desired, the \( N \times N \) matrix \( z \) is input as the identity matrix. If the eigenvectors of a matrix that has been reduced by \texttt{tred2} are required, then \( z \) is input as the matrix output by \texttt{tred2}. In either case, the \( k \)th column of \( z \) returns the normalized eigenvector corresponding to \( d(k) \).

```fortran
INTEGER(I4B) :: i,iter,l,m,n,ndum
REAL(SP) :: b,c,dd,f,g,p,r,s
REAL(SP), DIMENSION(size(e)) :: ff
n=assert_eq(size(d),size(e),'tqli:n')
if (present(z)) ndum=assert_eq(n,size(z,1),size(z,2),'tqli:ndum')
e(:)=eoshift(e(:),1) Convenient to renumber the elements of
do l=1,n
iter=0
iterate: do
  do m=l,n-1
    dd=abs(d(m))+abs(d(m+1))
    if (abs(e(m))+dd == dd) exit
  end do
  if (m == l) exit iterate
  if (iter == 30) call nrerror('too many iterations in tqli')
  iter=iter+1
  g=(d(l+1)-d(l))/(2.0_sp*e(l)) Form shift.
  r=pythag(g,1.0_sp)
  g=d(m)-d(l)+e(l)/(g+sign(r,g))
  s=1.0
  c=1.0
  p=0.0
  do i=m-1,l,-1
    f=s*e(i)
    b=c*e(i)
    r=pythag(f,g)
    e(i+1)=r
    if (r == 0.0) then
      d(i+1)=d(i+1)-p
      e(m)=0.0
      cycle iterate
    end if
    s=f/r
    c=g/r
    g=d(i+1)-p
    r=(d(i)-g)*s+2.0_sp*c*b
    p=s*r
d(i+1)=g+p
    g=c*r-b
    if (present(z)) then
      ff(1:n)=z(1:n,i+1)
      z(1:n,i+1)=s*z(1:n,i)+c*ff(1:n)
      z(1:n,i)=c*z(1:n,i)-s*ff(1:n)
    end if
  end do
  d(l)=d(l)-p
  e(l)=g
  e(m)=0.0
  end do iterate
end do
END SUBROUTINE tqli
```

The routine \texttt{tqli} is intrinsically serial. A parallel replacement based on a divide and conquer algorithm has been proposed \cite{1,2}. The idea is to split the tridiagonal matrix recursively into two tridiagonal matrices of
half the size plus a correction. Given the eigensystems of the two smaller tridiagonal matrices, it is possible to join them together and add in the effect of the correction. When some small size of tridiagonal matrix is reached during the recursive splitting, its eigensystem is found directly with a routine like tqli. Each of these small problems is independent and can be assigned to an independent processor. The procedures for sewing together can also be done independently. For very large matrices, this algorithm can be an order of magnitude faster than tqli even on a serial machine, and no worse than a factor of 2 or 3 slower, depending on the matrix. Unfortunately the parallelism is not well expressed in Fortran 90. Also, the sewing together requires quite involved coding. For an implementation see the LAPACK routine SSTEDC. Another parallel strategy for eigensystems uses inverse iteration, where each eigenvalue and eigenvector can be found independently [3].

This routine uses z as an optional argument that is required only if eigenvectors are being found as well as eigenvalues.

iterate: do  See discussion of named do loops after simp1x on p. 1219.

* * *

SUBROUTINE balanc(a)
USE nrtyp; USE nrtutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a
REAL(SP), PARAMETER :: RADX=radix(a), SQRADX=RADX**2

Given an N x N matrix a, this routine replaces it by a balanced matrix with identical eigenvalues. A symmetric matrix is already balanced and is unaffected by this procedure.

The parameter RADX is the machine's floating-point radix.

INTEGER(I4B) :: i, last, ndum
REAL(SP) :: c, f, g, r, s
ndum=assert_eq(size(a,1),size(a,2),'balanc')
do
last=1
  do i=1,size(a,1)
    c=sum(abs(a(:,i)))-a(i,i)
    r=sum(abs(a(i,:)))-a(i,i)
    if (c /= 0.0 .and. r /= 0.0) then
      g=r/RADX
      f=1.0
      s=c+r
      do
        if (c >= g) exit
        g=r*RADX
        f=f*RADX
        c=c*SQRADX
      end do
      g=r*RADX
      do
        if (c <= g) exit
        f=f/RADX
        c=c/SQRADX
      end do
      if ((c+r)/f < 0.95_sp*s) then
        last=0
        g=1.0_sp/f
        a(i,:)=a(i,:)*g
        a(:,i)=a(:,i)*f
      end if
    end if
  end do
end do


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end do
if (last /= 0) exit
end do
END SUBROUTINE balanc

REAL(SP), PARAMETER :: RADX=radix(a)... Fortran 90 provides a nice collection of numeric inquiry intrinsic functions. Here we find the machine’s floating-point radix. Note that only the type of the argument a affects the returned function value.

SUBROUTINE elmhes(a)
USE nrtype; USE nrutil, ONLY : assert_eq, imaxloc, outerprod, swap
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a
Reduction to Hessenberg form by the elimination method. The real, nonsymmetric, \(N \times N\) matrix a is replaced by an upper Hessenberg matrix with identical eigenvalues. Recommended, but not required, is that this routine be preceded by balanc. On output, the Hessenberg matrix is in elements \(a(i,j)\) with \(i \leq j + 1\). Elements with \(i > j + 1\) are to be thought of as zero, but are returned with random values.

INTEGER(I4B) :: i,m,n
REAL(SP) :: x
REAL(SP), DIMENSION(size(a,1)) :: y
n=assert_eq(size(a,1),size(a,2),'elmhes')
do m=2,n-1
   m is called \(r + 1\) in the text.
   i=imaxloc(abs(a(m:n,m-1)))+m-1
   Find the pivot.
   x=a(i,m-1)
   if (i /= m) then
      Interchange rows and columns.
      call swap(a(i,m-1:n),a(m,m-1:n))
      call swap(a(:,i),a(:,m))
   end if
   if (x /= 0.0) then
      Carry out the elimination.
      y(m+1:n)=a(m+1:n,m-1)/x
      a(m+1:n,m-1)=y(m+1:n)
      a(m+1:n,m:n)=a(m+1:n,m:n)-outerprod(y(m+1:n),a(m,m:n))
      a(:,m)=a(:,m)+matmul(a(:,m+1:n),y(m+1:n))
   end if
end do
END SUBROUTINE elmhes

If the four lines of code starting here were all coded for a serial machine in a single do-loop starting with do \(i=m+1,n\) (see Volume 1), it would pay to test whether \(y\) was zero because the next three lines could then be skipped for that value of \(i\). There is no convenient way to do this here, even with a where, since the shape of the arrays on each of the three lines is different. For a parallel machine it is probably best just to do a few unnecessary multiplies and skip the test for zero values of \(y\).
SUBROUTINE hqr(a,wr,wi)
USE nrtype; USE nrutil, ONLY : assert_eq, diagadd, nrerror, upper_triangle
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(OUT) :: wr,wi
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a

Finds all eigenvalues of an \( N \times N \) upper Hessenberg matrix \( a \). On input \( a \) can be exactly as output from elmhes \( \S 11.5 \); on output it is destroyed. The real and imaginary parts of the \( N \) eigenvalues are returned in \( wr \) and \( wi \), respectively.

INTEGER(I4B) :: i,its,k,l,m,n,nn,mnnk
REAL(SP) :: anorm,p,q,r,s,t,u,v,w,x,y,z
REAL(SP), DIMENSION(size(a,1)) :: pp
n=assert_eq(size(a,1),size(a,2),size(wr),size(wi),'hqr')
anorm=sum(abs(a),mask=upper_triangle(n,n,extra=2))
Compute matrix norm for possible use in locating single small subdiagonal element.
nn=n
t=0.0

Do while \( nn \geq 1 \).

if (nn < 1) exit
its=0

Begin iteration.
do l=nn,2,-1
Look for single small subdiagonal element.
s=abs(a(l-1,l-1))+abs(a(l,l))
if (s == 0.0) s=anorm
if (abs(a(l,l-1))+s == s) exit
end do
x=a(nn,nn)
if (l == nn) then
One root found.
wr(nn)=x+t
wi(nn)=0.0
nn=nn-1
exit iterate
Go back for next eigenvalue.
end if
y=a(nn-1,nn-1)
w=a(nn,nn-1)*a(nn-1,nn)
if (l == nn-1) then
Two roots found ...
p=0.5_sp*(y-x)
q=p**2+w
z=sqrt(abs(q))
x=x+t
if (q >= 0.0) then
... a real pair ...
z=p*sign(z,p)
wr(nn)=x+z
wr(nn-1)=wr(nn)
if (z /= 0.0) wr(nn)=x-w/z
wi(nn)=0.0
wi(nn-1)=0.0
else
... a complex pair.
wr(nn)=x+p
wr(nn-1)=wr(nn)
wi(nn)=z
wi(nn-1)=-z
end if
nn=nn-2
exit iterate
Go back for next eigenvalue.
end if

No roots found. Continue iteration.
if (its == 30) call nrerror(’too many iterations in hqr’)
if (its == 10 .or. its == 20) then
Form exceptional shift.
t=t+x
call diagadd(a(1:nn,1:nn),-x)
s=abs(a(nn,nn-1))*abs(a(nn-1,nn-2))
x=0.75_sp*s
y=x
w=-0.4375_sp*p**2
end if
its=its+1
end do

Form shift and then look for 2 consecutive small subdiagonal elements.

do m=nn-2,l,-1
Form shift and then look for 2 consecutive small subdiagonal elements.

z=a(m,m)

r=x-z

s=y-z

p=(r*s-w)/a(m+1,m)+a(m,m+1)  
\text{Equation (11.6.23).}

q=a(m+1,m+1)-z-r-s

r=a(m+2,m+1)

s=abs(p)+abs(q)+abs(r)

p=p/s

q=q/s

r=r/s

if (m == l) exit

u=abs(a(m,m-1))*(abs(q)+abs(r))

v=abs(p)*(abs(a(m-1,m-1))+abs(z)+abs(a(m+1,m+1)))

if (u+v == v) exit  
\text{Equation (11.6.26).}

end do

end do

i=m+2,nn

a(i,i-2)=0.0

if (i == m+2) a(i,i-3)=0.0

end do

end do

k=nn-1

if (k == m) then

Begin setup of Householder vector.

p=a(k,k-1)

q=a(k+1,k-1)

r=0.0

if (k == nn-1) r=a(k+2,k-1)

x=abs(p)+abs(q)+abs(r)

if (x /= 0.0) then

p=p/x  
\text{Scale to prevent overflow or underflow.}

q=q/x

r=r/x

end if

end if

u=abs(a(m,m-1))*(abs(q)+abs(r))

v=abs(p)*(abs(a(m-1,m-1))+abs(z)+abs(a(m+1,m+1)))

if (u+v == v) exit

end if

end if

\text{Equations (11.6.24).}

knnk=min(nn,k+3)

Column modification.

pp(l:mmnk)=a(l:mmnk)*p+a(l+1:mmnk)*q

if (k == nn-1) then

pp(l:mmnk)=pp(l:mmnk)+r*pp(l+2:mmnk)

y=a(k+2,k+1)*pp(k:mmnk)

mnnk=min(nn,k+3)

Column modification.

pp(l:mmnk)=x*a(l:mmnk)+y*a(l:mmnk,1)+a(l:mmnk,k+1)

if (k == nn-1) then

pp(l:mmnk)=pp(l:mmnk)+z*a(l:mmnk,k+2)

a(l:mmnk,k+2)=pp(l:mmnk)*r

end if

a(l:mmnk,k+1)=a(l:mmnk,k+1)-pp(l:mmnk)*q

a(l:mmnk,k)=a(l:mmnk,k)-pp(l:mmnk)

end if
end do iterate
end do

END SUBROUTINE hqr

\[ \texttt{anorm=\texttt{sum(abs(a)}, \texttt{mask=upper\_triangle(n,n,extra=2)}} \]

See the discussion of \texttt{upper\_triangle} after \texttt{jacobi} on p. 1226. Setting \texttt{extra=2} here picks out the upper Hessenberg part of the matrix.

iterate: do

We use a named loop to improve the readability and structuring of the routine. The if-blocks that test for one or two roots end with \texttt{exit iterate}, transferring control back to the outermost loop and thus starting a search for the next root.

call diagadd... The routines that operate on the diagonal of a matrix are collected in \texttt{nrutil} partly so you can write clear code and partly in the hope that compiler writers will provide parallel library routines. \texttt{Fortran 90} does not provide convenient parallel access to the diagonal of a matrix.

CITED REFERENCES AND FURTHER READING:
Chapter B12. Fast Fourier Transform

The algorithms underlying the parallel routines in this chapter are described in §22.4. As described there, the basic building block is a routine for simultaneously taking the FFT of each row of a two-dimensional matrix:

```fortran
SUBROUTINE fourrow_sp(data,isign)
USE nrtype; USE nrutil, ONLY : assert, swap
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:,,:), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign

INTEGER(I4B) :: n,i,istep,j,m,mmax,n2
REAL(DP) :: theta
COMPLEX(SPC), DIMENSION(size(data,1)) :: temp
COMPLEX(DPC) :: w,wp

n=size(data,2)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in fourrow_sp')
n2=n/2
j=n2

This is the bit-reversal section of the routine.

do i=1,n-2
   if (j > i) call swap(data(:,j+1),data(:,i+1))
   n=n2
   do
      if (m < 2 .or. j < m) exit
      j=j-m
      m=m/2
   end do
   j=j+m
end do
mmax=1

Here begins the Danielson-Lanczos section of the routine.

Outer loop executed \( \log_2 N \) times.

do if (n <= mmax) exit
   istep=2*mmax
   theta=PI_D/(isign*mmax)
   wp=cmplx(-2.0_dp*sin(0.5_dp*theta)**2,sin(theta),kind=dpc)
   w=cmplx(1.0_dp,0.0_dp,kind=dpc)
   do m=1,mmax
      ws=w
      do i=m,n,istep
         j=i+mmax
         temp=ws*data(:,j)
         data(:,j)=data(:,i)-temp
         data(:,i)=data(:,i)+temp
      end do
      w=w*wp+w
   end do
end do
mmax=istep
```

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All the Fourier routines in this chapter require the dimension $N$ of the data to be a power of 2. This is easily tested for by AND'ing $N$ and $N - 1$: $N$ should have the binary representation 10000..., in which case $N - 1 = 01111....
replaces each third-index section of data by \( N \) times its inverse discrete Fourier transform, if \( \text{isign} \) is input as \(-1\). \( N \) must be an integer power of 2. Parallelism is \( L \times M \)-fold on the first and second indices of data.

```fortran
INTEGER(14B) :: n, i, istep, j, m, mmax, n2
REAL(DP) :: theta
COMPLEX(SPC), DIMENSION(size(data,1), size(data,2)) :: temp
COMPLEX(DPC) :: w, wp

Call assert(iband(n-1)==0, 'n must be a power of 2 in fourrow_3d')
n2=n/2
j=n2

This is the bit-reversal section of the routine.

do i=1,n-2
    if (j > i) call swap(data(:,:,j+1),data(:,:,i+1))
    m=n2
    do
        if (m < 2 .or. j < m) exit
        j=j-m
        m=m/2
    end do
    j=j+m
end do

mmax=1

Here begins the Danielson-Lanczos section of the routine.

do
    if (n <= mmax) exit
    istep=2**mmax
    theta=pi_d/(isign*mmax)    ! Initialize for the trigonometric recurrence.
    wp=cmplx(-2.0_dp*sin(0.5_dp*theta)**2,sin(theta),kind=dpc)
    w=cmplx(1.0_dp,0.0_dp,kind=dpc)

    do m=1,mmax
        ws=w
        do i=m,n,istep
            j=i+mmax
            temp=ws*data(:,:,j)
            data(:,:,j)=data(:,:,i)-temp
            data(:,:,i)=data(:,:,i)+temp
        end do
        w=w*wp+w
    end do
    mmax=istep
end do
END SUBROUTINE fourrow_3d
```

```
⋆⋆⋆

Exactly as in the preceding routines, we can take the FFT of each column of a two-dimensional matrix, and for each first-index section of a three-dimensional array.

SUBROUTINE fourcol(data,isign)
USE nrtypes; USE nrutil, ONLY : assert, swap
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:,,:), INTENT(INOUT) :: data
INTEGER(14B), INTENT(IN) :: isign

Replaces each column (constant second index) of data(1:N,1:M) by its discrete Fourier transform (transform on first index), if isign is input as 1; or replaces each row of data

```
by \( N \) times its inverse discrete Fourier transform, if \( \text{isign} \) is input as \(-1\). \( N \) must be an integer power of 2. Parallelism is \( M \)-fold on the second index of \( \text{data} \).

```fortran
INTEGER(14B) :: n, i, istep, j, m, mmax, n2
REAL(DP) :: theta
COMPLEX(SPC), DIMENSION(size(data,2)) :: temp
COMPLEX(DPC) :: w, wp
COMPLEX(SPC) :: ws
DOUBLE PRECISION FOR THE TRIGONOMETRIC RECURRENCES.

n = size(data,1)
call assert(land(n,n-1)==0, 'n must be a power of 2 in fourcol')
n2 = n/2
j = n2
This is the bit-reversal section of the routine.
do i = 1, n-2
  if (j > i) call swap(data(j+1,:),data(i+1,:))
m = n2
  do
    if (m < 2 .or. j < m) exit
    j = j-m
    m = m/2
  end do
  j = j+m
end do
mmax = 1
Here begins the Danielson-Lanczos section of the routine.
do
  Outer loop executed \( \log_2 N \) times.
  if (n <= mmax) exit
  istep = 2*mmax
  theta = PI_D/(isign*mmax)
  wp = cmplx(-2.0_dp*sin(0.5_dp*theta)**2, sin(theta), kind=dpc)
  w = cmplx(1.0_dp, 0.0_dp, kind=dpc)
  do m = 1, mmax
    ws = w
    do i = m, n, istep
      j = i+mmax
      temp = ws*data(j,:)
      This is the Danielson-Lanczos formula.
      data(j,:) = data(i,:) - temp
      data(i,:) = data(i,:) + temp
    end do
    w = w*wp + w
  end do
  mmax = istep
end do
END SUBROUTINE fourcol
```

```fortran
SUBROUTINE fourcol_3d(data, isign)
USE nrtypes; USE nrutil, ONLY : assert, swap
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:, :, :), INTENT(INOUT) :: data
INTEGER(14B), INTENT(IN) :: isign
If \( \text{isign} \) is input as 1, replaces each first-index section (constant second and third indices) of \( \text{data}(1:N, 1:M, 1:L) \) by its discrete Fourier transform (transform on first index); or replaces each first-index section of \( \text{data} \) by \( N \) times its inverse discrete Fourier transform, if \( \text{isign} \) is input as \(-1\). \( N \) must be an integer power of 2. Parallelism is \( M \times L \)-fold on the second and third indices of \( \text{data} \).

INTEGER(14B) :: n, i, istep, j, m, mmax, n2
REAL(DP) :: theta
COMPLEX(SPC), DIMENSION(size(data,2), size(data,3)) :: temp
COMPLEX(DPC) :: w, wp
COMPLEX(SPC) :: ws
DOUBLE PRECISION FOR THE TRIGONOMETRIC RECURRENCES.

n = size(data,1)
 call assert(land(n,n-1)==0, 'n must be a power of 2 in fourcol_3d')
n2 = n/2
```
This is the bit-reversal section of the routine.

\[
do \ i = 1, n-2 \\
\text{if } (j > i) \text{ call swap(data(j+1,:,:),data(i+1,:,:))} \\
\text{m=n2} \\
\text{do} \\
\text{if } (m < 2 \text{ or } j < m) \text{ exit} \\
\quad j=j-m \\
\quad m=m/2 \\
\text{end do} \\
\text{j=j+m} \\
\end do
\]

Here begins the Danielson-Lanczos section of the routine.

\[
\text{do} \\
\quad \text{Outer loop executed } \log_2 N \text{ times.} \\
\quad \text{if } (n <= \text{ mmax} ) \text{ exit} \\
\quad \text{istep=2*\text{mmax}} \\
\quad \text{theta} = \pi / \text{ dint} \cdot \text{mmax} \\
\quad \text{wp=} \text{cxplx}(-2.0, \sin(0.5\cdot \text{theta})\cdot 2, \sin(\text{theta}), \text{kind=dpc}) \\
\quad \text{w=} \text{cxplx}(1.0, 0.0, \text{kind=dpc}) \\
\text{do } m=1, \text{mmax} \\
\quad \text{Here are the two nested inner loops.} \\
\quad \text{ws=} \text{w} \\
\quad \text{do } i=m, n, \text{istep} \\
\quad \text{j=} i+\text{mmax} \\
\quad \text{temp=} \text{ws} \cdot \text{data(j,:,:)} \\
\quad \text{data(j,:,:)} = \text{data(i,:,:)} - \text{temp} \\
\quad \text{data(i,:,:)} = \text{data(i,:,:)} + \text{temp} \\
\text{end do} \\
\quad \text{w=} \text{w} \cdot \text{wp} + \text{w} \\
\text{Trigonometric recurrence.} \\
\text{end do} \\
\text{mmax=} \text{istep} \\
\text{end do}
\]

END SUBROUTINE fourcol_3d

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Here now are implementations of the method of §22.4 for the FFT of one-dimensional single- and double-precision complex arrays:

SUBROUTINE four1_sp(data,isign)
USE nrtype; USE nrutil, ONLY: arth, assert
USE nr, ONLY: fourrow
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:,), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign

Replaces a complex array data by its discrete Fourier transform, if isign is input as 1; or replaces data by its inverse discrete Fourier transform times the size of data, if isign is input as -1. The size of data must be an integer power of 2. Parallelism is achieved by internally reshaping the input array to two dimensions. (Use this version if fourrow is faster than fourcol on your machine.)

COMPLEX(SPC), DIMENSION(:,,:), ALLOCATABLE :: dat,temp
COMPLEX(DPC), DIMENSION(:,,:), ALLOCATABLE :: w,wp
REAL(DP), DIMENSION(:,,:), ALLOCATABLE :: theta
INTEGER(I4B) :: n,m1,m2,j
n=size(data)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in four1_sp')
Find dimensions as close to square as possible, allocate space, and reshape the input array.
m1=2**ceiling(0.5*real(n,sp)/0.693147_sp)
m2=n/m1
allocate(dat(m1,m2),theta(m1),w(m1),wp(m1),temp(m2,m1))
dat=reshape(data,shape(dat))
call fourrow(dat,isign) 
Transform on second index.
Chapter B12. Fast Fourier Transform

The above routines use fourrow exclusively, on the assumption that it is faster than its sibling fourcol. When that is the case (as we typically find), it is likely that four1_sp is also faster than Volume 1’s scalar four1. The reason, on scalar machines, is that fourrow’s parallelism is taking better advantage of cache memory locality.

If fourrow is not faster than fourcol on your machine, then you should instead try the following alternative FFT version that uses fourcol only.
input as $-1$. The size of data must be an integer power of 2. Parallelism is achieved by internally reshaping the input array to two dimensions. (Use this version only if fourcol is faster than fourrow on your machine.)

```fortran
COMPLEX(SPC), DIMENSION(:,,:), ALLOCATABLE :: dat,temp
COMPLEX(DPC), DIMENSION(:,), ALLOCATABLE :: w,wp
REAL(DP), DIMENSION(:,), ALLOCATABLE :: theta
INTEGER(I4B) :: n,m1,m2,j
n=size(data)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in four1_alt')
Find dimensions as close to square as possible, allocate space, and reshape the input array.
m1=2**ceiling(0.5_dp*log(real(n,sp))/0.693147_sp)
m2=n/m1
allocate(dat(m1,m2),theta(m1),w(m1),wp(m1),temp(m2,m1))
dat=reshape(data,shape(dat))
temp=transpose(dat)    
Transpose and transform on (original) second index.
call fourcol(temp,isign)
theta=arth(0,isign,m1)*TWOPI_D/n    
Set up recurrence.
w=cmplx(-2.0_dp*sin(0.5_dp*theta)**2,sin(theta),kind=dpc)
wp=cmplx(-2.0_dp,sin(0.5_dp*theta)**2,kind=dpc)
do j=2,m2
   w=w*wp+w
   temp(:,j)=temp(:,j)*w
end do
data=transpose(temp)    
Transpose and then reshape the result back to one dimension.
dat=reshape(temp,shape(data))
deallocate(dat,w,wp,theta,temp)
END SUBROUTINE four1_alt
```

With all the machinery of fourrow and fourcol, two-dimensional FFTs are extremely straightforward. Again there is an alternative version provided in case your hardware favors fourcol (which would be, we think, unusual).

```fortran
SUBROUTINE four2(data,isign)
USE nrtype
USE nr, ONLY : fourrow
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:,:), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign

Replaces a 2-d complex array data by its discrete 2-d Fourier transform, if isign is input as 1; or replaces data by its inverse 2-d discrete Fourier transform times the product of its two sizes, if isign is input as $-1$. Both of data's sizes must be integer powers of 2 (this is checked for in fourrow). Parallelism is by use of fourrow.

```fortran
CALL fourrow(data,isign)    
Transform in second dimension.
temp=transpose(data)    
Transpose.
call fourrow(temp,isign)    
Transform in (original) first dimension.
data=transpose(temp)    
Transpose into data.
END SUBROUTINE four2
```
Chapter B12. Fast Fourier Transform

SUBROUTINE four2_alt(data,isign)
USE nrtype
USE nr, ONLY : fourcol
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:,,:), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign

Replaces a 2-d complex array data by its discrete 2-d Fourier transform, if isign is input as 1; or replaces data by its inverse 2-d discrete Fourier transform times the product of its two sizes, if isign is input as -1. Both of data's sizes must be integer powers of 2 (this is checked for in fourcol). Parallelism is by use of fourcol. (Use this version only if fourcol is faster than fourrow on your machine.)

COMPLEX(SPC), DIMENSION(size(data,2),size(data,1)) :: temp

temp=transpose(data)
TRanpose.
call fourcol(temp,isign)  Transform in (original) second dimension.
data=transpose(temp)
TRanpose.
call fourcol(data,isign)  Transform in (original) first dimension.
END SUBROUTINE four2_alt

⋆ ⋆ ⋆

Most of the remaining routines in this chapter simply call one or another of the above FFT routines, with a small amount of auxiliary computation, so they are fairly straightforward conversions from their Volume 1 counterparts.

SUBROUTINE twofft(data1,data2,fft1,fft2)
USE nrtype; USE nrutil, ONLY: assert,assert_eq
USE nr, ONLY: four1
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: data1,data2
COMPLEX(SPC), DIMENSION(:), INTENT(OUT) :: fft1,fft2

Given two real input arrays data1 and data2 of length N, this routine calls four1 and returns two complex output arrays, fft1 and fft2, each of complex length N, that contain the discrete Fourier transforms of the respective data arrays. N must be an integer power of 2.

INTEGER(I4B) :: n,n2
COMPLEX(SPC), PARAMETER :: C1=(0.5_sp,0.0_sp), C2=(0.0_sp,-0.5_sp)
COMPLEX, DIMENSION(size(data1)/2+1) :: h1,h2
n=assert_eq(size(data1),size(data2),size(fft1),size(fft2),'twofft')
call assert(iand(n,n-1)==0, 'n must be a power of 2 in twofft')
fft1=cmplx(data1,data2,kind=spc)
Pack the two real arrays into one complex array.
call four1(fft1,1)  Transform the complex array.
fft2(1)=cmplx(aimag(fft1(1)),0.0_sp,kind=spc)
fft1(1)=cmplx(real(fft1(1)),0.0_sp,kind=spc)
n2=n/2+1
h1(2:n2)=C1*(fft1(2:n2)+conjg(fft1(n:n2:-1)))
Use symmetries to separate the two transforms.
h2(2:n2)=C2*(fft1(2:n2)-conjg(fft1(n:n2:-1)))
fft1(2:n2)=h1(2:n2)
Ship them out in two complex arrays.
fft1(n:n2:-1)=conjg(h1(2:n2))
fft2(2:n2)=h2(2:n2)
fft2(n:n2:-1)=conjg(h2(2:n2))
END SUBROUTINE twofft

⋆ ⋆ ⋆
SUBROUTINE realft_sp(data,isign,zdata)
USE nttype; USE nrutil, ONLY : assert,assert_eq,zroots_unity
USE nr, ONLY : four1
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign
COMPLEX(SPC), DIMENSION(:), OPTIONAL, TARGET :: zdata

When isign = 1, calculates the Fourier transform of a set of $N$ real-valued data points, input in the array data. If the optional argument zdata is not present, the data are replaced by the positive frequency half of its complex Fourier transform. The real-valued first and last components of the complex transform are returned as elements data(1) and data(2), respectively. If the complex array zdata of length $N/2$ is present, data is unchanged and the transform is returned in zdata. $N$ must be a power of 2. If isign = −1, this routine calculates the inverse transform of a complex data array if it is the transform of real data. (Result in this case must be multiplied by $2/N$.) The data can be supplied either in data, with zdata absent, or in zdata.

INTEGER(I4B) :: n,ndum,nh,nq
COMPLEX(SPC), DIMENSION(size(data)/4) :: w
COMPLEX(SPC), DIMENSION(size(data)/4-1) :: h1,h2
COMPLEX(SPC), DIMENSION(:,), POINTER :: cdata
REAL(SP) :: c1=0.5_sp,c2

n=size(data)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in realft_sp')
nh=n/2
nq=n/4
if (present(zdata)) then
ndum=assert_eq(n/2,size(zdata),'realft_sp')
cdata=>zdata
else
allocate(cdata(n/2))
endif
if (isign == 1) cdata=cmplx(data(1:n-1:2),data(2:n:2),kind=spc)
else
allocate(cdata(n/2))
cdata=cmplx(data(1:n-1:2),data(2:n:2),kind=spc)
done if
if (isign == 1) c2=-0.5_sp
else
c2=0.5_sp
endif
w=zroots_unity(sign(n,isign),n/4)
w=cmplx(-aimag(w),real(w),kind=spc)
h1=c1*(cdata(2:nq)+conjg(cdata(nh:nq+2:-1)))
h2=c2*(cdata(2:nq)-conjg(cdata(nh:nq+2:-1)))

Next they are recombined to form the true transform of the original real data:
cdata(2:nq)=h1*w(2:nq)*h2
cdata(nh:nq+2:-1)=conjg(h1-w(2:nq)*h2)
end if
z=cdata(1) Squeeze the first and last data together to get them all within the original array.
if (isign == 1) then
end if
cdata(1)=cmplx(real(z)+aimag(z),real(z)-aimag(z),kind=spc)
call four1(cdata,-1)
done if
if (present(zdata)) then
if (isign /= 1) then
data(1:n-1:2)=real(cdata)
data(2:n:2)=aimag(cdata)
done if
else
data(1:n-1:2)=real(cdata)
data(2:n:2)=aimag(cdata)
deallocate(cdata)
done if

END SUBROUTINE realft_sp

SUBROUTINE realft_dp(data,isign,zdata)
USE nrtype; USE nrutil, ONLY : assert, assert_eq, zroots_unity
USE nr, ONLY : four1
IMPLICIT NONE
REAL(DP), DIMENSION(:,), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign
COMPLEX(DPC), DIMENSION(:,), OPTIONAL, TARGET :: zdata
INTEGER(I4B) :: n, nq, nh
COMPLEX(DPC), DIMENSION(size(data)/4) :: w
COMPLEX(DPC), DIMENSION(size(data)/4-1) :: h1, h2
COMPLEX(DPC), DIMENSION(:,), POINTER :: cdata
REAL(DP) :: c1=0.5_dp, c2

n=size(data)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in realft_dp')
nh=n/2
nq=n/4
if (present(zdata)) then
    ndum=assert_eq(n/2,size(zdata),'realft_dp')
cdata=>zdata
if (isign == 1) cdata=cmplx(data(1:n-1:2),data(2:n:2),kind=spc)
else
    allocate(cdata(n/2))
cdata=cmplx(data(1:n-1:2),data(2:n:2),kind=spc)
end if
if (isign == 1) then
    c2=-0.5_dp
    call four1(cdata,+1)
else
    c2=0.5_dp
end if
w=zroots_unity(sign(n,isign),n/4)
w=cmplx(=aimag(w),real(w),kind=dpc)
h1=c1*(cdata(2:nq)+conjg(cdata(nh:nq+2:-1))
h2=c2*(cdata(2:nq)-conjg(cdata(nh:nq+2:-1)))

end if
if (present(zdata)) then
    if (isign /= 1) then
        data(1:n-1:2)=real(cdata)
data(2:n:2)=aimag(cdata)
deallocate(cdata)
    end if
    data(1:n-1:2)=real(cdata)
data(2:n:2)=aimag(cdata)
deallocate(cdata)
end if
END SUBROUTINE realft_dp

* * *
SUBROUTINE sinft(y)
USE nrtype; USE nrutil, ONLY : assert,cumsum,zroots_unity
USE nr, ONLY : realft
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
Calculates the sine transform of a set of \( N \) real-valued data points stored in array \( y \). The number \( N \) must be a power of 2. On exit \( y \) is replaced by its transform. This program, without changes, also calculates the inverse sine transform, but in this case the output array should be multiplied by \( 2/N \).
REAL(SP), DIMENSION(size(y)/2+1) :: wi
REAL(SP), DIMENSION(size(y)/2) :: y1,y2
INTEGER(I4B) :: n,nh
n=size(y)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in sinft')

nh=n/2
wi=aimag(zroots_unity(n+n,nh+1))

y(1)=0.0
y1=y(2:nh+1)*y(2:nh+1)+y(n:nh+1:-1)

y2=0.5_sp*y(2:nh+1)-y(n:nh+1:-1))

y(2:nh+1)=y1+y2
y(n:nh+1:-1)=y1-y2

call realft(y,+1)

y(1)=0.5_sp*y(1)

y(2)=0.0
y1=cumsum(y(1:n-1:2))

y(1:n-1:2)=y(2:n:2)

y(2:n:2)=y1

END SUBROUTINE sinft

SUBROUTINE cosft1(y)
USE nrtype; USE nrutil, ONLY : assert,cumsum,zroots_unity
USE nr, ONLY : realft
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
Calculates the cosine transform of a set of \( N + 1 \) real-valued data points \( y \). The transformed data replace the original data in array \( y \). \( N \) must be a power of 2. This program, without changes, also calculates the inverse cosine transform, but in this case the output array should be multiplied by \( 2/N \).
COMPLEX(SPC), DIMENSION((size(y)-1)/2) :: w
REAL(SP), DIMENSION((size(y)-1)/2-1) :: y1,y2
REAL(SP) :: summ
INTEGER(I4B) :: n,nh
n=size(y)-1
call assert(iand(n,n-1)==0, 'n must be a power of 2 in cosft1')

nh=n/2
w=zroots_unity(n+n,nh)
summ=0.5_sp*y(1)-y(n+1)
y(1)=0.5_sp*y(1)+y(n+1)
y1=0.5_sp*y(2:nh)+y(n:nh+2:-1)
y2=2:nh)-y(n:nh+2:-1)
summ=summ+sum(real(w(2:nh))*y2)
y2=2:nh)+y(2:nh)*y2
y(2:nh)=y1=y2
y(n:nh+2:-1)=y1+y2

call realft(y(1:n),1)

y(n+1)=y(2)
y(2)=summ
y(2:n:2)=cumsum(y(2:n:2))

END SUBROUTINE cosft1
SUBROUTINE cosft2(y,isign)
USE nrtype; USE nrutil, ONLY : assert,cumsum,zroots_unity
USE nr, ONLY: realft
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: y
INTEGER(I4B), INTENT(IN) :: isign
Calculates the 'staggered' cosine transform of a set of \( N \) real-valued data points \( y \). The transformed data replace the original data in array \( y \). \( N \) must be a power of 2. Set \( isign \) to +1 for a transform, and to -1 for an inverse transform. For an inverse transform, the output array should be multiplied by \( 2/N \).

COMPLEX(SPC), DIMENSION(size(y)) :: w
REAL(SP), DIMENSION(size(y)/2) :: y1,y2
REAL(SP) :: ytemp
INTEGER(I4B) :: n,nh
n=size(y)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in cosft2')

w=zroots_unity(4*n,n)

if (isign == 1) then
  Forward transform.
y1=0.5_sp*(y(1:nh)+y(n:nh+1:-1))
y2=aimag(w(2:n:2))*(y(1:nh)-y(n:nh+1:-1))
y(1:nh)=y1+y2
y(n:nh+1:-1)=y1-y2
call realft(y,1)
y1(1:nh-1)=y(3:n-1:2)*real(w(3:n-1:2)) &
          -y(4:n:2)*aimag(w(3:n-1:2))
y2(1:nh-1)=y(4:n:2)*real(w(3:n-1:2)) &
          +y(3:n-1:2)*aimag(w(3:n-1:2))
y(3:n-1:2)=y1(1:nh-1)
y(4:n:2)=y2(1:nh-1)
ytemp=0.5_sp*y(2)
y(n-2:2:-2)=cumsum(y(n:4:-2),ytemp)
y(n)=ytemp
else if (isign == -1) then
  Inverse transform.
ytemp=y(n)
y(4:n:2)=y(2:n-2:2)-y(4:n:2)
y(2)=2.0_sp*ytemp
y1(1:nh-1)=y(3:n-1:2)*real(w(3:n-1:2)) &
          +y(4:n:2)*aimag(w(3:n-1:2))
y2(1:nh-1)=y(4:n:2)*real(w(3:n-1:2)) &
          -y(3:n-1:2)*aimag(w(3:n-1:2))
y(3:n-1:2)=y1(1:nh-1)
y(4:n:2)=y2(1:nh-1)
call realft(y,-1)
y1=y(1:nh)*y(n:nh+1:-1)
y2=(0.5_sp/aimag(w(2:n:2)))*(y(1:nh)-y(n:nh+1:-1))
y(1:nh)=0.5_sp*(y1+y2)
y(n:nh+1:-1)=0.5_sp*(y1-y2)
end if
END SUBROUTINE cosft2

SUBROUTINE four3(data,isign)
USE nrtype
USE nr, ONLY : fourrow_3d
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:,,:), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign
Replaces a 3-d complex array \( data \) by its discrete 3-d Fourier transform, if \( isign \) is input as 1; or replaces \( data \) by its inverse 3-d discrete Fourier transform times the product of its
three sizes, if isign is input as $-1$. All three of data's sizes must be integer powers of 2
(this is checked for in `fourrow_3d`). Parallelism is by use of `fourrow_3d`.

```fortran
COMPLEX(SPC), DIMENSION(:, :, ::), ALLOCATABLE :: dat2, dat3

call fourrow_3d(data, isign)  ! Transform in third dimension.
allocate(dat2(size(data, 2), size(data, 3), size(data, 1)))
dat2=reshape(data, shape=shape(dat2), order=(/3, 1, 2/))  ! Transpose.
call fourrow_3d(dat2, isign)  ! Transform in (original) first dimension.
allocate(dat3(size(data, 3), size(data, 1), size(data, 2)))
dat3=reshape(dat2, shape=shape(dat3), order=(/3, 1, 2/))  ! Transpose.
deallocate(dat2)
call fourrow_3d(dat3, isign)  ! Transform in (original) second dimension.
data=reshape(dat3, shape=shape(data), order=(/3, 1, 2/))  \ Transpose back to output order.
deallocate(dat3)
END SUBROUTINE four3
```

The `reshape` intrinsic, used with an `order=` parameter, is the multidimensional generalization of the two-dimensional `transpose` operation. The line

```fortran
dat2=reshape(data, shape=shape(dat2), order=(/3, 1, 2/))
```

is equivalent to the do-loop

```fortran
do j=1, size(data, 1)
dat2(:,:,j)=data(j, :, :)
end do
```

Incidentally, we have found some Fortran 90 compilers that (for scalar machines) are significantly slower executing the `reshape` than executing the equivalent do-loop. This, of course, shouldn't happen, since the `reshape` basically is an implicit do-loop. If you find such inefficient behavior on your compiler, you should report it as a bug to your compiler vendor! (Only thus will Fortran 90 compilers be brought to mature states of efficiency.)

```fortran
SUBROUTINE four3_alt(data, isign)
USE nrtype
USE nr, ONLY : fourcol_3d
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:, :, ::), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign

Replaces a 3-d complex array data by its discrete 2-d Fourier transform, if isign is input
as 1; or replaces data by its inverse 3-d discrete Fourier transform times the product
of its three sizes, if isign is input as $-1$. All three of data's sizes must be integer powers
of 2 (this is checked for in `fourcol_3d`). Parallelism is by use of `fourcol_3d`. (Use this
version only if `fourcol_3d` is faster than `fourrow_3d` on your machine.)

```fortran
COMPLEX(SPC), DIMENSION(:, :, ::), ALLOCATABLE :: dat2, dat3

call fourcol_3d(data, isign)  ! Transform in first dimension.
allocate(dat2(size(data, 2), size(data, 3), size(data, 1)))
dat2=reshape(data, shape=shape(dat2), order=(/3, 1, 2/))  ! Transpose.
call fourcol_3d(dat2, isign)  ! Transform in (original) second dimension.
allocate(dat3(size(data, 3), size(data, 1), size(data, 2)))
dat3=reshape(dat2, shape=shape(dat3), order=(/3, 1, 2/))  ! Transpose.
deallocate(dat2)
call fourcol_3d(dat3, isign)  ! Transform in (original) third dimension.
data=reshape(dat3, shape=shape(data), order=(/3, 1, 2/))  ! Transpose back to output order.
deallocate(dat3)
END SUBROUTINE four3_alt
```
Note that \texttt{four3} uses \texttt{fourrow\_3d}, the three-dimensional counterpart of \texttt{fourrow}, while \texttt{four3\_alt} uses \texttt{fourcol\_3d}, the three-dimensional counterpart of \texttt{fourcol}. You may want to time these programs to see which is faster on your machine.

\begin{center}
\begin{tabular}{c}
\texttt{* \text{* \text{*}}
\end{tabular}
\end{center}

In Volume 1, a single routine named \texttt{rlft3} was able to serve both as a three-dimensional real FFT, and as a two-dimensional real FFT. The trick is that the Fortran 77 version doesn’t care whether the input array data is dimensioned as two- or three-dimensional. Fortran 90 is not so indifferent, and better programming practice is to have two separate versions of the algorithm:

\begin{verbatim}
SUBROUTINE rlft2(data,spe,,speq,isign)
USE nrtype; USE nrutil, ONLY: assert,assert_eq
USE nr, ONLY: four2
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: data
COMPLEX(SPC), DIMENSION(:,,:), INTENT(INOUT) :: spec
COMPLEX(SPC), DIMENSION(:,), INTENT(INOUT) :: speq
INTEGER(I4B), INTENT(IN) :: isign
Given a two-dimensional real array data(1:M,1:N), this routine returns (for isign=1) the complex fast Fourier transform as two complex arrays: On output, spec(1:M/2,1:N) contains the zero and positive frequency values of the first frequency component, while speq(1:N) contains the Nyquist critical frequency values of the first frequency component. The second frequency components are stored for zero, positive, and negative frequencies, in standard wrap-around order. For isign=-1, the inverse transform (times M×N/2 as a constant multiplicative factor) is performed, with output data deriving from input spec and speq. For inverse transforms on data not generated first by a forward transform, make sure the complex input data array satisfies property (12.5.2). The size of all arrays must always be integer powers of 2.

INTEGER :: i1,j1,nn1,nn2
REAL(DP) :: theta
COMPLEX(SPC) :: c1=(0.5_sp,0.0_sp),c2,h1,h2,w
COMPLEX(SPC), DIMENSION(size(data,2)-1) :: h1a,h2a
COMPLEX(DPC) :: ww,wp
nn1=assert_eq(size(data,1),2*size(spec,1),'rlft2: nn1')
nn2=assert_eq(size(data,2),size(spec,2),size(speq),'rlft2: nn2')
call assert((/nn1,nn2/),(/nn1,nn2/)-1)==0, &
'dimensions must be powers of 2 in rlft2')
c2=cmplx(0.0_sp,-0.5_sp*isign,kind=spc)
theta=TWOPI_D/(isign*nn1)
wp=cmplx(-2.0_dp*sin(0.5_dp*theta)**2,sin(theta),kind=spc)
if (isign == 1) then
Case of forward transform.
spec(:,:)=cmplx(data(1:nn1:2,:),data(2:nn1:2,:),kind=spc)
call four2(spec,isign)
Here is where most all of the compute time
speq=spec(1,:,)
is spent.
end if
h1=c1*(spec(1,1)+conjg(speq(1)))
h1a=c1*(spec(1,2:nn2)+conjg(spec(nn2:2:-1)))
h2=c2*(spec(1,1)-conjg(spec(1)))
h2a=c2*(spec(1,2:nn2)-conjg(spec(nn2:2:-1)))
spec(1,1)=h1+h2
spec(1,2:nn2)=h1a+h2a
speq(1)=conjg(h1-h2)
speq(nn2:2:-1)=conjg(h1a-h2a)
ww=cmplx(1.0_dp,0.0_dp,kind=dpc)
Initialize trigonometric recurrence.
do i1=2,nn1/4+1
j1=nn1/2-i1+2

Corresponding negative frequency.
ww=ww*wp+ww
Do the trig recurrence.
ww

Equation (12.3.5).

h1=c1*(spec(1,1)+conjg(spec(1,1)))
h1a=c1*(spec(1,2:nn2)+conjg(spec(nn2:2:-1)))
end if
\end{verbatim}
h2 = c2 * (spec(i1,1) - conjg(spec(j1,1)))
h2a = c2 * (spec(i1,2:nn2) - conjg(spec(j1,nn2:2:-1)))
spec(i1,1) = h1 + w * h2
spec(i1,2:nn2) = h1a + w * h2a
spec(j1,1) = conjg(h1 - w * h2)
spec(j1,nn2:2:-1) = conjg(h1a - w * h2a)
end do
if (isign == -1) then
   Case of reverse transform.
call four2(spec,isign)
data(1:nn1:2,:) = real(spec)
data(2:nn1:2,:) = imag(spec)
end if
END SUBROUTINE rlft2

Here an overloaded version of assert that takes vector arguments is used to check that each dimension is a power of 2. Note that iand acts element-by-element on an array.

SUBROUTINE rlft3(data,spec,speq,isign)
USE nrtype; USE nrutil, ONLY: assert, assert_eq
USE nr, ONLY: four3
REAL(SP), DIMENSION(:, :, :), INTENT(INOUT) :: data
COMPLEX(SPC), DIMENSION(:, :, :), INTENT(INOUT) :: spec
COMPLEX(SPC), DIMENSION(:, :, :), INTENT(INOUT) :: speq
INTEGER(I4B), INTENT(IN) :: isign
Given a three-dimensional real array \( \text{data}(1:L,1:M,1:N) \), this routine returns (for \( \text{isign} = 1 \)) the complex Fourier transform as two complex arrays: On output, the zero and positive frequency values of the first frequency component are in \( \text{spec}(1:L/2,1:M,1:N) \), while \( \text{spec}(1:M,1:N) \) contains the Nyquist critical frequency values of the first frequency component. The second and third frequency components are stored for zero, positive, and negative frequencies, in standard wrap-around order. For \( \text{isign} = -1 \), the inverse transform (times \( L \times M \times N/2 \) as a constant multiplicative factor) is performed, with output data deriving from input \( \text{spec} \) and \( \text{speq} \). For inverse transforms on data not generated first by a forward transform, make sure the complex input data array satisfies property (12.5.2). The size of all arrays must always be integer powers of 2.

INTEGER :: i1,i3,j1,j3,nn1,nn2,nn3
REAL(DP) :: theta
COMPLEX(SPC) :: c1=(0.5_sp,0.0_sp),c2,h1,h2,w
COMPLEX(SPC), DIMENSION(size(data,2)-1) :: h1a,h2a
COMPLEX(DPC) :: ww,wp
c2 = cmplx(0.0_sp,-0.5_sp*isign,kind=spc)
nn1 = assert_eq(size(data,1),2*size(spec,1),'rlft2: nn1')
nn2 = assert_eq(size(data,2),size(spec,2),size(speq,1),'rlft2: nn2')
nn3 = assert_eq(size(data,3),size(spec,3),size(speq,2),'rlft2: nn3')
call assert(iand((/nn1,nn2,nn3/),(/nn1,nn2,nn3/)-1)==0, &
   'dimensions must be powers of 2 in rlft3')
theta = TWOPI_D/(isign*nn1)
wp = cmplx(-2.0_dp*sin(0.5_dp*theta)**2,sin(theta),kind=dpc)
nn1 = assert_eq(size(data(1,1:2,:,1)),size(spec,1),&
   'size mismatch in rlft3')
nn2 = assert_eq(size(data(1,:,nn2:2:-1,1)),size(spec,2),&
   'size mismatch in rlft3')
nn3 = assert_eq(size(data(:,1,1:2,:),nn3),size(spec,3),&
   'size mismatch in rlft3')
call assert(iand((/nn1,nn2,nn3/),(/nn1,nn2,nn3/)-1)==0, &
   'dimensions must be powers of 2 in rlft3')
theta = TWOPI_D/(isign*nn1)
wp = cmplx(-2.0_dp*sin(0.5_dp*theta)**2,sin(theta),kind=dpc)
if (isign == 1) then
   Case of forward transform.
spec(:, :) = cmplx(data(1:nn1:2,:,:),data(2:nn1:2,:,:),kind=dpc)
call four3(spec,isign)
speq = spec(1,:,:)
end if
do i3 = 1,nn3
   if (i3 /= 1) j3 = nn3-i3+2
   h1 = c1 * (spec(1,1:i3)+conjg(speq(1,j3)))
h1a = c1 * (spec(1,2:i3)+conjg(speq(nn2:2:-1,j3)))
h2 = c2 * (spec(1,1:i3)-conjg(spec(1,j3)))
h2a = c2 * (spec(1,2:i3)-conjg(spec(nn2:2:-1,j3)))
   spec(1,1:i3) = h1 + h2
   spec(1,2:i3) = h1a + h2a
end do
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spec(1,j3)=conjg(h1-h2)
spec(nn2:2:-1,j3)=conjg(h1a-h2a)
ww=cmplx(1.0_dp,0.0_dp,kind=dpc)  \[ \text{Initialize trigonometric recurrence.} \]
do i1=2,nn1/4+1
  j1=nn1/2-i1+2
  Corresponding negative frequency.
  ww=ww*wp+ww  \[ \text{Do the trig recurrence.} \]
  w=ww
  h1=c1*(spec(i1,1,i3)+conjg(spec(j1,1,j3)))  \[ \text{Equation (12.3.5).} \]
  h1a=c1*(spec(i1,2:nn2,i3)+conjg(spec(j1,nn2:2:-1,j3)))
  h2=c2*(spec(i1,1,i3)-conjg(spec(j1,1,j3)))
  h2a=c2*(spec(i1,2:nn2,i3)-conjg(spec(j1,nn2:2:-1,j3)))
  spec(i1,1,i3)=h1+w*h2
  spec(i1,2:nn2,i3)=h1a+w*h2a
  spec(j1,1,j3)=conjg(h1-w*h2)
  spec(j1,nn2:2:-1,j3)=conjg(h1a-w*h2a)
end do

if (isign == -1) then  \[ \text{Case of reverse transform.} \]
call four3(spec,isign)
data(1:nn1:2,:,:)=real(spec)
data(2:nn1:2,:,:)=aimag(spec)
end if
END SUBROUTINE rlft3

⋆⋆⋆

Referring back to the discussion of parallelism, §22.4, that led to four1’s implementation with $\sqrt{N}$ parallelism, you might wonder whether Fortran 90 provides sufficiently powerful high-level constructs to enable an FFT routine with $N$-fold parallelism. The answer is, “It does, but you wouldn’t want to use them!” Access to arbitrary interprocessor communication in Fortran 90 is through the mechanism of the “vector subscript” (one-dimensional array of indices in arbitrary order). When a vector subscript is on the right-hand side of an assignment statement, the operation performed is effectively a “gather”; when it is on the left-hand side, the operation is effectively a “scatter.”

It is quite possible to write the classic FFT algorithm in terms of gather and scatter operations. In fact, we do so now. The problem is efficiency: The computations involved in constructing the vector subscripts for the scatter/gather operations, and the actual scatter/gather operations themselves, tend to swamp the underlying very lean FFT algorithm. The result is very slow, though theoretically perfectly parallelizable, code. Since small-scale parallel (SSP) machines can saturate their processors with $\sqrt{N}$ parallelism, while massively multiprocessor (MMP) machines inevitably come with architecture-optimized FFT library calls, there is really no niche for these routines, except as pedagogical demonstrations. We give here a one-dimensional routine, and also an arbitrary-dimensional routine modeled on Volume 1’s four.

Note the complete absence of do-loops of size $N$; the loops that remain are over $\log N$ stages, or over the number of dimensions.

SUBROUTINE four1_gather(data,isign)
USE nrtype; USE nrutil, ONLY : arth, assert
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:,::), INTENT(INOUT) :: data
INTEGER(1B), INTENT(IN) :: isign

Replaces a complex array data by its discrete Fourier transform, if isign is input as 1; or replaces data by size(data) times its inverse discrete Fourier transform, if isign is input as −1. The size of data must be an integer power of 2. This routine demonstrates coding the FFT algorithm in high-level Fortran 90 constructs. Generally the result is very
much slower than library routines coded for specific architectures, and also significantly slower
than the parallelization-by-rows method used in the routine four1.

INTEGER(14B) :: n,n2,m,mm
REAL(DP) :: theta
COMPLEX(SPC) :: wp
INTEGER(14B), DIMENSION(size(data)) :: jarr
INTEGER(14B), DIMENSION(:,), ALLOCATABLE :: jrev
COMPLEX(SPC), DIMENSION(:,), ALLOCATABLE :: wtab,dtemp
n=size(data)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in four1_gather')
if (n <= 1) RETURN
allocate(jrev(n)) Begin bit-reversal section of the routine.
jarr=arth(0,1,n)
jrev=0
n2=n/2
m=n2
do where (iand(jarr,1) /= 0) jrev=jrev+m to its bit-reverse.
jarr=jarr/2
m=m/2
if (m == 0) exit
end do
data=data(jrev+1) Move all data to bit-reversed location by a
allocate(jrev)
data=data(jrev+1)
decall(jrev)
decall(dtemp(n),wtab(n2)) Begin Danielson-Lanczos section of the rou-
tine.
jarr=arth(0,1,n)
m=1
mm=n2
wtab(1)=(1.0_sp,0.0_sp) Seed the roots-of-unity table.
do where (iand(jarr,m) /= 0) The basic idea is to address the correct root-of-unity for each Danielson-Lanczos
jarr=jarr/2
m=m/2
if (m == 0) exit
end do
 data=data(jrev+1) This is half of Danielson-Lanczos.
decall(jrev)
nn(1:M) is an integer array containing
mm=mm/2
theta=PI_D/(isign*m) Ready for trigonometry?
wp=cmplx(-2.0_dp*sin(0.5_dp*theta)**2, sin(theta),kind=spc)
elsewhere
data=eoshift(dtemp,n) This is the other half. The referenced
mm=mm/2
wn=wn/2
wtab(1:n2:mm:mm*mm)=wtab(1:n2:mm:mm*mm)*wp+wtab(1:n2:mm:mm*mm)
decall(dtemp,wtab)
END SUBROUTINE four1_gather

SUBROUTINE fourn_gather(data,nn,isign)
USE nrtype; USE nrutil, ONLY : arth,assert
IMPLICIT NONE
COMPLEX(SPC), DIMENSION(:,), INTENT(INOUT) :: data
INTEGER(14B), DIMENSION(:,), :: nn
INTEGER(14B), INTENT(IN) :: isign
For data a one-dimensional complex array containing the values (in Fortran normal
ordering) of an M-dimensional complex array, this routine replaces data by its M-dimensional
discrete Fourier transform, if isign is input as 1. nn(1:M) is an integer array containing
the lengths of each dimension (number of complex values), each of which must be a power
of 2. If isign is input as −1, data is replaced by its inverse transform times the product of
the lengths of all dimensions. This routine demonstrates coding the multidimensional FFT
algorithm in high-level Fortran 90 constructs. Generally the result is very much slower than
library routines coded for specific architectures, and significantly slower than routines four2 and four3 for the two- and three-dimensional cases.

```
INTEGER(I4B), DIMENSION(:,), ALLOCATABLE :: jarr
INTEGER(I4B) :: ndim,idim,ntot,n2,msk0,msk1,msk2,m,mm,mn
REAL(DP) :: theta
COMPLEX(SPC) :: wp
COMPLEX(SPC), DIMENSION(:,), ALLOCATABLE :: wtab,dtemp
call assert(iand(nn,nn-1)==0, &
'each dimension must be a power of 2 in fourn_gather')
ndim=size(nn)
ntot=product(nn)
nprev=1
allocate(jarr(ntot))
do idim=1,ndim
  jarr=arth(0,1,ntot)
  n=nn(idim)
n2=n/2
  msk0=nprev
  msk1=nprev*n2
  msk2=msk0+msk1
  do if (msk1 <= msk0) exit
     where (iand(jarr,msk0) == 0 .neqv. iand(jarr,msk1) == 0) &
        jarr=ieor(jarr,msk2)
     msk0=msk0*2
     msk1=msk1/2
     msk2=msk0+msk1
  end do
  data=data(jarr+1)
allocate(dtemp(ntot),wtab(n2))
jarr=iand(n-1,arth(0,1,ntot)/nprev)
m=1
mn=n2
mm=n2*nprev
wtab(1)=(1.0_SPC,0.0_SPC)
do if (mm == 0) exit
   where (iand(jarr,m) /= 0)
      The basic idea is to address the correct root-of-unity for each Danielson-Lanczos
      multiplication by tricky bit manipulations.
   dtemp=data+tab(m;iand(jarr,m-1)+1)
data=eoshift(data,-mn)-dtemp
elsewhere
   data=data+eoshift(dtemp,mm)
end where
m=m*2
if (m >= n) exit
mn=mn+1
mm=mn/2
theta=PI_D/(isign*m)
wp=cmplx(-2.0_dp*sin(0.5_dp*theta)**2,sin(theta),kind=dpc)
Add entries to the table for the next iteration.
   wtab(mm+1:n2:2*mm)=wtab(1:n2-mm:2*mm)*wp &
   wtab(1:n2-mm:2*mm)
end do
deallocate(dtemp,wtab)
nprev=n*nprev
end do
deallocate(jarr)
END SUBROUTINE fourn_gather
call assert(iand(nn,nn-1)==0 ... Once again the vector version of
assert is used to test all the dimensions stored in nn simultaneously.
```
Chapter B13. Fourier and Spectral Applications

FUNCTION convlv(data, respns, isign)
USE nrtype; USE nrutil, ONLY : assert, nrerror
USE nr, ONLY : realft
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: data
REAL(SP), DIMENSION(:), INTENT(IN) :: respns
INTEGER(I4B), INTENT(IN) :: isign
REAL(SP), DIMENSION(size(data)) :: convlv

Convolves or deconvolves a real data set (of length N, including any user-supplied zero padding) with a response function respns, stored in wrap-around order in a real array of length \( M \leq N \). (M should be an odd integer, N a power of 2.) Wrap-around order means that the first half of the array respns contains the impulse response function at positive times, while the second half of the array contains the impulse response function at negative times, counting down from the highest element respns(M). On input isign is +1 for convolution, −1 for deconvolution. The answer is returned as the function convlv, an array of length \( N \). data has INTENT(INOUT) for consistency with realft, but is actually unchanged.

INTEGER(I4B) :: no2, n, m
COMPLEX(SPC), DIMENSION(size(data)/2) :: tmpd, tmpr
n = size(data)
m = size(respns)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in convlv')
call assert(mod(m,2)==1, 'm must be odd in convlv')
convlv(1:m) = respns(:)
Put respns in array of length n.
convlv(n-(m-3)/2:n) = convlv((m+3)/2:m)
Pad with zeros.
no2 = n/2
call realft(data,1,tmpd)
FFT both arrays.
call realft(convlv,1,tmpr)

if (isign == 1) then
  Multiply FFTs to convolve.
  tmpr(1) = cmplx(real(tmpd(1))*real(tmpr(1))/no2, &
  aimag(tmpd(1))*aimag(tmpr(1))/no2, kind=spc)
  tmpr(2:) = tmpd(2:)/tmpr(2:)/no2
else if (isign == -1) then
  Divide FFTs to deconvolve.
  if (any(abs(tmpr(2:)) == 0.0) .or. real(tmpr(1)) == 0.0 &
  .or. aimag(tmpr(1)) == 0.0) call nrerror &
  ("deconvolving at response zero in convlv")
  tmpr(1) = cmplx(real(tmpd(1))/real(tmpr(1))/no2, &
  aimag(tmpd(1))/aimag(tmpr(1))/no2, kind=spc)
  tmpr(2:) = tmpd(2:)/tmpr(2:)/no2
else
  call nrerror("no meaning for isign in convlv")
end if

END FUNCTION convlv
The intrinsic function `cmplx` returns a quantity of type default complex unless the kind argument is present. It is therefore a good idea always to include this argument. The intrinsic functions `real` and `aimag`, on the other hand, when called with a complex argument, return the same kind as their argument. So it is a good idea not to put in a kind argument for these. (In fact, `aimag` doesn’t allow one.) Don’t confuse these situations, regarding complex variables, with the completely unrelated use of `real` to convert a real or integer variable to a real value of specified kind. In this latter case, kind should be specified.

```fortran
FUNCTION correl(data1, data2)
USE nrtype; USE nrutil, ONLY : assert, assert_eq
USE nr, ONLY : realft
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: data1, data2
REAL(SP), DIMENSION(size(data1)) :: correl

Computes the correlation of two real data sets `data1` and `data2` of length `N` (including any user-supplied zero padding). `N` must be an integer power of 2. The answer is returned as the function `correl`, an array of length `N`. The answer is stored in wrap-around order, i.e., correlations at increasingly negative lags are in `correl(N)` on down to `correl(N/2 + 1)`, while correlations at increasingly positive lags are in `correl(1)` (zero lag) on up to `correl(N/2)`. Sign convention of this routine: if `data1` lags `data2`, i.e., is shifted to the right of it, then `correl` will show a peak at positive lags.

```
FUNCTION correl(data1, data2)
USE nrtype; USE nrutil, ONLY : assert, assert_eq
USE nr, ONLY : realft
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: data1, data2
REAL(SP), DIMENSION(size(data1)) :: correl

Computes the correlation of two real data sets `data1` and `data2` of length `N` (including any user-supplied zero padding). `N` must be an integer power of 2. The answer is returned as the function `correl`, an array of length `N`. The answer is stored in wrap-around order, i.e., correlations at increasingly negative lags are in `correl(N)` on down to `correl(N/2 + 1)`, while correlations at increasingly positive lags are in `correl(1)` (zero lag) on up to `correl(N/2)`. Sign convention of this routine: if `data1` lags `data2`, i.e., is shifted to the right of it, then `correl` will show a peak at positive lags.

COMPLEX(SPC), DIMENSION(size(data1)/2) :: cdat1, cdat2
INTEGER(I4B) :: no2, n

Normalization for inverse FFT.

`n` must be a power of 2 in `correl`

`call assert(iand(n, n-1) == 0, 'n must be a power of 2 in correl')`

`no2 = n/2`

`call realft(data1, 1, cdat1)` Transform both data vectors.

`call realft(data2, 1, cdat2)`

`cdat1(1) = cmplx(real(cdat1(1))*real(cdat2(1))/no2, &
aimag(cdat1(1))*aimag(cdat2(1))/no2, kind=spc)` Multiply to find FFT of their correlation.

`cdat1(2:)=cdat1(2:)*conjg(cdat2(2:))/no2`

`call realft(correl,-1,cdat1)` Inverse transform gives correlation.

END FUNCTION correl

SUBROUTINE spctrm(p, k, ovrlap, unit, n_window)
USE nrtype; USE nrutil, ONLY : arith, nrerror
USE nr, ONLY : four1
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: p
INTEGER(I4B), INTENT(IN) :: k
LOGICAL(LGT), INTENT(IN) :: ovrlap
INTEGER(I4B), OPTIONAL, INTENT(IN) :: n_window, unit

Reads data from input unit `9`, or if the optional argument `unit` is present, from that input unit. The output is an array `p` of length `M` that contains the data’s power (mean square amplitude) at frequency 

\[(j - 1)/2M \text{ cycles per grid point, for } j = 1, 2, \ldots, M,\]

based on \(2(k+1)M\) data points (if `ovrlap` is set .true.) or \(4kM\) data points (if `ovrlap` is false.). The number of segments of the data is \(2k\) in both cases. The routine calls `four1` times, each call with 2 partitions each of \(2M\) real data points. If the optional argument `n_window` is present, the routine uses the Bartlett window, the square window,
or the Welch window for \( n_{\text{window}} = 1, 2, 3 \) respectively. If \( n_{\text{window}} \) is not present, the Bartlett window is used.

```fortran
INTEGER(I4B) :: j, joff, joffn, kk, m, m4, m43, m44, mm, iunit, nn_window
REAL(SP) :: den, facm, facp, sumw
REAL(SP), DIMENSION(2*size(p)) :: w
REAL(SP), DIMENSION(4*size(p)) :: w1
REAL(SP), DIMENSION(size(p)) :: w2
COMPLEX(SPC), DIMENSION(2*size(p)) :: cv1
m = size(p)
if (present(n_window)) then
  nn_window = n_window
else
  nn_window = 1
end if
if (present(unit)) then
  iunit = unit
else
  iunit = 9
end if
mm = m + m
m4 = mm + m
m44 = m4 + 4
m43 = m4 + 3
den = 0.0
facm = m
facp = 1.0_sp/m
w1(1:mm) = window(arth(1,1,mm), facm, facp, nn_window)
sumw = dot_product(w1(1:mm), w1(1:mm))
p(:) = 0.0
if (ovrlap) read (iunit, *) (w2(j), j = 1, m)
do kk = 1, k
  Loop over data segments in groups of two.
do joff = -1, 0, 1
  Get two complete segments into workspace.
  if (ovrlap) then
    w1(joff + 2:joff + mm:2) = w2(1:m)
    read (iunit, *) (w2(j), j = 1, m)
    joffn = joff + mm
    w1(joffn + 2:joffn + mm:2) = w2(1:m)
  else
    read (iunit, *) (w1(j), j = joff + 2, m4, 2)
  end if
  end do
w = window(arth(1,1,mm), facm, facp, nn_window)
w1(1:mm) = window(arth(1,1,mm), facm, facp, nn_window)
w1(2:m4:2) = w1(2:m4:2)*w
w1(1:m4:2) = w1(1:m4:2)*w
cv1(1:mm) = cmplx(w1(1:m4:2), w1(2:m4:2), kind=spc)
call four1(cv1(1:mm))
p(1) = p(1) + w1(1:mm)*2
p(2:m) = p(2:m) + w1(4:2:m:2)*2 + w1(3:2:m-1:2)*2 + &w1(m44-4:m44-2:m:-2)*2 + w1(m43-4:m43-2:m:-2)*2
den = den + sumw
end do
p(:) = p(:)/(m4*den)
CONTAINS
FUNCTION window(j, facm, facp, nn_window)
IMPLICIT NONE
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: j
INTEGER(I4B), INTENT(IN) :: nn_window
REAL(SP), INTENT(IN) :: facm, facp
REAL(SP), DIMENSION(size(j)) :: window
select case(nn_window)
case(1)
```

---

**Useful factors.**

<table>
<thead>
<tr>
<th>Factor</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>Size of the window</td>
</tr>
<tr>
<td>( mm )</td>
<td>Sum of ( m ) and ( m )</td>
</tr>
<tr>
<td>( m4 )</td>
<td>Sum of ( mm ) and ( m )</td>
</tr>
<tr>
<td>( m44 )</td>
<td>Sum of ( m4 ) and ( m )</td>
</tr>
<tr>
<td>( m43 )</td>
<td>Sum of ( m44 ) and ( m )</td>
</tr>
<tr>
<td>( facm )</td>
<td>Factor used by the window function</td>
</tr>
<tr>
<td>( facp )</td>
<td>Factor used by the window function</td>
</tr>
<tr>
<td>( w )</td>
<td>Window function</td>
</tr>
<tr>
<td>( w1 )</td>
<td>Window function with weights</td>
</tr>
<tr>
<td>( sumw )</td>
<td>Sum of squared weights</td>
</tr>
<tr>
<td>( p )</td>
<td>Spectrum</td>
</tr>
</tbody>
</table>

---

**Factors used by the window function.**

- \( facm = m \)
- \( facp = 1.0_{sp}/m \)

---

**Apply the window to the data.**

\[
\begin{align*}
  w1(1:mm) &= \text{window}(arth(1,1,mm), facm, facp, nn_window) \\
  w1(2:m4:2) &= w1(2:m4:2) * w \\
  w1(1:m4:2) &= w1(1:m4:2) * w \\
  cv1(1:mm) &= \text{cmplx}(w1(1:m4:2), w1(2:m4:2), kind=spc) \\
\end{align*}
\]

**Fourier transform the windowed data.**

\[
\begin{align*}
  \text{call four1}(cv1(1:mm)) \\
  \text{p}(1) &= \text{p}(1) + w1(1:mm) * 2 \\
  \text{p}(2:m) &= \text{p}(2:m) + w1(4:2:m:2) * 2 + w1(3:2:m-1:2) * 2 + &w1(m44-4:m44-2:m:-2) * 2 + w1(m43-4:m43-2:m:-2) * 2 \\
  \text{den} &= \text{den} + \text{sumw} \\
\end{align*}
\]

**Normalize the output.**

\[
\begin{align*}
  \text{p}(:) &= \text{p}(:) / (m4 * \text{den}) \\
\end{align*}
\]
The Fortran 90 optional argument feature allows us to make unit 9
the default output unit in this routine, but leave the user the option of
specifying a different output unit by supplying an actual argument for
unit. We also use an optional argument to allow the user the option of overriding
the default selection of the Bartlett window function.

FUNCTION window(j,facm,facp,nn_window)
In Fortran 77 we coded this as a statement function. Here the internal function is equivalent, but allows full
specification of the interface and so is preferred.

SUBROUTINE memcof(data,xms,d)

Given a real vector data of length N, this routine returns M linear prediction coefficients
in a vector d of length M, and returns the mean square discrepancy as xms.

The algorithm is recursive, although it is implemented as an iteration. It builds up the
answer for larger and larger values of m until the desired value is reached. At this point
in the algorithm, one could return the vector d and scalar xms for a set of LP coefficients
with k (rather than m) terms.

if (k == m) RETURN
end do
call nrerror('never get here in memcof')
END SUBROUTINE memcof

* * *
SUBROUTINE fixrts(d)
USE nrtype
USE nr, ONLY : zroots
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: d

Given the LP coefficients d, this routine finds all roots of the characteristic polynomial
\(13.6.14\), reflects any roots that are outside the unit circle back inside, and then returns
a modified set of coefficients in d.

INTEGER(14B) :: i,m
LOGICAL(LGT) :: polish
COMPLEX(SPC), DIMENSION(size(d)+1) :: a
COMPLEX(SPC), DIMENSION(size(d)) :: roots
m=size(d)
a(m+1)=cmplx(1.0_sp,kind=spc)
\(\text{Set up complex coefficients for polynomial root finder.}\)
a(m:1:-1)=cmplx(-d(1:m),kind=spc)
\(\text{Find all the roots.}\)

where (abs(roots) > 1.0) roots=1.0_sp/conjg(roots)
\(\text{Reflect all roots outside the unit circle back inside.}\)
a(1)=-roots(1)
\(\text{Now reconstruct the polynomial coefficients,}\)
a(2:m+1)=cmplx(1.0_sp,kind=spc)
\(\text{by looping over the roots}\)
do i=2,m
a(2:i)=a(1:i-1)-roots(i)*a(2:i)
\(\text{and synthetically multiplying.}\)
a(i)=-roots(i)*a(1)
end do
d(m:1:-1)=-real(a(1:m))
\(\text{The polynomial coefficients are guaranteed}\)
\(\text{to be real, so we need only return the}\)
\(\text{real part as new LP coefficients.}\)
END SUBROUTINE fixrts

FUNCTION predic(data,d,nfut)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: data,d
INTEGER(I4B), INTENT(IN) :: nfut
REAL(SP), DIMENSION(nfut) :: predic

Gi v e na na r r a y data, and given the data's LP coefficients d in an array of length \(M\), this routine applies equation \(13.6.11\) to predict the next \(nfut\) data points, which it returns in
an array as the function value predic. Note that the routine references only the last \(M\)
values of data, as initial values for the prediction.

INTEGER(I4B) :: j,ndata,m
REAL(SP) :: discrp,sm
REAL(SP), DIMENSION(size(d)) :: reg
m=size(d)
ndata=size(data)
reg(1:m)=data(ndata:ndata+1-m:-1)
do j=1,nfut
   discrp=0.0
   \(\text{This is where you would put in a known discrepancy if you were reconstructing a function}\)
   \(\text{by linear predictive coding rather than extrapolating a function by linear prediction. See}\)
   \(\text{text.}\)
   sm=discrp+dot_product(d,reg)
   reg=eoshift(reg,-1,sm)
   \(\text{[If you want to implement circular arrays, you can}\)
   \(\text{avoid this shifting of coefficients!]}\)
predic(j)=sm
end do
END FUNCTION predic

* * *

FUNCTION predic(data,d,nfut)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: data,d
INTEGER(I4B), INTENT(IN) :: nfut
REAL(SP), DIMENSION(nfut) :: predic

Gi v e na na r r a y data, and given the data's LP coefficients d in an array of length \(M\), this routine applies equation \(13.6.11\) to predict the next \(nfut\) data points, which it returns in
an array as the function value predic. Note that the routine references only the last \(M\)
values of data, as initial values for the prediction.

INTEGER(I4B) :: j,ndata,m
REAL(SP) :: discrp,sm
REAL(SP), DIMENSION(size(d)) :: reg
m=size(d)
ndata=size(data)
reg(1:m)=data(ndata:ndata+1-m:-1)
do j=1,nfut
   discrp=0.0
   \(\text{This is where you would put in a known discrepancy if you were reconstructing a function}\)
   \(\text{by linear predictive coding rather than extrapolating a function by linear prediction. See}\)
   \(\text{text.}\)
   sm=discrp+dot_product(d,reg)
   reg=eoshift(reg,-1,sm)
   \(\text{[If you want to implement circular arrays, you can}\)
   \(\text{avoid this shifting of coefficients!]}\)
predic(j)=sm
end do
END FUNCTION predic

* * *
FUNCTION evlmem(fdt,d,xms)
USE nrtype; USE nrutil, ONLY : poly
IMPLICIT NONE
REAL(SP), INTENT(IN) :: fdt,xms
REAL(SP), DIMENSION(:), INTENT(IN) :: d
REAL(SP) :: evlmem

Given d and xms as returned by memcof, this function returns the power spectrum estimate
\[ P(f) \] as a function of \( fdt = f \Delta \).

COMPLEX(SPC) :: z,zz
REAL(DP) :: theta

Trigonometric recurrences in double precision.

\[ \theta = \text{TWOPI}_D \times fdt \]
\[ z = \text{cmplx}(\cos(\theta), \sin(\theta), \text{kind}=\text{spc}) \]
\[ zz = 1.0_{\text{sp}} - z \times \text{poly}(z,d) \]
\[ \text{evlmem} = xms / |zz|^2 \]

Equation (13.7.4).

END FUNCTION evlmem

\[ zz = \ldots \text{poly}(z,d) \] The poly function in nrutil returns the value of the polynomial with coefficients \( d(\cdot) \) at \( z \). Here a version that takes real coefficients and a complex argument is actually invoked, but all the different versions have been overloaded onto the same name poly.

⋆⋆⋆

SUBROUTINE period(x,y,ofac,hifac,px,py,jmax,prob)
USE nrtype; USE nrutil, ONLY : assert_eq,imaxloc
USE nr, ONLY : avevar
IMPLICIT NONE
INTEGER(I4B), INTENT(OUT) :: jmax
REAL(SP), INTENT(IN) :: ofac,hifac
REAL(SP), INTENT(OUT) :: prob
REAL(SP), INTENT(IN) :: x(:),y(:)
REAL(SP), POINTER :: px(:),py(:)

Input is a set of \( N \) data points with abscissas \( x \) (which need not be equally spaced) and ordinates \( y \), and a desired oversampling factor \( \text{ofac} \) (a typical value being 4 or larger). The routine returns pointers to internally allocated arrays \( px \) and \( py \). \( px \) is filled with an increasing sequence of frequencies (not angular frequencies) up to \( \text{hifac} \) times the "average" Nyquist frequency, and \( py \) is filled with the values of the Lomb normalized periodogram at those frequencies. The length of these arrays is \( 0.5 \times \text{ofac} \times \text{hifac} \times N \). The arrays \( x \) and \( y \) are not altered. The routine also returns \( jmax \) such that \( py(jmax) \) is the maximum element in \( py \), and \( prob \), an estimate of the significance of that maximum against the hypothesis of random noise. A small value of \( prob \) indicates that a significant periodic signal is present.

INTEGER(I4B) :: i,n,nout
REAL(SP) :: ave,cwtau,effm,expy,pnow,sumc,sumcy,sumsh,sumsy,swtau,var,wtau,xave,xdif,xmax,xmin
REAL(DP), DIMENSION(size(x)) :: tmp1,tmp2,wi,wpi,wpr,wr
LOGICAL(LGT), SAVE :: init=.true.

n=assert_eq(size(x),size(y),'period')
if (init) then
  init=.false.
nullify(px,py)
else
  if (associated(px)) deallocate(px)
  if (associated(py)) deallocate(py)
end if
nout=0.5_sp*ofac*hifac*n
allocate(px(nout),py(nout))
call avevar(y(:),ave,var)
xmax=maxval(x(:))
xmin=minval(x(:))
xdif=xmax-xmin

Get mean and variance of the input data.
Go through data to get the range of abscissas.
xave=0.5_sp*(xmax+xmin)  
pnow=1.0_sp/(dif*ofac)

Starting frequency.

tm1(:)=TWOPI_D*(x(:)-xave)*pnow

Initialize values for the trigonometric recurrences at each data point. The recurrences are done in double precision.
wpr(:)=-2.0_dp*sin(0.5_dp*tmp1)**2

wr(:)=cos(tmp1(:))

wp1(:)=sin(tmp1(:))

wi(:)=wp1(:)

do i=1,nout  
Main loop over the frequencies to be evaluated.
px(i)=pnow

First, loop over the data to get τ and related quantities.
sumsh=dot_product(wi,wr)

wtau=0.5_sp*atan2(2.0_sp*sumsh,sumc)

swtau=sin(wtau)
cwtau=cos(wtau)

Then, loop over the data again to get the periodogram value.

sumc=dot_product(tmp1,tmp1)

sums=dot_product(y(:)-ave,tmp1)

sumsy=dot_product(y(:)-ave,tmp2)

sumcy=dot_product(y(:)-ave,tmp3)

tmp1(:)=wi(:)*cwtau-wr(:)*swtau

Update the trigonometric recurrences.

wr(:)=(wr(:)*wpr(:)-wi(:)*wp1(:))+wr(:)

wi(:)=(wi(:)*wp1(:)+tmp1(:)*wp1(:))+wi(:)

py(i)=0.5_sp*(sumcy**2/sumc+sumsy**2/sums)/var

pnow=pnow+1.0_sp/(ofac*xdif)

The next frequency.

end do

jmax=imaxloc(py(1:nout))

Evaluate statistical significance of the maximum.

effm=2.0_sp*nout/ofac

prob=effm*exp(-py(jmax))

if (prob > 0.01_sp) prob=1.0_sp-(1.0_sp-expy)**effm
END SUBROUTINE period

This routine shows another example of how to return arrays whose size is not known in advance (cf. zbrac in Chapter B9). The coding is explained in the subsection on pointers in §21.5. The size of the output arrays, nout in the code, is available as size(px).

jmax=imaxloc... See discussion of imaxloc on p. 1017.
periodic signal is present.
Parameter: MACC is the number of interpolation points per 1/4 cycle of highest frequency.

```fortran
INTEGER(14B) :: j,k,n,ndim,nfreq,nfreqt,nout
REAL(SP) :: ave,ck,ckk,cterm,cvt,den,df,effm,expy,fac,fndim,hc2wt,&
hs2wt,hypo,term,swt,var,xdif,xmax,xmin
REAL(SP), DIMENSION(:,), ALLOCATABLE :: wk1,wk2
LOGICAL(LGT), SAVE :: init=.true.
n=assert_eq(size(x),size(y),'fasper')
if (init) then
  init=.false.
  nullify(px,py)
else
  if (associated(px)) deallocate(px)
  if (associated(py)) deallocate(py)
end if
nfreqt=ofac*hifac*n*MACC
nfreq=64
do if (nfreq >= nfreqt) exit
  nfreq=nfreq*2
end do
ndim=2*nfreq
allocate(wk1(ndim),wk2(ndim))
call avevar(y(1:n),ave,var)
  Compute the mean, variance, and range of the data.
xmax=maxval(x(:))
xmin=minval(x(:))
xdif=xmax-xmin
wk1(1:ndim)=0.0
  Zero the workspaces.
wk2(1:ndim)=0.0
fac=nadm/(xdif*ofac)
fndim=ndim
do j=1,n
  Extrapolate the data into the workspaces.
    ck=1.0_sp+mod((x(j)-xmin)*fac,fndim)
    ckk=1.0_sp+mod(2.0_sp*(ck-1.0_sp),fndim)
    call spreadval(y(j)-ave,wk1,ck,MACC)
    call spreadval(1.0_sp,wk2,ckk,MACC)
end do
call realft(wk1(1:ndim),1)
  Take the fast Fourier transforms.
call realft(wk2(1:ndim),1)
df=1.0_sp/(xdif*ofac)
nout=0.5_sp*ofac*hifac*n
allocate(px(nout),py(nout))
allocate(wk1(nout),wk2(nout))
k=3
do j=1,nout
  Compute the Lomb value for each frequency.
    hypo=sqrt(wk2(k)**2+wk2(k+1)**2)
    hc2wt=0.5_sp*wk2(k)/hypo
    hs2wt=0.5_sp*wk2(k+1)/hypo
    cvt=sqrt(0.5_sp*hc2wt)
    swt=sqrt(0.5_sp*hs2wt)
    den=0.5_sp*n+hc2wt*wk2(k)+hs2wt*wk2(k+1)
    cterm=(cvt*wk1(k)+swt*wk1(k+1))**2/den
    sterm=(cvt*wk1(k+1)-swt*wk1(k))**2/(n-den)
    px(j)=j*df
    py(j)=(cterm+sterm)/(2.0_sp*var)
    k=k+2
end do
deallocate(wk1,wk2)
jmax=imaxloc(py(1:nout))
expy=exp(-py(jmax))
  Estimate significance of largest peak value.
effm=2.0_sp*nout/ofac
prob=effm*expy
if (prob > 0.01_sp) prob=1.0_sp-(1.0_sp-expy)**effm
```

CONTAINS
SUBROUTINE spreadval(y,yy,x,m)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: y,x
REAL(SP), DIMENSION(:), INTENT(INOUT) :: yy
INTEGER(I4B), INTENT(IN) :: m

Given an array yy of length N, extrapolate (spread) a value y into m actual array elements that best approximate the “fictional” (i.e., possibly noninteger) array element number x.

The weights used are coefficients of the Lagrange interpolating polynomial.

INTEGER(I4B) :: ihi,ilo,ix,n,nden
REAL(SP) :: fac
INTEGER(I4B), DIMENSION(10) :: nfac = (/ &
1,1,2,6,24,120,720,5040,40320,362880 /)
if (m > 10) call nrerror('factorial table too small in spreadval')
n=size(yy)
ix=x
if (x == real(ix,sp)) then
  yy(ix)=yy(ix)+y
else
  ilo=min(max(int(x-0.5_sp*m+1.0_sp),1),n-m+1)
  ihi=ilo+m-1
  nden=nfac(m)
  fac=product((x-arith(ilo,1,m))
  yy(ihi)=yy(ihi)+y*fac/(nden*(x-ihi))
  do j=ihi-1,ilo,-1
    nden=(nden/(j+1-ilo))*(j-ihi)
    yy(j)=yy(j)+y*fac/(nden*(x-j))
  end do
end if
END SUBROUTINE spreadval
END SUBROUTINE fasper

This routine shows another example of how to return arrays whose size is not known in advance (cf. zbrac in Chapter B9). The coding is explained in the subsection on pointers in §21.5. The size of the output arrays, nout in the code, is available as size(px).

i=imaxloc... See discussion of imaxloc on p. 1017.

if (x == real(ix,sp)) then Without the explicit kind type parameter sp, real returns a value of type default real for an integer argument. This prevents automatic conversion of the routine from single to double precision. Here all you have to do is redefine sp in nrtype to get double precision.

*  *  *

SUBROUTINE dftcor(w,delta,a,b,endpts,corre,corim,corfac)
USE nrtype; USE nrutil, ONLY : assert
IMPLICIT NONE
REAL(SP), INTENT(IN) :: w,delta,a,b
REAL(SP), INTENT(OUT) :: corre,corim,corfac
REAL(SP), DIMENSION(:), INTENT(IN) :: endpts

For an integral approximated by a discrete Fourier transform, this routine computes the correction factor that multiplies the DFT and the endpoint correction to be added. Input is the angular frequency w, stepsize delta, lower and upper limits of the integral a and b, while the array endpts of length 8 contains the first 4 and last 4 function values. The
correction factor \( W(\theta) \) is returned as \( \text{corfac} \), while the real and imaginary parts of the endpoint correction are returned as \( \text{corre} \) and \( \text{corim} \).

```fortran
REAL(SP) :: a0i,a0r,a1i,a1r,a2i,a2r,a3i,a3r,\( w \),c,cl,cr,s,sl,sr,t,t2,t4,t6
REAL(DP) :: cth,ctth,spth2,sth,sth4i,stth,th,th2,th4,\( \text{th} \)&

corfac=1.0_sp-(11.0_sp/720.0_sp)*\( t^4 \)+
(23.0_sp/15120.0_sp)*\( t^6 \)
a0r=(-2.0_sp/3.0_sp)*\( t^2 \)+
(103.0_sp/15120.0_sp)*\( t^4 \)-
(169.0_sp/226800.0_sp)*\( t^6 \)
a1r=(7.0_sp/24.0_sp)-
(7.0_sp/180.0_sp)*\( t^2 \)+
(5.0_sp/3456.0_sp)*\( t^4 \)-
(169.0_sp/226800.0_sp)*\( t^6 \)
a2r=(-1.0_sp/6.0_sp)*\( t^2 \)+
(5.0_sp/6048.0_sp)*\( t^4 \)+
(5.0_sp/24192.0_sp)*\( t^6 \)
a3r=(1.0_sp/24.0_sp)-
(5.0_sp/6048.0_sp)*\( t^2 \)+
(5.0_sp/24192.0_sp)*\( t^4 \)+
(1.0_sp/24.0_sp)*\( t^6 \)
a0i=\( \text{th} \)*(2.0_sp/45.0_sp+\( \text{th} \)*
(2.0_sp/105.0_sp)*\( \text{th}^2 \)+
(8.0_sp/2835.0_sp)*\( \text{th}^4 \)+
(86.0_sp/467775.0_sp)*\( \text{th}^6 \)+
(86.0_sp/467775.0_sp)*\( \text{th}^8 \)+
(11.0_sp/72576.0_sp)*\( \text{th}^{10} \)+
(1.0_sp/24.0_sp)*\( \text{th}^{12} \)
a1i=\( \text{th} \)*
(7.0_sp/72.0_sp-\( \text{th} \)*
(11.0_sp/72576.0_sp)*\( \text{th}^{10} \)+
(13.0_sp/120.0_sp)*\( \text{th}^{12} \)
a2i=\( \text{th} \)*
(7.0_sp/90.0_sp)+
(2.0_sp/210.0_sp)*\( \text{th}^{12} \)+
(11.0_sp/90720.0_sp)*\( \text{th}^{14} \)+
(13.0_sp/120.0_sp)*\( \text{th}^{16} \)
a3i=\( \text{th} \)*
(7.0_sp/360.0_sp)+
(2.0_sp/840.0_sp)*\( \text{th}^{12} \)+
(11.0_sp/362880.0_sp)*\( \text{th}^{14} \)+
(13.0_sp/29937600.0_sp)*\( \text{th}^{16} \)
else
Use trigonometric formulas in double precision.
cth=cos(\( \text{th} \))
sth=sin(\( \text{th} \))
ctth=cth**2-sth**2
sthth=2.0_dp*sth*cth
th2=th*th
th4=th2*th2
th2*6.0_dp=th2
sth4i=1.0_sp/(6.0_dp*th4)
cth4i=2.0_dp*sth4i
cthth=cth4i+sth**2*(3.0_sp-4.0_dp*cth+cth2)
a0r=stth=\( \text{th} \)*
(14.0_dp*tmth2-7.0_dp*pth2*cth)
a1i=stth=\( \text{th} \)*
(30.0_dp*th-5.0_dp*ctth2)*sth
\end{verbatim}

```

The `dftcor` subroutine is defined here with the necessary mathematical expressions for calculating the various corrections. The subroutine takes a series of inputs and outputs the corrected values for `cl`, `cr`, `sl`, `sr`, and `arg` in both single and double precision. The code snippet includes various mathematical formulas for calculating these corrections, emphasizing the use of trigonometric identities and series expansions to achieve the necessary accuracy.
SUBROUTINE dftint(func,a,b,w,cosint,sinint)
USE nrtype; USE nrutil, ONLY : arth
USE nr, ONLY : dftcor,polint,realft
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b,w
REAL(SP), INTENT(OUT) :: cosint,sinint
INTERFACE
FUNCTION func(x)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
INTEGER(I4B), PARAMETER :: M=64,NDFT=1024,MPOL=6
Examplesubroutineillustratinghowtousetheroutine
dftcor. Theusersuppliesanexter-
nalfunction func thatreturnsthequantity
h(t). Therooutinethenreturns
\int_{a}^{b} \cos(\omega t) h(t) dt
as cosint and
\int_{a}^{b} \sin(\omega t) h(t) dt
as sinint.
Parameters: Thevalues of M, NDFT, and MPOL are merely illustrative and should be opti-
nimized for your particular application. M is the number of subintervals, NDFT is the length of
the FFT (a power of 2), and MPOL is the degree of polynomial interpolation used to obtain
the desired frequency from the FFT.
INTEGER(14B) :: nn
INTEGER(14B), SAVE :: init=0
INTEGER(14B), DIMENSION(MPOL) :: nnmpol
REAL(SP) :: c,cdft,cerr,corfac,corim,corre,en,s,sdft,serr
REAL(SP), SAVE :: delta
REAL(SP), DIMENSION(MPOL) :: cpol,spol,xpol
REAL(SP), DIMENSION(NDFT), SAVE :: data
REAL(SP), DIMENSION(8), SAVE :: endpts
REAL(SP), SAVE :: aold=-1.0e30_sp,bold=-1.0e30_sp
if (init /= 1 .or. a /= aold .or. b /= bold) then
  Do we need to initialize, or
  is only \omega changed?
  init=1
  aold=a
  bold=b
  delta=(b-a)/M
  data(1:M+1)=func(a+arth(0,1,M+1)*delta)
  Load the function values into the data array.
  data(M+2:NDFT)=0.0
  Zero pad the rest of the data array.
  endpts(1:4)=data(1:4)
  Load the endpoints.
  endpts(5:8)=data(M-2:M+1)
  call realft(data(1:NDFT),1)
  realft returns the unused value corresponding to \omega_{N/2} in data(2). We actually want
  this element to contain the imaginary part corresponding to \omega_{0}, which is zero.
  data(2)=0.0
end if
Now interpolate on the DFT result for the desired frequency. If the frequency is an \omega_n, i.e.,
the quantity en is an integer, then cdft=data(2*en-1), sdft=data(2*en), and you could
omit the interpolation.
en=\omega\Delta=NDFT/\pi\Delta=0
nn=min(\max(int(\omega/\pi/2/2),\omega-\pi/2/2)+1),\omega-\pi/2/2)+1)
Leftmost point for the inter-
npol=arth(nn,1,MPOL)
mpol(1:MPOL)=data(2*nnmpol(:)-1)
spol(1:MPOL)=data(2*nnmpol(:))
xpol(1:MPOL)=nnmpol(:)
call polint(xpol,spol,cdft,cerr)
call polint(xpol,spol,cdft,cerr)
call dftcor(w,\Delta,a,b,\omega,\omega,\omega,\omega)
Now get the endpoint cor-
cdft*corfac+corre
sdft*corfac+corim
Finally multiply by \Delta and \exp(i\omega a).
c=\Delta*\cos(\omega a)
s=\Delta*\sin(\omega a)
cosint=c*cdft-s*sdft
\[
\sin int = s \cdot \text{cdft} + c \cdot \text{sdft}
\]
END SUBROUTINE dftint

* * *

SUBROUTINE wt1(a,isign,wtstep)
USE nrtype; USE nrutil, ONLY : assert
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: isign
INTERFACE
SUBROUTINE wtstep(a,isign)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE wtstep
END INTERFACE
One-dimensional discrete wavelet transform. This routine implements the pyramid algorithm, replacing \(a\) by its wavelet transform (for \(\text{isign}=1\)), or performing the inverse operation (for \(\text{isign}=-1\)). The length of \(a\) is \(N\), which must be an integer power of 2. The subroutine \(\text{wtstep}\), whose actual name must be supplied in calling this routine, is the underlying wavelet filter. Examples of \(\text{wtstep}\) are \(\text{daub4}\) and (preceded by \(\text{pwtset}\)) \(\text{pwt}\).

INTEGER(I4B) :: n,nn
n=size(a)
call assert(iand(n,n-1)==0, 'n must be a power of 2 in wt1')
if (n < 4) RETURN
if (isign >= 0) then
Wavelet transform.
nn=n
Start at largest hierarchy,
do
if (nn < 4) exit
call \text{wtstep}(a(1:nn),isign)
nn=nn/2
and work towards smallest.
end do
else
Inverse wavelet transform.
nn=4
Start at smallest hierarchy,
do
if (nn > n) exit
call \text{wtstep}(a(1:nn),isign)
nn=nn*2
and work towards largest.
end do
end if
END SUBROUTINE wt1

SUBROUTINE daub4(a,isign)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: isign
Applies the Daubechies 4-coefficient wavelet filter to data vector \(a\) (for \(\text{isign}=1\)) or applies its transpose (for \(\text{isign}=-1\)). Used hierarchically by routines \(\text{wt1}\) and \(\text{wt3}\).
REAL(SP), DIMENSION(size(a)) :: wksp
REAL(SP), PARAMETER :: C0=0.4829629131445341_sp,&
C1=0.8365163037378079_sp,C2=0.2241438680420134_sp,&
C3=-0.1294095225512604_sp
INTEGER(I4B) :: n,nh,nhp,nhm
n=size(a)
if (n < 4) RETURN
nh=n/2
nhp=nh+1
nhm=nh-1
if (isign >= 0) then  Apply filter.
    wksp(1:nhm) = C0*a(1:n-3:2)+C1*a(2:n-2:2) &
                 +C2*a(3:n-1:2)+C3*a(4:n:2)
    wksp(nh)=C0*a(n-1)+C1*a(n)+C2*a(1)+C3*a(2)
    wksp(nhp:n-1) = C3*a(1:n-3:2)-C2*a(2:n-2:2) &
                     +C1*a(3:n-1:2)-C0*a(4:n:2)
    wksp(n)=C3*a(n-1)-C2*a(n)+C1*a(1)-C0*a(2)
else  
    Apply transpose filter.
    wksp(1)=C2*a(nh)+C1*a(n)+C0*a(1)+C3*a(nhp)
    wksp(2)=C3*a(nh)-C0*a(n)+C1*a(1)-C2*a(nhp)
    wksp(3:n-1:2) = C2*a(1:nhm)+C1*a(nhp:n-1) &
                    +C0*a(2:nh)+C3*a(nh+2:n)
    wksp(4:n:2) = C3*a(1:nhm)-C0*a(nhp:n-1) &
                  +C1*a(2:nh)-C2*a(nh+2:n)
end if
a(1:n)=wksp(1:n)
END SUBROUTINE daub4

MODULE pwtcom
USE nrtype
INTEGER(I4B), SAVE :: ncof=0,ioff,joff
Thesemodulevariablescommunicatethe
filterto
pwt
REAL(SP), DIMENSION(:), ALLOCATABLE, SAVE :: cc,cr
END MODULE pwtcom

SUBROUTINE pwtset(n)
USE nrtype; USE nrutil, ONLY : nrerror
USE pwtcom
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
Initializing routine for
pwt, here implementing the Daubechies wavelet filters with 4, 12,
and 20 coefficients, as selected by the input value n. Further wavelet filters can be included
in the obvious manner. This routine must be called (once) before the first use of
pwt. (For
the case n=4, the specific routine daub4 is considerably faster than
pwt.)
REAL(SP) :: sig
REAL(SP), PARAMETER ::
  c4(4)=(/0.4829629131445341_sp, 0.8365163037378079_sp, &
         0.2241438680420134_sp, 0.111540743350_sp /), &
  c12(12)=(/0.111840743350_sp, 0.494623890398_sp, 0.751133908021_sp, &
           0.315250351709_sp,-0.226264693965_sp,-0.129766867567_sp, &
           0.097501605587_sp, 0.027522865530_sp,-0.031582039318_sp, &
           0.000553842201_sp, 0.004777257511_sp,-0.001077301085_sp /), &
  c20(20)=(/0.026670057901_sp, 0.188176800078_sp, 0.527201188932_sp, &
            0.689459039484_sp, 0.281172343661_sp,-0.249946424327_sp, &
            0.195946274377_sp, 0.127369340336_sp, 0.093057364604_sp, &
            0.011394147166_sp,-0.029457536822_sp, 0.033212674059_sp, &
            0.033606553567_sp,-0.010733175483_sp, 0.00139531747_sp, &
            0.001992405295_sp,-0.000685886695_sp,-0.000116466855_sp, &
            0.000093588670_sp,-0.000013264203_sp /)
if (allocated(cc)) deallocate(cc)
if (allocated(cr)) deallocate(cr)
allocate(cc(n),cr(n))
ncof=n
ioff=-n/2  These values center the “support” of the wavelets at each
joff=-n/2  level. Alternatively, the “peaks” of the wavelets can
sig=-1.0  be approximately centered by the choices ioff=-2
select case(n)
  case(4)  and joff=-n+2. Note that daub4 and pwtset with
  n=4 use different default centerings.
Here we need to have as global variables arrays whose dimensions are known only at run time. At first sight the situation is the same as with the module fminln in newt on p. 1197. If you review the discussion there and in §21.5, you will recall that there are two good ways to implement this: with allocatable arrays (“Method 1”) or with pointers (“Method 2”). There is a difference here that makes allocatable arrays simpler. We do not wish to deallocate the arrays on exiting pwtset. On the contrary, the values in cc and cr need to be preserved for use in pwt. Since allocatable arrays are born in the well-defined state of “not currently allocated,” we can declare the arrays here as

```fortran
REAL(SP), DIMENSION(:), ALLOCATABLE, SAVE :: cc, cr
```

and test whether they were used on a previous call with

```fortran
if (allocated(cc)) deallocate(cc)
if (allocated(cr)) deallocate(cr)
```

We are then ready to allocate the new storage:

```fortran
allocate(cc(n), cr(n))
```

With pointers, we would need the additional machinery of nullifying the pointers on the initial call, since pointers are born in an undefined state (see §21.5).

There is an additional important point in this example. The module variables need to be used by a “sibling” routine, pwt. We need to be sure that they do not become undefined when we exit pwtset. We could ensure this by putting a USE pwtcom in the main program that calls both pwtset and pwt, but it’s easy to forget to do this. It is preferable to put explicit SAVEs on all the module variables.

```fortran
SUBROUTINE pwt(a, isign)
USE nrtype; USE nrutil, ONLY : arth, nrerror
USE pwtcom
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
INTEGER(4) :: isign

Partial wavelet transform: applies an arbitrary wavelet filter to data vector a (for isign=1) or applies its transpose (for isign=-1). Used hierarchically by routines wt1 and wtn. The actual filter is determined by a preceding (and required) call to pwtset, which initializes the module pwtcom.

REAL(SP), DIMENSION(size(a)) :: vbsp
INTEGER(4) :: jf, jr
INTEGER(4) :: k, n, nh, nmod

n=size(a)
if (n < 4) RETURN
if (ncof == 0) call nrerror('pwt: must call pwtset before pwt')
nmod=ncof*n
A positive constant equal to zero mod n.
nh=n/2
wksp(:)=0.0
jf=land(n-1,arithmetic(2+nmod+ioff,2,nh))  Use bitwise AND to wrap-around the point-
jr=land(n-1,arithmetic(2+nmod+ioff,2,nh))  ers. n-1 is a mask of all bits, since n is
do k=1,ncof
   if (isign >= 0) then
      Apply filter.
      wksp(1:nh)=wksp(1:nh)+cc(k)*a(jf+1)
      wksp(nh+1:n)=wksp(nh+1:n)+cr(k)*a(jr+1)
   else
      Apply transpose filter.
      wksp(jf+1)=wksp(jf+1)+cc(k)*a(1:nh)
      wksp(jr+1)=wksp(jr+1)+cr(k)*a(nh+1:n)
   end if
   if (k == ncof) exit
   jf=land(n-1,jf+1)
   jr=land(n-1,jr+1)
do
   a(:)=wksp(:)
end do

SUBROUTINE pwt

SUBROUTINE wtn(a,nn,isign,wtstep)
USE nrtype; USE nrutil, ONLY : arithmetic, assert
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: nn
INTEGER(I4B), INTENT(IN) :: isign
INTERFACE
SUBROUTINE wtstep(a,isign)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE wtstep
END INTERFACE

Replaces a by its N-dimensional discrete wavelet transform, if isign is input as 1. nn is an
integer array of length N, containing the lengths of each dimension (number of real values),
which must all be powers of 2. a is a real array of length equal to the product of these
lengths, in which the data are stored as in a multidimensional real FORTRAN array. If isign
is input as -1, a is replaced by its inverse wavelet transform. The subroutine wtstep,
whose actual name must be supplied in calling this routine, is the underlying wavelet filter.
Examples of wtstep are daub4 and (preceded by pwtset) pwt.

INTEGER(I4B) :: i1,i2,i3,idim,n,ndim,nnew,nprev,nt,ntot
REAL(SP), DIMENSION(:), ALLOCATABLE :: wksp
call assert(land(nn,nn-1)==0, 'each dimension must be a power of 2 in wtn')
allocate(wksp(maxval(nn)))
nndim=size(nn)
ntot=product(nn(:))
nprev=1
do idim=1,ndim
   n=nn(idim)
nnew=n*nprev
   if (n > 4) then
      do i2=0,ntot-1,nnew
         do i1=1,nprev
            i3=i1+i2
            wksp(1:n)=wksp(1:n)+a(arithmetic(i3,nprev,n))
         end do
      end do
   else
      do i1=1,n
         wksp(1:n)=wksp(1:n)+a(i1)
      end do
   end if
   if (isign >= 0) then
      copy the relevant row or column
      or etc. into workspace.
   else
      do i1=1,n
         wksp(i1)=wksp(i1)+a(n+1:n)
      end do
   end if
end do

* * *
if (nt < 4) exit
   call wtstep(wksp(1:nt),isign)
   nt=nt/2
end do
else
   nt=4
   do
      if (nt > n) exit
      call wtstep(wksp(1:nt),isign)
      nt=nt*2
   end do
end if
i3=i1+i2
a(arth(i3,nprev,n))=wksp(1:n)  Copy back from workspace.
i3=i3+n*nprev
end do
end if
nprev=nnew
deallocate(wksp)
END SUBROUTINE wtn
Chapter B14. Statistical Description of Data

SUBROUTINE moment(data, ave, adev, sdev, var, skew, curt)
USE nrtype; USE nrutil, ONLY : nrerror
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: ave, adev, sdev, var, skew, curt
REAL(SP), DIMENSION(:), INTENT(IN) :: data

Given an array of data, this routine returns its mean ave, average deviation adev, standard deviation sdev, variance var, skewness skew, and kurtosis curt.

INTEGER(I4B) :: n
REAL(SP) :: ep
REAL(SP), DIMENSION(size(data)) :: p, s
n=size(data)
if (n <= 1) call nrerror('moment: n must be at least 2')
ave=sum(data(:))/n  \, First pass to get the mean.
s(:)=data(:)-ave
ep=sum(abs(s(:)))  \, Second pass to get the first (absolute), second, third, and fourth moments of the deviation from the mean.
adev=sum(abs(s(:)))/n
p(:)=s(:)*s(:)
var=sum(p(:))  \, Corrected two-pass formula.
sdev=sqrt(var)
if (var /= 0.0) then
  skew=sum(p(:))/n  \, Student's t as t, and its significance as prob, small values of prob indicating that
  curt=curt/(n*sdev**2)-3.0_sp
else
  call nrerror('moment: no skew or kurtosis when zero variance')
end if
END SUBROUTINE moment

* * *

SUBROUTINE ttest(data1, data2, t, prob)
USE nrtype
USE nr, ONLY : avevar, betai
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: data1, data2
REAL(SP), INTENT(OUT) :: t, prob

Given the arrays data1 and data2, which need not have the same length, this routine returns Student's t as t, and its significance as prob, small values of prob indicating that
the arrays have significantly different means. The data arrays are assumed to be drawn from populations with the same true variance.

```fortran
INTEGER(I4B) :: n1, n2
REAL(SP) :: ave1, ave2, df, var, var1, var2
n1 = size(data1)
n2 = size(data2)
call avevar(data1, ave1, var1)
call avevar(data2, ave2, var2)
df = n1 + n2 - 2
var = ((n1 - 1) * var1 + (n2 - 1) * var2) / df
prob = betai(0.5_sp * df, 0.5_sp, df / (df + t**2))
END SUBROUTINE ttest
```

```
SUBROUTINE avevar(data, ave, var)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: data
REAL(SP), INTENT(OUT) :: ave, var
INTEGER(I4B) :: n
REAL(SP), DIMENSION(size(data)) :: s
n = size(data)
ave = sum(data(:)) / n
s(:) = data(:) - ave
var = dot_product(s, s) / n
var = (var - sum(s)**2 / n) / (n - 1)
END SUBROUTINE avevar
```

```
SUBROUTINE tutest(data1, data2, t, prob)
USE nrtype
USE nr, ONLY : avevar, betai
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: data1, data2
REAL(SP), INTENT(OUT) :: t, prob
INTEGER(I4B) :: n1, n2
REAL(SP) :: ave1, ave2, df, var1, var2
n1 = size(data1)
n2 = size(data2)
call avevar(data1, ave1, var1)
call avevar(data2, ave2, var2)
t = (ave1 - ave2) / sqrt(var1 / n1 + var2 / n2)
df = (var1 / n1 + var2 / n2) / ((var1 / n1)**2 / (n1 - 1) + (var2 / n2)**2 / (n2 - 1))
prob = betai(0.5_sp * df, 0.5_sp, df / (df + t**2))
END SUBROUTINE tutest
```
SUBROUTINE tptest(data1, data2, t, prob)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : avevar, betai
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: data1, data2
REAL(SP), INTENT(OUT) :: t, prob
Given the paired arrays data1 and data2 of the same length, this routine returns Student’s
\( t \) for paired data as \( t \), and its significance as \( prob \), small values of \( prob \) indicating a
significant difference of means.
INTEGER(I4B) :: n
REAL(SP) :: ave1, ave2, cov, df, sd, var1, var2
n = assert_eq(size(data1), size(data2), ’tptest’)
call avevar(data1, ave1, var1)
call avevar(data2, ave2, var2)
cov = dot_product(data1(:) - ave1, data2(:) - ave2)
df = n - 1
sd = sqrt((var1 + var2 - 2.0_sp * cov) / n)
t = (ave1 - ave2) / sd
prob = betai(0.5_sp * df, 0.5_sp, df / (df + t**2))
END SUBROUTINE tptest

SUBROUTINE ftest(data1, data2, f, prob)
USE nrtype
USE nr, ONLY : avevar, betai
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: f, prob
REAL(SP), DIMENSION(:), INTENT(IN) :: data1, data2
Given the arrays data1 and data2, which need not have the same length, this routine
returns the value of \( f \), and its significance as \( prob \). Small values of \( prob \) indicate that
the two arrays have significantly different variances.
INTEGER(I4B) :: n1, n2
REAL(SP) :: ave1, ave2, df1, df2, var1, var2
n1 = size(data1)
n2 = size(data2)
call avevar(data1, ave1, var1)
call avevar(data2, ave2, var2)
if (var1 > var2) then
   Make \( F \) the ratio of the larger variance to the smaller one.
   f = var1 / var2
   df1 = n1 - 1
   df2 = n2 - 1
else
   f = var2 / var1
   df1 = n1 - 1
   df2 = n2 - 1
end if
prob = 2.0_sp * betai(0.5_sp * df2, 0.5_sp * df1, df2 / (df2 + df1 * f))
if (prob > 1.0) prob = 2.0_sp - prob
END SUBROUTINE ftest
SUBROUTINE chsone(bins,ebins,knstrn,df,chsq,prob)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : gammq
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: knstrn
REAL(SP), INTENT(OUT) :: df,chsq,prob
REAL(SP), DIMENSION(:), INTENT(IN) :: bins,ebins

Given the same-size arrays bins containing the observed numbers of events, and ebins containing the expected numbers of events, and given the number of constraints knstrn (normally one), this routine returns (trivially) the number of degrees of freedom df, and (nontrivially) the chi-square chsq and the significance prob. A small value of prob indicates a significant difference between the distributions bins and ebins. Note that bins and ebins are both real arrays, although bins will normally contain integer values.

INTEGER(I4B) :: ndum
ndum=assert_eq(size(bins),size(ebins),'chsone')
if (any(ebins(:) <= 0.0)) call nrerror('bad expected number in chsone')

df=size(bins)-knstrn
chsq=sum((bins(:)-ebins(:))**2/ebins(:))

prob=gammq(0.5_sp*df,0.5_sp*chsq)  
Chi-square probability function. See §6.2.

END SUBROUTINE chsone

SUBROUTINE chstwo(bins1,bins2,knstrn,df,chsq,prob)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : gammq
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: knstrn
REAL(SP), INTENT(OUT) :: df,chsq,prob
REAL(SP), DIMENSION(:), INTENT(IN) :: bins1,bins2

Given the same-size arrays bins1 and bins2, containing two sets of binned data, and given the number of constraints knstrn (normally 1 or 0), this routine returns the number of degrees of freedom df, the chi-square chsq, and the significance prob. A small value of prob indicates a significant difference between the distributions bins1 and bins2. Note that bins1 and bins2 are both real arrays, although they will normally contain integer values.

INTEGER(I4B) :: ndum
LOGICAL(LGT), DIMENSION(size(bins1)) :: nzeromask
ndum=assert_eq(size(bins1),size(bins2),'chstwo')

nzeromask = bins1(:) /= 0.0 .or. bins2(:) /= 0.0

chsq=sum((bins1(:)-bins2(:))**2/(bins1(:)+bins2(:)),mask=nzeromask)

df=count(nzeromask)-knstrn  
No data means one less degree of freedom.
prob=gammq(0.5_sp*df,0.5_sp*chsq)  
Chi-square probability function. See §6.2.

END SUBROUTINE chstwo

We use the optional argument mask in sum to select out the elements to be summed over. In this case, at least one of the elements of bins1 or bins2 is not zero for each term in the sum.
SUBROUTINE ksone(data,func,d,prob)
USE nrtype; USE nrutil, ONLY : arth
USE nr, ONLY : probks,sort
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: d,prob
REAL(SP), DIMENSION(:), INTENT(INOUT) :: data
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
Given an array data, and given a user-supplied function of a single variable func which
is a cumulative distribution function ranging from 0 (for smallest values of its argument)
to 1 (for largest values of its argument), this routine returns the K-S statistic d, and the
significance level prob. Small values of prob show that the cumulative distribution function
of data is significantly different from func. The array data is modified by being sorted
into ascending order.
INTEGER(I4B) :: n
REAL(SP) :: en
REAL(SP), DIMENSION(size(data)) :: fvals
REAL(SP), DIMENSION(size(data)+1) :: temp
call sort(data)
If the data are already sorted into as-
cending order, then this call can be
omitted.
n=size(data)
en=n
fvals(:)=func(data(:))
temp=arth(0,1,n+1)/en
d=maxval(max(abs(temp(1:n)-fvals(:)), &
abs(temp(2:n+1)-fvals(:))))
en=sqrt(en)
prob=probks((en+0.12_sp+0.11_sp/en)*d)
END SUBROUTINE ksone

\( d = \text{maxval}(\text{max...} \text{ Note the difference between \text{max} and \text{maxval}: \text{max}} \text{ takes two or more arguments and returns the maximum. If the arguments}
\text{are two arrays, it returns an array each of whose elements is the maximum}
of the corresponding elements in the two arrays. \text{maxval} \text{ takes a single array}
\text{argument and returns its maximum value.} \\

SUBROUTINE kstwo(data1,data2,d,prob)
USE nrtype; USE nrutil, ONLY : cumsum
USE nr, ONLY : probks,sort2
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: d,prob
REAL(SP), DIMENSION(:), INTENT(IN) :: data1,data2
Given arrays data1 and data2, which can be of different length, this routine returns the
K-S statistic d, and the significance level prob for the null hypothesis that the data sets
are drawn from the same distribution. Small values of prob show that the cumulative
distribution function of data1 is significantly different from that of data2. The arrays
data1 and data2 are not modified.
INTEGER(I4B) :: n1,n2
REAL(SP) :: en1,en2,en
REAL(SP), DIMENSION(size(data1)+size(data2)) :: dat,org
n1=size(data1)
n2=size(data2)
en1=n1
en2=en2
dat(1:n1)=data1
Copy the two data sets into a single ar-
dat(n1+1:)=data2
ray.
Define an array that contains 0 when the corresponding element comes from data1, 1 from data2.

call sort2(dat,org)
Sort the array of 1’s and 0’s into the order of the merged data sets.

d=maxval(abs(cumsum(org)/en2-cumsum(1.0_sp-org)/en1))
Now use cumsum to get the c.d.f. corresponding to each set of data.
en=sqrt(en1*en2/(en1+en2))
prob=probks((en+0.12_sp+0.11_sp/en)*d) Compute significance.

FUNCTION probks(alam)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: alam
REAL(SP) :: probks
REAL(SP), PARAMETER :: EPS1=0.001_sp,EPS2=1.0e-8_sp
INTEGER(I4B), PARAMETER :: NITER=100
Kolmogorov-Smirnov probability function.
INTEGER(I4B) :: j
REAL(SP) :: a2,fac,term,termbf
a2=-2.0_sp*alam**2
fac=2.0
probks=0.0
probks=0.0

do j=1,NITER
    term=fac*exp(a2*j**2)
    probks=probks+term
    if (abs(term) <= EPS1*termbf .or. abs(term) <= EPS2*probks) RETURN
    fac=-fac
    termbf=abs(term)
end do
probks=1.0

END FUNCTION probks

The problem here is how to compute the cumulative distribution function (c.d.f.) corresponding to each set of data, and then find the corresponding KS statistic, without a serial loop over the data. The trick is to define an array that contains 0 when the corresponding element comes from the first data set and 1 when it’s from the second data set. Sort the array of 1’s and 0’s into the same order as the merged data sets. Now tabulate the partial sums of the array. Every time you encounter a 1, the partial sum increases by 1. So if you normalize the partial sums by dividing by the number of elements in the second data set, you have the c.d.f. of the second data set.

If you subtract the array of 1’s and 0’s from an array of all 1’s, you get an array where 1 corresponds to an element in the first data set, 0 the second data set. So tabulating its partial sums and normalizing gives the c.d.f. of the first data set. As we’ve seen before, tabulating partial sums can be done with a parallel algorithm (cumsum in nrutil1). The KS statistic is just the maximum absolute difference of the c.d.f.’s, computed in parallel with Fortran 90’s maxval function.
SUBROUTINE cntab1(nn,chisq,df,prob,cramrv,ccc)
USE nrtype; USE nrutil, ONLY : outerprod
USE nr, ONLY : gammq
IMPLICIT NONE
INTEGER(I4B), DIMENSION(:,,:), INTENT(IN) :: nn
REAL(SP), INTENT(OUT) :: chisq,df,prob,cramrv,ccc
REAL(SP), PARAMETER :: TINY=1.0e-30_sp
Given a two-dimensional contingency table in the form of a rectangular integer array nn,
this routine returns the chi-square chisq, the number of degrees of freedom df, the signi-
ficance level prob (small values indicating a significant association), and two measures of
association, Cramer's V (cramrv), and the contingency coefficient \( C \) (ccc).
INTEGER(I4B) :: nni,nnj
REAL(SP) :: sumn
REAL(SP), DIMENSION(size(nn,1)) :: sumi
REAL(SP), DIMENSION(size(nn,2)) :: sumj
REAL(SP), DIMENSION(size(nn,1),size(nn,2)) :: expctd
sumi(:)=sum(nn(:,:),dim=2) Get the row totals.
sumj(:)=sum(nn(:,:),dim=1) Get the column totals.
sumn=sum(sumi(:)) Get the grand total.
nni=size(sumi)-count(sumi(:) == 0.0)
Eliminate any zero rows by reducing the number of rows.
nnj=size(sumj)-count(sumj(:) == 0.0)
Corrected number of degrees of freedom.
chisq=-(sum(nn(:,:),dim=2)*log(nn(:,:)/sumn)
df=nni*nnj-nni-nnj+1
chi=-(sum(sumi(:)**2)*TINY)
prob=gammq(0.5_sp*df,0.5_sp*chisq)
Chi-square probability function.
cramrv=sqrt(chisq/(sumn*min(nni-1,nnj-1)))
ccc=sqrt(chisq/(chisq+sumn))
END SUBROUTINE cntab1

sumi(:)=sum(...dim=2) sumj(:)=sum(...dim=1) We use the optional arg-
ument dim of sum to sum first over the columns (dim=2) to get the row
totals, and then to sum over the rows (dim=1) to get the column totals.

This is a direct implementation of equation (14.4.2) using
outerprod from nrutil.

SUBROUTINE cntab2(nn,h,hx,hy,hygx,hxgy,uygx,uxgy,uxy)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), DIMENSION(:,,:), INTENT(IN) :: nn
REAL(SP), INTENT(OUT) :: h,hx,hy,hygx,hxgy,uygx,uxgy,uxy
REAL(SP), PARAMETER :: TINY=1.0e-30_sp
Given a two-dimensional contingency table in the form of a rectangular integer array nn,
where the first index labels the \( x \)-variable and the second index labels the \( y \) variable,
this routine returns the entropy \( h \) of the whole table, the entropy \( hx \) of the \( x \)-distribution, the
entropy \( hy \) of the \( y \)-distribution, the entropy \( hygx \) of \( y \) given \( x \), the entropy \( hxgy \) of \( x \)
given \( y \), the dependency \( uygx \) of \( y \) on \( x \) (eq. 14.4.15), the dependency \( uxgy \) of \( x \) on \( y \)
(eq. 14.4.16), and the symmetrical dependency \( uxy \) (eq. 14.4.17).
REAL(SP) :: sum
REAL(SP), DIMENSION(size(nn,1)) :: sumi
REAL(SP), DIMENSION(size(nn,2)) :: sumj
sumi(:)=sum(nn(:,:),dim=2) Get the row totals.
sumj(:)=sum(nn(:,:),dim=1) Get the column totals.
sumn=sum(sumi(:))
hx=sum(sumi(:))*log(sumi(:)/sumn), mask=(sumi(:) /= 0.0) )/sumn
Entropy of the \( x \) distribution.

This is a direct implementation of equation (14.4.3).
and of the y distribution.

\[
h = \frac{\text{sum}(\text{nn}(;:)) \cdot \log(\text{nn}(;:)/\text{sumn})}{\text{sumn}} \quad \text{mask=(nn(;:) /= 0) } / \text{sumn}
\]

Total entropy: loop over both x and y.

\[
h_{yx} = h - h_x \quad \text{Uses equation (14.4.18)}.
\]

\[
h_{xy} = h - h_y \quad \text{as does this.}
\]

\[
u_{yx} = \frac{(h_y-h_{yx})}{(h_y+TINY)} \quad \text{Equation (14.4.15)}.
\]

\[
u_{xy} = \frac{(h_x-h_{xy})}{(h_x+TINY)} \quad \text{Equation (14.4.16)}.
\]

\[
u_{xy} = 2.0\_sp \cdot \frac{(h_x+h_y-h)}{(h_x+h_y+TINY)} \quad \text{Equation (14.4.17)}.
\]

END SUBROUTINE cntab2

This code exploits both the dim feature of sum (see discussion after cntab1) and the mask feature to restrict the elements to be summed over.

⋆⋆⋆

SUBROUTINE pearsn(x,y,r,prob,z)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : betai
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: r,prob,z
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y
REAL(SP), PARAMETER :: TINY=1.0e-20_sp

Given two arrays x and y of the same size, this routine computes their correlation coefficient r (returned as r), the significance level at which the null hypothesis of zero correlation is disproved (prob whose small value indicates a significant correlation), and Fisher’s z (returned as z), whose value can be used in further statistical tests as described above the routine in Volume 1.

Parameter: TINY will regularize the unusual case of complete correlation.

REAL(SP), DIMENSION(size(x)) :: xt,yt
REAL(SP) :: ax,ay,df,sxx,sxy,syy,t
INTEGER(I4B) :: n
n=assert_eq(size(x),size(y),'pearsn')

ax=sum(x)/n
ay=sum(y)/n
xt(:)=x(:)-ax
Compute the correlation coefficient.
yt(:)=y(:)-ay
sxx=dot_product(xt,xt)
syy=dot_product(yt,yt)
sxy=dot_product(xt,yt)
r=sxy/(sqrt(sxx*syy)+TINY)

z=0.5_sp*log(((1.0_sp-r)+TINY)/((1.0_sp-r)+TINY)) \quad \text{Fisher’s z transformation.}

df=n-2

t=r*sqrt(df/((1.0_sp-r)+TINY)*((1.0_sp-r)+TINY)) \quad \text{Equation (14.5.5).}

prob=betai(0.5_sp*df,0.5_sp,df/(df+df**2)) \quad \text{Student’s t probability.}

\!

For large n, this easier computation of prob, using the short routine erfcc, would give approximately the same value.

END SUBROUTINE pearsn

⋆⋆⋆
SUBROUTINE spear(data1, data2, d, zd, probd, rs, probrs)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : betai, erfcc, sort2
IMPLICIT NONE
REAL(SP), DIMENSION(::), INTENT(IN) :: data1, data2
REAL(SP), INTENT(OUT) :: d, zd, probd, rs, probrs

Given two data arrays of the same size, data1 and data2, this routine returns their sum-squared difference of ranks as $D$, the number of standard deviations by which $D$ deviates from its null-hypothesis expected value as $zd$, the two-sided significance level of this deviation as probd, Spearman’s rank correlation $r_s$ as $rs$, and the two-sided significance level of its deviation from zero as probrs. data1 and data2 are not modified. A small value of either probd or probrs indicates a significant correlation ($rs$ positive) or anticorrelation ($rs$ negative).

INTEGER(14B) :: n
REAL(SP) :: aved, df, en, en3n, fac, sf, sg, t, vard
REAL(SP), DIMENSION(size(data1)) :: wksp1, wksp2
n = assert_eq(size(data1), size(data2), 'spear')
wksp1(:) = data1(:)
wksp2(:) = data2(:)
call sort2(wksp1, wksp2)
call sort2(wksp2, wksp1)
call crank(wksp1, wksp2)
call crank(wksp2, wksp1)
wksp1(:) = wksp1(:) - wksp2(:)

d = dot_product(wksp1, wksp1)

d Sum the squared difference of ranks.

en = n
en3n = en ** 3 - en
aved = en3n / 6.0_sp - (sf + sg) / 12.0_sp
fac = (1.0_sp - sf / en3n) * (1.0_sp - sg / en3n)
zd = (d - aved) / sqrt(vard)
probd = erfcc(abs(zd) / SQRT2)

rs = (1.0_sp - (6.0_sp / en3n) * (d + (sf + sg) / 12.0_sp) / sqrt(fac))
fac = (1.0_sp + rs) * (1.0_sp - rs)
if (fac > 0.0) then
  t = rs * sqrt((en - 2.0_sp) / fac)
df = en - 2.0_sp
  probrs = betai(0.5_sp * df, 0.5_sp * df / (df + t ** 2))
give its significance.
else
  probrs = 0.0
end if

SUBROUTINE crank(w, s)
USE nrtype; USE nrutil, ONLY : arth, array_copy
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: s
REAL(SP), DIMENSION(::), INTENT(INOUT) :: w

Given a sorted array w, replaces the elements by their rank, including midranking of ties, and returns as s the sum of $f^3 - f$, where $f$ is the number of elements in each tie.

INTEGER(14B) :: i, n, ndum, nties
INTEGER(14B), DIMENSION(size(w)) :: tstart, tend, tie, idx
n = size(w)
idx(:) = arth(1, n)
tie(:) = merge(1, 0, w == eoshift(w, -1))
Look for ties: Compare each element to the one before. If it’s equal, it’s part of a tie, and we put 1 into tie. Otherwise we put 0.
tie(1) = 0
w(:) = idx(:)
Boundary: the first element must be zero.
if (all(tie == 0)) then
  No ties—we’re done.
s = 0.0
else
  RETURN
end if
call array_copy(pack(idx(:), tie(:) < eoshift(tie(:), 1)), tstart, nties, ndum)
Look for $0 \rightarrow 1$ transitions in tie, which mean that the 0 element is the start of a tie run. Store index of each transition in tstart. nties is the number of ties found.

tend(1:nties)=pack(idx(:,),tie(:,)>eoshift(tie(:,),1))

Look for $1 \rightarrow 0$ transitions in tie, which mean that the 1 element is the end of a tie run.
do i=1,nties
w(tstart(i):tend(i))=(tstart(i)+tend(i))/2.0_sp
end do

tend(1:nties)=tend(1:nties)-tstart(1:nties)+1

Now calculate $s$.
s=sum(tend(1:nties)**3-tend(1:nties))

END SUBROUTINE crank
END SUBROUTINE spear

To understand how the parallel version of crank works, let's consider an example of 9 elements in the array $w$, which is input in sorted order to crank. The elements in our example are given in the second line of the following table:

<table>
<thead>
<tr>
<th>index</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>data in w</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>shift right</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>compare</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>tie array</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>shift left</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$0 \rightarrow 1$</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>start index</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1 \rightarrow 0$</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>stop index</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We look for ties by comparing this array with itself, right shifted by one element ("shift right" in table). We record a 1 for each element that is the same, a 0 for each element that is different ("compare"). A 1 indicates the element is part of a tie with the preceding element, so we always set the first element to 0, even if it was a 1 as in our example. This gives the "tie array." Now wherever the tie array makes a transition $0 \rightarrow 1$ indicates the start of a tie run, while a $1 \rightarrow 0$ transition indicates the end of a tie run. We find these transitions by comparing the tie array to itself left shifted by one ("shift left"). If the tie array element is smaller than the shifted array element, we have a $0 \rightarrow 1$ transition and we record the corresponding index as the start of a tie. Similarly if the tie array element is larger we record the index as the end of a tie. Note that the shifts must be end-off shifts with zeros inserted in the gaps for the boundary conditions to work.

call array_copy(pack(idx(:,),tie(:,)>eoshift(tie(:,),1)),
tstart,nties,ndum)

The start indices (1, 3, and 8 in our example above) are here packed into the first few elements of tstart. array_copy is a useful routine in nrutil for copying elements from one array to another, when the number of elements to be copied is not known in advance. This line of code is equivalent to

tstart(:)=0

nties=count(tstart(:) > 0)

The point is that we don’t know how many elements pack is going to select. We have to make sure the dimensions of both sides of the pack statement are the same,
so we set the optional third argument of pack to \texttt{tstart}. We then make a separate pass through \texttt{tstart} to count how many elements we copied. Alternatively, we could have used an additional logical array \texttt{mask} and coded this as

\begin{verbatim}
  mask(:)=tie(:) < eoshift(tie(:),1)
  nties=count(mask)
  tstart(1:nties)=pack(idx(:),mask)
\end{verbatim}

But we still need two passes through the \texttt{mask} array. The beauty of the \texttt{array\_copy} routine is that \texttt{nties} is determined from the \texttt{size} of the first argument, without the necessity for a second pass through the array.

\begin{verbatim}
  *   *   *
\end{verbatim}

\begin{verbatim}
SUBROUTINE kend1(data1,data2,tau,z,prob)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : erfcc
IMPLICIT NONE
REAL(SP), INTENT(OUT) :: tau,z,prob
REAL(SP), DIMENSION(:,:), INTENT(IN) :: data1, data2

Given same-size data arrays \texttt{data1} and \texttt{data2}, this program returns Kendall's $\tau$ as \texttt{tau}, its number of standard deviations from zero as \texttt{z}, and its two-sided significance level as \texttt{prob}. Small values of \texttt{prob} indicate a significant correlation (\texttt{tau} positive) or anticorrelation (\texttt{tau} negative).

INTEGER(I4B) :: is,j,n,n1,n2
REAL(SP) :: var
REAL(SP), DIMENSION(size(data1)) :: a1,a2

\texttt{n=assert\_eq(size(data1),size(data2),'kend1')} \texttt{This will be the argument of one square root in (14.6.8), and this the other.}
\texttt{n1=0} \texttt{This will be the numerator in (14.6.8).}
\texttt{n2=0}
\texttt{is=0} \texttt{For each first member of pair,}
\texttt{do j=1,n-1} \texttt{loop over second member.}
\texttt{   a1(j+1:n)=data1(j)-data1(j+1:n)}
\texttt{   a2(j+1:n)=data2(j)-data2(j+1:n)}
\texttt{   n1=n1+count(a1(j+1:n) /= 0.0)}
\texttt{   n2=n2+count(a2(j+1:n) /= 0.0)}
\texttt{   is=is+count((a1(j+1:n) > 0.0 .and. a2(j+1:n) > 0.0) .or. (a1(j+1:n) < 0.0 .and. a2(j+1:n) < 0.0)) - &
   count((a1(j+1:n) > 0.0 .and. a2(j+1:n) < 0.0) .and. a2(j+1:n) < 0.0) .and. a2(j+1:n) > 0.0))} \texttt{Now accumulate the numerator in (14.6.8):}
\texttt{   end do}
\texttt{tau=real(is,sp)/sqrt(real(n1,sp)*real(n2,sp))} \texttt{Equation (14.6.8).}
\texttt{var=(4.0_sp*n+10.0_sp)/(9.0_sp*n*(n-1.0_sp))} \texttt{Equation (14.6.9).}
\texttt{z=tau/sqrt(var)}
\texttt{prob=erfcc(abs(z)/SQRT2)} \texttt{Significance.}

END SUBROUTINE kend1
\end{verbatim}

\begin{verbatim}
SUBROUTINE kend2(tab,tau,z,prob)
USE nrtype; USE nrutil, ONLY : cumsum
USE nr, ONLY : erfcc
IMPLICIT NONE
REAL(SP), DIMENSION(:,:), INTENT(IN) :: tab
REAL(SP), INTENT(OUT) :: tau,z,prob

Given a two-dimensional table \texttt{tab} such that \texttt{tab(k,l)} contains the number of events falling in bin \texttt{k} of one variable and bin \texttt{l} of another, this program returns Kendall's $\tau$ as \texttt{tau}, its number of standard deviations from zero as \texttt{z}, and its two-sided significance level as \texttt{prob}. Small values of \texttt{prob} indicate a significant correlation (\texttt{tau} positive) or anticorrelation (\texttt{tau} negative).

INTEGER(I4B) :: is,j,k,n1,n2
REAL(SP) :: var
REAL(SP) :: a1,a2,
REAL(SP), DIMENSION(size(data1)) :: a1,a2
n=assert\_eq(size(data1),size(data2),'kend2')

\texttt{n1=0} \texttt{This will be the argument of one square root in (14.6.8),}
\texttt{n2=0} \texttt{and this the other.}
\texttt{is=0} \texttt{This will be the numerator in (14.6.8).}
\texttt{do j=1,n-1} \texttt{For each first member of pair,}
\texttt{   a1(j+1:n)=data1(j)-data1(j+1:n)}
\texttt{   a2(j+1:n)=data2(j)-data2(j+1:n)}
\texttt{   n1=n1+count(a1(j+1:n) /= 0.0)}
\texttt{   n2=n2+count(a2(j+1:n) /= 0.0)}
\texttt{   is=is+count((a1(j+1:n) > 0.0 .and. a2(j+1:n) > 0.0) .or. (a1(j+1:n) < 0.0 .and. a2(j+1:n) < 0.0)) - &
   count((a1(j+1:n) > 0.0 .and. a2(j+1:n) < 0.0) .and. a2(j+1:n) < 0.0) .and. a2(j+1:n) > 0.0))} \texttt{Now accumulate the numerator in (14.6.8):}
\texttt{   end do}
\texttt{tau=real(is,sp)/sqrt(real(n1,sp)*real(n2,sp))} \texttt{Equation (14.6.8).}
\texttt{var=(4.0_sp*n+10.0_sp)/(9.0_sp*n*(n-1.0_sp))} \texttt{Equation (14.6.9).}
\texttt{z=tau/sqrt(var)}
\texttt{prob=erfcc(abs(z)/SQRT2)} \texttt{Significance.}

END SUBROUTINE kend2
\end{verbatim}
negative) between the two variables. Although \texttt{tab} is a real array, it will normally contain integral values.

```fortran
REAL(SP), DIMENSION(size(tab,1),size(tab,2)) :: cum,cumt
INTEGER(I4B) :: i,j,ii,jj
REAL(SP) :: sc,sd,en1,en2,points,var
ii=size(tab,1)
jj=size(tab,2)
do i=1,ii
   cumt(i,jj:1:-1)=cumsum(tab(i,jj:1:-1))
   do j=1,jj
      cum(ii:1:-1,j)=cumsum(cumt(ii:1:-1,j))
   end do
   points=cum(1,1)
do j=1,jj
   cumt(ii:1:-1,j)=cumsum(tab(ii:1:-1,j))
   end do
   en1=sum(tab(1:ii-1,1:jj)*cumt(2:ii,1:jj))
   do j=1,jj
      cumt(ii:1:-1,j)=cumsum(tab(ii:1:-1,j))
   end do
   tau=(sc-sd)/sqrt((en1+sc+sd)*(en2+sc+sd))
end do
var=(4.0_sp*points+10.0_sp)/(9.0_sp*points*(points-1.0_sp))
z=tau/sqrt(var)
prob=erfcc(abs(z)/SQRT2)
END SUBROUTINE kendl2
```

The underlying algorithm in \texttt{kendl2} might seem to require looping over all pairs of cells in the two-dimensional table \texttt{tab}. Actually, however, clever use of the \texttt{cumsum} utility function reduces this to a simple loop over all the cells; moreover this “loop” parallelizes into a simple parallel product and call to the \texttt{sum} intrinsic. The basic idea is shown in the following table:

```
<table>
<thead>
<tr>
<th></th>
<th>d</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>y</td>
<td>y</td>
</tr>
<tr>
<td></td>
<td>x</td>
<td>c</td>
</tr>
<tr>
<td></td>
<td>x</td>
<td>c</td>
</tr>
<tr>
<td></td>
<td>x</td>
<td>c</td>
</tr>
</tbody>
</table>
```

Relative to the cell marked \textit{t} (which we use to denote the numerical value it contains), the cells marked \textit{d} contribute to the “discordant” tally in Volume 1’s equation (14.6.8),
while the cells marked \( c \) contribute to the “concordant” tally. Likewise, the cells marked \( x \) and \( y \) contribute, respectively, to the “extra-\( x \)” and “extra-\( y \)” tallies. What about the cells left blank? Since we want to count pairs of cells only once, without duplication, these cells will be counted, relative to the location shown as \( t \), when \( t \) itself moves into the blank-cell area.

Symbolically we have

\[
\begin{align*}
\text{concordant} & = \sum_n t_n \left( \sum_{\text{lower right}} c_m \right) \\
\text{discordant} & = \sum_n t_n \left( \sum_{\text{upper right}} d_m \right) \\
\text{extra-}x & = \sum_n t_n \left( \sum_{\text{below}} x_m \right) \\
\text{extra-}y & = \sum_n t_n \left( \sum_{\text{to the right}} y_m \right)
\end{align*}
\]

(B14.1)

Here \( n \) varies over all the positions in the table, while the limits of the inner sums are relative to the position of \( n \). (The letters \( t_n, c_m, d_m, x_m, y_m \) all represent the value in a cell; we use different letters only to make the relation with the above table clear.) Now the final trick is to recognize that the inner sums, over cells to the lower-or upper-right, below, and to the right can be done in parallel by cumulative sums (\texttt{cumsum}) sweeping to the right and up. The routine does these in a nonintuitive order merely to be able to reuse maximally the scratch spaces \texttt{cum} and \texttt{cumt}.

Subroutine \texttt{ks2d1s(x1,y1,quadvl,d1,prob)}

This subroutine performs a two-dimensional Kolmogorov-Smirnov test of one sample against a model. Given the \( x \)- and \( y \)-coordinates of a set of data points in arrays \( x1 \) and \( y1 \) of the same length, and given a user-supplied function \texttt{quadvl} that exemplifies the model, this routine returns the two-dimensional K-S statistic as \( d1 \), and its significance level as \( prob \). Small values of \( prob \) show that the sample is significantly different from the model. Note that the test is slightly distribution-dependent, so \( prob \) is only an estimate.

```fortran
SUBROUTINE ks2d1s(x1,y1,quadvl,d1,prob)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : pearsn,probks,quadct
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x1,y1
REAL(SP), INTENT(OUT) :: d1,prob
INTERFACE
  SUBROUTINE quadvl(x,y,fa,fb,fc,fd)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), INTENT(IN) :: x,y
    REAL(SP), INTENT(OUT) :: fa,fb,fc,fd
  END SUBROUTINE quadvl
END INTERFACE

Two-dimensional Kolmogorov-Smirnov test of one sample against a model. Given the \( x \)- and \( y \)-coordinates of a set of data points in arrays \( x1 \) and \( y1 \) of the same length, and given a user-supplied function \texttt{quadvl} that exemplifies the model, this routine returns the two-dimensional K-S statistic as \( d1 \), and its significance level as \( prob \). Small values of \( prob \) show that the sample is significantly different from the model. Note that the test is slightly distribution-dependent, so \( prob \) is only an estimate.

INTEGER(1) :: j,n1
REAL(SP) :: dum,dumm,fa,fb,fc,fd,ga,gb,gc,gd,r1,rr,sqen
n1=assert_eq(size(x1),size(y1),'ks2d1s')
d1=0.0
```
do j=1,n1
  Loop over the data points.
  call quadct(x1(j),y1(j),x1,y1,fa,fb,fc,fd)
  call quadvl(x1(j),y1(j),fa,fb,fc,fd)
  d1=max(di,abs(fa-ga),abs(fb-gb),abs(fc-gc),abs(fd-gd))
  For both the sample and the model, the distribution is integrated in each of four quadrants, and the maximum difference is saved.
end do

call pearson(x1,y1,r1,dum,dumm)  Get the linear correlation coefficient r1.
sqen=sqrt(real(n1,sp))
rr=sqrt(1.0_sp-r1**2)
Estimate the probability using the K-S probability function probks.
prob=probks(d1*sqen/(1.0_sp+rr*(0.25_sp-0.75_sp/sqen)))
END SUBROUTINE ks2d1s

SUBROUTINE quadct(x,y,xx,yy,fa,fb,fc,fd)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,y
REAL(SP), DIMENSION(:), INTENT(IN) :: xx,yy
REAL(SP), INTENT(OUT) :: fa,fb,fc,fd
Given an origin (x,y), and an array of points with coordinates xx and yy, count how many of them are in each quadrant around the origin, and return the normalized fractions. Quadrants are labeled alphabetically, counterclockwise from the upper right. Used by ks2d1s and ks2d2s.
INTEGER(I4B) :: na,nb,nc,nd,nn
REAL(SP) :: ff
nn=assert_eq(size(xx),size(yy),'quadct')
na=count(yy(:) > y .and. xx(:) > x)
b=COUNT(yy(:) > y .and. xx(:) <= x)
c=count(yy(:) <= y .and. xx(:) <= x)
d=nn-na-nb-nc
ff=1.0_sp/nn
fa=ff*na
fb=ff*nb
fc=ff*nc
fd=ff*nd
END SUBROUTINE quadct

SUBROUTINE quadvl(x,y,fa,fb,fc,fd)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x,y
REAL(SP), INTENT(OUT) :: fa,fb,fc,fd
This is a sample of a user-supplied routine to be used with ks2d1s. In this case, the model distribution is uniform inside the square \(-1 < x < 1, -1 < y < 1\). In general this routine should return, for any point (x,y), the fraction of the total distribution in each of the four quadrants around that point. The fractions, fa, fb, fc, and fd, must add up to 1. Quadrants are alphabetical, counterclockwise from the upper right.
REAL(SP) :: qa,qb,qc,qd
qa=min(2.0_sp,max(0.0_sp,1.0_sp-x))
qb=min(2.0_sp,max(0.0_sp,1.0_sp-y))
qc=min(2.0_sp,max(0.0_sp,x+1.0_sp))
qd=min(2.0_sp,max(0.0_sp,y+1.0_sp))
fa=0.25_sp*qa*qb
fb=0.25_sp*qb*qc
fc=0.25_sp*qc*qd
fd=0.25_sp*qd*qa
END SUBROUTINE quadvl
SUBROUTINE ks2d2s(x1,y1,x2,y2,d,prob)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : pearsn, probks, quadct
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x1,y1,x2,y2
REAL(SP), INTENT(OUT) :: d,prob
Compute two-dimensional Kolmogorov-Smirnov test on two samples. Input are the x- and y-coordinates of the first sample in arrays x1 and y1 of the same length, and of the second sample in arrays x2 and y2 of the same length (possibly different from the length of the first sample). The routine returns the two-dimensional, two-sample K-S statistic as d, and its significance level as prob. Small values of prob show that the two samples are significantly different. Note that the test is slightly distribution-dependent, so prob is only an estimate.

INTEGER(14B) :: j,n1,n2
REAL(SP) :: d1,d2,dum,dumm,fa,fb,fc,fd,ga,gb,gc,gd,r1,r2,rr,sqen
n1=assert_eq(size(x1),size(y1),'ks2d2s: n1')
n2=assert_eq(size(x2),size(y2),'ks2d2s: n2')
d1=0.0
do j=1,n1
   First, use points in the first sample as origins.
   call quadct(x1(j),y1(j),x1,y1,fa,fb,fc,fd)
   call quadct(x1(j),y1(j),x2,y2,ga,gb,gc,gd)
   d1=max(d1,abs(fa-ga),abs(fb-gb),abs(fc-gc),abs(fd-gd))
end do
d2=0.0
do j=1,n2
   Then, use points in the second sample as origins.
   call quadct(x2(j),y2(j),x1,y1,fa,fb,fc,fd)
   call quadct(x2(j),y2(j),x2,y2,ga,gb,gc,gd)
   d2=max(d2,abs(fa-ga),abs(fb-gb),abs(fc-gc),abs(fd-gd))
end do
d=0.5_sp*(d1+d2)  Average the K-S statistics.
sqen=sqrt(real(n1,sp)*real(n2,sp)/real(n1+n2,sp))
call pearsn(x1,y1,r1,dum,dumm)  Get the linear correlation coefficient for each sample.
call pearsn(x2,y2,r2,dum,dumm)
rr=sqrt(1.0_sp-0.5_sp*(r1**2+r2**2))  Estimate the probability using the K-S probability function probks.
prob=probks(d*sqen/(1.0_sp+rr*(0.25_sp-0.75_sp/sqen)))
END SUBROUTINE ks2d2s

FUNCTION savgol(nl,nrr,ld,m)
USE nrtype; USE nrutil, ONLY : arth, assert, poly
USE nr, ONLY : lubksb, ludcmp
IMPLICIT NONE
INTEGER(14B), INTENT(IN) :: nl,nrr,ld,m
Returns in array c, in wrap-around order (N.B.!) consistent with the argument respns in routine convlv, a set of Savitzky-Golay filter coefficients. nl is the number of leftward (past) data points used, while nrr is the number of rightward (future) data points, making the total number of data points used nl+nrr+1. ld is the order of the derivative desired (e.g., ld = 0 for smoothed function). m is the order of the smoothing polynomial, also equal to the highest conserved moment; usual value is m = 2 or m = 4.
REAL(SP), DIMENSION(nl+nrr+1) :: savgol
INTEGER(14B) :: imj, ipj, mm, np
INTEGER(14B), DIMENSION(m+1) :: index
REAL(SP) :: d, sm
REAL(SP), DIMENSION(m+1) :: b
REAL(SP), DIMENSION(m+1,m+1) :: a
INTEGER(14B) :: irng(nl+nrr+1)
call assert(nl >= 0, nrr >= 0, ld <= m, nl+nrr >= m, 'savgol args')
do ipj=0,2*m
   Set up the normal equations of the desired least squares fit.
sm=sm+sum(arth(1.0_sp,1.0_sp,nr,*ipj))&
   squares fit.
sum(arth(-1.0_sp,-1.0_sp,nl,*ipj))
if (ipj == 0) sm=sm+1.0_sp
mm=min(ipj,2*m-ipj)
do imj=-mm,mm,2
   a(1+(ipj+imj)/2,1+(ipj-imj)/2)=sm
end do
call ludcmp(a(:,,:),indx(:,),d) Solve them: LU decomposition.
  b(:)=0.0 Right-hand-side vector is unit vector, depending on which derivative we want.
call lubksb(a(:,,:),indx(:,),b(:))
savgol(:)=0.0 Zero the output array (it may be bigger than number of coefficients).
irng(:)=arth(-nl,1,nrr+nl+1)
np=nl+nrr+1
savgol(mod(np-irng(:),np)+1)=poly(real(irng(:),sp),b(:)) Each Savitzky-Golay coefficient is the value of the polynomial in (14.8.6) at the corresponding integer. The polynomial coefficients are a row of the inverse matrix. The mod function takes care of the wrap-around order.
END FUNCTION savgol

do imj=-mm,mm,2 Here is an example of a loop that cannot be parallelized in the framework of Fortran 90: We need to access “skew” sections of the matrix $a$.
savgol...=poly(real(irng(:,sp),b(:))) The poly function in nrutil returns the value of a polynomial, here the one in equation (14.8.6). We need the explicit kind type parameter sp in the real function, otherwise it would return type default real for the integer argument and would not automatically convert to double precision if desired.
Chapter B15. Modeling of Data

SUBROUTINE fit(x,y,a,b,siga,sigb,chi2,q,sig)
USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : gammq
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y
REAL(SP), INTENT(OUT) :: a,b,siga,sigb,chi2,q
REAL(SP), DIMENSION(:), OPTIONAL, INTENT(IN) :: sig

Given a set of data points in same-size arrays x and y, fit them to a straight line $y = a + bx$
by minimizing $\chi^2$. sig is an optional array of the same length containing the individual
standard deviations. If it is present, then a, b are returned with their respective probable
uncertainties siga and sigb, the chi-square chi2, and the goodness-of-fit probability q
(that the fit would have $\chi^2$ this large or larger). If sig is not present, then q is returned
as 1.0 and the normalization of chi2 is to unit standard deviation on all points.

INTEGER(I4B) :: ndata
REAL(SP) :: sigdat,ss,sx,sxoss,sy,st2
REAL(SP), DIMENSION(size(x)), TARGET :: t
REAL(SP), DIMENSION(:), POINTER :: wt

if (present(sig)) then
  ndata=assert_eq(size(x),size(y),size(sig),'fit')
  wt=>t
  Use temporary variable t to store weights.
  wt(:)=1.0_sp/(sig(:)**2)
  Accumulate sums with weights.
  ss=sum(wt(:))
  sx=dot_product(wt,x)
  sy=dot_product(wt,y)
else
  ndata=assert_eq(size(x),size(y),'fit')
  ss=real(size(x),sp)
  Accumulate sums without weights.
  sx=sum(x)
  sy=sum(y)
end if

sxoss=sx/ss
if (present(sig)) then
  t(:)=x(:)-sxoss
  b=dot_product(t/sig,y)
  Solve for a, b, $\sigma_a$, and $\sigma_b$.
else
  b=dot_product(t,y)
end if

st2=dot_product(t,t)

a=(sy-sx*b)/ss
siga=sqrt((1.0_sp+sx*sx/(ss*st2))/ss)
sigb=sqrt(1.0_sp/st2)
t(:)=y(:)-a-b*x(:)

q=1.0
if (present(sig)) then
  t(:)=t(:)/sig(:)
  chi2=dot_product(t,t)
else
  chi2=dot_product(t,t)

if (ndata > 2) q=gammq(0.5_sp*(size(x)-2),0.5_sp*chi2)
Equation (15.2.12).
else
Chapter B15. Modeling of Data

For unweighted data evaluate typical
\[ \text{sig}_{\text{dat}} = \sqrt{\chi^2/(\text{size}(x)-2)} \]
\[ \text{sig}_a = \text{sig}_a \cdot \text{sig}_{\text{dat}} \]
\[ \text{sig}_b = \text{sig}_b \cdot \text{sig}_{\text{dat}} \]
end if

**SUBROUTINE fit**

REAL(SP), DIMENSION(:,), POINTER :: wt...wt=t When standard deviations are supplied in \( \text{sig} \), we need to compute the weights for the least squares fit in a temporary array \( \text{wt} \). Later in the routine, we need another temporary array, which we call \( t \) to correspond to the variable in equation (15.2.15). It would be confusing to use the same name for both arrays. In Fortran 77 the arrays could share storage with an EQUIVALENCE declaration, but that is a deprecated feature in Fortran 90. We accomplish the same thing by making \( \text{wt} \) a pointer alias to \( t \).

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SUBROUTINE fitexy(x,y,sigx,sigy,a,b,siga,sigb,chi2,q)
USE nrtype; USE nrutil, ONLY : assert_eq,swap
USE nr, ONLY : avevar,brent,fit,gammq,mnbrak,zbrent
USE chixyfit
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y,sigx,sigy
REAL(SP), INTENT(OUT) :: a,b,siga,sigb,chi2,q
REAL(SP), PARAMETER :: POTN=1.571000_sp,BIG=1.0e30_sp,ACC=1.0e-3_sp
Straight-line fit to input data \( x \) and \( y \) with errors in both \( x \) and \( y \), the respective standard deviations being the input quantities \( \text{sigx} \) and \( \text{sigy} \). \( x \), \( y \), \( \text{sigx} \), and \( \text{sigy} \) are all arrays of the same length. Output quantities are \( a \) and \( b \) such that \( y = a + bx \) minimizes \( \chi^2 \), whose value is returned as \( \text{chi2} \). The \( \chi^2 \) probability is returned as \( q \), a small value indicating a poor fit (sometimes indicating underestimated errors). Standard errors on \( a \) and \( b \) are returned as \( \text{siga} \) and \( \text{sigb} \). These are not meaningful if either (i) the fit is poor, or (ii) \( b \) is so large that the data are consistent with a vertical (infinite \( b \)) line. If \( \text{siga} \) and \( \text{sigb} \) are returned as \( \text{BIG} \), then the data are consistent with all values of \( b \).

INTEGER(I4B) :: j,n
REAL(SP), DIMENSION(size(x)), TARGET :: xx,yy,sx,sy,ww
REAL(SP), DIMENSION(6) :: ang,ch
REAL(SP) :: amx,amn,varx,vary,scale,bmn,bmx,d1,d2,r2,&
            dum1,dum2,dum3,dum4,dum5
n=assert_eq(size(x),size(y),size(sigx),size(sigy),'fitexy')
xxp=>xx  \text{Set up communication with function chixy through global variables in the module chixyfit.}
yyp=>yy
sxp=>sx
sy=>sy
wwp=>ww
call avevar(x,dum1,varx)
call avevar(y,dum1,vary)
scale=sqrt(varx/vary)
xx(:)=x(:)
yy(:)=y(:)*scale
sx(:)=sigx(:)
sy(:)=sigy(:)*scale
ww(:)=sqrt(sx(:)**2+sy(:)**2) \text{Use both } x \text{ and } y \text{ weights in first trial fit.}
call fit(xx,yy,dum1,b,dum2,ww)
offs=0.0
ang(1)=0.0  \text{Construct several angles for reference points.}
ang(2)=atan(b)
ang(4)=0.0
ang(5)=ang(2)
ang(6)=POTN
do j=4,6
    ch(j)=chixy(ang(j))
end do

call mnbrak(ang(1),ang(2),ang(3),ch(1),ch(2),ch(3),chixy)

Bracket the $\chi^2$ minimum and then locate it with brent.

chi2=brent(ang(1),ang(2),ang(3),chixy,ACC,b)

chi2=chixy(b)
a=aa

=gamma(0.5_sp*(n-2),0.5_sp*chi2) Compute $\chi^2$ probability.
r2=1.0_sp/sum(ww(:)) Save inverse sum of weights at the minimum.

bmx=big Now, find standard errors for $b$ as points where
bmn=big $\Delta\chi^2 = 1$.

offs=ch2+1.0_sp Go through saved values to bracket the desired roots. Note periodicity in slope angles.

do j=1,6

if (ch(j) > offs) then

d1=mod(abs(ang(j))=b),PI)
d2=PI-d1
if (ang(j) < b) call swap(d1,d2)
if (d1 < bmx) bmx=d1
if (d2 < bmn) bmn=d2
end if

end do

if (bmx < big) then Call zbrent to find the roots.

bmx=zbrent(chixy,b,b+bmx,ACC)-b
amx=aa-a
bmn=zbrent(chixy,b,b-bmn,ACC)-b
amm=aa-a
sigb=sqrt(0.5_sp*(bmx**2+bmn**2))/(scale*cos(b)**2)
siga=sqrt(0.5_sp*(amx**2+amm**2)+r2)/scale Error in a has additional piece

else

sigb=big
siga=big
end if

a=a/scale Unsacle the answers.
b=tan(b)/scale

END SUBROUTINE fitexy

USE chixyfit We need to pass arrays and other variables to chixy, but
not as arguments. See §21.5 and the discussion of fminln on p. 1197
for two good ways to do this. The pointer construction here is analogous
to the one used in fminln.

MODULE chixyfit
USE nrtype; USE nrutil, ONLY : nrerror
REAL(sp) , dimension(:), pointer :: xxp,yyp,sxp,syp,wwp
REAL(sp) :: aa,offs
CONTAINS
FUNCTION chixy(bang)
IMPLICIT NONE
REAL(sp), intent(in) :: bang
REAL(sp) :: chixy
REAL(sp) , parameter :: big=1.0e30_sp
Captive function of fitexy, returns the value of ($\chi^2 - offs$) for the slope $b=tan(bang)$.
Scaled data and offs are communicated via the module chixyfit.
REAL(sp) :: avex,avey,sumw,b
if (.not. associated(wwp)) call nrerror("chixy: bad pointers")
b=tan(bang)
wwp(:)=(b*sxp(:))**2+syp(:)**2
where (wwp(:) < 1.0/big)
wwp(:)=big
elsewhere
wwp(:)=1.0_sp/wwp(:)
end where
sumw=sum(wp)
ax=dot_product(wp,xp)/sumw
ay=dot_product(wp,yp)/sumw
aa=ay-b*ax
chixy=sum(wp(:)*(yp(:)-aa-b*xp(:)**2)-offs)
END FUNCTION chixy
END MODULE chixyfit

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SUBROUTINE lfit(x,y,sig,a,maska,covar,chisq,funcs)
USE nrtype; USE nrutil, ONLY : assert_eq, diagmult, nrerror
USE nr, ONLY : covsrt, gaussj
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: covar
REAL(SP), INTENT(OUT) :: chisq
INTERFACE
SUBROUTINE funcs(x,arr)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(OUT) :: arr
END SUBROUTINE funcs
END INTERFACE

Given a set of \(N\) data points \(x, y\) with individual standard deviations \(\sigma\), all arrays of length \(N\), use \(\chi^2\) minimization to fit for some or all of the \(M\) coefficients \(a_i\) of a function that depends linearly on \(a_i\), \(y = \sum_{i=1}^{M} a_i \times \text{afunc}(x)\). The input logical array \(\text{maska}\) of length \(M\) indicates by true entries those components of \(a\) that should be fitted for, and by false entries those components that should be held fixed at their input values. The program returns values for \(a\), \(\chi^2 = \text{chisq}\), and the \(M \times M\) covariance matrix \(\text{covar}\). (Parameters held fixed will return zero covariances.) The user supplies a subroutine \(\text{funcs}(x,\text{afunc})\) that returns the \(M\) basis functions evaluated at \(x = x_i\) in the array \(\text{afunc}\).

```
INTEGER(I4B) :: i,j,k,l,ma,mfit,n
REAL(SP) :: sig2i,wt,ym
REAL(SP), DIMENSION(size(maska)) :: afunc
REAL(SP), DIMENSION(size(maska),1) :: beta
n=assert_eq(size(x),size(y),size(sig),'lfit: n')
ma=assert_eq(size(maska),size(a),size(covar,1),size(covar,2),'lfit: ma')
mfit=count(maska)
if (mfit == 0) call nrerror('lfit: no parameters to be fitted')
covar(1:mfit,1:mfit)=0.0
beta(1:mfit,1)=0.0
do i=1,n
    call funcs(x(i),afunc)
    ym=y(i)
    if (mfit < ma) ym=ym-sum(a(1:ma)*afunc(1:ma), mask=.not. maska)
        Subtract off dependences on known pieces of the fitting function.
    sig2i=1.0_sp/sig(i)**2
    do j=0,m
        do l=1,ma
            if (maska(l)) then
                j=j+1
            if (mfit < ma) ym=ym-sum(a(1:ma)*afunc(1:ma), mask=.not. maska)
                Subtract off dependences on known pieces of the fitting function.
            sig2i=1.0_sp/sig(i)**2
            do j=0,m
                do l=1,ma
                    if (maska(l)) then
                        j=j+1
                    end if
                    wt=afunc(l)*sig2i
                    x=pack(afunc(1:l),maska(1:l))
                    covar(j,1:k)=covar(j,1:k)+wt*x
                    beta(j,1)=beta(j,1)+wt
                    end if
                end do
            end do
        end do
    end do
```


end do
call diagmult(covar(1:mfit,1:mfit),0.5_sp)
covar(1:mfit,1:mfit)= &
   Fill in above the diagonal from symmetry.
covar(1:mfit,1:mfit)+transpose(covar(1:mfit,1:mfit))
call gaussj(covar(1:mfit,1:mfit),beta(1:mfit,1:1))
Matrix solution.
a(1:ma)=unpack(beta(1:ma,1),maska,a(1:ma))
Partition solution to appropriate coefficients a.
chisq=0.0 Evaluate $\chi^2$ of the fit.
do i=1,n
   call funcs(x(i),afunc)
   chisq=chisq+((y(i)-dot_product(a(1:ma),afunc(1:ma)))/sig(i))**2
end do
call covsrt(covar,maska) Sort covariance matrix to true order of fitting
coefficients.

if (mfit < ma) ym=ym-sum(a(1:ma)*afunc(1:ma), mask=.not. maska)
This is the first of several uses of maska in this routine to control
which elements of an array are to be used. Here we include in the sum
only elements for which maska is false, i.e., elements corresponding to parameters
that are not being fitted for.
covar(j,1:k)=covar(j,1:k)+wt*pack(afunc(1:l),maska(1:l)) Here maska
controls which elements of afunc get packed into the covariance matrix.
call diagmult(covar(1:mfit,1:mfit),0.5_sp) See discussion of diagadd
after hqr on p. 1234.
a(1:ma)=unpack(beta(1:ma,1),maska,a(1:ma)) And here maska controls which
elements of beta get unpacked into the appropriate slots in a. Where maska is
false, corresponding elements are selected from the third argument of unpack, here
a itself. The net effect is that those elements remain unchanged.

* * *

SUBROUTINE covsrt(covar,maska)
USE nrtype; USE nrutil, ONLY : assert_eq,swap
IMPLICIT NONE
REAL(SP), DIMENSION(:,::), INTENT(INOUT) :: covar
LOGICAL(LGT), DIMENSION(:,), INTENT(IN) :: maska
Expand in storage the covariance matrix covar, so as to take into account parameters
that are being held fixed. (For the latter, return zero covariances.)
INTEGER(14B) :: ma,mfit,j,k
ma=assert_eq(size(covar,1),size(covar,2),size(maska),'covsrt')
mfit=count(maska)
covar(mfit+1:ma,1:ma)=0.0
covar(1:ma,mfit+1:ma)=0.0
k=mfit
do j=ma,1,-1
   if (maska(j)) then
      call swap(covar(1:ma,k),covar(1:ma,j))
call swap(covar(k,1:ma),covar(j,1:ma))
      k=k-1
   end if
end do
END SUBROUTINE covsrt

* * *
SUBROUTINE svdfit(x,y,sig,a,v,w,chisq,funcs)
USE nrtype; USE nrutil, ONLY : assert_eq,vabs
USE nr, ONLY : svbksb,svdcmp
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(OUT) :: a,w
REAL(SP), DIMENSION(:,:), INTENT(OUT) :: v
REAL(SP), INTENT(OUT) :: chisq

INTERFACE
FUNCTION funcs(x,n)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(n) :: funcs
END FUNCTION funcs
END INTERFACE

REAL(SP), PARAMETER :: TOL=1.0e-5_sp
Given a set of \( N \) data points \( x, y \) with individual standard deviations \( \text{sig} \), all arrays of length \( N \), use \( \chi^2 \) minimization to determine the \( M \) coefficients \( a \) of a function that depends linearly on \( a, y = \sum_{i=1}^{M} a_i \times \text{afunc}(x) \). Here we solve the fitting equations using singular value decomposition of the \( N \times M \) matrix, as in §2.6. On output, the \( M \times M \) array \( v \) and the vector \( w \) of length \( M \) define part of the singular value decomposition, and can be used to obtain the covariance matrix. The program returns values for the \( M \) fit parameters \( a \), and \( \chi^2 \), \( \text{chisq} \). The user supplies a subroutine \( \text{funcs}(x,\text{afunc}) \) that returns the \( M \) basis functions evaluated at \( x = X \) in the array \( \text{afunc} \).

INTEGER(I4B) :: i,ma,n
REAL(SP), DIMENSION(size(x)) :: b,sigi
REAL(SP), DIMENSION(size(x),size(a)) :: u,usav
n=assert_eq(size(x),size(y),size(sig),'svdfit: n')
ma=assert_eq(size(a),size(v,1),size(v,2),size(w),'svdfit: ma')
sigi=1.0_sp/sig
Accumulate coefficients of the fitting matrix in \( u \).
b=y*sigi
do i=1,n
   usav(i,:)=funcs(x(i),ma)
end do
u=usav*spread(sigi,dim=2,ncopies=ma)
usav=u
call svdcmp(u,w,v) \hspace{1cm} \text{Singular value decomposition.}
where \( (w < \text{TOL} \times \text{maxval}(w)) \) \( \Rightarrow 0.0 \)
call svbksb(u,w,v,b,a) \hspace{1cm} \text{Edit the singular values, given TOL from the parameter statement.}
chisq=vabs(matmul(usav,a)-b)**2 \hspace{1cm} \text{Evaluate chi-square.}
END SUBROUTINE svdfit

\texttt{spread} works: the vector \( \text{sigi} \) is copied \textit{along} the dimension 2, making a matrix whose columns are each a copy of \( \text{sigi} \). The multiplication here is element by element, so each row of \( \text{usav} \) is multiplied by the corresponding element of \( \text{sigi} \).

chisq=vabs(matmul(usav,a)-b)**2 \hspace{1cm} \text{Fortran 90's} \texttt{matmul} intrinsic allows us to evaluate \( \chi^2 \) from the mathematical definition in terms of matrices. \text{vabs in nrutil} returns the length of a vector (\( L_2 \) norm).

SUBROUTINE svdvar(v,w,cvm)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: v
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: w
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: cvm

To evaluate the covariance matrix \( \text{cvm} \) of the fit for \( M \) parameters obtained by \texttt{svdfit}, call this routine with matrices \( v, w \) as returned from \texttt{svdfit}. The dimensions are \( M \) for \( w \) and \( M \times M \) for \( v \) and \( \text{cvm} \).

\begin{verbatim}
INTEGER(4B) :: ma
REAL(SP), DIMENSION(size(w)) :: wti
ma=assert_eq((/size(v,1),size(v,2),size(w),size(cvm,1),size(cvm,2)/),'svdvar')
where (w /= 0.0)
wti=1.0_sp/(w*w)
elsewhere
wti=0.0
end where
cvm=v*spread(wti,dim=1,ncopies=ma)
cvm=matmul(cvm,transpose(v))
\end{verbatim}

Covariance matrix is given by (15.4.20).

\section{Fitting routines for polynomials}

\subsection{Fitting routine for a polynomial of degree \( n-1 \)}

\begin{verbatim}
FUNCTION fpoly(x,n)
USE nrtype; USE nrutil, ONLY : geop
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(n) :: fpoly
    Fitting routine for a polynomial of degree \( n-1 \), returning \( n \) coefficients in \text{fpoly}.
fpoly=geop(1.0_sp,x,n)
END FUNCTION fpoly
\end{verbatim}

\subsection{Fitting routine for Legendre polynomials}

\begin{verbatim}
FUNCTION fleg(x,nl)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
INTEGER(I4B), INTENT(IN) :: nl
REAL(SP), DIMENSION(nl) :: fleg
    Fitting routine for an expansion with \( nl \) Legendre polynomials evaluated at \( x \) and returned in the array \text{fleg} of length \( nl \). The evaluation uses the recurrence relation as in \S 5.5.
INTEGER(I4B) :: j
REAL(SP) :: d,f1,f2,twox
fleg(1)=1.0
fleg(2)=x
if (nl > 2) then
twox=2.0_sp*x
f2=x
d=1.0
do j=3,nl
    f1=d
    f2=f2+twox
d=d+1.0_sp
end do
fleg(j)=d
\end{verbatim}
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\[
\text{fleg}(j) = \frac{(f2 \times \text{fleg}(j-1) - f1 \times \text{fleg}(j-2))}{d}
\]
end do
end if
END FUNCTION fleg

SUBROUTINE mrqmin(x,y,sig,a,maska,covar,alpha,chisq,funcs,alamda)
USE nrtype; USE nrutil, ONLY : assert_eq,diagmult
USE nr, ONLY : covsrt,gaussj
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: covar,alpha
REAL(SP), INTENT(OUT) :: chisq
REAL(SP), INTENT(INOUT) :: alamda
LOGICAL(LGT), DIMENSION(:,), INTENT(IN) :: maska

INTERFACE
SUBROUTINE funcs(x,a,yfit,dyda)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,a
REAL(SP), DIMENSION(:), INTENT(OUT) :: yfit
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: dyda
END SUBROUTINE funcs
END INTERFACE

Levenberg-Marquardt method, attempting to reduce the value $\chi^2$ of a fit between a set of $N$ data points $x$, $y$ with individual standard deviations $\text{sig}$, and a nonlinear function dependent on $M$ coefficients $a$. The input logical array $\text{maska}$ of length $M$ indicates by true entries those components of $a$ that should be fitted for, and by false entries those components that should be held fixed at their input values. The program returns current best-fit values for the parameters $a$, and $\chi^2 = \text{chisq}$. The $M \times M$ arrays $\text{covar}$ and $\text{alpha}$ are used as working space during most iterations. Supply a subroutine $\text{funcs}(x,a,yfit,dyda)$ that evaluates the fitting function $yfit$, and its derivatives $dyda$ with respect to the fitting parameters $a$ at $x$. On the first call provide an initial guess for the parameters $a$, and set $alamda<0$ for initialization (which then sets $alamda=.001$). If a step succeeds $\text{chisq}$ becomes smaller and $alamda$ decreases by a factor of 10. If a step fails $alamda$ grows by a factor of 10. You must call this routine repeatedly until convergence is achieved. Then, make one final call with $alamda=0$, so that $\text{covar}$ returns the covariance matrix, and $\text{alpha}$ the curvature matrix. (Parameters held fixed will return zero covariances.)

INTEGER(I4B) :: ma,ndata
INTEGER(I4B), SAVE :: mfit
call mrqmin_private
CONTAINS
SUBROUTINE mrqmin_private
REAL(SP), SAVE :: ochisq
REAL(SP), DIMENSION(:,), ALLOCATABLE, SAVE :: atry,beta
REAL(SP), DIMENSION(:,,:), ALLOCATABLE, SAVE :: da
ndata=assert_eq(size(x),size(y),size(sig),'mrqmin: ndata')
ma=assert_eq((/size(a),size(maska),size(covar,1),size(covar,2),&
size(alpha,1),size(alpha,2)/),'mrqmin: ma')
mfit=count(maska)
if ($alamda < 0.0$) then
    Initialization.
    allocate(atry(ma),beta(ma),da(ma,1))
    alamda=0.001_sp
    call mrqcof(a,alpha,beta)
    ochisq=chisq
    atry=a
end if
chisq=mrqfit(x,y,sig,a,maska,alamda,ochisq,funcs)
end subroutine mrqmin_private
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Alter linearized fitting matrix, by augmenting diagonal elements.

da(1:mfit,1)=beta(1:mfit)
call gaussj(covar(1:mfit,1:mfit),da(1:mfit,1:1)) Matrix solution.

if (alamda == 0.0) then
    Once converged, evaluate covariance ma-
    call covsrt(covar,maska)
    call covsrt(alpha,maska)
    deallocate(atry,beta,da)
    RETURN
end if

atry=a+unpack(da(1:mfit,1),maska,0.0_sp) Did the trial succeed?
call mrqcof(atry,covar,da(1:mfit,1))
if (chisq < ochisq) then
    Success, accept the new solution.
    alamda=0.1_sp*alamda
    ochisq=chisq
    alpha(1:mfit,1:mfit)=covar(1:mfit,1:mfit)
    beta(1:mfit)=da(1:mfit,1)
    a=atry
else
    Failure, increase alamda and return.
    alamda=10.0_sp*alamda
    chisq=ochisq
end if
END SUBROUTINE mrqmin_private

SUBROUTINE mrqcof(a,alpha,beta)
REAL(SP), DIMENSION(:), INTENT(IN) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:,:), INTENT(OUT) :: alpha

Used by mrqmin to evaluate the linearized fitting matrix alpha, and vector beta as in
(15.5.8), and calculate $\chi^2$.

INTEGER(I4B) :: j,k,l,m
REAL(SP), DIMENSION(size(x),size(a)) :: dyda
REAL(SP), DIMENSION(size(x)) :: dy,sig2i,wt,ymod

call funcs(x,a,ymod,dyda)
Loop over all the data.
sig2i=1.0_sp/(sig**2)
dy=y-ymod
j=0
do l=1,ma
    if (maska(l)) then
        j=j+1
        wt=dyda(:,l)*sig2i
        k=0
        do m=1,l
            if (maska(m)) then
                k=k+1
                alpha(j,k)=dot_product(wt,dyda(:,m))
                alpha(k,j)=alpha(j,k)
            end if
        end do
        beta(j)=dot_product(dy,wt)
    end if
end do

chisq=dot_product(dy**2,sig2i) Find $\chi^2$.
END SUBROUTINE mrqcof
END SUBROUTINE mrqmin

The organization of this routine is similar to that of amoeba, discussed on p. 1209. We want to keep the argument list of mrqcof to a minimum, but we want to make clear what global variables it accesses, and protect mrqmin_private's name space.

REAL(SP), DIMENSION(:,), ALLOCATABLE, SAVE :: atry,beta These arrays, as well as da, are allocated with the correct dimensions on the first call to mrqmin.
They need to retain their values between calls, so they are declared with the SAVE attribute. They get deallocated only on the final call when $\lambda = 0$.

```
call diagmult(...)  See discussion of diagadd after hqr on p. 1234.

atry=a+ unpack(da(1:mfit,1),maska,0.0_sp)  maska controls which elements of a get incremented by da and which by 0.

* * *
```

**SUBROUTINE fgauss(x,a,y,dyda)**

USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,a
REAL(SP), DIMENSION(:,), INTENT(OUT) :: y,dyda

$y(x; a)$ is the sum of $N/3$ Gaussians (15.5.16). Here $N$ is the length of the vectors $x$, $y$ and $a$, while $dyda$ is an $N \times N$ matrix. The amplitude, center, and width of the Gaussians are stored in consecutive locations of $a$: $a(i) = B_k$, $a(i+1) = E_k$, $a(i+2) = G_k$, $k = 1, \ldots, N/3$.

```
INTEGER(I4B) :: i,na,nx
REAL(SP), DIMENSION(size(x)) :: arg,ex,fac
nx=assert_eq(size(x),size(y),size(dyda,1),'fgauss: nx')
na=assert_eq(size(a),size(dyda,2),'fgauss: na')
y(:)=0.0

do i=1,na-1,3
  arg(:)=(x(:)-a(i+1))/a(i+2)
  ex(:)=exp(-arg(:)**2)
  fac(:)=a(i)*ex(:)*2.0_sp*arg(:)
  y(:)=y(:)+a(i)*ex(:)
  dyda(:,i)=ex(:)
  dyda(:,i+1)=fac(:)/a(i+2)
  dyda(:,i+2)=fac(:)*arg(:)/a(i+2)
end do
```

```
END SUBROUTINE fgauss

* * *
```

**SUBROUTINE medfit(x,y,a,b,abdev)**

USE nrtype; USE nrutil, ONLY : assert_eq
USE nr, ONLY : select
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y
REAL(SP), INTENT(OUT) :: a,b,abdev

Fits $y = a + bx$ by the criterion of least absolute deviations. The same-size arrays $x$ and $y$ are the input experimental points. The fitted parameters $a$ and $b$ are output, along with $abdev$, which is the mean absolute deviation (in $y$) of the experimental points from the fitted line.

```
INTEGER(I4B) :: ndata
REAL(SP) :: aa
ndata=assert_eq(size(x),size(y),'medfit')
sx=sum(x)
As a first guess for $a$ and $b$, we will find the least

```
sy=sum(y)
sxy=dot_product(x,y)
```

squares fitting line.
sxx=dot_product(x,x)
del=ndata*sxx-sx**2
aa=(sxx*sy-sx*sxy)/del
bb=(ndata*sxy-sx*sy)/del
tmp(:)=y(:)-(aa+bb*x(:))
chisq=dot_product(tmp,tmp)
sigb=sqrt(chisq/del)
b1=bb
f1=rofunc(b1)
b2=bb+sign(3.0_sp*sigb,f1)
f2=rofunc(b2)
if (b2 == b1) then
  a=aa
  b=bb
  RETURN
endif
do
  if (f1*f2 <= 0.0) exit
  bb=b2+1.6_sp*(b2-b1)
  b1=b2
  f1=f2
  b2=bb
  f2=rofunc(b2)
end do
sigb=0.01_sp*sigb
do
  if (abs(b2-b1) <= sigb) exit
  bb=b1+0.5_sp*(b2-b1)
  if (bb == b1 .or. bb == b2) exit
  f=rofunc(bb)
  if (f*f1 >= 0.0) then
    f1=f
    b1=bb
  else
    f2=f
    b2=bb
  end if
end do
a=aa
b=bb
abdev=abdev/ndata
END SUBROUTINE medfit_private

FUNCTION rofunc(b)
IMPLICIT NONE
REAL(SP), INTENT(IN) :: b
REAL(SP) :: rofunc
REAL(SP), PARAMETER :: EPS=epsilon(b)
INTEGER(I4B) :: j
REAL(SP), DIMENSION(size(x)) :: arr,d
arr(:)=y(:)-b*x(:)
if (mod(ndata,2) == 0) then
  j=ndata/2
  aa=0.5_sp*(select(j,arr)+select(j+1,arr))
else
  aa=select((ndata+1)/2,arr)
end if
d(:)=y(:)-(b*x(:))+aa
abdev=sum(abs(d))
where (y(:) /= 0.0) d(:)=d(:)/abs(y(:))
rofunc=sum(x(:)*sign(1.0_sp,d(:)), mask=(abs(d(:)) > EPS )
END FUNCTION rofunc
END SUBROUTINE medfit
The organization of this routine is similar to that of amoeba discussed on p. 1209. We want to keep the argument list of rofunc to a minimum, but we want to make clear what global variables it accesses and protect medfit_private's name space. In the Fortran 77 version, we kept the only argument as b by passing the global variables in a common block. This required us to make copies of the arrays x and y. An alternative Fortran 90 implementation would be to use a module with pointers to the arguments of medfit like x and y that need to be passed to rofunc. We think the medfit_private construction is simpler.
Chapter B16. Integration of Ordinary Differential Equations

SUBROUTINE rk4(y,dydx,x,h,yout,derivs)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: y,dydx
REAL(SP), INTENT(IN) :: x,h
REAL(SP), DIMENSION(:), INTENT(OUT) :: yout
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE

Given values for the \( N \) variables \( y \) and their derivatives \( dydx \) known at \( x \), use the fourth-order Runge-Kutta method to advance the solution over an interval \( h \) and return the incremented variables as \( yout \), which need not be a distinct array from \( y \). \( y \), \( dydx \) and \( yout \) are all of length \( N \). The user supplies the subroutine \( \text{derivs}(x,y,dydx) \), which returns derivatives \( dydx \) at \( x \).

INTEGER(I4B) :: ndum
REAL(SP) :: h6, hh, xh
REAL(SP), DIMENSION(size(y)) :: dym, dyt, yt
ndum=assert_eq(size(y),size(dydx),size(yout),'rk4')
hh=h*0.5_sp
h6=h/6.0_sp
xh=x+hh
yt=y+hh*dydx
call derivs(xh,yt,dyt)  \text{First step.}
yt=y+hh*dyt
call derivs(xh,yt,dym)  \text{Second step.}
yt=y+h*dym
dym=dyt+dym
call derivs(x+h,yt,dym)  \text{Third step.}
yout=y+h*(dydx+dyt+2.0_sp*dym)  \text{Accumulate increments with proper weights.}
END SUBROUTINE rk4

⋆⋆⋆

MODULE rkdumb_path
Storage of results.
USE nrtype
REAL(SP), DIMENSION(:), ALLOCATABLE:: xx
REAL(SP), DIMENSION(:,,:), ALLOCATABLE :: y
END MODULE rkdumb_path
SUBROUTINE rkdumb(vstart,x1,x2,nstep,derivs)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : rk4
USE rkdumb_path
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: vstart
REAL(SP), INTENT(IN) :: x1,x2
INTEGER(I4B), INTENT(IN) :: nstep
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE

Starting from \( N \) initial values \( v_{\text{start}} \) known at \( x_1 \), use fourth-order Runge-Kutta to advance \( n_{\text{step}} \) equal increments to \( x_2 \). The user-supplied subroutine \( \text{derivs}(x,y,dydx) \) evaluates derivatives. Results are stored in the module variables \( xx \) and \( y \).

INTEGER(I4B) :: k
REAL(SP) :: h,x
REAL(SP), DIMENSION(size(vstart)) :: dv,v
v(:)=vstart(:) Load starting values.
if (allocated(xx)) deallocate(xx) Clear out old stored variables if necessary.
if (allocated(y)) deallocate(y)
allocate(xx(nstep+1)) Allocate storage for saved values.
allocate(y(size(vstart),nstep+1))
y(:,1)=v(:)
xx(1)=x1
x=x1
h=(x2-x1)/nstep

DO k=1,nstep Take \( n_{\text{step}} \) steps.
   call derivs(x,v,dydx)
   call rk4(v,dv,x,h,v,derivs)
   if (x+h == x) call nrerror('stepsize not significant in rkdumb')
   x=x+h
   xx(k+1)=x Store intermediate steps.
   y(:,k+1)=v(:)
END DO

END SUBROUTINE rkdumb

MODULE rkdumb_path
This routine needs straightforward communication of arrays with the calling program. The dimension of the arrays is not known in advance, and if the routine is called a second time we need to throw away the old array information. The Fortran 90 construction for this is to declare allocatable arrays in a module, and then test them at the beginning of the routine with \( \text{if (allocated...)} \).

* * *

SUBROUTINE rkqs(y,dydx,x,htry,eps,yscal,hdid,hnext,derivs)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : rkck
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:,), INTENT(IN) :: dydx,yscal
REAL(SP), INTENT(INOUT) :: x
REAL(SP), INTENT(IN) :: htry,eps
REAL(SP), INTENT(INOUT) :: hdid,hnext
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE

Fifth order Runge-Kutta step with monitoring of local truncation error to ensure accuracy and adjust stepsize. Input are the dependent variable vector \( y \) and its derivative \( dydx \) at the starting value of the independent variable \( x \). Also input are the stepsize to be attempted \( h_{\text{try}} \), the required accuracy \( \varepsilon \), and the vector \( y_{\text{scal}} \) against which the error is scaled. \( y \), \( dydx \), and \( y_{\text{scal}} \) are all of the same length. On output, \( y \) and \( x \) are replaced by their new values, \( h_{\text{did}} \) is the stepsize that was actually accomplished, and \( h_{\text{next}} \) is the estimated next stepsize. \( \text{derivs} \) is the user-supplied subroutine that computes the right-hand-side derivatives.

INTEGER(I4B) :: ndum
REAL(SP) :: errmax,h,htemp,xnew
REAL(SP), DIMENSION(size(y)) :: yerr,ytemp
REAL(SP), PARAMETER :: SAFETY=0.9_sp,PGRW=-0.2_sp,PSHRNK=-0.25_sp,&
ERRCON=1.89e-4
The value \( \text{ERRCON} \) equals \( (5/\text{SAFETY})^{(1/PGRW)} \), see use below.

ndum=assert_eq(size(y),size(dydx),size(yscal),'rkqs')
h=htry Set stepsize to the initial trial value.
do
  call rkck(y,dydx,x,h,yout,yerr,derivs) Take a step.
  errmax=maxval(abs(yerr(:)/yscal(:)))/eps Evaluate accuracy.
  if (errmax <= 1.0) exit Step succeeded.
  htemp=SAFETY*h*(errmax**PSHRNK) Truncation error too large, reduce stepsize.
  h=sign(max(abs(htemp),0.1_sp*abs(h)),h) No more than a factor of 10.
  xnew=x+h
  if (xnew == x) call nrerror('stepsize underflow in rkqs')
  end do Go back for another try.
  if (errmax > ERRCON) then Compute size of next step.
    hnext=SAFETY*h*(errmax**PGRW)
  else
    hnext=5.0_sp*h
  end if
  hdid=h
  x=x+h
  y(:)=ytemp(:)
END SUBROUTINE rkqs

SUBROUTINE rkck(y,dydx,x,h,yout,yerr,derivs)
USE nrtype; USE nrutil, ONLY: assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: y,dydx
REAL(SP), INTENT(IN) :: x,h
REAL(SP), DIMENSION(:), INTENT(OUT) :: yout,yerr
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE

Given values for \( N \) variables \( y \) and their derivatives \( dydx \) known at \( x \), use the fifth order Cash-Karp Runge-Kutta method to advance the solution over an interval \( h \) and return...
the incremented variables as \( y_{out} \). Also return an estimate of the local truncation error in \( y_{out} \) using the embedded fourth order method. The user supplies the subroutine \( \text{derivs}(x,y,dydx) \), which returns derivatives \( dydx \) at \( x \).

```
INTEGER(I4B) :: ndum
REAL(SP), DIMENSION(size(y)) :: ak2,ak3,ak4,ak5,ak6,ytemp
REAL(SP), PARAMETER :: A2=0.2_sp,A3=0.3_sp,A4=0.6_sp,A5=1.0_sp,&
B21=0.2_sp,B31=3.0_sp/40.0_sp,B32=9.0_sp/40.0_sp,&
B41=0.3_sp,B42=0.9_sp,B43=1.2_sp,B51=11.0_sp/54.0_sp,&
B52=2.5_sp,B53=70.0_sp/27.0_sp,B64=35.0_sp/27.0_sp,&
B61=1631.0_sp/B5296.0_sp,B62=175.0_sp/B512.0_sp,&
B63=575.0_sp/13624.0_sp,B64=44275.0_sp/110592.0_sp,&
B65=253.0_sp/4096.0_sp,C1=37.0_sp/378.0_sp,&
C3=250.0_sp/621.0_sp,C4=125.0_sp/594.0_sp,&
C6=512.0_sp/1771.0_sp,DC1=C1-2825.0_sp/27648.0_sp,&
DC3=C3-18575.0_sp/48384.0_sp,DC4=C4-13525.0_sp/55296.0_sp,&
DC5=-277.0_sp/14336.0_sp,DC6=C6-0.25_sp

ndum=assert_eq(size(y),size(dydx),size(yout),size(yerr),'rkck')
ytemp=y+B21*h*dydx     \hspace{1cm} \text{First step.}
call derivs(x+B21*h,ytemp,ak2)
ytemp=y+h*(B31*dydx+B32*ak2)     \hspace{1cm} \text{Second step.}
call derivs(x+B31*h,ytemp,ak3)
ytemp=y+h*(B41*dydx+B42*ak2+B43*ak3)     \hspace{1cm} \text{Third step.}
call derivs(x+B41*h,ytemp,ak4)
ytemp=y+h*(B51*dydx+B52*ak2+B53*ak3+B54*ak4)     \hspace{1cm} \text{Fourth step.}
call derivs(x+B51*h,ytemp,ak5)
ytemp=y+h*(B61*dydx+B62*ak2+B63*ak3+B64*ak4+B65*ak5)     \hspace{1cm} \text{Fifth step.}
call derivs(x+B61*h,ytemp,ak6)  \hspace{1cm} \text{Sixth step.}
yout=y+h*(C1*dydx+C3*ak3+C4*ak4+C6*ak6) \hspace{1cm} \text{Accumulate increments with proper weights.}
yerr=h*(DC1*dydx+DC3*ak3+DC4*ak4+DC6*ak6) \hspace{1cm} \text{Estimate error as difference between fourth and fifth order methods.}
```

### MODULE ode_path

```
USE nrtype
MODULE ode_path
INTEGER(I4B) :: nok,nbad,kount
LOGICAL(LGT), SAVE :: save_steps=.false.
REAL(SP) :: dxsav
REAL(SP), DIMENSION(:,), POINTER :: xp
REAL(SP), DIMENSION(:,,:), POINTER :: yp
END MODULE ode_path
```

SUBROUTINE odeint(ystart,x1,x2,eps,h1,hmin,derivs,rkqs)
USE nrtype; USE nrutil, ONLY : nrerror,reallocate
USE ode_path
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: ystart
REAL(SP), INTENT(IN) :: x1,x2,eps,h1,hmin
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
END SUBROUTINE derivs
```

SUBROUTINE rkqs(y,dydx,x,htry,eps,yscal,hdid,hnext,derivs)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:,), INTENT(IN) :: dydx, yscal
REAL(SP), INTENT(INOUT) :: x
REAL(SP), INTENT(IN) :: htry, eps
REAL(SP), INTENT(OUT) :: hdid, hnext
INTERFACE
  SUBROUTINE derivs(x,y,dydx)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), INTENT(IN) :: x
    REAL(SP), DIMENSION(:,), INTENT(IN) :: y
    REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
  END SUBROUTINE derivs
END INTERFACE
END SUBROUTINE rkqs
END INTERFACE
REAL(SP), PARAMETER :: TINY=1.0e-30_sp
INTEGER(I4B), PARAMETER :: MAXSTP=10000
Runge-Kutta driver with adaptive stepsize control. Integrate the array of starting values 
ystart from x1 to x2 with accuracy eps, storing intermediate results in the module 
variables in ode path. h1 should be set as a guessed first stepsize, hmin as the minimum 
allowed stepsize (can be zero). On output ystart is replaced by values at the end of the 
integration interval. derivs is the user-supplied subroutine for calculating the right-hand-
side derivative, while rkqs is the name of the stepper routine to be used.
INTEGER(I4B) :: nstp
REAL(SP) :: h, hdid, hnext, x, xsav
REAL(SP), DIMENSION(size(ystart)) :: dydx, y, yscal
x=x1
h=sign(h1, x2-x1)
nok=0
nbad=0
kount=0
y(:)=ystart(:)
if (save_steps) then
  xsav=x-2.0_sp*dxsav
  nullify(xp, yp)
  allocate(xp(256))
  allocate(yp(size(ystart), size(xp)))
end if
do nstp=1, MAXSTP
  call derivs(x, y, dydx)
yescal(:)=abs(y(:))+abs(h*dydx(:))+TINY
  Scaling used to monitor accuracy. This general purpose choice can be modified if need 
  be.
  if (save_steps .and. (abs(x-xsav) > abs(dxsav))) &
    Store intermediate results.
    call save_a_step
    if ((x+h-x2)* (x+h-x1) > 0.0) h=x2-x
    If stepsize can overshoot, decrease.
    call rkqs(y, dydx, x, h, eps, yscal, hdid, hnext, derivs)
    if (hdid == h) then
      nok=nok+1
    else
      nbad=nbad+1
    end if
    if ((x-x2)*(x2-x1) >= 0.0) then
      Are we done?
      ystart(:)=y(:)
      if (save_steps) call save_a_step
      Save final step.
      RETURN
    end if
    if (abs(hnext) < hmin) &
      call nrerror('stepsize smaller than minimum in odeint')
    h=hnext
  end if
end do
call nrerror('too many steps in odeint')
CONTAINS

SUBROUTINE save_a_step
  kount=kount+1
  if (kount > size(xp)) then
    xp=reallocate(xp,2*size(xp))
    yp=reallocate(yp,size(yp,1),size(xp))
  end if
  xp(kount)=x
  yp(:,kount)=y(:)
xsav=x
END SUBROUTINE save_a_step
END SUBROUTINE odeint

MODULE ode_path
The situation here is similar to rkdumb_path, except we don’t know at run time how much storage to allocate. We may need to use reallocate from nrutil to increase the storage. The solution is pointers to arrays, with a nullify to be sure the pointer status is well-defined at the beginning of the routine.

SUBROUTINE save_a_step An internal subprogram with no arguments is like a macro in C: you could imagine just copying its code wherever it is called in the parent routine.

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SUBROUTINE mmid(y,dydx,xs,htot,nstep,yout,derivs)
USE nrtype; USE nrutil, ONLY : assert_eq, swap
IMPLICIT NONE
INTEGER(14B), INTENT(IN) :: nstep
REAL(SP), INTENT(IN) :: xs,htot
REAL(SP), DIMENSION(:), INTENT(IN) :: y,dydx
REAL(SP), DIMENSION(:), INTENT(OUT) :: yout
INTERFACE
  SUBROUTINE derivs(x,y,dydx)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), INTENT(IN) :: x
    REAL(SP), DIMENSION(:), INTENT(IN) :: y
    REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
  END SUBROUTINE derivs
END INTERFACE

Modified midpoint step. Dependent variable vector \( y \) and its derivative vector \( \text{dydx} \) are input at \( xs \). Also input is \( htot \), the total step to be taken, and \( nstep \), the number of substeps to be used. The output is returned as \( yout \), which need not be a distinct array from \( y \); if it is distinct, however, then \( y \) and \( dydx \) are returned undamaged. \( y \), \( dydx \), and \( yout \) must all have the same length.

INTEGER(14B) :: n,ndum
REAL(SP) :: h,h2,x
REAL(SP), DIMENSION(size(y)) :: ym,yn
ndum=assert_eq(size(y),size(dydx),size(yout),'mmid')
h=htot/nstep Stepsize this trip.
yn=y
yn=y+h*dydx First step.
x=x+h
call derivs(x,yn,yout)

h2=2.0_sp*h Will use yout for temporary storage of derivatives.
do n=2,nstep
  call swap(ym,yn)
  yn=yn+h2*yout General step.
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\[ x = x + h \]

end do

\[ y_{\text{out}} = 0.5 \times (y_{\text{m}} + y_{\text{n}} + h \times y_{\text{out}}) \]

Last step.

END SUBROUTINE mmid

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SUBROUTINE bsstep(y, dydx, x, htry, eps, yscal, hdid, hnext, derivs)

USE nrtype; USE nrutil, ONLY : arth, assert_eq, cumsum, iminloc, nrerror, & outerdiff, outerprod, upper_triangle

USE nr, ONLY : mmid, pzextr

IMPLICIT NONE

REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:), INTENT(IN) :: dydx, yscal
REAL(SP), INTENT(INOUT) :: x
REAL(SP), INTENT(IN) :: htry, eps
REAL(SP), INTENT(OUT) :: hdid, hnext

INTERFACE

SUBROUTINE derivs(x, y, dydx)

USE nrtype

IMPLICIT NONE

REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx

END SUBROUTINE derivs

END INTERFACE

INTEGER(I4B), PARAMETER :: IMAX=9, KMAXX=IMAX-1
REAL(SP), PARAMETER :: SAFE1=0.25_sp, SAFE2=0.7_sp, REDMAX=1.0e-5_sp, & REDMIN=0.7_sp, TINY=1.0e-30_sp, SCALMX=0.1_sp

Bulirsch-Stoer step with monitoring of local truncation error to ensure accuracy and adjust stepsize. Input are the dependent variable vector \( y \) and its derivative \( \text{dydx} \) at the starting value of the independent variable \( x \). Also input are the stepsize to be attempted \( h_{\text{try}} \), the required accuracy \( \text{eps} \), and the vector \( \text{yscal} \) against which the error is scaled. On output, \( y \) and \( x \) are replaced by their new values, \( \text{hdid} \) is the stepsize that was actually accomplished, and \( \text{hnext} \) is the estimated next stepsize. \( \text{derivs} \) is the user-supplied subroutine that computes the right-hand-side derivatives. \( y \), \( \text{dydx} \), and \( \text{yscal} \) must all have the same length. Be sure to set \( h_{\text{try}} \) on successive steps to the value of \( \text{hnext} \) returned from the previous step, as is the case if the routine is called by \( \text{odeint} \).

Parameters: \( KMAXX \) is the maximum row number used in the extrapolation; \( IMAX \) is the next row number; \( \text{SAFE1} \) and \( \text{SAFE2} \) are safety factors; \( \text{REDMAX} \) is the maximum factor used when a stepsize is reduced, \( \text{REDMIN} \) the minimum; \( \text{TINY} \) prevents division by zero; \( 1/\text{SCALMX} \) is the maximum factor by which a stepsize can be increased.

INTEGER(1) :: k, km, ndum
INTEGER(1), DIMENSION(IMAX) :: nseq = (/ 2, 4, 6, 8, 10, 12, 14, 16, 18 /)
INTEGER(1), SAVE :: kopt, kmax
REAL(SP), DIMENSION(KMAXX, KMAXX), SAVE :: alf
REAL(SP), DIMENSION(KMAXX) :: err
REAL(SP), DIMENSION(KMAXX) :: a
REAL(SP), SAVE :: epsold = -1.0_sp, xnew
REAL(SP), SAVE :: eps1, ermax, fact, h, red, scale, wrkmin, xest
REAL(SP), DIMENSION(size(y)) :: yerr, ysav, yseq
LOGICAL(LGT) :: reduct
LOGICAL(LGT), SAVE :: first=.true.

ndum=assert_eq(size(y), size(dydx), size(yscal), 'bsstep')

if (eps /= epsold) then
  A new tolerance, so reinitialize.
  hnext=1.0e29_sp
  'Impossible' values.
  xnew=1.0e29_sp
  eps1=SAFE1*eps
  a(1)=cumsum(nseq,1)
  Compute \( \alpha(k, q) \):
  where (upper_triangle(KMAXX,KMAXX)) \( \alpha \) =eps1** &
  (outerdiff(a(2:),a(2:)))outerprod(arth( &
3.0_sp,2.0_sp,KMAXX),(a(2:)-a(1)+1.0_sp))
epsold=eps
do kopt=2,KMAXX-1
   Determine optimal row number for con-
   vergence.
   if (a(kopt+1) > a(kopt)*alf(kopt-1,kopt)) exit
end do
kmax=kopt
end if
h=htry
ysav(:)=y(:)
Save the starting values.
if (h /= hnext .or. x /= xnew) then
   A new stepsize or a new integration: Re-
   establish the order window.
   first=.true.
   kopt=kmax
   end if
reduct=.false.
main_loop: do
   do k=1,kmax
      Evaluate the sequence of modified mid-
      point integrations.
xnew=x+h
      if (xnew == x) call nrerror('step size underflow in bsstep')
call maids(ysav,dydx,x,h,nseq(k),yseq,derivs)
xest=(h/nseq(k))**2
   Compute normalized error estimate \( \epsilon \).
call pxtextr(k,xest,yseq,y,yerr)
   Perform extrapolation.
   if (k /= 1) then
      errmax=maxval(abs(yerr(:)/yscal(:)))
      Compute normalized error estimate \( \epsilon \).
      errmax=max(TINY,errmax)/eps
      Scale error relative to tolerance.
   end if
   if (k /= 1 .and. (k >= kopt-1 .or. first)) then
      In order window.
      errmax=max(TINY,errmax)**(1.0_sp/(2*km+1))
      err(km)=(errmax/SAFE1)**(1.0_sp/(2*km+1))
      err(km)=(errmax/SAFE1)**(1.0_sp/(2*km+1))
   end if
   if (k /= 1 .and. (k >= kopt-1 .or. first)) then
      In order window.
      if (errmax < 1.0) exit main_loop
      Converged.
      if (k == kmax .or. (kopt = km)) then
         Check for possible step-
         size reduction.
         red=SAFE2/err(km)
         exit
      end if
   end if
   if (kopt == km) then
      if (alf(km,km) < err(km)) then
         red=alf(km,km)/err(km)
         exit
      end if
   else if (alf(km,km) < err(km)) then
      red=alf(km,km)/err(km)
      exit
   end if
   if (alf(km,km) < err(km)) then
      red=alf(km,km)/err(km)
      exit
   end if
end do
red=max(min(red,REDMIN),REDMAX)
Reduce stepsize by at least REDMIN and
h=h*red
at most REDMAX.
end do main_loop
Try again.
x=xnew
Successful step taken.
hdid=h
first=.false.
kopt=1+iminloc(a(2:km+1)*max(err(1:km),SCALMX))
Compute optimal row for convergence and corresponding stepsize.
scale=max(err(kopt-1),SCALMX)
wrkmin=scale*a(kopt)
hnext=h/scale
if (kopt >= k .and. kopt +/- km .or. not. reduct) then
   Check for possible or-
   der increase, but
   fact=max(scale/alf(kopt-1,kopt),SCALMX)
   if (a(kopt+1)*fact <= wrkmin) then
      not if stepsize was
      hnext=h/fact
      just reduced.
a(:)=cumsum(nseq,1)  The function cumsum in nrutil with the optional argument seed=1 gives a direct implementation of equation (16.4.6).

where (upper_triangle(KMAXX,KMAXX))... The upper_triangle function in nrutil returns an upper triangular logical mask. As used here, the mask is true everywhere in the upper triangle of a \( KMAXX \times KMAXX \) matrix, excluding the diagonal. An optional integer argument extra allows additional diagonals to be set to true. With extra=1 the upper triangle including the diagonal would be true.

main_loop: do Using a named do-loop provides clear structured code that required goto's in the Fortran 77 version.

kopt=1+iminloc(...)  See the discussion of iminloc on p. 1017.

SUBROUTINE pzextr(iest,xest,yest,yz,dy)
USE nrtype; USE nrutil, ONLY: assert_eq,nrerror
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: iest
REAL(SP), INTENT(IN) :: xest
REAL(SP), DIMENSION(:), INTENT(IN) :: yest
REAL(SP), DIMENSION(:), INTENT(OUT) :: yz,dy
Use polynomial extrapolation to evaluate \( N \) functions at \( x = 0 \) by fitting a polynomial to a sequence of estimates with progressively smaller values \( x = xest \), and corresponding function vectors \( yest \). This call is number \( iest \) in the sequence of calls. Extrapolated function values are output as \( yz \), and their estimated error is output as \( dy \). \( yest \), \( yz \), and \( dy \) are arrays of length \( N \).

INTEGER(I4B), PARAMETER :: IEST_MAX=16
INTEGER(I4B) :: j,nv
INTEGER(I4B), SAVE :: nvold=-1
REAL(SP) :: delta,f1,f2
REAL(SP), DIMENSION(size(yz)) :: d,tmp,q
REAL(SP), DIMENSION(IEST_MAX), SAVE :: x
REAL(SP), DIMENSION(::,:), ALLOCATABLE, SAVE :: qcol
nv=assert_eq(size(yz),size(yest),size(dy),'pzextr')
if (iest > IEST_MAX) call & nrerror('pzextr: probable misuse, too much extrapolation')
if (nv /= nvold) then
  Set up internal storage.
  if (allocated(qcol)) deallocate(qcol)
  allocate(qcol(nv,UEST_MAX))
  nvold=nv
end if
x(iest)=xest
Save current independent variable.
dy(:)=yest(:)
yz(:)=yest(:)
if (iest == 1) then
  Store first estimate in first column.
  qcol(:,1)=yest(:)
else
d(:)=yest(:)
do j=1,iest-1
delta=1.0_sp/(x(iest-j)-xest)
f1=xest*delta
f2=x(iest-j)*delta
q(:)=qcol(:,j)
Propagate tableau 1 diagonal more.
REAL(SP), DIMENSION(:,,:), ALLOCATABLE, SAVE :: qcol  
The second dimension of qcol is known at compile time to be IEST_MAX, but the first 
dimension is known only at run time, from size(yz). The language 
requires us to have all dimensions allocatable if any one of them is.

if (nv /= nvold) then...  
This routine generally gets called many times with 
iest cycling repeatedly through the values 1, 2, ..., up to some value less than 
IEST_MAX. The number of variables, nv, is fixed during the solution of the problem. 
The routine might be called again in solving a different problem with a new value of 
nv. This if block ensures that qcol is dimensioned correctly both for the first 
and subsequent problems, if any.

SUBROUTINE rzextr(iest,xest,yest,yz,dy) 
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: iest
REAL(SP), INTENT(IN) :: xest
REAL(SP), DIMENSION(:), INTENT(IN) :: yest
REAL(SP), DIMENSION(:), INTENT(OUT) :: yz,dy

Exact substitute for pzextr, but uses diagonal rational function extrapolation instead of 
polynomial extrapolation.

INTEGER(I4B), PARAMETER :: IEST_MAX=16
INTEGER(I4B) :: k,nv
INTEGER(I4B), SAVE :: nvold=-1
REAL(SP), DIMENSION(size(yz)) :: yy,v,c,b,b1,ddy
REAL(SP), DIMENSION(:,,:), ALLOCATABLE, SAVE :: d
REAL(SP), DIMENSION(IEST_MAX), SAVE :: fx,x

nv=assert_eq(size(yz),size(dy),size(yest),'rzextr')
if (iest > IEST_MAX) call &
nrerror('rzextr: probable misuse, too much extrapolation')
if (nv /= nvold) then
  if (allocated(d)) deallocate(d)
  allocate(d(nv,IFEST_MAX))
nvold=nv
end if

x(iest)=xest  
Save current independent variable.
if (iest == 1) then
  yz=yest
d(:,1)=yest
dy=yest
else
  fx(2:iest)=x(iest-1:1:-1)/xest
  yy=yest
  v=d(1:nv,1)
  c=yy
  d(1:nv,1)=yy
  do k=2,iest
    bl=fx(k)*v
    b=b1+c
    where (b /= 0.0)
b=(c-v)/b
ddy=c*b
c=b1*b
elsewhere

Care needed to avoid division by 0.

ddy=v
end where

if (k /= iest) v=d(1:nv,k)
d(1:nv,k)=ddy
yy=yy+ddy
end do

dy=ddy
z2=yy
end if

END SUBROUTINE rzextr

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SUBROUTINE stoerm(y,d2y,xs,htot,nstep,yout,derivs)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: y,d2y
REAL(SP), INTENT(IN) :: xs,htot
INTEGER(I4B), INTENT(IN) :: nstep
REAL(SP), DIMENSION(:,), INTENT(OUT) :: yout
 INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE

Stoermer’s rule for integrating $y'' = f(x, y)$ for a system of $n$ equations. On input $y$ contains $y$ in its first $n$ elements and $y'$ in its second $n$ elements, all evaluated at $xs$. $d2y$ contains the right-hand-side function $f$ (also evaluated at $xs$) in its first $n$ elements. Its second $n$ elements are not referenced. Also input is $htot$, the total step to be taken, and $nstep$, the number of substeps to be used. The output is returned as $yout$, with the same storage arrangement as $y$. $derivs$ is the user-supplied subroutine that calculates $f$.

INTEGER(I4B) :: neqn,neqn1,nn,nv
REAL(SP) :: h,h2,halfh,x
REAL(SP), DIMENSION(size(y)) :: ytemp
nv=assert_eq(size(y),size(d2y),size(yout),'stoerm')
neqn=nv/2
neqn1=neqn+1
h=htot/nstep
halfh=0.5_sp*h
h2=h*h

ytemp(neqn1:nv)=h*(y(neqn1:nv)+halfh*d2y(1:neqn))
ytemp(1:neqn)=y(1:neqn)+ytemp(neqn1:nv)
x=xs+h
call derivs(x,ytemp,yout)

do nn=2,nstep

ytemp(neqn1:nv)=ytemp(neqn1:nv)+h2*yout(1:neqn)
ytemp(1:neqn)=ytemp(1:neqn)+ytemp(neqn1:nv)
x=x+h
call derivs(x,ytemp,yout)
end do

yout(neqn1:nv)=ytemp(neqn1:nv)/h+halfh*yout(1:neqn)
yout(1:neqn)=ytemp(1:neqn)

END SUBROUTINE stoerm

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SUBROUTINE stiff(y, dydx, x, htry, eps, yscal, hdid, hnext, derivs)
USE nrtype; USE nrutil, ONLY : assert_eq, diagadd, nrerror
USE nr, ONLY : lubksb, ludcmp
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:), INTENT(IN) :: dydx, yscal
REAL(SP), INTENT(INOUT) :: x
REAL(SP), INTENT(IN) :: htry, eps
REAL(SP), INTENT(OUT) :: hdid, hnext
INTERFACE
SUBROUTINE derivs(x, y, dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
SUBROUTINE jacobn(x, y, dfdx, dfdy)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dfdx
REAL(SP), DIMENSION(:,:), INTENT(OUT) :: dfdy
END SUBROUTINE jacobn
END INTERFACE
INTEGER(I4B), PARAMETER :: MAXTRY=40
REAL(SP), PARAMETER :: SAFETY=0.9_sp, GROW=1.5_sp, PGROW=-0.25_sp, &
SHRNK=0.5_sp, PSHRKX=1.0_sp/3.0_sp, ERRCON=0.1296_sp, &
GAM=1.0_sp/2.0_sp, &
A21=2.0_sp, A31=48.0_sp/25.0_sp, A32=6.0_sp/25.0_sp, C21=-8.0_sp, &
C31=372.0_sp/25.0_sp, C32=12.0_sp/5.0_sp, &
C41=-112.0_sp/125.0_sp, C42=-54.0_sp/125.0_sp, E1=17.0_sp/54.0_sp, &
E2=7.0_sp/36.0_sp, E3=0.0_sp, E4=125.0_sp/108.0_sp, &
C1X=1.0_sp/2.0_sp, C2X=-3.0_sp/2.0_sp, &
C3X=121.0_sp/50.0_sp, C4X=29.0_sp/250.0_sp, A2X=1.0_sp, A3X=3.0_sp/5.0_sp
Fourth order Rosenbrock step for integrating stiff ODEs, with monitoring of local truncation error to adjust stepsize. Input are the dependent variable vector $y$ and its derivative $dydx$ at the starting value of the independent variable $x$. Also input are the stepsize to be attempted $htry$, the required accuracy $eps$, and the vector $yscal$ against which the error is scaled. On output, $y$ and $x$ are replaced by their new values, $hdid$ is the stepsize that was actually accomplished, and $hnext$ is the estimated next stepsize. $derivs$ is a user-supplied subroutine that computes the derivatives of the right-hand side with respect to $x$, while $jacobn$ (a fixed name) is a user-supplied subroutine that computes the Jacobi matrix of derivatives of the right-hand side with respect to the components of $y$, $y$, $dydx$, and $yscal$ must have the same length.
Parameters: $GROW$ and $SHRNK$ are the largest and smallest factors by which stepsize can change in one step; $ERRCON=(GROW/SAFETY)**(1/PGROW)$ and handles the case when $errmax \approx 0$.

INTEGER(I4B) :: jtry, ndum
INTEGER(I4B), DIMENSION(size(y)) :: indx
REAL(SP), DIMENSION(size(y)) :: dfdx, dytmp, err, g1, g2, g3, g4, ysav
REAL(SP), DIMENSION(size(y), size(y)) :: a, dfdy
REAL(SP) :: d, ermax, h, xsav
ndum=assert_eq(size(y), size(dydx), size(yscal), 'stiff')
xsav=x \ Save initial values.
ysav(:)=y(:)
call jacobn(xsav, ysav, dfdx, dfdy)
The user must supply this subroutine to return the $n \times n$ matrix $dfdy$ and the vector $dfdx$.

h=htry \ Set stepsize to the initial trial value.
do jtry=1, MAXTRY
a(:, :) = -dfdy(:, :)
call diagadd(a, 1.0_sp/(GAM*h))
call ludcmp(a, indx, d)
g1 = dydx + C1X*dfdx
call lubksb(a, indx, g1)
y = ysav + A21*g1
x = xsav + A2X*h
call derive(x, y, dytmp)
g2 = dytmp + C2X*dfdx + C21*g1/h
call lubksb(a, indx, g2)
y = ysav + A31*g1 + A32*g2
x = xsav + A3X*h
call derive(x, y, dytmp)
g3 = dytmp + C3X*dfdx + (C31*g1 + C32*2) / h
call lubksb(a, indx, g3)
g4 = dytmp + C4X*dfdx + (C41*g1 + C42*g2 + C43*g3) / h
call lubksb(a, indx, g4)
y = ysav + B11*g1 + B22*g2 + B3*g3 + B4*g4
err = E1*g1 + E2*g2 + E3*g3 + E4*g4
x = xsav + h

if (x == xsav) call &
nrerror('stepsize not significant in stiff')
errmax = maxval(abs(err) / yscal) / eps
if (errmax <= 1.0) then
    Compute size of next step
    hnext = merge(SAFETY*h*errmax**PGROW, GROW*h, &
        errmax > ERRCON)
RETURN
else
    Truncation error too large, reduce stepsize.
    hnext = SAFETY*h*errmax**PSHRNK
    h = sign(max(abs(hnext), SHRNK*abs(h)), h)
end if
end do

END SUBROUTINE stiff

SUBROUTINE jacobn(x, y, dfdx, dfdy)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:, ::), INTENT(IN) :: y
REAL(SP), DIMENSION(:, ::), INTENT(OUT) :: dfdx
REAL(SP), DIMENSION(:, ::), INTENT(OUT) :: dfdy
Routine for Jacobian matrix corresponding to example in equations (16.6.27).
dfdx(:, :) = 0.0
dfdx(1, 1) = -0.013_sp - 1000.0_sp*y(3)
dfdx(1, 2) = 0.0
dfdx(1, 3) = -1000.0_sp*y(1)
dfdx(2, 1) = 0.0
dfdx(2, 2) = -2500.0_sp*y(3)
dfdx(2, 3) = -2500.0_sp*y(2)
dfdx(3, 1) = -0.013_sp - 1000.0_sp*y(3)
dfdx(3, 2) = -2500.0_sp*y(3)
dfdx(3, 3) = -1000.0_sp*y(1) - 2500.0_sp*y(2)
END SUBROUTINE jacobn

call diagadd(...) See discussion of diagadd after hqr on p. 1234.
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
Routine for right-hand side of example in equations (16.6.27).
   dydx(1)=-0.013_sp*y(1)-1000.0_sp*y(1)*y(3)
   dydx(2)=-2500.0_sp*y(2)*y(3)
   dydx(3)=-0.013_sp*y(1)-1000.0_sp*y(1)*y(3)-2500.0_sp*y(2)*y(3)
END SUBROUTINE derivs

SUBROUTINE simpr(y,dydx,dfdx,dfdy,xs,htot,nstep,yout,derivs)
USE nrtype; USE nrutil, ONLY: assert_eq, diagadd
USE nr, ONLY: lubksb, ludcmp
IMPLICIT NONE
REAL(SP), INTENT(IN) :: xs,htot
REAL(SP), DIMENSION(:), INTENT(IN) :: y,dydx,dfdx
REAL(SP), DIMENSION(:,:), INTENT(IN) :: dfdy
INTEGER(I4B), INTENT(IN) :: nstep
REAL(SP), DIMENSION(:), INTENT(OUT) :: yout
INTERFACE
  SUBROUTINE derivs(x,y,dydx)
  USE nrtype
  IMPLICIT NONE
  REAL(SP), INTENT(IN) :: x
  REAL(SP), DIMENSION(:), INTENT(IN) :: y
  REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
  END SUBROUTINE derivs
  END INTERFACE
Performs one step of semi-implicit midpoint rule. Input are the dependent variable y, its derivative dydx, the derivative of the right-hand side with respect to x, dfdx, which are all vectors of length N, and the N x N Jacobian dfdy at xs. Also input are htot, the total step to be taken, and nstep, the number of substeps to be used. The output is returned as yout, a vector of length N. derivs is the user-supplied subroutine that calculates dydx.
INTEGER(I4B) :: ndum,nn
INTEGER(I4B), DIMENSION(size(y)) :: indx
REAL(SP) :: d,h,x
REAL(SP), DIMENSION(size(y)) :: del,ytemp
REAL(SP), DIMENSION(size(y),size(y)) :: a
ndum=assert_eq((/size(y),size(dydx),size(dfdx),size(dfdy,1),&
   size(dfdy,2),size(yout)/),'simpr')
   h=htot/nstep
   a(:,:,)=h*dfdy(:,:,)
   call diagadd(a,1.0_sp)
   call ludcmp(a,indx,d)
   yout=h*(dydx+h*dfdx)
   call lubksb(a,indx,yout)
   del=yout
   ytemp=yout+2.0_sp*del
   x=x+h
   call derivs(x,ytemp,yout)
   do nn=2,nstep
      yout=h*yout-del
      call lubksb(a,indx,yout)
      del=del+2.0_sp*yout
      ytemp=yout+del
      x=x+h
      call derivs(x,ytemp,yout)
end do  
  yout=h*yout-del  
  Set up right-hand side for last step.  
  call lubksb(a,indx,yout)  
  Take last step.  
END SUBROUTINE simpr

SUBROUTINE stifbs(y,dydx,x,htry,eps,yscal,hdid,hnext,derivs)
USE nrtype; USE nrutil, ONLY : arth,assert_eq,cumsum,iminloc,nerror,&
  outerdiff,outerprod,upper_triangle
USE nr, ONLY : simpr,pzextr
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:,), INTENT(IN) :: dydx,yscal
REAL(SP), INTENT(IN) :: htry,eps
REAL(SP), INTENT(INOUT) :: x
REAL(SP), INTENT(OUT) :: hdid,hnext
INTERFACE
  SUBROUTINE derivs(x,y,dydx)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), INTENT(IN) :: x
    REAL(SP), DIMENSION(:,), INTENT(IN) :: y
    REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
  END SUBROUTINE derivs
  SUBROUTINE jacobn(x,y,dfdx,dfdy)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), INTENT(IN) :: x
    REAL(SP), DIMENSION(:,), INTENT(IN) :: y
    REAL(SP), DIMENSION(:,), INTENT(OUT) :: dfdx
    REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: dfdy
  END SUBROUTINE jacobn
END INTERFACE
INTEGER(I4B), PARAMETER :: IMAX=8, KMAXX=IMAX-1
REAL(SP), PARAMETER :: SAFE1=0.25_sp,SAFE2=0.7_sp,REDMAX=1.0e-5_sp,&
  REDMIN=0.7_sp,TINY=1.0e-30_sp,SCALMAX=0.1_sp
Semi-implicit extrapolation step for integrating stiff ODEs, with monitoring of local truncation error to adjust stepsize. Input are the dependent variable vector \( y \) and its derivative \( dydx \) at the starting value of the independent variable \( x \). Also input are the stepsize to be attempted \( htry \), the required accuracy \( \epsilon \), and the vector \( yscal \) against which the error is scaled. On output, \( y \) and \( x \) are replaced by their new values, \( hdid \) is the stepsize that was actually accomplished, and \( hnext \) is the estimated next stepsize. derivs is a user-supplied subroutine that computes the derivatives of the right-hand side with respect to \( x \), while jacobn (a fixed name) is a user-supplied subroutine that computes the Jacobian matrix of derivatives of the right-hand side with respect to the components of \( y \). \( y \), \( dydx \), and \( yscal \) must all have the same length. Be sure to set \( htry \) on successive steps to the value of \( hnext \) returned from the previous step, as is the case if the routine is called by odeint.  
INTEGER(I4B) :: k,km,ndum
INTEGER(I4B), DIMENSION(IMAX) :: nseq = (/ 2,6,10,14,22,34,50,70 /)
Sequence is different from bsstep.  
INTEGER(I4B), SAVE :: kopt,kmax,nvold=-1
REAL(SP), DIMENSION(KMAXX,KMAXX), SAVE :: alf
REAL(SP), DIMENSION(KMAXX) :: err
REAL(SP), DIMENSION(IMAX), SAVE :: a
REAL(SP), SAVE :: epsold = -1.0
REAL(SP) :: eps1,ermax,fact,h,red,scale,wrkmin,xest
REAL(SP), SAVE :: xnew
REAL(SP), DIMENSION(size(y)) :: dfdx, yerr, ysav, yseq
REAL(SP), DIMENSION(size(y), size(y)) :: dfdy
LOGICAL(LGT) :: reduct
LOGICAL(LGT), SAVE :: first=.true., ndum=assert_eq(size(y), size(dydx), size(yscal), 'stifbs')

if (eps /= epsold .or. nvold /= size(y)) then
  Reinitialize also if number of variables has changed.
  hnext=-1.0e29_sp
  xnew=-1.0e29_sp
  eps1=SAFE1*eps
  a(:)=csum(nseq, 1)
  where (upper_triangle(KMAXX, KMAXX)) alf=eps1** &
  (outerdiff(a(2:), a(2:))/outerprod(arth( &
  3.0_sp, 2.0_sp, KMAXX), (a(2:)-a(1)+1.0_sp)))
  epsold=eps
  nvold=size(y)
  Add number of Jacobian evaluations to work coefficients.
  kopt=2, KMAXX-1
  if (a(kopt+1) > a(kopt)*alf(kopt-1,kopt)) exit
  kmax=kopt
  end if
end if

h=htry
ysav(:)=y(:)
call jacobn(x, y, dfdx, dfdy) Evaluate Jacobian.
if (h /= hnext .or. x /= xnew) then
  first=.true.
  kopt=kmax
end if

reduct=.false.

main_loop: do
  do k=1, kmax
    nextr=x+h
    if (nextr == x) call nrerror('step size underflow in stifbs')
    call simpri(y, dydx, dfdx, dfdy, x, h, nseq(k), yseq, derivs)
    xest=(h/nseq(k))**2
    The rest of the routine is identical to bsstep.
    call pzextr(k, xest, yseq, y, yerr)
    if (k /= 1) then
      errmax=maxval(abs(yerr(:)/yscal(:)))
      errmax=max(TINY, errmax)/eps
      km=k-1
      err(km)=(errmax/SAFE1)**(1.0_sp/(2*km+1))
    end if
    if (k /= 1 .and. (k >= kopt-1 .or. first)) then
      if (errmax < 1.0) exit main_loop
      if (k == kmax .or. k == kopt+1) then
        red=SAFE2/err(km)
        exit
      else if (k == kopt) then
        if (alf(kopt-1,kopt) < err(km)) then
          red=1.0_sp/err(km)
          exit
        end if
      end if
    end if
    if (kopt == kmax) then
      if (alf(km, kmax-1) < err(km)) then
        red=alf(km, kmax-1)*SAFE2/err(km)
        exit
      end if
    end if
  end do
This routine is very similar to bsstep, and the same remarks about Fortran 90 constructions on p. 1305 apply here.
Chapter B17. Two Point Boundary Value Problems

! FUNCTION shoot(v) is named "funcv" for use with "newt"
FUNCTION funcv(v)
USE nrtype
USE nr, ONLY : odeint, rkqs
USE sshoot_caller, ONLY : nvar, x1, x2; USE ode_path, ONLY : xp, yp
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: v
REAL(SP), DIMENSION(size(v)) :: funcv
REAL(SP), PARAMETER :: EPS=1.0e-6_sp

Routine for use with newt to solve a two point boundary value problem for N coupled ODEs by shooting from x1 to x2. Initial values for the ODEs at x1 are generated from the n2 input coefficients v, using the user-supplied routine load. The routine integrates the ODEs to x2 using the Runge-Kutta method with tolerance EPS, initial stepsize h1, and minimum stepsize hmin. At x2 it calls the user-supplied subroutine score to evaluate the n2 functions funcv that ought to be zero to satisfy the boundary conditions at x2. The functions funcv are returned on output. newt uses a globally convergent Newton's method to adjust the values of v until the functions funcv are zero. The user-supplied subroutine derivs(x,y,dydx) supplies derivative information to the ODE integrator (see Chapter 16). The module sshoot_caller receives its values from the main program so that funcv can have the syntax required by newt. Set nvar = N in the main program.

REAL(SP) :: h1, hmin
REAL(SP), DIMENSION(nvar) :: y
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
END SUBROUTINE derivs
SUBROUTINE load(x1,v,y)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1
REAL(SP), DIMENSION(:,), INTENT(IN) :: v
REAL(SP), DIMENSION(:,), INTENT(OUT) :: y
END SUBROUTINE load
SUBROUTINE score(x2,y,f)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x2
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: f
END SUBROUTINE score
END INTERFACE

h1=(x2-x1)/100.0_sp
FUNCTION shootf(v) is named "funcv" for use with "newt"

FUNCTION funcv(v)
USE nrtype
USE nr, ONLY : odeint, rkqs
USE sphfpt_caller, ONLY : x1, x2, xf, nn2; USE ode_path, ONLY : xp, yp
IMPLICIT NONE
REAL(SP), DIMENSION(:), INTENT(IN) :: v
REAL(SP), DIMENSION(size(v)) :: funcv
REAL(SP), PARAMETER :: EPS=1.0e-6_sp

Routine for use with newt to solve a two point boundary value problem for N coupled ODEs by shooting from x1 and x2 to a fitting point xf. Initial values for the ODEs at x1 (x2) are generated from the n2 (n1) coefficients V1 (V2), using the user-supplied routine load1 (load2). The coefficients V1 and V2 should be stored in a single array v of length N in the main program, and referenced by pointers as v1=>v(1:n2), v2=>v(n2+1:N). Here N = n1 + n2. The routine integrates the ODEs to xf using the Runge-Kutta method with tolerance EPS, initial stepsize h1, and minimum stepsize hmin. At xf it calls the user-supplied subroutine score to evaluate the N functions f1 and f2 that ought to match at xf. The differences funcv are returned on output.

newt uses a globally convergent Newton's method to adjust the values of v until the functions funcv are zero. The user-supplied subroutine derivs(x,y,dydx) supplies derivative information to the ODE integrator (see Chapter 16). The module sphfpt_caller receives its values from the main program so that funcv can have the syntax required by newt.

Set nn2 = n2 in the main program.

REAL(SP) :: h1, hmin
REAL(SP), DIMENSION(size(v)) :: f1, f2, y

INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs

SUBROUTINE load1(x1,v1,y)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1
REAL(SP), DIMENSION(:), INTENT(IN) :: v1
REAL(SP), DIMENSION(:), INTENT(OUT) :: y
END SUBROUTINE load1

SUBROUTINE load2(x2,v2,y)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x2
REAL(SP), DIMENSION(:), INTENT(IN) :: v2
REAL(SP), DIMENSION(:), INTENT(OUT) :: y
END SUBROUTINE load2

SUBROUTINE score(x2,y,f)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x2
REAL(SP), DIMENSION(:), INTENT(IN) :: y

hmin=0.0
call load(x1,v,y)
if (associated(xp)) deallocate(xp,yp)
Prevent memory leak if save steps set
call odeint(y,x1,x2,EPS,h1,hmin,derivs,rkqs) to true.
call score(x2,y,funcv)
END FUNCTION funcv
REAL(SP), DIMENSION(:, :), INTENT(OUT) :: f
END SUBROUTINE score

END INTERFACE

h1=(x2-x1)/100.0_sp
hmin=0.0

call load1(x1,v,y)
Path from x1 to xf with best trial values V1.
if (associated(xp)) deallocate(xp,yp)
Prevent memory leak if save_steps set
call odeint(y,x1,xf,EPS,h1,hmin,derivs,rkqs) to .true.
call score(xf,y,f1)
call load2(x2,v(nn2+1:),y)
Path from x2 to xf with best trial values V2.
call odeint(y,x2,xf,EPS,h1,hmin,derivs,rkqs)
call score(xf,y,f2)
cfuncv(:)=f1(:)-f2(:)

END FUNCTION funcv

⋆⋆⋆

SUBROUTINE solvde(itmax,conv,slowc,scalv,indexv,nb,y)
USE nrtype; USE nrutil, ONLY : assert_eq,imaxloc,nrerror
USE nr, ONLY : difeq
IMPLICIT NONE

INTEGER(I4B), INTENT(IN) :: itmax, nb
REAL(SP), INTENT(IN) :: conv, slowc
REAL(SP), DIMENSION(:, :), INTENT(IN) :: scalv
INTEGER(I4B), DIMENSION(:, :), INTENT(IN) :: indexv
REAL(SP), DIMENSION(:, :, :, :), INTENT(INOUT) :: y

Driver routine for solution of two point boundary value problems with N equations by relaxation. itmax is the maximum number of iterations. conv is the convergence criterion (see text). slowc controls the fraction of corrections actually used after each iteration. scalv, a vector of length N, contains typical sizes for each dependent variable, used to weight errors. indexv, also of length N, lists the column ordering of variables used to construct the matrix s of derivatives. (The nb boundary conditions at the first mesh point must contain some dependence on the first nb variables listed in indexv.) There are a total of M mesh points. y is the N × M array that contains the initial guess for all the dependent variables at each mesh point. On each iteration, it is updated by the calculated correction.

INTEGER(I4B) :: ic1,ic2,ic3,ic4,ic5,ic6,ic7,ic8,ic9,ic10,ic11,ic12,ic13,ic14,ic15,ic16,ic17,ic18,ic19,ic20,ic21,ic22,ic23,ic24,ic25,ic26,ic27,ic28,ic29,ic30
REAL(SP) :: err,fac
REAL(SP), DIMENSION(size(scalv)) :: ermax
REAL(SP), DIMENSION(size(scalv),2*size(scalv)+1) :: s
REAL(SP), DIMENSION(size(scalv),size(indexv),size(y,2)+1) :: c
ne=assert_eq(size(scalv),size(indexv),size(y,2),'solvde: ne')
m=size(y,2)
k1=1
k2=m
nvars=ne+m
j1=1
j2=nb
j3=nb+1
j4=ne
j5=j4+j1
j6=j4+j2
j7=j4+j3
j8=j4+j4
j9=j8+j1
ic1=1
ic2=ne-nb
ic3=ic2+1
ic4=ne
jc1=1
jc2=ic3

do it=1,itmax
k=k1
Primary iteration loop.
Boundary conditions at first point.
call difeq(k,k1,k2,j9,ic3,ic4,indexv,s,y)  
call pinnv(ic3,ic4,j5,j9,jc1,k1,c,s)  
do k=k1+1,k2  
    kp=k-1  
call difeq(k,k1,k2,j9,ic1,ic4,indexv,s,y)  
call red(ic1,ic4,j5,j6,j7,j8,j9,ic3,jc1,jcf,k1,c,s)  
call pinnv(ic1,ic4,j3,j9,jc1,k1,c,s)  
end do  
k=k2+1

call difeq(k,k1,k2,j9,ic1,ic2,indexv,s,y)  
call red(ic1,ic2,j5,j6,j7,j8,j9,ic3,jc1,jcf,k2,c,s)  
call pinnv(ic1,ic2,j7,j9,jcf,k2+1,c,s)  
call bksub(ne,nb,jcf,k1,k2,c)  
end do  

Finite difference equations at all point pairs.

kp=k-1  
call difeq(k,k1,k2,j9,ic1,ic4,indexv,s,y)  
call red(ic1,ic4,j5,j6,j7,j8,j9,ic3,jc1,jcf,k1,c,s)  
call pinnv(ic1,ic4,j3,j9,jc1,k1,c,s)  
end do  
k=k2+1

call difeq(k,k1,k2,j9,ic1,ic2,indexv,s,y)  
call red(ic1,ic2,j5,j6,j7,j8,j9,ic3,jc1,jcf,k2,c,s)  
call pinnv(ic1,ic2,j7,j9,jcf,k2+1,c,s)  
call bksub(ne,nb,jcf,k1,k2,c)  
end do  

Finite boundary conditions.

call difeq(k,k1,k2,j9,ic1,ic2,indexv,s,y)  
call red(ic1,ic2,j5,j6,j7,j8,j9,ic3,jc1,jcf,k2,c,s)  
call pinnv(ic1,ic2,j7,j9,jcf,k2+1,c,s)  
end do  

k=k2+1

Call for Convergence Check and Average Error.
if (any(pscl(ie1:ie2) == 0.0)) &
call nrerror('singular matrix, row all 0 in pinvs')
pscl(ie1:ie2)=1.0_sp.pscl(ie1:ie2)
indxr(ie1:ie2)=0
do id=ie1,ie2
  piv=0.0
  do i=ie1,ie2
    Find pivot element.
    if (indxr(i) == 0) then
      jp=imaxloc(abs(s(i,je1:je2)))+je1-1
      big=abs(s(i,jp))
      if (big*pscl(i) > piv) then
        ipiv=i
        jpiv=jp
        piv=big*pscl(i)
      end if
    end if
  end do
  if (s(ipiv,jpiv) == 0.0) call nrerror('singular matrix in pinvs')
  indxr(ipiv)=jpiv
  Do in place reduction. Save column ordering.
  pivinv=1.0_sp/s(ipiv,jpiv)
  s(ipiv,je1:jsf)=s(ipiv,je1:jsf)*pivinv
  Normalize pivot row.
  s(ipiv,jpiv)=1.0
  do i=ie1,ie2
    Reduce nonpivot elements in column.
    if (indxr(i) /= jpiv .and. s(i,jpiv) /= 0.0) then
      s(i,je1:jsf)=s(i,je1:jsf)-s(i,jpiv)*s(ipiv,je1:jsf)
      s(i,jpiv)=0.0
    end if
  end do
end do
jcoff=jc1-js1
Sort and store unreduced coefficients.
icoff=ie1-jel
 END SUBROUTINE pinvs

SUBROUTINE red(iz1,iz2,jz1,jz2,jm1,jm2,jmf,ic1,jc1,jcf,kc,c,s)
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: iz1,iz2,jz1,jz2,jm1,jm2,jmf,ic1,jc1,jcf,kc
REAL(SP),DIMENSION(:, :, :),INTENT(INOUT) :: s
REAL(SP),DIMENSION(:, :, :),INTENT(IN) :: c
Reduce columns jm1-jm2,jmf of the s matrix, using previous results as stored in the c matrix.
Only columns jm1-jm2,jmf are affected by the prior results. red is used internally by solvde.
INTEGER(I4B) :: ic,l,loff
loff=jc1-jm1
ic=ic1
do j=jz1,jz2
  Loop over columns to be zeroed.
do l=jm1,jm2
    Loop over columns altered.
    s(iz1:iz2,l)=s(iz1:iz2,l)-s(iz1:iz2,j)*c(ic,l*loff,kc)
  end do
  Loop over rows.
do ic=ic+1
  Plus final element.
end do
END SUBROUTINE red
END SUBROUTINE solvde

km=imaxloc... See discussion of imaxloc on p. 1017.
*    *    *
MODULE sfroid_data
Communicates with difeq.
USE nrtype
INTEGER (I4B), PARAMETER :: M=41
INTEGER (I4B) :: mm, n
REAL (SP) :: anorm, c2, h
REAL (SP), DIMENSION (M) :: x
END MODULE sfroid_data

PROGRAM sfroid
USE nrtype; USE nrutil, ONLY : arth
USE nr, ONLY : plgnr, solvde
USE sfroid_data
IMPLICIT NONE
INTEGER (I4B), PARAMETER :: NE=3, NB=1

Sample program using solvde. Computes eigenvalues of spheroidal harmonics $S_{mn}(x;c)$ for $m \geq 0$ and $n \geq m$. In the program, $m$ is mm, $c^2$ is c2, and $\gamma$ of equation (17.4.20) is anorm.

INTEGER (I4B) :: itmax
INTEGER (I4B), DIMENSION (NE) :: indexv
REAL (SP) :: conv, slowc
REAL (SP), DIMENSION (M) :: deriv, fac1, fac2
REAL (SP), DIMENSION (NE) :: scalv
REAL (SP), DIMENSION (NE, M) :: y
itmax = 100
conv = 5.0e-6_sp
slowc = 1.0_sp
h = 1.0_sp/(M-1)
c2 = 0.0
write(*,*) 'ENTER M,N'
read(*,*) mm, n
indexv (1:3) = merge ( (/ 1, 2, 3 /), (/, 2, 1, 3 /), (mod(n+mm, 2) == 1) )
No interchanges necessary if n+mm is odd; otherwise interchange $y_1$ and $y_2$.
anorm = 1.0
Compute $\gamma$.
if (mm /= 0) then
  anorm = (-0.5_sp)**mm*product(&
             arth(n+1,1,mm)*arth(real(n,sp),-1.0_sp,mm)/arth(1,1,mm))
end if

x(1:M-1) = arth(0,1,M-1)*h
fac1(1:M-1) = 1.0_sp-x(1:M-1)**2
fac2(1:M-1) = fac1(1:M-1)**(-mm/2.0_sp)
y(1,1:M-1) = plgnr(n,mm,x(1:M-1))*fac2(1:M-1)
$P_{mn}^m$ from \S 6.8.
deriv(1:M-1) = (n-mm+1)*plgnr(n+1,mm,x(1:M-1))-(n+1)*&
             x(1:M-1)*plgnr(n,mm,x(1:M-1))/fac1(1:M-1)
Derivative of $P_{mn}^m$ from a recurrence relation.
y(2,1:M-1) = mm*x(1:M-1)+deriv(1:M-1)*fac2(1:M-1)
y(3,1:M-1) = n*(n+1)-mm*(mm+1)
x(M) = 1.0
Initial guess at $x = 1$ done separately.
y(1,M) = anorm
y(3,M) = n*(n+1)-mm*(mm+1)
y(2,M) = y(3,M)-c2*y(1,M)/(2.0_sp-mm+1.0_sp)
scalv (1:3) = (/ abs(anorm), max(abs(anorm),y(2,M)), max(1.0_sp,y(3,M)) /)
do
write (*,*) 'ENTER C**2 OR 999 TO END'
read (*,*) c2
if (c2 == 999.0) exit
call solvde(itmax, conv, slowc, scalv, indexv, NB, y)
write ('*',*) ' C**2 = ', c2, ' LAMBDA = ', y(3,1)+mm*(mm+1)
end do
Go back for another value of $c^2$.
END PROGRAM sfroid
This module functions just like a common block to communicate variables with difeq. The advantage of a module is that it allows complete specification of the variables.

\[
anorm = (-0.5_{sp})^{mm} \times \text{product}(...)\]

This statement computes equation (17.4.20) by direct multiplication.

```
SUBROUTINE difeq(k,k1,k2,jsf,is1,isf,indexv,s,y)
USE nrtypenone
USE sfroid_data
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: is1,isf,jsf,k,k1,k2
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: indexv
REAL(SP), DIMENSION(:,), INTENT(IN) :: s
REAL(SP), DIMENSION(:,), INTENT(OUT) :: y

Returns matrix \( s(i,j) \) for solve.

REAL(SP) :: temp,temp2
INTEGER(I4B), DIMENSION(3) :: indexv3
indexv3(1:3)=3+indexv(1:3)
if (k == k1) then
  Boundary condition at first point.
  if (mod(n+mm,2) == 1) then
    \( s(3, indexv3(1:3)) = (1.0_{sp}, 0.0_{sp}, 0.0_{sp}) \) \hspace{1cm} Equation (17.4.32).
    \( s(3, jsf) = y(1,1) \) \hspace{1cm} Equation (17.4.31).
  else
    \( s(3, indexv3(1:3)) = (0.0_{sp}, 1.0_{sp}, 0.0_{sp}) \) \hspace{1cm} Equation (17.4.32).
    \( s(3, jsf) = y(2,1) \) \hspace{1cm} Equation (17.4.31).
  end if
else if (k > k2) then
  Boundary conditions at last point.
  \( s(1, indexv3(1:3)) = (-y(3,M)-c2)/(2.0_{sp}*(mm+1.0_{sp})), 1.0_{sp}, -y(1, M)/(2.0_{sp}*(mm+1.0_{sp})) \) \hspace{1cm} Equation (17.4.35).
  \( s(1, jsf) = y(2,M)-(y(3,M)-c2)*y(1,M)/(2.0_{sp}*(mm+1.0_{sp})) \) \hspace{1cm} Equation (17.4.33).
  \( s(2, indexv3(1:3)) = (1.0_{sp}, 0.0_{sp}, 0.0_{sp}) \) \hspace{1cm} Equation (17.4.36).
  \( s(2, jsf) = y(1,M)-anorm \) \hspace{1cm} Equation (17.4.34).
else
  Interior point.
  \( s(1, indexv3(1:3)) = (-y(3,M)-c2)/(2.0_{sp}*(mm+1.0_{sp})), -1.0_{sp}, -0.5_{sp}*h, 0.0_{sp} \) \hspace{1cm} Equation (17.4.28).
  \( s(1, jsf) = y(3,k)+y(3,k-1)-c2*0.25_{sp}*(x(k)+x(k-1))**2 \)
  \( temp = h/(1.0_{sp}-(x(k)+x(k-1))**2*0.25_{sp}) \)
  \( temp2 = 0.5_{sp}*(y(3,k)+y(3,k-1)) - c2*0.25_{sp}*(x(k)+x(k-1))**2 \)
  \( s(2, indexv3(1:3)) = temp*temp2*0.5_{sp}, 1.0_{sp} - 0.5_{sp}*temp*(mm+1.0_{sp})*(x(k)+x(k-1)), 0.25_{sp}*
    temp*(y(1,k)+y(1,k-1)) \) \hspace{1cm} Equation (17.4.29).
  \( s(2, jsf) = y(2,k)-y(2,k-1)-temp*((x(k)+x(k-1))**2*0.5_{sp}+0.25_{sp}*
    temp*(y(1,k)+y(1,k-1))) \) \hspace{1cm} Equation (17.4.24).
  \( s(3, indexv3(1:3)) = (0.0_{sp}, 0.0_{sp}, 1.0_{sp}) \) \hspace{1cm} Equation (17.4.30).
  \( s(3, jsf) = y(3,k)-y(3,k-1) \) \hspace{1cm} Equation (17.4.27).
end if
END SUBROUTINE difeq
```
**Module sphoot_data**

Communicates with load, score, and deriv.

```fortran
MODULE sphoot_data
USE nrttype
INTEGER(14B) :: m,n
REAL(SP) :: c2,dx,gamma
END MODULE sphoot_data
```

**Module sphoot_caller**

Communicates with shoot.

```fortran
MODULE sphoot_caller
USE nrttype
INTEGER(14B) :: nvar
REAL(SP) :: x1,x2
END MODULE sphoot_caller
```

**Program sphoot**

Sample program using shoot. Computes eigenvalues of spheroidal harmonics \( S_{mn}(x;c) \) for \( m \geq 0 \) and \( n \geq m \). Be sure that routine funcv for newt is provided by shoot (§17.1).

```fortran
USE nrttype
USE nr, ONLY : newt
USE sphoot_data
USE sphoot_caller
IMPLICIT NONE
INTEGER(I4B), PARAMETER :: NV=3,N2=1
REAL(SP), DIMENSION(N2) :: v
LOGICAL(LGT) :: check
nvar=NV
Number of equations.
dx=1.0e-4_sp
Avoid evaluating derivatives exactly at \( x = -1 \).
do
    write(*,*) 'input m,n,c-squared (999 to end)'
    read(*,*) m,n,c2
    if (c2 == 999.0) exit
    if ((n < m) .or. (m < 0)) cycle
    gamma=(-0.5_sp)**m*product(&
        Compute \( \gamma \) of equation (17.4.20).
        arth(n+1,1,m)*(arth(real(n,sp),-1.0_sp,m)/arth(1,1,m)))
    v(1)=n*(n+1)-m*(m+1)+c2/2.0_sp
    Initial guess for eigenvalue.
x1=-1.0_sp+dx
    Set range of integration.
x2=0.0
    call newt(v,check)
    Find v that zeros function f in score.
    if (check) then
        write(*,*)'shoot failed; bad initial guess'
        exit
    else
        write(*,'(1x,t6,a)') 'mu(m,n)'
        write(*,'(1x,f12.6)') v(1)
    end if
end do
END PROGRAM sphoot
```

**Subroutine load(x1,v,y)**

```fortran
SUBROUTINE load(x1,v,y)
USE nrttype
USE sphoot_data
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x1
REAL(SP), DIMENSION(:,), INTENT(IN) :: v
REAL(SP), DIMENSION(:,), INTENT(OUT) :: y
Supplies starting values for integration at \( x = -1 + dx \).
REAL(SP) :: y1
y(3)=v(1)
y1=merge(gamma,-gamma, mod(n-m,2) == 0 )
y(2)=-(y(3)-c2)*y1/(2*(m+1))
y(1)=y1*y(2)*dx
END SUBROUTINE load
```
SUBROUTINE score(x2,y,f)
USE nrtype
USE sphoot_data
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x2
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: f
Tests whether boundary condition at $x = 0$ is satisfied.
f(1)=merge(y(2),y(1), mod(n-m,2) == 0 )
END SUBROUTINE score

MODULE sphoot_data...MODULE sphoot_caller These modules function
just like common blocks to communicate variables from sphoot to the
various subsidiary routines. The advantage of a module is that it allows
complete specification of the variables.

SUBROUTINE derivs(x,y,dydx)
USE nrtype
USE sphoot_data
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
Evaluates derivatives for odeint.
dydx(1)=y(2)
dydx(2)=(2.0_sp*x*(m+1.0_sp)*y(2)-(y(3)-c2*x*x)*y(1))/(1.0_sp-x*x)
dydx(3)=0.0
END SUBROUTINE derivs

* * *

MODULE sphfpt_data
Communicates with load1, load2, score, and deriv.
USE nrtype
INTEGER(I4B) :: m,n
REAL(SP) :: c2,dx, gamma
END MODULE sphfpt_data

MODULE sphfpt_caller
Communicates with shootf.
USE nrtype
INTEGER(I4B) :: nn2
REAL(SP) :: x1,x2,xf
END MODULE sphfpt_caller
PROGRAM sphfpt
Sample program using shootf. Computes eigenvalues of spheroidal harmonics $S_{mn}(x;c)$ for $m \geq 0$ and $n \geq m$. Be sure that routine funcv for newt is provided by shootf (§17.2).

The routine derivs is the same as for sphoot.

USE nrtype; USE nrutil, ONLY : arth
USE nr, ONLY : newt
USE sphfpt_data
USE sphfpt_caller
IMPLICIT NONE

INTEGER(14B), PARAMETER :: N1=2,N2=1,NTOT=N1+N2
REAL(SP), PARAMETER :: DXX=1.0e-4_sp
REAL(SP), DIMENSION(:), POINTER :: v1, v2
REAL(SP), DIMENSION(NTOT), TARGET :: v
LOGICAL(LGT) :: check

v1=>v(1:N2)
v2=>v(N2+1:NTOT)
nn2=N2

Avoid evaluating derivatives exactly at $x = \pm 1$. 
do
write(*,*) 'input m,n,c-squared (999 to end)' 
read(*,*) m,n,c2 
if (c2 == 999.0) exit 
if ((m < n) .or. (m < 0)) cycle 

gamma=(-0.5_sp)**m*product(&
 Compute $\gamma$ of equation (17.4.20).
arth(n+1,1,m)*(arth(real(n,sp),-1.0_sp,m)/arth(1,1,m))) 
v1(1)=n*(n+1)-m*(m+1)+c2/2.0_sp 
Initial guess for eigenvalue and function value.
v2(2)=v1(1) 
v2(1)=gamma*(1.0_sp-(v2(2)-c2)*dx/(2*(m+1))) 
x1=-1.0_sp+dx 
Set range of integration.
x2=1.0_sp-dx 
xf=0.0 
Fitting point.
call newt(v,check) 
Find $v$ that zeros function $f$ in score.
if (check) then
write(*,*) 'shootf failed; bad initial guess'
exit 
else
write(*,'(1x,t6,a)') 'mu(m,n)' 
write(*,'(1x,f12.6)') v1(1)
end if 
end do
END PROGRAM sphfpt

SUBROUTINE load1(x1,v1,y)
USE nrtype
USE sphfpt_data
IMPLICIT NONE

REAL(SP), INTENT(IN) :: x1 
REAL(SP), DIMENSION(:), INTENT(IN) :: v1 
REAL(SP), DIMENSION(:,), INTENT(OUT) :: y 
Supplies starting values for integration at $x = -1 + dx$. 
REAL(SP) :: y1 
y(3)=v1(1) 
y1=merge(gamma,-gamma,mod(n-m,2) == 0) 
y(2)=-(y(3)-c2)*y1/(2*(n+1)) 
y(1)=y1+y(2)*dx 
END SUBROUTINE load1
SUBROUTINE load2(x2,v2,y)
USE nrtype
USE sphfpt_data
IMPLICIT NONE
REAL(SP), INTENT(IN) :: x2
REAL(SP), DIMENSION(:), INTENT(IN) :: v2
REAL(SP), DIMENSION(:), INTENT(OUT) :: y
    Supplies starting values for integration at \( x = 1 - dx \).
    y(3)=v2(2)
y(1)=v2(1)
y(2)=(y(3)-c2)*y(1)/(2*(m+1))
END SUBROUTINE load2

SUBROUTINE score(xf,y,f)
USE nrtype
USE sphfpt_data
IMPLICIT NONE
REAL(SP), INTENT(IN) :: xf
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: f
    Tests whether solutions match at fitting point \( x = 0 \).
    f(1:3)=y(1:3)
END SUBROUTINE score

MODULE sphfpt_data...MODULE sphfpt_caller
These modules function just like common blocks to communicate variables from sphfpt to the various subsidiary routines. The advantage of a module is that it allows complete specification of the variables.
Chapter B18. Integral Equations and Inverse Theory

SUBROUTINE fred2(a,b,t,f,w,g,ak)
USE nrtype; USE nrutil, ONLY : assert_eq,unit_matrix
USE nr, ONLY : gauleg,lubksb,ludcmp
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP), DIMENSION(:,), INTENT(OUT) :: t,f,w
INTERFACE
FUNCTION g(t)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: t
REAL(SP), DIMENSION(size(t)) :: g
END FUNCTION g
FUNCTION ak(t,s)
USE nrtype
IMPLICIT NONE
REAL(SP), DIMENSION(:,), INTENT(IN) :: t,s
REAL(SP), DIMENSION(size(t),size(s)) :: ak
END FUNCTION ak
END INTERFACE
Solves a linear Fredholm equation of the second kind by \( N \)-point Gaussian quadrature. On input, a and b are the limits of integration. \( g \) and \( \text{ak} \) are user-supplied external functions. \( g \) returns \( g(t) \) as a vector of length \( N \) for a vector of \( N \) arguments, while \( \text{ak} \) returns \( \lambda K(t,s) \) as an \( N \times N \) matrix. The routine returns arrays \( t \) and \( f \) of length \( N \) containing the abscissas \( t_i \) of the Gaussian quadrature and the solution \( f \) at these abscissas. Also returned is the array \( w \) of length \( N \) of Gaussian weights for use with the Nystrom interpolation routine \text{fredin}.

INTEGER(I4B) :: n
INTEGER(I4B), DIMENSION(size(f)) :: indx
REAL(SP) :: d
REAL(SP), DIMENSION(size(f),size(f)) :: omk
n=assert_eq(size(f),size(t),size(w),'fred2')
call gauleg(a,b,t,w)
Replace \text{gauleg} with another routine if not using Gauss-Legendre quadrature.
Form \( \lambda = 1 - K \).
call unit_matrix(omk)
call lubksb(omk,indx,f)
call lubcmp(omk,indx,d)
call unit_matrix(omk)
\text{The unit_matrix routine in nrutil does exactly what its name suggests.}

END SUBROUTINE fred2
omk=omk-ak(t,t)*spread(w,dim=1,ncopies=n)  By now this idiom should be
second nature: the first column of ak gets multiplied by the first element of w,
and so on.

*  *  *

FUNCTION fredin(x,a,b,t,f,w,g,ak)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(SP), INTENT(IN) :: a,b
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,t,f,w
REAL(SP), DIMENSION(size(x)) :: fredin
INTERFACE
  FUNCTION g(t)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), DIMENSION(:,), INTENT(IN) :: t
    REAL(SP), DIMENSION(size(t)) :: g
    END FUNCTION g
  FUNCTION ak(t,s)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), DIMENSION(:,), INTENT(IN) :: t,s
    REAL(SP), DIMENSION(size(t),size(s)) :: ak
    END FUNCTION ak
END INTERFACE
Input are arrays t and w of length N containing the abscissas and weights of the N-point
Gaussian quadrature, and the solution array f of length N from fred2. The function
fredin returns the array of values of f at an array of points x using the Nyström interpolation formula. On input, a and b are the limits of integration. g and ak are user-supplied
external functions. g returns g(t) as a vector of length N for a vector of N arguments,
while ak returns $\lambda K(t,s)$ as an $N \times N$ matrix.
INTEGER(I4B) :: n
n=assert_eq(size(f),size(t),size(w),'fredin')
fredin=g(x)+matmul(ak(x,t),w*f)
END FUNCTION fredin

Fredin=g(x)+matmul...  Fortran 90 allows very concise coding here,
which also happens to be much closer to the mathematical formulation
than the loops required in Fortran 77.

*  *  *

SUBROUTINE voltra(t0,h,t,f,g,ak)
USE nrtype; USE nrutil, ONLY : array_copy,assert_eq,unit_matrix
USE nr, ONLY : lubksb,ludcmp
IMPLICIT NONE
REAL(SP), INTENT(IN) :: t0,h
REAL(SP), DIMENSION(:,), INTENT(OUT) :: t
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: f
INTERFACE
  FUNCTION g(t)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), INTENT(IN) :: t
    REAL(SP), DIMENSION(:,), POINTER :: g
    END FUNCTION g
  FUNCTION ak(t,s)
    USE nrtype
    IMPLICIT NONE
    REAL(SP), DIMENSION(:,), POINTER :: g
    END FUNCTION ak
END INTERFACE
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: t,s
REAL(SP), DIMENSION(:,,:), POINTER :: ak
END FUNCTION ak
END INTERFACE

Solves a set of $M$ linear Volterra equations of the second kind using the extended trapezoidal rule. On input, $t_0$ is the starting point of the integration. The routine takes $N - 1$ steps of size $h$ and returns the abscissas in $t$, a vector of length $N$. The solution at these points is returned in the $M \times N$ matrix $f$. $g$ is a user-supplied external function that returns a pointer to the $M$-dimensional vector of functions $g_k(t)$, while $ak$ is another user-supplied external function that returns a pointer to the $M \times M$ matrix $K(t,s)$.

INTEGER(I4B) :: i,j,n,ncop,nerr,m
INTEGER(I4B), DIMENSION(size(f,1)) :: indx
REAL(SP) :: d
REAL(SP), DIMENSION(size(f,1),size(f,1)) :: a

n=assert_eq(size(f,2),size(t),'voltra: n')
t(1)=t0
initialize.
call array_copy(g(t(1)),f(:,1),ncop,nerr)
m=assert_eq(size(f,1),ncop,ncop+nerr,'voltra: m')
take a step $h$.
do i=2,n
  t(i)=t(i-1)+h
  b=g(t(i))+0.5_sp*h*matmul(ak(t(i),t(1)),f(:,1))
  accumulate right-hand side
  of linear equations in b.
do j=2,i-1
    b=b+h*matmul(ak(t(i),t(j)),f(:,j))
  end do
left-hand side goes in matrix $a$.
call unit_matrix(a)
a=a-0.5_sp*h*ak(t(i),t(i))
call ludcmp(a,indx,d)
left-hand side goes in matrix $a$.
call lubksb(a,indx,b)
solve linear equations.
f(:,i)=b(:)
end do
end SUBROUTINE voltra

FUNCTION g(t) REAL(SP), DIMENSION(:,), POINTER :: g
The routine $\text{voltra}$ requires an argument that is a function returning a vector, but we don't know the dimension of the vector at compile time. The solution is to make the function return a pointer to the vector. This is not the same thing as a pointer to a function, which is not allowed in Fortran 90. When you use the pointer in the routine, Fortran 90 figures out from the context that you want the vector of values, so the code remains highly readable. Similarly, the argument $ak$ is a function returning a pointer to a matrix.

The coding of the user-supplied functions $g$ and $ak$ deserves some comment: functions returning pointers to arrays are potential memory leaks if the arrays are allocated dynamically in the functions. Here the user knows in advance the dimension of the problem, and so there is no need to use dynamical allocation in the functions. For example, in a two-dimensional problem, you can code $g$ as follows:

FUNCTION g(t)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: t
REAL(SP), DIMENSION(:,), POINTER :: g
REAL(SP), DIMENSION(2), TARGET, SAVE :: gg
g=>gg
g(1)=...
g(2)=...
END FUNCTION g
and similarly for \(a_k\).

Suppose, however, we coded \(g\) with dynamical allocation:

```fortran
FUNCTION g(t)
USE nrtype
IMPLICIT NONE
REAL(SP), INTENT(IN) :: t
REAL(SP), DIMENSION(:), POINTER :: g
allocate(g(2))
g(1)=...
g(2)=...
END FUNCTION g
```

Now \(g\) never gets deallocated; each time we call the function fresh memory gets consumed. If you have a problem that really does require dynamical allocation in a pointer function, you have to be sure to deallocate the pointer in the calling routine. In `voltra`, for example, we would declare pointers `gtemp` and `aktemp`.

Then instead of writing simply

\[
b = g(t(i)) + \ldots
\]

we would write

\[
gtemp = g(t(i))
b = gtemp + \ldots
deallocate(gtemp)
\]

and similarly for each pointer function invocation.

The routine would work if we replaced this statement with simply \(f(:,i) = g(t(i))\). The purpose of using `array_copy` from `nrutil` is that we can check that \(f\) and \(g\) have consistent dimensions with a call to `assert_eq`.

\[
\star \quad \star \quad \star
\]

```fortran
FUNCTION wweights(n,h,kermom)
USE nrtype; USE nrutil, ONLY: geop
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: h
REAL(SP), DIMENSION(n) :: wweights
INTERFACE
FUNCTION kermom(y,m)
USE nrtype
IMPLICIT NONE
REAL(DP), INTENT(IN) :: y
INTEGER(I4B), INTENT(IN) :: m
REAL(DP), DIMENSION(m) :: kermom
END FUNCTION kermom
END INTERFACE
RETURN wweights(1:n)
weights for the \(n\)-point equal-interval quadrature from 0 to \((n-1)h\) of a function \(f(x)\) times an arbitrary (possibly singular) weight function \(w(x)\) whose indefinite-integral moments \(F_n(y)\) are provided by the user-supplied function \(kermom\).
```
\begin{verbatim}
if (n >= 4) then
  Use highest available order.
b=0.0
  For another problem, you might change this lower limit.
do j=1,n-3
  c=j-1
  a=b
  This is called $k$ in equation (18.3.5).
  Set upper and lower limits for this step.
  b=a+hh
  if (j == n-3) b=(n-1)*hh
  Last interval: go all the way to end.
  wnew(1:4)=kermom(b,4)
  w(1:4)=(wnew(1:4)-wold(1:4))*geop(1.0_dp,hi,4)  \hspace{1cm} Equation (18.3.4).
  wghts(j:j+3)=wghts(j:j+3)+(/(\(c+1.0_dp)*(c+2.0_dp)*(c+3.0_dp)*w(1)&
                -(11.0_dp+12.0_dp*c+3.0_dp)*w(2)&
                +3.0_dp*(c+2.0_dp)*w(3)-w(4))/6.0_dp,&
                (-c*(c+2.0_dp)*(c+3.0_dp)*w(1)&
                  +(6.0_dp+10.0_dp*c+3.0_dp)*w(2)&
                  -(3.0_dp+c+5.0_dp)*w(3)+w(4))*0.50_dp,&
                  (c*(c+1.0_dp)*(c+3.0_dp)*w(1)&
                    -(3.0_dp+c+4.0_dp)*w(2)&
                    +3.0_dp*(c+1.0_dp)*w(3)-w(4))*0.50_dp,&
                    (-c*(c+1.0_dp)*(c+2.0_dp)*w(1)&
                      +2.0_dp*c+6.0_dp*c+3.0_dp)*w(2)&
                      -3.0_dp*(c+1.0_dp)*w(3)+w(4))/6.0_dp /)
  wold(1:4)=wnew(1:4)
  Reset lower limits for moments.
end do
else if (n == 3) then
  Lower-order cases; not recommended.
  wnew(1:3)=kermom(hh+hh,3)
  w(1:3)= (/ wnew(1)-wold(1), hi*(wnew(2)-wold(2)),&
        hi**2*(wnew(3)-wold(3)) /)
  wghts(1:3)= (/ w(1)-1.50_dp*w(2)+0.50_dp*w(3),&
                  2.0_dp*w(2)-w(3), 0.50_dp*(w(3)-w(2)) /)
else if (n == 2) then
  wnew(1:2)=kermom(hh,2)
  wghts(2)=hi*(wnew(2)-wold(2))
  wghts(1)=wnew(1)-wold(1)-wghts(2)
end if
END FUNCTION wghts

** ** **

\textit{MODULE kermom_info}
\begin{verbatim}
USE nrtype
REAL(DP) :: kermom_x
END MODULE kermom_info
\end{verbatim}
\begin{verbatim}
FUNCTION kermom(y,m)
   USE nrtype
   USE kermom_info
   IMPLICIT NONE
   REAL(DP), INTENT(IN) :: y
   INTEGER(14B), INTENT(IN) :: m
   REAL(DP) :: x,d,df,clog,x2,x3,x4
   x=kermom_x
   We can take x as the lower limit of integration. Thus, we return the moment integrals either purely to the left or purely to the right of the diagonal.
   if (y >= x) then
     d=y-x
     df=2.0_dp*sqrt(d)*d
     kermom(1:4) = ( / df*sqrt(d), &
                   f(x/3.0_dp+d/5.0_dp), &
                   f(x/3.0_dp+d/5.0_dp), &
                   f(x/3.0_dp+d/5.0_dp) /)
   end if
END FUNCTION kermom
\end{verbatim}
\end{verbatim}
df*(((x/3.0_dp + 0.4_dp*d)*x + d**2/7.0_dp),
 df*(((x/3.0_dp + 0.6_dp*d)*x + 3.0_dp*d**2/7.0_dp)*x&
 + d**3/9.0_dp) /)
else
x2=x**2
x3=x2*x
x4=x2*x2
d=x-y
clog=log(d)
kermom(1:4) = (/ d*(clog-1.0_dp),&
 -0.25_dp*(3.0_dp*x+y-2.0_dp*clog*(x+y))*d,&
 (-11.0_dp*x3+y*(6.0_dp*x2+y*(3.0_dp*x+2.0_dp*y))&
 +6.0_dp*clog*(x3-y**3))/18.0_dp,&
 (-25.0_dp*x4+y*(12.0_dp*x3+y*(6.0_dp*x2+y*&
 (4.0_dp*x*x3-0.0_dp*y)))+12.0_dp*clog*(x4-y**4))/48.0_dp /)
end if
END FUNCTION kermom

MODULE kermom_info
This module functions just like a common block to
share the variable kermom_x with the routine quadmx.

SUBROUTINE quadmx(a)
USE nrtype; USE nrutil, ONLY: arth,assert_eq,diagadd,outerprod
USE nr, ONLY: wwghts,kermom
USE kermom_info
IMPLICIT NONE
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: a
Constructs in the N X N array a the quadrature matrix for an example Fredholm equation of
the second kind. The nonsingular part of the kernel is computed within this routine, while
the quadrature weights that integrate the singular part of the kernel are obtained via calls
to wwghts. An external routine kermom, which supplies indefinite-integral moments of the
singular part of the kernel, is passed to wwghts.
INTEGER(I4B) :: j,n
REAL(SP) :: h,x
REAL(SP), DIMENSION(size(a,1)) :: wt
n=assert_eq(size(a,1),size(a,2),'quadmx')
h=PI/(n-1)
do j=1,n
x=(j-1)*h
kermom_x=x
wt(:)=wwghts(n,h,kermom)  \* Put x in the module kermom_info for use by kermom.
a(j,:)=wt(:) \* Part of nonsingular kernel.
end do
wt(:)=cos(arth(0,1,n)*h)
a(:,:)=a(:,:)*outerprod(wt(:),wt(:)) \* Put together all the pieces of the kernel.
call diagadd(a,1.0_sp) \* Since equation of the second kind, there is diagonal
END SUBROUTINE quadmx

call diagadd... See discussion of diagadd after hqr on p. 1234.
PROGRAM fredex
USE nrtype; USE nrutil, ONLY : arth
USE nr, ONLY : quadmx,ludcmp,lubksb
IMPLICIT NONE
INTEGER(I4B), PARAMETER :: N=40
INTEGER(I4B) :: j
INTEGER(I4B), DIMENSION(N) :: indx
REAL(SP) :: d
REAL(SP), DIMENSION(N) :: g,x
REAL(SP), DIMENSION(N,N) :: a

This sample program shows how to solve a Fredholm equation of the second kind using the product Nystrom method and a quadrature rule especially constructed for a particular, singular, kernel.

Parameter: N is the size of the grid.
call quadmx(a) Make the quadrature matrix; all the action is here.
call ludcmp(a,indx,d) Decompose the matrix.
g(:,)=sin(x(:,)) Construct the right-hand side, here sin x.
call lubksb(a,indx,g) Backsubstitute.
do j=1,N Write out the solution.
write (*,*) j,x(j),g(j)
end do
write (*,*) 'normal completion'
END PROGRAM fredex
Chapter B19. Partial Differential Equations

SUBROUTINE sor(a,b,c,d,e,f,u,rjac)
USE nrtype; USE nrutil, ONLY: assert_eq,nrerror
IMPLICIT NONE
REAL(DP), DIMENSION(:,::), INTENT(IN) :: a,b,c,d,e,f
REAL(DP), DIMENSION(:,::), INTENT(INOUT) :: u
REAL(DP), INTENT(IN) :: rjac
INTEGER(14B), PARAMETER :: MAXITS=1000
REAL(DP), PARAMETER :: EPS=1.0e-5_dp

Successive overrelaxation solution of equation (19.5.25) with Chebyshev acceleration. a, b, c, d, e, and f are input as the coefficients of the equation, each dimensioned to the grid size $J \times J$. u is input as the initial guess to the solution, usually zero, and returns with the final value. rjac is input as the spectral radius of the Jacobi iteration, or an estimate of it. Double precision is a good idea for $J$ bigger than about 25.

REAL(DP), DIMENSION(size(a,1),size(a,1)) :: resid
INTEGER(14B) :: jmax,jm1,jm2,jm3,n
REAL(DP) :: anorm,anormf,omega
jmax=assert_eq((/size(a,1),size(a,2),size(b,1),size(b,2), &
size(c,1),size(c,2),size(d,1),size(d,2),size(e,1), &
size(e,2),size(f,1),size(f,2),size(u,1),size(u,2)/),'sor')
jm1=jmax-1
jm2=jmax-2
jm3=jmax-3
anormf=sum(abs(f(2:jm1,2:jm1)))

Compute initial norm of residual and terminate iteration when norm has been reduced by a factor EPS. This computation assumes initial u is zero.
omega=1.0_dp

do n=1,MAXITS
  First do the even-even and odd-odd squares of the grid, i.e., the red squares of the checkerboard:
  resid(2:jm1:2,2:jm1:2)=a(2:jm1:2,2:jm1:2)*u(3:jmax:2,2:jm1:2)+&
  b(2:jm1:2,2:jm1:2)*u(1:jm2:2,2:jm1:2)+&
  c(2:jm1:2,2:jm1:2)*u(2:jm1:2,3:jmax:2)+&
  d(2:jm1:2,2:jm1:2)*u(2:jm1:2,1:jm2:2)+&
  e(2:jm1:2,2:jm1:2)*u(2:jm1:2,2:jm1:2)-f(2:jm1:2,2:jm1:2)
  u(2:jm1:2,2:jm1:2)=u(2:jm1:2,2:jm1:2)-omega*&
  resid(2:jm1:2,2:jm1:2)/e(2:jm1:2,2:jm1:2)
  resid(3:jm2:2,2:jm1:2)=a(3:jm2:2,2:jm1:2)*u(4:jm1:2,2:jm1:2)+&
  b(3:jm2:2,2:jm1:2)*u(2:jm3:2,2:jm1:2)+&
  c(3:jm2:2,2:jm1:2)*u(3:jm2:2,4:jm1:2)+&
  d(3:jm2:2,2:jm1:2)*u(3:jm2:2,2:jm3:2)+&
  e(3:jm2:2,2:jm1:2)*u(3:jm2:2,2:jm1:2)+e(3:jm2:2,2:jm1:2)-f(3:jm2:2,2:jm1:2)
  u(3:jm2:2,2:jm1:2)=u(3:jm2:2,2:jm1:2)-omega*&
  resid(3:jm2:2,2:jm1:2)/e(3:jm2:2,2:jm1:2)

  Now do even-odd and odd-even squares of the grid, i.e., the black squares of the checkerboard:
  resid(3:jm2:2,3:jm1:2)=a(3:jm2:2,3:jm1:2)*u(4:jm1:2,3:jm2:2)+&
  b(3:jm2:2,3:jm1:2)*u(2:jm3:2,3:jm2:2)+&
  c(3:jm2:2,3:jm1:2)*u(3:jm2:2,4:jm1:2)+&
  d(3:jm2:2,3:jm1:2)*u(3:jm2:2,2:jm3:2)+&
  e(3:jm2:2,3:jm1:2)*u(3:jm2:2,2:jm1:2)+e(3:jm2:2,3:jm1:2)+e(3:jm2:2,3:jm1:2)-f(3:jm2:2,3:jm1:2)
  u(3:jm2:2,3:jm1:2)=u(3:jm2:2,3:jm1:2)-omega*&
  resid(3:jm2:2,3:jm1:2)/e(3:jm2:2,3:jm1:2)

omega=merge(1.0_dp/(1.0_dp-0.5_dp*rjac**2), &
1.0_dp/(1.0_dp-0.25_dp*rjac**2*omega), n == 1)

enddo
Red-black iterative schemes like the one used in sor are easily parallelizable. Updating the red grid points requires information only from the black grid points, so they can all be updated independently. Similarly the black grid points can all be updated independently. Since nearest neighbors are involved in the updating, communication costs can be kept to a minimum.

There are several possibilities for coding the red-black iteration in a data parallel way using only Fortran 90 and no parallel language extensions. One way is to define an $N \times N$ logical mask red that is true on the red grid points and false on the black. Then each iteration consists of an update governed by a where(red)...end where block and a where(.not. red)...end where block. We have chosen a more direct coding that avoids the need for storage of the array red. The red update corresponds to the even-even and odd-odd grid points, the black to the even-odd and odd-even points. We can code each of these four cases directly with array sections, as in the routine above.

The array section notation used in sor is rather dense and hard to read. We could use pointer aliases to try to simplify things, but since each array section is different, we end up merely giving names to each term that was there all along. Pointer aliases do help if we code sor using a logical mask. Since there may be machines on which this version is faster, and since it is of some pedagogic interest, we give the alternative code:

```
SUBROUTINE sor_mask(a,b,c,d,e,f,u,rjac)
USE nrtype; USE nrutil, ONLY: assert_eq,nrerror
IMPLICIT NONE
REAL(DP), DIMENSION(::,:), TARGET, INTENT(IN) :: a,b,c,d,e,f
REAL(DP), DIMENSION(:,:), TARGET, INTENT(INOUT) :: u
REAL(DP), INTENT(IN) :: rjac
INTEGER(I4B), PARAMETER :: MAXITS=1000
REAL(DP), PARAMETER :: EPS=1.0e-5_dp
REAL(DP), DIMENSION(::,:), ALLOCATABLE :: resid
REAL(DP), DIMENSION(:,:), POINTER :: u_int,u_down,u_up,u_left,&
    u_right,a_int,b_int,c_int,d_int,e_int,f_int
INTEGER(I4B) :: jmax,jm1,jm2,jm3,jm4,n
REAL(DP) anorm,anormf,omega
LOGICAL, DIMENSION(:,,:), ALLOCATABLE :: red
jmax=assert_eq((/size(a,1),size(a,2),size(b,1),size(b,2), &
```

---

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---
size(c,1),size(c,2),size(d,1),size(d,2),size(e,1), &
size(e,2),size(f,1),size(f,2),size(u,1),size(u,2)/),'sor')
jm1=jmax-1
jm2=jmax-2
jm3=jmax-3
allocate(resid(jm2,jm2),red(jm2,jm2)) \quad \text{Interior is (jmax – 2) × (jmax – 2).}
red=.false.
red(1:jm2:2,1:jm2:2)=true.
red(2:jm3:2,2:jm3:2)=true.

u_int=>u(2:jm3,2:jm1)
ul_{}\text{down}=>u(3:jmax,2:jm1)
ul_{}\text{up}=>u(1:jm2,2:jm1)
u_{}\text{left}=>u(2:jm1,1:jm2)
u_{}\text{right}=>u(2:jm1,3:jmax)
a_{}\text{int}=>a(2:jm1,2:jm1)
b_{}\text{int}=>b(2:jm1,2:jm1)
c_{}\text{int}=>c(2:jm1,2:jm1)
d_{}\text{int}=>d(2:jm1,2:jm1)
e_{}\text{int}=>e(2:jm1,2:jm1)
f_{}\text{int}=>f(2:jm1,2:jm1)
anormf=sum(abs(f_{}\text{int}))
omega=1.0
do n=1,MAXITS
where(red)
\quad \text{resid}={a_{}\text{int}}*u_{}\text{down}+{b_{}\text{int}}*u_{}\text{up}+{c_{}\text{int}}*u_{}\text{right}+&
\quad {d_{}\text{int}}*u_{}\text{left}+{e_{}\text{int}}*u_{}\text{int}+f_{}\text{int}
\quad u_{}\text{int}=u_{}\text{int}-\omega*\text{resid}/e_{}\text{int}
end where
\quad omega=merge(1.0_dp/(1.0_dp-0.5_dp*rjac**2), &
\quad 1.0_dp/(1.0_dp-0.25_dp*rjac**2*omega), n == 1)
where(.not.red)
\quad resid={a_{}\text{int}}*u_{}\text{down}+{b_{}\text{int}}*u_{}\text{up}+{c_{}\text{int}}*u_{}\text{right}+&
\quad {d_{}\text{int}}*u_{}\text{left}+{e_{}\text{int}}*u_{}\text{int}+f_{}\text{int}
\quad u_{}\text{int}=u_{}\text{int}+\omega*\text{resid}/e_{}\text{int}
end where
\quad omega=1.0_dp/(1.0_dp-0.25_dp*rjac**2*omega)
anorm=sum(abs(resid))
if(anorm < EPS*anormf)exit
end do
deallocate(resid,red)
if (n > MAXITS) call nrerror('MAXITS exceeded in sor')
END SUBROUTINE sor_mask

SUBROUTINE mglin(u,ncycle)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : interp,rstrct,slvsml
IMPLICIT NONE
REAL(DP), DIMENSION(:,,:), INTENT(INOUT) :: u
INTEGER(4B), INTENT(IN) :: ncycle
INTEGER(4B) :: j,jcycle,n,ng,ngrid,nn
TYPE ptr2d
\quad \text{Define a type so we can have an array of pointers}
\quad \text{to arrays of grid variables.}
\quad \text{REAL(DP), POINTER :: a(:, :)}
END TYPE ptr2d
TYPE(ptr2d), ALLOCATABLE :: rho(:)
REAL(DP), DIMENSION(:,::), POINTER :: uj, uj_1
n=assert_eq(size(u,1),size(u,2),'mglin')
ng=int((n-1)/log(2.0))
if (n /= 2**ng+1) call nrerror('n-1 must be a power of 2 in mglin')
allocate(rho(ng))
n=n
ngrid=ng
allocate(rho(nggrid)%a(nn,nn))
rho(nggrid)%a=u
allocate(rho(nggrid)%a(nn,nn))
rho(nggrid)%a=rstrct(rho(nggrid+1)%a)
end do
nn=3
allocate(uj(nn,nn))
call slvsml(uj,rho(1)%a)
do j=2,ng
nn=2*nn-1
uj_1=uj
allocate(uj(nn,nn))
uj=interp(uj_1)
deallocate(uj_1)
do jcycle=1,ncycle
V-cycle loop.
call mg(j,uj,rho(j)%a)
end do
end do
u=uj
dallocate(rho)
deallocate(uj)
deallocate(rho(nggrid)%a)
deallocate(rho(nggrid))
deallocate(uj)
CONTAINS
RECURSIVE SUBROUTINE mg(j,u,rhs)
USE nrtype
USE nr, ONLY: interp, relax, resid, rstrct, slvsml
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: j
REAL(DP), DIMENSION(:,::), INTENT(INOUT) :: u
REAL(DP), DIMENSION(:,::), INTENT(IN) :: rhs
INTEGER(I4B), PARAMETER :: NPRE=1,NPOST=1

Recursive multigrid iteration. On input, j is the current level, u is the current value of the solution, and rhs is the right-hand side. On output u contains the improved solution at the current level.

Parameters: NPRE and NPOST are the number of relaxation sweeps before and after the coarse-grid correction is computed.

INTEGER(I4B) :: jpost, jpre
REAL(DP), DIMENSION((size(u,1)+1)/2,(size(u,1)+1)/2) :: res, v
if (j == 1) then
Bottom of V: Solve on coarsest grid.
call slvsml(u,rhs)
else
On downward stroke of the V.
do jpre=1,NPRE
Pre-smoothing.
call relax(u,rhs)
end do
res=rstrct(resid(u,rhs))
v=0.0
Restriction of the residual is the next r.h.s.
call mg(j-1,v,res)
Zero for initial guess in next relaxation.
u=u+interp(v)
On upward stroke of V.
do jpost=1,NPOST
Post-smoothing.
call relax(u,rhs)
end do
end if
END SUBROUTINE mg
The Fortran 90 version of mglin (and of mgfas below) is quite different from the Fortran 77 version, although the algorithm is identical. First, we use a recursive implementation. This makes the code much more transparent. It also makes the memory management much better: we simply define the new arrays res and v as automatic arrays of the appropriate dimension on each recursive call to a coarser level. And a third benefit is that it is trivial to change the code to increase the number of multigrid iterations done at level \( j - 1 \) by each iteration at level \( j \), i.e., to set the quantity \( \gamma \) in §19.6 to a value greater than one. (Recall that \( \gamma = 1 \) as chosen in mglin gives V-cycles, \( \gamma = 2 \) gives W-cycles.) Simply enclose the recursive call in a do-loop:

```fortran
    do i=1,merge(gamma,1,j /= 2)
        call mg(j-1,v,res)
    end do
```

The merge expression ensures that there is no more than one call to the coarsest level, where the problem is solved exactly.

A second improvement in the Fortran 90 version is to make the procedures resid, interp, and rstrct functions instead of subroutines. This allows us to code the algorithm exactly as written mathematically.

```fortran
    TYPE(ptr2d... The right-hand-side quantity \( \rho \) is supplied initially on the finest grid in the argument \( u \). It has to be defined on the coarser grids by restriction, and then supplied as the right-hand side to mg in the nested iteration loop. This loop starts at the coarsest level and progresses up to the finest level. We thus need a data structure to store \( \rho \) on all the grid levels. A convenient way to implement this in Fortran 90 is to define a type ptr2d, a pointer to a two-dimensional array \( a \) that represents a grid. (In three dimensions, \( a \) would of course be three-dimensional.) We then declare the variable \( \rho \) as an allocatable array of type ptr2d:

```fortran
    TYPE(ptr2d), ALLOCATABLE :: rho(:)
```

Next we allocate storage for \( \rho \) on each level. The number of levels or grids, \( ng \), is known only at run time:

```fortran
    allocate(rho(ng))
```

Then we allocate storage as needed on particular sized grids. For example,

```fortran
    allocate(rho(ngrid)%a(nn,nn))
```

allocates an \( nn \times nn \) grid for \( \rho \) on grid number \( ngrid \).

The various subsidiary routines of mglin such as rstrct and interp are written to accept two-dimensional arrays as arguments. With the data structure we’ve employed, using these routines is simple. For example,

```fortran
    rho(ngrid)%a=rstrct(rho(ngrid+1)%a)
```

will restrict \( \rho \) from the grid \( ngrid+1 \) to the grid \( ngrid \). The statement is even more readable if we mentally ignore the \%a that is tagged onto each variable. (If
we actually did omit \( \frac{1}{2}a \) in the code, the compiler would think we meant the array of type ptr2d instead of the grid array.)

Note that while Fortran 90 does not allow you to declare an array of pointers directly, you can achieve the same effect by declaring your own type, as we have done with ptr2d in this example.

FUNCTION rstrct(uf)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(DP), DIMENSION(:,,:), INTENT(IN) :: uf
REAL(DP), DIMENSION((size(uf,1)+1)/2,(size(uf,1)+1)/2) :: rstrct

Half-weighting restriction. If \( N_c \) is the coarse-grid dimension, the fine-grid solution is input in the \((2N_c - 1) \times (2N_c - 1)\) array \( uf \), the coarse-grid solution is returned in the \( N_c \times N_c \) array \( rstrct \).

INTEGER(14B) :: nc,nf
nf=assert_eq(size(uf,1),size(uf,2),'rstrct')
nc=(nf+1)/2
rstrct(2:nc-1,2:nc-1)=0.5_dp*uf(3:nf-2:2,3:nf-2:2)+0.125_dp*(
   uf(4:nf-1:2,3:nf-2:2)+uf(2:nf-3:2,3:nf-2:2)+
   uf(3:nf-2:2,4:nf-1:2)+uf(3:nf-2:2,2:nf-3:2))
rstrct(1:nc,1)=uf(1:nf:2,1)
Boundary points.
rstrct(1:nc,nc)=uf(1:nf:2,nf)
rstrct(1,1:nc)=uf(1,1:nf:2)
rstrct(nc,1:nc)=uf(nf,1:nf:2)
END FUNCTION rstrct

FUNCTION interp(uc)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(DP), DIMENSION(:,,:), INTENT(IN) :: uc
REAL(DP), DIMENSION(2*size(uc,1)-1,2*size(uc,1)-1) :: interp

Coarse-to-fine prolongation by bilinear interpolation. If \( N_f \) is the fine-grid dimension and \( N_c \) the coarse-grid dimension, then \( N_f = 2N_c - 1 \). The coarse-grid solution is input as \( uc \), the fine-grid solution is returned in \( interp \).

INTEGER(14B) :: nc,nf
nc=assert_eq(size(uc,1),size(uc,2),'interp')
nc=2*nc-1
interp(1:nf,1:nf)=uc(1:nc,1:nc)
Do elements that are copies.
interp(2:nf-1:2,1:nf:2)=0.5_dp*(interp(3:nf:2,1:nf:2)+
   interp(1:nf,3:nf:2)+interp(1:nf,1:nf-2:2))
Do odd-numbered columns, interpolating vertically.
interp(1:nf,2:nf-1:2)=0.5_dp*(interp(1:nf,3:nf:2)+interp(1:nf,1:nf-2:2))
Do even-numbered columns, interpolating horizontally.
END FUNCTION interp

SUBROUTINE slvsml(u,rhs)
USE nrtype
IMPLICIT NONE
REAL(DP), DIMENSION(3,3), INTENT(OUT) :: u
REAL(DP), DIMENSION(3,3), INTENT(IN) :: rhs

Solution of the model problem on the coarsest grid, where \( h = \frac{1}{2} \). The right-hand side is input in \( rhs(1:3,1:3) \) and the solution is returned in \( u(1:3,1:3) \).

REAL(DP) :: h
u=0.0
h=0.5_dp
u(2,2)=-h*h*rhs(2,2)/4.0_dp
END SUBROUTINE slvsml
SUBROUTINE relax(u,rhs)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(DP), DIMENSION(:,::), INTENT(INOUT) :: u
REAL(DP), DIMENSION(:,::), INTENT(IN) :: rhs

Red-black Gauss-Seidel relaxation for model problem. The current value of the solution \( u \) is updated, using the right-hand-side function \( rhs \). \( u \) and \( rhs \) are square arrays of the same odd dimension.

INTEGER(I4B) :: n
REAL(DP) :: h,h2
n=assert_eq(size(u,1),size(u,2),size(rhs,1),size(rhs,2),'relax')
h=1.0_dp/(n-1)
h2=h*h

First do the even-even and odd-odd squares of the grid, i.e., the red squares of the checkerboard:

\[
\begin{align*}
  u_{2:n-1,2:n-1} &= 0.25_dp\cdot(u_{3:n,2:n-1}+u_{1:n-2,2:n-1}+u_{2:n-1,3:n}+u_{2:n-1,1:n-2}-h^2\cdot rhs_{2:n-1,2:n-1}) \\
  u_{3:n-2,3:n-2} &= 0.25_dp\cdot(u_{4:n-1,3:n-2}+u_{2:n-3,3:n-2}+u_{3:n-2,4:n-1}+u_{3:n-2,2:n-3}-h^2\cdot rhs_{3:n-2,3:n-2})
\end{align*}
\]

Now do even-odd and odd-even squares of the grid, i.e., the black squares of the checkerboard:

\[
\begin{align*}
  u_{3:n-2,2:n-1} &= 0.25_dp\cdot(u_{4:n-1,2:n-1}+u_{2:n-3,2:n-1}+u_{3:n-2,3:n}+u_{3:n-2,1:n-2}-h^2\cdot rhs_{3:n-2,2:n-1}) \\
  u_{2:n-1,3:n-2} &= 0.25_dp\cdot(u_{3:n,3:n-2}+u_{1:n-2,3:n-2}+u_{2:n-1,4:n-1}+u_{2:n-1,2:n-3}-h^2\cdot rhs_{2:n-1,3:n-2})
\end{align*}
\]

END SUBROUTINE relax

FUNCTION resid(u,rhs)
USE nrtype; USE nrutil, ONLY : assert_eq
IMPLICIT NONE
REAL(DP), DIMENSION(:,::) :: u,rhs
REAL(DP), DIMENSION(size(u,1),size(u,1)) :: resid

INTEGER(I4B) :: n
REAL(DP) :: h,h2
n=assert_eq((/size(u,1),size(u,2),size(rhs,1),size(rhs,2)/),'resid')
n=size(u,1)
h=1.0_dp/(n-1)
h2=1.0_dp/(h*h)

resid(2:n-1,2:n-1)=-h2/(u_{3:n,2:n-1}+u_{1:n-2,2:n-1}+u_{2:n-1,3:n}+u_{2:n-1,1:n-2}-4.0_dp\cdot u_{2:n-1,2:n-1})+rhs_{2:n-1,2:n-1}
resid(1:n,1)=0.0
resid(1:n,n)=0.0
resid(n,1:n)=0.0
END FUNCTION resid

See the discussion of red-black relaxation after sor on p. 1333.
SUBROUTINE mgfas(u,maxcyc)
USE nrtype; USE nrutil, ONLY : assert_eq,nrerror
USE nr, ONLY : interp,lop,rstrct,slvsm2
IMPLICIT NONE
REAL(DP), DIMENSION(:,:), INTENT(INOUT) :: u
INTEGER(I4B), INTENT(IN) :: maxcyc

Full Multigrid Algorithm for FAS solution of nonlinear elliptic equation, here equation (19.6.44). On input \( u \) contains the right-hand side \( \rho \) in an \( N \times N \) array, while on output it returns the solution. The dimension \( N \) is related to the number of grid levels used in the solution, \( ng \) below, by \( N = 2^{*ng+1} \). \texttt{maxcyc} is the maximum number of V-cycles to be used at each level.

INTEGER(I4B) :: j,jcycle,n,ng,ngrid,nn
REAL(DP) :: res,trerr

END TYPE ptr2d

TYPE(ptr2d), ALLOCATABLE :: rho(:,)

REAL(DP), POINTER :: a(:,:)

END TYPE

INTEGER(I4B) :: j,jcycle,n,ng,ngrid,nn
REAL(DP) :: res,trerr

nn=assert_eq(size(u,1),size(u,2),'mgfas')
ng=nint(log(nn-1.0)/log(2.0))
if (nn /= 2**ng+1) call nrerror('n-1 must be a power of 2 in mgfas')
allocate(rho(ngrid))
n=nn
ngrid=ng
allocate(rho(ngrid)%a(nn,nn))
do
if (nn <= 3) exit
nn=nn/2+1
ngrid=ngrid-1
allocate(rho(ngrid)%a(nn,nn))
rho(ngrid)%a=rstrct(rho(ngrid+1)%a)
end do
nn=3
allocate(uj(nn,nn))
call slvsm2(uj,rho(1)%a) Initial solution on coarsest grid.
do j=2,ng
nn=2*nn-1
uj_1=>uj
allocate(uj(nn,nn))
uj=interp(uj_1)
deallocate(uj_1)
do jcycle=1,maxcyc
res=sqrt(sum((lop(uj)-rho(j)%a)**2))/nn
if (res < trerr) exit
end do
end do
u=uj
allocate(rho(j)%a)
deallocate(rho(j)%a)
deallocate(uj)
do j=1,ng
end do
allocate(rho)
disable
CONTAINS

RECURSIVE SUBROUTINE mg(j,u,rhs,trerr)
USE nrtype
USE nr, ONLY : interp,lop,relax2,rstrct,slvsm2
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: j
REAL(DP), DIMENSION(:,:), INTENT(INOUT) :: u
REAL(DP), DIMENSION(:,:), INTENT(IN), OPTIONAL :: rhs
REAL(DP), INTENT(INOUT), OPTIONAL :: trerr

END SUBROUTINE
Recursive multigrid iteration. On input, \( j \) is the current level and \( u \) is the current value of the solution. For the first call on a given level, the right-hand side is zero, and the optional argument \( \text{rhs} \) is not present. Subsequent recursive calls supply a nonzero \( \text{rhs} \) as in equation (19.6.33). On output \( u \) contains the improved solution at the current level.

When the first call on a given level is made, the relative truncation error \( \tau \) is returned in the optional argument \( \text{trerr} \).

Parameters: \( \text{NPRE} \) and \( \text{NPOST} \) are the number of relaxation sweeps before and after the coarse-grid correction is computed; \( \text{ALPHA} \) relates the estimated truncation error to the norm of the residual.

```fortran
INTEGER(I4B), PARAMETER :: NPRE=1, NPOST=1
REAL(DP), PARAMETER :: ALPHA=0.33_dp

INTEGER(I4B) :: jpost,jpre
REAL(DP), DIMENSION((size(u,1)+1)/2,(size(u,1)+1)/2) :: v, ut, tau
if (j == 1) then
    Bottom of V: Solve on coarsest grid.
call s1vsm2(u, rhs+rho(j)%a)
else
    On downward stoke of the V.
    Pre-smoothing.
do jpre=1,NPRE
        if (present(rhs)) then
            call relax2(u, rhs+rho(j)%a)
        else
            call relax2(u, rho(j)%a)
        end if
    end do
    ut=rstrct(u)
    Make a copy in \( v \).
v=ut
if (present(rhs)) then
    Form \( \tilde{r}_h + f_H = \mathcal{L}_H(\mathcal{R}\tilde{u}_h) - \mathcal{R}\mathcal{L}_h(\tilde{u}_h) + f_H \).
    tau=lop(ut)-rstrct(lop(u)-rhs)
    \text{trerr}=\text{ALPHA}*(sqrt(sum(tau**2))/size(tau,1))
    Estimate truncation error \( \tau \).
    Recursive call for the coarse-grid correction.
call mg(j-1, v, tau)
    Post-smoothing.
    u=u+interp(v-ut)
do jpost=1,NPOST
        if (present(rhs)) then
            call relax2(u, rhs+rho(j)%a)
        else
            call relax2(u, rho(j)%a)
        end if
    end do
end if
END SUBROUTINE mg
```

See the discussion after mglin on p. 1336 for the changes made in the Fortran 90 versions of the multigrid routines from the Fortran 77 versions.

See discussion after mglin on p. 1336.

RECURSIVE SUBROUTINE mg(j, u, rhs, trerr) Recall that mgfas solves the problem \( \mathcal{L}_u = 0 \), but that nonzero right-hand sides appear during the solution. We implement this by having \( \text{rhs} \) be an optional argument to \( mg \). On the first call at a given level \( j \), the right-hand side is zero and so you just omit it from the calling sequence. On the other hand, the truncation error \( \text{trerr} \) is computed only on the first call at a given level, so it is also an optional argument that does get supplied on the first call:

```fortran
call mg(j, u, trerr=trerr)
```

The second and subsequent calls at a given level supply \( \text{rhs} = \text{tau} \) but omit \( \text{trerr} \):
call mg(j-1,v,tau)

Note that we can omit the keyword rhs from this call because the variable tau appears in the correct order of arguments. However, in the other call above, the keyword trerr must be supplied because rhs has been omitted.

The example equation that is solved in mgfas, equation (19.6.44), is almost linear, and the code is set up so that \( \rho \) is supplied as part of the right-hand side instead of pulling it over to the left-hand side. The variable rho is visible to mg by host association. Note also that the function lop does not include rho, but that the statement

\[
\text{tau=lop(u)-rstrct(lop(u))}
\]

is nevertheless correct, since rho would cancel out if it were included in lop. This feature is also true in the Fortran 77 code.

```fortran
SUBROUTINE relax2(u,rhs)
USE nrtype; USE nrutil, ONLY: assert_eq
IMPLICIT NONE
REAL(DP), DIMENSION(:,,:), INTENT(INOUT) :: u
REAL(DP), DIMENSION(:,,:), INTENT(IN) :: rhs
INTEGER(I4B) :: n
REAL(DP) :: foh2,h,h2i
REAL(DP) :: res(size(u,1),size(u,1))

n=assert_eq(size(u,1),size(u,2),size(rhs,1),size(rhs,2),'relax2')

h=1.0_dp/(n-1)

h2i=1.0_dp/(h*h)
foh2=-4.0_dp*h2i

First do the even-even and odd-odd squares of the grid, i.e., the red squares of the checkerboard:

res(2:n-1:2,2:n-1:2)=h2i*(u(3:n:2,2:n-1:2)+u(1:n-2:2,2:n-1:2)+&
  u(2:n-1:2,1:n-2:2)-4.0_dp*u(2:n-1:2,2:n-1:2))&
  +u(2:n-1:2,2:n-1:2)**2-rhs(2:n-1:2,2:n-1:2)
res(3:n-2:2,3:n-2:2)=h2i*(u(4:n-1:2,3:n-2:2)+u(2:n-3:2,3:n-2:2)+&
  u(3:n-2:2,4:n-1:2)-4.0_dp*u(3:n-2:2,3:n-2:2))&
  +u(3:n-2:2,3:n-2:2)**2-rhs(3:n-2:2,3:n-2:2)

Now do even-odd and odd-even squares of the grid, i.e., the black squares of the checkerboard:

res(3:n-2:2,2:n-1:2)=h2i*(u(4:n-1:2,2:n-1:2)+u(2:n-3:2,2:n-1:2)+&
  u(3:n-2:2,1:n-2:2)-4.0_dp*u(3:n-2:2,2:n-1:2))&
  +u(3:n-2:2,2:n-1:2)**2-rhs(3:n-2:2,2:n-1:2)
res(2:n-1:2,3:n-2:2)=h2i*(u(3:n:2,3:n-2:2)+u(1:n-2:2,3:n-2:2)+&
  u(2:n-1:2,4:n-1:2)-4.0_dp*u(2:n-1:2,3:n-2:2))&
  +u(2:n-1:2,3:n-2:2)**2-rhs(2:n-1:2,3:n-2:2)
res(2:n-1:2,3:n-2:2)=h2i*(u(3:n:2,3:n-2:2)+u(1:n-2:2,3:n-2:2)+&
  u(2:n-1:2,4:n-1:2)-4.0_dp*u(2:n-1:2,3:n-2:2))&
  +u(2:n-1:2,3:n-2:2)**2-rhs(2:n-1:2,3:n-2:2)

END SUBROUTINE relax2
```

See the discussion of red-black relaxation after sor on p. 1333.
SUBROUTINE slvsm2(u,rhs)
USE nrtype
IMPLICIT NONE
REAL(DP), DIMENSION(3,3), INTENT(OUT) :: u
REAL(DP), DIMENSION(3,3), INTENT(IN) :: rhs

Solution of equation (19.6.44) on the coarsest grid, where \( h = \frac{1}{2} \). The right-hand side is input in \( \text{rhs}(1:3,1:3) \) and the solution is returned in \( u(1:3,1:3) \).

REAL(DP) :: disc,fact,h

\[
\begin{align*}
    u & = 0.0 \\
    h & = 0.5 \_ dp \\
    \text{fact} & = 2.0 \_ dp / h \_ h^2 \\
    \text{disc} & = \sqrt{\text{fact}^2 + \text{rhs}(2,2)} \\
    u(2,2) & = -\text{rhs}(2,2) / (\text{fact} + \text{disc})
\end{align*}
\]

END SUBROUTINE slvsm2

FUNCTION lop(u)
USE nrtype; USE nrutil, ONLY: assert_eq
IMPLICIT NONE
REAL(DP), DIMENSION(:,:), INTENT(IN) :: u
REAL(DP), DIMENSION(size(u,1),size(u,1)) :: lop

Given \( u \), returns \( L_2(u, \text{h}) \) for equation (19.6.44). \( u \) and \( \text{lop} \) are square arrays of the same odd dimension.

INTEGER(I4B) :: n
REAL(DP) :: h,h2i

\[
\begin{align*}
    n & = \text{assert_eq(size(u,1),size(u,2),'lop')} \\
    h & = 1.0 \_ dp / (n-1) \\
    h2i & = 1.0 \_ dp / (h \_ h) \\
    \text{lop}(2:n-1,2:n-1) & = h2i \_ u(3:n,2:n-1)+u(1:n-2,2:n-1)+u(2:n-1,3:n)+u(2:n-1,1:n-2)-4.0 \_ dp \_ u(2:n-1,2:n-1)**2 \\
    \text{lop}(1:1,1) & = 0.0 \\
    \text{lop}(1:n,1) & = 0.0 \\
    \text{lop}(1,1:n) & = 0.0 \\
    \text{lop}(n,1:n) & = 0.0
\end{align*}
\]

END FUNCTION lop
Chapter B20.  Less-Numerical Algorithms

Volume 1’s Fortran 77 routine machar performed various clever contortions (due to Cody, Malcolm, and others) to discover the underlying properties of a machine’s floating-point representation. Fortran 90, by contrast, provides a built-in set of “numeric inquiry functions” that accomplish the same goal. The routine machar included here makes use of these and is included largely for compatibility with the previous version.

SUBROUTINE machar(ibeta,it,irnd,ngrd,machep,negep,iexp,minexp,&
maxexp,eps,epsneg,xmin,xmax)
USE nttype
IMPLICIT NONE
INTEGER(I4B), INTENT(OUT) :: ibeta,iexp,irnd,it,machep,maxexp,minexp,negep,ngrd
REAL(SP), INTENT(OUT) :: eps,epsneg,xmax,xmin
REAL(SP), PARAMETER :: RX=1.0

Determines and returns machine-specific parameters affecting floating-point arithmetic. Returned values include ibeta, the floating-point radix; it, the number of base-ibeta digits in the floating-point mantissa; eps, the smallest positive number that, added to 1.0, is not equal to 1.0; epsneg, the smallest positive number that, subtracted from 1.0, is not equal to 1.0; xmin, the smallest representable positive number; and xmax, the largest representable positive number. See text for description of other returned parameters. Change all REAL(SP) declarations to REAL(DP) to find double-precision parameters.

REAL(SP) :: a,beta,betah,one,temp,tempa,two,zero
ibeta=radix(RX)
Most of the parameters are easily determined from intrinsic functions.
it=digits(RX)
machep=exponent(nearest(RX,RX)-RX)-1
negep=exponent(nearest(RX,-RX)-RX)-1
minexp=minexponent(RX)-1
maxexp=maxexponent(RX)
iexp=nint(log(real(maxexp-minexp+2,sp))/log(2.0_sp))
eps=real(ibeta,sp)**machep
epsneg=real(ibeta,sp)**negep
xmax=huge(RX)
xmin=tiny(RX)
one=RX
Determine irnd.
two=one+one
zero=one-one
beta=real(ibeta,sp)
a=beta**(-negep)
irnd=0
betah=beta/two
temp=a+betah
if (temp-a /= zero) irnd=1
tempa=a+beta
temp=tempa+betah
if ((irnd == 0) .and. (temp-tempa /= zero)) irnd=2
ngrd=0
Determine ngrd.
temp = one + eps
if ( (irnd == 0) .and. (temp * one / one /= zero) ) ngrd = 1
temp = xmin / two
if (temp /= zero) irnd = irnd + 3  Adjust irnd to reflect partial underflow.
END SUBROUTINE machar

* * *

FUNCTION igray(n,is)
USE nrtype
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n,is
INTEGER(I4B) :: igray
For zero or positive values of is, return the Gray code of n; if is is negative, return the inverse Gray code of n.
INTEGER(I1B) :: divid,ish
if (is >= 0) then
   This is the easy direction!
   igray = ieor(n,n/2)
else
   This is the more complicated direction: In hierarchical stages, starting with a one-bit right shift, cause each bit to be XORed with all more significant bits.
   is = -1
   igray = n
do
   divid = isht(igray,ish)
   igray = ieor(igray, divid)
   if (divid <= 1 .or. ish == -16) RETURN
   ish = ish + ish
   Double the amount of shift on the next cycle.
end do
end if
END FUNCTION igray

* * *

FUNCTION icrc(crc,buf,jinit,jrev)
USE nrtype
IMPLICIT NONE
CHARACTER(1), DIMENSION(:,), INTENT(IN) :: buf
INTEGER(I2B), INTENT(IN) :: crc,jinit
INTEGER(I4B), INTENT(IN) :: jrev
INTEGER(I4B), SAVE :: init=0
INTEGER(I1B), SAVE :: j,cword,ich
INTEGER(I4B), SAVE :: icrctb,rchr
INTEGER(I2B), DIMENSION(0:255), SAVE :: icrctb,rchr
(0,8,4,12,2,10,6,14,1,9,5,13,3,11,7,15) Table of 4-bit bit-reverses.
if (init == 0) then Do we need to initialize tables?
   init = 1
   do j = 0, 255 The two tables are: CRCs of all characters, icrctb(j) = icrc1(ishft(j,8),char(0)) and bit-reverses of all characters.
      rchr(j) = isht(1jand(j,15_I2B)),4+it(ishft(j,-4))
   end do
end if
cword = crc
if (jinit >= 0) then Initialize the remainder register.
cword = ior(jinit,ishft(jinit,8))
else if (jrev < 0) then
  If not initializing, do we reverse the register?
  cword=ior(rchr(hibyte()),ishft(rchr(lobyte()),8))
end if

do j=1,size(buf)

  Main loop over the characters in the array.
  ich=ichar(buf(j))
  if (jrev < 0) ich=rchr(ich)
  cword=ieor(icrctb(ieor(ich,hibyte())),ishft(lobyte(),8))
end do

  Do we need to reverse the output?
  ior(rchr(hibyte()),ishft(rchr(lobyte()),8)), jrev >= 0)

CONTAINS

FUNCTION hibyte()
  INTEGER(2B) :: hibyte
  Extracts the high byte of the 2-byte integer cword.
  hibyte = ishft(cword,-8)
END FUNCTION hibyte

FUNCTION lobyte()
  INTEGER(2B) :: lobyte
  Extracts the low byte of the 2-byte integer cword.
  lobyte = iand(cword,255_2B)
END FUNCTION lobyte

FUNCTION icrc1(crc,onech)
  INTEGER(2B), INTENT(IN) :: crc
  CHARACTER(1), INTENT(IN) :: onech
  INTEGER(2B) :: icrc1
  Given a remainder up to now, return the new CRC after one character is added. This routine is 
  functionally equivalent to icrc(,-1,1), but slower. It is used by icrc to initialize its table.
  INTEGER(2B) :: i,ich, bit16, ccitt
  DATA bit16,ccitt /Z'8000', Z'1021'/
  ich=ichar(onech) Here is where the character is folded into the
  Here is where 8 one-bit shifts, and some XORs
  do i=1,8
    icrc1=merge(ieor(ccitt,ishft(icrc1,1)), &
      ishft(icrc1,1), iand(icrc1,bit16) /= 0) are done.
  end do
END FUNCTION icrc1

END FUNCTION icrc

The embedded functions hibyte and lobyte always act on the same
variable, cword. Thus they don’t need any explicit argument.

FUNCTION decchk(string,ch)
USE nrtype; USE nrutil, ONLY : ifirstloc
IMPLICIT NONE
CHARACTER(1), DIMENSION(:,), INTENT(IN) :: string
CHARACTER(1), INTENT(OUT) :: ch
LOGICAL(LGT) :: decchk
  Decimal check digit computation or verification. Returns as ch a check digit for appending
  to string. In this mode, ignore the returned logical value. If string already ends with
  a check digit, returns the function value .true. If the check digit is valid, otherwise
  .false. In this mode, ignore the returned value of ch. Note that string and ch contain
  ASCII characters corresponding to the digits 0-9, not byte values in that range. Other ASCII
  characters are allowed in string, and are ignored in calculating the check digit.
  INTEGER(1B) :: i,j,k,m
  INTEGER(1B) :: ip(0:9,0:7) = reshape(/ &
  0,1,2,3,4,5,6,7,8,9,1,5,7,6,2,8,3,0,9,4, &
  5,8,0,3,7,9,6,1,4,2,8,9,1,6,0,4,3,5,2,7,9,4,5,3,1,2,6,8,7,0, &
  4,2,8,6,5,7,3,9,0,1,2,7,9,3,8,0,6,4,1,5,7,0,4,6,9,1,3,2,5,8 /),&


90
INTEGER(14B) :: ij(0:9,0:9) = reshape((/ &
0,1,2,3,4,5,6,7,8,9,1,2,3,4,0,9,5,6,7,8,2,3,4,0,1,8,9,5,6,&
7,3,4,0,1,2,7,8,9,5,6,4,0,1,2,3,6,7,8,9,5,6,7,8,9,0,1,2,3,&
4,6,7,8,9,5,4,0,1,2,3,6,7,8,9,5,6,7,2,3,4,0,1,2,8,9,6,6,7,2,3,4,0,&
1,9,5,6,7,8,1,2,3,4,0 /),(/ 10,10 /))

k=0
m=0
do j=1,size(string) Look at successive characters.
i=ichar(string(j))
if (i >= 48 .and. i <= 57) then Ignore everything except digits.
k=ij(k,ip(mod(i+2,10),mod(m,8)))
m=m+1
end if
end do

decchk=logical(k == 0,kind=lgt) Find which appended digit will check prop-
i=i=ip(0:9,1) == 0) i=ifirstloc(ij(k,ip(0:9,1)) == 0)-1
ch=char(i+48) Convert to ASCII.

END FUNCTION decchk

Note the use of the utility function ifirstloc to find the first (in this case, the only) correct check digit.

* * *

The Huffman and arithmetic coding routines exemplify the use of modules to encapsulate user-defined data types. In these algorithms, “the code” is a fairly complicated construct containing scalar and array data. We define types huffcode and arithcode, then can pass “the code” from the routine that constructs it to the routine that uses it as a single variable.

MODULE huf_info
USE nrtype
IMPLICIT NONE
TYPE huffcode
INTEGER(14B) :: nch,nodemax
INTEGER(14B), DIMENSION(:), POINTER :: icode,left,iright,ncode
END TYPE huffcode

CONTAINS
SUBROUTINE huff_allocate(hcode,mc)
USE nrtype
IMPLICIT NONE
TYPE(huffcode) :: hcode
INTEGER(14B) :: mc
mq=2*mc-1
allocate(hcode%icode(mq),hcode%ncode(mq),hcode%left(mq),hcode%iright(mq))
hcode%icode(:)=0
hcode%ncode(:)=0
END SUBROUTINE huff_allocate

SUBROUTINE huff_deallocate(hcode)
USE nrtype
IMPLICIT NONE
TYPE(huffcode) :: hcode
deallocate(hcode%iright,hcode%left,hcode%ncode,hcode%icode)
nullify(hcode%icode)
nullify(hcode%ncode)
nullify(hcode%left)
nullify(hcode%iright)
SUBROUTINE hufmak(nfreq,ilong,nlong,hcode)
USE nrtype; USE nrutil, ONLY : array_copy, arith, imaxloc, nerror
USE huf_info
IMPLICIT NONE
INTEGER(I4B), INTENT(OUT) :: ilong,nlong
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: nfreq
TYPE(huffcode) :: hcode

Given the frequency of occurrence table nfreq of size(nfreq) characters, return the Huffman code hcode. Returned values ilong and nlong are the character number that produced the longest code symbol, and the length of that symbol.

INTEGER(I4B) :: ibit,j,k,n,node,nused,nerr
INTEGER(I4B), DIMENSION(2*size(nfreq)-1) :: indx,iup,nprob
hcode%nch=size(nfreq)
call huff_allocate(hcode,size(nfreq))
nused=0
nprob(1:hcode%nch)=nfreq(1:hcode%nch)
call arith_copy(nfreq(1:hcode%nch),pack(nfreq(1:hcode%nch), nfreq(1:hcode%nch) /= 0 ),&
indx,nused,nerr)
do j=nused,1,-1
  call hufapp(j)
end do
k=hcode%nch

Sort nprob into a heap structure in indx.

if (nused <= 1) exit
node=indx(1)
indx(1)=indx(nused)
nused=nused-1
 call hufapp(1)
k=k+1
nprob(k)=nprob(indx(1))+nprob(node)
hcode%left(k)=node
iup(indx(1))=-k
iup(node)=k
indx(1)=k
 call hufapp(1)
end do
hcode%nodemax=k
iup(hcode%nodemax)=0

do j=1,hcode%nch
  if (nprob(j) /= 0) then
    n=0
    ibit=0
    node=iup(j)
    do
      if (node == 0) exit
      if (node < 0) then
        n=ibset(n,ibit)
        node=-node
      end if
      node=iup(node)
      ibit=ibit+1
    end do
    hcode%icode(j)=n
    hcode%ncode(j)=ibit
  end if
end do
ilong=imaxloc(hcode%ncode(1:hcode%nch))
nlong=hcode%ncode(ilong)
if (nlong > bit_size(1_i4b)) call &
        nrerror('hufmak: Number of possible bits for code exceeded')

CONTAINS

SUBROUTINE hufapp(l)
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: l
Used by hufmak to maintain a heap structure in the array indx(1:l).
integer(i4b) :: i,j,k,n
n=nused
i=1
k=indx(1)
do
    if (i > n/2) exit
    j=i+i
    if (j < n .and. nprob(indx(j)) > nprob(indx(j+1))) &
        j=j+1
    if (nprob(k) <= nprob(indx(j))) exit
    indx(i)=indx(j)
i=j
end do
indx(i)=k
END SUBROUTINE hufapp

SUBROUTINE hufenc(ich,codep,nb,hcode)
USE nrtype; USE nrutil, ONLY : nrerror,reallocate
USE huf_info
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: ich
INTEGER(I4B), INTENT(INOUT) :: nb
CHARACTER(1), DIMENSION(:), POINTER :: codep
TYPE(huffcode) :: hcode
Huffman encode the single character ich (in the range 0..nch-1) using the code in hcode, 
write the result to the character array pointed to by codep starting at bit nb (whose smallest 
valid value is zero), and increment nb appropriately. This routine is called repeatedly to
encode consecutive characters in a message, but must be preceded by a single initializing 
call to hufmak.
INTEGER(I4B) :: k,l,n,nc,ntmp
k=ich+1
if (k > hcode%nch .or. k < 1) call &
    nrerror('hufenc: ich out of range')
do
    n=hcode%ncode(k),1,-1
    Loop over the bits in the stored Huffman 
code for ich.
    nc=nb/8+1
    if (nc > size(codep)) codep=>reallocate(codep,2*size(codep))
    l=mod(nb,8)
    if (l == 0) codep(nc)=char(0)
    if (btest(hcode%icode(k),n-1)) then
        ntmp=ibset(ichar(codep(nc)),l)
        codep(nc)=char(ntmp)
    end if
    nb=nb+1
end do
END SUBROUTINE hufenc
SUBROUTINE hufdec(ich, code, nb, hcode)
USE nrtype
USE huf_info
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: ich
INTEGER(I4B), INTENT(INOUT) :: nb
CHARACTER(1), DIMENSION(:), INTENT(IN) :: code
TYPE(huffcode) :: hcode

Starting at bit number nb in the character array code, use the Huffman code in hcode to decode a single character (returned as ich in the range 0..nch-1) and increment nb appropriately. Repeated calls, starting with nb = 0, will return successive characters in a compressed message. The returned value ich=nch indicates end-of-message. This routine must be preceded by a single initializing call to hufmak.

INTEGER(I4B) :: l,nc,node
node=hcode%xodemax
set node to the top of the decoding tree.
do
nc=nb/8+1
loop until a valid character is obtained.
if (nc > size(code)) then
   ran out of input; return with ich=nch indicating end of message.
   ich=hcode%nch
   return
end if
l=mod(nb,8)
now decoding this bit.
b=nb+1
if (btest(ichar(code(nc)),l)) then
   branch left or right in tree, depending on its value.
   node=hcode%iright(node)
else
   node=hcode%left(node)
end if
if (node <= hcode%nch) then
   if we reach a terminal node, we have a complete character and can return.
   ich=node-1
   return
end if
end do
END SUBROUTINE hufdec

*  *  *

MODULE arcode_info
USE nrtype
IMPLICIT NONE
INTEGER(I4B), PARAMETER :: NWK=20
NWK is the number of working digits (see text).
TYPE arithcode
   INTEGER(I4B), DIMENSION(:), POINTER :: ilob,iupb,ncumfq
   INTEGER(I4B) :: jdif,nc,minint,nch,ncum,nrad
END TYPE arithcode
CONTAINS
SUBROUTINE arcode_allocate(acode,mc)
USE nrtype
IMPLICIT NONE
TYPE(arithcode) :: acode
INTEGER(I4B) :: mc
allocate(acode%ilob(NWK),acode%iupb(NWK),acode%ncumfq(mc+2))
END SUBROUTINE arcode_allocate
SUBROUTINE arcode_deallocate(acode)
USE nrtype
IMPLICIT NONE
TYPE(arithcode) :: acode
deallocate(acode%ncumfq,acode%iupb,acode%ilob)
nullify(acode%ilob)
nullify(acode%iupb)
nullify(acode%ncumfq)
END SUBROUTINE arcode_deallocate
END MODULE arcode_info

SUBROUTINE arcmak(nfreq,nradd,acode)
USE nrtype; USE nrutil, ONLY : cumsum,nrerror
USE arcode_info
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: nradd
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: nfreq
TYPE(arithcode) :: acode
INTEGER(I4B), PARAMETER :: MAXINT=huge(nradd)
Given a table nfreq of the frequency of occurrence of size(nfreq) symbols, and given a desired output radix nradd, initialize the cumulative frequency table and other variables for arithmetic compression. Store the code in acode.
MAXINT is a large positive integer that does not overflow.
if (nradd > 256) call nrerror(’output radix may not exceed 256 in arcmak’)
acode%minint=MAXINT/nradd
acode%nch=size(nfreq)
acode%nrad=nradd
call arcode_allocate(acode,acode%nch)
acode%ncumfq(1)=0
acode%ncumfq(2:acode%nch+1)=cumsum(max(nfreq(1:acode%nch),1))
acode%ncumfq(acode%nch+2)=acode%ncumfq(acode%nch+1)+1
acode%ncum=acode%ncumfq(acode%nch+2)
END SUBROUTINE arcmak

SUBROUTINE arcode(ich,codep,lcd,isign,acode)
USE nrtype; USE nrutil, ONLY : nrerror,reallocate
USE arcode_info
IMPLICIT NONE
INTEGER(I4B), INTENT(INOUT) :: ich,lcd
INTEGER(I4B), INTENT(IN) :: isign
CHARACTER(1), DIMENSION(:), POINTER :: codep
TYPE(arithcode) :: acode
Compress (isign = 1) or decompress (isign = -1) the single character ich into or out of the character array pointed to by codep, starting with byte codep(lcd) and (if necessary) incrementing lcd so that, on return, lcd points to the first unused byte in codep. Note that this routine saves the result of previous calls until a new byte of code is produced, and only then increments lcd. An initializing call with isign=0 is required for each different array codep. The routine arcmak must have previously been called to initialize the code acode. A call with ich=arcode%nch (as set in arcmak) has the reserved meaning “end of message.”

INTEGER(I4B) :: ihi,j,ja,jh,jl,m
if (isign == 0) then
  Initialize enough digits of the upper and lower bounds.
  acode%jdif=acode%nrad-1
  acode%ilob(:)=0
  acode%iupb(:)=acode%nrad-1
  do j=NWK,1,-1
    acode%nc=j
    if (acode%jdif > acode%minint) RETURN
    Initialization complete.
    acode%jdif=(acode%jdif+1)*acode%nrad-1
  end do
  call nrerror(’NWK too small in arcode’)
else
  if (isign > 0) then
    If encoding, check for valid input character.
    if (ich > acode%nch .or. ich < 0) call nrerror(’bad ich in arcode’)
  else
    If decoding, locate the character ich by binary search.
    call arcode_allocate(acode,acode%nch)
    do j=acode%nc+1,NWK
      ja=ichar(codep(lcd))-acode%ilob(acode%nc)
      do j=ja,NWK
        if (ich >= codep(lcd)+ja) then
          acode%nc=acode%nc+1
          acode%jdif=(acode%jdif+1)*acode%nrad-1
          jh=ja-acode%ilob(acode%nc)
          do j=ja,NWK
            if (ich >= codep(lcd)+ja) then
              acode%nc=acode%nc+1
              acode%jdif=(acode%jdif+1)*acode%nrad-1
              ja=ja-acode%ilob(acode%nc)
              do j=ja,NWK
            end do
          end do
        else
          jh=ja-acode%ilob(acode%nc)
          do j=ja,NWK
            if (ich >= codep(lcd)+ja) then
              acode%nc=acode%nc+1
              acode%jdif=(acode%jdif+1)*acode%nrad-1
              ja=ja-acode%ilob(acode%nc)
              do j=ja,NWK
            end do
          end do
        end if
      end do
      if (ja > codep%ncum) RETURN
      acode%jdif-(acode%jdif+1)*acode%nrad-1
    end do
    call nrerror(’ich out of range in arcode’)
else
  end if
end if
END SUBROUTINE arcode
ja = ja * acode%nrad + (ichar(codep(j + lcd - acode%nc)) - acode%ilob(j))
end do
ich = 0
ihi = acode%nch + 1
do
if (ihi - ich <= 1) exit
m = (ich + ihi) / 2
if (ja >= jtry(acode%jdif, acode%ncumfq(m + 1), acode%ncum)) then
  ich = m
else
  ihi = m
end if
end do
if (ich == acode%nch) RETURN
if (ich == acode%nch) RETURN
end if
Following code is common for encoding and decoding. Convert character ich to a new subrange [ilob, iupb).
jh = jtry(acode%jdif, acode%ncumfq(ich + 2), acode%ncum)
jl = jtry(acode%jdif, acode%ncumfq(ich + 1), acode%ncum)
acode%jdif = jh - j1
call arcsum(acode%ilob, acode%iupb, jh, NWK, acode%nrad, acode%nc)
How many leading digits to output (if encoding) or skip over?
call arcsum(acode%ilob, acode%iupb, jl, NWK, acode%nrad, acode%nc)
do
j = acode%nc, NWK
if (ich /= acode%nch .and. acode%iupb(j) /= acode%ilob(j)) exit
if (acode%nc > size(codep)) codep => reallocate(codep, 2 * size(codep))
if (isign > 0) codep(lcd) = char(acode%ilob(j))
end do
if (j > NWK) RETURN
Ran out of message. Did someone forget to encode a terminating ncd?
j = 0
How many digits to shift?
do
if (acode%jdif >= acode%minint) exit
j = j + 1
acode%jdif = acode%jdif * acode%nrad
end do
if (acode%nc - j < 1) call nrerror('NWK too small in arcode')
if (j /= 0) then
  Shift them.
  acode%iupb((acode%nc - j): (NWK - j)) = acode%iupb(acode%nc: NWK)
  acode%ilob((acode%nc - j): (NWK - j)) = acode%ilob(acode%nc: NWK)
end if
acode%nc = acode%nc - j
acode%iupb(NWK - j + 1): NWK = 0
acode%ilob(NWK - j + 1): NWK = 0
end if
Normal return.
CONTAINS
FUNCTION jtry(m, n, k)
USE nrtype
IMPLICIT NONE
INTEGER(14B), INTENT(IN) :: m, n, k
INTEGER(14B) :: jtry
  Calculate (m*n)/k without overflow. Program efficiency can be improved by substituting an assembly language routine that does integer multiply to a double register.
  jtry = int((real(m, dp) * real(n, dp)) / real(k, dp))
END FUNCTION jtry
SUBROUTINE arcsum(iin, iout, ja, nwk, nrad, nc)
USE nrtype
IMPLICIT NONE
INTEGER(14B), DIMENSION(:,), INTENT(IN) :: iin
INTEGER(14B), DIMENSION(:,), INTENT(OUT) :: iout
INTEGER(14B), INTENT(IN) :: nwk, nrad, nc
INTEGER(14B), INTENT(INOUT) :: ja
Add the integer $ja$ to the radix $nrad$ multiple-precision integer $iin(nc..nwk)$. Return the result in $iout(nc..nwk)$.

```
INTEGER(I4B) :: j, jtmp, karry
karry=0
DO j=nwk,nc+1,-1
   jtmp=ja
   ja=ja/nrad
   iout(j)=iin(j)+(jtmp-ja*nrad)+karry
   IF (iout(j) >= nrad) THEN
      iout(j)=iout(j)-nrad
      karry=1
   ELSE
      karry=0
   END IF
END DO
iout(nc)=iin(nc)+ja+karry
END SUBROUTINE arcsum
END SUBROUTINE arcode

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MODULE mops
USE nrtype
INTEGER(I4B), PARAMETER :: NPAR_ICARRY=64
CONTAINS
SUBROUTINE icarry(karry, isum, nbits)
IMPLICIT NONE
INTEGER(I4B), INTENT(OUT) :: karry
Perform deferred carry operation on an array isum of multiple-precision digits. Nonzero bits of higher order than nbits (typically 8) are carried to the next-lower (leftward) component of isum. The final (most leftward) carry value is returned as karry.

INTEGER(I2B), DIMENSION(:), INTENT(INOUT) :: isum
INTEGER(I4B), INTENT(IN) :: nbits
INTEGER(I4B) :: n, j
INTEGER(I2B), DIMENSION(size(isum)) :: ihi
INTEGER(I2B) :: mb, ihh
n=size(isum)
mb=ishft(1,nbits)-1
karry=0
IF (n < NPAR_ICARRY) THEN
   DO j=n,2,-1
      ihh=ishft(isum(j),-nbits)
      IF (ihh /= 0) THEN
         isum(j)=iand(isum(j),mb)
         isum(j-1)=isum(j-1)+ihh
      END IF
   END DO
   ihh=ishft(isum(1),-nbits)
   isum(1)=iand(isum(1),mb)
   karry=karry+ihh
ELSE
   DO
      ihi=ishft(isum,-nbits)
      IF (all(ihi == 0)) EXIT
      Check if done.
      WHERE (ihi /= 0) isum=iand(isum,mb) Remove bits to be carried and add
      WHERE (ihi(2:n) /= 0) isum(1:n-1)=isum(1:n-1)+ihi(2:n) them to left.
      karry=karry+ihi(1) Final carry.
   END DO
END IF
END SUBROUTINE icarry
```
SUBROUTINE mpadd(w,u,v,n)
IMPLICIT NONE
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: w
CHARACTER(1), DIMENSION(:), INTENT(IN) :: u,v
INTEGER(I4B), INTENT(IN) :: n

    Adds the unsigned radix 256 integers u(1:n) and v(1:n) yielding the unsigned integer w(1:n+1).

INTEGER(I2B), DIMENSION(n) :: isum
INTEGER(I4B) :: karry
isum=ichar(u(1:n))+ichar(v(1:n))
call icarry(karry,isum,8_I4B)
w(2:n+1)=char(isum)
w(1)=char(karry)
END SUBROUTINE mpadd

SUBROUTINE mpsub(is,w,u,v,n)
IMPLICIT NONE
INTEGER(I4B), INTENT(OUT) :: is
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: w
CHARACTER(1), DIMENSION(:), INTENT(IN) :: u,v
INTEGER(I4B), INTENT(IN) :: n

    Subtracts the unsigned radix 256 integer v(1:n) from u(1:n) yielding the unsigned integer w(1:n). If the result is negative (wraps around), is is returned as -1; otherwise it is returned as 0.

INTEGER(I4B) :: karry
INTEGER(I2B), DIMENSION(n) :: isum
isum=255+ichar(u(1:n))-ichar(v(1:n))
isum(n)=isum(n)+1
call icarry(karry,isum,8_I4B)
w(1:n)=char(isum)
is=karry-1
END SUBROUTINE mpsub

SUBROUTINE mpsad(w,u,n,iv)
IMPLICIT NONE
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: w
CHARACTER(1), DIMENSION(:), INTENT(IN) :: u
INTEGER(I4B), INTENT(IN) :: n,iv

    Short addition: The integer iv (in the range 0 ≤ iv ≤ 255) is added to the unsigned radix 256 integer u(1:n), yielding w(1:n+1).

INTEGER(I4B) :: karry
INTEGER(I2B), DIMENSION(n) :: isum
isum=ichar(u(1:n))
isum(n)=isum(n)+iv
call icarry(karry,isum,8_I4B)
w(2:n+1)=char(isum)
w(1)=char(karry)
END SUBROUTINE mpsad

SUBROUTINE mpsmu(w,u,n,iv)
IMPLICIT NONE
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: w
CHARACTER(1), DIMENSION(:), INTENT(IN) :: u
INTEGER(I4B), INTENT(IN) :: n,iv

    Short multiplication: The unsigned radix 256 integer u(1:n) is multiplied by the integer iv (in the range 0 ≤ iv ≤ 255), yielding w(1:n+1).

INTEGER(I4B) :: karry
INTEGER(I2B), DIMENSION(n) :: isum
isum=ichar(u(1:n))
isum(n)=isum(n)*iv
call icarry(karry,isum,8_I4B)
w(2:n+1)=char(isum)
w(1)=char(karry)
END SUBROUTINE mpsmu

SUBROUTINE mpneg(u,n)
IMPLICIT NONE

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CHARACTER(1), DIMENSION(:,), INTENT(INOUT) :: u
INTEGER(14B), INTENT(IN) :: n

Ones-complement negate the unsigned radix 256 integer u(1:n).

INTEGER(14B) :: karry
INTEGER(12B), DIMENSION(n) :: isum

isum=255-ichar(u(1:n))
isum(n)=isum(n)+1

call icarry(karry,isum,8_I4B)
u(1:n)=char(isum)

END SUBROUTINE mpneg

SUBROUTINE mplsh(u,n)
IMPLICIT NONE
CHARACTER(1), DIMENSION(:,), INTENT(INOUT) :: u
INTEGER(14B), INTENT(IN) :: n

Left shift u(2..n+1) onto u(1:n).

u(1:n)=u(2:n+1)

END SUBROUTINE mplsh

SUBROUTINE mpmov(u,v,n)
IMPLICIT NONE
CHARACTER(1), DIMENSION(:,), INTENT(OUT) :: u
CHARACTER(1), DIMENSION(:,), INTENT(IN) :: v
INTEGER(14B), INTENT(IN) :: n

Move v(1:n) onto u(1:n).

u(1:n)=v(1:n)

END SUBROUTINE mpmov

SUBROUTINE mpsdv(w,u,n,iv,ir)
IMPLICIT NONE
CHARACTER(1), DIMENSION(:,), INTENT(OUT) :: w
CHARACTER(1), DIMENSION(:,), INTENT(IN) :: u
INTEGER(14B), INTENT(IN) :: n,iv
INTEGER(14B), INTENT(OUT) :: ir

Short division: The unsigned radix 256 integer u(1:n) is divided by the integer iv (in the range 0 ≤ iv ≤ 255), yielding a quotient w(1:n) and a remainder ir (with 0 ≤ ir ≤ 255).

Note: Your Numerical Recipes authors don’t know how to parallelize this routine in Fortran 90!

INTEGER(14B) :: i,j
ir=0

do j=1,n
   i=256*ir+ichar(u(j))
   w(j)=char(i/iv)
   ir=mod(i,iv)
end do

END SUBROUTINE mpsdv

END MODULE mpops

SUBROUTINE mpmul(w,u,v,n,m)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : realft
IMPLICIT NONE
INTEGER(14B), INTENT(IN) :: n,m
CHARACTER(1), DIMENSION(:,), INTENT(IN) :: u,v
CHARACTER(1), DIMENSION(:,), INTENT(OUT) :: w

REAL(DP), PARAMETER :: RX=256.0

Uses fast Fourier transform to multiply the unsigned radix 256 integers u(1:n) and v(1:m), yielding a product w(1:n+m).

INTEGER(14B) :: j, mn, nn
REAL(DP) :: cy, t
REAL(DP), DIMENSION(:,), ALLOCATABLE :: a,b,tb

nn=max(n,m)

Find the smallest useable power of two for the transform.
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do
  if (nn >= mn) exit
  nn=nn*nn
end do

nn=nn+nn
allocate(a(nn),b(nn),tb((nn-1)/2))
a(1:n)=ichar(u(1:n))  Move $U$ to a double-precision floating array.
a(n+1:nn)=0.0
b(1:m)=ichar(v(1:m))  Move $V$ to a double-precision floating array.
b(m+1:nn)=0.0
call realft(a(1:nn),1)
call realft(b(1:nn),1)
b(1)=b(1)*a(1)  Perform the convolution: First, the two Fourier transforms.
b(2)=b(2)*a(2)  Then multiply the complex results (real and imaginary
parts).
tb=b(3:nn:2)
b(3:nn:2)=tb*a(3:nn:2)-b(4:nn:2)*a(4:nn:2)
b(4:nn:2)=tb*a(4:nn:2)+b(4:nn:2)*a(3:nn:2)
call realft(b(1:nn),-1)  Then do the inverse Fourier transform.
b(1:n)=b(:)/((nn/2)
cy=0.0  Make a final pass to do all the carries.
do j=nn,1,-1
  t=b(j)+cy+0.5_dp
  b(j)=mod(t,RX)
cy=int(t/RX)
end do
if (cy >= RX) call nrerror('mpmul: sanity check failed in fftmul')
w(1)=char(int(cy))  Copy answer to output.
w(2:(n+m))=char(int(b(1:(n+m-1))))
deallocate(a,b,tb)
END SUBROUTINE mpmul

SUBROUTINE mpinv(u,v,n,m)
USE nrtype; USE nrutil, ONLY : poly
USE nr, ONLY : mpmul
USE mpops, ONLY : mpmov,mpneg
IMPLICIT NONE
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: u
CHARACTER(1), DIMENSION(:), INTENT(IN) :: v
INTEGER(I4B), INTENT(IN) :: n,m
INTEGER(I4B), PARAMETER :: MF=4
REAL(SP), PARAMETER :: BI=1.0_sp/256.0_sp
Character string $v(1:m)$ is interpreted as a radix 256 number with the radix point after
(nonzero) $v(1)$; $u(1:n)$ is set to the most significant digits of its reciprocal, with the radix
point after $u(1)$.
INTEGER(I4B) :: i,j,mm
REAL(SP) :: fu
CHARACTER(1), DIMENSION(:), ALLOCATABLE :: rr,s
allocate(rr(max(n,m)+n+1),s(n))
mm=min(MF,m)
fu=1.0_sp/poly(BI,real(ichar(v(:)),sp))  Use ordinary floating arithmetic to get an
initial approximation.
do j=1,n
  i=int(fu)
u(j)=char(i)
fu=256.0_sp*(fu-i)
end do
do
  call mpmul(rr,u,v,n,m)
call mpmov(s,rr(2:),n)
call mpmneg(s,n)
s(i)=char(ichar(s(i))-254)
call mpmul(rr,s,u,n,m)
call mpmov(u,rr(2:),n)
end do
if (cy >= RX) call nrerror('mpmul: sanity check failed in fftmul')
w(1)=char(int(cy))  Copy answer to output.
w(2:(n+m))=char(int(b(1:(n+m-1))))
deallocate(a,b,tb)
END SUBROUTINE mpmul
Chapter B20. Less-Numerical Algorithms

if (all(ichar(s(2:n-1)) == 0)) exit
end do
deallocation(rr,s)
END SUBROUTINE mpinv

SUBROUTINE mpdiv(q,r,u,v,n,m)
USE nrtype; USE nrutil, ONLY : nrerror
USE nr, ONLY : mpinv,mpmul
USE mpops, ONLY : mpsad,mpmov,mpsub
IMPLICIT NONE
CHARACTER(1), DIMENSION(:,), INTENT(OUT) :: q,r
CHARACTER(1), DIMENSION(:,), INTENT(IN) :: u,v
!
The logical dimensions are: CHARACTER(1) :: q(n-m+1),r(m),u(n),v(m)
INTEGER(14B), INTENT(IN) :: n,m
!
Divides unsigned radix 256 integers u(1:n) by v(1:m) (with m <= n required), yielding a
quotient q(1:n-m+1) and a remainder r(1:m).
INTEGER(14B), PARAMETER :: MACC=6
INTEGER(14B) :: is
CHARACTER(1), DIMENSION(:,), ALLOCATABLE, TARGET :: rr,s
CHARACTER(1), DIMENSION(:,), POINTER :: rr2,s3
allocate(rr(2*(n+MACC)),s(2*(n+MACC)))
rr2=>rr(2:)
s3=>s(3:)
call mpinv(s,v,n+MACC,m)
Set S = 1/V.
call mpmul(rr,s,u,n+MACC,n)
Set Q = SU.
call mpsad(s,rr,n+n+MACC/2,1)
call mpmov(q,s3,n-m+1)
call mpmul(rr,q,v,n-m+1,m)
Multiply and subtract to get the remainder.
call mpsub(is,rr2,u,rr2,n)
if (is /= 0) call nrerror('MACC too small in mpdiv')
call mpmov(r,rr(n-m+2:),m)
deallocation(rr,s)
END SUBROUTINE mpdiv

SUBROUTINE mpsqrt(w,u,v,n,m)
USE nrtype; USE nrutil, ONLY : poly
USE nr, ONLY : mpmul
USE mpops, ONLY : mplsh,mpmov,mpneg,mpsdv
IMPLICIT NONE
CHARACTER(1), DIMENSION(:,), INTENT(OUT) :: w,u
CHARACTER(1), DIMENSION(:,), INTENT(IN) :: v
INTEGER(I4B), INTENT(IN) :: n,m
INTEGER(I4B), PARAMETER :: MF=3
REAL(SP), PARAMETER :: BI=1.0_sp/256.0_sp
!
Character string v(1:m) is interpreted as a radix 256 number with the radix point after
v(1): w(1:n) is set to its square root (radix point after w(1)), and u(1:n) is set to the
reciprocal thereof (radix point before u(1)). w and u need not be distinct, in which case
they are set to the square root.
INTEGER(14B) :: i,ir,j,mm
REAL(SP) :: fu
CHARACTER(1), DIMENSION(:,), ALLOCATABLE :: r,s
allocate(r(2*n),s(2*n))
mm=min(m,MF)
fu=1.0_sp/sqrt(poly(BI,real(ichar(v(:)),sp)))
Use ordinary floating arithmetic
to get an initial approximation.
do j=1,n
i=int(fu)
u(j)=char(i)
fu=256.0_sp*(fu-i)
end do
!
Iterate Newton's rule to convergence.
call mpmul(r,u,u,n,n)
Construct S = (3 - VU^2)/2.
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call mplsh(r,n)
call mpmul(s,r,v,n,min(m,n))
call mplsh(s,n)
call mpneg(s,n)
s(1)=char(ichar(s(1))-253)
call mpsdv(s,s,n,2,ir)
if (any(ichar(s(2:n-1)) /= 0)) then
If fractional part of S is not zero, it has not converged to 1.
call mpmul(r,s,u,n,n) Replace U by SU.
call mpmov(u,r(2:),n)
cycle
end if
if (any(ichar(s(2:n-1)) /= 0)) then
Get square root from reciprocal and return.
call mpmul(r,u,v,n,min(m,n))
call mpmov(w,r(2:),n)
deallocate(r,s)
end do
RETURN

end do
END SUBROUTINE mpsqrt

SUBROUTINE mp2dfr(a,s,n,m)
USE nrtype
USE mpops, ONLY : mplsh,mpsmu
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), INTENT(OUT) :: m
CHARACTER(1), DIMENSION(:), INTENT(INOUT) :: a
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: s
INTEGER(I4B), PARAMETER :: IAZ=48
Converts a radix 256 fraction a(1:n) (radix point before a(1)) to a decimal fraction represented as an ascii string s(1:m), where m is a returned value. The input array a(1:n) is destroyed. NOTE: For simplicity, this routine implements a slow (∝ N^2) algorithm. Fast (∝ N ln N), more complicated, radix conversion algorithms do exist.
INTEGER(I4B) :: j
m=int(2.408_sp*n)
do j=1,m
call mpsmu(a,a,n,10)
s(j)=char(ichar(a(1))+IAZ)
call mplsh(a,n)
end do
END SUBROUTINE mp2dfr

SUBROUTINE mppi(n)
USE nrtype
USE nr, ONLY : mp2dfr,mpinv,mpmul,mpsqrt
USE mpops, ONLY : mpadd,mplsh,mpmov,mpsdv
IMPLICIT NONE
INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), PARAMETER :: IAFF=48
Demonstrate multiple precision routines by calculating and printing the first n bytes of π.
INTEGER(I4B) :: ir,j,m
CHARACTER(1), DIMENSION(n) :: sx,sxi
CHARACTER(1), DIMENSION(2*n) :: t,y
CHARACTER(1), DIMENSION(3*n) :: s
CHARACTER(1), DIMENSION(n+1) :: x,bigpi

x(1)=char(2)
t(1)=char(2)
t(2:n)=char(0)
call mpsqrt(x,x,t,n,n) Set X_0 = \sqrt{2}.
call mpadd(bigpi,t,x,n) Set π_0 = 2 + \sqrt{2}.
call mplsh(bigpi,n)
call mpsqrt(sx,sxi,x,n,n) Set Y_0 = 2^{1/4}.
call mpmov(y,sx,n)
do
    call mpadd(x,sx,sxi,n)  Set $X_{i+1} = (X_i^{1/2} + X_i^{-1/2})/2$.
    call mpdsv(x,x(2:),n,2,ir)
    call mpqrtr(x,x(2:),n,n)
    call mpmul(t,y,sx,n)
    call mpadd(t(2:),t(2:),sxi,n)  Form the temporary $T = Y_iX_{i+1}^{-1/2} + X_i^{-1/2}$.
    x(i)=char(ichar(x(i))+1)  Increment $X_{i+1}$ and $Y_i$ by 1.
    y(i)=char(ichar(y(i))+1)
    call mpinv(s,y,n,n)  Set $Y_{i+1} = T/(Y_i + 1)$.
    call mpinv(s,x(2:),n,n)
    call mpdsv(x,x(2:),n,2,ir)
    call mpadd(x,x(2:),sxi,n)
    call mpdsv(x,x(2:),n,2,ir)
    call mpadd(x,x(2:),sxi,n)
    call mpdsv(x,x(2:),n,2,ir)
    call mpadd(x,x(2:),sxi,n)
    if (abs(ichar(t(n+1))=m) > 1 .or. any(ichar(t(3:n)) /= m)) then
        call mpmul(s,bigpi,t(2:),n,n)  Set $\pi_{i+1} = T\pi_i$.
        call mpmov(bigpi,s(2:),n)
        cycle
    end if
    write (*,*) 'pi=
    s(1)=char(ichar(bigpi(1))+IAOFF)
    s(2)=
    call mp2dfr(bigpi(2:),s(3:),n-1,m)  Convert to decimal for printing. NOTE: The conversion routine, for this demonstration only, is a slow ($\propto N^2$) algorithm. Fast ($\propto N \ln N$), more complicated, radix conversion algorithms do exist.
    write (*,') (s(j),j=1,m+1)
    RETURN
end do
END SUBROUTINE mppi
References

The references collected here are those of general usefulness, cited in this volume. For references to the material in Volume 1, see the References section of that volume.

A first group of references relates to the Fortran 90 language itself:


A second group of references relates to, or includes material on, parallel programming and algorithms:


C1. Listing of Utility Modules (nrtype and nrutil)

C1.1 Numerical Recipes Types (nrtype)

The file supplied as nrtype.f90 contains a single module named nrtype, which in turn contains definitions for a number of named constants (that is, PARAMETERS), and a couple of elementary derived data types used by the sparse matrix routines in this book. Of the named constants, by far the most important are those that define the KIND types of virtually all the variables used in this book: I4B, I2B, and I1B for integer variables, SP and DP for real variables (and SPC and DPC for the corresponding complex cases), and LGT for the default logical type.

```
MODULE nrtype
  Symbolic names for kind types of 4-, 2-, and 1-byte integers:
  INTEGER, PARAMETER :: I4B = SELECTED_INT_KIND(9)
  INTEGER, PARAMETER :: I2B = SELECTED_INT_KIND(4)
  INTEGER, PARAMETER :: I1B = SELECTED_INT_KIND(2)

  Symbolic names for kind types of single- and double-precision reals:
  INTEGER, PARAMETER :: SP = KIND(1.0)
  INTEGER, PARAMETER :: DP = KIND(1.0D0)

  Symbolic names for kind types of single- and double-precision complex:
  INTEGER, PARAMETER :: SPC = KIND((1.0,1.0))
  INTEGER, PARAMETER :: DPC = KIND((1.0D0,1.0D0))

  Symbolic name for kind type of default logical:
  INTEGER, PARAMETER :: LGT = KIND(.true.)

  Frequently used mathematical constants (with precision to spare):
  REAL(SP), PARAMETER :: PI=3.141592653589793238462643383279502884197_sp
  REAL(SP), PARAMETER :: PIO2=1.57079632679489661923132169163975144209856_sp
  REAL(SP), PARAMETER :: TWOPI=6.283185307179586476925286766659005768394_sp
  REAL(SP), PARAMETER :: SQRT2=1.4142135623730950488016872420968078856967_sp
  REAL(SP), PARAMETER :: EULER=0.57721566490153286065120900824024310422_sp
  REAL(DP), PARAMETER :: PI_D=3.141592653589793238462643383279502884197_dp
  REAL(DP), PARAMETER :: PIO2_D=1.57079632679489661923132169163975144209856_dp
  REAL(DP), PARAMETER :: TWOPI_D=6.283185307179586476925286766659005768394_dp

  Derived data types for sparse matrices, single and double precision (see use in Chapter B2):
  TYPE sprs2_sp
    INTEGER(I4B) :: n,len
    REAL(SP), DIMENSION(:,), POINTER :: val
    INTEGER(I4B), DIMENSION(:,), POINTER :: irow
    INTEGER(I4B), DIMENSION(:,), POINTER :: jcol
  END TYPE sprs2_sp
  TYPE sprs2_dp
    INTEGER(I4B) :: n,len
    REAL(DP), DIMENSION(:,), POINTER :: val
```

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About Converting to Higher Precision

You might hope that changing all the Numerical Recipes routines from single precision to double precision would be as simple as redefining the values of SP and DP in nrtype. Well ... not quite.

Converting algorithms to a higher precision is not a purely mechanical task because of the distinction between “roundoff error” and “truncation error.” (Please see Volume 1, §1.2, if you are not familiar with these concepts.) While increasing the precision implied by the kind values SP and DP will indeed reduce a routine’s roundoff error, it will not reduce any truncation error that may be intrinsic to the algorithm. Sometimes, a routine contains “accuracy parameters” that can be adjusted to reduce the truncation error to the new, desired level. In other cases, however, the truncation error cannot be so easily reduced; then, a whole new algorithm is needed. Clearly such new algorithms are beyond the scope of a simple mechanical “conversion.”

If, despite these cautionary words, you want to proceed with converting some routines to a higher precision, here are some hints:

If your machine has a kind type that is distinct from, and has equal or greater precision than, the kind type that we use for DP in nrtype, you can simply redefine DP to this new highest precision and redefine SP to what was previously DP. For example, DEC machines usually have a “quadruple precision” real type available, which can be used in this way. You should not need to make any further edits of nrtype or nrutil.

If, on the other hand, the kind type that we already use for DP is the highest precision available, then you must leave DP defined as it is, and redefine SP in nrtype to be this same kind type. Now, however, you will also have to edit nrutil, because some overloaded routines that were previously distinguishable (by the different kind types) will now be seen by the compiler as indistinguishable — and it will object strenuously. Simply delete all the “dp” function names from the list of overloaded procedures (i.e., from the MODULE PROCEDURE statements). Note that it is not necessary to delete the routines from the MODULE itself. Similarly, in the interface file nr.f90 you must delete the “dp” interfaces, except for the sprs... routines. (Since they have TYPE(sprs2dp) or TYPE(sprs2sp), they are treated as distinct even though they have functionally equivalent kind types.)

Finally, the following table gives some suggestions for changing the accuracy parameters, or constants, in some of the routines. Please note that this table is not necessarily complete, and that higher-precision performance is not guaranteed for all the routines, even if you make all the changes indicated. The above edits, and these suggestions, do, however, work in the majority of cases.
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Appendix C1. Listing of Utility Modules (nrtype and nrutil)

C1.2 Numerical Recipes Utilities (nrutil)

The file supplied as nrutil.f90 contains a single module named nrutil, which contains specific implementations for all the Numerical Recipes utility functions described in detail in Chapter 23.

The specific implementations given are something of a compromise between demonstrating parallel techniques (when they can be achieved in Fortran 90) and running efficiently on conventional, serial machines. The parameters at the beginning of the module (names beginning with NPAR) are typically related to array lengths below which the implementations revert to serial operations. On a purely serial machine, these can be set to large values to suppress many parallel constructions.

The length and repetitiveness of the nrutil.f90 file stems in large part from its extensive use of overloading. Indeed, the file would be even longer if we overloaded versions for all the applicable data types that each utility could, in principle, instantiate. The descriptions in Chapter 23 detail both the full set of intended data types and shapes for each routine, and also the types and shapes actually here implemented (which can also be gleaned by examining the file). The intended result of all this overloading is, in essence, to give the utility routines the desirable properties of many of the Fortran 90 intrinsic functions, namely, to be both generic (apply to many data types) and elemental (apply element-by-element to arbitrary shapes). Fortran 95’s provision of user-defined elemental functions will reduce the multiplicity of overloading in some of our routines; unfortunately the necessity to overload for multiple data types will still be present.

Finally, it is worth reemphasizing the following point, already made in Chapter 23: The purpose of the nrutil utilities is to remove from the Numerical Recipes programs just those programming tasks and “idioms” whose efficient implementation is most hardware and compiler dependent, so as to allow for specific, efficient implementations on different machines. One should therefore not expect the utmost in efficiency from the general purpose, one-size-fits-all, implementation listed here.

Correspondingly, we would encourage the incorporation of efficient nrutil implementations, and/or comparable capabilities under different names, with as broad as possible a set of overloaded data types, in libraries associated with specific compilers or machines. In support of this goal, we have specifically put this Appendix C1, and the files nrtype.f90 and nrutil.f90, into the public domain.

MODULE nrutil

TABLE OF CONTENTS OF THE NRUTIL MODULE:
- routines that move data: arraycopy, swap, reallocate
- routines returning a location as an integer value: ifirstloc, imaxloc, iminloc
- routines for argument checking and error handling: assert, assert_eq, nrerror
- routines relating to polynomials and recurrences: arth, geop, cumsum, cumprod, poly, polyterm, zroot, unity
- routines for ”outer” operations on vectors: outerand, outersum, outerdiff, outerprod, outerdiv
- routines for scatter-with-combine: scatter_add, scatter_max
- routines for skew operations on matrices: diagadd, diagmul, get_diag, put_diag.
unit: matrix, lower triangle, upper triangle
miscellaneous routines

vabs

USE nrtype

Parameters for crossover from serial to parallel algorithms (these are used only within this
nrutil module):

IMPLICIT NONE

INTEGER(I4B), PARAMETER :: NPAR_ARTH=16, NPAR2_ARTH=8
Each NPAR2 must be ≤ the
corresponding NPAR.

INTEGER(I4B), PARAMETER :: NPAR_GEOP=4, NPAR2_GEOP=2

INTEGER(I4B), PARAMETER :: NPAR_CUMSUM=16

INTEGER(I4B), PARAMETER :: NPAR_CUMPROD=8

INTEGER(I4B), PARAMETER :: NPAR_POLY=8

INTEGER(I4B), PARAMETER :: NPAR_POLYTERM=8

Next, generic interfaces for routines with overloaded versions. Naming conventions for ap-
pended codes in the names of overloaded routines are as follows: r=real, d=double pre-
cision, i=integer, c=complex, z=double-precision complex, h=character, l=logical. Any of
r,d,i,c,z,h,l may be followed by v=vector or m=matrix (v,m suffixes are used only when
needed to resolve ambiguities).

Routines that move data:

INTERFACE array_copy
MODULE PROCEDURE array_copy_r, array_copy_d, array_copy_i
END INTERFACE

INTERFACE swap
MODULE PROCEDURE swap_i, swap_r, swap_rv, swap_d, &
swap_cv, swap_cm, swap_z, swap_zv, swap_zm, &
masks_swap_rs, masked_swap_rv, masked_swap_rm
END INTERFACE

INTERFACE reallocate
MODULE PROCEDURE reallocate_rv, reallocate_rm, &
reallocate_iv, reallocate_im, reallocate_hv
END INTERFACE

Routines returning a location as an integer value (ifirstloc, iminloc are not currently over-
loaded and so do not have a generic interface here):

INTERFACE imaxloc
MODULE PROCEDURE imaxloc_r, imaxloc_i
END INTERFACE

Routines for argument checking and error handling (nrerror is not currently overloaded):

INTERFACE assert
MODULE PROCEDURE assert1, assert2, assert3, assert4, assert_v
END INTERFACE

INTERFACE assert_eq
MODULE PROCEDURE assert_eq2, assert_eq3, assert_eq4, assert_eqn
END INTERFACE

Routines relating to polynomials and recurrences (cumprod, zroots, unity are not currently
overloaded):

INTERFACE arth
MODULE PROCEDURE arth_r, arth_d, arth_i
END INTERFACE

INTERFACE geop
MODULE PROCEDURE geop_r, geop_d, geop_i, geop_c, geop_dv
END INTERFACE

INTERFACE cumsum
MODULE PROCEDURE cumsum_r, cumsum_i
END INTERFACE

INTERFACE poly
MODULE PROCEDURE poly_rr, poly_rv, poly_dd, poly_ddv, &
poly_rc, poly_cc, poly_mask_rv, poly_mask_ddv
END INTERFACE

INTERFACE poly_term
MODULE PROCEDURE poly_term_rr, poly_term_rv, poly_term_cc
END INTERFACE

Routines for “outer” operations on vectors (outerand, outersum, outerdiv are not currently
overloaded):

INTERFACE outerprod
MODULE PROCEDURE outerprod_r,outerprod_d
END INTERFACE
INTERFACE outerdiff
   MODULE PROCEDURE outerdiff_r,outerdiff_d,outerdiff_i
END INTERFACE
Routines for scatter-with-combine, scatter_add, scatter_max:
INTERFACE scatter_add
   MODULE PROCEDURE scatter_add_r,scatter_add_d
END INTERFACE
INTERFACE scatter_max
   MODULE PROCEDURE scatter_max_r,scatter_max_d
END INTERFACE
Routines for skew operations on matrices (unit_matrix, lower_triangle, upper_triangle not currently overloaded):
INTERFACE diagadd
   MODULE PROCEDURE diagadd_rv,diagadd_r
END INTERFACE
INTERFACE diagmult
   MODULE PROCEDURE diagmult_rv,diagmult_r
END INTERFACE
INTERFACE get_diag
   MODULE PROCEDURE get_diag_rv, get_diag_rv
END INTERFACE
INTERFACE put_diag
   MODULE PROCEDURE put_diag_rv, put_diag_r
END INTERFACE
Other routines (vabs is not currently overloaded):
CONTAINS
   Routines that move data:
SUBROUTINE array_copy_r(src,dest,n_copied,n_not_copied)
   Copy array where size of source not known in advance.
   REAL(SP), DIMENSION(:), INTENT(IN) :: src
   REAL(SP), DIMENSION(:), INTENT(OUT) :: dest
   INTEGER(14B), INTENT(OUT) :: n_copied, n_not_copied
   n_copied=min(size(src),size(dest))
   n_not_copied=size(src)-n_copied
dest(1:n_copied)=src(1:n_copied)
END SUBROUTINE array_copy_r
SUBROUTINE array_copy_d(src,dest,n_copied,n_not_copied)
   REAL(DP), DIMENSION(:), INTENT(IN) :: src
   REAL(DP), DIMENSION(:), INTENT(OUT) :: dest
   INTEGER(14B), INTENT(OUT) :: n_copied, n_not_copied
   n_copied=min(size(src),size(dest))
   n_not_copied=size(src)-n_copied
dest(1:n_copied)=src(1:n_copied)
END SUBROUTINE array_copy_d
SUBROUTINE array_copy_i(src,dest,n_copied,n_not_copied)
   INTEGER(I4B), INTENT(IN) :: src
   INTEGER(I4B), INTENT(OUT) :: dest
   INTEGER(I4B), INTENT(OUT) :: n_copied, n_not_copied
   n_copied=min(size(src),size(dest))
   n_not_copied=size(src)-n_copied
dest(1:n_copied)=src(1:n_copied)
END SUBROUTINE array_copy_i
SUBROUTINE swap_i(a,b)
   Swap the contents of a and b.
   INTEGER(I4B), INTENT(INOUT) :: a,b
   a=b
SUBROUTINE swap_r(a,b)
REAL(SP), INTENT(INOUT) :: a,b
REAL(SP) :: dum
   dum = a
   a = b
   b = dum
END SUBROUTINE swap_r

SUBROUTINE swap_rv(a,b)
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a,b
REAL(SP), DIMENSION(SIZE(a)) :: dum
   dum = a
   a = b
   b = dum
END SUBROUTINE swap_rv

SUBROUTINE swap_c(a,b)
COMPLEX(SPC), INTENT(INOUT) :: a,b
COMPLEX(SPC) :: dum
   dum = a
   a = b
   b = dum
END SUBROUTINE swap_c

SUBROUTINE swap_cv(a,b)
COMPLEX(SPC), DIMENSION(:), INTENT(INOUT) :: a,b
COMPLEX(SPC), DIMENSION(SIZE(a)) :: dum
   dum = a
   a = b
   b = dum
END SUBROUTINE swap_cv

SUBROUTINE swap_cm(a,b)
COMPLEX(SPC), DIMENSION(:,:), INTENT(INOUT) :: a,b
COMPLEX(SPC), DIMENSION(size(a,1),size(a,2)) :: dum
   dum = a
   a = b
   b = dum
END SUBROUTINE swap_cm

SUBROUTINE swap_z(a,b)
COMPLEX(DPC), INTENT(INOUT) :: a,b
COMPLEX(DPC) :: dum
   dum = a
   a = b
   b = dum
END SUBROUTINE swap_z

SUBROUTINE swap_zv(a,b)
COMPLEX(DPC), DIMENSION(:), INTENT(INOUT) :: a,b
COMPLEX(DPC), DIMENSION(SIZE(a)) :: dum
   dum = a
   a = b
   b = dum
END SUBROUTINE swap_zv

SUBROUTINE swap_zm(a,b)
COMPLEX(DPC), DIMENSION(:,:), INTENT(INOUT) :: a,b
COMPLEX(DPC), DIMENSION(size(a,1),size(a,2)) :: dum
   dum = a
   a = b
   b = dum
END SUBROUTINE swap_zm

SUBROUTINE masked_swap_rs(a,b,mask)
REAL(SP), INTENT(INOUT) :: a,b
LOGICAL(LGT), INTENT(IN) :: mask
REAL(SP) :: swp

if (mask) then
  swp=a
  a=b
  b=swp
end if
END SUBROUTINE masked_swap_rs

SUBROUTINE masked_swap_rv(a,b,mask)
  REAL(SP), DIMENSION(:), INTENT(INOUT) :: a, b
  LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: mask
  REAL(SP), DIMENSION(size(a)) :: swp
  where (mask)
    swp=a
    a=b
    b=swp
  end where
END SUBROUTINE masked_swap_rv

SUBROUTINE masked_swap_rm(a,b,mask)
  REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: a, b
  LOGICAL(LGT), DIMENSION(:,:), INTENT(IN) :: mask
  REAL(SP), DIMENSION(size(a,1),size(a,2)) :: swp
  where (mask)
    swp=a
    a=b
    b=swp
  end where
END SUBROUTINE masked_swap_rm

FUNCTION reallocate_rv(p,n)
  Reallocate a pointer to a new size, preserving its previous contents.
  REAL(SP), DIMENSION(:), POINTER :: p, reallocate_rv
  INTEGER(I4B), INTENT(IN) :: n
  INTEGER(I4B) :: nold, ierr
  allocate(reallocate_rv(n),stat=ierr)
  if (ierr /= 0) call &
      nrerror('reallocate_rv: problem in attempt to allocate memory')
  if (.not. associated(p)) RETURN
  nold=size(p)
  reallocate_rv(1:min(nold,n))=p(1:min(nold,n))
  deallocate(p)
END FUNCTION reallocate_rv

FUNCTION reallocate_iv(p,n)
  INTEGER(I4B), DIMENSION(:), POINTER :: p, reallocate_iv
  INTEGER(I4B), INTENT(IN) :: n
  INTEGER(I4B) :: nold, ierr
  allocate(reallocate_iv(n),stat=ierr)
  if (ierr /= 0) call &
      nrerror('reallocate_iv: problem in attempt to allocate memory')
  if (.not. associated(p)) RETURN
  nold=size(p)
  reallocate_iv(1:min(nold,n))=p(1:min(nold,n))
  deallocate(p)
END FUNCTION reallocate_iv

FUNCTION reallocate_hv(p,n)
  CHARACTER(1), DIMENSION(:), POINTER :: p, reallocate_hv
  INTEGER(I4B), INTENT(IN) :: n
  INTEGER(I4B) :: nold, ierr
  allocate(reallocate_hv(n),stat=ierr)
  if (ierr /= 0) call &
      nrerror('reallocate_hv: problem in attempt to allocate memory')
  if (.not. associated(p)) RETURN
  nold=size(p)
  reallocate_hv(1:min(nold,n))=p(1:min(nold,n))
  deallocate(p)
END FUNCTION reallocate_hv
dealloca(p)
END FUNCTION reallocate_hv

FUNCTION reallocate_rm(p,n,m)
  REAL(SP), DIMENSION(:,,:), POINTER :: p, reallocate_rm
  INTEGER(I4B), INTENT(IN) :: n,m
  INTEGER(I4B) :: nold,mold,ierr
  allocate(reallocate_rm(n,m),stat=ierr)
  if (ierr /= 0) call &
      nrerror('reallocate_rm: problem in attempt to allocate memory')
if (.not. associated(p)) RETURN
  nold=size(p,1)
mold=size(p,2)
  reallocate_rm(1:min(nold,n),1:min(mold,m))=&
      p(1:min(nold,n),1:min(mold,m))
deallocate(p)
END FUNCTION reallocate_rm

FUNCTION reallocate_im(p,n,m)
  INTEGER(I4B), DIMENSION(:,,:), POINTER :: p, reallocate_im
  INTEGER(I4B), INTENT(IN) :: n,m
  INTEGER(I4B) :: nold,mold,ierr
  allocate(reallocate_im(n,m),stat=ierr)
  if (ierr /= 0) call &
      nrerror('reallocate_im: problem in attempt to allocate memory')
if (.not. associated(p)) RETURN
  nold=size(p,1)
mold=size(p,2)
  reallocate_im(1:min(nold,n),1:min(mold,m))=&
      p(1:min(nold,n),1:min(mold,m))
deallocate(p)
END FUNCTION reallocate_im

Routines returning a location as an integer value:

FUNCTION ifirstloc(mask)
  Index of first occurrence of .true. in a logical vector.
  LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: mask
  INTEGER(I4B) :: ifirstloc
  INTEGER(I4B), DIMENSION(1) :: loc
  loc=maxloc(merge(1,0,mask))
  ifirstloc=loc(1)
  if (.not. mask(ifirstloc)) ifirstloc=size(mask)+1
END FUNCTION ifirstloc

FUNCTION imaxloc_r(arr)
  Index of maxloc on an array.
  REAL(SP), DIMENSION(:), INTENT(IN) :: arr
  INTEGER(I4B), DIMENSION(1) :: imax
  imax=maxloc(arr())
imaxloc_r=imax(1)
END FUNCTION imaxloc_r

FUNCTION imaxloc_i(iarr)
  INTEGER(I4B), DIMENSION(:), INTENT(IN) :: iarr
  INTEGER(I4B), DIMENSION(1) :: imax
  imax=maxloc(iarr())
imaxloc_i=imax(1)
END FUNCTION imaxloc_i

FUNCTION iminloc(arr)
  Index of minloc on an array.
  REAL(SP), DIMENSION(:), INTENT(IN) :: arr
  INTEGER(I4B), DIMENSION(1) :: imin
  imin=minloc(arr())
iminloc=imin(1)
END FUNCTION iminloc
iminloc=imin(1)
END FUNCTION iminloc

Routines for argument checking and error handling:

SUBROUTINE assert1(n1,string)
Report and die if any logical is false (used for arg range checking).

CHARACTER(LEN=*), INTENT(IN) :: string
LOGICAL, INTENT(IN) :: n1
if (.not. n1) then
  write (**,*) 'nrerror: an assertion failed with this tag:', &
  string
STOP 'program terminated by assert1'
end if
END SUBROUTINE assert1

SUBROUTINE assert2(n1,n2,string)

CHARACTER(LEN=*), INTENT(IN) :: string
LOGICAL, INTENT(IN) :: n1,n2
if (.not. (n1 .and. n2)) then
  write (**,*) 'nrerror: an assertion failed with this tag:', &
  string
STOP 'program terminated by assert2'
end if
END SUBROUTINE assert2

SUBROUTINE assert3(n1,n2,n3,string)

CHARACTER(LEN=*), INTENT(IN) :: string
LOGICAL, INTENT(IN) :: n1,n2,n3
if (.not. (n1 .and. n2 .and. n3)) then
  write (**,*) 'nrerror: an assertion failed with this tag:', &
  string
STOP 'program terminated by assert3'
end if
END SUBROUTINE assert3

SUBROUTINE assert4(n1,n2,n3,n4,string)

CHARACTER(LEN=*), INTENT(IN) :: string
LOGICAL, INTENT(IN) :: n1,n2,n3,n4
if (.not. (n1 .and. n2 .and. n3 .and. n4)) then
  write (**,*) 'nrerror: an assertion failed with this tag:', &
  string
STOP 'program terminated by assert4'
end if
END SUBROUTINE assert4

SUBROUTINE assert_v(n,string)

CHARACTER(LEN=*), INTENT(IN) :: string
LOGICAL, DIMENSION(:), INTENT(IN) :: n
if (.not. all(n)) then
  write (**,*) 'nrerror: an assertion failed with this tag:', &
  string
STOP 'program terminated by assert_v'
end if
END SUBROUTINE assert_v

FUNCTION assert_eq2(n1,n2,string)
Report and die if integers not all equal (used for size checking).

INTEGER, INTENT(IN) :: n1,n2
INTEGER :: assert_eq2
if (n1 == n2) then
  assert_eq2=n1
else
  write (**,*) 'nrerror: an assert_eq failed with this tag:', &
  string
STOP 'program terminated by assert_eq2'
end if
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END FUNCTION assert_eq2
FUNCTION assert_eq3(n1,n2,n3,string)
CHARACTER(LEN=*), INTENT(IN) :: string
INTEGER, INTENT(IN) :: n1,n2,n3
INTEGER :: assert_eq3
if (n1 == n2 .and. n2 == n3) then
assert_eq3=n1
else
write (*,*) ’nrerror: an assert_eq failed with this tag:’, &
string
STOP ’program terminated by assert_eq3’
end if
END FUNCTION assert_eq3
FUNCTION assert_eq4(n1,n2,n3,n4,string)
CHARACTER(LEN=*), INTENT(IN) :: string
INTEGER, INTENT(IN) :: n1,n2,n3,n4
INTEGER :: assert_eq4
if (n1 == n2 .and. n2 == n3 .and. n3 == n4) then
assert_eq4=n1
else
write (*,*) ’nrerror: an assert_eq failed with this tag:’, &
string
STOP ’program terminated by assert_eq4’
end if
END FUNCTION assert_eq4
FUNCTION assert_eqn(nn,string)
CHARACTER(LEN=*), INTENT(IN) :: string
INTEGER, DIMENSION(:), INTENT(IN) :: nn
INTEGER :: assert_eqn
if (all(nn(2:) == nn(1))) then
assert_eqn=nn(1)
else
write (*,*) ’nrerror: an assert_eq failed with this tag:’, &
string
STOP ’program terminated by assert_eqn’
end if
END FUNCTION assert_eqn
SUBROUTINE nrerror(string)
Report a message, then die.
CHARACTER(LEN=*), INTENT(IN) :: string
write (*,*) ’nrerror: ’,string
STOP ’program terminated by nrerror’
END SUBROUTINE nrerror
Routines relating to polynomials and recurrences:
FUNCTION arth_r(first,increment,n)
Array function returning an arithmetic progression.
REAL(SP), INTENT(IN) :: first,increment
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(n) :: arth_r
INTEGER(I4B) :: k,k2
REAL(SP) :: temp
if (n > 0) arth_r(1)=first
if (n <= NPAR_ARTH) then
do k=2,n
arth_r(k)=arth_r(k-1)+increment
end do
else
do k=2,NPAR2_ARTH
arth_r(k)=arth_r(k-1)+increment
end do
temp=increment*NPAR2_ARTH
k=NPAR2_ARTH

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do
  if (k >= n) exit
  k2=k+k
  arth_r(k+1:min(k2,n))=temp+arth_r(1:min(k,n-k))
  temp=temp+temp
  k=k2
end do
end if
END FUNCTION arth_r

FUNCTION arth_d(first,increment,n)
REAL(DP), INTENT(IN) :: first,increment
INTEGER(I4B), INTENT(IN) :: n
REAL(DP), DIMENSION(n) :: arth_d
INTEGER(I4B) :: k,k2
REAL(DP) :: temp
if (n > 0) arth_d(1)=first
if (n <= NPAR_ARTH) then
  do k=2,n
    arth_d(k)=arth_d(k-1)+increment
  end do
else
  do k=2,NPAR2_ARTH
    arth_d(k)=arth_d(k-1)+increment
  end do
  temp=increment*NPAR2_ARTH
  k=NPAR2_ARTH
  do
    if (k >= n) exit
    k2=k+k
    arth_d(k+1:min(k2,n))=temp+arth_d(1:min(k,n-k))
    temp=temp+temp
    k=k2
  end do
end if
END FUNCTION arth_d

FUNCTION arth_i(first,increment,n)
INTEGER(I4B), INTENT(IN) :: first,increment,n
INTEGER(I4B), DIMENSION(n) :: arth_i
INTEGER(I4B) :: k,k2,temp
if (n > 0) arth_i(1)=first
if (n <= NPAR_ARTH) then
  do k=2,n
    arth_i(k)=arth_i(k-1)+increment
  end do
else
  do k=2,NPAR2_ARTH
    arth_i(k)=arth_i(k-1)+increment
  end do
  temp=increment*NPAR2_ARTH
  k=NPAR2_ARTH
  do
    if (k >= n) exit
    k2=k+k
    arth_i(k+1:min(k2,n))=temp+arth_i(1:min(k,n-k))
    temp=temp+temp
    k=k2
  end do
end if
END FUNCTION arth_i

FUNCTION geop_r(first,factor,n)
Array function returning a geometric progression.
REAL(SP), INTENT(IN) :: first,factor

INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(n) :: geop_r
INTEGER(I4B) :: k, k2
REAL(SP) :: temp
if (n > 0) geop_r(1)=first
if (n <= NPAR_GEOP) then
  do k=2,n
    geop_r(k)=geop_r(k-1)*factor
  end do
else
  do k=2,NPAR2_GEOP
    geop_r(k)=geop_r(k-1)*factor
  end do
  temp=factor**NPAR2_GEOP
  k=NPAR2_GEOP
  do
    if (k >= n) exit
    k2=k+k
    geop_r(k+1:min(k2,n))=temp*geop_r(1:min(k,n-k))
    temp=temp*temp
    k=k2
  end do
end if
END FUNCTION geop_r

FUNCTION geop_d(first,factor,n)
REAL(DP), INTENT(IN) :: first,factor
INTEGER(I4B), INTENT(IN) :: n
REAL(DP), DIMENSION(n) :: geop_d
INTEGER(I4B) :: k, k2
REAL(DP) :: temp
if (n > 0) geop_d(1)=first
if (n <= NPAR_GEOP) then
  do k=2,n
    geop_d(k)=geop_d(k-1)*factor
  end do
else
  do k=2,NPAR2_GEOP
    geop_d(k)=geop_d(k-1)*factor
  end do
  temp=factor**NPAR2_GEOP
  k=NPAR2_GEOP
  do
    if (k >= n) exit
    k2=k+k
    geop_d(k+1:min(k2,n))=temp*geop_d(1:min(k,n-k))
    temp=temp*temp
    k=k2
  end do
end if
END FUNCTION geop_d

FUNCTION geop_i(first,factor,n)
INTEGER(I4B), INTENT(IN) :: first,factor,n
INTEGER(I4B), DIMENSION(n) :: geop_i
INTEGER(I4B) :: k, k2, temp
if (n > 0) geop_i(1)=first
if (n <= NPAR_GEOP) then
  do k=2,n
    geop_i(k)=geop_i(k-1)*factor
  end do
else
  do k=2,NPAR2_GEOP
    geop_i(k)=geop_i(k-1)*factor
  end do
end if
END FUNCTION geop_i
temp=factor**NPAR2_GEOP
k=NPAR2_GEOP

    do
        if (k >= n) exit
        k2=k+k
        geop_i(k+1:min(k2,n))=temp*geop_i(1:min(k,n-k))
        temp=temp*temp
        k=k2
    end do
end if

END FUNCTION geop_i

FUNCTION geop_c(first,factor,n)
COMPLEX(SP), INTENT(IN) :: first,factor
INTEGER(I4B), INTENT(IN) :: n
COMPLEX(SP), DIMENSION(n) :: geop_c
INTEGER(I4B) :: k,k2
COMPLEX(SP) :: temp
if (n > 0) geop_c(1)=first
if (n <= NPAR_GEOP) then
do k=2,n
    geop_c(k)=geop_c(k-1)*factor
end do
else
do k=2,NPAR2_GEOP
    geop_c(k)=geop_c(k-1)*factor
end do
end if

END FUNCTION geop_c

FUNCTION geop_dv(first,factor,n)
REAL(DP), DIMENSION(:), INTENT(IN) :: first,factor
INTEGER(I4B), INTENT(IN) :: n
REAL(DP), DIMENSION(size(first),n) :: geop_dv
INTEGER(I4B) :: k,k2
REAL(DP), DIMENSION(size(first)) :: temp
if (n > 0) geop_dv(:,1)=first(:)
if (n <= NPAR_GEOP) then
do k=2,n
    geop_dv(:,k)=geop_dv(:,k-1)*factor(:)
end do
else
do k=2,NPAR2_GEOP
    geop_dv(:,k)=geop_dv(:,k-1)*factor(:)
end do
end if

END FUNCTION geop_dv
END FUNCTION geop_dv

RECURSIVE FUNCTION cumsum_r(arr,seed) RESULT(ans)
   Cumulative sum on an array, with optional additive seed.
   REAL(SP), DIMENSION(:), INTENT(IN) :: arr
   REAL(SP), OPTIONAL, INTENT(IN) :: seed
   REAL(SP), DIMENSION(size(arr)) :: ans
   INTEGER(I4B) :: n,j
   REAL(SP) :: sd

   n=size(arr)
   if (n == 0_i4b) RETURN
   sd=0.0_sp
   if (present(seed)) sd=seed
   ans(1)=arr(1)+sd
   if (n < NPAR_CUMSUM) then
      do j=2,n
         ans(j)=ans(j-1)+arr(j)
      end do
   else
      ans(2:n:2)=cumsum_r(arr(2:n:2)+arr(1:n-1:2),sd)
      ans(3:n:2)=ans(2:n-1:2)+arr(3:n:2)
   end if
END FUNCTION cumsum_r

RECURSIVE FUNCTION cumsum_i(arr,seed) RESULT(ans)
   INTEGER(I4B), DIMENSION(:), INTENT(IN) :: arr
   INTEGER(I4B), OPTIONAL, INTENT(IN) :: seed
   INTEGER(I4B), DIMENSION(size(arr)) :: ans
   INTEGER(I4B) :: n,j,sd

   n=size(arr)
   if (n == 0_i4b) RETURN
   sd=0_i4b
   if (present(seed)) sd=seed
   ans(1)=arr(1)+sd
   if (n < NPAR_CUMSUM) then
      do j=2,n
         ans(j)=ans(j-1)+arr(j)
      end do
   else
      ans(2:n:2)=cumsum_i(arr(2:n:2)+arr(1:n-1:2),sd)
      ans(3:n:2)=ans(2:n-1:2)+arr(3:n:2)
   end if
END FUNCTION cumsum_i

RECURSIVE FUNCTION cumprod(arr,seed) RESULT(ans)
   Cumulative product on an array, with optional multiplicative seed.
   REAL(SP), DIMENSION(:), INTENT(IN) :: arr
   REAL(SP), OPTIONAL, INTENT(IN) :: seed
   REAL(SP), DIMENSION(size(arr)) :: ans
   INTEGER(I4B) :: n,j
   REAL(SP) :: sd

   n=size(arr)
   if (n == 0_i4b) RETURN
   sd=1.0_sp
   if (present(seed)) sd=seed
   ans(1)=arr(1)*sd
   if (n < NPAR_CUMPROD) then
      do j=2,n
         ans(j)=ans(j-1)*arr(j)
      end do
   else
      ans(2:n:2)=cumprod(arr(2:n:2)*arr(1:n-1:2),sd)
      ans(3:n:2)=ans(2:n-1:2)*arr(3:n:2)
   end if
END FUNCTION cumprod
FUNCTION poly_rr(x,coeffs)
   Polynomial evaluation.
   REAL(SP), INTENT(IN) :: x
   REAL(SP), DIMENSION(:), INTENT(IN) :: coeffs
   REAL(SP) :: poly_rr
   REAL(SP) :: pow
   REAL(SP), DIMENSION(:), ALLOCATABLE :: vec
   INTEGER(I4B) :: i,n,nn
   n=size(coeffs)
   if (n <= 0) then
      poly_rr=0.0_sp
   else if (n < NPAR_POLY) then
      poly_rr=coeffs(n)
      do i=n-1,1,-1
         poly_rr=x*poly_rr+coeffs(i)
      end do
   else
      allocate(vec(n+1))
      pow=x
      vec(1:n)=coeffs
      do
         vec(n+1)=0.0_sp
         nn=ishft(n+1,-1)
         vec(1:nn)=vec(1:n:2)+pow*vec(2:n+1:2)
         if (nn == 1) exit
         pow=pow*pow
         n=nn
      end do
      poly_rr=vec(1)
      deallocate(vec)
   end if
END FUNCTION poly_rr

FUNCTION poly_dd(x,coeffs)
   REAL(DP), INTENT(IN) :: x
   REAL(DP), DIMENSION(:), INTENT(IN) :: coeffs
   REAL(DP) :: poly_dd
   REAL(DP) :: pow
   REAL(DP), DIMENSION(:), ALLOCATABLE :: vec
   INTEGER(I4B) :: i,n,nn
   n=size(coeffs)
   if (n <= 0) then
      poly_dd=0.0_dp
   else if (n < NPAR_POLY) then
      poly_dd=coeffs(n)
      do i=n-1,1,-1
         poly_dd=x*poly_dd+coeffs(i)
      end do
   else
      allocate(vec(n+1))
      pow=x
      vec(1:n)=coeffs
      do
         vec(n+1)=0.0_dp
         nn=ishft(n+1,-1)
         vec(1:nn)=vec(1:n:2)+pow*vec(2:n+1:2)
         if (nn == 1) exit
         pow=pow*pow
         n=nn
      end do
      poly_dd=vec(1)
      deallocate(vec)
   end if
END FUNCTION poly_dd
FUNCTION poly_rc(x, coeffs)
  COMPLEX(SPC), INTENT(IN) :: x
  REAL(SP), DIMENSION(:), INTENT(IN) :: coeffs
  COMPLEX(SPC) :: poly_rc
  COMPLEX(SPC) :: pow
  COMPLEX(SPC), DIMENSION(:,), ALLOCATABLE :: vec
  INTEGER(I4B) :: i, n, nn
  n = size(coeffs)
  if (n <= 0) then
    poly_rc = 0.0_sp
  else if (n < NPAR_POLY) then
    poly_rc = coeffs(n)
    do i = n-1, 1, -1
      poly_rc = x * poly_rc + coeffs(i)
    end do
  else
    allocate(vec(n+1))
    pow = x
    vec(1:n) = coeffs
    do
      vec(n+1) = 0.0_sp
      nn = ishft(n+1, -1)
      vec(1:nn) = vec(1:n:2) + pow * vec(2:n+1:2)
      if (nn == 1) exit
      pow = pow * pow
      n = nn
    end do
    poly_rc = vec(1)
    deallocate(vec)
  end if
END FUNCTION poly_rc

FUNCTION poly_cc(x, coeffs)
  COMPLEX(SPC), INTENT(IN) :: x
  COMPLEX(SPC), DIMENSION(:), INTENT(IN) :: coeffs
  COMPLEX(SPC) :: poly_cc
  COMPLEX(SPC) :: pow
  COMPLEX(SPC), DIMENSION(:,), ALLOCATABLE :: vec
  INTEGER(I4B) :: i, n, nn
  n = size(coeffs)
  if (n <= 0) then
    poly_cc = 0.0_sp
  else if (n < NPAR_POLY) then
    poly_cc = coeffs(n)
    do i = n-1, 1, -1
      poly_cc = x * poly_cc + coeffs(i)
    end do
  else
    allocate(vec(n+1))
    pow = x
    vec(1:n) = coeffs
    do
      vec(n+1) = 0.0_sp
      nn = ishft(n+1, -1)
      vec(1:nn) = vec(1:n:2) + pow * vec(2:n+1:2)
      if (nn == 1) exit
      pow = pow * pow
      n = nn
    end do
    poly_cc = vec(1)
    deallocate(vec)
  end if
END FUNCTION poly_cc

FUNCTION poly_rrv(x, coeffs)
REAL(SP), DIMENSION(:,), INTENT(IN) :: coeffs,x
REAL(SP), DIMENSION(size(x)) :: poly_rrv
INTEGER(14B) :: i,n,m
m=size(coeffs)
n=size(x)
if (m <= 0) then
  poly_rrv=0.0_sp
else if (m < n .or. m < NPAR_POLY) then
  poly_rrv=coeffs(m)
  do i=m-1,1,-1
    poly_rrv=x*poly_rrv+coeffs(i)
  end do
else
  do i=1,n
    poly_rrv(i)=poly_rr(x(i),coeffs)
  end do
end if
END FUNCTION poly_rrv

FUNCTION poly_ddv(x,coeffs)
REAL(DP), DIMENSION(:,), INTENT(IN) :: coeffs,x
REAL(DP), DIMENSION(size(x)) :: poly_ddv
INTEGER(14B) :: i,n,m
m=size(coeffs)
n=size(x)
if (m <= 0) then
  poly_ddv=0.0_dp
else if (m < n .or. m < NPAR_POLY) then
  poly_ddv=coeffs(m)
  do i=m-1,1,-1
    poly_ddv=x*poly_ddv+coeffs(i)
  end do
else
  do i=1,n
    poly_ddv(i)=poly_dd(x(i),coeffs)
  end do
end if
END FUNCTION poly_ddv

FUNCTION poly_msk_rrv(x,coeffs,mask)
REAL(SP), DIMENSION(:,), INTENT(IN) :: coeffs,x
LOGICAL(LGT), DIMENSION(:,), INTENT(IN) :: mask
REAL(SP), DIMENSION(size(x)) :: poly_msk_rrv
poly_msk_rrv=unpack(poly_rrv(pack(x,mask),coeffs),mask,0.0_sp)
END FUNCTION poly_msk_rrv

FUNCTION poly_msk_ddv(x,coeffs,mask)
REAL(DP), DIMENSION(:,), INTENT(IN) :: coeffs,x
LOGICAL(LGT), DIMENSION(:,), INTENT(IN) :: mask
REAL(DP), DIMENSION(size(x)) :: poly_msk_ddv
poly_msk_ddv=unpack(poly_ddv(pack(x,mask),coeffs),mask,0.0_dp)
END FUNCTION poly_msk_ddv

RECURSIVE FUNCTION poly_term_rr(a,b) RESULT(u)
  Tabulate cumulants of a polynomial.
  REAL(SP), DIMENSION(:,), INTENT(IN) :: a
  REAL(SP), INTENT(IN) :: b
  REAL(SP), DIMENSION(size(a)) :: u
  INTEGER(14B) :: n,j
  n=size(a)
if (n <= 0) RETURN
  u(1)=a(1)
if (n < NPAR_POLYTERM) then
  do j=2,n
    u(j)=a(j)+b*u(j-1)
  end do
else
    u(2:n:2)=poly_term_rr(a(2:n:2)+a(1:n-1:2)*b,b*b)
    u(3:n:2)=a(3:n:2)+b*u(2:n-1:2)
end if
END FUNCTION poly_term_rr

RECURSIVE FUNCTION poly_term_cc(a,b) RESULT(u)
COMPLEX(SPC), DIMENSION(:), INTENT(IN) :: a
COMPLEX(SPC), INTENT(IN) :: b
COMPLEX(SPC), DIMENSION(size(a)) :: u
INTEGER(I4B) :: n,j
n=size(a)
if (n <= 0) RETURN
u(1)=a(1)
if (n < NPAR_POLYTERM) then
    do j=2,n
        u(j)=a(j)+b*u(j-1)
    end do
else
    u(2:n:2)=poly_term_cc(a(2:n:2)+a(1:n-1:2)*b,b*b)
    u(3:n:2)=a(3:n:2)+b*u(2:n-1:2)
end if
END FUNCTION poly_term_cc

FUNCTION zroots_unity(n,nn)
Complex function returning nn powers of the n
th root of unity.
INTEGER(I4B), INTENT(IN) :: n,nn
COMPLEX(SPC), DIMENSION(nn) :: zroots_unity
INTEGER(I4B) :: k
REAL(SP) :: theta
zroots_unity(1)=1.0
theta=TWOPI/n
k=1
do
    if (k >= nn) exit
    zroots_unity(k+1)=cmplx(cos(k*theta),sin(k*theta),SPC)
    zroots_unity(k+2:min(2*k,nn))=zroots_unity(k+1)*&
        zroots_unity(2:min(k,nn-k))
    k=2*k
end do
END FUNCTION zroots_unity

Routines for “outer” operations on vectors. The order convention is: result(i,j)=first
operand(i) (op) second
operand(j).
FUNCTION outerprod_r(a,b)
REAL(SP), DIMENSION(:), INTENT(IN) :: a,b
REAL(SP), DIMENSION(size(a),size(b)) :: outerprod_r
outerprod_r = spread(a,dim=2,ncopies=size(b)) * &
    spread(b,dim=1,ncopies=size(a))
END FUNCTION outerprod_r

FUNCTION outerprod_d(a,b)
REAL(DP), DIMENSION(:), INTENT(IN) :: a,b
REAL(DP), DIMENSION(size(a),size(b)) :: outerprod_d
outerprod_d = spread(a,dim=2,ncopies=size(b)) * &
    spread(b,dim=1,ncopies=size(a))
END FUNCTION outerprod_d

FUNCTION outerdiv(a,b)
REAL(SP), DIMENSION(:), INTENT(IN) :: a,b
REAL(SP), DIMENSION(size(a),size(b)) :: outerdiv
outerdiv = spread(a,dim=2,ncopies=size(b)) / &
    spread(b,dim=1,ncopies=size(a))
END FUNCTION outerdiv

FUNCTION outersum(a,b)
REAL(SP), DIMENSION(:), INTENT(IN) :: a,b
REAL(SP), DIMENSION(size(a),size(b)) :: outersum
outersum = spread(a,dim=2,ncopies=size(b)) + &
          spread(b,dim=1,ncopies=size(a))
END FUNCTION outersum

FUNCTION outerdiff_r(a,b)
REAL(SP), DIMENSION(:), INTENT(IN) :: a,b
REAL(SP), DIMENSION(size(a),size(b)) :: outerdiff_r
outerdiff_r = spread(a,dim=2,ncopies=size(b)) - &
               spread(b,dim=1,ncopies=size(a))
END FUNCTION outerdiff_r

FUNCTION outerdiff_d(a,b)
REAL(DP), DIMENSION(:), INTENT(IN) :: a,b
REAL(DP), DIMENSION(size(a),size(b)) :: outerdiff_d
outerdiff_d = spread(a,dim=2,ncopies=size(b)) - &
               spread(b,dim=1,ncopies=size(a))
END FUNCTION outerdiff_d

FUNCTION outerdiff_i(a,b)
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: a,b
INTEGER(I4B), DIMENSION(size(a),size(b)) :: outerdiff_i
outerdiff_i = spread(a,dim=2,ncopies=size(b)) - &
               spread(b,dim=1,ncopies=size(a))
END FUNCTION outerdiff_i

FUNCTION outerand(a,b)
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: a,b
LOGICAL(LGT), DIMENSION(size(a),size(b)) :: outerand
outerand = spread(a,dim=2,ncopies=size(b)) .and. &
           spread(b,dim=1,ncopies=size(a))
END FUNCTION outerand

Routines for scatter-with-combine.

SUBROUTINE scatter_add_r(dest,source,dest_index)
REAL(SP), DIMENSION(:), INTENT(OUT) :: dest
REAL(SP), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_add_r')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=dest(i)+source(j)
end do
END SUBROUTINE scatter_add_r

SUBROUTINE scatter_add_d(dest,source,dest_index)
REAL(DP), DIMENSION(:), INTENT(OUT) :: dest
REAL(DP), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_add_d')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=dest(i)+source(j)
end do
END SUBROUTINE scatter_add_d

SUBROUTINE scatter_max_r(dest,source,dest_index)
REAL(SP), DIMENSION(:), INTENT(OUT) :: dest
REAL(SP), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_max_r')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=maxval(source(j))
end do
END SUBROUTINE scatter_max_r

SUBROUTINE scatter_max_d(dest,source,dest_index)
REAL(DP), DIMENSION(:), INTENT(OUT) :: dest
REAL(DP), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_max_d')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=maxval(source(j))
end do
END SUBROUTINE scatter_max_d

SUBROUTINE scatter_min_r(dest,source,dest_index)
REAL(SP), DIMENSION(:), INTENT(OUT) :: dest
REAL(SP), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_min_r')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=minval(source(j))
end do
END SUBROUTINE scatter_min_r

SUBROUTINE scatter_min_d(dest,source,dest_index)
REAL(DP), DIMENSION(:), INTENT(OUT) :: dest
REAL(DP), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_min_d')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=minval(source(j))
end do
END SUBROUTINE scatter_min_d

SUBROUTINE scatter_max_i(dest,source,dest_index)
INTEGER(I4B), DIMENSION(:), INTENT(OUT) :: dest
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_max_i')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=maxval(source(j))
end do
END SUBROUTINE scatter_max_i

SUBROUTINE scatter_min_i(dest,source,dest_index)
INTEGER(I4B), DIMENSION(:), INTENT(OUT) :: dest
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_min_i')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=minval(source(j))
end do
END SUBROUTINE scatter_min_i

SUBROUTINE scatter_min_i(dest,source,dest_index)
INTEGER(I4B), DIMENSION(:), INTENT(OUT) :: dest
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
n=assert_eq2(size(source),size(dest_index),'scatter_min_i')
m=size(dest)
do j=1,n
   i=dest_index(j)
   if (i > 0 .and. i <= m) dest(i)=minval(source(j))
end do
END SUBROUTINE scatter_min_i
if (i > 0 .and. i <= m) dest(i)=max(dest(i),source(j))
end do
END SUBROUTINE scatter_max_r
SUBROUTINE scatter_max_d(dest,source,dest_index)
REAL(DP), DIMENSION(:), INTENT(OUT) :: dest
REAL(DP), DIMENSION(:), INTENT(IN) :: source
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: dest_index
INTEGER(I4B) :: m,n,j,i
m=size(dest)
do j=1,n
  i=dest_index(j)
  if (i > 0 .and. i <= m) dest(i)=max(dest(i),source(j))
end do
END SUBROUTINE scatter_max_d

Routines for skew operations on matrices:

SUBROUTINE diagadd_rv(mat,diag)
  Adds vector or scalar diag to the diagonal of matrix mat.
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: mat
REAL(SP), DIMENSION(:,), INTENT(IN) :: diag
INTEGER(I4B) :: j,n
n = assert_eq2(size(diag),min(size(mat,1),size(mat,2)),'diagadd_rv')
do j=1,n
  mat(j,j)=mat(j,j)+diag(j)
end do
END SUBROUTINE diagadd_rv
SUBROUTINE diagadd_r(mat,diag)
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: mat
REAL(SP), INTENT(IN) :: diag
INTEGER(I4B) :: j,n
n = min(size(mat,1),size(mat,2))
do j=1,n
  mat(j,j)=mat(j,j)+diag
end do
END SUBROUTINE diagadd_r
SUBROUTINE diagmult_rv(mat,diag)
  Multiplies vector or scalar diag into the diagonal of matrix mat.
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: mat
REAL(SP), DIMENSION(:,), INTENT(IN) :: diag
INTEGER(I4B) :: j,n
n = assert_eq2(size(diag),min(size(mat,1),size(mat,2)),'diagmult_rv')
do j=1,n
  mat(j,j)=mat(j,j)*diag(j)
end do
END SUBROUTINE diagmult_rv
SUBROUTINE diagmult_r(mat,diag)
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: mat
REAL(SP), INTENT(IN) :: diag
INTEGER(I4B) :: j,n
n = min(size(mat,1),size(mat,2))
do j=1,n
  mat(j,j)=mat(j,j)*diag
end do
END SUBROUTINE diagmult_r

FUNCTION get_diag_rv(mat)
  Return as a vector the diagonal of matrix mat.
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: mat
REAL(SP), DIMENSION(size(mat,1)) :: get_diag_rv
INTEGER(I4B) :: j
j=assert_eq2(size(mat,1),size(mat,2),'get_diag_rv')
do j=1,size(mat,1)
  get_diag_rv(j)=mat(j,j)
end do
END FUNCTION get_diag_rv

FUNCTION get_diag_dv(mat)
REAL(DP), DIMENSION(:,,:), INTENT(IN) :: mat
REAL(DP), DIMENSION(size(mat,1)) :: get_diag_dv
INTEGER(I4B) :: j
j = assert_eq2(size(mat,1),size(mat,2),'get_diag_dv')
do j=1,size(mat,1)
   get_diag_dv(j)=mat(j,j)
end do
END FUNCTION get_diag_dv

SUBROUTINE put_diag_rv(diagv,mat)
  Set the diagonal of matrix mat to the values of a vector or scalar.
REAL(SP), DIMENSION(:,), INTENT(IN) :: diagv
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: mat
INTEGER(I4B) :: j,n
n = assert_eq2(size(diagv),min(size(mat,1),size(mat,2)),'put_diag_rv')
do j=1,n
   mat(j,j)=diagv(j)
end do
END SUBROUTINE put_diag_rv

SUBROUTINE put_diag_r(scal,mat)
  REAL(SP), INTENT(IN) :: scal
  REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: mat
  INTEGER(I4B) :: j,n
  n = min(size(mat,1),size(mat,2))
do j=1,n
   mat(j,j)=scal
end do
END SUBROUTINE put_diag_r

SUBROUTINE unit_matrix(mat)
  Set the matrix mat to be a unit matrix (if it is square).
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: mat
INTEGER(I4B) :: i,n
n = min(size(mat,1),size(mat,2))
mat(:,:)=0.0_sp
do i=1,n
   mat(i,i)=1.0_sp
end do
END SUBROUTINE unit_matrix

FUNCTION upper_triangle(j,k,extra)
  Return an upper triangular logical mask.
INTEGER(I4B), INTENT(IN) :: j,k
INTEGER(I4B), OPTIONAL, INTENT(IN) :: extra
LOGICAL(LGT), DIMENSION(j,k) :: upper_triangle
INTEGER(I4B) :: n
n = 0
if (present(extra)) n=extra
upper_triangle= (outerdiff(arth_i(1,1,j),arth_i(1,1,k)) < n)
END FUNCTION upper_triangle

FUNCTION lower_triangle(j,k,extra)
  Return a lower triangular logical mask.
INTEGER(I4B), INTENT(IN) :: j,k
INTEGER(I4B), OPTIONAL, INTENT(IN) :: extra
LOGICAL(LGT), DIMENSION(j,k) :: lower_triangle
INTEGER(I4B) :: n
n = 0
if (present(extra)) n=extra
lower_triangle= (outerdiff(arth_i(1,1,j),arth_i(1,1,k)) > -n)
END FUNCTION lower_triangle

Other routines:
FUNCTION vabs(v)
    RETURN the length (ordinary $L_2$ norm) of a vector.
REAL(SP), DIMENSION(:), INTENT(IN) :: v
REAL(SP) :: vabs
vabs=sqrt(dot_product(v,v))
END FUNCTION vabs

END MODULE nrutil
C2. Alphabetical Listing of Explicit Interfaces

The file supplied as nr.f90 contains explicit interfaces for all the Numerical Recipes routines (except those already in the module nrutil). The interfaces are in alphabetical order, by the generic interface name, if one exists, or by the specific routine name if there is no generic name.

The file nr.f90 is normally invoked via a USE statement within a main program or subroutine that references a Numerical Recipes routine. See §21.1 for an example.

```fortran
MODULE nr
    INTERFACE
        SUBROUTINE airy(x,ai,bi,aip,bip)
            USE nrtype
            REAL(SP), INTENT(IN) :: x
            REAL(SP), INTENT(OUT) :: ai,bi,aip,bip
        END SUBROUTINE airy
    END INTERFACE
    INTERFACE
        SUBROUTINE amebsa(p,y,pb,yb,ftol,func,iter,temptr)
            USE nrtype
            INTEGER(I4B), INTENT(INOUT) :: iter
            REAL(SP), INTENT(INOUT) :: yb
            REAL(SP), INTENT(IN) :: ftol,temptr
            REAL(SP), DIMENSION(:), INTENT(INOUT) :: y,pb
            REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: p
            INTERFACE
                FUNCTION func(x)
                    USE nrtype
                    REAL(SP), DIMENSION(:), INTENT(IN) :: x
                    REAL(SP) :: func
                END FUNCTION func
            END INTERFACE
        END SUBROUTINE amebsa
    END INTERFACE
    INTERFACE
        SUBROUTINE amoeba(p,y,ftol,func,iter)
            USE nrtype
            INTEGER(I4B), INTENT(OUT) :: iter
            REAL(SP), INTENT(IN) :: ftol
            REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
            REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: p
            INTERFACE
                FUNCTION func(x)
                    USE nrtype
                    REAL(SP), DIMENSION(:), INTENT(IN) :: x
                    REAL(SP) :: func
                END FUNCTION func
            END INTERFACE
        END SUBROUTINE amoeba
    END INTERFACE
END MODULE nr
```
END SUBROUTINE amoeba
END INTERFACE

INTERFACE
SUBROUTINE anneal(x,y,iorder)
USE nrtype
INTEGER(I4B), DIMENSION(:), INTENT(INOUT) :: iorder
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y
END SUBROUTINE anneal
END INTERFACE

INTERFACE
SUBROUTINE asolve(b,x,itrnsp)
USE nrtype
REAL(DP), DIMENSION(:), INTENT(IN) :: b
REAL(DP), DIMENSION(:), INTENT(OUT) :: x
INTEGER(I4B), INTENT(IN) :: itrnsp
END SUBROUTINE asolve
END INTERFACE

INTERFACE
SUBROUTINE atimes(x,r,itrnsp)
USE nrtype
REAL(DP), DIMENSION(:), INTENT(IN) :: x
REAL(DP), DIMENSION(:), INTENT(OUT) :: r
INTEGER(I4B), INTENT(IN) :: itrnsp
END SUBROUTINE atimes
END INTERFACE

INTERFACE
SUBROUTINE avevar(data,ave,var)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: data
REAL(SP), INTENT(OUT) :: ave,var
END SUBROUTINE avevar
END INTERFACE

INTERFACE
SUBROUTINE balanc(a)
USE nrtype
REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: a
END SUBROUTINE balanc
END INTERFACE

INTERFACE
SUBROUTINE banbks(a,m1,m2,al,indx,b)
USE nrtype
INTEGER(I4B), INTENT(IN) :: m1,m2
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: indx
REAL(SP), DIMENSION(:,:), INTENT(IN) :: a,al
REAL(SP), DIMENSION(:), INTENT(INOUT) :: b
END SUBROUTINE banbks
END INTERFACE

INTERFACE
SUBROUTINE bandec(a,m1,m2,al,indx,d)
USE nrtype
INTEGER(I4B), INTENT(IN) :: m1,m2
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: indx
REAL(SP), DIMENSION(:,:), INTENT(IN) :: a,al
REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: d
REAL(SP), DIMENSION(:,:), INTENT(OUT) :: a
END SUBROUTINE bandec
END INTERFACE

INTERFACE
SUBROUTINE banmul(a,m1,m2,x,b)
USE nrtype
INTEGER(I4B), INTENT(IN) :: m1,m2
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(INOUT) :: b
REAL(SP), DIMENSION(:), INTENT(IN) :: a
END SUBROUTINE banmul
END INTERFACE
END SUBROUTINE bannul
END INTERFACE

INTERFACE
  SUBROUTINE bcucof(y,y1,y2,y12,d1,d2,c)
    USE nrtype
    REAL(SP), INTENT(IN) :: d1,d2
    REAL(SP), DIMENSION(4), INTENT(IN) :: y,y1,y2,y12
    REAL(SP), DIMENSION(4,4), INTENT(OUT) :: c
  END SUBROUTINE bcucof
END INTERFACE

INTERFACE
  SUBROUTINE bcuint(y,y1,y2,y12,x1l,x1u,x2l,x2u,x1,x2,ansy,&
                    ansy1,ansy2)
    USE nrtype
    REAL(SP), DIMENSION(4), INTENT(IN) :: y,y1,y2,y12
    REAL(SP), INTENT(IN) :: x1l,x1u,x2l,x2u,x1,x2
    REAL(SP), INTENT(OUT) :: ansy,ansy1,ansy2
  END SUBROUTINE bcuint
END INTERFACE

INTERFACE beschb
  SUBROUTINE beschb_s(x,gam1,gam2,gampl,gammi)
    USE nrtype
    REAL(DP), INTENT(IN) :: x
    REAL(DP), INTENT(OUT) :: gam1,gam2,gampl,gammi
  END SUBROUTINE beschb_s
  SUBROUTINE beschb_v(x,gam1,gam2,gampl,gammi)
    USE nrtype
    REAL(DP), DIMENSION(:), INTENT(IN) :: x
    REAL(DP), DIMENSION(:), INTENT(OUT) :: gam1,gam2,gampl,gammi
  END SUBROUTINE beschb_v
END INTERFACE

INTERFACE bessi
  FUNCTION bessi_s(n,x)
    USE nrtype
    INTEGER(I4B), INTENT(IN) :: n
    REAL(SP), INTENT(IN) :: x
    REAL(SP) :: bessi_s
  END FUNCTION bessi_s
  FUNCTION bessi_v(n,x)
    USE nrtype
    INTEGER(I4B), INTENT(IN) :: n
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: bessi_v
  END FUNCTION bessi_v
END INTERFACE

INTERFACE bessi0
  FUNCTION bessi0_s(x)
    USE nrtype
    REAL(SP), INTENT(IN) :: x
    REAL(SP) :: bessi0_s
  END FUNCTION bessi0_s
  FUNCTION bessi0_v(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: bessi0_v
  END FUNCTION bessi0_v
END INTERFACE

INTERFACE bessi1
  FUNCTION bessi1_s(x)
    USE nrtype
    REAL(SP), INTENT(IN) :: x
    REAL(SP) :: bessi1_s
  END FUNCTION bessi1_s
END INTERFACE
FUNCTION bessi1_v(x)
USE nrttype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessi1_v
END FUNCTION bessi1_v
END INTERFACE

INTERFACE
SUBROUTINE bessik(x,xnu,ri,rk,rip,rkp)
USE nrttype
REAL(SP), INTENT(IN) :: x,xnu
REAL(SP), INTENT(OUT) :: ri,rk,rip,rkp
END SUBROUTINE bessik
END INTERFACE

INTERFACE bessj
FUNCTION bessj_s(n,x)
USE nrttype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessj_s
END FUNCTION bessj_s
FUNCTION bessj_v(n,x)
USE nrttype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessj_v
END FUNCTION bessj_v
END INTERFACE

INTERFACE bessj0
FUNCTION bessj0_s(x)
USE nrttype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessj0_s
END FUNCTION bessj0_s
FUNCTION bessj0_v(x)
USE nrttype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessj0_v
END FUNCTION bessj0_v
END INTERFACE

INTERFACE bessj1
FUNCTION bessj1_s(x)
USE nrttype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessj1_s
END FUNCTION bessj1_s
FUNCTION bessj1_v(x)
USE nrttype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessj1_v
END FUNCTION bessj1_v
END INTERFACE

INTERFACE bessjy
SUBROUTINE bessjy_s(x,xnu,rj,ry,rjp,ryp)
USE nrttype
REAL(SP), INTENT(IN) :: x,xnu
REAL(SP), INTENT(OUT) :: rj,ry,rjp,ryp
END SUBROUTINE bessjy_s
SUBROUTINE bessjy_v(x,xnu,rj,ry,rjp,ryp)
USE nrttype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(OUT) :: rj,rjp,ry,ryp
END INTERFACE
SUBROUTINE bessjy_v
END SUBROUTINE

INTERFACE bessk
FUNCTION bessk_s(n,x)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessk_s
END FUNCTION bessk_s

FUNCTION bessk_v(n,x)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessk_v
END FUNCTION bessk_v
END INTERFACE

INTERFACE bessk0
FUNCTION bessk0_s(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessk0_s
END FUNCTION bessk0_s

FUNCTION bessk0_v(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessk0_v
END FUNCTION bessk0_v
END INTERFACE

INTERFACE bessk1
FUNCTION bessk1_s(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessk1_s
END FUNCTION bessk1_s

FUNCTION bessk1_v(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessk1_v
END FUNCTION bessk1_v
END INTERFACE

INTERFACE bessy
FUNCTION bessy_s(n,x)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessy_s
END FUNCTION bessy_s

FUNCTION bessy_v(n,x)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessy_v
END FUNCTION bessy_v
END INTERFACE

INTERFACE bessy0
FUNCTION bessy0_s(x)
USE nrwtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: bessy0_s
END FUNCTION bessy0_s

FUNCTION bessy0_v(x)
USE nrwtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: bessy0_v
END FUNCTION bessy0_v
END INTERFACE

INTERFACE bessy1
  FUNCTION bessy1_s(x)
  USE nrtype
  REAL(SP), INTENT(IN) :: x
  REAL(SP) :: bessy1_s
  END FUNCTION bessy1_s
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: x
  REAL(SP), DIMENSION(size(x)) :: bessy1_v
  END FUNCTION bessy1_v
END INTERFACE

INTERFACE beta
  FUNCTION beta_s(z,w)
  USE nrtype
  REAL(SP), INTENT(IN) :: z,w
  REAL(SP) :: beta_s
  END FUNCTION beta_s
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: z,w
  REAL(SP), DIMENSION(size(z)) :: beta_v
  END FUNCTION beta_v
END INTERFACE

INTERFACE betacf
  FUNCTION betacf_s(a,b,x)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b,x
  REAL(SP) :: betacf_s
  END FUNCTION betacf_s
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: a,b,x
  REAL(SP), DIMENSION(size(x)) :: betacf_v
  END FUNCTION betacf_v
END INTERFACE

INTERFACE betai
  FUNCTION betai_s(a,b,x)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b,x
  REAL(SP) :: betai_s
  END FUNCTION betai_s
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: a,b,x
  REAL(SP), DIMENSION(size(a)) :: betai_v
  END FUNCTION betai_v
END INTERFACE

INTERFACE bico
  FUNCTION bico_s(n,k)
  USE nrtype
  INTEGER(I4B), INTENT(IN) :: n,k
  REAL(SP) :: bico_s
  END FUNCTION bico_s
  USE nrtype
  INTEGER(I4B), DIMENSION(:), INTENT(IN) :: n,k
  REAL(SP), DIMENSION(size(n)) :: bico_v
  END FUNCTION bico_v
END INTERFACE
END FUNCTION bico_v
END INTERFACE

INTERFACE
  FUNCTION bnldev(pp,n)
  USE nrtype
  REAL(SP), INTENT(IN) :: pp
  INTEGER(I4B), INTENT(IN) :: n
  REAL(SP) :: bnldev
  END FUNCTION bnldev
END INTERFACE

INTERFACE
  FUNCTION brent(ax,bx,cx,func,tol,xmin)
  USE nrtype
  REAL(SP), INTENT(IN) :: ax,bx,cx,tol
  REAL(SP), INTENT(OUT) :: xmin
  REAL(SP) :: brent
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), INTENT(IN) :: x
    REAL(SP) :: func
    END FUNCTION func
  END INTERFACE
END INTERFACE

INTERFACE
  SUBROUTINE broydn(x,check)
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(INOUT) :: x
  LOGICAL(LGT), INTENT(OUT) :: check
  END SUBROUTINE broydn
END INTERFACE

INTERFACE
  SUBROUTINE bsstep(y,dydx,x,htry,eps,yscal,hdid,hnext,derivs)
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
  REAL(SP), DIMENSION(:), INTENT(IN) :: dydx,yscal
  REAL(SP), INTENT(INOUT) :: x
  REAL(SP), INTENT(IN) :: htry,eps
  REAL(SP), INTENT(OUT) :: hdid,hnext
  INTERFACE
    SUBROUTINE derivs(x,y,dydx)
    USE nrtype
    REAL(SP), INTENT(IN) :: x
    REAL(SP), DIMENSION(:), INTENT(IN) :: y
    REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
    END SUBROUTINE derivs
  END INTERFACE
END SUBROUTINE bsstep
END INTERFACE

INTERFACE
  SUBROUTINE caldat(julian,mm,id,iyyy)
  USE nrtype
  INTEGER(I4B), INTENT(IN) :: julian
  INTEGER(I4B), INTENT(OUT) :: mm,id,iyyy
  END SUBROUTINE caldat
END INTERFACE

INTERFACE
  FUNCTION chder(a,b,c)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP), DIMENSION(:), INTENT(IN) :: c
  REAL(SP), DIMENSION(size(c)) :: chder
  END FUNCTION chder
END INTERFACE

INTERFACE chebev
  FUNCTION chebev_s(a,b,c,x)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b,c
  REAL(SP), DIMENSION(:), INTENT(IN) :: c
  REAL(SP) :: chebev_s
  END FUNCTION chebev_s

  FUNCTION chebev_v(a,b,c,x)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b,c
  REAL(SP), DIMENSION(:), INTENT(IN) :: c,x
  REAL(SP), DIMENSION(size(x)) :: chebev_v
  END FUNCTION chebev_v

END INTERFACE

INTERFACE
  FUNCTION chebft(a,b,n,func)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b
  INTEGER(I4B), INTENT(IN) :: n
  REAL(SP), DIMENSION(n) :: chebft

  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: func
    END FUNCTION func

  END INTERFACE

  END FUNCTION chebft

END INTERFACE

INTERFACE
  FUNCTION chebpc(c)
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: c
  REAL(SP), DIMENSION(size(c)) :: chebpc
  END FUNCTION chebpc

END INTERFACE

INTERFACE
  FUNCTION chint(a,b,c)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b,c
  REAL(SP), DIMENSION(size(c)) :: chint
  END FUNCTION chint

END INTERFACE

INTERFACE
  SUBROUTINE choldc(a,p)
  USE nrtype
  REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: a
  REAL(SP), DIMENSION(:), INTENT(OUT) :: p
  END SUBROUTINE choldc

END INTERFACE

INTERFACE
  SUBROUTINE cholsl(a,p,b,x)
  USE nrtype
  REAL(SP), DIMENSION(:,:), INTENT(IN) :: a
  REAL(SP), DIMENSION(:,:), INTENT(IN) :: p,b
  REAL(SP), DIMENSION(:), INTENT(INOUT) :: x
  END SUBROUTINE cholsl

END INTERFACE

INTERFACE
  SUBROUTINE chsone(bins,ebins,knstrn,df,chsq,prob)
  USE nrtype

INTEGER(I4B), INTENT(IN) :: knstrn
REAL(SP), INTENT(OUT) :: df, chsq, prob
REAL(SP), DIMENSION(:,), INTENT(IN) :: bins, ebins
END SUBROUTINE chsone
END INTERFACE

INTERFACE
SUBROUTINE chstwo(bins1, bins2, knstrn, df, chsq, prob)
USE nrtype
INTEGER(I4B), INTENT(IN) :: knstrn
REAL(SP), INTENT(OUT) :: df, chsq, prob
REAL(SP), DIMENSION(:,), INTENT(IN) :: bins1, bins2
END SUBROUTINE chstwo
END INTERFACE

INTERFACE
SUBROUTINE cisi(x, ci, si)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: ci, si
END SUBROUTINE cisi
END INTERFACE

INTERFACE
SUBROUTINE cntab1(nn, chisq, df, prob, cramrv, ccc)
USE nrtype
INTEGER(I4B), DIMENSION(:,,:), INTENT(IN) :: nn
REAL(SP), INTENT(OUT) :: chisq, df, prob, cramrv, ccc
END SUBROUTINE cntab1
END INTERFACE

INTERFACE
SUBROUTINE cntab2(nn, h, hx, hy, hygx, hxgy, uygx, uxy)
USE nrtype
INTEGER(I4B), DIMENSION(:,,:), INTENT(IN) :: nn
REAL(SP), INTENT(OUT) :: h, hx, hy, hygx, hxgy, uygx, uxy
END SUBROUTINE cntab2
END INTERFACE

INTERFACE
FUNCTION convlv(data, respns, isign)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: data
REAL(SP), DIMENSION(:), INTENT(IN) :: respns
INTEGER(I4B), INTENT(IN) :: isign
REAL(SP), DIMENSION(size(data)) :: convlv
END FUNCTION convlv
END INTERFACE

INTERFACE
FUNCTION correl(data1, data2)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: data1, data2
REAL(SP), DIMENSION(size(data1)) :: correl
END FUNCTION correl
END INTERFACE

INTERFACE
SUBROUTINE cosft1(y)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
END SUBROUTINE cosft1
END INTERFACE

INTERFACE
SUBROUTINE cosft2(y, isign)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE cosft2
END INTERFACE
SUBROUTINE covsrt(covar,maska)
USE nrtype
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: covar
LOGICAL(LGT), DIMENSION(:,), INTENT(IN) :: maska
END SUBROUTINE covsrt
END INTERFACE

INTERFACE
SUBROUTINE cyclic(a,b,c,alp,bet,r,x)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: a,b,c,r
REAL(SP), INTENT(IN) :: alp,bet
REAL(SP), DIMENSION(:,), INTENT(OUT) :: x
END SUBROUTINE cyclic
END INTERFACE

INTERFACE
SUBROUTINE daub4(a,isign)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE daub4
END INTERFACE

INTERFACE
dawson
  FUNCTION dawson_s(x)
  USE nrtype
  REAL(SP) :: dawson_s
  END FUNCTION dawson_s

  FUNCTION dawson_v(x)
  USE nrtype
  REAL(SP), DIMENSION(size(x)) :: dawson_v
  END FUNCTION dawson_v
END INTERFACE

INTERFACE
FUNCTION dbrent(ax,bx,cx,func,dbrent_dfunc,tol,xmin)
USE nrtype
REAL(SP), INTENT(IN) :: ax,bx,cx,tol
REAL(SP), INTENT(OUT) :: xmin
REAL(SP) :: dbrent
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
FUNCTION dbrent_dfunc(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: dbrent_dfunc
END FUNCTION dbrent_dfunc
END INTERFACE

FUNCTION dbrent
END FUNCTION dbrent
END INTERFACE

INTERFACE
SUBROUTINE ddpoly(c,x,pd)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: c
REAL(SP), DIMENSION(:,), INTENT(OUT) :: pd
END SUBROUTINE ddpoly
END INTERFACE

INTERFACE
FUNCTION decchk(string,ch)
END FUNCTION decchk
END INTERFACE
USE nrtype
CHARACTER(1), DIMENSION(:), INTENT(IN) :: string
CHARACTER(1), INTENT(OUT) :: ch
LOGICAL(LGT) :: decchk
END FUNCTION decchk
END INTERFACE
END INTERFACE
INTERFACE
SUBROUTINE dfpmin(p,gtol,iter,fret,func,dfunc)
USE nrtype
INTEGER(I4B), INTENT(OUT) :: iter
REAL(SP), INTENT(IN) :: gtol
REAL(SP), INTENT(OUT) :: fret
REAL(SP), DIMENSION(:), INTENT(INOUT) :: p
INTERFACE
FUNCTION func(p)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: p
REAL(SP) :: func
END FUNCTION func
FUNCTION dfunc(p)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: p
REAL(SP), DIMENSION(size(p)) :: dfunc
END FUNCTION dfunc
END INTERFACE
END SUBROUTINE dfpmin
END INTERFACE
INTERFACE
SUBROUTINE dfridr(func,x,h,err)
USE nrtype
REAL(SP), INTENT(IN) :: x,h
REAL(SP), INTENT(OUT) :: err
REAL(SP) :: dfridr
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
END SUBROUTINE dfridr
END INTERFACE
INTERFACE
SUBROUTINE dftcor(w,delta,a,b,endpts,corre,corim,corfac)
USE nrtype
REAL(SP), INTENT(IN) :: w,delta,a,b
REAL(SP), INTENT(OUT) :: corre,corim,corfac
REAL(SP), DIMENSION(:) :: endpts
END SUBROUTINE dftcor
END INTERFACE
INTERFACE
SUBROUTINE dftint(func,a,b,w,cosint,sinint)
USE nrtype
REAL(SP), INTENT(IN) :: a,b,w
REAL(SP), INTENT(OUT) :: cosint,sinint
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
END SUBROUTINE dftint
END INTERFACE
SUBROUTINE difeq(k,k1,k2,jsf,is1,isf,indexv,s,y)
USE nrtype
INTEGER(I4B), INTENT(IN) :: is1,isf,jsf,k,k1,k2
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: indexv
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: s
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: y
END SUBROUTINE difeq

FUNCTION eclass(lista,listb,n)
USE nrtype
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: lista,listb
INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), DIMENSION(n) :: eclass
END FUNCTION eclass

FUNCTION eclazz(equiv,n)
USE nrtype
INTERFACE
FUNCTION equiv(i,j)
USE nrtype
LOGICAL(LGT) :: equiv
INTEGER(I4B), INTENT(IN) :: i,j
END FUNCTION equiv
END INTERFACE
INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), DIMENSION(n) :: eclazz
END FUNCTION eclazz

FUNCTION ei(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: ei
END FUNCTION ei

SUBROUTINE eigsrt(d,v)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: d
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: v
END SUBROUTINE eigsrt

FUNCTION elle_s(phi,ak)
USE nrtype
REAL(SP), INTENT(IN) :: phi,ak
REAL(SP) :: elle_s
END FUNCTION elle_s

FUNCTION elle_v(phi,ak)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: phi,ak
REAL(SP), DIMENSION(size(phi)) :: elle_v
END FUNCTION elle_v

FUNCTION ellef_s(phi,ak)
USE nrtype
REAL(SP), INTENT(IN) :: phi,ak
REAL(SP) :: ellef_s
END FUNCTION ellef_s

FUNCTION ellf_v(phi,ak)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: phi,ak
REAL(SP), DIMENSION(size(phi)) :: ellf_v
END FUNCTION ellf_v

FUNCTION ellf(phi,ak)
USE nrtype
REAL(SP), INTENT(IN) :: phi,ak
REAL(SP) :: ellf
END FUNCTION ellf

SUBROUTINE ellf(d,v)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: d
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: v
END SUBROUTINE ellf

SUBROUTINE ellf_s(phi,ak)
USE nrtype
REAL(SP), INTENT(IN) :: phi,ak
REAL(SP) :: ellf_s
END SUBROUTINE ellf_s

SUBROUTINE ellf_v(phi,ak)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: phi,ak
REAL(SP), DIMENSION(size(phi)) :: ellf_v
END SUBROUTINE ellf_v

SUBROUTINE ellf(phi,ak)
USE nrtype
REAL(SP), INTENT(IN) :: phi,ak
REAL(SP) :: ellf
END SUBROUTINE ellf

FUNCTION ellf_s(phi,ak)
USE nrtype
REAL(SP), INTENT(IN) :: phi,ak
REAL(SP) :: ellf_s
END FUNCTION ellf_s

FUNCTION ellf_v(phi,ak)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: phi,ak
REAL(SP), DIMENSION(size(phi)) :: ellf_v
END FUNCTION ellf_v
END FUNCTION ellf_s
FUNCTION ellf_v(phi,ak)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: phi,ak
REAL(SP), DIMENSION(size(phi)) :: ellf_v
END FUNCTION ellf_v
END INTERFACE

INTERFACE ellpi
FUNCTION ellpi_s(phi,en,ak)
USE nrtype
REAL(SP), INTENT(IN) :: phi,en,ak
REAL(SP) :: ellpi_s
END FUNCTION ellpi_s
FUNCTION ellpi_v(phi,en,ak)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: phi,en,ak
REAL(SP), DIMENSION(size(phi)) :: ellpi_v
END FUNCTION ellpi_v
END INTERFACE

INTERFACE
SUBROUTINE elmhes(a)
USE nrtype
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a
END SUBROUTINE elmhes
END INTERFACE

INTERFACE erf
FUNCTION erf_s(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: erf_s
END FUNCTION erf_s
FUNCTION erf_v(x)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: erf_v
END FUNCTION erf_v
END INTERFACE

INTERFACE erfc
FUNCTION erfc_s(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: erfc_s
END FUNCTION erfc_s
FUNCTION erfc_v(x)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: erfc_v
END FUNCTION erfc_v
END INTERFACE

INTERFACE erfcc
FUNCTION erfcc_s(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: erfcc_s
END FUNCTION erfcc_s
FUNCTION erfcc_v(x)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: erfcc_v
END FUNCTION erfcc_v
END INTERFACE

INTERFACE
SUBROUTINE eulsum(sum, term, jterm)
USE nrtype
REAL(SP), INTENT(INOUT) :: sum
REAL(SP), INTENT(IN) :: term
INTEGER(I4B), INTENT(IN) :: jterm
END SUBROUTINE eulsum
END INTERFACE

INTERFACE
FUNCTION evlmem(fdt, d, xms)
USE nrtype
REAL(SP), INTENT(IN) :: fdt, xms
REAL(SP), DIMENSION(:), INTENT(IN) :: d
REAL(SP) :: evlmem
END FUNCTION evlmem
END INTERFACE

INTERFACE expdev
SUBROUTINE expdev_s(harvest)
USE nrtype
REAL(SP), INTENT(OUT) :: harvest
END SUBROUTINE expdev_s
SUBROUTINE expdev_v(harvest)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(OUT) :: harvest
END SUBROUTINE expdev_v
END INTERFACE

INTERFACE
FUNCTION expint(n, x)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: x
REAL(SP) :: expint
END FUNCTION expint
END INTERFACE

INTERFACE factln
FUNCTION factln_s(n)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP) :: factln_s
END FUNCTION factln_s
FUNCTION factln_v(n)
USE nrtype
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: n
REAL(SP), DIMENSION(size(n)) :: factln_v
END FUNCTION factln_v
END INTERFACE

INTERFACE factrl
FUNCTION factrl_s(n)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP) :: factrl_s
END FUNCTION factrl_s
FUNCTION factrl_v(n)
USE nrtype
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: n
REAL(SP), DIMENSION(size(n)) :: factrl_v
END FUNCTION factrl_v
END INTERFACE

INTERFACE
SUBROUTINE fasper(x, y, ofac, hifac, px, py, jmax, prob)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x, y
REAL(SP), INTENT(IN) :: ofac, hifac
INTEGER(I4B), INTENT(OUT) :: jmax
REAL(SP), INTENT(OUT) :: prob
REAL(SP), DIMENSION(:), POINTER :: px,py
END SUBROUTINE fasper
END INTERFACE
INTERFACE
SUBROUTINE fdjac(x,fvec,df)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: fvec
REAL(SP), DIMENSION(:), INTENT(INOUT) :: x
REAL(SP), DIMENSION(:,:), INTENT(OUT) :: df
END SUBROUTINE fdjac
END INTERFACE
INTERFACE
SUBROUTINE fgauss(x,a,y,dyda)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,a
REAL(SP), DIMENSION(:), INTENT(OUT) :: y
REAL(SP), DIMENSION(:,:), INTENT(OUT) :: dyda
END SUBROUTINE fgauss
END INTERFACE
INTERFACE
SUBROUTINE fit(x,y,a,b,siga,sigb,chi2,q,sig)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y
REAL(SP), INTENT(OUT) :: a,b,siga,sigb,chi2,q
REAL(SP), INTENT(INOUT) :: sig
END SUBROUTINE fit
END INTERFACE
INTERFACE
SUBROUTINE fitexy(x,y,sigx,sigy,a,b,siga,sigb,chi2,q)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sigx,sigy
REAL(SP), INTENT(OUT) :: a,b,siga,sigb,chi2,q
END SUBROUTINE fitexy
END INTERFACE
INTERFACE
SUBROUTINE fixrts(d)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: d
END SUBROUTINE fixrts
END INTERFACE
INTERFACE
FUNCTION fleg(x,n)
USE nrtype
REAL(SP), INTENT(IN) :: x
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(n) :: fleg
END FUNCTION fleg
END INTERFACE
INTERFACE
SUBROUTINE flmoon(n,nph,jd,frac)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n,nph
INTEGER(I4B), INTENT(OUT) :: jd
REAL(SP), INTENT(OUT) :: frac
END SUBROUTINE flmoon
END INTERFACE
INTERFACE
SUBROUTINE four1_dp(data,isign)
USE nrtype
COMPLEX(DPC), DIMENSION(:), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE four1_dp
END INTERFACE
SUBROUTINE four1_sp(data,isign)
  USE nrtype
  COMPLEX(SPC), DIMENSION(:), INTENT(INOUT) :: data
  INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE four1_sp

END INTERFACE

INTERFACE
  SUBROUTINE four1_alt(data,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE four1_alt
END INTERFACE

INTERFACE
  SUBROUTINE four1_gather(data,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE four1_gather
END INTERFACE

INTERFACE
  SUBROUTINE four2(data,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:,:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE four2
END INTERFACE

INTERFACE
  SUBROUTINE four2_alt(data,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:,:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE four2_alt
END INTERFACE

INTERFACE
  SUBROUTINE four3(data,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:,:,:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE four3
END INTERFACE

INTERFACE
  SUBROUTINE four3_alt(data,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:,:,:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE four3_alt
END INTERFACE

INTERFACE
  SUBROUTINE fourcol(data,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:,:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE fourcol
END INTERFACE

INTERFACE
  SUBROUTINE fourcol_3d(data,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:,:,:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE fourcol_3d
END INTERFACE

INTERFACE
  SUBROUTINE fourn_gather(data,nn,isign)
    USE nrtype
    COMPLEX(SPC), DIMENSION(:,:), INTENT(INOUT) :: data
    INTEGER(I4B), INTENT(IN) :: isign
  END SUBROUTINE fourn_gather
END INTERFACE
USE nrtype  
COMPLEX(SPC), DIMENSION(:), INTENT(INOUT) :: data  
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: nn  
INTEGER(I4B), INTENT(IN) :: isign  
END SUBROUTINE fourn_gather  
END INTERFACE

INTERFACE fourrow  
SUBROUTINE fourrow_dp(data,isign)  
USE nrtype  
COMPLEX(DPC), DIMENSION(:,,:), INTENT(INOUT) :: data  
INTEGER(I4B), INTENT(IN) :: isign  
END SUBROUTINE fourrow_dp  
SUBROUTINE fourrow_sp(data,isign)  
USE nrtype  
COMPLEX(SPC), DIMENSION(:,,:), INTENT(INOUT) :: data  
INTEGER(I4B), INTENT(IN) :: isign  
END SUBROUTINE fourrow_sp  
END INTERFACE

INTERFACE  
SUBROUTINE fourrow_3d(data,isign)  
USE nrtype  
COMPLEX(SPC), DIMENSION(:,:,:), INTENT(INOUT) :: data  
INTEGER(I4B), INTENT(IN) :: isign  
END SUBROUTINE fourrow_3d  
END INTERFACE

INTERFACE  
FUNCTION fpoly(x,n)  
USE nrtype  
REAL(SP), INTENT(IN) :: x  
INTEGER(I4B), INTENT(IN) :: n  
REAL(SP), DIMENSION(n) :: fpoly  
END FUNCTION fpoly  
END INTERFACE

INTERFACE  
SUBROUTINE fred2(a,b,t,f,w,g,ak)  
USE nrtype  
REAL(SP), INTENT(IN) :: a,b  
REAL(SP), DIMENSION(:,), INTENT(OUT) :: t,f,w  
INTERFACE  
FUNCTION g(t)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t  
REAL(SP), DIMENSION(size(t)) :: g  
END FUNCTION g  
FUNCTION ak(t,s)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t,s  
REAL(SP), DIMENSION(size(t),size(s)) :: ak  
END FUNCTION ak  
END INTERFACE  
END SUBROUTINE fred2  
END INTERFACE

INTERFACE  
FUNCTION fredin(x,a,b,t,f,w,g,ak)  
USE nrtype  
REAL(SP), INTENT(IN) :: a,b  
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,t,f,w  
REAL(SP), DIMENSION(size(x)) :: fredin  
INTERFACE  
FUNCTION g(t)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t  
REAL(SP), DIMENSION(size(t)) :: g  
END FUNCTION g  
END INTERFACE  
END SUBROUTINE fredin  
END INTERFACE

SUBROUTINE fred2(a,b,t,f,w,g,ak)  
USE nrtype  
REAL(SP), INTENT(IN) :: a,b  
REAL(SP), DIMENSION(:,), INTENT(OUT) :: t,f,w  
INTERFACE  
FUNCTION g(t)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t  
REAL(SP), DIMENSION(size(t)) :: g  
END FUNCTION g  
FUNCTION ak(t,s)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t,s  
REAL(SP), DIMENSION(size(t),size(s)) :: ak  
END FUNCTION ak  
END INTERFACE  
END SUBROUTINE fred2  
END INTERFACE

FUNCTION fredin(x,a,b,t,f,w,g,ak)  
USE nrtype  
REAL(SP), INTENT(IN) :: a,b  
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,t,f,w  
REAL(SP), DIMENSION(size(x)) :: fredin  
INTERFACE  
FUNCTION g(t)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t  
REAL(SP), DIMENSION(size(t)) :: g  
END FUNCTION g  
END INTERFACE  
END SUBROUTINE fredin  
END INTERFACE

FUNCTION fred2(a,b,t,f,w,g,ak)  
USE nrtype  
REAL(SP), INTENT(IN) :: a,b  
REAL(SP), DIMENSION(:,), INTENT(OUT) :: t,f,w  
INTERFACE  
FUNCTION g(t)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t  
REAL(SP), DIMENSION(size(t)) :: g  
END FUNCTION g  
FUNCTION ak(t,s)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t,s  
REAL(SP), DIMENSION(size(t),size(s)) :: ak  
END FUNCTION ak  
END INTERFACE  
END SUBROUTINE fred2  
END INTERFACE

FUNCTION fredin(x,a,b,t,f,w,g,ak)  
USE nrtype  
REAL(SP), INTENT(IN) :: a,b  
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,t,f,w  
REAL(SP), DIMENSION(size(x)) :: fredin  
INTERFACE  
FUNCTION g(t)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: t  
REAL(SP), DIMENSION(size(t)) :: g  
END FUNCTION g  
END INTERFACE  
END SUBROUTINE fredin  
END INTERFACE
Appendix C2. Alphabetical Listing of Explicit Interfaces

END FUNCTION g
FUNCTION ak(t,s)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: t,s
REAL(SP), DIMENSION(size(t),size(s)) :: ak
END FUNCTION ak
END INTERFACE
END FUNCTION fredin
END INTERFACE
INTERFACE
SUBROUTINE frenel(x,s,c)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: s,c
END SUBROUTINE frenel
END INTERFACE
INTERFACE
SUBROUTINE frprmn(p,ftol,iter,fret)
USE nrtype
INTEGER(I4B), INTENT(OUT) :: iter
REAL(SP), INTENT(IN) :: ftol
REAL(SP), INTENT(OUT) :: fret
REAL(SP), DIMENSION(:), INTENT(INOUT) :: p
END SUBROUTINE frprmn
END INTERFACE
INTERFACE
SUBROUTINE ftest(data1,data2,f,prob)
USE nrtype
REAL(SP), INTENT(OUT) :: f,prob
REAL(SP), DIMENSION(:), INTENT(IN) :: data1,data2
END SUBROUTINE ftest
END INTERFACE
INTERFACE
FUNCTION gamdev(ia)
USE nrtype
INTEGER(I4B), INTENT(IN) :: ia
REAL(SP) :: gamdev
END FUNCTION gamdev
END INTERFACE
INTERFACE gammln
FUNCTION gammln_s(xx)
USE nrtype
REAL(SP), INTENT(IN) :: xx
REAL(SP) :: gammln_s
END FUNCTION gammln_s
FUNCTION gammln_v(xx)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: xx
REAL(SP), DIMENSION(size(xx)) :: gammln_v
END FUNCTION gammln_v
END INTERFACE
INTERFACE gammp
FUNCTION gammp_s(a,x)
USE nrtype
REAL(SP), INTENT(IN) :: a,x
REAL(SP) :: gammp_s
END FUNCTION gammp_s
FUNCTION gammp_v(a,x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: a,x
REAL(SP), DIMENSION(size(a)) :: gammp_v
END FUNCTION gammp_v
END INTERFACE
INTERFACE gammq
  FUNCTION gammq_s(a,x)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,x
  REAL(SP) :: gammq_s
  END FUNCTION gammq_s
  FUNCTION gammq_v(a,x)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(IN) :: a,x
  REAL(SP), DIMENSION(size(a)) :: gammq_v
  END FUNCTION gammq_v
END INTERFACE

INTERFACE gasdev
  SUBROUTINE gasdev_s(harvest)
  USE nrtype
  REAL(SP), INTENT(OUT) :: harvest
  END SUBROUTINE gasdev_s
  SUBROUTINE gasdev_v(harvest)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(OUT) :: harvest
  END SUBROUTINE gasdev_v
END INTERFACE

INTERFACE
  SUBROUTINE gaucof(a,b,amu0,x,w)
  USE nrtype
  REAL(SP), INTENT(IN) :: amu0
  REAL(SP), DIMENSION(:), INTENT(INOUT) :: a,b
  REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
  END SUBROUTINE gaucof
END INTERFACE

INTERFACE
  SUBROUTINE gauher(x,w)
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(INOUT) :: x,w
  END SUBROUTINE gauher
END INTERFACE

INTERFACE
  SUBROUTINE gaujac(x,w,alf,bet)
  USE nrtype
  REAL(SP), INTENT(IN) :: alf,bet
  REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
  END SUBROUTINE gaujac
END INTERFACE

INTERFACE
  SUBROUTINE gaulag(x,w,alf)
  USE nrtype
  REAL(SP), INTENT(IN) :: alf
  REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
  END SUBROUTINE gaulag
END INTERFACE

INTERFACE
  SUBROUTINE gauleg(x1,x2,x,w)
  USE nrtype
  REAL(SP), INTENT(IN) :: x1,x2
  REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
  END SUBROUTINE gauleg
END INTERFACE

INTERFACE
  SUBROUTINE gaussj(a,b)
  USE nrtype
  REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: a,b
  END SUBROUTINE gaussj
END INTERFACE
Appendix C2. Alphabetical Listing of Explicit Interfaces

INTERFACE gcf
  FUNCTION gcf_s(a,x,gln)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,x
  REAL(SP), OPTIONAL, INTENT(OUT) :: gln
  REAL(SP) :: gcf_s
  END FUNCTION gcf_s
  
  FUNCTION gcf_v(a,x,gln)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(IN) :: a,x
  REAL(SP), DIMENSION(:,), OPTIONAL, INTENT(OUT) :: gln
  REAL(SP), DIMENSION(size(a)) :: gcf_v
  END FUNCTION gcf_v
END INTERFACE

INTERFACE
  FUNCTION golden(ax,bx,cx,func,tol,xmin)
  USE nrtype
  REAL(SP), INTENT(IN) :: ax,bx,cx,tol
  REAL(SP), INTENT(OUT) :: xmin
  REAL(SP) :: golden
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), INTENT(IN) :: x
    REAL(SP) :: func
    END FUNCTION func
  END INTERFACE
  END FUNCTION golden
END INTERFACE

INTERFACE gser
  FUNCTION gser_s(a,x,gln)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,x
  REAL(SP), OPTIONAL, INTENT(OUT) :: gln
  REAL(SP) :: gser_s
  END FUNCTION gser_s
  
  FUNCTION gser_v(a,x,gln)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(IN) :: a,x
  REAL(SP), DIMENSION(:,), OPTIONAL, INTENT(OUT) :: gln
  REAL(SP), DIMENSION(size(a)) :: gser_v
  END FUNCTION gser_v
END INTERFACE

INTERFACE
  SUBROUTINE hqr(a,wr,wi)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(OUT) :: wr,wi
  REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
  END SUBROUTINE hqr
END INTERFACE

INTERFACE
  SUBROUTINE hunt(xx,x,jlo)
  USE nrtype
  INTEGER(I4B), INTENT(INOUT) :: jlo
  REAL(SP), INTENT(IN) :: x
  REAL(SP), DIMENSION(:,), INTENT(IN) :: xx
  END SUBROUTINE hunt
END INTERFACE

INTERFACE
  SUBROUTINE hypdrv(s,ry,rdyds)
  USE nrtype
  REAL(SP), INTENT(IN) :: s
  REAL(SP), DIMENSION(:,), INTENT(IN) :: ry
END INTERFACE
REAL(SP), DIMENSION(:), INTENT(OUT) :: rdyds
END SUBROUTINE hypdrv
END INTERFACE

INTERFACE
  FUNCTION hypgeo(a,b,c,z)
  USE nrtype
  COMPLEX(SPC), INTENT(IN) :: a,b,c,z
  COMPLEX(SPC) :: hypgeo
  END FUNCTION hypgeo
END INTERFACE

 INTERFACE
  SUBROUTINE hypser(a,b,c,z,series,deriv)
  USE nrtype
  COMPLEX(SPC), INTENT(IN) :: a,b,c,z
  COMPLEX(SPC), INTENT(OUT) :: series,deriv
  END SUBROUTINE hypser
END INTERFACE

INTERFACE
  FUNCTION icrc(crc,buf,jinit,jrev)
  USE nrtype
  CHARACTER(1), DIMENSION(:), INTENT(IN) :: buf
  INTEGER(I2B), INTENT(IN) :: crc,jinit
  INTEGER(I4B), INTENT(IN) :: jrev
  INTEGER(I2B) :: icrc
  END FUNCTION icrc
END INTERFACE

INTERFACE
  SUBROUTINE index_bypack(arr,index,partial)
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: arr
  INTEGER(I4B), DIMENSION(:), INTENT(INOUT) :: index
  INTEGER, OPTIONAL, INTENT(IN) :: partial
  END SUBROUTINE index_bypack
END INTERFACE

INTERFACE indexx
  SUBROUTINE indexx_sp(arr,index)
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: arr
  INTEGER(I4B), DIMENSION(:), INTENT(OUT) :: index
  END SUBROUTINE indexx_sp
END INTERFACE

INTERFACE
  SUBROUTINE indexx_i4b(iarr,index)
  USE nrtype
  INTEGER(I4B), DIMENSION(:), INTENT(IN) :: iarr
  INTEGER(I4B), DIMENSION(:), INTENT(OUT) :: index
  END SUBROUTINE indexx_i4b
END INTERFACE

INTERFACE
  FUNCTION interp(uc)
  USE nrtype
  REAL(DP), DIMENSION(:,:), INTENT(IN) :: uc
  REAL(DP), DIMENSION(2*size(uc,1)-1,2*size(uc,1)-1) :: interp
  END FUNCTION interp
END INTERFACE

INTERFACE
  FUNCTION rank(indx)
  USE nrtype
  INTEGER(I4B), DIMENSION(:), INTENT(IN) :: indx
  END FUNCTION rank
END INTERFACE
INTEGER(4B), DIMENSION(size(indx)) :: rank
END FUNCTION rank

END INTERFACE

INTERFACE
FUNCTION irbit1(iseed)
USE nrtype
INTEGER(4B), INTENT(INOUT) :: iseed
INTEGER(4B) :: irbit1
END FUNCTION irbit1
END INTERFACE

INTERFACE
FUNCTION irbit2(iseed)
USE nrtype
INTEGER(4B), INTENT(INOUT) :: iseed
INTEGER(4B) :: irbit2
END FUNCTION irbit2
END INTERFACE

INTERFACE
SUBROUTINE jacobi(a,d,v,nrot)
USE nrtype
INTEGER(4B), INTENT(OUT) :: nrot
REAL(SP), DIMENSION(:,), INTENT(OUT) :: d
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: v
END SUBROUTINE jacobi
END INTERFACE

INTERFACE
SUBROUTINE jacobn(x,y,dfdx,dfdy)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dfdx
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: dfdy
END SUBROUTINE jacobn
END INTERFACE

INTERFACE
FUNCTION julday(mm,id,iyyy)
USE nrtype
INTEGER(4B), INTENT(IN) :: mm,id,iyyy
INTEGER(4B) :: julday
END FUNCTION julday
END INTERFACE

INTERFACE
SUBROUTINE kendl1(data1,data2,tau,z,prob)
USE nrtype
REAL(SP), INTENT(OUT) :: tau,z,prob
REAL(SP), DIMENSION(:,), INTENT(IN) :: data1,data2
END SUBROUTINE kendl1
END INTERFACE

INTERFACE
SUBROUTINE kendl2(tab,tau,z,prob)
USE nrtype
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: tab
REAL(SP), INTENT(OUT) :: tau,z,prob
END SUBROUTINE kendl2
END INTERFACE

INTERFACE
FUNCTION kermom(y,m)
USE nrtype
REAL(DP), INTENT(IN) :: y
INTEGER(4B), INTENT(IN) :: m
REAL(DP), DIMENSION(m) :: kermom
END FUNCTION kermom
END INTERFACE
INTERFACE
  SUBROUTINE ks2d1s(x1,y1,quadvl,d1,prob)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x1,y1
    REAL(SP), INTENT(OUT) :: d1,prob
  END SUBROUTINE ks2d1s
END INTERFACE
END SUBROUTINE quadvl
END INTERFACE
INTERFACE
  SUBROUTINE ks2d2s(x1,y1,x2,y2,d,prob)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x1,y1,x2,y2
    REAL(SP), INTENT(OUT) :: d,prob
  END SUBROUTINE ks2d2s
END INTERFACE
INTERFACE
  SUBROUTINE ksone(data,func,d,prob)
    USE nrtype
    REAL(SP), INTENT(OUT) :: d,prob
    REAL(SP), DIMENSION(:), INTENT(INOUT) :: data
    INTERFACE
      FUNCTION func(x)
        USE nrtype
        REAL(SP), DIMENSION(:), INTENT(IN) :: x
        REAL(SP), DIMENSION(size(x)) :: func
      END FUNCTION func
    END INTERFACE
  END SUBROUTINE ksone
END INTERFACE
INTERFACE
  SUBROUTINE kstwo(data1,data2,d,prob)
    USE nrtype
    REAL(SP), INTENT(OUT) :: d,prob
    REAL(SP), DIMENSION(:), INTENT(IN) :: data1,data2
  END SUBROUTINE kstwo
END INTERFACE
INTERFACE
  SUBROUTINE laguer(a,x,its)
    USE nrtype
    INTEGER(I4B), INTENT(OUT) :: its
    COMPLEX(SPC), INTENT(INOUT) :: x
    COMPLEX(SPC), DIMENSION(:), INTENT(IN) :: a
  END SUBROUTINE laguer
END INTERFACE
INTERFACE
  SUBROUTINE lfit(x,y,sig,a,maska,covar,chisq,funcs)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
    REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
    LOGICAL(LGT), DIMENSION(:), INTENT(INOUT) :: maska
    REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: covar
    REAL(SP), INTENT(OUT) :: chisq
    INTERFACE
      SUBROUTINE funcs(x,arr)
        USE nrtype
        REAL(SP), INTENT(IN) :: x
        REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
      END SUBROUTINE funcs
      SUBROUTINE funcs(x,arr)
        USE nrtype
        REAL(SP), INTENT(IN) :: x
        REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
      END SUBROUTINE funcs
    END INTERFACE
  END SUBROUTINE lfit
END INTERFACE
END INTERFACE
END SUBROUTINE lfit
END INTERFACE
INTERFACE
SUBROUTINE linbcg(b,x,itol,tol,itmax,iter,err)
USE nrtype
REAL(DP), DIMENSION(:), INTENT(IN) :: b
REAL(DP), DIMENSION(:), INTENT(INOUT) :: x
INTEGER(I4B), INTENT(IN) :: itol,itmax
REAL(DP), INTENT(IN) :: tol
INTEGER(I4B), INTENT(OUT) :: iter
REAL(DP), INTENT(OUT) :: err
END SUBROUTINE linbcg
END INTERFACE
INTERFACE
SUBROUTINE linmin(p,xi,fret)
USE nrtype
REAL(SP), INTENT(OUT) :: fret
REAL(SP), DIMENSION(:), TARGET, INTENT(INOUT) :: p,xi
END SUBROUTINE linmin
END INTERFACE
INTERFACE
SUBROUTINE lnsrch(xold,fold,g,p,x,stpmax,check,func)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: xold,g
REAL(SP), DIMENSION(:), INTENT(INOUT) :: p
REAL(SP), INTENT(IN) :: fold,stpmax
REAL(SP), DIMENSION(:), INTENT(OUT) :: x
REAL(SP), INTENT(OUT) :: f
LOGICAL(LGT), INTENT(OUT) :: check
FUNCTION func(x)
USE nrtype
REAL(SP) :: func
REAL(SP), DIMENSION(:), INTENT(IN) :: x
END FUNCTION func
END INTERFACE
END SUBROUTINE lnsrch
END INTERFACE
INTERFACE
FUNCTION locate(xx,x)
USE nrtype
REAL(SP), DIMENSION(:) :: xx
REAL(SP), INTENT(IN) :: x
INTEGER(I4B) :: locate
END FUNCTION locate
END INTERFACE
INTERFACE
FUNCTION lop(u)
USE nrtype
REAL(DP), DIMENSION(:,:), INTENT(IN) :: u
REAL(DP), DIMENSION(size(u,1),size(u,1)) :: lop
END FUNCTION lop
END INTERFACE
INTERFACE
SUBROUTINE lubksb(a,indx,b)
USE nrtype
REAL(SP), DIMENSION(:,:) :: a
INTEGER(I4B), DIMENSION(:) :: indx
REAL(SP), DIMENSION(:), INTENT(INOUT) :: b
END SUBROUTINE lubksb
END INTERFACE
INTERFACE
SUBROUTINE ludcmp(a,indx,d)
USE nrtype
REAL(SP), DIMENSION(:,:) :: a
INTEGER(I4B), DIMENSION(:) :: indx
REAL(SP), DIMENSION(:), INTENT(INOUT) :: d
END SUBROUTINE ludcmp
END INTERFACE
Appendix C2. Alphabetical Listing of Explicit Interfaces

USE nrtype
REAL(SP), DIMENSION(:,::), INTENT(INOUT) :: a
INTEGER(14B), DIMENSION(:,::), INTENT(OUT) :: indx
REAL(SP), INTENT(OUT) :: d
END SUBROUTINE ludcmp
END INTERFACE

INTERFACE
SUBROUTINE machar(ibeta,it,irnd,ngrd,machep,negep,iexp,&
maxexp,eps,epsneg,xmin,xmax)
USE nrtype
INTEGER(I4B), INTENT(OUT) :: ibeta,iexp,irnd,it,machep,maxexp,&
negep,ngrd
REAL(SP), INTENT(OUT) :: eps,epsneg,xmax,xmin
END SUBROUTINE machar
END INTERFACE

INTERFACE
SUBROUTINE medfit(x,y,a,b,abdev)
USE nrtype
REAL(SP), DIMENSION(:,::), INTENT(IN) :: x,y
REAL(SP), INTENT(OUT) :: a,b,abdev
END SUBROUTINE medfit
END INTERFACE

INTERFACE
SUBROUTINE memcof(data,xms,d)
USE nrtype
REAL(SP), INTENT(OUT) :: xms
REAL(SP), DIMENSION(:,::), INTENT(IN) :: data
REAL(SP), DIMENSION(:,::), INTENT(OUT) :: d
END SUBROUTINE memcof
END INTERFACE

INTERFACE
SUBROUTINE mgfas(u,maxcyc)
USE nrtype
REAL(DP), DIMENSION(:,::), INTENT(INOUT) :: u
INTEGER(I4B), INTENT(IN) :: maxcyc
END SUBROUTINE mgfas
END INTERFACE

INTERFACE
SUBROUTINE mglin(u,ncycle)
USE nrtype
REAL(DP), DIMENSION(:,::), INTENT(INOUT) :: u
INTEGER(I4B), INTENT(IN) :: ncycle
END SUBROUTINE mglin
END INTERFACE

INTERFACE
SUBROUTINE midexp(funk,aa,bb,s,n)
USE nrtype
REAL(SP), INTENT(IN) :: aa,bb
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
FUNCTION funk(x)
USE nrtype
REAL(SP), DIMENSION(:,::), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: funk
END FUNCTION funk
END INTERFACE
END SUBROUTINE midexp
END INTERFACE

INTERFACE
SUBROUTINE midinf(funk,aa,bb,s,n)
USE nrtype
REAL(SP), INTENT(IN) :: aa,bb
REAL(SP), INTENT(INOUT) :: s
END SUBROUTINE midinf
END INTERFACE
INTEGER(I4B), INTENT(IN) :: n

INTERFACE
  FUNCTION funk(x)
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: x
  REAL(SP), DIMENSION(size(x)) :: funk
  END FUNCTION funk
END INTERFACE
END SUBROUTINE midinf
END INTERFACE

INTERFACE
  SUBROUTINE midpnt(func,a,b,s,n)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP), INTENT(INOUT) :: s
  INTEGER(I4B), INTENT(IN) :: n
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: funk
    END FUNCTION func
  END INTERFACE
END SUBROUTINE midpnt
END INTERFACE

INTERFACE
  SUBROUTINE midsq1(func,a,b,s,n)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP), INTENT(INOUT) :: s
  INTEGER(I4B), INTENT(IN) :: n
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: funk
    END FUNCTION func
  END INTERFACE
END SUBROUTINE midsq1
END INTERFACE

INTERFACE
  SUBROUTINE midsq2(func,a,b,s,n)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP), INTENT(INOUT) :: s
  INTEGER(I4B), INTENT(IN) :: n
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: funk
    END FUNCTION func
  END INTERFACE
END SUBROUTINE midsq2
END INTERFACE

INTERFACE
  SUBROUTINE midsq3(func,a,b,s,n)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP), INTENT(INOUT) :: s
  INTEGER(I4B), INTENT(IN) :: n
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: funk
    END FUNCTION func
  END INTERFACE
END SUBROUTINE midsq3
END INTERFACE

INTERFACE
  RECURSIVE SUBROUTINE miser(func,regn,ndim,npts,dith,ave,var)
  USE nrtype
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP) :: func
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: funk
    END FUNCTION func
  END INTERFACE
END SUBROUTINE miser
END INTERFACE
Appendix C2. Alphabetical Listing of Explicit Interfaces

END INTERFACE
REAL(SP), DIMENSION(:,), INTENT(IN) :: regn
INTEGER(4B), INTENT(IN) :: ndim,npts
REAL(SP), INTENT(IN) :: dith
REAL(SP), INTENT(OUT) :: ave,var
END SUBROUTINE miser
END INTERFACE
INTERFACE
SUBROUTINE mmid(y,dydx,xs,htot,nstep,yout,derivs)
USE nrtype
INTEGER(I4B), INTENT(IN) :: nstep
REAL(SP), INTENT(IN) :: xs,htot
REAL(SP), DIMENSION(:,), INTENT(IN) :: y,dydx
REAL(SP), DIMENSION(:,), INTENT(OUT) :: yout
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE
END SUBROUTINE mmid
END INTERFACE
INTERFACE
SUBROUTINE mnbrak(ax,bx,cx,fa,fb,fc,func)
USE nrtype
REAL(SP), INTENT(INOUT) :: ax,bx
REAL(SP), INTENT(OUT) :: cx,fa,fb,fc
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
END SUBROUTINE mnbrak
END INTERFACE
INTERFACE
SUBROUTINE mnewt(ntrial,x,tolx,tolf,usrfun)
USE nrtype
INTEGER(I4B), INTENT(IN) :: ntrial
REAL(SP), INTENT(IN) :: tolx,tolf
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: x
INTERFACE
SUBROUTINE usrfun(x,fvec,fjac)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(OUT) :: fvec
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: fjac
END SUBROUTINE usrfun
END INTERFACE
END SUBROUTINE mnewt
END INTERFACE
INTERFACE
SUBROUTINE moment(data,ave,adev,sdev,var,skew,curt)
USE nrtype
REAL(SP), INTENT(OUT) :: ave,adev,sdev,var,skew,curt
REAL(SP), DIMENSION(:,), INTENT(IN) :: data
END SUBROUTINE moment
END INTERFACE
INTERFACE
SUBROUTINE mp2dfr(a,s,n,m)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), INTENT(OUT) :: m
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: a
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: s
END SUBROUTINE mp2dfr
END INTERFACE
INTERFACE
SUBROUTINE mpdiv(q,r,u,v,n,m)
USE nrtype
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: q,r
CHARACTER(1), DIMENSION(:), INTENT(IN) :: u,v
INTEGER(I4B), INTENT(IN) :: n,m
END SUBROUTINE mpdiv
END INTERFACE
INTERFACE
SUBROUTINE mpinv(u,v,n,m)
USE nrtype
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: u
CHARACTER(1), DIMENSION(:), INTENT(IN) :: v
INTEGER(I4B), INTENT(IN) :: n,m
END SUBROUTINE mpinv
END INTERFACE
INTERFACE
SUBROUTINE mpmul(w,u,v,n,m)
USE nrtype
CHARACTER(1), DIMENSION(:), INTENT(IN) :: u,v
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: w
INTEGER(I4B), INTENT(IN) :: n,m
END SUBROUTINE mpmul
END INTERFACE
INTERFACE
SUBROUTINE mppi(n)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
END SUBROUTINE mppi
END INTERFACE
INTERFACE
SUBROUTINE mprove(a,alud,indx,b,x)
USE nrtype
REAL(SP), DIMENSION(:,:), INTENT(IN) :: a,alud
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: indx
REAL(SP), DIMENSION(:), INTENT(IN) :: b
REAL(SP), DIMENSION(:), INTENT(INOUT) :: x
END SUBROUTINE mprove
END INTERFACE
INTERFACE
SUBROUTINE mpsqrt(w,u,v,n,m)
USE nrtype
CHARACTER(1), DIMENSION(:), INTENT(OUT) :: w,u
CHARACTER(1), DIMENSION(:), INTENT(IN) :: v
INTEGER(I4B), INTENT(IN) :: n,m
END SUBROUTINE mpsqrt
END INTERFACE
INTERFACE
SUBROUTINE mrqcof(x,y,sig,a,maska,alpha,beta,chisq,funcs)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
INTERFACE
SUBROUTINE funcs(x,a,yfit,dyda)
USE nrtype
SUBROUTINE mrqcof(x,y,sig,a,maska,alpha,beta,chisq,funcs)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
INTERFACE
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USE nrtype
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
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REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
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USE nrtype
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
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USE nrtype
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
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USE nrtype
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
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USE nrtype
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
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USE nrtype
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
INTERFACE
SUBROUTINE funcs(x,a,yfit,dyda)
USE nrtype
SUBROUTINE mrqcof(x,y,sig,a,maska,alpha,beta,chisq,funcs)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
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REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
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USE nrtype
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USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: beta
REAL(SP), DIMENSION(:), INTENT(OUT) :: alpha
REAL(SP), INTENT(OUT) :: chisq
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
INTERFACE
SUBROUTINE funcs(x,a,yfit,dyda)
REAL(SP), DIMENSION(:), INTENT(IN) :: x, a
REAL(SP), DIMENSION(:), INTENT(OUT) :: yfit
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: dyda
END SUBROUTINEfuncs
END INTERFACE
END SUBROUTINE mrqcof
END INTERFACE
SUBROUTINE mrqmin(x, y, sig, a, maska, covar, alpha, chisq, funcs, alamda)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x, y, sig
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: covar, alpha
REAL(SP), INTENT(OUT) :: chisq
REAL(SP), INTENT(INOUT) :: alamda
LOGICAL(LGT), DIMENSION(:), INTENT(IN) :: maska
INTERFACE
SUBROUTINEfuncs(x, a, yfit, dyda)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x, a
REAL(SP), DIMENSION(:), INTENT(OUT) :: yfit
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: dyda
END SUBROUTINEfuncs
END INTERFACE
END SUBROUTINE mrqmin
END INTERFACE
SUBROUTINE newt(x, check)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: x
LOGICAL(LGT), INTENT(OUT) :: check
END SUBROUTINEnewt
END INTERFACE
SUBROUTINE odeint(ystart, x1, x2, eps, h1, hmin, derivs, rkqs)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: ystart
REAL(SP), INTENT(IN) :: x1, x2, eps, h1, hmin
INTERFACE
SUBROUTINE derivs(x, y, dydx)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
END SUBROUTINEderivs
SUBROUTINE rkqs(y, dydx, x, htry, eps, yscal, hdid, hnext, derivs)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:,), INTENT(IN) :: dydx, yscal
REAL(SP), INTENT(OUT) :: htry, eps
REAL(SP), INTENT(IN) :: hdid, hnext
END SUBROUTINE rkqs
INTERFACE
SUBROUTINE derivs(x, y, dydx)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END SUBROUTINE odeint
END SUBROUTINE rkqs
END INTERFACE
END SUBROUTINE odeint
END INTERFACE
INTERFACE
    SUBROUTINE orthog(anu,alpha,beta,a,b)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(IN) :: anu,alpha,beta
    REAL(SP), DIMENSION(:,), INTENT(OUT) :: a,b
    END SUBROUTINE orthog
END INTERFACE

END INTERFACE
INTERFACE
    SUBROUTINE pade(cof,resid)
    USE nrtype
    REAL(DP), DIMENSION(:,), INTENT(INOUT) :: cof
    REAL(SP), INTENT(OUT) :: resid
    END SUBROUTINE pade
END INTERFACE

END INTERFACE
INTERFACE
    FUNCTION pccheb(d)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(IN) :: d
    REAL(SP), DIMENSION(size(d)) :: pccheb
    END FUNCTION pccheb
END INTERFACE

END INTERFACE
INTERFACE
    SUBROUTINE pcshft(a,b,d)
    USE nrtype
    REAL(SP), INTENT(IN) :: a,b
    REAL(SP), DIMENSION(:,), INTENT(INOUT) :: d
    END SUBROUTINE pcshft
END INTERFACE

END INTERFACE
INTERFACE
    SUBROUTINE pearsn(x,y,r,prob,z)
    USE nrtype
    REAL(SP), INTENT(OUT) :: r,prob,z
    REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y
    END SUBROUTINE pearsn
END INTERFACE

END INTERFACE
INTERFACE
    SUBROUTINE period(x,y,ofac,hifac,px,py,jmax,prob)
    USE nrtype
    INTEGER(I4B), INTENT(OUT) :: jmax
    REAL(SP), INTENT(IN) :: ofac,hifac
    REAL(SP), INTENT(OUT) :: prob
    REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y
    REAL(SP), DIMENSION(:,), POINTER :: px,py
    END SUBROUTINE period
END INTERFACE

INTERFACE
    FUNCTION plgndr_s(l,m,x)
    USE nrtype
    INTEGER(I4B), INTENT(IN) :: l,m
    REAL(SP), INTENT(IN) :: x
    REAL(SP) :: plgndr_s
    END FUNCTION plgndr_s

    FUNCTION plgndr_v(l,m,x)
    USE nrtype
    INTEGER(I4B), INTENT(IN) :: l,m
    REAL(SP), DIMENSION(:,), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: plgndr_v
    END FUNCTION plgndr_v
END INTERFACE

END INTERFACE
INTERFACE
    FUNCTION poidev(xm)
    USE nrtype
    REAL(SP), INTENT(IN) :: xm
    END FUNCTION poidev
END INTERFACE
REAL(SP), INTENT(IN) :: xm
REAL(SP) :: poidev
END FUNCTION poidev
END INTERFACE

INTERFACE
FUNCTION polcoe(x,y)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y
REAL(SP), DIMENSION(size(x)) :: polcoe
END FUNCTION polcoe
END INTERFACE

INTERFACE
FUNCTION polcof(xa,ya)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: xa,ya
REAL(SP), DIMENSION(size(xa)) :: polcof
END FUNCTION polcof
END INTERFACE

INTERFACE
SUBROUTINE poldiv(u,v,q,r)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: u,v
REAL(SP), DIMENSION(:,), INTENT(OUT) :: q,r
END SUBROUTINE poldiv
END INTERFACE

INTERFACE
SUBROUTINE polin2(x1a,x2a,ya,x1,x2,y,dy)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x1a,x2a
REAL(SP), DIMENSION(:,:), INTENT(IN) :: ya
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP), INTENT(OUT) :: y,dy
END SUBROUTINE polin2
END INTERFACE

INTERFACE
SUBROUTINE polint(xa,ya,x,y,dy)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: xa,ya
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: y,dy
END SUBROUTINE polint
END INTERFACE

INTERFACE
SUBROUTINE powell(p,xi,ftol,iter,fret)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: p
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: xi
INTEGER(I4B), INTENT(OUT) :: iter
REAL(SP), INTENT(IN) :: ftol
REAL(SP), INTENT(OUT) :: fret
END SUBROUTINE powell
END INTERFACE

INTERFACE
FUNCTION predic(data,d,nfut)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: data,d
INTEGER(I4B), INTENT(IN) :: nfut
REAL(SP), DIMENSION(nfut) :: predic
END FUNCTION predic
END INTERFACE

INTERFACE
FUNCTION probks(alam)
USE nrtype
REAL(SP), INTENT(IN) :: alam
REAL(SP) :: probks
END FUNCTION probks
END INTERFACE

INTERFACE psdes
    SUBROUTINE psdes_s(lword,rword)
        USE nrtype
        INTEGER(I4B), INTENT(INOUT) :: lword,rword
    END SUBROUTINE psdes_s
    SUBROUTINE psdes_v(lword,rword)
        USE nrtype
        INTEGER(I4B), DIMENSION(:), INTENT(INOUT) :: lword,rword
    END SUBROUTINE psdes_v
END INTERFACE

INTERFACE
    SUBROUTINE pwt(a,isign)
        USE nrtype
        REAL(SP), DIMENSION(:), INTENT(INOUT) :: a
        INTEGER(I4B), INTENT(IN) :: isign
    END SUBROUTINE pwt
END INTERFACE

INTERFACE
    SUBROUTINE pwtset(n)
        USE nrtype
        INTEGER(I4B), INTENT(IN) :: n
    END SUBROUTINE pwtset
END INTERFACE

INTERFACE pythag
    FUNCTION pythag_dp(a,b)
        USE nrtype
        REAL(DP), INTENT(IN) :: a,b
    END FUNCTION pythag_dp
    FUNCTION pythag_sp(a,b)
        USE nrtype
        REAL(SP), INTENT(IN) :: a,b
    END FUNCTION pythag_sp
END INTERFACE

INTERFACE
    SUBROUTINE pzextr(iest,xest,yest,yz,dy)
        USE nrtype
        INTEGER(I4B), INTENT(IN) :: iest
        REAL(SP), INTENT(IN) :: xest
        REAL(SP), DIMENSION(:), INTENT(IN) :: yest
        REAL(SP), DIMENSION(:), INTENT(OUT) :: yz,dy
    END SUBROUTINE pzextr
END INTERFACE

INTERFACE
    SUBROUTINE qrdcmp(a,c,d,sing)
        USE nrtype
        REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
        REAL(SP), DIMENSION(:,), INTENT(IN) :: c,d
        LOGICAL(LGT), INTENT(OUT) :: sing
    END SUBROUTINE qrdcmp
END INTERFACE

INTERFACE
    FUNCTION qromb(func,a,b)
        USE nrtype
        REAL(SP), INTENT(IN) :: a,b
    END FUNCTION qromb
END INTERFACE

FUNCTION func(x)
USE nrtype
END FUNCTION func(x)
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
END FUNCTION qromb
END INTERFACE

INTERFACE
FUNCTION qromo(func,a,b,choose)
USE nrtype
REAL(SP), INTENT(IN) :: a,b
REAL(SP) :: qromo
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
INTERFACE
SUBROUTINE choose(funk,aa,bb,s,n)
USE nrtype
REAL(SP) :: aa,bb
REAL(SP), INTENT(INOUT) :: s
INTEGER(I4B), INTENT(IN) :: n
INTERFACE
FUNCTION funk(x)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x
REAL(SP), DIMENSION(size(x)) :: funk
END FUNCTION funk
END INTERFACE
END SUBROUTINE choose
END INTERFACE
END FUNCTION qromo
END INTERFACE

INTERFACE
SUBROUTINE qroot(p,b,c,eps)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: p
REAL(SP), INTENT(INOUT) :: b,c
REAL(SP), INTENT(IN) :: eps
END SUBROUTINE qroot
END INTERFACE

INTERFACE
SUBROUTINE qrsolv(a,c,d,b)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: a
REAL(SP), DIMENSION(:,), INTENT(IN) :: c,d
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: b
END SUBROUTINE qrsolv
END INTERFACE

INTERFACE
SUBROUTINE qrupdt(r,qt,u,v)
USE nrtype
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: r,qt
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: u
REAL(SP), DIMENSION(:,), INTENT(IN) :: v
END SUBROUTINE qrupdt
END INTERFACE

INTERFACE
FUNCTION qsimp(func,a,b)
USE nrtype
REAL(SP), INTENT(IN) :: a,b
REAL(SP) :: qsimp
END FUNCTION qsimp
END INTERFACE
INTERFACE
  FUNCTION func(x)
  USE nrtype
  REAL(SP), DIMENSION(:), INTENT(IN) :: x
  REAL(SP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
END FUNCTION qsimp
END INTERFACE
INTERFACE
  FUNCTION qtrap(func,a,b)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP) :: qtrap
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), DIMENSION(:), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: func
  END FUNCTION func
END INTERFACE
END FUNCTION qtrap
END INTERFACE
INTERFACE
  SUBROUTINE quadct(x,y,xx,yy,fa,fb,fc,fd)
  USE nrtype
  REAL(SP), INTENT(IN) :: x,y
  REAL(SP), DIMENSION(:), INTENT(IN) :: xx,yy
  REAL(SP), INTENT(OUT) :: fa,fb,fc,fd
END SUBROUTINE quadct
END INTERFACE
END SUBROUTINE quadmx
INTERFACE
  SUBROUTINE quadmx(a)
  USE nrtype
  REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: a
END SUBROUTINE quadmx
END INTERFACE
INTERFACE
  SUBROUTINE quadvl(x,y,fa,fb,fc,fd)
  USE nrtype
  REAL(SP), INTENT(IN) :: x,y
  REAL(SP), INTENT(OUT) :: fa,fb,fc,fd
END SUBROUTINE quadvl
END INTERFACE
INTERFACE
  FUNCTION ran(idum)
  INTEGER(selected_int_kind(9)), INTENT(INOUT) :: idum
  REAL :: ran
END FUNCTION ran
END INTERFACE
INTERFACE ran0
  SUBROUTINE ran0_s(harvest)
  USE nrtype
  REAL(SP), INTENT(OUT) :: harvest
END SUBROUTINE ran0_s
  SUBROUTINE ran0_v(harvest)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(OUT) :: harvest
END SUBROUTINE ran0_v
END INTERFACE
INTERFACE ran1
  SUBROUTINE ran1_s(harvest)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(OUT) :: harvest
END SUBROUTINE ran1_s
END INTERFACE
REAL(SP), INTENT(OUT) :: harvest
END SUBROUTINE ran1_s
SUBROUTINE ran1_v(harvest)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(OUT) :: harvest
END SUBROUTINE ran1_v
END INTERFACE

INTERFACE ran2
SUBROUTINE ran2_s(harvest)
USE nrtype
REAL(SP), INTENT(OUT) :: harvest
END SUBROUTINE ran2_s
SUBROUTINE ran2_v(harvest)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(OUT) :: harvest
END SUBROUTINE ran2_v
END INTERFACE

INTERFACE ran3
SUBROUTINE ran3_s(harvest)
USE nrtype
REAL(SP), INTENT(OUT) :: harvest
END SUBROUTINE ran3_s
SUBROUTINE ran3_v(harvest)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(OUT) :: harvest
END SUBROUTINE ran3_v
END INTERFACE

INTERFACE
SUBROUTINE ratint(xa,ya,x,y,dy)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: xa,ya
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: y,dy
END SUBROUTINE ratint
END INTERFACE

INTERFACE
SUBROUTINE ratlsq(func,a,b,mm,kk,cof,dev)
USE nrtype
REAL(DP), INTENT(IN) :: a,b
INTEGER(I4B), INTENT(IN) :: mm,kk
REAL(DP), DIMENSION(:,), INTENT(OUT) :: cof
REAL(DP), INTENT(OUT) :: dev
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(DP), DIMENSION(:,), INTENT(IN) :: x
REAL(DP), DIMENSION(size(x)) :: func
END FUNCTION func
END INTERFACE
END SUBROUTINE ratlsq
END INTERFACE

INTERFACE ratval
FUNCTION ratval_s(x,cof,mm,kk)
USE nrtype
REAL(DP), INTENT(IN) :: x
INTEGER(I4B), INTENT(IN) :: mm,kk
REAL(DP), DIMENSION(mm+kk+1), INTENT(IN) :: cof
REAL(DP) :: ratval_s
END FUNCTION ratval_s
FUNCTION ratval_v(x,cof,mm,kk)
USE nrtype
REAL(DP), DIMENSION(:,), INTENT(IN) :: x
REAL(DP), DIMENSION(size(x)) :: func
END FUNCTION ratval_v
END INTERFACE
END SUBROUTINE ratval
END INTERFACE
INTEGER(I4B), INTENT(IN) :: mm, kk
REAL(DP), DIMENSION(mm+kk+1), INTENT(IN) :: cof
REAL(DP), DIMENSION(size(x)) :: ratval_v
END FUNCTION ratval_v
END INTERFACE

INTERFACE rc
FUNCTION rc_s(x,y)
USE nrtype
REAL(SP), INTENT(IN) :: x,y
REAL(SP) :: rc_s
END FUNCTION rc_s
FUNCTION rc_v(x,y)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y
REAL(SP), DIMENSION(size(x)) :: rc_v
END FUNCTION rc_v
END INTERFACE

INTERFACE rd
FUNCTION rd_s(x,y,z)
USE nrtype
REAL(SP), INTENT(IN) :: x,y,z
REAL(SP) :: rd_s
END FUNCTION rd_s
FUNCTION rd_v(x,y,z)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y,z
REAL(SP), DIMENSION(size(x)) :: rd_v
END FUNCTION rd_v
END INTERFACE

INTERFACE realft
SUBROUTINE realft_dp(data,isign,zdata)
USE nrtype
REAL(DP), DIMENSION(:,), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign
COMPLEX(DPC), DIMENSION(:,), OPTIONAL, TARGET :: zdata
END SUBROUTINE realft_dp

SUBROUTINE realft_sp(data,isign,zdata)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: data
INTEGER(I4B), INTENT(IN) :: isign
COMPLEX(SPC), DIMENSION(:,), OPTIONAL, TARGET :: zdata
END SUBROUTINE realft_sp
END INTERFACE

INTERFACE recur1
FUNCTION recur1(a,b) RESULT(u)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: a,b
REAL(SP), DIMENSION(size(a)) :: u
END FUNCTION recur1
END INTERFACE

INTERFACE recur2
FUNCTION recur2(a,b,c)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: a,b,c
REAL(SP), DIMENSION(size(a)) :: recur2
END FUNCTION recur2
END INTERFACE

INTERFACE relax
SUBROUTINE relax(u,rhs)
USE nrtype
REAL(DP), DIMENSION(:,,:), INTENT(INOUT) :: u
REAL(DP), DIMENSION(:,,:), INTENT(IN) :: rhs
END SUBROUTINE relax
END INTERFACE
INTERFACE
   SUBROUTINE relax2(u,rhs)
   USE nrtype
   REAL(DP), DIMENSION(:,:,), INTENT(INOUT) :: u
   REAL(DP), DIMENSION(:,:,), INTENT(IN) :: rhs
   END SUBROUTINE relax2
END INTERFACE

INTERFACE
   FUNCTION resid(u,rhs)
   USE nrtype
   REAL(DP), DIMENSION(:,:,), INTENT(IN) :: u,rhs
   REAL(DP), DIMENSION(size(u,1),size(u,1)) :: resid
   END FUNCTION resid
END INTERFACE

INTERFACE rf
   FUNCTION rf_s(x,y,z)
   USE nrtype
   REAL(SP), INTENT(IN) :: x,y,z
   REAL(SP) :: rf_s
   END FUNCTION rf_s
   FUNCTION rf_v(x,y,z)
   USE nrtype
   REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,z
   REAL(SP), DIMENSION(size(x)) :: rf_v
   END FUNCTION rf_v
END INTERFACE

INTERFACE rj
   FUNCTION rj_s(x,y,z,p)
   USE nrtype
   REAL(SP), INTENT(IN) :: x,y,z,p
   REAL(SP) :: rj_s
   END FUNCTION rj_s
   FUNCTION rj_v(x,y,z,p)
   USE nrtype
   REAL(SP), DIMENSION(:), INTENT(IN) :: x,y,z,p
   REAL(SP), DIMENSION(size(x)) :: rj_v
   END FUNCTION rj_v
END INTERFACE

INTERFACE
   SUBROUTINE rk4(y,dydx,x,h,yout,derivs)
   USE nrtype
   REAL(SP), DIMENSION(:), INTENT(IN) :: y,dydx
   REAL(SP), INTENT(IN) :: x,h
   REAL(SP), DIMENSION(:), INTENT(OUT) :: yout
   INTERFACE
      SUBROUTINE derivs(x,y,dydx)
      USE nrtype
      REAL(SP), INTENT(IN) :: x
      REAL(SP), DIMENSION(:), INTENT(IN) :: y
      REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
      END SUBROUTINE derivs
   END INTERFACE
   END SUBROUTINE rk4
END INTERFACE

INTERFACE
   SUBROUTINE rkck(y,dydx,x,h,yout,yerr,derivs)
   USE nrtype
   REAL(SP), DIMENSION(:,), INTENT(IN) :: y,dydx
   REAL(SP), INTENT(IN) :: x,h
   REAL(SP), DIMENSION(:,), INTENT(OUT) :: yout,yerr
   INTERFACE
      SUBROUTINE derivs(x,y,dydx)
      USE nrtype
      REAL(SP), INTENT(IN) :: x
      REAL(SP), DIMENSION(:), INTENT(IN) :: y
      REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
      END SUBROUTINE derivs
   END INTERFACE
   END SUBROUTINE rkck
END INTERFACE
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE
END SUBROUTINE rkck
END INTERFACE
INTERFACE
SUBROUTINE rkdumb(vstart,x1,x2,nstep,derivs)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: vstart
REAL(SP), INTENT(IN) :: x1,x2
INTEGER(I4B), INTENT(IN) :: nstep
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE
END SUBROUTINE rkdumb
END INTERFACE
INTERFACE
SUBROUTINE rkqs(y,dydx,x,htry,eps,yscal,hdid,hnext,derivs)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:), INTENT(IN) :: dydx,yscal
REAL(SP), INTENT(INOUT) :: x
REAL(SP), INTENT(IN) :: htry,eps
REAL(SP), INTENT(OUT) :: hdid,hnext
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE
END SUBROUTINE rkqs
END INTERFACE
INTERFACE
SUBROUTINE rlft2(data,spec,speq,isign)
USE nrtype
REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: data
COMPLEX(SPC), DIMENSION(:,:), INTENT(OUT) :: spec
COMPLEX(SPC), DIMENSION(:), INTENT(OUT) :: speq
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE rlft2
END INTERFACE
INTERFACE
SUBROUTINE rlft3(data,spec,speq,isign)
USE nrtype
REAL(SP), DIMENSION(:,:,:), INTENT(INOUT) :: data
COMPLEX(SPC), DIMENSION(:,:,:), INTENT(OUT) :: spec
COMPLEX(SPC), DIMENSION(:,:), INTENT(OUT) :: speq
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE rlft3
END INTERFACE
INTERFACE
SUBROUTINE rotate(r,qt,i,a,b)
USE nrtype
REAL(SP), DIMENSION(:,:), TARGET, INTENT(INOUT) :: r,qt
INTEGER(I4B), INTENT(IN) :: i
REAL(SP), INTENT(IN) :: a, b
END SUBROUTINE rotate

END INTERFACE

INTERFACE
SUBROUTINE rsolv(a,d,b)
USE nrtype
REAL(SP), DIMENSION(:,::), INTENT(IN) :: a
REAL(SP), DIMENSION(:,::), INTENT(IN) :: d
REAL(SP), DIMENSION(:,::), INTENT(INOUT) :: b
END SUBROUTINE rsolv
END INTERFACE

INTERFACE
FUNCTION rstrct(uf)
USE nrtype
REAL(DP), DIMENSION(:,::), INTENT(IN) :: uf
REAL(DP), DIMENSION((size(uf,1)+1)/2,(size(uf,1)+1)/2) :: rstrct
END FUNCTION rstrct
END INTERFACE

INTERFACE
FUNCTION rtbis(func,x1,x2,xacc)
USE nrtype
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtbis
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
END FUNCTION rtbis
END INTERFACE

INTERFACE
FUNCTION rtflsp(func,x1,x2,xacc)
USE nrtype
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtflsp
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
END FUNCTION rtflsp
END INTERFACE

INTERFACE
FUNCTION rtnewt(funcd,x1,x2,xacc)
USE nrtype
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtnewt
INTERFACE
SUBROUTINE funcd(x,fval,fderiv)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: fval,fderiv
END SUBROUTINE funcd
END INTERFACE
END FUNCTION rtnewt
END INTERFACE

INTERFACE
FUNCTION rtsafe(funcd,x1,x2,xacc)
USE nrtype
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtsafe
INTERFACE
SUBROUTINE funcd(x,fval,fderiv)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: fval,fderiv
END SUBROUTINE funcd
END INTERFACE
END FUNCTION rtsafe
END INTERFACE

INTERFACE
FUNCTION rtnewt(funcd,x1,x2,xacc)
USE nrtype
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtnewt
INTERFACE
SUBROUTINE funcd(x,fval,fderiv)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: fval,fderiv
END SUBROUTINE funcd
END INTERFACE
END FUNCTION rtnewt
END INTERFACE

INTERFACE
FUNCTION rtsafe(funcd,x1,x2,xacc)
USE nrtype
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: rtsafe
INTERFACE
SUBROUTINE funcd(x,fval,fderiv)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), INTENT(OUT) :: fval,fderiv
END SUBROUTINE funcd
END INTERFACE
END FUNCTION rtsafe
END INTERFACE
REAL(SP), INTENT(IN) :: x1, x2, xacc
REAL(SP) :: rtsafe
INTERFACE
SUBROUTINE funcd(x, fval, fderiv)
  USE nrtype
  REAL(SP), INTENT(IN) :: x
  REAL(SP), INTENT(OUT) :: fval, fderiv
END SUBROUTINE funcd
END INTERFACE
END FUNCTION rtsafe
END INTERFACE
INTERFACE
FUNCTION rtsec(func, x1, x2, xacc)
  USE nrtype
  REAL(SP), INTENT(IN) :: x1, x2, xacc
  REAL(SP) :: rtsec
  INTERFACE
    FUNCTION func(x)
      USE nrtype
      REAL(SP), INTENT(IN) :: x
      REAL(SP) :: func
    END FUNCTION func
  END INTERFACE
END FUNCTION rtsec
END INTERFACE
INTERFACE
SUBROUTINE rzextr(iest, xest, yest, yz, dy)
  USE nrtype
  INTEGER(I4B), INTENT(IN) :: iest
  REAL(SP), INTENT(IN) :: xest
  REAL(SP), DIMENSION(:), INTENT(IN) :: yest
  REAL(SP), DIMENSION(:), INTENT(OUT) :: yz, dy
END SUBROUTINE rzextr
END INTERFACE
INTERFACE
FUNCTION savgol(nl, nrr, ld, m)
  USE nrtype
  INTEGER(I4B), INTENT(IN) :: nl, nrr, ld, m
  REAL(SP), DIMENSION(nl+nrr+1) :: savgol
END FUNCTION savgol
END INTERFACE
INTERFACE
SUBROUTINE scrsho(func)
  USE nrtype
  INTERFACE
    FUNCTION func(x)
      USE nrtype
      REAL(SP), INTENT(IN) :: x
      REAL(SP) :: func
    END FUNCTION func
  END INTERFACE
END SUBROUTINE scrsho
END INTERFACE
INTERFACE
FUNCTION select(k, arr)
  USE nrtype
  INTEGER(I4B), INTENT(IN) :: k
  REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
  REAL(SP) :: select
END FUNCTION select
END INTERFACE
INTERFACE
FUNCTION select_bypack(k, arr)
  USE nrtype
INTEGER(I4B), INTENT(IN) :: k  
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: arr  
REAL(SP) :: select_bypack  
END FUNCTION select_bypack  
END INTERFACE  

INTERFACE  
SUBROUTINE select_heap(arr,heap)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(IN) :: arr  
REAL(SP), DIMENSION(:,), INTENT(OUT) :: heap  
END SUBROUTINE select_heap  
END INTERFACE  

INTERFACE  
FUNCTION select_inplace(k,arr)  
USE nrtype  
INTEGER(I4B), INTENT(IN) :: k  
REAL(SP), DIMENSION(:,), INTENT(IN) :: arr  
REAL(SP) :: select_inplace  
END FUNCTION select_inplace  
END INTERFACE  

INTERFACE  
SUBROUTINE simplx(a,m1,m2,m3,icase,izrov,iposv)  
USE nrtype  
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a  
INTEGER(I4B), INTENT(IN) :: m1,m2,m3  
INTEGER(I4B), INTENT(OUT) :: icase  
INTEGER(I4B), DIMENSION(:,), INTENT(OUT) :: izrov,iposv  
END SUBROUTINE simplx  
END INTERFACE  

INTERFACE  
SUBROUTINE simpr(y,dydx,dfdx,dfdy,xs,htot,nstep,yout,derivs)  
USE nrtype  
REAL(SP), INTENT(IN) :: xs,htot  
REAL(SP), DIMENSION(:,), INTENT(IN) :: y,dydx,dfdx  
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: dfdy  
INTEGER(I4B), INTENT(IN) :: nstep  
REAL(SP), DIMENSION(:,), INTENT(OUT) :: yout  
INTERFACE  
SUBROUTINE derivs(x,y,dydx)  
USE nrtype  
REAL(SP), INTENT(IN) :: x  
REAL(SP), DIMENSION(:,), INTENT(IN) :: y  
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx  
END SUBROUTINE derivs  
END INTERFACE  
END SUBROUTINE simpr  
END INTERFACE  

INTERFACE  
SUBROUTINE sinft(y)  
USE nrtype  
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: y  
END SUBROUTINE sinft  
END INTERFACE  

INTERFACE  
SUBROUTINE slvsm2(u,rhs)  
USE nrtype  
REAL(DP), DIMENSION(3,3), INTENT(OUT) :: u  
REAL(DP), DIMENSION(3,3), INTENT(IN) :: rhs  
END SUBROUTINE slvsm2  
END INTERFACE  

INTERFACE  
SUBROUTINE slvsm1(u,rhs)  
USE nrtype  
REAL(DP), DIMENSION(3,3), INTENT(OUT) :: u  
END SUBROUTINE slvsm1  
END INTERFACE
REAL(DP), DIMENSION(3,3), INTENT(IN) :: rhs
END SUBROUTINE slvsml

INTERFACE
SUBROUTINE sncndn(uu, emmc, sn, cn, dn)
USE nrtype
REAL(SP), INTENT(IN) :: uu, emmc
REAL(SP), INTENT(OUT) :: sn, cn, dn
END SUBROUTINE sncndn

INTERFACE
FUNCTION snrm(sx, itol)
USE nrtype
REAL(DP), DIMENSION(:), INTENT(IN) :: sx
INTEGER(I4B), INTENT(IN) :: itol
REAL(DP) :: snrm
END FUNCTION snrm

INTERFACE
SUBROUTINE sobseq(x, init)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(OUT) :: x
INTEGER(I4B), OPTIONAL, INTENT(IN) :: init
END SUBROUTINE sobseq

INTERFACE
SUBROUTINE solvde(itmax, conv, slowc, scalv, indexv, nb, y)
USE nrtype
INTEGER(I4B), INTENT(IN) :: itmax, nb
REAL(SP), INTENT(IN) :: conv, slowc
REAL(SP), DIMENSION(:), INTENT(IN) :: scalv
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: indexv
REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: y
END SUBROUTINE solvde

INTERFACE
SUBROUTINE sor(a, b, c, d, e, f, u, rjac)
USE nrtype
REAL(DP), DIMENSION(:,:), INTENT(IN) :: a, b, c, d, e, f
REAL(DP), INTENT(INOUT) :: u
REAL(DP), INTENT(IN) :: rjac
END SUBROUTINE sor

INTERFACE
SUBROUTINE sort(arr)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
END SUBROUTINE sort

INTERFACE
SUBROUTINE sort2(arr, slave)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr, slave
END SUBROUTINE sort2

INTERFACE
SUBROUTINE sort3(arr, slave1, slave2)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr, slave1, slave2
END SUBROUTINE sort3

INTERFACE
SUBROUTINE sort_bypack(arr)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
END SUBROUTINE sort_bypack

INTERFACE
SUBROUTINE slvsm1
END INTERFACE

INTERFACE
SUBROUTINE slvsm1
END INTERFACE

INTERFACE
SUBROUTINE sncndn(uu, emmc, sn, cn, dn)
USE nrtype
REAL(SP), INTENT(IN) :: uu, emmc
REAL(SP), INTENT(OUT) :: sn, cn, dn
END SUBROUTINE sncndn

INTERFACE
FUNCTION snrm(sx, itol)
USE nrtype
REAL(DP), DIMENSION(:), INTENT(IN) :: sx
INTEGER(I4B), INTENT(IN) :: itol
REAL(DP) :: snrm
END FUNCTION snrm

INTERFACE
SUBROUTINE sobseq(x, init)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(OUT) :: x
INTEGER(I4B), OPTIONAL, INTENT(IN) :: init
END SUBROUTINE sobseq

INTERFACE
SUBROUTINE solvde(itmax, conv, slowc, scalv, indexv, nb, y)
USE nrtype
INTEGER(I4B), INTENT(IN) :: itmax, nb
REAL(SP), INTENT(IN) :: conv, slowc
REAL(SP), DIMENSION(:), INTENT(IN) :: scalv
INTEGER(I4B), DIMENSION(:), INTENT(IN) :: indexv
REAL(SP), DIMENSION(:,:), INTENT(INOUT) :: y
END SUBROUTINE solvde

INTERFACE
SUBROUTINE sor(a, b, c, d, e, f, u, rjac)
USE nrtype
REAL(DP), DIMENSION(:,:), INTENT(IN) :: a, b, c, d, e, f
REAL(DP), INTENT(INOUT) :: u
REAL(DP), INTENT(IN) :: rjac
END SUBROUTINE sor

INTERFACE
SUBROUTINE sort(arr)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
END SUBROUTINE sort

INTERFACE
SUBROUTINE sort2(arr, slave)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr, slave
END SUBROUTINE sort2

INTERFACE
SUBROUTINE sort3(arr, slave1, slave2)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr, slave1, slave2
END SUBROUTINE sort3

INTERFACE
SUBROUTINE sort_bypack(arr)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: arr
END SUBROUTINE sort_bypack
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: arr
END SUBROUTINE sort_bypack
END INTERFACE

INTERFACE
  SUBROUTINE sort_byreshape(arr)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(INOUT) :: arr
  END SUBROUTINE sort_byreshape
END INTERFACE

INTERFACE
  SUBROUTINE sort_heap(arr)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(INOUT) :: arr
  END SUBROUTINE sort_heap
END INTERFACE

INTERFACE
  SUBROUTINE sort_pick(arr)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(INOUT) :: arr
  END SUBROUTINE sort_pick
END INTERFACE

INTERFACE
  SUBROUTINE sort_radix(arr)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(INOUT) :: arr
  END SUBROUTINE sort_radix
END INTERFACE

INTERFACE
  SUBROUTINE sort_shell(arr)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(INOUT) :: arr
  END SUBROUTINE sort_shell
END INTERFACE

INTERFACE
  SUBROUTINE spctrm(p,k,ovrlap,unit,n_window)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(OUT) :: p
    INTEGER(I4B), INTENT(IN) :: k
    LOGICAL(LGT), INTENT(IN) :: ovrlap
    INTEGER(I4B), OPTIONAL, INTENT(IN) :: n_window,unit
  END SUBROUTINE spctrm
END INTERFACE

INTERFACE
  SUBROUTINE spear(data1,data2,d,zd,probd,rs,probrs)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(IN) :: data1,data2
    REAL(SP), INTENT(OUT) :: d,zd,probd,rs,probrs
  END SUBROUTINE spear
END INTERFACE

INTERFACE sphbes
  SUBROUTINE sphbes_s(n,x,sj,sy,sjp,syp)
    USE nrtype
    INTEGER(I4B), INTENT(IN) :: n
    REAL(SP), INTENT(IN) :: x
    REAL(SP), INTENT(OUT) :: sj,sy,sjp,syp
  END SUBROUTINE sphbes_s

  SUBROUTINE sphbes_v(n,x,sj,sy,sjp,syp)
    USE nrtype
    INTEGER(I4B), INTENT(IN) :: n
    REAL(SP), INTENT(IN) :: x
    REAL(SP), INTENT(OUT) :: sj,sy,sjp,syp
  END SUBROUTINE sphbes_v
END INTERFACE
Appendix C2. Alphabetical Listing of Explicit Interfaces

INTERFACE
SUBROUTINE splie2(x1a,x2a,ya,y2a)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x1a,x2a
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: ya
REAL(SP), DIMENSION(:,,:), INTENT(OUT) :: y2a
END SUBROUTINE splie2
END INTERFACE

INTERFACE
FUNCTION splin2(x1a,x2a,ya,y2a,x1,x2)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x1a,x2a
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: ya,y2a
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP) :: splin2
END FUNCTION splin2
END INTERFACE

INTERFACE
SUBROUTINE spline(x,y,yp1,ypn,y2)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: x,y
REAL(SP), INTENT(IN) :: yp1,ypn
REAL(SP), DIMENSION(:), INTENT(OUT) :: y2
END SUBROUTINE spline
END INTERFACE

INTERFACE
FUNCTION splint(xa,ya,y2a,x)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: xa,ya,y2a
REAL(SP), INTENT(IN) :: x
REAL(SP) :: splint
END FUNCTION splint
END INTERFACE

INTERFACE sprsax
SUBROUTINE sprsax_dp(sa,x,b)
USE nrtype
TYPE(sprs2_dp), INTENT(IN) :: sa
REAL(DP), DIMENSION (:), INTENT(IN) :: x
REAL(DP), DIMENSION (:), INTENT(OUT) :: b
END SUBROUTINE sprsax_dp

SUBROUTINE sprsax_sp(sa,x,b)
USE nrtype
TYPE(sprs2_sp), INTENT(IN) :: sa
REAL(SP), DIMENSION (:), INTENT(IN) :: x
REAL(SP), DIMENSION (:), INTENT(OUT) :: b
END SUBROUTINE sprsax_sp
END INTERFACE

INTERFACE sprsdiag
SUBROUTINE sprsdiag_dp(sa,b)
USE nrtype
TYPE(sprs2_dp), INTENT(IN) :: sa
REAL(DP), DIMENSION (:), INTENT(OUT) :: b
END SUBROUTINE sprsdiag_dp

SUBROUTINE sprsdiag_sp(sa,b)
USE nrtype
TYPE(sprs2_sp), INTENT(IN) :: sa
REAL(SP), DIMENSION (:), INTENT(OUT) :: b
END SUBROUTINE sprsdiag_sp
END INTERFACE

INTERFACE sprsin
SUBROUTINE sprsin_sp(a,thresh,sa)
USE nrtype
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: a
REAL(SP), INTENT(IN) :: thresh
TYPE(sprs2_sp), INTENT(OUT) :: sa
END SUBROUTINE sprsin_sp

SUBROUTINE sprsin_dp(a,thresh,sa)
USE nrtype
REAL(DP), DIMENSION(:,:), INTENT(IN) :: a
REAL(DP), INTENT(IN) :: thresh
TYPE(sprs2_dp), INTENT(OUT) :: sa
END SUBROUTINE sprsin_dp
END INTERFACE

INTERFACE
SUBROUTINE sprstp(sa)
USE nrtype
TYPE(sprs2_sp), INTENT(INOUT) :: sa
END SUBROUTINE sprstp
END INTERFACE

INTERFACE sprstx
SUBROUTINE sprstx_dp(sa,x,b)
USE nrtype
TYPE(sprs2_dp), INTENT(INOUT) :: sa
REAL(DP), DIMENSION (:), INTENT(IN) :: x
REAL(DP), DIMENSION (:), INTENT(OUT) :: b
END SUBROUTINE sprstx_dp
SUBROUTINE sprstx_sp(sa,x,b)
USE nrtype
TYPE(sprs2_sp), INTENT(IN) :: sa
REAL(SP), DIMENSION (:), INTENT(IN) :: x
REAL(SP), DIMENSION (:), INTENT(OUT) :: b
END SUBROUTINE sprstx_sp
END INTERFACE

INTERFACE
SUBROUTINE stifbs(y,dydx,x,htry,eps,yscal,hdid,hnext,derivs)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:), INTENT(IN) :: dydx,yscal
REAL(SP), INTENT(IN) :: htry,eps
REAL(SP), INTENT(INOUT) :: x
REAL(SP), INTENT(OUT) :: hdid,hnext
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE
END SUBROUTINE stifbs
END INTERFACE

INTERFACE
SUBROUTINE stiff(y,dydx,x,htry,eps,yscal,hdid,hnext,derivs)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(INOUT) :: y
REAL(SP), DIMENSION(:), INTENT(IN) :: dydx,yscal
REAL(SP), INTENT(INOUT) :: x
REAL(SP), INTENT(IN) :: htry,eps
REAL(SP), INTENT(OUT) :: hdid,hnext
INTERFACE
SUBROUTINE derivs(x,y,dydx)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:), INTENT(IN) :: y
REAL(SP), DIMENSION(:), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE
END SUBROUTINE stiff
END INTERFACE
END SUBROUTINE stiff
END INTERFACE

INTERFACE
SUBROUTINE stoerm(y,d2y,xs,htot,nstep,yout,derivs)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: y, d2y
REAL(SP), INTENT(IN) :: xs, htot
INTEGER(I4B), INTENT(IN) :: nstep
REAL(SP), DIMENSION(:,), INTENT(OUT) :: yout
SUBROUTINE derivs(x,y,dydx)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP), DIMENSION(:,), INTENT(IN) :: y
REAL(SP), DIMENSION(:,), INTENT(OUT) :: dydx
END SUBROUTINE derivs
END INTERFACE
END SUBROUTINE stoerm
END INTERFACE

INTERFACE
SUBROUTINE svbksb_dp(u,w,v,b,x)
USE nrtype
REAL(DP), DIMENSION(:,,:), INTENT(IN) :: u,v
REAL(DP), DIMENSION(:,), INTENT(IN) :: w,b
REAL(DP), DIMENSION(:,), INTENT(OUT) :: x
SUBROUTINE svbksb_sp(u,w,v,b,x)
USE nrtype
REAL(SP), DIMENSION(:,,:), INTENT(IN) :: u,v
REAL(SP), DIMENSION(:,), INTENT(IN) :: w,b
REAL(SP), DIMENSION(:,), INTENT(OUT) :: x
END SUBROUTINE svbksb_dp
END SUBROUTINE svbksb_sp
END INTERFACE

INTERFACE
SUBROUTINE svdcmp_dp(a,w,v)
USE nrtype
REAL(DP), DIMENSION(:,,:), INTENT(INOUT) :: a
REAL(DP), DIMENSION(:,), INTENT(OUT) :: w,v
SUBROUTINE svdcmp_sp(a,w,v)
USE nrtype
REAL(SP), DIMENSION(:,,:), INTENT(INOUT) :: a
REAL(SP), DIMENSION(:,), INTENT(OUT) :: w,v
END SUBROUTINE svdcmp_dp
END SUBROUTINE svdcmp_sp
END INTERFACE

INTERFACE
SUBROUTINE svdfit(x,y,sig,a,v,w,chisq,funcs)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(IN) :: x,y,sig
REAL(SP), DIMENSION(:,), INTENT(OUT) :: a,v,w
REAL(SP), DIMENSION(:,), INTENT(OUT) :: chisq
FUNCTION funcs(x,n)
USE nrtypes
REAL(SP), INTENT(IN) :: x
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), DIMENSION(n) :: funcs
END SUBROUTINE svdfit
END INTERFACE
END SUBROUTINE svdfit
END INTERFACE
INTERFACE
  SUBROUTINE svdvar(v,w,cvm)
  USE nrtype
  REAL(SP), DIMENSION(:,::), INTENT(IN) :: v
  REAL(SP), DIMENSION(:,::), INTENT(IN) :: w
  REAL(SP), DIMENSION(:,::), INTENT(OUT) :: cvm
END SUBROUTINE svdvar
END INTERFACE
INTERFACE
  SUBROUTINE svdvar(v,w,cvm)
  USE nrtype
  REAL(SP), DIMENSION(:,::), INTENT(IN) :: v
  REAL(SP), DIMENSION(:,::), INTENT(IN) :: w
  REAL(SP), DIMENSION(:,::), INTENT(OUT) :: cvm
END SUBROUTINE svdvar
END INTERFACE
INTERFACE
  FUNCTION toeplz(r,y)
  USE nrtype
  REAL(SP), DIMENSION(:,::), INTENT(IN) :: r,y
  REAL(SP), DIMENSION(size(y)) :: toeplz
END FUNCTION toeplz
END INTERFACE
INTERFACE
  SUBROUTINE tptest(data1,data2,t,prob)
  USE nrtype
  REAL(SP), DIMENSION(:,::), INTENT(IN) :: data1,data2
  REAL(SP), INTENT(OUT) :: t,prob
END SUBROUTINE tptest
END INTERFACE
INTERFACE
  SUBROUTINE tqli(d,e,z)
  USE nrtype
  REAL(SP), DIMENSION(:,::), INTENT(INOUT) :: d,e
  REAL(SP), DIMENSION(:,::), OPTIONAL, INTENT(INOUT) :: z
END SUBROUTINE tqli
END INTERFACE
INTERFACE
  SUBROUTINE trapzd(func,a,b,s,n)
  USE nrtype
  REAL(SP), INTENT(IN) :: a,b
  REAL(SP), INTENT(INOUT) :: s
  INTEGER(I4B), INTENT(IN) :: n
  INTERFACE
    FUNCTION func(x)
    USE nrtype
    REAL(SP), DIMENSION(:,::), INTENT(IN) :: x
    REAL(SP), DIMENSION(size(x)) :: func
  END FUNCTION func
END INTERFACE
END SUBROUTINE trapzd
END INTERFACE
INTERFACE
  SUBROUTINE tred2(a,d,e,novectors)
  USE nrtype
  REAL(SP), DIMENSION(:,::), INTENT(INOUT) :: a
  REAL(SP), DIMENSION(:,::), INTENT(OUT) :: d,e
  LOGICAL(LGT), OPTIONAL, INTENT(IN) :: novectors
END SUBROUTINE tred2
END INTERFACE
INTERFACE
  SUBROUTINE tridag_par(a,b,c,r,u)
  USE nrtype
  REAL(SP), DIMENSION(:,::), INTENT(INOUT) :: a
  REAL(SP), DIMENSION(:,::), INTENT(INOUT) :: b,c,r
  REAL(SP), DIMENSION(:,::), INTENT(OUT) :: u
END SUBROUTINE tridag_par
END INTERFACE
INTERFACE
  SUBROUTINE tridag_ser(a,b,c,r,u)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(IN) :: a,b,c,r
  REAL(SP), DIMENSION(:,), INTENT(OUT) :: u
  END SUBROUTINE tridag_ser
END INTERFACE

INTERFACE
  SUBROUTINE ttest(data1,data2,t,prob)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(IN) :: data1,data2
  REAL(SP), INTENT(OUT) :: t,prob
  END SUBROUTINE ttest
END INTERFACE

INTERFACE
  SUBROUTINE tutest(data1,data2,t,prob)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(IN) :: data1,data2
  REAL(SP), INTENT(OUT) :: t,prob
  END SUBROUTINE tutest
END INTERFACE

INTERFACE
  SUBROUTINE twofft(data1,data2,fft1,fft2)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(IN) :: data1,data2
  COMPLEX(SPC), DIMENSION(:,), INTENT(OUT) :: fft1,fft2
  END SUBROUTINE twofft
END INTERFACE

INTERFACE
  FUNCTION vander(x,q)
  USE nrtype
  REAL(DP), DIMENSION(:,), INTENT(IN) :: x,q
  REAL(DP), DIMENSION(size(x)) :: vander
  END FUNCTION vander
END INTERFACE

INTERFACE
  SUBROUTINE vegas(region,func,init,ncall,itmx,nprn,tgral,sd,chi2a)
  USE nrtype
  REAL(SP), DIMENSION(:,), INTENT(IN) :: region
  INTEGER(I4B), INTENT(IN) :: init,ncall,itmx,nprn
  REAL(SP), INTENT(OUT) :: tgral,sd,chi2a
  INTERFACE
    FUNCTION func(pt,wgt)
    USE nrtype
    REAL(SP), DIMENSION(:,), INTENT(IN) :: pt
    REAL(SP), INTENT(IN) :: wgt
    REAL(SP) :: func
    END FUNCTION func
  END INTERFACE
  END SUBROUTINE vegas
END INTERFACE

INTERFACE
  SUBROUTINE voltra(t0,h,t,f,g,ak)
  USE nrtype
  REAL(SP), INTENT(IN) :: t0,h
  REAL(SP), DIMENSION(:,), INTENT(OUT) :: t
  REAL(SP), DIMENSION(:,), INTENT(OUT) :: f
  INTERFACE
    FUNCTION g(t)
    USE nrtype
    REAL(SP), INTENT(IN) :: t
    REAL(SP), DIMENSION(:,), POINTER :: g
    END FUNCTION g
  END INTERFACE
END INTERFACE
FUNCTION ak(t,s)
USE nrtype
REAL(SP), INTENT(IN) :: t,s
REAL(SP), DIMENSION(:,::), POINTER :: ak
END FUNCTION ak
END INTERFACE
END SUBROUTINE voltra

INTERFACE
SUBROUTINE wt1(a,isign,wtstep)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: isign
INTERFACE
SUBROUTINE wtstep(a,isign)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE wtstep
END INTERFACE
END SUBROUTINE wt1
END INTERFACE

INTERFACE
SUBROUTINE wtn(a,nn,isign,wtstep)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
INTEGER(I4B), DIMENSION(:,), INTENT(IN) :: nn
INTEGER(I4B), INTENT(IN) :: isign
INTERFACE
SUBROUTINE wtstep(a,isign)
USE nrtype
REAL(SP), DIMENSION(:,), INTENT(INOUT) :: a
INTEGER(I4B), INTENT(IN) :: isign
END SUBROUTINE wtstep
END INTERFACE
END SUBROUTINE wtn
END INTERFACE

INTERFACE
FUNCTION wwghts(n,h,kermom)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
REAL(SP), INTENT(IN) :: h
REAL(SP), DIMENSION(n) :: wwghts
INTERFACE
FUNCTION kermom(y,m)
USE nrtype
REAL(DP), INTENT(IN) :: y
INTEGER(I4B), INTENT(IN) :: m
REAL(DP), DIMENSION(m) :: kermom
END FUNCTION kermom
END INTERFACE
END FUNCTION wwghts
END INTERFACE

INTERFACE
SUBROUTINE zbrac(func,x1,x2,succes)
USE nrtype
REAL(SP), INTENT(INOUT) :: x1,x2
LOGICAL(LGT), INTENT(OUT) :: succes
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
END INTERFACE
END SUBROUTINE zbrac
END INTERFACE

INTERFACE
SUBROUTINE zbrak(func,x1,x2,n,xb1,xb2,nb)
USE nrtype
INTEGER(I4B), INTENT(IN) :: n
INTEGER(I4B), INTENT(OUT) :: nb
REAL(SP), INTENT(IN) :: x1,x2
REAL(SP), DIMENSION(:), POINTER :: xb1,xb2
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
END SUBROUTINE zbrak
END INTERFACE

INTERFACE
FUNCTION zbrent(func,x1,x2,tol)
USE nrtype
REAL(SP), INTENT(IN) :: x1,x2,tol
REAL(SP) :: zbrent
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
END FUNCTION zbrent
END INTERFACE

INTERFACE
SUBROUTINE zrhqr(a,rtr,rti)
USE nrtype
REAL(SP), DIMENSION(:), INTENT(IN) :: a
REAL(SP), DIMENSION(:), INTENT(OUT) :: rtr,rti
END SUBROUTINE zrhqr
END INTERFACE

INTERFACE
FUNCTION zriddr(func,x1,x2,xacc)
USE nrtype
REAL(SP), INTENT(IN) :: x1,x2,xacc
REAL(SP) :: zriddr
INTERFACE
FUNCTION func(x)
USE nrtype
REAL(SP), INTENT(IN) :: x
REAL(SP) :: func
END FUNCTION func
END INTERFACE
END FUNCTION zriddr
END INTERFACE

INTERFACE
SUBROUTINE zroots(a,roots,polish)
USE nrtype
COMPLEX(SPC), DIMENSION(:), INTENT(IN) :: a
COMPLEX(SPC), DIMENSION(:), INTENT(OUT) :: roots
LOGICAL(LGT), INTENT(IN) :: polish
END SUBROUTINE zroots
END INTERFACE

END MODULE nr
## C3. Index of Programs and Dependencies

The following table lists, in alphabetical order, all the routines in Volume 2 of *Numerical Recipes*. When a routine requires subsidiary routines, either from this book or else user-supplied, the full dependency tree is shown: A routine calls directly all routines to which it is connected by a solid line in the column immediately to its right; it calls indirectly the connected routines in all columns to its right. Typographical conventions: Routines from this book are in typewriter font (e.g., `eulsum`, `gammln`). The smaller, slanted font is used for the second and subsequent occurrences of a routine in a single dependency tree. (When you are getting routines from the *Numerical Recipes* machine-readable media or hypertext archives, you need specify names only in the larger, upright font.) User-supplied routines are indicated by the use of text font and square brackets, e.g., `[funcv]`. Consult the text for individual specifications of these routines. The right-hand side of the table lists chapter and page numbers for each program.

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