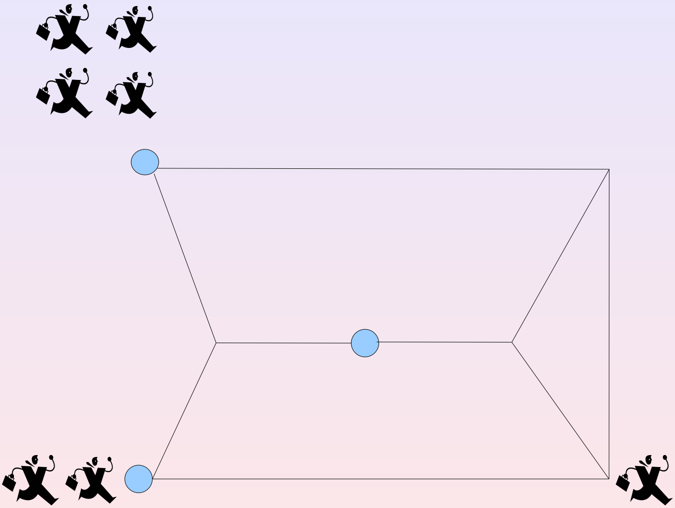
Deux problèmes sur les réseaux

1er octobre 2008

Ecole des Ponts, France

Affectation dynamique du trafic : il y a un équilibre !
avec Nicolas Wagner.

Traffic assignment



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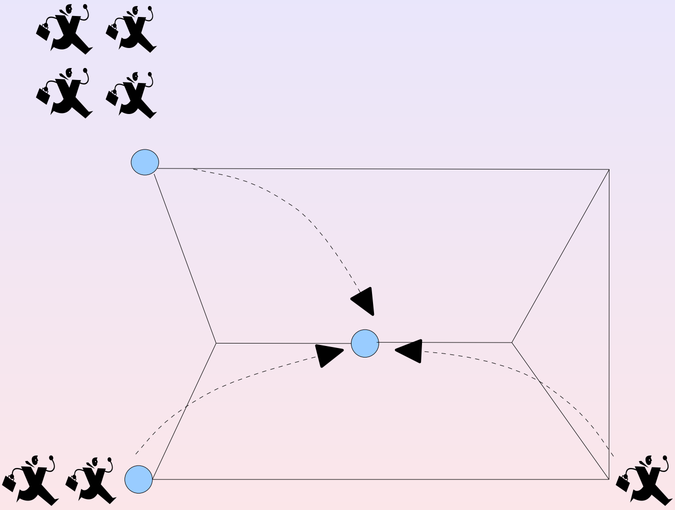

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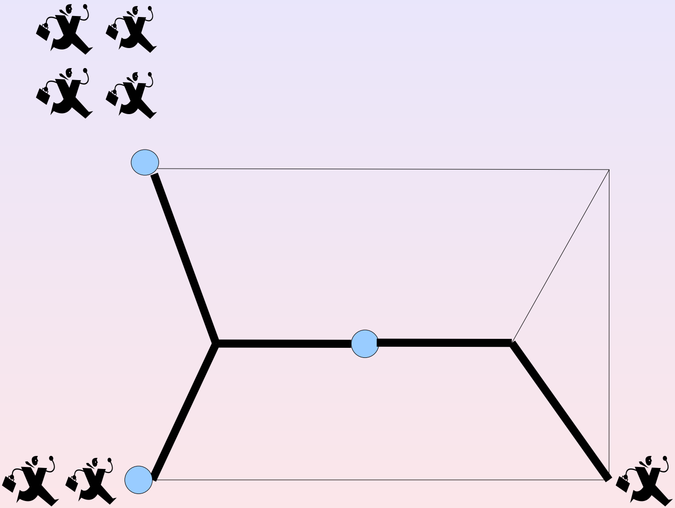

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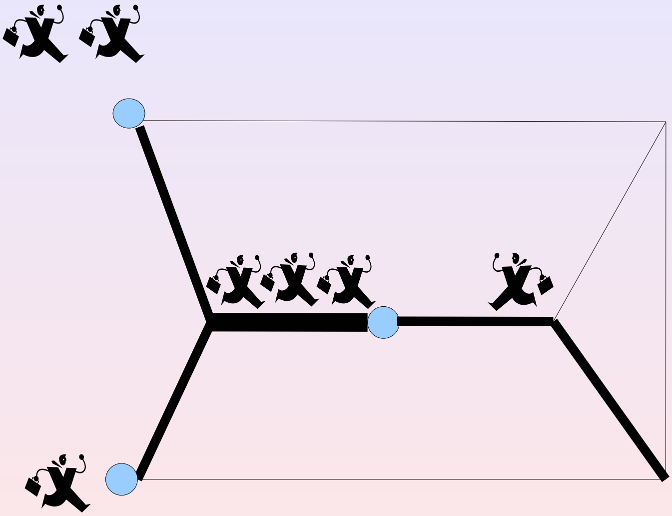

Static traffic assignment

Supply A directed graph $G = (V, A)$, a *cost function* $c_a : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ for each $a \in A$. Assumptions : c_a is non-decreasing and continuous and G is strongly connected.

Demand A *demand function* $b : V \times V \longrightarrow \mathbb{R}_+$.

Realization A *flow* $x : R \longrightarrow \mathbb{R}_+$ where $R = (R_{o,d})_{(o,d)\in V\times V}$, such that $\sum_{r\in R_{o,d}} x(r) = b(o,d)$, with $R_{o,d}$ being the set of all o–d route of G.

Equilibrium Whenever $r, r' \in R_{o,d}$, we have

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where $x_a := \sum_{r\in R:\, a\in r} x(r)$ (*Wardrop*).

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Static traffic assignment : results

Theorem

There is always an equilibrium. Moreover, the values $\boldsymbol{c_a}\left(\sum_{\boldsymbol{r}\in\boldsymbol{R}:\,\boldsymbol{a}\in\boldsymbol{r}}\boldsymbol{x}(\boldsymbol{r})\right)$ at equilibrium are unique.

Formulation as a convex program $\longrightarrow$ computable.

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This model has a limited significance :

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Dynamic traffic assignment

Since the 70's, several models has been proposed, for instance :

- Vickrey (1969)
- Merchant and Nemhauser (1978)
- Friez and al. (1989)
- 90's : Leurent (LADTA), Bellei, Gentile and Papola, Akamatsu and Kuwahara

Roughly speaking, the models are such that

Dynamic traffic assignment model

Time interval $I = [0, H]$.

Users are *dynamic flows* : $x : R \times I$. The quantity $x(r, h)$ is the number of users choosing the route r at time h.

$y_a : I \longrightarrow \mathbb{R}_+$. The quantity $y_a(h)$ is the number of users entering the arc a at time h.

Useful notation :

$$X_r(h) := \int_{h'=0}^{h} x(r, h') \quad \text{and} \quad Y_a(h) := \int_{h'=0}^{h} y_a(h')$$

(*cumulated flow*).

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Existence of an equilibrium ?

In general, it is an open question.

- Zhu and Marcotte prove an equilibrium for Friez's model (with a stronger assumptions on travel time).
- Ladta : Unknown.
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Results

Our result

- a general model with minimal assumption : continuity, fifoness (in a weak form), causality, no infinite speed.
- existence of an equilibrium that contains all previous results.

Traffic assignement as a continuous game

R : set of *routes*, $I := [0, H]$ time interval.

$\mathcal{M}(R \times I)$: set of measures on $R \times I$ (choice of time departure allowed).
$\mathcal{T}_r$: set of continuous maps $\mathcal{M}(R \times I) \to \mathcal{C}(\mathbb{R}, \mathbb{R})$ ($t_r \in \mathcal{T}_r$ is such that $t_r(\vec{X})(h)$ gives the time needed to traverse the route r).

user : characterized by a function $u : \left(\bigcup_{r \in R} \mathcal{T}_r\right) \times R \times I \to \bar{\mathbb{R}}$ (that is upper semicontinuous in r and h, and continuous in the travel times, (the *utility-function*) : $u(\vec{t}, r, h)$: a choice $(r, h) \mapsto$ the payoff, when route travel times are $\vec{t} := t_{r_1}, t_{r_2}, \ldots,$.

$\mathcal{U}$: space of users u.

A *traffic game* is a measure U on $\mathcal{U}$. *Number of users* : $N := U(\mathcal{U})$.

Traffic equilibrium

A *realization* of the traffic game is a measure on $\mathcal{U} \times \boldsymbol{R} \times \boldsymbol{I}$.
Denote $\boldsymbol{t}_{r_1}, \boldsymbol{t}_{r_2}, \ldots$ by $\vec{\boldsymbol{t}}$.

Given a traffic game $\boldsymbol{U}$, a realization $\boldsymbol{D}$ is a *Cournot-Nash equilibrium* if we have $\boldsymbol{D}_{\mathcal{U}} = \boldsymbol{U}$ and

$$\boldsymbol{D} \quad \Big\{(u, r, h) : \text{ for each } (r', h') \in \boldsymbol{R} \times \boldsymbol{I}, \\ u(\vec{\boldsymbol{t}}(\boldsymbol{D}_{\boldsymbol{R}\times\boldsymbol{I}}), r, h) \geq u(\vec{\boldsymbol{t}}(\boldsymbol{D}_{\boldsymbol{R}\times\boldsymbol{I}}), r', h')\Big\} = \boldsymbol{N}$$

$\boldsymbol{D}_{\boldsymbol{R}\times\boldsymbol{I}}$ is exactly the cumulated flows $\boldsymbol{X}$. Set
$\boldsymbol{X}_r(\boldsymbol{J}) := \boldsymbol{D}_{\boldsymbol{R}\times\boldsymbol{I}}(\{r\} \times \boldsymbol{J})$.

Khan-Mas-Colell's theorem ?

Khan-Mas-Colell's theorem tells us that there is a Nash equilibrium, provided that the utility function are continuous in the realization of the game (=here the travel times).

For traffic assignement : $D_{R\times I} \mapsto u(\vec{t}(D_{R\times I}), \cdot, \cdot)$ must be continuous.

Reformulation : Our purpose : prove that $\vec{X} \mapsto \vec{t}(\vec{X})$ is continuous.

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Arc exit time

Given $\boldsymbol{t_a}$, the *arc exit time* function is

$$H_a(Y)(h) := h + t_a(Y)(h) \quad \text{for } Y \in \mathcal{M}(\mathbb{R}) \text{ and } h \in \mathbb{R}$$

Assumptions on travel time

Continuity $H_a : \mathcal{M}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ is continuous.

Limited speed there exists $t_{\min} > 0$ such that for all $Y \in \mathcal{M}(\mathbb{R})$ and all $h \in \mathbb{R}$, we have $H_a(Y)(h) > h + t_{\min}$.

Fifo Let $h_1 < h_2$ in $\mathbb{R}$ and let $Y \in \mathcal{M}(\mathbb{R})$. Whenever $Y[h_1, h_2] \neq 0$, we have $H_a(Y)(h_1) < H_a(Y)(h_2)$.

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Arc flowing function

Let $\vec{Y}_a$ be in $\mathcal{M}(R \times \mathbb{R})$; the flow of users following route r and leaving the arc a on the time subset J is :

$$\psi_a^r(\vec{Y}_a)(J) := \psi_a(\vec{Y}_a)(\{r\} \times J) := \begin{cases} Y_a^r(H_a(Y_a)^{-1}(J)) & \text{if } a \in r \\ 0 & \text{if not.} \end{cases} \quad (1)$$

Flows on routes $\longrightarrow$ flow on arcs

An *outflow* of $\vec{X}$ is a measure $Y_a := \sum_r Y_a^r$ on $\mathbb{R}$ such that for every $r = a_1 a_2 .. a_n$:

1. $Y_{a_1}^r = X_r$
2. $Y_{a_i}^r = \psi_r^{a_{i-1}}(\vec{Y}_{a_{i-1}})$ for $i = 2..n$
3. $Y_a^r = 0$ if $a \notin r$

Uniqueness and continuity of the outflow

Proposition

Given $\vec{\boldsymbol{X}}$, there exists a unique outflow $\boldsymbol{Y_a}$. Moreover, the map $\phi_{\boldsymbol{a}} : \vec{\boldsymbol{X}} \mapsto \boldsymbol{Y_a}$ is continuous.

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Indeed

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Theorem

Given a directed graph $G = (V, A)$ with arc travel time functions $(t_a)_{a \in A}$ satisfying assumptions of causality, fifoness, limited speed and continuity and given a measure U on the set of possible users (identified with their utility function), there is a Nash equilibrium.

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A versatile model

The model is really versatile and contains previous models. Playing with the map $\boldsymbol{u(\vec{t}(\vec{X}), r, h)}$,

1. we cover also the case when there is no possible choice of time departure.
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